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Contents

Abstract

Chapter 1

16 September

1.1 Gunnar Carlsson — Topology and Data

D'Arcy Thompson was interested in shape and form. Then there was statistical shape theory (Kendall, Bookstein etc.). Define Σ_n^k as $(\mathbf{R}^n)^k/\mathcal{R}_n^k$ where $\mathcal{R}_n^k$ is the rigid modtions of $\mathbf{R}^n$, acting diagonally on the k-fold product. Gromov has shown that there's a metric all spaces, so the original asumption of Euclideanness is no longer required. Even Euclidean data have a choice of metric. Genomic data have a combination of Hamming distance and others. We are also interested in large k. The metrics in genomics are pretty subjective, so distance between points are not very comparable either (and hence ideas such as curvature aren't very useful). If geometry is the study of metrics, topology is the study of that which doesn't change under change of metric. We might wish to import holmology, e.g. $\beta_1 \neq 0$ implies periodicity, and higher Betti numbers tend to mean something. "Clustering is the statistician's equivalent of finding connected components". Hence we get the dendrogram, and hierarchical clustering. A similar summary can be found for homology in all degrees.

We assume a data set X witha distance function d, and scale parameter ϵ. We get a *persistence Boolean vector space* by applying H_k to the increasing sequences of the Vietoris–Rips complexes. The Betti numbers are the doimensions of H_k, and can be computed by Gaussian elimination. Betti numbers are the same for Klein bottle and torus, but we can also do modulo 3 homology. Consider$f = q(\lambda(x))$ where q is a single variable quadratic andλ is linear, with $\int_D f = 0$, $\int_d f^2 = 1$.

It turns out that there are neurons which fire on particular patterns, as seen in his diagrams.

1.1.1 Mappin

Algebraic topology can produce signatures which can help in pointing out the data set. Can one obtain flexible topological mapping methods? Yes: joint work with Singh and others. Let X be a space and

1.2 Jonathan Pila — A model-theoretic approach to certain Diophantine problems

1.2.1 Diophantine problems

The basic idea is that there must be a reason why a Diophantine problem has a lot of solutions. Example: Mordell conjecture. In C is not rational or elliptic, then $C(\mathbf{Q})$ is finite — proved by Faltings in 1983. Bombieri–Lang conjecture: If a projetive variety V has infinitelymanyrational points then V must contain a non-trivial rational image of an abelian variety on $\mathbf{P}^n$. Implies Modrdell. Hence

> Geometry govens arithmetic.

Manin–Mumford conjecture on the finiteness of the torsion points on curves of genus ≥ 2 embedded in their jacobians. Lang's extension to $V \subset A$ proved by Raynaud in 1984. Then an extension to ....

The multiplicative case of MM considers an algebraic variety $V \subset (\mathbf{C}^*)^n$. Torsion points on V are those whose ocordinates are all roots of unity. We conjecture that this contains onlyfinitely many torsion points unless V contains a subtorus of positive dimension, or a translate of that by a torsion point.. A subtours might be $x^2y^3z = 1$, and a coset might be $x^2y^3z = \exp(2\pi i/7)$.

Abelian varieties $A \equiv \mathbf{C}^n/\Lambda$. Raynaud 1983/4, Hrushovski via model theory.

1.2.2 Model-theoretic analogue

Look at rational points on definable sets. JP+Bombieri 1989 — upper bounds for integer points on plane curves. Then rational points on transcendental algebaric curves. Heath-Brown has p-asdic version for projective algebaric varieties (2002). JP+Wilkie (2006) A **reasonable** set in $\mathbf{R}^n$ has **not many** rational points unless it has a semi-algebraic subset of positive dimension.

For $Z \subset \mathbf{R}^n$ define $Z(\mathbf{Q}, T) = \{x \in Z : x \in \mathbf{Q}^n, H(x) \leq T\}$. Then "not many" means that $|Z(\mathbf{Q}, T)| <$ any polynomial in T. Given ϵ, there is a $d(\epsilon)$ such that $Z(\mathbf{Q}, T)$ is contained in at most $c(||f||_{d^2/2})T^\epsilon$ intersections $\cup C$ with real algebraic curves of degree $\leq d(\epsilon)$ (possiby reducible).

So the constant $c(f, \epsilon)$ depends n the maximum size of the derivatives of f, and the maximal number of intersection points of Z with algebraic curves up to some degree depending on ϵ. Gabrielov's theorem gives the required bounds.

It turns out that P-minimal structures are the natural setting. **reasonable** will mean "deinfablein an O-minimal structure over $\mathbf{R}$. A collection $\mathcal{S}$ of sets in $\mathbf{R}^n$ which contains all semi-akgebraic sets:

- is closed under Boolean operations, cartiesian products and ocordinate projections
- and is **o-minimal**, i.e. the definable subsets of $\mathbf{R}$ in S and unions of points and intervals.

Grew from Tarski–Seidenberg. $(\mathbf{R}, 0, 1, +, \cdot, <)$ is decidable, and asked if $(\mathbf{R}, 0, 1, +, \cdot, \exp, <)$ is decidable. Note that $(\mathbf{R}, 0, 1, +, \cdot, \sin, <)$ is not decidable, since we get the integers immediately, and hence Gödel.

Khovanskii's theory of fewnomials shows that there are only finitely many connected components. Wilkie proved that $\mathbf{R}_{\exp}$ is o-minimal. Macintyre–Wilkie shows decidability of $\mathbf{R}_{\exp}$ modulo Schanuel's conjecture (over $\mathbf{R}$).

Examples of o-minimal structures include

- the semi-algebraic sets
- globally sub-analytic sets, i.e. all bounded semi-analytic sets
- sets definable with $y = e^x$
- $\mathbf{R}_{an,\exp}$ generated by both of these (van den Dries, Miller).

The last will do for us (but not $\mathbf{R}_{an}$.

Theorem 1 (Yomdin–Gromov) *Let $Y = V \cap [0, 1]^n$ where V is a closed algebraic subset of degree d, dimension k, For each $b = 1, 2, \ldots m$ there is an integer $N(n, d, b)$ such that Y can be paratemrized as the union of images of at most N maps $\psi : [0, 1]^k \to [0, 1]^n$.*

Definition 1 *The* algebraic part Z^{alg} *of a set set $Z \subset \mathbf{R}^n$ is the union of all connected positive-dimensional semialgebraic subsets of Z.*

Theorem 2 (JP,Wilkie, 2006) *Let $Z subset \mathbf{R}^n$ be deifniable in an o-minimal structure over $\mathbf{R}^n$, and $\epsilon > 0$. Then*

$$N(Z - Z^{\text{alg}}, T) \leq c(Z, \epsilon)T^{\epsilon}.$$

This is Archimedean in nature

> O-minimal geometry givens arithmetic.

1.2.3 Application

Zannier proposed a strategy for a new proof of MM for abelian varieties $A/\overline{Q}$. First consider the multiplicative case. Torsion cosets are known as special subvarieties, and torsionpoints as special coordinates. Our real coordinates on $\mathbf{C}^n$ will be $\Re(z), \Im(z)/2\pi$. Map $\mathbf{C}^n \to (\mathbf{C}^*)^n$ by exp (in each coordinate). Then pre-images of torsion points are rational points. The uniformization is $2\pi i\mathbf{Z}$-periodic, so cannot be deifnable. But its restriction to fundamental domain F is definable in $\mathbf{R}_{an,\exp}$. Let $Z = \pi^{-1}(V) \cap F$. Let W be an irreducible complex variety in $\pi^{-1}(V) \cap \mathbf{C}^n$. Then Ax–Lindemann-Weierstrass shows that the $\exp(a_i$ are algebaric independent unless the a_i are linearly algebraic over $\mathbf{Q}$ modulo constants.

To close this one needs Masser's lower bound.

Conjecture 1 *André–Oort conjecture is the analogue of MM for Shimura varieties. Then V contains only finitely many special points unless it contains a special subvariety of positive dimension.*

We have an unconditional proof of André–Oort for $\mathbf{C}^n$. We uniformise by j (instead of exp). Special points are then elliptic curves with complex multiplications. $j(\tau)$ is special iff τ is imaginary quadratic. As a lower bound, we use Siegel's unconditional (though ineffective) lower bound. To characterise we need Ax–Lindemann-Weierstrass for j.

1.3 Reinhard Diestel — Infinite graphs with ends: a topological approach

Infinite grapphs produce new phenomena. Most finite path/cycle theorems fail for infinite graphs: there are "too few" paths/cycles. Examples

1. A finite graph of high minimal degree has to contain a complete graph, as large as one wants. False for infinite graphs — consider a tree.
2. **Theorem 3 (Whitney)** *A graph G is planar iff there is a G^* (whose edges are the bonds of G) such that ....*

   Can a collection of edges be both the bonds of G_1 and the edge-sets of circuits in G_2? Whitney says this is possible iff we have a planar graph. But this too fails for infinite graphs — consider the grid, where bonds are infinite, but circults have to be finite.
3. **Theorem 4 (MacLane)** *G is planar iff $C(G)$ has a 2-basis (i.e. each edge occurs in at most two generators).*

   This too is wrong for infinite graphs.
4. Hamilton and other extremal theorems

The speaker drew examples (replacing vertices by "ends" and edges by "rays", which are infinite paths from one 'end' to another) of what might make sense as 'infinite cycles', but even these aren't enough. Even iterating this construction (even transfinitely) doesn't help.

We need a topological definition rather than a combinatorial one: Freudenthal compactification. We replace 'cycle' by 'circuit', and 'cycle space' by 'circuit space'. In this notation, one can define a 'topological spanning tree'. With these topological generalisations, all the standard theorems are either true, or too hard to prove! True includes Whitney, MacLane, Fleischner dots.

We can ask about Hamiltonian circuits (as opposed to cycles). But a self-respecting graph has $2^{\aleph_0}$ ends! He has an example of 'the wild circle'.

Having done the first homology, we can ask about higher ones. Example of a semi-infinite ladder with one end. There is a loop here which is *not* null-homogenous, even though its finite analogues are.

Theorem 5 (RD+Spoüssel 2009) *for all connected locally finite graphs G there are subsgroup embeddings $\pi_n(|G|) \to F_\infty \ldots$.*

It turns out we need to treat ends differently, only as 'glue' to patch holes (i.e. missing inner points of simplices)'.

As an example, we could take a hyperbolic graph/group, with a hyperbolic boundary.

1.3.1 Infinite matroids

In finite graph theory, tree-packing is a consquence of theorems in finite matroids. So are these topological spanning trees something to do with infinite matroids. Whitney's theorem also has a matroid version:

Theorem 6 (Whiteny: matroid) *G is planar iff the matroid dual of its cycle matriod is the cycle matroid of some graph, i.e. is graphic.*

Usually infinite matroids are deined as finitematroids plus

I4 an infinite set in independent as soon as its finite subsets are independent.

But this forces circuits to be finite. Hence Rado (1966) wanted an alternative.

Theorem 7 (Oxley 1992) *The models of any theory of infinte matroids with duality must be the structures proposed as "B-matroids" by Higgs i 1969.*

But there is no theory with this model (according to Wikipedia). However

Theorem 8 (RD+others, 16 March 2010) *There are 5 equivalent theories of infinite matroids with duality, in terms of*

1. *independent sets*
2. *bases*
3. *circuits*
4. *closure*
5. *???*

For finite sets, these reduce to the usual definition.

Examples include cycle and bond matroids in infinite graphs. The dual of finite bonds turn out to be infinite circuits, and of infinite bonds to be finite cycles.

In questions, there was a connection between the semi-infite ladder and the Hawaiian earring.

Chapter 2

17 September

2.1 Sanju Velani — The Badly Approximables

2.1.1 The classical case

[His talk on the schedule was marked as "TBA"]

What is the analogue of the classical set of badly approximable numbers

$$\mathrm{Bad} := \{\alpha \in \mathbf{R} : \liminf_{q\to\infty} q||q\alpha|| > 0\}$$

(where $||\cdot||$ means 'distance to nearest integer').

Theorem 9 (Dirichlet) *For every $\alpha \in \mathbf{R}$, there exist infinitely many q with $q||q\alpha|| \leq 1$.*

Theorem 10 (Jarnik, 1928) *The set*

$$\mathrm{Bad} := \{\alpha \in \mathbf{R} : \liminf_{q\to\infty} q||q\alpha|| > 0\}$$

is of full Hausdorff dimension ($\dim \mathrm{Bad} = 1$*).*

Theorem 11 (Khintchine 1925) *Measure theoretically we can improve on Dirichlet by a log: for almost all $\alpha \in \mathbf{R}$,*

$$q \log q||q\alpha|| \leq 1 \quad \text{i.o..}$$

What about

$$\overset{\lambda}{\mathrm{Bad}} := \{\alpha \in \mathbf{R} : \liminf_{q\to\infty} q(\log q)^{\lambda}||q\alpha||\} :$$

this has full measure if $\lambda \leq 1$, and 0 otherwise.

2.1.2 Multiplicative

If we take $(\alpha, \beta) \in \mathbf{R}^2$, then $q||q\alpha||||q\beta|| \leq 1$ infinitely often (Dirichlet on α, and $||q\beta|| < 1$ trivially).

Conjecture 2 (Littlewood)

$$\forall(\alpha, \beta) \in \mathbf{R}^2 \liminf_{q\to\infty} q||q\alpha||||q\beta|| = 0.$$

It's true of at least one of $\alpha, \beta \notin \text{Bad}$. Khintchine implies $\forall\alpha$ and almost all β

$$q \log q||q\alpha||||q\beta|| \leq 1 \qquad (*)$$

i.o.

Theorem 12 (Cassels–Swinnerton-Dyer 1955) *True if α, β cubic irrationals in the same field.*

Note that we do not know if cubic irrationals are badly approximable.

Theorem 13 (Peck 1961) *Same with an extra* log*.*

Theorem 14 (V+Pollington) *Given $\alpha \in \text{Bad}$,*

$$\dim\{\beta \in \text{Bad} : (\text{*}) \text{ holds}\} = 1$$

Theorem 15 (Gallagher 1962) *For ψ real positive decreasing function,*

$$|\left\{(\alpha, \beta) \in \mathbf{R}^2 : ||q\alpha|||q\beta|| \leq \psi(q) \text{ i.o.}\right\}| = \begin{cases} 0 & \sum \psi(q) \log(q) < \infty \\ \mathit{full} & \sum \psi(q) \log(q) = \infty \end{cases}.$$

Let

$$\overset{\lambda}{\text{Mad}} = \{(\alpha, \beta) \in \mathbf{R}^2 : \liminf_{q\to\infty} q(\log q)^\lambda ||q\alpha||||q\beta|| > 0\}$$

Then, conjecturally[1],

$$|\overset{\lambda}{\text{Mad}}| = \begin{cases} o & \lambda \leq 2 \\ \mathit{full} & \lambda > 2 \end{cases}$$

We know that

1. $\text{Mad}^\lambda = \emptyset$ if $\lambda < 1$;
2. $\dim(\text{Mad}^\lambda) = 2$ if $\lambda \geq 1$.

How can we improve on Gallagher? We know that For almost all $(\alpha, .\beta) \in |R^2$

$$\liminf_{q\to\infty} q(\log q)^2 ||q\alpha||||q\beta|| \qquad (**).$$

V asserts this is rue for **all** α and almost all β.

[1] This is what JHD understood.

Theorem 16 (Schmidt 1964) *For $\epsilon > p$ and almost all α,*

$$N \log N \ll \sum_{q=1}^{N} \frac{1}{||q\alpha||} \ll N(logN)^{1+\epsilon}$$

Recent work V+others make the first $\ll$ for all α, and the second for $\epsilon = 0$.

Conjecture 3 (Duffin–Schaefer 1941) *For ψ, almost all α $|q\alpha - p| < \psi(q)$ i.m. $(p, q) \in \mathbf{Z} \times \mathbf{N}$ with $(p, q) = 1$ if $\sum \psi(q)\frac{\phi(q)}{q} = \infty$.*

2.2 Peter Keevash — Restricted intersections and their applications

Example 1 *Eventown has 100 inhabitants, in clubs*

1. *Every club has to have an even number of members.*
2. *Any two clubs shae an even number of memebrs*
3. *No two clubs have identical membership.*

Trivial if eventown has 50 couples, who like to be in the same clubs. 2^{50} *possible clubs*

Change rule 1 to be "Every club has to have an odd number of members", to get OddTown.

More formally, label the inhabitants by $[n] = \{1, \dots, n\}$. Identify clubs with subsets of $[n]$. Clearly each inhabitant in a singleton club will do. Can we do better?

Theorem 17 (Berlekamp 1969) *Oddtown can have at most n clubs.*

Each club A_i gives a characteristic vector $v_i \in \mathbf{F}_2^n$. Then $\{v_i\}$ is linearly independent because $v_i \cdot v_i = |A_i| = 1 \pmod 2$, but $v_i \cdot v_j = |A_i| = 0 \pmod 2$ if $i \neq j$.

Suppose all clubs are the same size, and two differnet clubs share precisely one memebr. Then the Fano plane has 7 inhabitatnts and 7 clubs. Note there is a dual system by interchanging people and clubs.

Suppose $L \subset \mathbf{N}$. We say A is L-intersecting if $|A \cap B| \in L$ if $A \neq B$. For L-intersecting mod p require $|A \cap B| \in L \pmod p$ and $|A| \neq 0 \pmod p$.

Theorem 18 (Chaudhuri–Wilson–...) *Supose p prime or 0, and $|L| = s$ Then any L-intersecting family ....*

Conjecture 4 (Nelson 1950) *How many colours are needed to colour the points of $\mathbf{R}^2$ such that no two points at distance 1 get the same colour.*

The answer is between 4 and 7 (hexagonal tiling).

Theorem 19 (Frankl–Wilson 1081) *In d dimensions, need at least 1.1^d colours.*

Let p be rpime, $k = 2p - 1$, $d = 4p$. Rescale to look at points $\sqrt{2p}$ apart. Let $V = \{x \in \{0,1\}^d : \sum x_i = k\}$. How many coours des V need? Suppose $S \subset V$ with no two points at distance $\sqrt{2p}$. Then identify S witha k-uniform set system $\mathcal{A}$ on $[d]$. For $A, B \in \mathcal{A}$ we have $d(V_A, v_B)^2 = 2(k - |A \cap B| \neq 2p$, i.e. $|AcapB| \neq k - p = p - 1$. Then apply previous.

2.2.1 Explicit Ramsey Graphs

$R(s)$ is the smallest n such that any red/blue colouring of the edges of the complete graph K_n contains a monochromiatic K_s. Ramsey [1930] showed they are finite. We now $\sqrt{2}^s \leq R(s) \leq 4^2$. The upper bound is not hard, but the lower bound (Erdős) was an early use of the probabilistic method.

A Frankl–Wilson construction has p prime, $k = p^2 - 1$, $v = p^3$, $n = \binom{v}{k}$. Identify verticesof K_n with k-subset of $[v]$. Colour AB red if $|A \cap B| = 1$, blue otherwise. This bound is super-polynomial ($R(s) > s^{w(s)}$ for $w \ldots$) but far from optimal.

Computer Scientists care about randomness, e.g. Miller–Rabin, probabilistic zero testing of polynomials etc.

How do we generate randomness. Consider a two-source bit extractor $f : \{0,1\}^s \times \{0,1\}^s \to \{0,1\}$ such that $\mathbf{P}(f(X,Y)) = \frac{1}{2} \pm \epsilon$. Even for a much weaker probelm (dispersion), we would need bipartite Ramsey graphs, which seems much harder.

2.2.2 Forbidden Intersections

Conjecture 5 (Erdős 1970s) *There is a $c > 0$ such that if $\mathcal{A}$ is a set system on $[n]$ with $|A \cap B| \neq n/4$ for any $A, B \in \mathcal{A}$ then $|\mathcal{A}| \leq (2-c)^n$?*

Hard to have good intuition here.

Theorem 20 (Frankl–Rödl 1987) *For any $0 < d < 1/2$ there is a $c > 0$ such that if $\mathcal{A}, \mathcal{B}$ are set systems on $[n]$ with $|AcapB| \neq dn$ for an $A \in \mathcal{A}$, $B \in \mathcal{B}$ then $|\mathcal{A}|, |\mathcal{B}| \leq (2-c)^n$?*

This has applications in communications complexity. Alice and Bob each have n bits, and Bob has to output a function of both. Clearly Alice could send all her bits to Bob — can we do better?

Theorem 21 (Buhrman, Cleve & Widgerson 1998) *If $f(a,b)$ is 1 when $a = b$, 0 when Hamming distance is $n/1$ and undefined otherwise. Then Alice has to send Bob cn bits.*

Quantum communications only needs $c \log n$ qubits! The Hamming distance condition is one on $|A \cap B|$, hence Frankl–Rödl can be used.

We say that $\mathcal{A}, \mathcal{B}$ on $[n]$ are L-cross-intersecting if $|A \cap B| \in L$ forall A, B. Problem: deltermine $P_L(n) = \max(|\mathcal{A}|, |\mathcal{B}|)$. Some presults, but

Biplane problem — is there a $\{2\}$- inttersecting $\mathcal{A}$ on $[n]$ for large n? Projectove planes give an infinite family if 2 is replaced by 1. All known biplanes are symmetric.

Let $m(n, L)$ be the maximum size of an L-interstecing set system on $[n]$, and $m(n, k, L)$ is laos k-uniform.

Conjecture 6 (Frankl) *$m(n, k, L) = \Theta(n^\alpha)$ for rational α*

Conjecture 7 (Deza–Erdős–Frankl 1978) *Is $m(n, 13, \{0, 1, 3\})$ quadratic or cubic in n?*

Even less is known for non-uniform problems. Füredi showed that $m(n, \{0, 2, 3\})$ is quadratic (but the upper and lower bounds are quite far apart). For modular problems, with prime powers m is polynomial (of currently unknown degree), but [Gromulsz 2000] $\exists \mathcal{A} \ldots$ modulo $6 \ldots |\mathcal{A}| > n^{c \log n / \log \log n}$, i.e. superpolynomial (just).

2.3 Nick Trefethen —Impossibility of fast stable approximation of analytic functions from equispaced samples

[Joint work with Platte (ASU) and Kuijlaars (KU Leuven): [PTK10]] — alternatively "you can't beat Gibbs and Rugne" Mathematicians split pure/applied and discrete continuous: I am applied/discrete.

Given analytic function f on $[-1, 1]$, can we recover f with exponential accurary from samples $x_1, \ldots, x_n$. If the points are equispaced and f is periodic, we can use trigonometric interpolation. But if f is not periodic, we don't have convergence — Gibbs. Fpr polynomial inerpolation, we can say yes if the x_i are Chebyshev, but if they are equispaced, we get exponential divergence — Runge phenomenon.

What if the x_i are equispaced but f is not periodic?

2.3.1 Trigonometric Interpolation

Suppose f is 2-periodic, analytic and bounded is a strip of half-width a based on $[-1, 1]$. $||f - p_n|| = O(e^{-\pi a n/2})$ and the same is true for $f' - p' \ldots$. There are two approaches to computing p_n — space and dual space. We have FFT (Gauss 1805) in Fourier space, and in direct space there is Barycentric interpolation [Henrici1979]. The Lebesgue constant $\Lambda_n \sim (2/\pi) \log n$. This is $||L||$ where L is the map date $\to$ interpolant.

In the non-periodic case we have Gibbs phenomenon $||f - p_n|| = O(1)||$ (actually due to [Wilbraham1848] — Cambridge & Dublin mathematical journal)

2.3.2 Polynomial Interpolation

Choose $x_j = \cos(\frac{j\pi}{n})$. We now look at an ellipse with foci $-1, 1$, and the sum of semi-axes ρ. $||f - p_n|| = O(\rho^{-n})$. In FFT space see [Ors71] and barycentric [Salzer1972],[Riesz1916]. [Runge1901] $||f - p_n|| \approx O(2^n)$ is the points are equispaced. The Lebesgue constant is $\Lambda_n \sim 2^n / \log n$ [Turetski1948], [Schonhage1961].

2.3.3 `chebfun`

A MatLab package that overloads basic operations.

```
x=chebfun('x');
f=sin(3*x)+sin(33*x)+sin(333*x);
plot(f)
max(f)
```

This package easily and accurately manipulates polynomials of degrees in the thousands.

```
h=max(f,x);
std(h);
```

[Boyd1992] used Tikhonov regularization. [Chandrasekharanetal2009] Minimum Sobolev norms. [Ber07] attacked the problem. [HR87] essentially ignored some points to get a Chebyshev-like grid, so is $O(2^{n'})$, where $n' \approx \sqrt{n}$ is the number of points actually used. [FH07] have other ideas. ...

There are also people trying to beat Gibbs. [Huybrechs2009] 'Onthe Fourier extension of non-periodic functions'. He *does* get exponentially-conergent approximatin. [DriscollFornberg2001] used a Padé approximation. Gottleib and colleagues did projection onto Gegenbauer basis, which they claim completely removes Gibbs phenomenon. [GelbTanner] 'completely removes the Gobbs phenomenon'. [TodmorTanner2002] use edge detcection.

These are all impressive in practice, with a few hundred data points (where naïve methods are disastrous). However, we claim that the ones that converge exponentialy all run into stability/precision problems if you push them far enough. This happens because the theorems in these papers, while correct, somehow miss the point, e.g. geometric convergence may be demonstrated, but must then be unstable.

So let $E \subset \mathbf{C}$ be compact with $[-1, 1]$ in its interior. Let $B(E)$ be the set of functions continuous on E, analytic on its interior. Let $\phi(n) : C[-1, 1] \to C[-1, 1]$ which depends only on the values on the n-grid.

Definition 2

$$\kappa(\phi_n) = \sup_f \ldots$$

Theorem 22 *Suppose* $\forall f \in B(E)$ *and* $\forall n \geq 1$

$$||f - \phi_n(f)||_{[-1,1]} \leq M\sigma^{-n}||f||_E$$

with $M < \infty$ and $\sigma > 1$. Then for some $C > 1$ and all sufficiently large n, $\kappa(\phi_n) \geq C^n$.

[i.e. exponential convergence implies exponential ill-conditioning]. Note that this applies to *arbitrary* approximation schemes, not necessarily polynomial, and indeed not necessarily linear.

Theorem 23 (Bernstein1912) *Let E be a ρ-ellipse. $||p||_E \leq \rho^d ||p||_{[-1,1]}$ if p is a polynomial of dgeree at most d.*

This leads to

$$\kappa(\phi_n) \geq \frac{1}{2} \sum_{\deg p \leq \alpha n} \frac{||p||_{[-1,1]}}{||p||_{n\text{-grid}}}.$$

Then use [CR92]. In their [PTK10] paper, the proof is the caption to figure 3.1!

Graphical illustration of Coppersmith–Rivlin. Can now quantify this phenomenon.

Q. Grid or equi-spacing?

A. Potential theory implies that one needs the Chebyshev distribution.

Q. Can you say anything about the degree of a polynomial needed?

A. Good question.

2.4 Chantal David — The fluctuations in the number of points of curves over finite fields

Let $\mathbf{F}_q$ be a finite field with q elements, and $\mathcal{F}$ a family of smooth (in an appropriate model) curves. Examples: hyperelliptic curves, cyclic curves etc. Zeta functions of curves

$$P_C(T) = \frac{P_C(T)}{(1-t)(1-qT)}$$

where P is a polynomial of degree $2g$ — Weil's Riemann hypothesis. Then we count the number of points.

2.4.1 Riemann Zeta function over $\mathbf{F}_q$

$\mathbf{Q}$	$\mathbf{F}_q(X)$
$\mathbf{Z}$	$\mathbf{F}_q[X]$
n integer	$F(X)$ polynomial
p positive prime	$P(X)$ monic irreducible

Then $\zeta(s, \mathbf{F}_q(X)) = \sum_{d\geq 0} q^d q^{--ds} = (1-q^{1-s})^d$. Then $P_C(T) = \prod_{j=1}^{2g}(1-T_{\alpha_j})$ and ....

1. Hyperelliptic curves

2. Cyclic p-covers: $C_f : Y^p = F(X)$ withgenus ...

3. Smooth place curves has genus $g = (d-1)(d-2)/2$

If the let g be fixed as $q \to \infty$m then $P_C(T) = \det(1 - P\sqrt{q}\Theta_C)$, where Θ_C is a unitary symplectic matrix in $USp(2g)$ with eigenvalues ....

Theorem 24 (Katz–Sarnak)

$$\lim_{q\to\infty} \frac{\sum_{C\in H_g(\mathbf{F}_q)} \frac{f(\Theta_c)}{Aut}}{\ldots} = \ldots$$

Why do we get certain probabilities when we count the number of points? For $Y^p = F(X)$. Let p be a prime dividing $q-1$. Then there are $(q-1)/p$ p-th powers in $\mathbf{F}_q^*$. We need to understand how character sums are distributed. If we want to count the number of F of degree d with a certain zero, there are q^{d-1}, so with probability $1/d$.

As a trivial example, probability $p^2 \not| n = 1 - 1/p^2$, so $\zeta_{Riemann}(2)\frac{6}{\pi^2}$ is the fraction of square-free integers.. So for square-free polynomials, I get a probability of $1 - \frac{1}{q^{2\deg P}}$. Hence we get $\zeta(2, \mathbf{F}_q) = \frac{q-1}{q}$ as the proportion of square-free polynomials. Then we get the Kurlberg–Rudnick sieve for square-free polynomials taking prescribed values.

Want we want is to count the polynomials F in $S_{d_1,\ldots,d_{p-1}}$. This can be done (hairier in details!).

2.5 Richard Kenyon — Dimers and Clusters

[Joint work with A. Goncharov]

Shows a dimer model on $\mathbf{Z}^2$ — each vertex chooses one of its nearest neighbours. So a random dimer model is a random perfect matching. Similarly a honeycomb grid.

Theorem 25 (???) *For any subset of honeycomb grid with adjacency matrix $K = k_{i,j}$, then the number of dimer covering is $\sqrt{|\det K|}$*

It is possible to weight the tilings (not sure how these are related to the dimers).

Goal is to study all possible dimer models (combinations of weights) on all periodic dimer graphs. Label the faces of the graph on the torus (= periodic graph on plane) w_i and let z_1, z_2 be the two homotopies of the torus. We can think of w_i as being the homotopy clas of going round that face. We have skew-symmetric pairing

$$\epsilon : H_1(\Gamma) \wedge H_1(\Gamma) \to \mathbf{Z}$$

(not the usual intersection pairing) as the sum of local contributions from each vertex. For each geodesic, there is a corresponding zig-zag path in the kernel of ϵ. We want a Poisson structure on $\mathbf{C}[w_1^\pm, \ldots, w_n^\pm, z_1^\pm, z_2^\pm]$ by $\{w_i, w_j\} =$

Table 2.1: Dimer⇔Teichmüller dictionary

Dimers	Riemann surface
bipartite graph on torus	ideal triangulation
mutation/urban renewal	flip move
face weights	cross ratois
Harnack curve+divisor	conformal structure
Space of face weights	Teichmüller space $\equiv \mathbf{R}^{6g-6+2n}$
tropical harnack curve	measured lamination
cluster algebra	cluster algebra
commutating Hamiltonians	[left blank]

$\epsilon(\gamma_i, \gamma_j) w_i w_j$, with similar rules for z, and extended by Leibniz to the whole algebra ($\{ab, c\} = a\{b, c\} + b\{a, c\}$).

Fix a base cover $m_0 \in M(G)$ the set of all dimer covers. Then $[m - m_0] \in H_1(G, \mathbf{R})$. But this is generated by $[w_i]$ and $[z_1]$ and $[z_2]$. Hence $[m - m_0] = \sum \alpha_i [w_i] + \beta_1 [z_1] + \beta_2 [z_2]$. Then define the weight of m to be $\prod w_i^{\alpha_i} z_1^{\beta_1} z_2^{\beta_2}$.

Theorem 26 (Goncharov–K) *The Poisson bracket defined a completely integrable system of dimension 2+2Area(N), with symplectic leaves of dimension 2int(N). A basis for the Casimir elements is given by (ratios of) boundary coefficients of P. The commuting Hamiltonian elements is given by ....* The proof of commutativity is basically combinatorial.

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