The unreasonable effectiveness of algebra

(but don't take it for granted)

James Davenport

University of Bath
(visiting Waterloo)

11 June 2009

What is the derivative of $\sin x$?

What is the derivative of $\sin x$?

$$f(x) = \lim_{\delta \to 0} \left(\frac{\sin(x+\delta) - \sin(x)}{\delta} \right)$$

What is the derivative of $\sin x$?

$$f(x) = \lim_{\delta \to 0} \left(\frac{\sin(x+\delta) - \sin(x)}{\delta} \right)$$

$$\forall \epsilon > 0 \exists D_\epsilon : \delta < D_\epsilon \Rightarrow \left| f(x) - \frac{\sin(x+\delta) - \sin(x)}{\delta} \right| < \epsilon$$

What is the derivative of $\sin x$?

$$f(x) = \lim_{\delta \to 0} \left(\frac{\sin(x+\delta) - \sin(x)}{\delta} \right)$$

$$\forall \epsilon > 0 \exists D_\epsilon : \delta < D_\epsilon \Rightarrow \left| f(x) - \frac{\sin(x+\delta) - \sin(x)}{\delta} \right| < \epsilon$$

$\sin(x+\delta) - \sin(x) = \cos(x)\sin(\delta) - \sin(x)(1-\cos(\delta))$ so we can make the choice $f(x) = \cos(x)$.

What is the derivative of $\sin x$?

$$f(x) = \lim_{\delta \to 0} \left(\frac{\sin(x+\delta) - \sin(x)}{\delta} \right)$$

$$\forall \epsilon > 0 \exists D_\epsilon : \delta < D_\epsilon \Rightarrow \left| f(x) - \frac{\sin(x+\delta) - \sin(x)}{\delta} \right| < \epsilon$$

$\sin(x+\delta) - \sin(x) = \cos(x)\sin(\delta) - \sin(x)(1 - \cos(\delta))$ so we can make the choice $f(x) = \cos(x)$. We can then take $D_\epsilon = \frac{1}{2}\min(\sqrt{\epsilon}, 1)$, and some simple algebra shows the result.

What is the derivative of $\sin x$?

$$f(x) = \lim_{\delta \to 0} \left(\frac{\sin(x+\delta) - \sin(x)}{\delta} \right)$$

$$\forall \epsilon > 0 \exists D_\epsilon : \delta < D_\epsilon \Rightarrow \left| f(x) - \frac{\sin(x+\delta) - \sin(x)}{\delta} \right| < \epsilon$$

$\sin(x+\delta) - \sin(x) = \cos(x)\sin(\delta) - \sin(x)(1 - \cos(\delta))$ so we can make the choice $f(x) = \cos(x)$. We can then take $D_\epsilon = \frac{1}{2}\min(\sqrt{\epsilon}, 1)$, and some simple algebra shows the result. Did any of you do that?

What is the derivative of $\sin x$?

$$f(x) = \lim_{\delta \to 0} \left(\frac{\sin(x+\delta) - \sin(x)}{\delta} \right)$$

$$\forall \epsilon > 0 \exists D_\epsilon : \delta < D_\epsilon \Rightarrow \left| f(x) - \frac{\sin(x+\delta) - \sin(x)}{\delta} \right| < \epsilon$$

$\sin(x+\delta) - \sin(x) = \cos(x)\sin(\delta) - \sin(x)(1-\cos(\delta))$ so we can make the choice $f(x) = \cos(x)$. We can then take $D_\epsilon = \frac{1}{2}\min(\sqrt{\epsilon}, 1)$, and some simple algebra shows the result.

Did any of you do that?

Did any of you take $D_\epsilon = \frac{\min(\sqrt{\epsilon},1)}{2\max(|\cos(x)|,|\sin(x)|,1)}$ to allow for $x \in \mathbf{C}$?

What is the derivative of $\sin x$?

$$f(x) = \lim_{\delta \to 0} \left(\frac{\sin(x+\delta) - \sin(x)}{\delta} \right)$$

$$\forall \epsilon > 0 \exists D_\epsilon : \delta < D_\epsilon \Rightarrow \left| f(x) - \frac{\sin(x+\delta) - \sin(x)}{\delta} \right| < \epsilon$$

$\sin(x+\delta) - \sin(x) = \cos(x)\sin(\delta) - \sin(x)(1-\cos(\delta))$ so we can make the choice $f(x) = \cos(x)$. We can then take $D_\epsilon = \frac{1}{2}\min(\sqrt{\epsilon}, 1)$, and some simple algebra shows the result.

Did any of you do that?

Did any of you take $D_\epsilon = \frac{\min(\sqrt{\epsilon},1)}{2\max(|\cos(x)|,|\sin(x)|,1)}$ to allow for $x \in \mathbf{C}$?

Or did you "just know it"?

What is the integral: $\int_0^{\pi/2} \cos x$?

What is the integral: $\int_0^{\pi/2} \cos x$?

$$I = \lim_{|\Delta| \to 0} \overline{S}_\Delta [\cos(x)] = \lim_{|\Delta| \to 0} \underline{S}_\Delta [\cos(x)]$$

(Δ ranges over all dissections of $[0, \pi/2]$)

What is the integral: $\int_0^{\pi/2} \cos x$?

$$I = \lim_{|\Delta| \to 0} \overline{S}_\Delta \left[\cos(x)\right] = \lim_{|\Delta| \to 0} \underline{S}_\Delta \left[\cos(x)\right]$$

(Δ ranges over all dissections of $[0, \pi/2]$)

Does anyone actually do this?

What is the integral: $\int_0^{\pi/2} \cos x$?

$$I = \lim_{|\Delta|\to 0} \overline{S}_\Delta [\cos(x)] = \lim_{|\Delta|\to 0} \underline{S}_\Delta [\cos(x)]$$

(Δ ranges over all dissections of $[0, \pi/2]$)
Does anyone actually do this?
Can anyone actually do this?

What is the integral: $\int_0^{\pi/2} \cos x$?

$$I = \lim_{|\Delta| \to 0} \overline{S}_\Delta [\cos(x)] = \lim_{|\Delta| \to 0} \underline{S}_\Delta [\cos(x)]$$

(Δ ranges over all dissections of $[0, \pi/2]$)

Does anyone actually do this?

Can anyone actually do this?

Or do we say "well, we have proved that $\sin' = \cos$, so $\int \cos = \sin$, and therefore the answer is $\sin \frac{\pi}{2} - \sin 0 = 1$?

What is the integral: $\int_0^{\pi/2} \cos x$?

$$I = \lim_{|\Delta| \to 0} \overline{S}_\Delta [\cos(x)] = \lim_{|\Delta| \to 0} \underline{S}_\Delta [\cos(x)]$$

(Δ ranges over all dissections of $[0, \pi/2]$)

Does anyone actually do this?

Can anyone actually do this?

Or do we say "well, we have proved that $\sin' = \cos$, so $\int \cos = \sin$, and therefore the answer is $\sin \frac{\pi}{2} - \sin 0 = 1$?

We might say "therefore, by the Fundamental Theorem of Calculus, ..."

What is the Fundamental Theorem of Calculus?

What is the Fundamental Theorem of Calculus?

Indeed, what is calculus?

What is the Fundamental Theorem of Calculus?

Indeed, what is calculus?

> *. . . the deism of Liebniz over the dotage of Newton . . .*
> *[Babbage, chapter 4]*

What is the Fundamental Theorem of Calculus?

Indeed, what is calculus?

> *. . . the deism of Liebniz over the dotage of Newton . . .*
> *[Babbage, chapter 4]*

I claim that calculus is actually the interesting fusion of *two, different* subjects.

What is the Fundamental Theorem of Calculus?

Indeed, what is calculus?

> *. . . the deism of Liebniz over the dotage of Newton . . .*
> *[Babbage, chapter 4]*

I claim that calculus is actually the interesting fusion of *two, different* subjects.

- What you learned in calculus, which *I* shall write as $D_{\epsilon\delta}$: the "differentiation of ϵ–δ analysis". Also $\frac{\mathrm{d}}{\mathrm{d}_{\epsilon\delta} x}$, and its inverse ${}_{\epsilon\delta}\int$.

What is the Fundamental Theorem of Calculus?

Indeed, what is calculus?

> *... the deism of Liebniz over the dotage of Newton ...*
> *[Babbage, chapter 4]*

I claim that calculus is actually the interesting fusion of *two, different* subjects.

- What you learned in calculus, which *I* shall write as $D_{\epsilon\delta}$: the "differentiation of ϵ–δ analysis". Also $\frac{\mathrm{d}}{\mathrm{d}_{\epsilon\delta}x}$, and its inverse ${}_{\epsilon\delta}\int$.
- What is taught in differential algebra, which *I* shall write as D_{DA}: the "differentiation of differential algebra". Also $\frac{\mathrm{d}}{\mathrm{d}_{\mathrm{DA}}x}$, and its inverse ${}_{\mathrm{DA}}\int$.

$D_{\epsilon\delta}$ (for functions $\mathbf{R} \to \mathbf{R}$)

$D_{\epsilon\delta}$ (for functions $\mathbf{R} \to \mathbf{R}$)

Define $\mathrm{CL}(f, x_0)$ (the "Cauchy Limit") as

$$\mathrm{CL}(f, x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and $D_{\epsilon\delta}(f) = \lambda x.\, \mathrm{CL}(f, x)$.

$D_{\epsilon\delta}$ (for functions $\mathbf{R} \to \mathbf{R}$)

Define $\mathrm{CL}(f, x_0)$ (the "Cauchy Limit") as

$$\mathrm{CL}(f, x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and $D_{\epsilon\delta}(f) = \lambda x.\, \mathrm{CL}(f, x)$. Then the following are theorems.

$D_{\epsilon\delta}$ (for functions $\mathbf{R} \to \mathbf{R}$)

Define $\mathrm{CL}(f, x_0)$ (the "Cauchy Limit") as

$$\mathrm{CL}(f, x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and $D_{\epsilon\delta}(f) = \lambda x.\, \mathrm{CL}(f, x)$. Then the following are theorems.

- $D_{\epsilon\delta}(c) = 0 \qquad \forall c$ constants.

$D_{\epsilon\delta}$ (for functions $\mathbf{R} \to \mathbf{R}$)

Define $\mathrm{CL}(f, x_0)$ (the "Cauchy Limit") as

$$\mathrm{CL}(f, x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and $D_{\epsilon\delta}(f) = \lambda x.\, \mathrm{CL}(f, x)$. Then the following are theorems.

- $D_{\epsilon\delta}(c) = 0 \qquad \forall c$ constants.
- $\mathrm{CL}(f + g, x_0) = \mathrm{CL}(f, x_0) + \mathrm{CL}(g, x_0)$, so $D_{\epsilon\delta}(f + g) = D_{\epsilon\delta}(f) + D_{\epsilon\delta}(g)$.

$D_{\epsilon\delta}$ (for functions $\mathbf{R} \to \mathbf{R}$)

Define $\mathrm{CL}(f, x_0)$ (the "Cauchy Limit") as

$$\mathrm{CL}(f, x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and $D_{\epsilon\delta}(f) = \lambda x.\, \mathrm{CL}(f, x)$. Then the following are theorems.

- $D_{\epsilon\delta}(c) = 0 \qquad \forall c$ constants.
- $\mathrm{CL}(f + g, x_0) = \mathrm{CL}(f, x_0) + \mathrm{CL}(g, x_0)$, so $D_{\epsilon\delta}(f + g) = D_{\epsilon\delta}(f) + D_{\epsilon\delta}(g)$.
- $\mathrm{CL}(fg, x_0) = \mathrm{CL}(f, x_0)g(x_0) + f(x_0)\,\mathrm{CL}(g, x_0)$, so $D_{\epsilon\delta}(fg) = D_{\epsilon\delta}(f)g + fD_{\epsilon\delta}(g)$.

$D_{\epsilon\delta}$ (for functions $\mathbf{R} \to \mathbf{R}$)

Define $\mathrm{CL}(f, x_0)$ (the "Cauchy Limit") as

$$\mathrm{CL}(f, x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and $D_{\epsilon\delta}(f) = \lambda x.\, \mathrm{CL}(f, x)$. Then the following are theorems.

- $D_{\epsilon\delta}(c) = 0 \qquad \forall c$ constants.
- $\mathrm{CL}(f + g, x_0) = \mathrm{CL}(f, x_0) + \mathrm{CL}(g, x_0)$, so $D_{\epsilon\delta}(f + g) = D_{\epsilon\delta}(f) + D_{\epsilon\delta}(g)$.
- $\mathrm{CL}(fg, x_0) = \mathrm{CL}(f, x_0)g(x_0) + f(x_0)\,\mathrm{CL}(g, x_0)$, so $D_{\epsilon\delta}(fg) = D_{\epsilon\delta}(f)g + fD_{\epsilon\delta}(g)$.
- $D_{\epsilon\delta}(\lambda x.f(g(x))) = D_{\epsilon\delta}(g)\lambda x.D_{\epsilon\delta}(f)(g(x))$. (Chain rule)

D_{DA} (for *any* ring R of characteristic 0)

Definition: $D_{\mathrm{DA}} : R \to R$ is a *derivation* on R if it satisfies:

D_{DA} (for *any* ring R of characteristic 0)

Definition: $D_{\mathrm{DA}} : R \longrightarrow R$ is a *derivation* on R if it satisfies:

- $D_{\mathrm{DA}}(f + g) = D_{\mathrm{DA}}(f) + D_{\mathrm{DA}}(g)$

D_{DA} (for *any* ring R of characteristic 0)

Definition: $D_{\mathrm{DA}} : R \longrightarrow R$ is a *derivation* on R if it satisfies:

- $D_{\mathrm{DA}}(f+g) = D_{\mathrm{DA}}(f) + D_{\mathrm{DA}}(g)$
- $D_{\mathrm{DA}}(fg) = D_{\mathrm{DA}}(f)g + fD_{\mathrm{DA}}(g)$

D_{DA} (for *any* ring R of characteristic 0)

Definition: $D_{\mathrm{DA}} : R \longrightarrow R$ is a *derivation* on R if it satisfies:

- $D_{\mathrm{DA}}(f+g) = D_{\mathrm{DA}}(f) + D_{\mathrm{DA}}(g)$
- $D_{\mathrm{DA}}(fg) = D_{\mathrm{DA}}(f)g + fD_{\mathrm{DA}}(g)$

D_{DA} (for *any* ring R of characteristic 0)

Definition: $D_{\mathrm{DA}} : R \to R$ is a *derivation* on R if it satisfies:

- $D_{\mathrm{DA}}(f+g) = D_{\mathrm{DA}}(f) + D_{\mathrm{DA}}(g)$
- $D_{\mathrm{DA}}(fg) = D_{\mathrm{DA}}(f)g + fD_{\mathrm{DA}}(g)$

Corollaries:

D_{DA} (for *any* ring R of characteristic 0)

Definition: $D_{\mathrm{DA}} : R \to R$ is a *derivation* on R if it satisfies:

- $D_{\mathrm{DA}}(f+g) = D_{\mathrm{DA}}(f) + D_{\mathrm{DA}}(g)$
- $D_{\mathrm{DA}}(fg) = D_{\mathrm{DA}}(f)g + fD_{\mathrm{DA}}(g)$

Corollaries:

- $D_{\mathrm{DA}}(c) = 0$ for all c algebraic over $\langle 1 \rangle$.

D_{DA} (for *any* ring R of characteristic 0)

Definition: $D_{\mathrm{DA}} : R \rightarrow R$ is a *derivation* on R if it satisfies:

- $D_{\mathrm{DA}}(f+g) = D_{\mathrm{DA}}(f) + D_{\mathrm{DA}}(g)$
- $D_{\mathrm{DA}}(fg) = D_{\mathrm{DA}}(f)g + fD_{\mathrm{DA}}(g)$

Corollaries:

- $D_{\mathrm{DA}}(c) = 0$ for all c algebraic over $\langle 1 \rangle$.
- $D_{\mathrm{DA}}(\frac{f}{g}) = \frac{D_{\mathrm{DA}}(f)g - fD_{\mathrm{DA}}(g)}{g^2}$.

D_{DA} (for *any* ring R of characteristic 0)

Definition: $D_{\mathrm{DA}} : R \to R$ is a *derivation* on R if it satisfies:

- $D_{\mathrm{DA}}(f+g) = D_{\mathrm{DA}}(f) + D_{\mathrm{DA}}(g)$
- $D_{\mathrm{DA}}(fg) = D_{\mathrm{DA}}(f)g + fD_{\mathrm{DA}}(g)$

Corollaries:

- $D_{\mathrm{DA}}(c) = 0$ for all c algebraic over $\langle 1 \rangle$.
- $D_{\mathrm{DA}}(\frac{f}{g}) = \frac{D_{\mathrm{DA}}(f)g - fD_{\mathrm{DA}}(g)}{g^2}$.
- D_{DA} extends uniquely to algebraic extensions.

D_{DA} (for *any* ring R of characteristic 0)

Definition: $D_{\mathrm{DA}} : R \rightarrow R$ is a *derivation* on R if it satisfies:

- $D_{\mathrm{DA}}(f + g) = D_{\mathrm{DA}}(f) + D_{\mathrm{DA}}(g)$
- $D_{\mathrm{DA}}(fg) = D_{\mathrm{DA}}(f)g + fD_{\mathrm{DA}}(g)$

Corollaries:

- $D_{\mathrm{DA}}(c) = 0$ for all c algebraic over $\langle 1 \rangle$.
- $D_{\mathrm{DA}}(\frac{f}{g}) = \frac{D_{\mathrm{DA}}(f)g - fD_{\mathrm{DA}}(g)}{g^2}$.
- D_{DA} extends uniquely to algebraic extensions.

D_{DA} (for *any* ring R of characteristic 0)

Definition: $D_{\mathrm{DA}} : R \rightarrow R$ is a *derivation* on R if it satisfies:

- $D_{\mathrm{DA}}(f+g) = D_{\mathrm{DA}}(f) + D_{\mathrm{DA}}(g)$
- $D_{\mathrm{DA}}(fg) = D_{\mathrm{DA}}(f)g + fD_{\mathrm{DA}}(g)$

Corollaries:

- $D_{\mathrm{DA}}(c) = 0$ for all c algebraic over $\langle 1 \rangle$.
- $D_{\mathrm{DA}}(\frac{f}{g}) = \frac{D_{\mathrm{DA}}(f)g - fD_{\mathrm{DA}}(g)}{g^2}$.
- D_{DA} extends uniquely to algebraic extensions.

Note that there is no Chain Rule as such, since composition is not necessarily a defined concept on R.

A note on the word "constant"

A note on the word "constant"

We said

- $D_{\mathrm{DA}}(c) = 0$ for all c algebraic over $\langle 1 \rangle$.

A note on the word “constant”

We said

- $D_{\mathrm{DA}}(c) = 0$ for all c algebraic over $\langle 1 \rangle$.
- “constants differentiate to 0”

A note on the word "constant"

We said

- $D_{\mathrm{DA}}(c) = 0$ for all c algebraic over $\langle 1 \rangle$.
- "constants differentiate to 0"

A note on the word "constant"

We said

- $D_{\mathrm{DA}}(c) = 0$ for all c algebraic over $\langle 1 \rangle$.
- "constants differentiate to 0"

By **abuse** of language, we say that anything that differentiates to zero is a "constant$_{\mathrm{DA}}$".

How are the two subjects related?

If we define $D_{\mathrm{DA}}(x) = 1$, then D_{DA} is defined on $\mathbf{Z}[x]$, and extends to $\mathbf{Q}(x)$ and indeed $\overline{\mathbf{Q}(x)}$.

How are the two subjects related?

If we define $D_{\mathrm{DA}}(x) = 1$, then D_{DA} is defined on $\mathbf{Z}[x]$, and extends to $\mathbf{Q}(x)$ and indeed $\overline{\mathbf{Q}(x)}$.

If we interpret (denoted $\mathcal{I}$) $\mathbf{Q}(x)$ as functions $\mathbf{R} \to \mathbf{R}$, then D_{DA} can be interpreted as $D_{\epsilon\delta}$, i.e.

$$\mathcal{I}(D_{\mathrm{DA}}(f)) = D_{\epsilon\delta}(\mathcal{I}(f))$$

How are the two subjects related?

If we define $D_{\mathrm{DA}}(x) = 1$, then D_{DA} is defined on $\mathbf{Z}[x]$, and extends to $\mathbf{Q}(x)$ and indeed $\overline{\mathbf{Q}(x)}$.
If we interpret (denoted $\mathcal{I}$) $\mathbf{Q}(x)$ as functions $\mathbf{R} \to \mathbf{R}$, then D_{DA} can be interpreted as $D_{\epsilon\delta}$, i.e.

$$\mathcal{I}(D_{\mathrm{DA}}(f)) = D_{\epsilon\delta}(\mathcal{I}(f))$$

(at least up to removable singularities).

An aside on interpretation

Consider $L := \frac{(x-1)^2}{x^2-1}$ and $R := \frac{x-1}{x+1}$.

An aside on interpretation

Consider $L := \frac{(x-1)^2}{x^2-1}$ and $R := \frac{x-1}{x+1}$.
As elements of $\mathbf{Q}(x)$ they are equal, since $L - R = \frac{0}{x^2-1} = 0$.

An aside on interpretation

Consider $L := \frac{(x-1)^2}{x^2-1}$ and $R := \frac{x-1}{x+1}$.
As elements of $\mathbf{Q}(x)$ they are equal, since $L - R = \frac{0}{x^2-1} = 0$.
But as formulae (viewed as algorithm specifications), $L(1) =$ "divide by zero error", whereas $R(1) = 0$.

An aside on interpretation

Consider $L := \frac{(x-1)^2}{x^2-1}$ and $R := \frac{x-1}{x+1}$.
As elements of $\mathbf{Q}(x)$ they are equal, since $L - R = \frac{0}{x^2-1} = 0$.
But as formulae (viewed as algorithm specifications), $L(1)=$"divide by zero error", whereas $R(1) = 0$.
Hence the warning about removable singularities!

An aside on interpretation

Consider $L := \frac{(x-1)^2}{x^2-1}$ and $R := \frac{x-1}{x+1}$.
As elements of $\mathbf{Q}(x)$ they are equal, since $L - R = \frac{0}{x^2-1} = 0$.
But as formulae (viewed as algorithm specifications), $L(1)=$"divide by zero error", whereas $R(1) = 0$.
Hence the warning about removable singularities!
Also, note that -1 is *not* a removable singularity!

An aside on interpretation

Consider $L := \frac{(x-1)^2}{x^2-1}$ and $R := \frac{x-1}{x+1}$.
As elements of $\mathbf{Q}(x)$ they are equal, since $L - R = \frac{0}{x^2-1} = 0$.
But as formulae (viewed as algorithm specifications), $L(1)$="divide by zero error", whereas $R(1) = 0$.
Hence the warning about removable singularities!
Also, note that -1 is *not* a removable singularity!
In the case of $\mathbf{Q}(x)$, every element has a "most continuous formula", so interpreting this as $\mathbf{R} \rightarrow \mathbf{R}$ can't go wrong.

An aside on interpretation

Consider $L := \frac{(x-1)^2}{x^2-1}$ and $R := \frac{x-1}{x+1}$.
As elements of $\mathbf{Q}(x)$ they are equal, since $L - R = \frac{0}{x^2-1} = 0$.
But as formulae (viewed as algorithm specifications), $L(1)=$"divide by zero error", whereas $R(1) = 0$.
Hence the warning about removable singularities!
Also, note that -1 is *not* a removable singularity!
In the case of $\mathbf{Q}(x)$, every element has a "most continuous formula", so interpreting this as $\mathbf{R} \to \mathbf{R}$ can't go wrong.
We use $\mathcal{I}^*$ to indicate "interpretation with removable singularities removed", since there isn't always a formulaic way of doing so.

An aside on interpretation

Consider $L := \frac{(x-1)^2}{x^2-1}$ and $R := \frac{x-1}{x+1}$.
As elements of $\mathbf{Q}(x)$ they are equal, since $L - R = \frac{0}{x^2-1} = 0$.
But as formulae (viewed as algorithm specifications), $L(1)=$"divide by zero error", whereas $R(1) = 0$.
Hence the warning about removable singularities!
Also, note that -1 is *not* a removable singularity!
In the case of $\mathbf{Q}(x)$, every element has a "most continuous formula", so interpreting this as $\mathbf{R} \rightarrow \mathbf{R}$ can't go wrong.
We use $\mathcal{I}^*$ to indicate "interpretation with removable singularities removed", since there isn't always a formulaic way of doing so.
Indeed, there may be *no* way of interpreting without singularities, as in $z \mapsto \sqrt{z} : \mathbf{C} \rightarrow \mathbf{C}$.

${}_{\epsilon\delta}\int$ (for functions $\mathbf{R} \to \mathbf{R}$)

${}_{\epsilon\delta}\int$ (for functions $\mathbf{R} \to \mathbf{R}$)

What is naturally defined is integration over an interval I. We let D stand for sub-divisions $d_1 = a < d_2 < \cdots < d_n = b$ of $I = [a, b]$, and $|D|$ for the largest distance between neighbouring points in D, i.e. $\max_i(d_{i+1} - d_i)$.

${}_{\epsilon\delta}\int$ (for functions $\mathbf{R} \to \mathbf{R}$)

What is naturally defined is integration over an interval I. We let D stand for sub-divisions $d_1 = a < d_2 < \cdots < d_n = b$ of $I = [a, b]$, and $|D|$ for the largest distance between neighbouring points in D, i.e. $\max_i(d_{i+1} - d_i)$. Let

- $\overline{S_D} = \sum_i (d_{i+1} - d_i) \max_{d_{i+1} \geq x \geq d_i} f(x)$;

${}_{\epsilon\delta}\int$ (for functions $\mathbf{R} \to \mathbf{R}$)

What is naturally defined is integration over an interval I. We let D stand for sub-divisions $d_1 = a < d_2 < \cdots < d_n = b$ of $I = [a, b]$, and $|D|$ for the largest distance between neighbouring points in D, i.e. $\max_i(d_{i+1} - d_i)$. Let

- $\overline{S_D} = \sum_i (d_{i+1} - d_i) \max_{d_{i+1} \geq x \geq d_i} f(x)$;
- $\underline{S_D} = \sum_i (d_{i+1} - d_i) \min_{d_{i+1} \geq x \geq d_i} f(x)$;

${}_{\epsilon\delta}\int$ (for functions $\mathbf{R} \to \mathbf{R}$)

What is naturally defined is integration over an interval I. We let D stand for sub-divisions $d_1 = a < d_2 < \cdots < d_n = b$ of $I = [a, b]$, and $|D|$ for the largest distance between neighbouring points in D, i.e. $\max_i(d_{i+1} - d_i)$. Let

- $\overline{S_D} = \sum_i (d_{i+1} - d_i) \max_{d_{i+1} \geq x \geq d_i} f(x)$;
- $\underline{S_D} = \sum_i (d_{i+1} - d_i) \min_{d_{i+1} \geq x \geq d_i} f(x)$;

Then ${}_{\epsilon\delta}\int_I f = \liminf_{|D| \to 0} \overline{S_D} = \limsup_{|D| \to 0} \underline{S_D}$ *if* both exist and are equal.

Consequences: Fundamental Theorem of Calculus (FTC$_{\epsilon\delta}$)

Define ${}_{\epsilon\delta}\int_a^b f = \begin{cases} {}_{\epsilon\delta}\int_{[a,b]} f & a \leq b \\ -{}_{\epsilon\delta}\int_{[b,a]} f & a > b \end{cases}$

Consequences: Fundamental Theorem of Calculus (FTC$_{\epsilon\delta}$)

Define $_{\epsilon\delta}\int_a^b f = \begin{cases} _{\epsilon\delta}\int_{[a,b]} f & a \leq b \\ -_{\epsilon\delta}\int_{[b,a]} f & a > b \end{cases}$

(If we're not careful, we wind up saying "[2,1] is the same set as [1,2] except that if you integrate over it you have to add a $-$ sign". What we really have here is the beginnings of contour integration.

Consequences: Fundamental Theorem of Calculus ($FTC_{\epsilon\delta}$)

Define ${}_{\epsilon\delta}\int_a^b f = \begin{cases} {}_{\epsilon\delta}\int_{[a,b]} f & a \leq b \\ -{}_{\epsilon\delta}\int_{[b,a]} f & a > b \end{cases}$

(If we're not careful, we wind up saying "[2,1] is the same set as [1,2] except that if you integrate over it you have to add a $-$ sign". What we really have here is the beginnings of contour integration. Apostol theorem 1.20 states that if $g(x) \leq f(x)$ for every x in $[a, b]$, then

$$\int_a^b g(x)dx \leq \int_a^b f(x)dx,$$

and here $a < b$ is implicit.)

Consequences: Fundamental Theorem of Calculus (FTC$_{\epsilon\delta}$)

Define ${}_{\epsilon\delta}\int_a^b f = \begin{cases} {}_{\epsilon\delta}\int_{[a,b]} f & a \leq b \\ -{}_{\epsilon\delta}\int_{[b,a]} f & a > b \end{cases}$

(If we're not careful, we wind up saying "[2,1] is the same set as [1,2] except that if you integrate over it you have to add a $-$ sign". What we really have here is the beginnings of contour integration. Apostol theorem 1.20 states that if $g(x) \leq f(x)$ for every x in $[a, b]$, then

$$\int_a^b g(x)dx \leq \int_a^b f(x)dx,$$

and here $a < b$ is implicit.)

FTC$_{\epsilon\delta}$: ${}_{\epsilon\delta}\int_{[a,b]} D_{\epsilon\delta} f = f(b) - f(a)$.

Consequences: Fundamental Theorem of Calculus ($\text{FTC}_{\epsilon\delta}$)

Define ${}_{\epsilon\delta}\int_a^b f = \begin{cases} {}_{\epsilon\delta}\int_{[a,b]} f & a \le b \\ -{}_{\epsilon\delta}\int_{[b,a]} f & a > b \end{cases}$

(If we're not careful, we wind up saying "[2,1] is the same set as [1,2] except that if you integrate over it you have to add a $-$ sign". What we really have here is the beginnings of contour integration. Apostol theorem 1.20 states that if $g(x) \le f(x)$ for every x in $[a, b]$, then

$$\int_a^b g(x)dx \le \int_a^b f(x)dx,$$

and here $a < b$ is implicit.)

$\text{FTC}_{\epsilon\delta}$: ${}_{\epsilon\delta}\int_{[a,b]} D_{\epsilon\delta} f = f(b) - f(a)$.

Or: $D_{\epsilon\delta}(\lambda x. {}_{\epsilon\delta}\int_a^x f) = f$ (if the ${}_{\epsilon\delta}\int$ exists).

Consequences: Fundamental Theorem of Calculus (FTC$_{\epsilon\delta}$)

Define ${}_{\epsilon\delta}\int_a^b f = \begin{cases} {}_{\epsilon\delta}\int_{[a,b]} f & a \leq b \\ -{}_{\epsilon\delta}\int_{[b,a]} f & a > b \end{cases}$

(If we're not careful, we wind up saying "[2,1] is the same set as [1,2] except that if you integrate over it you have to add a $-$ sign". What we really have here is the beginnings of contour integration. Apostol theorem 1.20 states that if $g(x) \leq f(x)$ for every x in $[a, b]$, then

$$\int_a^b g(x)dx \leq \int_a^b f(x)dx,$$

and here $a < b$ is implicit.)

FTC$_{\epsilon\delta}$: ${}_{\epsilon\delta}\int_{[a,b]} D_{\epsilon\delta} f = f(b) - f(a)$.

Or: $D_{\epsilon\delta}(\lambda x. {}_{\epsilon\delta}\int_a^x f) = f$ (if the ${}_{\epsilon\delta}\int$ exists).

(Of course, this is normally stated without the λ.)

$_{\mathrm{DA}}\int$: FTC becomes a definition

${}_{\mathrm{DA}}\int$: FTC becomes a definition

FTC_{DA}: **define** ${}_{\mathrm{DA}}\int f$ to be *any* g such that $f = D_{\mathrm{DA}} g$.

${}_{\mathrm{DA}}\int$: FTC becomes a definition

FTC$_{DA}$: **define** ${}_{\mathrm{DA}}\int f$ to be *any* g such that $f = D_{\mathrm{DA}} g$.
If g and h are two such integrals, then $D_{\mathrm{DA}}(g - h) = 0$, i.e. "$g$ and h differ by a constant".

$_{\mathrm{DA}}\int$: FTC becomes a definition

FTC$_{DA}$: **define** $_{\mathrm{DA}}\int f$ to be *any* g such that $f = D_{\mathrm{DA}}g$.
If g and h are two such integrals, then $D_{\mathrm{DA}}(g-h)=0$, i.e. "g and h differ by a constant".
One difficulty is that this is really a "constant$_{\mathrm{DA}}$" (something whose D_{DA} is zero), and, for example, a Heaviside function is a constant$_{\mathrm{DA}}$, though not a constant in the usual sense.

Will the real FTC please stand up?

Will the real FTC please stand up?

FTC (as it *should* be taught).

Will the real FTC please stand up?

FTC (as it *should* be taught).
If $g =_{\mathrm{DA}} \int f$, and $\mathcal{I}(g)$ is continuous on $[a, b]$, then

$$_{\epsilon\delta}\int_a^b \mathcal{I}(f) = \mathcal{I}(g)(b) - \mathcal{I}(g)(a).$$

Will the real FTC please stand up?

FTC (as it *should* be taught).
If $g =_{\mathrm{DA}} \int f$, and $\mathcal{I}(g)$ is continuous on $[a, b]$, then

$$_{\epsilon\delta}\int_a^b \mathcal{I}(f) = \mathcal{I}(g)(b) - \mathcal{I}(g)(a).$$

Note the caveat on continuity: $g : x \mapsto \arctan\left(\frac{1}{x}\right)$ is discontinuous at $x = 0$ ($\lim_{x\to 0^-} \arctan\left(\frac{1}{x}\right) = \frac{-\pi}{2}$ whereas $\lim_{x\to 0^+} \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$),

Will the real FTC please stand up?

FTC (as it *should* be taught).
If $g =_{\mathrm{DA}} \int f$, and $\mathcal{I}(g)$ is continuous on $[a, b]$, then

$$_{\epsilon\delta}\int_a^b \mathcal{I}(f) = \mathcal{I}(g)(b) - \mathcal{I}(g)(a).$$

Note the caveat on continuity: $g : x \mapsto \arctan\left(\frac{1}{x}\right)$ is discontinuous at $x = 0$ ($\lim_{x\to 0^-} \arctan\left(\frac{1}{x}\right) = \frac{-\pi}{2}$ whereas $\lim_{x\to 0^+} \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$), which accounts for the invalidity of deducing that the integral of a negative function is positive —

$$\int_{-1}^{1} \frac{-1}{x^2+1} = \mathcal{I}(g)(1) - \mathcal{I}(g)(-1) = \frac{\pi}{4} - \frac{-\pi}{4} = \frac{\pi}{2} > 0.$$

Rescuing the Fundamental Theorem of Calculus

Of course, another ${}_{\mathrm{DA}}\int \frac{-1}{x^2+1}$ is $h(x) = \arctan\left(\frac{1}{x}\right) + H(x)$ where $H(x) = \begin{cases} 0 & x < 0 \\ -\pi & x > 0 \end{cases}$ is a $\text{constant}_{\mathrm{DA}}$. This has a removable singularity at $x = 0$.

Rescuing the Fundamental Theorem of Calculus

Of course, another ${}_{\mathrm{DA}}\int \frac{-1}{x^2+1}$ is $h(x) = \arctan\left(\frac{1}{x}\right) + H(x)$ where $H(x) = \begin{cases} 0 & x < 0 \\ -\pi & x > 0 \end{cases}$ is a constant$_{\mathrm{DA}}$. This has a removable singularity at $x = 0$.

$\mathcal{I}^*(h)$ is continuous, so FTC is appropriate and

$$\int_{-1}^{1} \frac{-1}{x^2+1} = \mathcal{I}(h)(1) - \mathcal{I}(h)(-1) = \left(\frac{\pi}{4} - \pi\right) - \frac{-\pi}{4} = -\frac{\pi}{2} < 0.$$

“Most continuous expressions” revisited

$\arctan\left(\frac{1}{x}\right) + H(x)$ can also be rewritten as $\frac{\pi}{2} - \arctan(x)$.

“Most continuous expressions” revisited

$\arctan\left(\frac{1}{x}\right) + H(x)$ can also be rewritten as $\frac{\pi}{2} - \arctan(x)$.
In this case $\frac{\pi}{2} - \arctan(x)$ is a “most continuous formula”.

“Most continuous expressions” revisited

$\arctan\left(\frac{1}{x}\right) + H(x)$ can also be rewritten as $\frac{\pi}{2} - \arctan(x)$.
In this case $\frac{\pi}{2} - \arctan(x)$ is a “most continuous formula”.
(except at ∞, where the original $\arctan\left(\frac{1}{x}\right)$ is continuous!).

“Most continuous expressions” revisited

$\arctan\left(\frac{1}{x}\right) + H(x)$ can also be rewritten as $\frac{\pi}{2} - \arctan(x)$.
In this case $\frac{\pi}{2} - \arctan(x)$ is a “most continuous formula”.
(except at ∞, where the original $\arctan\left(\frac{1}{x}\right)$ is continuous!).
What about $\arctan\left(\frac{z-1}{z+1}\right)$?

“Most continuous expressions” revisited

$\arctan\left(\frac{1}{x}\right) + H(x)$ can also be rewritten as $\frac{\pi}{2} - \arctan(x)$.
In this case $\frac{\pi}{2} - \arctan(x)$ is a “most continuous formula”.
(except at ∞, where the original $\arctan\left(\frac{1}{x}\right)$ is continuous!).
What about $\arctan\left(\frac{z-1}{z+1}\right)$? An equivalent is

$$\arctan\left(\frac{z-1+\sqrt{z^2+6\,z-7}}{2(z-1)}\right) - \arctan\left(\frac{z-1+\sqrt{z^2+6\,z-7}}{2(z-1)}+1\right)$$

"Most continuous expressions" revisited

$\arctan\left(\frac{1}{x}\right) + H(x)$ can also be rewritten as $\frac{\pi}{2} - \arctan(x)$.
In this case $\frac{\pi}{2} - \arctan(x)$ is a "most continuous formula".
(except at ∞, where the original $\arctan\left(\frac{1}{x}\right)$ is continuous!).
What about $\arctan\left(\frac{z-1}{z+1}\right)$? An equivalent is

$$\arctan\left(\frac{z-1+\sqrt{z^2+6\,z-7}}{2(z-1)}\right) - \arctan\left(\frac{z-1+\sqrt{z^2+6\,z-7}}{2(z-1)}+1\right)$$

or $-\arctan\left(\frac{2\,z-1+z^2}{-2\,z-1+z^2}\right) - \arctan(z)$, where the singularities are at $-1 \pm \sqrt{2}$ (and ∞), or ....

“Most continuous expressions” revisited

$\arctan\left(\frac{1}{x}\right) + H(x)$ can also be rewritten as $\frac{\pi}{2} - \arctan(x)$.
In this case $\frac{\pi}{2} - \arctan(x)$ is a “most continuous formula”.
(except at ∞, where the original $\arctan\left(\frac{1}{x}\right)$ is continuous!).
What about $\arctan\left(\frac{z-1}{z+1}\right)$? An equivalent is

$$\arctan\left(\frac{z-1+\sqrt{z^2+6\,z-7}}{2(z-1)}\right) - \arctan\left(\frac{z-1+\sqrt{z^2+6\,z-7}}{2(z-1)}+1\right)$$

or $-\arctan\left(\frac{2\,z-1+z^2}{-2\,z-1+z^2}\right) - \arctan(z)$, where the singularities are at $-1\pm\sqrt{2}$ (and ∞), or ....
Still an unsolved problem (Rioboo, ...).

But I'm not interested in all this DA stuff

But I'm not interested in all this DA stuff

- Surely most people (even engineers) do $\epsilon\delta \int$ etc.

But I'm not interested in all this DA stuff

- Surely most people (even engineers) do ${}_{\epsilon\delta}\int$ etc.
- They may *think* they do, but in practice they *do* ${}_{\mathrm{DA}}\int$ even when intending to do ${}_{\epsilon\delta}\int$.

But I'm not interested in all this DA stuff

- Surely most people (even engineers) do ${}_{\epsilon\delta}\int$ etc.
- They may *think* they do, but in practice they *do* ${}_{\mathrm{DA}}\int$ even when intending to do ${}_{\epsilon\delta}\int$.
- What is $\lim_{h\to 0} \frac{1}{h}\left((1+h)\cos^2(1+h) - \cos^2 1\right)$?

But I'm not interested in all this DA stuff

- Surely most people (even engineers) do ${}_{\epsilon\delta}\int$ etc.
- They may *think* they do, but in practice they *do* ${}_{\mathrm{DA}}\int$ even when intending to do ${}_{\epsilon\delta}\int$.
- What is $\lim_{h\to 0} \frac{1}{h}\left((1+h)\cos^2(1+h) - \cos^2 1\right)$?
- If you immediately said $\cos^2(1) - 2\sin(1)\cos(1)$

But I'm not interested in all this DA stuff

- Surely most people (even engineers) do ${}_{\epsilon\delta}\int$ etc.
- They may *think* they do, but in practice they *do* ${}_{\mathrm{DA}}\int$ even when intending to do ${}_{\epsilon\delta}\int$.
- What is $\lim_{h\to 0} \frac{1}{h}\left((1+h)\cos^2(1+h) - \cos^2 1\right)$?
- If you immediately said $\cos^2(1) - 2\sin(1)\cos(1)$
- I bet you *actually* did $D_{\mathrm{DA}}(x\cos^2 x)|_{x=1}$.

But I'm not interested in all this DA stuff

- Surely most people (even engineers) do ${}_{\epsilon\delta}\int$ etc.
- They may *think* they do, but in practice they *do* ${}_{\mathrm{DA}}\int$ even when intending to do ${}_{\epsilon\delta}\int$.
- What is $\lim_{h\to 0} \frac{1}{h}\left((1+h)\cos^2(1+h) - \cos^2 1\right)$?
- If you immediately said $\cos^2(1) - 2\sin(1)\cos(1)$
- I bet you *actually* did $D_{\mathrm{DA}}(x\cos^2 x)|_{x=1}$.
- (If you said $-.617370845$, you probably had Maple on your Blackberry, and *it* did that.)

Conclusions

Conclusions

- We already know that ${}_{\epsilon\delta}\int$ is a powerful world model

Conclusions

- We already know that ${}_{\epsilon\delta}\int$ is a powerful world model
- ${}_{\mathrm{DA}}\int$ is well-implemented and very powerful

Conclusions

- We already know that ${}_{\epsilon\delta}\int$ is a powerful world model
- ${}_{\mathrm{DA}}\int$ is well-implemented and very powerful
- "e^{-x^2} has no integral" means "has no ${}_{\mathrm{DA}}\int$ in terms of already known functions"

Conclusions

- We already know that ${}_{\epsilon\delta}\int$ is a powerful world model
- ${}_{\mathrm{DA}}\int$ is well-implemented and very powerful
- "e^{-x^2} has no integral" means "has no ${}_{\mathrm{DA}}\int$ in terms of already known functions"
- $\mathcal{I}$ *does* map from one to the other, often very effectively

Conclusions

- We already know that ${}_{\epsilon\delta}\int$ is a powerful world model
- ${}_{\mathrm{DA}}\int$ is well-implemented and very powerful
- "e^{-x^2} has no integral" means "has no ${}_{\mathrm{DA}}\int$ in terms of already known functions"
- $\mathcal{I}$ *does* map from one to the other, often very effectively
- but not always!

$\mathcal{I}$ crops up elsewhere

$\mathcal{I}$ crops up elsewhere

Claim $\sqrt{1-z^2} = \sqrt{1-z}\sqrt{1+z}$.

$\mathcal{I}$ crops up elsewhere

Claim $\sqrt{1-z^2} = \sqrt{1-z}\sqrt{1+z}$.
Squaring both sides gives $1 - z^2 = (1-z)(1+z)$, so there is *some* interpretation in which it is true.

$\mathcal{I}$ crops up elsewhere

Claim $\sqrt{1-z^2} = \sqrt{1-z}\sqrt{1+z}$.
Squaring both sides gives $1 - z^2 = (1-z)(1+z)$, so there is *some* interpretation in which it is true.
What about $\sqrt{z^2-1} \stackrel{?}{=} \sqrt{z-1}\sqrt{z+1}$.

$\mathcal{I}$ crops up elsewhere

Claim $\sqrt{1-z^2} = \sqrt{1-z}\sqrt{1+z}$.
Squaring both sides gives $1-z^2 = (1-z)(1+z)$, so there is *some* interpretation in which it is true.
What about $\sqrt{z^2-1} \stackrel{?}{=} \sqrt{z-1}\sqrt{z+1}$.
The same arguments apply and there is *some* interpretation of $\sqrt{z^2-1}$, $\sqrt{z-1}$ and $\sqrt{z+1}$ in which it is true, *but*

$\mathcal{I}$ crops up elsewhere

Claim $\sqrt{1-z^2} = \sqrt{1-z}\sqrt{1+z}$.
Squaring both sides gives $1-z^2 = (1-z)(1+z)$, so there is *some* interpretation in which it is true.
What about $\sqrt{z^2-1} \stackrel{?}{=} \sqrt{z-1}\sqrt{z+1}$.
The same arguments apply and there is *some* interpretation of $\sqrt{z^2-1}$, $\sqrt{z-1}$ and $\sqrt{z+1}$ in which it is true, *but* when $z=-2$ we get $\sqrt{3} = \sqrt{-3}\sqrt{-1}$, so there is *no* interpretation of $\sqrt{\ }$ as such, consistent with $\sqrt{-n} = \sqrt{-1}\sqrt{n}$ even for $n \in \mathbf{N}$, for which it is true.

$\mathcal{I}$ crops up elsewhere

Claim $\sqrt{1-z^2} = \sqrt{1-z}\sqrt{1+z}$.
Squaring both sides gives $1-z^2 = (1-z)(1+z)$, so there is *some* interpretation in which it is true.
What about $\sqrt{z^2-1} \stackrel{?}{=} \sqrt{z-1}\sqrt{z+1}$.
The same arguments apply and there is *some* interpretation of $\sqrt{z^2-1}$, $\sqrt{z-1}$ and $\sqrt{z+1}$ in which it is true, *but* when $z=-2$ we get $\sqrt{3} = \sqrt{-3}\sqrt{-1}$, so there is *no* interpretation of $\sqrt{}$ as such, consistent with $\sqrt{-n} = \sqrt{-1}\sqrt{n}$ even for $n \in \mathbf{N}$, for which it is true.
There is *no* interpretation of $\sqrt{}$ as such for which both are true.

$\mathcal{I}$ crops up elsewhere

Claim $\sqrt{1-z^2} = \sqrt{1-z}\sqrt{1+z}$.
Squaring both sides gives $1-z^2 = (1-z)(1+z)$, so there is *some* interpretation in which it is true.
What about $\sqrt{z^2-1} \stackrel{?}{=} \sqrt{z-1}\sqrt{z+1}$.
The same arguments apply and there is *some* interpretation of $\sqrt{z^2-1}$, $\sqrt{z-1}$ and $\sqrt{z+1}$ in which it is true, *but* when $z=-2$ we get $\sqrt{3} = \sqrt{-3}\sqrt{-1}$, so there is *no* interpretation of $\sqrt{}$ as such, consistent with $\sqrt{-n} = \sqrt{-1}\sqrt{n}$ even for $n \in \mathbf{N}$, for which it is true.
There is *no* interpretation of $\sqrt{}$ as such for which both are true.
$\mathcal{I}$ interprets individual functions, such as $\sqrt{1+z}$, and there is no general composition.