A Comparison of Equality in Computer Algebra and Correctness in Mathematical Pedagogy

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- "Harness the power of technology to improve teaching and learning" [AMS Notices, June 2009]. [1]

Web-based Assessment and Testing Systems

"Homework software has the potential to handle the grading of homework at a low cost. While this software has the limitation of requiring a concise answer — an algebraic expression or a multiple-choice response — it also has an important advantage over hand grading. Namely, if a students answer to a problem is wrong, the student learns of the mistake immediately and can be allowed to try the problem or a similar problem repeatedly until the right answer is obtained." [AMS]

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Marking other than true/false wa snot discussed by the MS, but seems important to us.

Typical computer aided assessment

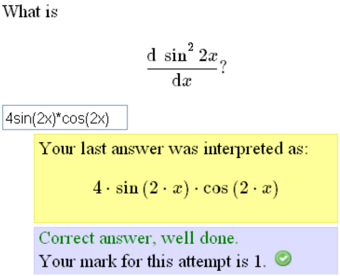

Figure: STACK system [2]

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- Probably no — a correct answer to "expand $(x+1)^2$" is

$$x^2 + \left(\max_{n\in\mathbf{N}} \exists x, y, z \in \mathbf{N}^* x^n + y^n = z^n\right) x + 1. \qquad (1)$$

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Both have their drawbacks.

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Note that answer 5 is marked wrong, since trigonometric contraction is not one of our rules. It probably should be, but we need a digression.

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- "simplify by removing brackets and collecting like terms".
- "factor and cancel like terms" (in the same book!)
- "do what I've just shown you" (often).
- "Give me the answer I want" (Classes préparatoires professors to JHD).
- [Carette 2004] "The/A shortest equivalent expression".

"The right answer" is "*a shortest expression under* $\equiv_{\mathcal{F}}$".

What is the “right” answer

Assuming we do not know about trigonometric contraction, most people would say $4 \sin 2x \cos 2x$ (or $4 \cos 2x \sin 2x$, which is equivalent under $\mathcal{U}$). But mathematically this is

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“The right answer” is “*a shortest expression under* $\equiv_{\mathcal{F}}$”. It had better be the case that only $\mathcal{U}$ can produce equivalent expressions of the same length.

Answers re-analysed

Add various fraction-simplifying rules to $\mathcal{V}$, and

U2 `sin a * cos a =` $\frac{1}{2}$`sin 2*a`.

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Table: Re-analysed answers: $\frac{\mathrm{d}\sin^2 2x}{\mathrm{d}x} = 2\sin 4x$

No.	Student's answer	relation	Score
1.	$4\sin 2x\cos 2x$	$\equiv_{\mathcal{U}}$	1
2.	$\frac{\mathrm{d}\sin^2 2x}{\mathrm{d}x}$	$\equiv_{\mathcal{F}}$	0
3.	$2\sin 2x\cos 2x$	F	buggy
4.	$2 \times 2\sin 2x\cos 2x$	$\equiv_{\mathcal{V}}$	0.8
5.	$2\sin 4x$	$=$	1
6.	$2\sin 2x\cos 2x + 2\sin 2x\cos 2x$	$\equiv_{\mathcal{V}}$	0.8
7.	$x/4 - \sin(4 * x)/8$	none	0

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Note that answer 5 is now precisely right.

Variations on a theme

- We could have added rule U2 to the set $\mathcal{V}$, rather than to $\mathcal{U}$. This would then mean that $2 \sin 4x$ was now right, but $4 \sin 2x \cos 2x$, although still mathematically correct, only scores 0.8, since it is only equivalent to the right answer under venial rules.

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- We could create a new class $\mathcal{V}'$ of "weakly venial" rules between $\mathcal{U}$ and $\mathcal{V}$, with U2 in it, and say that $\equiv_{\mathcal{V}'}$ was worth, say, 0.9 rather than 0.8.

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- Indeed, one could imagine a stronger form of $\mathcal{V}$, which cost 50% of the marks.

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Conclusions

- We can't let *any* algebra get at the student's input before we do!
- This is going to be even more important as we develop tests like 'factor', 'express as partial fractions' etc.
- This formalism may actually help a teacher *explain why*, rather than just say "I expected you to".
- We do *not* preclude use of the full power of a computer algebra system — "the system thinks your answer is right, but you'd better get it marked manually".

Future Work

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- "However, if a specified simplification of an expression is desired, as is often the case in college algebra and precalculus courses, WeBWorK cannot be used." [AMS]
- This will require a blend of syntactic analysis and the techniques mentioned above.

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