TheoryGuru: A Mathematica Package to Apply Quantifier Elimination Technology to Economics

C. Mulligan, **J.H. Davenport**, and Matthew England

Universities of Chicago, Bath, Coventry

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Speaker funded by EU Project 712689 (SC^2); M.E. drafted the slides.

Acknowledgement

Speaker is a researcher in Symbolic Computation (QE, CAD etc.) and not an economist.

The lead author of this (and related) papers is Casey Mulligan: Professor in the Department of Economics at the University of Chicago.

He cannot be here today but is happy to engage with the community.

`http://home.uchicago.edu/~cbm4/`

Key Message

Key Messages: Many problems in economics can be studied with technology for real closed fields, i.e.

- QE algorithms in Computer Algebra Systems;
- SMT Solvers that support the `NRA` and `QF_NRA` logics.

However (a) few economists are aware of this; (b) few are experienced with using mathematical software. We should make it more accessible for them!

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Side Message People like the speaker should pay more attention to this potentially large application domain and the specifics of examples within.

Outline

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Standard Framework

Determine whether, with variables $\boldsymbol{v} = (v_1, \ldots, v_n)$, the hypotheses $H(\boldsymbol{v})$ follow from the assumptions $A(\boldsymbol{v})$, i.e. answer

$$\forall \boldsymbol{v} \,.\, A(\boldsymbol{v}) \Rightarrow H(\boldsymbol{v})?$$

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Logically the answer must be `True` or `False` but economists may be also be interested in things like:

- Are the assumptions themselves contradictory?
- If `False`, can additional assumptions be made to give `True`?
- If `True`, can any assumptions be removed?

Such questions can be answered by technology for Quantifier Elimination over the reals.

Categorisation by a pair of existential statements (4/22)

Suppose we check both:

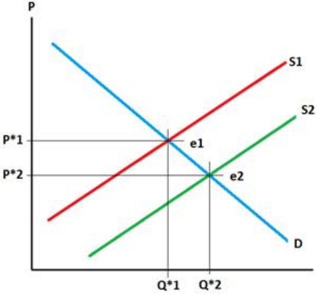
- the existence of an example
  $\exists \boldsymbol{v}[A \wedge H]$,
- and the existence of a counterexample
  $\exists \boldsymbol{v}[A \wedge \neg H]$.

Then we can categorize the theorem as follows:

	$\neg\exists \boldsymbol{v}[A \wedge \neg H]$	$\exists \boldsymbol{v}[A \wedge \neg H]$
$\exists \boldsymbol{v}[A \wedge H]$	`True`	`Mixed`
$\neg\exists \boldsymbol{v}[A \wedge H]$	Contradictory Assumptions	`False`

Simple Example: Marshall 1895

Marshall considered the effect of a reduction in supply costs (a) concluding that *for any supply-demand equilibrium in which the two curves have their usual slopes, a downward supply shift increases the equilibrium quantity and decreases the equilibrium price.*



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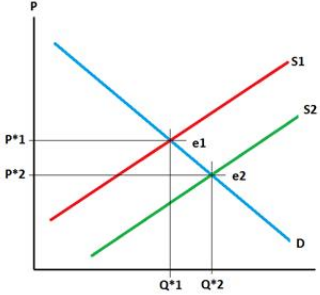

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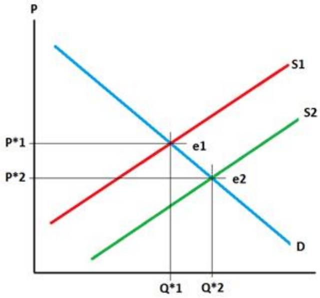

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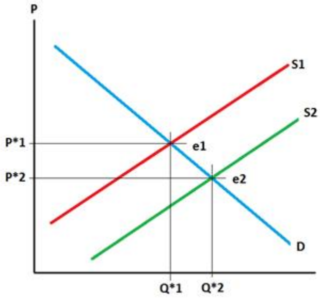

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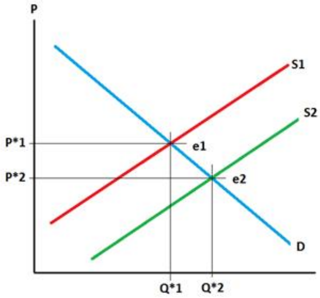

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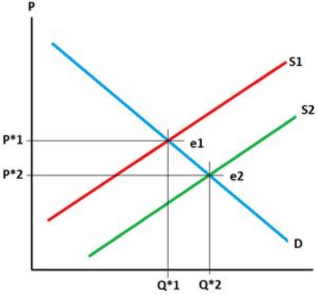

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From Marshall to Tarski

To study as a Tarski formula we set the "variables" $\boldsymbol{v}$ to be four real numbers (v_1, v_2, v_3, v_4):

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Then, after applying the chain rule, A and H may be understood as Boolean combinations of polynomial equalities and inequalities:

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From here it takes only a little reasoning by hand to see that $A \Rightarrow H$. Any of the tools mentioned earlier can conclude this almost instantly.

TheoryGuru

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To access need Mathematica and then simply run

```
Get["http://economicreasoning.com"]
```

Examples and documentation are online at:

`http://examples.economicreasoning.com/`

as both interactive Mathematica notebooks and static pdfs.

Main TheoryGuru Functionality (11/22)

Parse input: Identify whether variable is vector, scalar, or boolean; change notation to format acceptable for `Resolve`.
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Interpret Output: E.g. Identify table cell; show counterexample; offer alternative QE calls to explore possible related theorems.

Outline

1. Automated Reasoning in Economics
   - Standard Framework
   - TheoryGuru
2. Examples and Benchmarks
   - Theory Guru in Practice
   - Benchmarks

Tax Incidence Example

We will consider the effect of a tax on buyers and sellers in a market. Each transaction involves the buyer paying price, p to the seller and tax, t to the government. Symbolic functions `D(.)` and `S(.)` represent the quantities bought and sold. We have a market equilibrium price where

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$$D(p+t) = S(p).$$

We fix some standard assumptions:

- The slopes of the demand and supply curves are in the usual directions: $D'(p+t) > 0$ and $S'(p) < 0$.
- Changing the `tax` moves the market equilibrium.

$$\frac{d}{dt}D(p+t) = \frac{d}{dt}S(p)$$

We wish to draw conclusions about the effect of the tax on the market equilibrium price.

Solutions by hand

For this simple example we can analyse by hand.

We have $D'(p+t) > 0$, $S'(p) < 0$ and

$$\frac{d}{dt} D(p+t) = \frac{d}{dt} S(p)$$
$$D'(p+t) \cdot (p'+1) = S'(p) \cdot p'$$

So we cannot have $p' > 0$ or $p' < -1$ without incompatible signs on either side of the equation.

Let us see what `TheoryGuru` could do here. The same approach works on more substantial examples.

Testing a Theorem

Suppose first that, under the assumptions, we want to check the hypothesis that $p' \leq 0$.

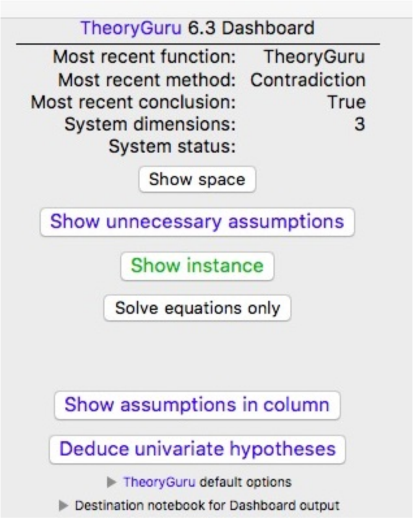
```
In[2]:= Equilibrium = demand[price + tax] == supply[price];

In[3]:= TheoryGuru[{ⅆEquilibrium/ⅆtax, demand'[price + tax] < 0, supply'[price] > 0}, ⅆprice/ⅆtax ≤ 0]

Out[3]= True
```

The call to `TheoryGuru` in cell two has the set of assumptions as first argument and hypothesis as second argument.

- The symbolic differentiation in the argument is performed by MATHEMATICA before `TheoryGuru` is called.
- `TheoryGuru` recognised the partially interpreted functions, treats them as variables and forms the two existential calls.
- No counterexample exists so the theorem is true: under these assumptions there cannot be a positive impact on price.

More can be done

After the evaluation a dashboard appears summarising the calculation so far and suggesting possible next steps.



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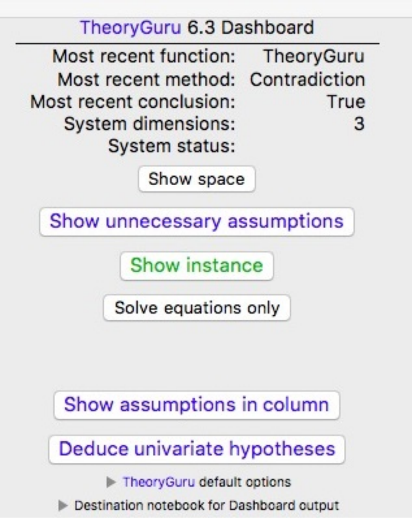
In this case **Deduce univariate hypotheses** ⟶ is useful . . . .



TheoryPossibilities

Pressing it generates a call to TheoryPossibilities. Here, free variables (can be chosen by user or software) are projected onto separately (i.e. QE eliminating existential quantifiers from all variables except that one) to look for useful information.

```
In[4]:= TheoryPossibilities[{(* optional: replace this comment with a list of variables *)}
        ]//OtherTools`TFPrintL;

        Using MostRecentAssumption and

        Forming hypotheses from {dprice/dtax}.
```

$$-1 < \frac{\text{dprice}}{\text{dtax}} < 0$$

Automatically typed by clicking dashboard

In this example it discovers the exact restriction on the price impact.

TheorySufficient

Consider the call below – same as before but user has forgotten to constrain the slope of the supply curve. In this case both examples and counterexamples of the theorem were found.

In[5]:= $\text{TheoryGuru}\left[\left\{\frac{d\,\text{Equilibrium}}{d\,\text{tax}},\ \text{demand}'[\text{price}+\text{tax}]<0\right\},\ \frac{d\,\text{price}}{d\,\text{tax}}\leq 0\right]$

Out[5]= True for some, False for others

In[6]:= TheorySufficient[]

Out[6]= $\text{supply}'[\text{price}]\geq 0$

The forgotten assumption can be discovered with `TheorySufficient`. It assembles the formula $A \wedge \neg H$ defining counterexamples; projects on each of the axes, in one case recovering the missing supply-slope restriction (the others added no new information).

Main Messages Recap

Key Message: Many problems in economics can be studied with technology for real closed fields. But for widespread use we must lower the cost to economists of using the technology

`TheoryGuru` does this – now used by classes of economics students at Chicago.

Side Message: People like the speaker should pay more attention to this application domain and the specifics of examples within.

[MDE18] C. Mulligan, J.H. Davenport, and M. England,
TheoryGuru: A Mathematica Package to Apply Quantifier Elimination Technology to Economics. Proc. Mathematical Software — ICMS 2018 (ed. Davenport,J.H., Kauers,M., Labahn,G. & Urban,J.), Springer Lecture Notes in Computer Science 10931, Springer, Cham, 2018, pp. 369-378.
https://arxiv.org/abs/1806.10925

New Benchmark Set

We have presented a benchmark set of 45 potential economic theorems. Each theorem creates three `QF_NRA` calls to check the compatibility of assumptions, the existence of an example, and the existence of a counterexample. Thus 135 problems in total.

Assumption and example checks are SAT for all 45; counterexample is UNSAT for 42/45.

[MDE18] C. Mulligan, R. Bradford, J.H. Davenport, M. England, and Z. Tonks
Non-linear Real Arithmetic Benchmarks derived from Automated Reasoning in Economics
Proc. SC^2 Workshop 2018 ceur-ws.org **2189**(2018) pp. 48–60.
https://arxiv.org/abs/1806.11447.

Note: further related problems may be derived from these by quantifying less of the variables to simulate theory exploration.

Benchmark Availability

The benchmark problems are hosted on the Zenodo data repository at URL:

`https://doi.org/10.5281/zenodo.1226892`

Available in SMT2 format and also in format suitable for REDLOG and MAPLE.

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Also available are MATHEMATICA notebooks which contain commands to solve the examples in MATHEMATICA and further information on the economic background of each problem:

`http://examples.economicreasoning.com`

The set includes examples from macro, micro, and econometrics (the three main economics subfields).

The End

Contact Details

Slides will be available from:

http://staff.bath.ac.uk/masjhd/Slides/2019Economics.pdf

Thanks for listening!

Trio of SMT problems

So every proposed economics theorem generates a pair of SAT problems to check its validity.

In practice actually a trio as it is simpler to check for contradictory assumptions separately first. If the theorem is correct then two of these will be SAT (easy) and one UNSAT (hard).

Such problems can of course be tackled by any SMT-solver that supports `QF_NRA`.

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But we may want to do more than check validity.

Theory Exploration

An economist could vary the question by strengthening the assumptions that led to a `Mixed` result in search of a `True` theorem; or weaken the assumptions that generated a `True` result to identify a theorem that can be applied more widely.

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This may discovered by quantifying more or less of the variables in $\boldsymbol{v}$. For example, partition $\boldsymbol{v}$ as $\boldsymbol{v}_1, \boldsymbol{v}_2$ and ask for

$$\{\boldsymbol{v}_1 : \forall \boldsymbol{v}_2 \, . \, A(\boldsymbol{v}_1, \boldsymbol{v}_2) \Rightarrow H(\boldsymbol{v}_1, \boldsymbol{v}_2)\}.$$

The result in these cases would be a formula in the free variables that can be used to modify the assumptions accordingly.

Technology that can address these problems (7/22)

The theory exploration is implemented by answering questions of **Quantifier Elimination** (QE): given a quantified Tarski formula produce an equivalent one without quantifiers.

QE possible over real closed fields thanks to the seminal work of Tarski. Practical implementations followed the work of Collins (Cylindrical Algebraic Decomposition) and Weispfenning (Virtual Substitution)

Modern implementations of real QE in MATHEMATICA, REDLOG, MAPLE (SYNRAC and the REGULARCHAINS Library) and QEPCAD-B.

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The core existential problems can be addressed by SMT solvers such as SMT-RAT, veriT, Yices2 and Z3.

Input Parsing

An important part of TheoryGuru is to recognise when more complicated mathematical objects can be represented by scalars suitable for study by QE (e.g. the derivatives and symbolic functions in previous example). Further examples of this:

- Second example in ICMS paper – involves probability density functions, however the logical reasoning depended only on them as summarised by various scalars.
- Second example in SC^2 Workshop paper – involves vectors of undetermined length, however the logical reasoning depended only on dot products between them (we must add to assumptions that the corresponding symmetric Gram matrix be positive semi-definite to ensure real solutions).