

The doubly-exponential problem in equation/inequality solving

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Theoretical versus Practical Complexity

Notation n variables, m polynomials of degree d (in each variable separately; \mathfrak{d} total degree: $d \leq \mathfrak{d} \leq nd$), coefficients length l

Theoretical **doubly exponential**, whether via Gröbner bases [MM82, Yap91, lower], [Dub90, upper] or Cylindrical Algebraic Decomposition [DH88, BD07, lower], [Col75, BDE⁺16, upper]

But this is doubly exponential in n , polynomial in everything else.

In practice we see very bad dependence on m, d, l , and n is often fixed

Anyway The Bézout bound says there are \mathfrak{d}^n solutions to such polynomial systems: **singly exponential** if the system is zero-dimensional

Gröbner bases: [MR13] versus [MM82]

Let r be the dimension of the variety of solutions. Focus on the degrees of the polynomials (more intrinsic than actual times)

[MR13] modified both lower and upper bounds to show $\mathfrak{d}^{n^{\Theta(1)}2^{\Theta(r)}}$

lower Essentially, use the r -variable [Yap91] ideal

which encodes an EXPSPACE-complete rewriting problem into a system of binomials

note that these ideals are definitely not radical (square-free)

upper A very significant improvement to [Dub90], again using r rather than n where possible

What we would like to do (but can't)

Show radical ideal problems are only singly-exponential in n

This ought to follow from [Kol88]

Show non-radical ideals are rare (non-square-free polynomials occur with density 0)

However there seems to be no theory of distribution of ideals

Deduce **weak worst-case complexity** (i.e. apart from an exponentially-rare subset: [AL15]) of Gröbner bases is singly exponential

A technical complication, and solution

Making sets of polynomials square-free, or even irreducible,

- is computationally nearly always advantageous
- is sometimes required by the theory

but might leave the degree alone, or might replace one polynomial by $O(\sqrt{d})$ polynomials

hard to control from the point of view of complexity theory.

Solution [McC84] Say that a set of polynomials has the (M, D) property if it can be partitioned into M sets, each with combined degree at most D (in each variable)

This is **preserved** by taking square-free decompositions etc.

Can Define (M, \mathcal{D}) analogously

Cylindrical Algebraic Decomposition for polynomials

Assume All CADs we encounter are **well-oriented** [McC84], i.e. no relevant polynomial vanishes identically on a cell

However there is no theory of distribution of CADs

And Bath has a family of examples which aren't well-oriented

And rescuing from failure is doable, but not well-studied

Note [MPP16] says this is no longer relevant

Then if A_n is the polynomials in n variables, with primitive irreducible basis B_n , the projection is

$$A_{n-1} := \text{cont}(A_n) \cup [\mathcal{P}(B_n) := \text{coeff}(B_n) \cup \text{disc}(B_n) \cup \text{res}(B_n)]$$

If A_n has (M, D) then A_{n-1} has $((M+1)^2/2, 2D^2)$

Hence **doubly-exponential** growth in n

The induction (on n) hypothesis is **order-invariant** decompositions

Cylindrical Algebraic Decomposition for propositions (1)

Suppose we are trying to understand (e.g. quantifier elimination) a proposition Φ (or set of propositions), and $f(\mathbf{x}) = 0$ is a consequence of Φ (either explicit or implicit), an **equational constraint**, and f involves x_n and is primitive

Then [Col98] we are only interested in $\mathbf{R}^n | f(\mathbf{x}) = 0$, not \mathbf{R}^n

So [McC99] let F be an irreducible basis for f , and use

$$\mathcal{P}_F(B) := \mathcal{P}(F) \cup \{\text{res}(f, b) | f \in F, b \in B \setminus F\}$$

This has $(2M, 2D^2)$ rather than $(O(M^2), 2D^2)$, but only produces a **sign-invariant** decomposition

Cylindrical Algebraic Decomposition for propositions (2)

Generalised to $\mathcal{P}_F^*(B) := \mathcal{P}_F(B) \cup \text{disc}(B \setminus F)$ [McC01], which produces an **order-invariant** decomposition, and has $(3M, 2D^2)$
If $f(\mathbf{x}) = 0$ and $g(\mathbf{x}) = 0$ are both equational constraints, then $\text{res}_{x_n}(f, g)$ is also an equational constraint

Suppose we have s equational constraints

And (after resultants) we have a constraint in each of the last s variables

And these constraints are all primitive

Then [EBD15] we get $O\left(m^{s2^{n-s}} d^{2^n}\right)$ behaviour

CASC 2016[ED16] Under the same assumptions,

$$O\left(m^{s2^{n-s}} d^{s2^{n-s}}\right) \text{ behaviour}$$

using Gröbner bases rather than resultants for the elimination, but multivariate resultants [BM09] for the bounds

ICMS 2016[DE16] The primitivity restriction is inherent: we can write [DH88] in this format, with $n - 1$ non-primitive equational constraints

it's not **R/C**: it's quantifiers (and alternations)

[DH88, BD07] Are really about the combinatorial complexity of

Let $S_k(x_k, y_k)$ be the statement $x_k = f(y_k)$ and then define recursively $S_{k-1}(x_{k-1}, y_{k-1}) := x_{k-1} = f(f(y_{k-1})) :=$

$$\underbrace{\exists z_k \forall x_k \forall y_k}_{Q_k} \underbrace{((y_{k-1} = y_k \wedge x_k = z_k) \vee (y_k = z_k \wedge x_{k-1} = x_k))}_{L_k} \Rightarrow S_k(x_k, y_k)$$

We can transpose this to the complexes, and get zero-dimensional QE examples in \mathbf{C}^n with $2^{2^{O(n)}}$ isolated point solutions, even though the equations are all linear and the solution set is zero-dimensional.

So let's not be mesmerised by the QE problem

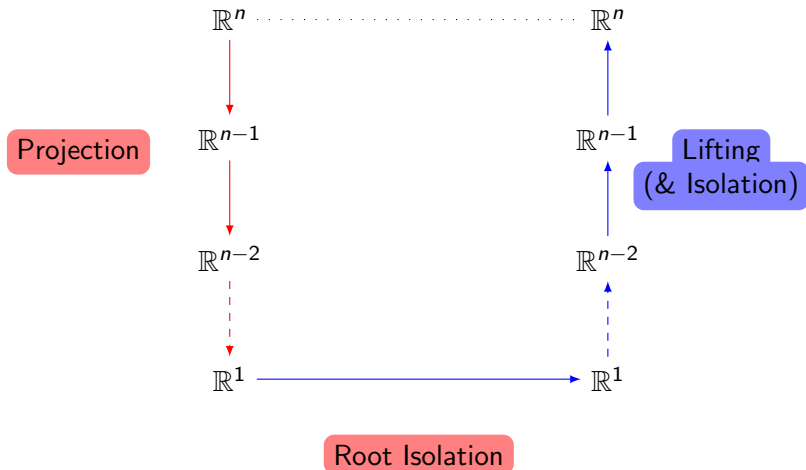
Consider (as we, TS and others have been doing) a single semi-algebraic set defined by

$$f_1(x_1, \dots, x_{n-1}, k_1) = 0 \wedge f_2(x_1, \dots, x_{n-1}, k_1) = 0 \wedge \dots \\ f_{n-1}(x_1, \dots, x_{n-1}, k_1) = 0 \wedge x_1 > 0 \wedge \dots \wedge x_{n-1} > 0$$

and ask the question “How does the number of solutions vary with k_1 ?” The f_i are multilinear ($d = 1$) and primitive, and are pretty “generic”.

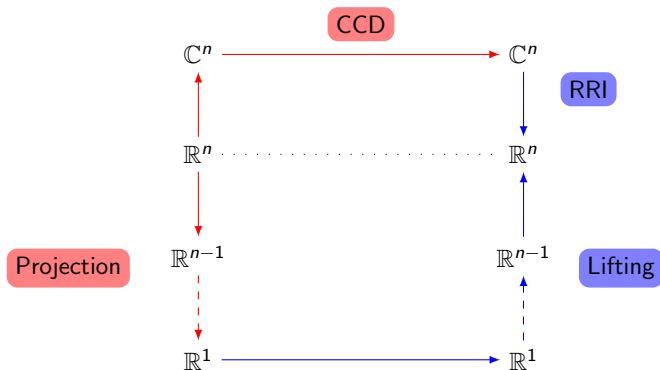
Of course, this doesn't guarantee that all the iterated resultants in [EBD15], or the Gröbner polynomials in [ED16], are primitive, but in practice they are.

The basic idea for CAD [Col75]



An alternative approach [CMXY09]

Proceed via the complex numbers,



Do a complex cylindrical decomposition via **Regular Chains**, then use Real Root Isolation

Regular Chain Decompositions

Fix an ordering of variables. The initial of f , $\text{init}(f)$, is the leading coefficient of f with respect to its main variable.

Definition

A list, or chain, of polynomials f_1, \dots, f_k is a *regular chain* if:

- 1 whenever $i < j$, $\text{mvar}(f_i) \prec \text{mvar}(f_j)$ (therefore the chain is triangular);
- 2 $\text{init}(f_i)$ is invertible modulo the ideal $(f_j : j < i)$.

The set of *regular zeros* $W(S)$ of a set S of polynomials is $V(S) \setminus V(\text{init}(S))$.

A (Complex) Regular Chain Decomposition of I is a set of regular chains T_i such that $V(I) = \bigcup W(T_i)$.

Normally (and I wish I knew what that meant) there is one RC of maximal (complex) dimension, and many of lower dimension.

RealTriangularize (assuming a pure conjunction)

- 1 Do a CCD of all the equations
- 2 Make the result SemiAlgebraic over the reals
- 3 Add all the inequalities, splitting chains as we need to

LazyRealTriangularize [CDM⁺13] doesn't bother with the lower (complex) dimensional components, but wraps then up as unevaluated calls to itself: "Here's the generic answers(s), and how to ask me for the special cases".

In the examples with TS, LazyRealTriangularize seems to produce the same answer as the [ED16] version of Projection CAD. This is good news, as what we want should be a geometric invariant.

Questions?



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