What does Mathematical Notation actually mean, and how can computers process it?

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[1] Thanks to many people: typesetters, editors, OpenMath and MathML colleagues, TEXnicians

Overview

Disclaimer: This is a brief introduction to a very large (and diverse) subject: however, I used to typeset mathematics at school, and have been in OpenMath for 23 years, and MathML for 17

1. Mathematical notation

   and some of its flaws
2. How it is currently displayed/ represented: TEX/PDF;=

   MathML (Presentation/Content); OpenMath
3. How it might be understood

The subjects do overlap

(The outsider's perception of) Mathematical Notation

Unambiguous, unchanging, precise, world-wide (or more so)

- "as clear as 2+2=4"
- Google the phrase "mathematically precise"
- Various science-fiction stories (e.g. Pythagoras' Theorem)
- And in real life — mathematicians *can* and *do* communicate via notation
- The computing discipline of "Formal Methods" tries to reduce computer programming to mathematics/logic

And indeed there's a lot of truth in this

Certainly not unchanging

- $+$ is less than 500 years old [Sti44] (also $-$ and $\sqrt{}$)
- $=$ is slightly younger [Rec57]
- Recorde wrote $2\overline{a+b}$: $2(a+b)$ is later
- (...) won because it is (much!) easier for manual typesetting
- Calculus had/has two conflicting notations $\dot{x}$ or $\frac{\mathrm{d}x}{\mathrm{d}t}$.
- Relativity introduced the summation convention: $\sum_{i=1}^{3} c_i x^i$ is just $c_i x^i$ (but $c_\mu x^\mu$ is short for $\sum_{\mu=0}^{3} c_\mu x^\mu$) [Ein16]

And practically every mathematician introduces some notation: natural selection (generally) applies

Not quite so international

Idea	Anglo-Saxon	French	German
half-open interval	$(0,1]$	$]0,1]$	varies
single-valued function	$\arctan$	Arctan	$\arctan$
multi-valued function	Arctan	$\arctan$	Arctan
$\{0,1,2,\ldots\}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}\cup\{0\}$
$\{1,2,3,\ldots\}$	$\mathbb{N}\setminus\{0\}$	$\mathbb{N}\setminus\{0\}$	$\mathbb{N}$

Or universal: $\sqrt{-1}$ is *i* to most people, but *j* to Electrical Engineers, and the MatLab system allows both

And these problems occur at an early age [Lop08]

MATHEMATICAL NOTATION COMPARISONS BETWEEN U.S. AND LATIN AMERICAN COUNTRIES

OPERATION DESCRIPTION
DIVISION

Many students come into the U.S. schools using algorithms learned in their country of origin. For example, students in many Latin American countries are expected to do and exhibit more mental computation as the following algorithm illustrates. To assist educators in recognizing different procedural knowledge as valid, we explain how this algorithm works

Format 1	Format 2	
$3\sqrt{74}$	**$74 \underline{\vert 3}$**	In this algorithm, students will divide 3 into 74 and may write it in one of two ways.
$\begin{array}{r} 2 \\ 3\overline{)74} \\ 1 \end{array}$	**$\begin{array}{ll} 74 & \underline{\vert 3} \\ 1 & \ \ 2 \end{array}$**	▪ Students typically begin to formulate and answer questions such as: How many times can 3 go into 7? Another way of asking is if we divide 70 into 3 sets, how many are in each set. ▪ Students write the 2 in the tens place, above the 7, on Format 1, but the 2 goes below the divisor when written in Format 2 style. Notice

	subtract. The only part that is written on paper is the remainder, 1 ten. Notice its location on both formats.
2 **3)74** **14** **74 |3** **14 2**	▪ The 4 is brought down and students consider 14 next. ▪ Notice where the 14 is written on both formats.
24 **3)74** **14** **74 |3** **14 24**	▪ Students now find that 3 will go into 14 three (3) times. They write 4 in the quotient's place.
24 **3)74** **14** **2** **74 |3** **14 24** **2**	▪ Students again mentally subtract 12 from 14 and write only the remainder: 2.

in fact there are many variations of long division

The MathML community know of 10, such as `stackedleftlinetop`: see `http://www.w3.org/Math/draft-spec/mathml.html#chapter3_presm.mlongdiv.ex`
Note the utility of being able to re-use one example with different presentations.

And it's certainly subject area specific

For example $(2,4)$ might be

- **Set Theory** The ordered pair "first 2, then 4"
- **(Geometry)** The point $x = 2, y = 4$
- **(Vectors)** The 2-vector of 2 and 4
- **Calculus** Open interval from 2 to 4
- **Group Theory** The transposition that swaps 2 and 4
- **Number Theory** The greatest common divisor of 2 and 4

In general, these are **spoken** differently: the written text "we draw a line from (2,4) to (3,5)" is spoken "we draw a line from the point two four to the point three five" . This makes "text to speech" very difficult for (advanced) mathematics
: consider "Since $H_i \leq G$ for $i \leq n$"

Our Notation isn't perfect I (Landau Notation)

Orders of growth (The "Landau Notation" [Bac94])

- √ $O(f(n))$ for $\{g(n) | \exists N, A : \forall n > N \ |g(n)| < Af(n)\}$
- √ And similarly Ω, Θ etc.
- But we write "$n = O(n^2)$" when we should write "$n \in O(n^2)$"

  Generally spoken "n is big-O of n squared", not equals

This isn't the traditional use of "$=$", for example "$n = O(n^2)$" but *not* "$O(n^2) = n$"

Causes grief every time I have to explain this (I lecture the first-year Maths course that introduces this), and many books don't give the simple definition $\Theta(f(n)) = O(f(n)) \cap \Omega(f(n))$

[Lev07] is the only text I know to be "correct"

Our Notation isn't perfect II: Iterated functions

√ $\sin(x^2)$: square x, then apply sin

√ $(\sin x)^2$: apply sin to x, then square the result

√ $\sin(\sin(x))$: apply sin to x, then apply sin again

$\sin^2 x$ is generally used to mean $(\sin x)^2$:
"[This] is by far the most objectionable of any" [Bab30]

If anything, it should mean $\sin(\sin(x))$:

since this is the sense in which we write $\sin^{-1}(x)$ — apply the inverse operation of sin, not $1/\sin(x)$

An example of mathematical notation?

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \ddots}}}}$$

which is nearly always written as

$$\pi = 3 + \frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{292+}\cdots$$

Much easier for (manual) typesetting, and uses less space

So how might a computer display mathematical notation?

Historically	Some kind of image: GIF/JPEG
Typesetting	Many attempts, then TEX [Knu84]
Principle	boxes with width, height and depth
depth	is vital: recall continued fraction
Since 1998	(at least in theory) MathML (Presentation) [Wor99]
But	back then browsers didn't have depth — still a significant problem, and Chrome, for example, sometimes does and sometimes doesn't support MathML (reasons vary)
And	the range of fonts is often inadequate, or nonstandard
MathJax	is a very pragmatic solution [Mat11]

Linebreaking: a major challenge

How should a mathematical expression be broken across across multiple lines?

Author TeX, and LaTeX, provide no support for breaking displayed equations, and not much for “in-line” equations

when I reformat a document, re-breaking equations is a significant part of the effort

System the author of a web page has no control over the screen-size of the browser, so the browser *has* to break the expression

The author can give hints, and the MathML standard provides suggestions, but this is an unsolved problem (and an important one for e-books!)

MathML (Presentation)

This specifies the ‘presentation’ elements of MathML, which can be used to describe the layout structure of mathematical notation. $f(x)$, `f(x)` in TEX, would (best) be represented in MathML as

```
<mrow>
  <mi> f </mi>
  <mo> &ApplyFunction; </mo>
  <mrow>
    <mo> ( </mo>
    <mi> x </mi>
    <mo> ) </mo>
  </mrow>
</mrow>
```

Note that it is clear precisely what the argument of f is: this matters for line breaking and speech rendering — “f of x”, as well as meaning

But it **is** presentation

and, I would argue, largely written presentation, though MathML→speech is definitely better than predecessors, and good for “K-12” (school) mathematics

```
<mrow>
  <mo> ( </mo>
  <mn> 2 </mn>
  <mo> , </mo>
  <mn> 4 </mn>
  <mo> ) </mo>
</mrow>
```

(spoken “open bracket, two, comma, four, close bracket”)
is just as ambiguous as $(2,4)$ (indeed, it’s really the same thing) To ask what the mathematics “means”, we need MathML (Content)

MathML (Content)

"an explicit encoding of the underlying mathematical meaning of an expression, rather than any particular rendering for the expression" [Wor12]

Consider $(F+G)x$: this could be

multiplication or function application

```
<apply><times/>
  <apply><plus/>
    <ci>F</ci>
    <ci>G</ci>
  </apply>
  <ci>x</ci>
</apply>
```

```
<apply>
  <apply><plus/>
    <ci>F</ci>
    <ci>G</ci>
  </apply>
  <ci>x</ci>
</apply>
```

No need for brackets, as `<apply>` groups, and the meaning is explicit: in the first we have application of `<times/>` while in the second we are applying $F+G$

OpenMath: 1993–

This grew out of the computer algebra community: exchanging mathematics between different algebra systems

Extensibility was key: very few basic concepts

Basic objects `OMI` integers, `OMF` (IEEE) floating point numbers, `OMSTR` (Unicode) strings, `OMB` byte arrays, `OMV` (mathematical) variables, `OMS` OpenMath symbols

`OMA` (the concept of) function application

`OMATTR` attributes of an object

`OMBIND` binding variables (λ, $\sum_i$ etc.)

`OMERR` error objects

All else is built from these: even addition is just a symbol

OpenMath symbols

A symbol (or several) is defined in a *Content Dictionary* (CD), which lists the symbols and, formally or informally, their meaning

- `<OMS name="plus" cd="arith1"/>` the “addition” operator
- `<OMS name="times" cd="arith1"/>` the “multiplication” operator
- `<OMS name="times" cd="arith2"/>` non-commutative multiplication
- `<OMS name="log" cd="transc1"/>` the complex logarithm, with an informal specification of the branch cut (following [AS64])
- `<OMS name="arctan" cd="transc1"/>` the inverse tangent, with a **formal** relationship with log.

Anyone can wrte a Content Dictionary: `private`, `experimental` and can become `official`

MathML (Content) evolution

MathML was the first XML application

1.0: 1998 "K–12" (Kindergarten to High School) Mathematics: 90 elements

2.0: 2000 rather more calculus: 127 elements

2.0 2nd ed: 2003 ability to extend via OpenMath

3.0: 2010 Full interoperability with OpenMath

3.0 2nd ed: 2014 (some bug fixes)

so now `<times/>` is just a shorthand for
`<OMS name="times" cd="arith1"/>`
OpenMath workshop at CICM 2016 next week
(`http://cicm-conference.org/2016/cicm.php`) will consider closer integration

How might a computer understand written mathematics?

The technical term is **parsing** and there are papers, books and numerous tools (`flex`, `bison` etc.) to do this, for over fifty years

But two-dimensional parsing? Little literature and no tools

It's not even clear what the specification would be

A few packages, both for reverse-engineering PDF [BSS12, Suz11] and for handwritten mathematics [HW13]

Generally a mass of heuristics, often with machine-learning

And some of the tables are indescribable

[AS64, p.78]

4.3.131

$$\int \frac{dz}{a+b\sin z}=\frac{2}{(a^2-b^2)^{\frac{1}{2}}}\arctan\frac{a\tan\left(\frac{z}{2}\right)+b}{(a^2-b^2)^{\frac{1}{2}}}\qquad (a^2>b^2)$$

$$=\frac{1}{(b^2-a^2)^{\frac{1}{2}}}\ln\left[\frac{a\tan\left(\frac{z}{2}\right)+b-(b^2-a^2)^{\frac{1}{2}}}{a\tan\left(\frac{z}{2}\right)+b+(b^2-a^2)^{\frac{1}{2}}}\right]\qquad (b^2>a^2)$$

4.3.132 $$\int \frac{dz}{1\pm\sin z}=\mp\tan\left(\frac{\pi}{4}\mp\frac{z}{2}\right)$$

4.3.133

$$\int \frac{dz}{a+b\cos z}=\frac{2}{(a^2-b^2)^{\frac{1}{2}}}\arctan\frac{(a-b)\tan\frac{z}{2}}{(a^2-b^2)^{\frac{1}{2}}}\qquad (a^2>b^2)$$

$$=\frac{1}{(b^2-a^2)^{\frac{1}{2}}}\ln\left[\frac{(b-a)\tan\frac{z}{2}+(b^2-a^2)^{\frac{1}{2}}}{(b-a)\tan\frac{z}{2}-(b^2-a^2)^{\frac{1}{2}}}\right]\qquad (b^2>a^2)$$

4.3.141 $$\int_0^{\pi}\sin^2 nt\,dt=\int_0^{\pi}\cos^2 nt\,dt=\frac{\pi}{2}$$

(n an integer,

4.3.142 $$\int_0^{\infty}\frac{\sin mt}{t}\,dt=\frac{\pi}{2}\qquad (m>0)$$

$$=0\qquad (m=0)$$

$$=-\frac{\pi}{2}\qquad (m<0)$$

4.3.143 $$\int_0^{\infty}\frac{\cos at-\cos bt}{t}\,dt=\ln(b/a)$$

4.3.144 $$\int_0^{\infty}\sin t^2\,dt=\int_0^{\infty}\cos t^2\,dt=\frac{1}{2}\sqrt{\frac{\pi}{2}}$$

4.3.145

$$\int_0^{\pi/2}\ln\sin t\,dt=\int_0^{\pi/2}\ln\cos t\,dt=-\frac{\pi}{2}\ln$$

4.3.146 $$\int_0^{\infty}\frac{\cos mt}{1+t^2}\,dt=\frac{\pi}{2}e^{-m}$$

[AS64, p.576]: metasymbols $p, q \in \{c, d, s, n\}$

16.25. Notation for the Integrals of the Squares of the Twelve Jacobian Elliptic Functions

16.25.1 $$\mathrm{Pq}\, u = \int_0^u \mathrm{pq}^2 t\, dt \text{ when } \mathrm{q} \neq \mathrm{s}$$

16.25.2 $$\mathrm{Ps}\, u = \int_0^u \left(\mathrm{pq}^2 t - \frac{1}{t^2}\right) dt - \frac{1}{u}$$

Examples

$$\mathrm{Cd}\, u = \int_0^u \mathrm{cd}^2 t\, dt, \quad \mathrm{Ns}\, u = \int_0^u \left(\mathrm{ns}^2 t - \frac{1}{t^2}\right) dt - \frac{1}{u}$$

Even the one-dimensional parsing is hard:

What does juxtaposition mean?

Number formation 23 ($2 \cdot 10 + 3$)

Word formation sin

function application $\sin x$ (<sin/>⁡x)

Multiplication xy (x⁢y)

Concatenation M_{ij} (i⁣j)

Addition $4\frac{1}{2}$ (4⁤...)

(for technical reasons, this isn't 4&InvisiblePlus;)

What is M_{12}? — "Em twelve" or "Em one two"?

Juxtaposition “explained” [Dav14, Table 1]

left weight	right weight	meaning	example
normal	normal	lexical	sin
normal	italic	application	$\sin x$
italic	italic	multiplication	xy (but M_{ij})
italic	normal	multiplication	$a \sin x$
digit	digit	lexical	42 (but M_{42})
digit	italic	multiplication	$2x$
digit	normal	multiplication	$2 \sin x$
normal	digit	application	$\sin 2$
	(but note the precedence in		$2 \sin 3x$)
italic	digit	error	$x2$
		(but reconsider)	x^2 or x_2?
digit	fraction	addition	$4\frac{1}{2}$
italic	greek	application^{-1}	$a\phi$
		(as in group theory)	i.e. $\phi(a)$
italic	(	unclear	$f(y+z)$ or $x(y+z)$

Consequences

- Compare “$\sin x$” (`$\sin x$`) with “$\rm sin x$” (`${\rm sin}x$`)
- The (trained!) eye is very sensitive to these differences of spacing
- Note also that the font drives the **meaning** of juxtaposition
- Hence the requirement to digitise mathematics more carefully than normal text (at least 400dpi, preferably 600dpi, whereas normal text is fine at 300dpi)
- Size of characters also matters: at least as important a clue as vertical alignment to sub/super scripts, especially prescripts

We’ve come a long way from just images, but there’s still a long way to go: in particular *searching* for formulae is still an unsolved problem (MathSearch workshops/challenges)

Conclusions

1. "Mathematical Notation" is like "Chinese cooking": all one to the outsider, but a wide variety of tastes with subtle fusions to the expert

   * Hence recognition of subject area is crucial (and hard to do automatically)

2. "All variables are equal" (α-conversion) isn't true in practice: $f(y+z)$ versus $x(y+z)$, or $n_i \leq n$ versus $G_i \leq G$ versus $R_i \leq R$, however, there's no theory here (except in relativistic summation)

   ? Is $N_i \leq N$ about numbers, or normalisers? Probably needs a lot of context.

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