Triangular Decomposition of Semi-algebraic Systems

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Overview

Regular chains and triangular decompositions are fundamental tools for describing the complex solutions of polynomial systems. We introduce adaptations of these tools to the real analogue: semi-algebraic systems. We show that any such system decomposes into finitely many *regular semi-algebraic systems*. We propose two specifications of such a decomposition, with corresponding algorithms. Under some assumptions, one algorithm runs in *singly exponential time* w.r.t. the number of variables. Our Maple implementation illustrates the effectiveness of our approach.

Basic Concepts

Let T be a regular chain of $\mathbb{Q}[x_1 < \cdots < x_n]$ with free variables $\mathbf{u} = u_1, \cdots, u_d$. Let $P \subseteq \mathbb{Q}[x_1 < \ldots < x_n]$ s.t. $[T, P]$ forms a regular system. Let $\mathcal{Q}$ be a quantifier-free formula of $\mathbb{Q}[\mathbf{u}]$. Let R be the *triple* $[\mathcal{Q}, T, P]$. Denote by $Z_{\mathbb{R}}(R)$ the set $\{(u,y) \mid \mathcal{Q}(u), t(u,y) = 0, p(u,y) > 0, \forall (t,p) \in T \times P\}$.

Definition. *R is a regular semi-algebraic system if*

(i) $\mathcal{Q}$ defines a non-empty open semi-algebraic set S in $\mathbb{R}^d$,

(ii) $[T, P]$ specializes well at each point u of S,

(iii) the system $[T(u), P(u)_>]$ has real zeros, for all $u \in S$.

Theorem. *Let F, $N_{\geq}$, $P_>$, and $H_{\neq}$ be respectively a set of polynomial equations, non-negtive inequalities, positive inequalities and inequations. Every semi-algebraic system $\mathfrak{S} = [F, N_{\geq}, P_>, H_{\neq}]$ of $\mathbb{Q}[\mathbf{x}]$ can be decomposed as a finite union of regular semi-algebraic systems $\mathcal{R}$ s.t. the union of their zero sets is that of $\mathfrak{S}$. We call $\mathcal{R}$ a (full) triangular decomposition of the semi-algebraic system $\mathfrak{S}$.*

Let d be the dimension of *the constructible set* $\{x \in \mathbb{C}^n \mid f(x) = 0, g(x) \neq 0, \text{ for all } f \in F, g \in P \cup H\}$.

Definition. *A finite set of regular semi-algebraic systems R_i is called a lazy triangular decomposition of $\mathfrak{S}$ if*

- *for each i, $Z_{\mathbb{R}}(R_i) \subseteq Z_{\mathbb{R}}(\mathfrak{S})$ holds, and*
- *there exists $G \subset \mathbb{Q}[\mathbf{x}]$ s. t. $Z_{\mathbb{R}}(\mathfrak{S}) \setminus (\cup_{i=1}^t Z_{\mathbb{R}}(R_i)) \subseteq Z_{\mathbb{R}}(G)$, where the complex zero set $V(G)$ has dimension less than d.*

We denote respectively by LazyRealTriangularize and RealTriangularize an algorithm to compute a lazy and full triangular decomposition of a semi-algebraic system.

Examples

Example 1. *Solve the following Quantifier Elimination problem:*

$$(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})[f = g = 0 \wedge y \neq 0 \wedge xy - 1 < 0],$$

where $f = x^3 - 3xy^2 + ax + b$, $g = 3x^2 - y^2 + a$

The related semi-algebraic system:

$$\mathfrak{S} := \begin{cases} f = 0, \\ g = 0, \\ y \neq 0, \\ xy - 1 < 0 \end{cases}$$

The triangular decomposition of $\mathfrak{S}$ can be computed as in the following Maple session:

```
> with(RegularChains) :
> F := [x^3-3*x*y^2 + a*x + b, 3*x^2-y^2 + a] : H := [y] : P := [1-x*y] : N := [ ] :
  R := PolynomialRing([y, x, b, a]) :
  st := time( ) :
  rtd := RealTriangularize(F, N, P, H, R, output = record);
  time( ) − st;
```

$$rtd := \left\{ \begin{array}{c} y^2 - 3x^2 - a = 0 \\ 0 < 1 - xy \\ 8x^3 + 2ax - b = 0 \\ 0 < 4a^3 + 27b^2 \\ 27b^4 + 4a^3b^2 - 16a^4 - 512a^2 - 4096 \neq 0 \end{array} \right., \left\{ \begin{array}{c} xy + 1 = 0 \\ 0 < 1 - xy \\ (2a^3 + 32a + 18b^2)x + b(-48 - a^2) = 0 \\ 27b^4 + 4a^3b^2 - 16a^4 - 512a^2 - 4096 = 0 \end{array} \right.$$

$$0.186 \qquad (1)$$

One can read the QE results directly from the decomposition as:

$$(0 < 4a^3 + 27b^2 \ \wedge\ 27b^4 + 4a^3b^2 - 16a^4 - 512a^2 - 4096 \neq 0) \quad \vee \quad 27b^4 + 4a^3b^2 - 16a^4 - 512a^2 - 4096 = 0,$$

which can be further reduced to $0 < 4a^3 + 27b^2$.

Example 2. *Triangular decomposition of the intersection of two surfaces: $Sofa = x^2 + y^3 + z^5 - 1$ and $Cyl = x^4 + z^2 - 1$.*

```
> Sofa := x^2 + y^3 + z^5 :
  Cyl := x^4 + z^2 − 1 :
  R := PolynomialRing([z, y, x]) :
  st := time( ) :
  RealTriangularize([Sofa, Cyl], [ ], [ ], [ ], R, output = record);
  time( ) − st;
```

$$\left\{ \begin{array}{c} (1 - 2x^4 + x^8)z + y^3 + x^2 = 0 \\ y^6 + 2x^2y^3 + 10x^{12} - 10x^8 + x^{20} - 5x^{16} + 6x^4 - 1 = 0 \\ x < 1 \\ 0 < x + 1 \\ x^{12} - 4x^8 + 5x^4 - 1 \neq 0 \end{array} \right., \left\{ \begin{array}{c} z = 0 \\ y + 1 = 0 \\ x - 1 = 0 \end{array} \right., \left\{ \begin{array}{c} z = 0 \\ y + 1 = 0 \\ x + 1 = 0 \end{array} \right.,$$

$$\left\{ \begin{array}{c} (1 - 2x^4 + x^8)z + x^2 = 0 \\ y = 0 \\ x^{12} - 4x^8 + 5x^4 - 1 = 0 \end{array} \right., \left\{ \begin{array}{c} (1 - 2x^4 + x^8)z - x^2 = 0 \\ y^3 + 2x^2 = 0 \\ x^{12} - 4x^8 + 5x^4 - 1 = 0 \end{array} \right.$$

$$0.115 \qquad (3)$$

Complexity Estimates

Assumptions:

- $V(F)$ is equi-dimensional of dimension d,
- $x_1, \ldots, x_d$ are algebraically independent modulo each associated prime ideal of the ideal generated by F in $\mathbb{Q}[\mathbf{x}]$,
- F consists of $m := n - d$ polynomials, $f_1, \ldots, f_m$.

Let δ, $\hbar$ be respectively the maximum total degree and height of polynomials in F.

Proposition. *Within $m^{O(1)}(\delta^{O(n^2)})^{d+1} + \delta^{O(m^4)O(n)}$ operations in $\mathbb{Q}$, one can compute a Kalkbrener triangular decomposition $E_1, \ldots, E_e$ of $V(F)$, where each polynomial of each E_i*

- *has total degree upper bounded by $O(\delta^{2m})$,*
- *has height upper bounded by $O(\delta^{2m}(m\hbar + dm\log(\delta) + n\log(n)))$.*

From $E_1, \ldots, E_e$, a lazy triangular decomposition of F can be computed in $\left(\delta^{n^2} n4^n\right)^{O(n^2)} \hbar^{O(1)}$ bit operations.

Experimental Results

Table 1 Timings for algebraic varieties

system	#v/#e/d	G	T	LR
Hairer-2-BGK	13/ 11/ 4	25	1.924	2.396
Collins-jsc02	5/ 4/ 3	876	0.296	0.820
Leykin-1	8/ 6/ 4	103	3.684	3.924
8-3-config-Li	12/ 7/ 2	109	5.440	6.360
Lichtblau	3/ 2/ 11	126	1.548	11
Cinquin-3-3	4/ 3/ 4	64	0.744	2.016
Cinquin-3-4	4/ 3/ 5	$> 1h$	10	22
DonatiTraverso-rev	4/ 3/ 8	154	7.100	7.548
Cheaters-homotopy-1	7/ 3/ 7	3527	174	$> 1h$
hereman-8.8	8/ 6/ 6	$> 1h$	33	62
L	12/ 4/ 3	$> 1h$	0.468	0.676
dgp6	17/19/ 2	27	60	63
dgp29	5/ 4/ 15	84	0.008	0.016

Table 2 Timings for semi-algebraic systems

system	#v/#e/d	T	LR	R	Q
BM05-1	4/ 2/ 3	0.008	0.208	0.568	86
BM05-2	4/ 2/ 4	0.040	2.284	$> 1h$	FAIL
Solotareff-4b	5/ 4/ 3	0.640	2.248	924	$> 1h$
Solotareff-4a	5/ 4/ 3	0.424	1.228	8.216	FAIL
putnam	6/ 4/ 2	0.044	0.108	0.948	$> 1h$
MPV89	6/ 3/ 4	0.016	0.496	2.544	$> 1h$
IBVP	8/ 5/ 2	0.272	0.560	12	$> 1h$
Lafferriere37	3/ 3/ 4	0.056	0.184	0.180	10
Xia	6/ 3/ 4	0.164	191	739	$> 1h$
SEIT	11/ 4/3	0.400	$> 1h$	$> 1h$	$> 1h$
p3p-isosceles	7/ 3/ 3	1.348	$> 1h$	$> 1h$	$> 1h$
p3p	8/ 3/ 3	210	$> 1h$	$> 1h$	FAIL
Ellipse	6/ 1/ 3	0.012	$> 1h$	$> 1h$	$> 1h$

Notations: $\#v$,$\#e$, d - the number of variables, equations and the dimension, Q - QEPCAD B
G - Groebner:-Basis with plex order, T - Triangularize, LR - LazyRealTriangularize , R - RealTriangularize