Formal Specifications of Analytic Functions (Possible PhD Project)

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It is relatively easy to specify functions such as z^2 or $\exp(z)$ as functions from the complex numbers to the complex numbers. Even square root is harder to specify, and much harder to reason with, essentially due to the branch cut. Logarithms, or other functions defined by analytic continuation but subject to branch cuts, have never been formally defined as complex functions. This project will build on previous work of the supervisor and colleagues to produce such a formal definition framework, and tools for manipulating such definitions.

1 Understanding of Functions

In computing, we are essentially forced to adopt "the table-makers view" [Dav10] of functions: a mapping $\mathbf{C} \to \mathbf{C} \cup \{\perp\}$.

Notation 1 *Let $f : \mathbf{C} \to \mathbf{C} \cup \{\perp\}$ be a function defined (in some manner to be specified — see section 3) such that it is locally analytic except at certain singularities (inherent in its definition) and ranch cuts (semi-arbitrarily imposed to sacrifice continuity for uniqueness of definition).* We essentially only consider functions of one argument in this note, though the same principles apply in greater generality, and many practically-interesting examples involve such functions.

See also [Dav07]

2 Hard-coded Branch Cuts

Assuming that we are prepared to handle expressions containing only a finite set of hard-coded functions (typically the "elementary" functions, fundamentally log and exp), then quite a lot has been written about the challenges of manipulating these [Bra93, DF94, BD02, BBD03, BBDP04, BBP05, BBDP05, BBDP07, Phi11]. However, even the most developed such algorithms [BBDP07] have limitations, both theoretical and practical.

1. f should have only a finite number of branch cuts. This is violated, for example, by $\log \sin z$, whose branch cuts are $((2n-1)\pi, 2n\pi) \subset \mathbf{R} : n \in \mathbf{Z}$. In this case, and for some purposes, it is possible to regard the whole of $\mathbf{R}$ as an "extended ranch cut", but, to the ets of our knowledge, this has not been explored at all.

2. f should have only algebraic branch cuts.

3. The branch cuts should be separated

4. ...

3 Open-ended Branch Cuts

There are various ways of specifying a function for use in Notation 1.

1. As the inverse function f of an analytic function g, as $\log(z)$ is the inverse of $\exp(z)$, or $\sqrt{z}$ of z^2. Such an inverse needs a starting point, e.g. $\sqrt{z}$ is that inverse of z^2 with $\sqrt{1} = 1$, and then it can be extended by analytic continuation. The zerosof g' are then the potential branch points.

2. As the solution of a linear differential equations. This is considered in [CDKS11], who propose some "natural" rules for determining, in many case, the branch cuts. These generalise the rules for the elementary functions stated in [Kah87], and which abstract the modern literative consensus [fST10].

3. As a definite integral, for example the Γ function [H$\ddot{8}$7]. To the best of our knowledge no work has been done on these, and indeed the literature contains several errors [Dav02].

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