On the Brown–Davenport construction

*In memory of Joos Heintz, 27 October 1945–3 October 2024 [1]

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Introduction: Quantifier Elimination

"Quantifier Elimination" is what it says: the problem of eliminating quantifiers, i.e. given a quantified expression (which we may as well assume is in prenex form), can we find an equivalent expression Ψ without quantifiers, i.e.

$$Q_l x_l Q_{l+1} x_{l+1} \cdots Q_n x_n \Phi(x_1, \ldots, x_n) \Leftrightarrow \Psi(x_1, \ldots, x_{l-1}), \qquad (1)$$

where $Q_i \in \{\forall, \exists\}$.
Some logics admit quantifier elimination, some do not.

Quantifier Elimination: Polynomials

Definition

An *elementary (polynomial) constraint* is a Boolean-valued function $p(x_1, \ldots)\sigma 0$ where p is a polynomial $\in \mathbf{Q}[x_1, \ldots, x_n]$ and $\sigma \in \{=, \neq, >, <, \leq, \geq\}$.

If Φ and Ψ are Boolean combinations of elementary polynomial constraints, then they define semi-algebraic sets in $\mathbf{R}^n$ (resp. $\mathbf{R}^{l-1}$), and the assertion that (1) is possible is equivalent to asserting that the projection of a semi-algebraic set is semi-algebraic [Sei54]. In fact a constructive process for (1) had already been given in the 1930s but published in 1951 [Tar51].

Useful Notation

- d Maximum degree (in each variable)
- k Number of iterations of a construction
- m Number of polynomials
- n Number of variables

The Heintz Construction

In [Hei83] Joos Heintz showed a simple construction to build $f(f(x))$ from $f(x)$ *without* using two (syntactic) copies of f. In other words, if $|\cdot|$ denotes formula length, we would normally expect $|f(f(x))| = 2|f(x)| + O(1)$, whereas Heintz has $|f(f(x))| = |f(x)| + O(1)$. Let $\Phi_i(y_i, x_i)$ be a formula defining $y_i = f_i(x_i)$. Then in order to build $f_k = f_{k-1}(f_{k-1})$ we consider

$$\Phi_k(x_k, y_k) := \exists z_k \forall x_{k-1} \forall y_{k-1} \left[\begin{array}{c} \left[\begin{array}{c} y_{k-1} = y_k \wedge x_{k-1} = z_k \\ \vee \\ y_{k-1} = z_k \wedge x_{k-1} = x_k \end{array} \right] \\ \Rightarrow \\ \Phi_{k-1}(y_{k-1}, x_{k-1}) \end{array} \right] \tag{2}$$

Expansion (Heintz)

If we move the $\vee$ symbol outside, and flatten the $\forall x_{k-1} \forall y_{k-1}$, we get $\exists z_k \Phi_{k-1}(y_k, z_k) \wedge \Phi_{k-1}(z_k, x_k)$: $y_k = f(z_k) \wedge z_k = f(x_k)$.

Using The Heintz Construction

This construction was applied in [DH88] to show that real (polynomial) quantifier elimination (RQE) had a doubly-exponential (in the number of variables) worst case. Strictly speaking, this is a doubly-exponential *bad* case. But [Col75] showed that RQE had a doubly-exponential *upper* bound on the complexity, hence, to within the crudeness of "doubly exponential", this is a worst case example. This was done by starting with the real and imaginary parts of $x_1^4 = x_2$ and building the real and imaginary parts of $x_1^{2^{2^k}} = x_2$. Such a high degree polynomial has a doubly-exponential number of roots, defined by a polynomial of degree 2^{2^n} which shows the result.

The Brown–Davenport construction

Heintz was applied in [BD07] to a linear starting point:

$$y_0 = f_0(x_0) := \begin{cases} 2x_0 & \text{for } x_0 \le 1/2 \\ 2 - 2x_0 & \text{for } x_0 > 1/2 \end{cases} \tag{3}$$

We consider $y_k = \frac{1}{2} \wedge y_k = f_k(x_k)$, where f_k is defined by applying (2) to (4). Graphs of f_0, f_1 and f_2 are shown in Figure 1.

The Brown–Davenport construction

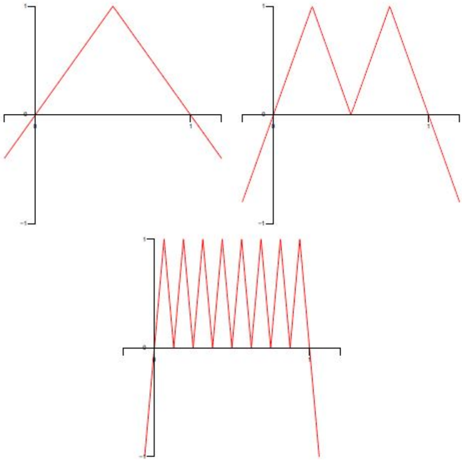

Figure 1: Plot of f_0, as defined by (2), followed by plots of the functions f_1 and f_2 defined by the recursion (1).

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f^k has 2^{2^k} lines (since each line in the "outer" f_{k-1} is replaced by $2^{2^{k-1}}$ lines from the "inner" f_{k-1}.
Then x_0 has values $\frac{1}{4}$, $\frac{3}{4}$, x_1 has values $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$ and $\frac{7}{8}$, x_2 has values $\frac{i}{32}$ for $i = 1, 3, \ldots, 31$ and so on.

An Apparent Conundrum (1)

Excluding the conditions $x \le 1/2$ and $x > 1/2$ from (4), which are there to ensure that f_0 (and hence all f_i) are functions, we see that f_0 has two equations $\underbrace{y_0 = 2x_0}_{H_1}$ and $\underbrace{y_0 = 2 - 2x_0}_{H_2}$ in the definition,

to which we add $\underbrace{y_k = \frac{1}{2}}_{H_0}$ with $k = 0$. Then our solutions are

defined by $H_0 \wedge H_1$ and $H_0 \wedge H_2$.

f_1 is obtained by adding (2) to (4) (and we change H_0 to be $y_1 = \frac{1}{2}$). Being explicit, the premise is

$$\underbrace{\left(\underbrace{y_0 = y_1}_{H_3} \wedge \underbrace{x_0 = z_1}_{H_4}\right)}_{C_A} \vee \underbrace{\left(\underbrace{y_0 = z_1}_{H_5} \wedge \underbrace{x_0 = x_1}_{H_6}\right)}_{C_B}, \tag{5}$$

seven equations in five variables. H_0 is always present, so are the solutions obtained from this and some subset of the other H_i?

An Apparent Conundrum (2)

Can subsets of $\mathcal{O}(n)$ (to be precise, $\frac{4}{3}n + \frac{1}{3}$) equations really generate a doubly exponential number of solutions? We tabulate k, the number of variables n, the number of equations E *including* the $y_k = \frac{1}{2}$ equation, the number of ways of producing a point solution by intersecting a subset of the equations (allowing for $y_k = \frac{1}{2}$ being always there), and the number of actual solutions.

Table: Figures for [BD07]

k	n	$E = \#$ equations	$\binom{E-1}{n-1}$	2^{2^k}
0	2	3	2	2
1	5	7	15	4
2	8	11	120	16
3	11	15	1001	256
4	14	19	8568	65536
5	17	23	74613	2^{32}

No, this doesn't work out!

The resolution (1)

Example

Consider the solution $y_1 = \frac{1}{2}, x_1 = \frac{1}{8}$ for the case $k = 1$. If we first consider the first disjunct of the premise in (5), we have $H_0 \wedge H_3 \wedge H_4 \wedge H_1$, we have $y_1 = \frac{1}{2}$: $y_0 = y_1 = \frac{1}{2}$; $x_0 = z_1$ and $x_0 = \frac{1}{4}$ (and therefore $z_1 = \frac{1}{4}$).
Then from the second disjunct, $H_5 \wedge H_6 \wedge H_1$, we have $y_0 = z_1 = \frac{1}{4}$, $x_1 = x_0$ and $x_0 = \frac{1}{8}$ (so $x_1 = \frac{1}{8}$).
We use H_1 twice here, with the uses coupled by z_1 occuring in both disjuncts. Replacing either or both of these with H_2 would give us the other three solutions.

The resolution (2)

We can now see what ignoring H_0, i.e. just considering

$$(5) \Rightarrow (4),$$

would be, viz.

$$f_1(x_1) := \begin{cases} 4x_1 & \text{for } x_1 \le 1/4 & H_1^* := (H_1, H_1) \\ 2 - 4x_1 & \text{for } 1/4 < x_1 \le 1/2 & H_2^* := (H_1, H_3) \\ 4x_1 - 2 & \text{for } 1/2 < x_1 \le 3/4 & H_3^* := (H_3, H_1) \\ 4 - 4x_1 & \text{for } x_1 > 3/4 & H_4^* := (H_3, H_3) \end{cases}, \quad (6)$$

where we have labelled each option with its origins in (4). This is indeed the graph for f_1 shown in Figure 1.

Then, when building the definition of f_2 (i.e. ignoring $y_2 = 1/2$), we are effectively applying the equivalent of (5) (with half-plane equation $H_7, \ldots, H_{10}$)) to (6), we can consider $H_7 \wedge H_8 \wedge H_i^*$ for four values of i, and similarly $H_9 \wedge H_{10} \wedge H_i^*$ for four values of i, giving 16 values, as in Table 1. Note that $y_2 = \frac{1}{2}, x_2 = \frac{1}{32}$ uses H_1 four times (and each of $H_3, \ldots, H_7$ twice), and so on.

The resolution (3)

Hence the Heintz construction is doing much more than allowing us different choices of the H_i (which would be the $\left(\begin{array}{c} E-1 \\ n-1 \end{array} \right)$ column in Table 1): it is also allowing re-use of the H_i (as H_1 was re-used in Example 2), to the extent that the points defined are not simple intersections of the H_i.

Variants on the Heintz construction (1)

Consider the following variation on (2).

$$\Phi_k(x_k, y_k) :=$$

$$\exists z_k, w_k \forall x_{k-1}, y_{k-1} \left[\begin{array}{c} \left[\begin{array}{c} y_{k-1} = y_k \wedge x_{k-1} = z_k \\ \vee \\ y_{k-1} = z_k \wedge x_{k-1} = w_k \\ \vee \\ y_{k-1} = w_k \wedge x_{k-1} = x_k \end{array} \right] \\ \Rightarrow \\ \Phi_{k-1}(y_{k-1}, x_{k-1}) \end{array} \right] \tag{7}$$

If we move the $\vee$ symbol outside, and flatten the $\forall x_{k-1} \forall y_{k-1}$, we get $\exists z_k, w_k \Phi_{k-1}(y_k, z_k) \wedge \Phi_{k-1}(z_k, w_k) \wedge \Phi_{k-1}(w_k, x_k)$.
If we apply (7) to (4), still interpreting $\Phi_k(y_k, x_k)$ as $y_k = f_k(x_k)$, we see that $f_1 = f_0(f_0(f_0(x_0)))$ etc. $y_1 = \frac{1}{2} \wedge \Phi_1(y_1, x_1)$ gives us eight solutions at $x_i = \frac{i}{16}$ for $i \in \{1, 3, \ldots, 15\}$. In general $y_k = \frac{k}{2} \wedge \Phi_k(y_k, x_k)$ will have 2^{3^k} solutions.

Variants on the Heintz construction (2, 3)

We can add a further variable v_k to the constuction in (7), and get $f_1 = f_0(f_0(f_0(f_0(x_0))))$, and similarly 16 solutions, and more generally 2^{4^k} solutions.
We can add a further variable u_k to the constuction in (7), and get $f_1 = f_0(f_0(f_0(f_0(f_0(x_0)))))$, and similarly 32 solutions, and more generally 2^{5^k} solutions.

Summary of Heintz variants

Table: Figures for various constructions

Mathod	Alternations	Variables	Solutions	Eff.
	a	n	s	e
[DH88, Theorem 2]	$2k-1$	$6k+2$	$2^{2^{k+1}}$	$\frac{1}{6}$
[DH88, Theorem 2']	$2k-1$	$10k+2$	$2^{2^{2k+1}}$	$\frac{1}{5}$
[BD07]	$2k-1$	$3k+2$	2^{2^k}	$\frac{1}{3}$
Variant 1	$2k-1$	$4k+2$	2^{3^k}	≈ 0.396
Variant 2	$2k-1$	$5k+2$	$2^{4^k} = 2^{2^{2k}}$	$\frac{2}{5}$
Variant 3	$2k-1$	$6k+2$	2^{5^k}	≈ 0.387

e, the "efficiency" is defined as the limit of $\log_2 \log_2(s)/n$.

Theorem 2' was about the limit of [DH88].
Similarly the second variant seems to be the limit of the [BD07] method.

Bounds

These are lower bounds.
[Col75] gave an upper bound of $e = 2$.
The Lazard projection [MPP19, BM20] gives an unconditional upper bound of $e = 1$ (whereas [McC85, Dav85], quoted in [DH88], were conditional on the system being "well-oriented").
The gap between upper $e = 1$ and lower $e = 0.4$ might not seem great, but it's in a double exponent, so corresponds to (more than) squaring the complexity.

Virtual Term Substitution [Wei88]

A direct method of quantifier elimination. Takes $\exists x_n \Phi(x_1, \ldots, x_n)$, where Φ is a pure conjunction of elementary constraints.
This covers all cases since we can transform $\forall x \Phi \Rightarrow \neg\exists x \neg\Phi$ and $\exists x \bigvee_i \Phi_i \Rightarrow \bigvee_i \exists x \Phi_i$. The degree of x in Φ is limited:

linear	[Wei88]
quadratic	[Wei97]
cubic	[Koš16]
quartic	Theoretically possible, but essentially unverifiable

For fixed $x_1, \ldots, x_{n-1}$, each constraint ϕ_i in Φ defines ≤ 1, ≤ 2 and ≤ 3 critical points. For example, if the constraint is $\lambda_i x_n + \mu_i = \geq 0$, the critical point is $x_n = c_i := -\mu_i/\lambda_i$ unless $\lambda = 0$.
Then ϕ_i is true when $x \geq c_i$ if $\lambda_i > 0$, and when $x \leq c_i$ if $\lambda_i < 0$.
If all the ϕ_i are of .this form, then Φ is feasible if all the c_i with $\lambda_i > 0$ are less than all the c_i with $\lambda_i < 0$ This leads to a (large) disjunction of possibilities on the λ_i and c_i, which are in $x_1, \ldots, x_{n-1}$.

Virtual Term Substitution [Wei88]

For higher degrees, we also need to know that the critical points are real.

For quadratic and cubic ϕ_i, the situation is similar, but more complicated in terms of the critical points if there is more than one for a ϕ_i.

The output is a large disjunction, which is not a problem for the case of $\exists y \exists x \Phi$ which becomes $\exists y \bigvee_i \Psi_i$ which is $\bigvee_i \exists y \Psi_i$.

Bot $\forall y \exists x \Phi$ becomes $\forall y \bigvee_i \Psi_i$ which is $\neg \exists y \neg \bigvee_i \Psi_i = \neg \exists y \bigwedge_i \overline{\Psi_i}$ and we have a (potentially exponential) CNF to DNF transformation before we can apply the next VTS.

From a different point of view, this also emphasises the importance of alternations in the complexity.

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