

Small Roots of Modular Equations

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(with thanks to Nick Howgrave-Graham, ex-Bath/NTL, and
Paul Crouch, ex-Bath now BT)

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Let $q_i = q_{u,v} = N^{h-1-v} x^u (p(x))^v$ where $v = \lfloor (i-1)/k \rfloor$ and u is the remainder. $q_{u,v}(x_0) \equiv 0 \pmod{N^{h-1}}$.

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Let M be the lower triangular $hk \times hk$ matrix $m_{i,j} = e_{i,j} X^{j-1}$ where $e_{i,j}$ is the coefficient of x^{j-1} in q_i .

$\text{Det}(M) = X^{hk(hk-1)/2} N^{hk(h-1)/2}$.

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Write the LLL-“shortest” vector as $r(xX)$, then $r(x_0) \equiv 0 \pmod{N^{h-1}}$ and

$$|x| \leq X \Rightarrow |r(x)| \leq \left(2^{\frac{hk-1}{4}} \sqrt{hk}\right) X^{\frac{hk-1}{2}} N^{\frac{h-1}{2}}.$$

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$$X = \left\lceil \left(2^{-1/2} (hk)^{-1/(hk-1)}\right) N^{(h-1)/(hk-1)} \right\rceil - 1$$

means that $r(x) < N^{h-1}$ for $|x| \leq X$. So $r(x_0) = 0$.

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As $h \rightarrow \infty$, $X \rightarrow 2^{-1/2} N^{1/k}$.

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Take polynomials $g_{ij}(x) = x^i(x - N + s_0X)^j \cdot U^{m-j}$, so $g_{ij}(x_0) \equiv 0 \pmod{\phi(N)^m}$: a congruence to an *unknown* modulus.

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($< \phi(N)^m / 2m$), we then have $h(x_0) = 0$.

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Complexity (LL) $O(\log^7 N)$ (I think!).

The general case

Write U for $de - 1$, ϕ for $\phi(N)$. Consider the polynomials

$$g_{ijk}(x, y) = x^i y^j U^{m-k} (x + y - (N + 1))^k \Big|_{xy \mapsto N}$$

- No mixed monomials
- All $\equiv 0 \pmod{\phi^m}$ when $(x, y) = (p, q)$.

Which equations?

$$g_{ijk}(x, y) = x^i y^j U^{m-k} (x + y - (N + 1))^k \Big|_{xy \mapsto N}$$

$$m + 1 \quad i = 0, j = 0, 0 \leq k \leq m$$

$$m + 1 \quad i = 1, j = 0, 0 \leq k \leq m$$

$$a - 1 \quad 1 < i \leq a, j = 0, k = m$$

$$b \quad i = 0, 1 \leq j \leq b, k = m$$

$2m + a + b + 1$ total.

Given that there are no mixed monomials, we have
 $(a + m) + (b + m) + 1$ monomials.

Assume high-order components known

Assume we know $p = p_0X + x$, $q = q_0Y + y$ with $0 \leq x < X$,
 $0 \leq y < Y$.

Recovery?

As in the univariate case, if $h(x_0, y_0) \equiv 0 \pmod{\phi^m}$ and $\|h(xX, yY)\| < \phi^m / \sqrt{w}$ where h has w monomials, then $h(x_0, y_0) = 0$ exactly.

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Roughly speaking the difference in complexity comes from the fact that we are doing 2-D guessing for p and q .

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Suppose $ed = 1 + k\phi(N)$, and approximate k by $k' = (ed - 1)/N$.

Then $0 \leq k - k' \leq 6$, and we test all: $O(\log^2 N)$.

A different scenario

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Let's consider the case of IP (an industry standard) packets being transmitted with low-exponent RSA encryption.

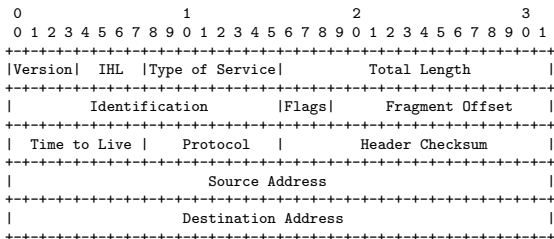
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Our aim now is to recover *individual messages* rather than break the key as such.

Figure: IP datagram, showing the fields in the IP header



Checksum = $-\sum w_i \pmod{65535}$: w_i the 16-bit words in the header.

Low-exponent RSA-encrypted IP [CD01]

Assume an IP packet m is sent as $m^d \pmod{N}$ for some small exponent d . If we can, e.g. denial of service, get two transmissions, where the identification differs by c , we have $m^d \pmod{N}$ and

$$(m + (2^{48} - 1)c2^{72})^d \pmod{N}$$

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Checksum wrapping gives us

$$(m + ((2^{48} - 1)c - 1)2^{72})^d \pmod{N}$$

again a degree d^2 equation in c , but this doesn't collapse.

Timings from [CD01]

NTL Timings in seconds to lattice reduce RedHat Linux 6.2 on 1Ghz Pentium III with 500Mb RAM							
Public exponent		e=3			e=5		
RSA-type	wrapping	512	1024	2048	512	1024	2048
IP	Without	2	9	27	8068**	177	1386
	With	653	3413	3976	†	793465	§

† Not implemented due to software restrictions.

** Taking $\alpha \leq 2^{11}$ allowed $h = 2$, with $e = 5$ this formed a 10×10 matrix which reduced in 19 seconds.

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Once we have c , we recover m by a resultant calculation.

References

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