Michael Kohlhase, OpenMath and Flexiformalism

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In order to be knowledge-driven, one must have knowledge. At this point one can become philosophical, and ask if the woodpecker "knows" that tapping on trees will bring food, etc. One can also talk about the transmission of knowledge.
Human beings used oral transmission of knowledge, and then writing came, and knowledge could be transmitted and even stored, by strange (to the uninitiated) symbols on clay tablets etc. Many symbols represented sounds (alphabets etc.), or word-like concepts (Chinese), but some represented numbers (e.g. quipu)

Origin of symbols

Until relatively recently, writing symbols just evolved.

- 1443 Invention of Hangul (한글) in Korea
- 1557 The $=$ sign [Rec57]
- 1837–1844 Morse code
- 1880s Much "set theory": $\cup, \cap$ from [Pea88, §2, pp. 1–2] and ϵ etc. from [Pea89]: see [LD24] for the use and dissemination of set theory notation
- N.B. Peano clearly had an inventive typesetter who was not a typefounder, for $\supset$ he used a rotated 'c' etc.
- 1963 ASCII, "more convenient sorting"
- 1980 TEX etc. [Knu84]
- 1991 Unicode
- 1997 MathML [Wor97, Wor99]

Overloading

Human beings will insist on using symbols to mean more than one thing. I grew up with English, whose letter↔sound mapping is ambiguous at best: consider "ough" in "though", "through", "cough", "bough" etc. (my mother had 13 variants).
This is particularly true in programming

2+3	arithmetic: 5
"2"+"3"	string concatenation: "23".
=, := or ==	A major source of errors (and cybersecurity issues)

Also in mathematics: we all know that $x^2 = x * x$ and $x^{-1} = 1/x$, but what about $\sin^2\theta$ and $\sin^{-1}\theta$?
Also deep issues with branch cuts: what is $\log(-i)$: $3\pi/2$, $-\pi/2$ or $\{(4k+3)\pi/2\}$?

OpenMath

OpenMath is an extensible standard for representing the semantics of mathematical objects.
From the beginning we have been "as formal as is sensible", rather than being fully formal like theorem-provers, but OpenMath owes Michael the coining of the word "flexiformal" and prompting Stephen Watt for its visual image.

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Types of formalism

in OpenMath — where flexiformalism comes in.

- CMP Commented Mathematical Property: text. Not formal at all
- FMP Formal Mathematical Property: OpenMath object.
-  This looks dangerously recursive (and it is if one is looking purely through the lens of formalism)
- FMP `type="defining"` A definition in the sense of *Principia Mathematica* [WR10] — "the definiendum can be replaced by the definiens"

<OMS name="log" cd="transc1">

```
<CDDefinition>
<Name>log</Name>
<Role>application</Role>
<Description>
This symbol represents a binary log function; the first arg
is the base to which the second argument is log'ed.
It is defined in Abramowitz and Stegun, Handbook of
Mathematical Functions, section 4.1
</Description>
<CMP>
  a^b = c implies log_a c = b
</CMP>
<FMP>
  <OMOBJ xmlns="http://www.openmath.org/OpenMath" version="
...
</FMP>
</CDDefinition>
```

<OMS name="arcsin" cd="transc1">

```
<Name>arcsin</Name>
<Role>application</Role>
<Description>
This symbol represents the arcsin function. This is the
inverse of the sin function as described in Abramowitz
and Stegun, section 4.4. It takes one argument.
</Description>
<CMP>
arcsin(z) = -i ln (sqrt(1-z^2)+iz)
</CMP>
<FMP>   <OMOBJ xmlns="http://www.openmath.org/OpenMath" ver
  </OMOBJ> </FMP>
<CMP> x in [-(pi/2),(pi/2)] implies arcsin(sin x) = x </CMP
<FMP><OMOBJ xmlns="http://www.openmath.org/OpenMath" versio
...
</OMOBJ></FMP>
</CDDefinition>
```

Flexiformalism and calculus functions

See [Koh08] (but it seems to be lost), [DK09].

log/ln and "the" branch cut.

Some applications (numerical tables and software, computer algebra) need a precise branch cut, much mathematics does not.

[AS64] (and Digital Library of Mathematical Functions) gives precise definitions, but informally.

Fully formal is still quite a challenge (grateful to MK for inviting me to Hausdorff last year)

However We can relate these definitions, so `<OMS name="ln" cd="transc1">` can be informal, and `<OMS name="arcsin" cd="transc1">` has an FMP in terms of this, which should be `type="defining"`.

OpenMath Society

1993	First OpenMath Workshop.
1999	Twelfth OpenMath Workshop, the first I have documented evidence of Michael attending.
2007	Elects Michael Kohlhase as President.
So	he has been OpenMath President for ≈30% of his life
N.B.	Also 450th birthday of $=$
2009–	Integration of MathML and OpenMath [DK09, etc.]: MathML has excellent presentation, but had fixed (therefore limited) semantics.
2015	Finally concluded as MathML 3.0 (2nd edition): an international standard [ISO15].
Ongoing	Growth of OpenMath [HKR11, Koh14a, Koh14b, etc.]

Knowledge snippets are not enough

We can represent a formal mathematical proof in OpenMath, as a sequence of well-formed formulae following from previous ones by rules of inference, but that's not how mathematicians actually *do*, or *represent* knowledge.

[Koh01] OMDoc: Towards an OpenMath Representation of mathematical knowledge

[KL08] OMDoc as an Ontology Language

[JK10] STEXIDE: An Integrated Development Environment for STEX Collections.

[GSMK11] The LaTeXML Daemon: Editable Math on the Collaborative Web.

[CHK+11] The LATIN Logic Atlas.

[DKL+14] Publishing and Browsing Legacy Mathematical Lecture Notes as Linked Open Data

[DKK+16] Interoperability in the OpenDreamKit Project: The Math-in-the-Middle Approach.

Beyond Knowledge representation to knowledge use in pedagogy

[SBF+00] *Adaptive course generation and presentation*

[DGKC10] eMath 3.0.

[MK14] *Semantic Technologies for Mathematical eLearning?*

[BBC+23] *Learning Support Systems Based on Mathematical Knowledge Management*

[KBB+23] *Learning with ALeA: Tailored experiences through annotated course material*

[BKM24] ALeA educational system.

[BLBK24] Examination correction at scale.

Very grateful to Michael for demonstrating his systems to me at Hausdorff last year, and I still need to develop my use of them, particularly pedagogic support such as "mouse over" symbols.

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