Simplification in Computer Algebra

(based on work by Davenport
/Bradford/Beaumont/Phisanbut)
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***** It doesn't help that these are called "rational functions"!

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Useful for which we generally read 'shorter'.

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If only "computer algebra' were just that — **algebra**.

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- $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$

* $x = y = 2$ gives $2 \arctan 2 = \arctan\left(\frac{-4}{3}\right)$.

Some History

- Moses (1971)
- Moses, Fitch, Fateman: simplification of algebraic expressions
- Caviness, Norman: The difficulties of simplifying transcendental expressions, the constant problem.
- 1990's Richardson: algorithms for testing zero equivalence of elementary functions.

- Fateman & Dingle (ISSAC 1994) "Branch Cuts in Computer Algebra" Introduced *The Decomposition Method*:
  we want to simplify $h = f - g$ to 0.
  (a) Calculate the branch cuts of the given function;
  (b) Choose a sample point s in each of the regions defined by the branches;
  (c) Decide $h(s) \stackrel{?}{=} 0$ numerically.

- Bradford & Davenport (ISSAC 2002) "Better Simplification of Elementary Functions".

Identified a restricted class of functions whose cuts are semi-algebraic sets.
Proposed the use of Cylindrical Algebraic Decomposition (CAD) for part (b). Examined in more detail the problem of (c).
Examined multivariate examples

- Beaumont, Bradford & Davenport (ISSAC 2003) "Better Simplification of Elementary Functions through Power Series"
Addressed problem of testing the formulae on the branch cuts numerically.

$$\sqrt{1-z}\sqrt{1+z} \stackrel{?}{=} \sqrt{1-z^2}$$

Clearly $\mathrm{Sqrt}(1-z)\,\mathrm{Sqrt}(1+z) = \mathrm{Sqrt}(1-z^2)$, so $\sqrt{1-z}\sqrt{1+z} \in \mathrm{Sqrt}(1-z^2)$.
The branch cut for $\sqrt{1-z}$ is along

$$\{z \mid \Re(z) > 1 \wedge \Im(z) = 0\}. \tag{1}$$

The branch cut for $\sqrt{1+z}$ is along

$$\{z \mid \Re(z) < -1 \wedge \Im(z) = 0\}. \tag{2}$$

Also the branch cut for $\sqrt{1-z^2}$ is along

$$\{z \mid \Re(z^2) > 1 \wedge \Im(z^2) = 0\} \tag{3}$$

Since (3) $=$ (1)$\cup$(2), there are three connected components: (1), (2) and their complement (which is connected).

On the complement, the identity is true (e.g. $z = 0$), as it is on each of the cuts (e.g. $z = \pm 2$).

$$\sqrt{z-1}\sqrt{z+1} \stackrel{?}{=} \sqrt{z^2-1}$$

Clearly $\mathrm{Sqrt}(z-1)\,\mathrm{Sqrt}(z+1) = \mathrm{Sqrt}(z^2+1)$, so $\sqrt{z-1}\sqrt{z+1} \in \mathrm{Sqrt}(z^2-1)$.
The branch cut for $\sqrt{z-1}$ is along

$$\{z \mid |\Re(z)| < 1 \wedge \Im(z) = 0\}. \tag{4}$$

The branch cut for $\sqrt{z+1}$ is along

$$\{z \mid \Re(z) < -1 \wedge \Im(z) = 0\}. \tag{5}$$

Also the branch cut for $\sqrt{z^2-1}$ is along

$$\{z \mid |\Re(z)| < 1 \wedge \Im(z) = 0\} \cup \{z \mid \Re(z) = 0\} \quad (6)$$

This last disconnects the complex plane. On $\Re(z) > 0$, the identity is true (e.g. $z = 2$). $\Re(z) \leq 0$ is itself disconnected by (4), but on each half the identity is false (e.g. $z = -1 \pm i$), except $\{\Re(z) = 0 \wedge \Im(z) > 0\}$.

The behaviour on (4) is more mysterious. The identity is false on (5), but true on (4) \ (5). In other words, the identity is false on the negative half-plane *except* on $\{z \mid -1 < \Re(z) < 0 \wedge \Im(z) = 0\}$.

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Worse: $\log(\Gamma(\overline{z})) \stackrel{?}{=} \overline{\log(\Gamma(z))}$ (A+S 6.1.23(b)).

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No, we don't have an algorithm for this one yet!

$$\arctan x + \arctan y \stackrel{?}{=} \arctan\left(\frac{x+y}{1-xy}\right)$$

False even for *real* x, y.

* But real arctan has no branch cuts
  Except at infinity!

Consider $\log(1/z) = -\log(z)$ closely

Branch cut: $B = \{z \mid \Re(z) < 0 \wedge \Im(z) = 0\}$.
On B, $\log(1/z)$ is upper-continuous, i.e.

$$\lim_{y \to 0^+} \log \frac{1}{x+iy} = \log \frac{1}{x}.$$

But $-\log(z)$ is lower-continuous, i.e.

$$\lim_{y \to 0^-} \log(x+iy) = -\log(x).$$

The branch cut is genuine, so they *must* differ on it.
The functions *adhere* differently on the branch cut.

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* Also, can prove incorrectness directly, as in $\log\frac{1}{z}\stackrel{?}{=}-\log z$.