Better Simplification of Elementary Functions Through Power Series

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ABSTRACT

In [5], we introduced an algorithm for deciding whether a proposed simplification of elementary functions was correct in the presence of branch cuts. This algorithm used multi-valued function simplification followed by verification that the branches were consistent.

In [14] an algorithm was presented for zero-testing functions defined by ordinary differential equations, in terms of their power series.

The purpose of the current paper is to investigate merging the two techniques. In particular, we will show an explicit reduction to the constant problem [16].

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms

General Terms

Algorithms, Theory

Keywords

Simplification Power Series

1. INTRODUCTION

In [5], we introduced an algorithm for deciding whether a proposed simplification of elementary functions was correct in the presence of branch cuts. This algorithm works by:

(a) verifying that the proposed simplification is correct as a simplification of multi-valued functions;

(b) decomposing $\mathbf{C}$ (or $\mathbf{C}^n$ in the case of multivariate simplifications) according to the branch cuts of the relevant functions;

(c) checking that the proposed identity is valid on each relevant[1] component of that decomposition, by evaluation at a sample point. The fact that it is possible to use a single representative of each connected region is a consequence of the Monodromy theorem. (see [12] for example)

The algorithm works for cases where the only nesting allowed is that of square roots inside other elementary functions, and inverse (potentially multivalued) functions may only occur in the numerator. The rationale for these restrictions is given in [5, section 7.1]: the relevant one here is that we need to be able to ensure that two different branches of a multi-valued function are not accidentally equal at the sample point in step (c) above, as in $(x-2)\ln x$, where all the multivalued forms of $(x-2)\,\mathrm{Ln}\,x$ are equal at $x=2$. If there are inverse (potentially multivalued) functions in the denominator, then we cannot distinguish this case from the case where arbitrarily close values are taken: consider the function $\mathrm{Arctan}(x)/\,\mathrm{Arctan}(2x)$ evaluated at $x=1$. Depending on the choices of branch in the numerator and denominator, we can (by the density of the rationals) approximate the principal value $\arctan(1)/\arctan(2)$ arbitrarily closely, as shown in Table 1 (taken from [5]).

Table 1: Values of $\mathrm{Arctan}(1)/\,\mathrm{Arctan}(2)$

numerator branch	denominator branch	Value
arctan	arctan	0.7093881343802613367895 9
arctan $+2\pi$	arctan $+3\pi$	0.6711576795311655911882 6
arctan $+7\pi$	arctan $+10\pi$	0.7003195903480992321789 6

Here we present a different algorithm, which replaces explicit manipulation of multivalued functions by the manipulation of power series and their coefficients, as described in [14]. It will turn out that this algorithm has different strengths and

*We thank Dr. Smyth for drawing our attention to the identities shown in [18]
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[1]Assumptions [19] made by the user, or an explicit lack of interest in lower-dimensional components, may mean that not all components need to be analysed.

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weaknesses — see the table in section 7. Furthermore, much of the machinery, notably step (b) above and its potential interfacing to an `assume` facility, is common to both, so that a system could have "two strings to its bow".

NOTATION 1. *Lower case letters, as in* arctan*, or symbols as in* $\sqrt{\ }$ *, denote single-valued functions* $\mathbf{C} \to \mathbf{C}$*, whereas capitalised functions, as in* Arctan *or* Sqrt*, denote multivalued functions* $\mathbf{C} \to 2^{\mathbf{C}}$*.*

NOTATION 2. *The meaning, and choice of branch cuts, for the single-valued functions is as defined in [7], which tightens up the behaviour on branch cuts from [1].*

2. A NEW ZERO–TEST FOR FORMAL POWER SERIES

This is the title of [14], which

> presents a new zero-test for expressions which are constructed from formal power series solutions to algebraic differential equations using the ring operations and differentiation. ... We will be concerned with expressions that represent formal power series (in fact this approach covers most elementary calculus on special functions, using analytic continuation if necessary).

Hence in fact the functions being considered in [14] are essentially the same multivalued functions as in [5].

A convenient survey of other approaches is found in [14]. In outline, the algorithm presented in that paper runs as follows.

Given a formula $h(z)$ to test whether $h = 0$ (in the sense of functions $\mathbf{C} \to$Riemann surface, defined by power series and analytic continuation):

1. Compute a suitable differential equation Q satisfied by h (i.e. $Q(h) = 0$), with coefficients in the power series ring $\mathbf{C}[[z]]$: in practice these coefficients will also be formulae that we can expand in $\mathbf{C}[[z]]$.
2. From the shape of Q, compute an integer s such that, if the coefficients of z^0, z^1, ..., z^s in the $\mathbf{C}[[z]]$ expansion of h are all zero, then h is identically zero.
3. Verify that these coefficients are all zero.

Step 3 is non-trivial and calls for more explanation. In general, the power series coefficients will be from $\mathbf{C}$. As is usual in this area, it was assumed in [14] that the coefficients belong to an *effective field of constants* and so that there exists algorithms for zero equivalence. For many function fields however, the problem is known to be undecidable [16]. In practice however, it is often the case that an numerical evaluation, together with guaranteed precision bounds, will reveal when a constant is *not* zero; see section c-2.5. One has to be careful however of evaluating functions on their branch cuts, where such precision bounds are useless, as the smallest error can cause a jump in the value. More sophisticated techniques exist, see [17] for example, but rely on the truth of the Schanuel conjecture.

3. OUR ALGORITHM

We assume that we have a proposed simplification $f(z) \to g(z)$, which we wish to check is valid

- everywhere; or
- everywhere except on sets of measure zero; or
- within the validity of some `assume` [19] conditions on z.

Such a simplification might be generated by a tool such as Maple's `simplify(...,symbolic)`.

Let $h = f - g$. Then we wish to check that $h = 0$ on the region as described above. The algorithm proceeds roughly as follows (we will see later on that there are some complications that need to be addressed):

(a) computing[2] the differential equation $Q(h) = 0$ satisfied by h;

(b) decomposing $\mathbf{C}$ into *Regions* R_i according to the branch cuts of f and g;

(c) checking that the proposed identity is valid on each relevant component of that decomposition, by evaluating the appropriate number of initial conditions for Q required to describe the single-valued function $f - g$ at a sample point.

Step (a) is described in [14], and step (b) in [5, 4]. Therefore we concentrate on step (c) here, following reasonably closely the notation of [14].

NOTATION 3. *Let* δ *be the differential operator* $z\frac{\partial}{\partial z}$*.* $F^{(r)}$ *will denote* $\delta^r F$*.*[3]

NOTATION 4. *Let* $\mathcal{R}$ *be an effective power series ring:* $\mathbf{C}[z] \subseteq \mathcal{R} \subset \mathbf{C}[[z]]$*. If* $f \in \mathcal{R}$*, let* f_k *denote the coefficient of* z^k *in the expansion of* f*.*

Note that this restriction to power series rings rules out Puiseux series or logarithmic transseries, where the monomials are $z^{a_0}(\log z)^{a_1}(\log\log z)^{a_2}\ldots$ ($a_i \in \mathbf{Q}$ and may be negative, as in Laurent series, but not unboundedly negative). The second restriction is not a problem for us[4], since if such

[2] As we will see in the examples, f and g often satisfy the same linear differential equation, in which case h satisfies the corresponding homogeneous equation.

[3] One referee noted that Knuth uses θ instead

[4] In the setting of only one independent variable. If we did generalise to more than one independent variable (see section 7.2), this might need to be re-thought.

logarithmic terms are needed when expanding about a sample point z_0, then there is an essential singularity in h at that sample point (which therefore must be an isolated point[5]), and we need merely check that $f(z_0) =$ "undefined" $= g(z_0)$. The Puiseux series limitation is more serious, but again can be circumvented — see step (c3) below. This restriction also rules out Laurent series, a restriction not explicitly noted in [14], but again this can be circumvented — see step (c4) below.

NOTATION 5. *Let Q be a non-zero differential polynomial of order r defining $h = f - g$, with H the formal differential indeterminate corresponding to h, so that $Q \in \mathcal{R}[H, H^{(1)}, \ldots, H^{(r)}] \subset \mathcal{R}\{H\}$. Let v be a minimal (over all i) valuation of $\frac{\partial Q}{\partial H^{(i)}}$, i.e. the exponent of the smallest power of z to have a non-zero coefficient in the series expansion.*

LEMMA 1. *[14] After a transformation of the form $h \to h_0 + h_1 z + \cdots h_v z^v + \hat{h} z^{v+1}$, and division by a suitable power of z, we may assume that*

$$Q = LH + zM, \tag{1}$$

where $L \in \mathbf{C}[\delta]$ is non-trivial, and $M \in R\{H\}$ is the differential polynomial extension of R by H and its derivatives.

NOTATION 6. *Let $\Lambda(k) \in \mathbf{C}[k]$ be the polynomial obtained from L in equation (1) by interpreting δ as the indeterminate k.*

LEMMA 2. *[14] Equation (1) gives the recurrence relation*

$$h_k = -\frac{1}{\Lambda(k)}(M(h))_{k-1}, \tag{2}$$

except when k is a root of Λ.

Let s be the largest root of Λ in $\mathbf{N}$ (or -1 if none such exist). Then h is the unique solution of $Q(h) = 0$ with given initial power series coefficients $h_0, \ldots, h_s$.

Step (c) of the main algorithm can then be fleshed out as follows.

(c) For each appropriate region R_i of the complex plane viewed as $\mathbf{R}^2$, with corresponding sample point z_i, do the following.

- **(c1)** Perform the substitution $z := z - z_i$ in Q to bring z_i to the origin (in practice one might not do this, but rather rewrite the theory of [14] to expand about arbitrary points).
- **(c2)** If the solution to Q requires logarithmic terms, then R_i is $\{z_i\}$. Verify directly that $f(z_i) = g(z_i)$ (which will normally be "undefined" = "undefined").
- **(c3)** If the solution to Q requires non-trivial Puiseux series, there are two cases.
  - **(c3.1)** $R_i = \{z_i\}$ is zero-dimensional. Verify directly that $f(z_i) = g(z_i)$ (which may well be "undefined" = "undefined").
  - **(c3.2)** R_i is not zero-dimensional. Pick a different sample point z_i and restart at step (c1).
- **(c4)** If the solution to Q requires non-trivial Laurent series, then replace[6] h by $\tilde{h}/z^k$, where the Laurent series begins $c_{-k}z^{-k} + \cdots$.
- **(c5)** Perform the appropriate translation to put Q in the form of equation (1).
- **(c6)** Let Λ be as in Notation 6, and let s be the largest root of Λ in $\mathbf{N}$ (or -1 if none such exist).
- **(c7)** Verify that all the coefficients $h_0, \ldots, h_s$ of the power series solution h of Q are all zero, evaluating them as coefficients of $f - g$ (as transformed), and taking account of the genuine $\mathbf{C} \to \mathbf{C}$ meanings of f and g.

[5]Again, since we only have one independent variable.

4. EXAMPLES

Throughout this section, we are manipulating functions and their values. The reader may think that these manipulations ought themselves to be checked by the methods of this paper, but in fact we only use:

- manipulation of values, where the side-conditions can be explicitly checked;
- $\left(\sqrt{F}\right)^2 \to F$, which is universally valid.

4.1 Combining square roots

We consider the example $\sqrt{1-z}\sqrt{1+z} \stackrel{?}{=} \sqrt{1-z^2}$ from [5]. Write $p = \sqrt{1-z}$, $q = \sqrt{1+z}$, $r = pq$ and $\hat{r} = \sqrt{1-z^2}$, so that we are testing $r \stackrel{?}{=} \hat{r}$, i.e. $h = 0$ where $h = r - \hat{r}$.

(a) $p' = \frac{-1}{2p} = \frac{-p}{2(1-z)}$, $p' + \frac{p}{2(1-z)} = 0$. Similarly, $q' - \frac{q}{2(1+z)} = 0$. Therefore

$$r' = p'q + pq' = \frac{-pq}{2(1-z)} + \frac{pq}{2(1+z)} = \frac{-zr}{1-z^2}.$$

It is also the case that $\hat{r}' = \frac{-z\hat{r}}{1-z^2}$.

The differential equation is then $Q(h) = h' + \frac{z}{1-z^2}h$. Though not part of the algorithm, we note that the general solution is $h = \frac{c}{\sqrt{1-z^2}}$ for constant c. Whenever this is the case, c determines the correction factor required to make the purported equation hold on the different regions.

(b) This was analysed in [5], and the regions are

$$R_1 := \{z \mid \Re(z) > 1 \wedge \Im(z) = 0\}, \tag{3}$$

$$R_2 := \{z \mid \Re(z) < -1 \wedge \Im(z) = 0\} \tag{4}$$

and R_3, the complement of these.

[6]This step is not explicit in [14], but seems to be necessary in practice — see step c-3'.4 in section 4.1: there is no reason why Q should not have a Laurent series for solution. See also footnote **??**.

(c-1) R_1 with $z_1 = 2$.

(c-1.1) Applying $z := z - 2$, the equation becomes $Q = (-3 + 4z - z^2)H^{(1)} + (z^2 - 2z)H^{(0)}$.

(c-1.5) $\frac{\partial Q}{\partial H^{(1)}} = -3+4z-z^2$, so the valuation is zero. We do not actually need to compute $\frac{\partial Q}{\partial H^{(0)}}$, but this is $z^2 - 2z$. Hence $v = 0$, and the necessary substitution is $h \to h_0 + \hat{h}z$. But $h_0 = h(0) = r(2) - \hat{r}(2) = \sqrt{-1}\sqrt{3} - \sqrt{-3} = 0$ (with the standard meaning of $\sqrt{\ }$). Hence we replace h by $z\hat{h}$, (and therefore h' by $z\hat{h}' + \hat{h}$ which is $\hat{H}^{(1)} + \hat{H}^{(0)}$) to get $(-3+2z)\hat{H}^{(0)} + (-3+4z-z^2)\hat{H}^{(1)}$. In the format of equation (1), this is $Q = (-3\delta - 3)\hat{H} + zM$ where $M = -z\hat{H}^{(1)} + (2\hat{H}^{(0)} + 4\hat{H}^{(1)})$.

(c-1.6) Therefore $\Lambda = -3k - 3$, so $s = -1$.

(c-1.7) There is nothing to check.

Thus we find that $h \equiv 0$ on this region.

(c-2) R_2 with $z_2 = -2$. This is similar, so we will not repeat the calculation here, but again we find that $h \equiv 0$ on this region.

(c-3) R_3 with $z_3 = 0$.

(c-3.1) There is nothing to do: $Q = (1 - z^2)H^{(1)} + z^2H^{(0)}$.

(c-3.5) Q is already in the form of (1): $Q = H^{(1)} + z(-zH^{(1)} + zH^{(0)})$.

(c-3.6) Therefore $\Lambda = k$ so $s = 0$.

(c-3.7) We only need verify that $h_0 = 0$. Indeed, $h_0 = h(0) = r(0) - \hat{r}(0) = 0$.

- Therefore the equation is globally valid.

We note that, at step c-1.1, the equation for Q is already in the required form of (1), as it can be written as $Q = (-3\delta)H + zM$ where $M = (4 - z)H^{(1)} + (z - 2)H^{(0)}$. Hence $\Lambda = -3k$, so $s = 0$, and we would have to verify that $h(0) = 0$. However, we previously followed all of the relevant algorithm steps for illustration purposes.

This was fairly straight-forward, however, we could have chosen a more complicated scenario.

(c-3') R_3 with $z_3{}' = 1$.

(c-3'.1) Applying $z := z - 1$, the equation becomes $Q(h) = (2z-z^2)h' + (z-1)h$. The solution to this is $h = c\sqrt{z(z-2)}$, and therefore involves Puiseux series. We then write it as $Q = (2 - z)H^{(1)} + (z - 1)H^{(0)}$ as usual.

(c-3'.3) Since R_3 is not zero-dimensional, we would normally choose a different point, e.g. as in (c-3) above. However, to illustrate how this problem can be solved, we will proceed anyway. We write $w^2 = z$, so that Q becomes $(2 - w^2)\frac{\partial w}{\partial z}\frac{\partial H}{\partial w} + (w^2 - 1)H$, which is $(\frac{1}{w^2} - \frac{1}{2})H^{(1)} + (w^2 - 1)H^{(0)}$, using the $^{(r)}$ notation to denote differentiation with respect to w, and δ now being $w\frac{\partial}{\partial w}$.

(c-3'.4) The indicial equation $w\frac{\partial}{\partial w} + 1$ implies that $h = O(w^{-1})$, i.e a non-trivial Laurent series. We could now proceed with step (c4), but in the interests of brevity, we stop here.

4.2 arcsin and arctan

Here we investigate the often-quoted identity

$$\arcsin z \stackrel{?}{=} \arctan \frac{z}{\sqrt{1 - z^2}}, \quad (5)$$

which we write in the standard notation as $f(z) \stackrel{?}{=} g(z)$. The definition of arcsin is[7]

$$\arcsin z = -i \ln\left(\sqrt{1 - z^2} + iz\right). \quad (6)$$

Hence

$$\begin{aligned} f' &= -i\frac{\frac{-z}{\sqrt{1-z^2}} + i}{\sqrt{1 - z^2} + iz} = -i\frac{-z + i\sqrt{1 - z^2}}{(1 - z^2) + iz\sqrt{1 - z^2}} \\ &= -i\frac{\left(-z + i\sqrt{1 - z^2}\right)\left((1 - z^2) - iz\sqrt{1 - z^2}\right)}{\left((1 - z^2) + iz\sqrt{1 - z^2}\right)\left((1 - z^2) - iz\sqrt{1 - z^2}\right)} \\ &= -i\frac{i\sqrt{1 - z^2}(1 - z^2) + iz^2\sqrt{1 - z^2}}{(1 - z^2)^2 + z^2(1 - z^2)} \\ &= \frac{\sqrt{1 - z^2}}{1 - z^2} = \frac{1}{\sqrt{1 - z^2}}. \end{aligned}$$

Since we know the series definition for $\frac{1}{\sqrt{1-z^2}}$, this is sufficient.

Similarly,

$$\arctan(z) = \frac{1}{2i}\left(\ln(1 + iz) - \ln(1 - iz)\right), \quad (7)$$

so

$$\begin{aligned} g' &= \frac{\sqrt{1 - z^2} + \frac{z^2}{\sqrt{1-z^2}}}{2i(1 - z^2)}\left(\frac{i}{1 + i\frac{z}{\sqrt{1-z^2}}} - \frac{-i}{1 - i\frac{z}{\sqrt{1-z^2}}}\right) \\ &= \frac{(1 - z^2) + z^2}{2i(1 - z^2)\sqrt{1 - z^2}}\frac{2i(1 - z^2)}{(\sqrt{1 - z^2} + iz)(\sqrt{1 - z^2} - iz)} \\ &= \frac{1}{(1 - z^2)\sqrt{1 - z^2}}\frac{(1 - z^2)}{(\sqrt{1 - z^2}^2 + z^2)} \\ &= \frac{1}{\sqrt{1 - z^2}}. \end{aligned}$$

(a) Since f and g satisfy the same linear differential equation, $h = f - g$ satisfies the corresponding homogeneous equation: $Q = h' = 0$.

(b) The branch cuts for f are $R_2 = (-\infty, -1)$ and $R_3 = (1, \infty)$, which are in fact also the branch cuts for g. Given this form of Q, which implies that $f - g$ is (locally) constant, we could in practice just check $f = g$ at non-trivial sample points as in [5].

(c) The regions are R_2, R_3 and their complement, R_1.

(c-1) R_1 with $z_1 = 0$.

[7] [7] following [15], which agrees with [1], but is more specific about the values taken on the branch cuts.

(c-1.5) $\frac{\partial Q}{\partial H^{(1)}} = 1$, so $v = 0$. Proceeding formally, $\arcsin(0) = \arctan(\frac{0}{1}) = 0$, so $h_0 = 0$ and we write $h = z\hat{h}$ whence $Q = z\hat{h}' + \hat{h}$. In the formalism of equation (1), this is $\hat{H}^{(1)} + \hat{H}^{(0)}$, or $(\delta + 1)\hat{H}$.

(c-1.6) $\Lambda = k + 1$, so $s = -1$.

(c-1.7) There is nothing to check.

(c-2) R_2 with $z_2 = -2$.

(c-2.7) We check $h_0 = \arcsin(-2) - \arctan(\frac{-2}{\sqrt{1-2^2}}) = 0$. This works out as

$$\begin{array}{ll} & -i\ln\left(\sqrt{1-2^2} - 2i\right) + \frac{i}{2}\left(\ln(1 + i\frac{-2}{\sqrt{-3}}) - \ln(1 + i\frac{2}{\sqrt{-3}})\right) \\ = & -i\ln(i\left(\sqrt{3} - 2\right)) + \frac{i}{2}\left(\ln(1 - \frac{2}{\sqrt{3}}) - \ln(1 + \frac{2}{\sqrt{3}})\right) \\ = & -i\ln(i\left(\sqrt{3} - 2\right)) + \frac{i}{2}\left(\ln(\sqrt{3} - 2) - \ln(\sqrt{3} + 2)\right) \\ = & -i\ln(2 - \sqrt{3}) - \frac{\pi}{2} + \frac{i}{2}\left(i\pi + \ln(2 - \sqrt{3}) - \ln(\sqrt{3} + 2)\right) \\ = & -i(\ln(2 - \sqrt{3}) - \pi + \frac{i}{2}\left(\ln(2 - \sqrt{3}) - \ln(\sqrt{3} + 2)\right) \\ = & -\pi + i(-2\ln(2 - \sqrt{3}) + \ln(2 - \sqrt{3}) - \ln(\sqrt{3} + 2)) \\ \neq & 0. \end{array}$$

Where we have adopted the strategy suggested in [4] of converting the proposed identity into one just involving complex logarithms; numerical evaluation of those whose arguments lie on the branch cut cannot be trusted in general, but where possible we can use the fact that the arguments are real to transform the logarithm away from the cut. We should then note that a guaranteed-precision floating-point evaluation would be sufficient to prove inequality.

(c-3) R_3 with $z_3 = 2$.

(c-3.5) Similar.

- So the equation is false on the branch cuts, but true elsewhere.

4.3 log and exp

Here we consider the proposed identity

$$z(\log(\exp(z)) - 2\pi i) - z^2 \stackrel{?}{=} 0. \qquad (8)$$

(a) Putting $h(z) = z(\log(\exp(z)) - 2\pi i) - z^2$, $h(z)$ satisfies the equation $Q(h) = h'z - h$.

(b) To find the branch cuts of h, we must determine where $\exp(z)$ maps onto the cut for $\log(z)$, the latter region being

$$R := \{z \mid \Re(z) < 0 \wedge \Im(z) = 0\}. \qquad (9)$$

A simple calculation shows that the cuts are of the form, $\{z \mid \Im(z) = (2n+1)\pi, n \in \mathbf{Z}\}$. We note that these no longer comprise a finite set of algebraic equations, so cannot be handled by the original cylindrical algebraic decomposition (CAD) method [6]. However it is easy to choose sample points by hand in this case; we return to these issues in a later paper. We consider the regions,

$$R_1 := \{z \mid -\pi < \Im(z) < \pi\}, \qquad (10)$$

$$R_2 := \{z \mid \Im(z) = -\pi\}. \qquad (11)$$

(c-1) R_1 with $z = 0$.

(c-1.1) $Q = H^{(1)} - H^{(0)}$.

(c-1.5) This is already in the form of (1) with $M \equiv 0$, $L = \delta - 1$.

(c-1.6) Therefore $\Lambda = k - 1$, so $s = 1$.

(c-1.7) Hence we need to verify that h_0 and h_1 are zero. Now $h_0 = h(0) = 0$ whilst $h_1 = h'(0) = -2\pi i$.

(c-2) R_2 with $z = -\pi i$.

(c-2.1) $Q = (z + \pi i)H^{(1)} - zH^{(0)}$.

(c-2.5) This is already in the form of (1) with $M = H^{(1)} - H^{(0)}$ and $L = \pi i\delta$.

(c-2.6) Therefore $\Lambda = \pi ik$, so $s = 0$.

(c-2.7) Hence we need only verify that h_0 is zero. Now $h_0 = h(-\pi i)$ which works out to be

$$\begin{array}{ll} & -\pi i\left(\log(e^{-\pi i} - 2\pi i\right) - (-\pi i)^2 \\ = & -\pi i(\pi i - 2\pi i) + \pi^2 \\ = & -\pi i(-\pi i) + \pi^2 \\ = & 0. \end{array}$$

- Hence in R_1 the identity is false, whilst in R_2 it is true.

4.4 Remarks regarding complexity

Regarding step (b), one possibility is to use CAD, as was suggested in [5]. However, it is well known this algorithm has complexity that is doubly exponential in the number of variables [10]. As for step (c), [14] states that it should be possible to give complexity bounds for the algorithm given there with some more work. For now, we remark that, unlike the examples given so far, it is possible to produce simple examples where we must check a large number of terms in order to conclude that the function is zero. For suppose that $r \in \mathbf{N}$. Then $h(z) = \sqrt{(z^{2r})} - z^r$ satisfies $Q = zh' - rh = 0$, and the algorithm then yields for Λ the equation $k - r = 0$, so that $s = r$. Hence we can make s arbitrarily large by our choice of r.

5. RESTRICTIONS

Since our algorithm is based on [14], it is necessarily restricted to analytic functions. This imposes two restrictions with respect to [5].

5.1 Complex Conjugation

Complex conjugation is a powerful operator. It is useful in various identities, as for example the relationship between Derive's definition of arctan and that of [7, 1]: $\underbrace{\arctan}_{\text{Derive}}(z) = \overline{\arctan \overline{z}}$ [7]. Furthermore, it can be hard to reason with manually, as in [1, 6.1.23 (part 2)], which states $\ln\Gamma(\overline{z}) = \overline{\ln\Gamma(z)}$, although this is false whenever z is real and in the range $(-(2n+1), -2n)$ for $n \in \{0, 1, 2 \ldots\}$, for then $\Gamma(z) = \Gamma(\overline{z})$ is real and negative, and on this branch cut $\overline{\log Z} = \log \overline{Z} - 2\pi i$ [9].

Unfortunately, complex conjugation is not an analytic operator, and hence cannot be brought within the framework of [14]. It remains to be seen how much of a practical limitation this is.

5.2 Absolute values

Although not explicitly stated in [5], that method can cope with absolute values in the definition of functions, by using arg and its multivalued equivalent. So, for example, the following identity from [1]:

$$|\tanh(z)| = \left(\frac{\cosh(2x)-\cos(2y)}{\cosh(2x)+\cos(2y)}\right)^{1/2}$$

would be rewritten as

$$\tanh(z)\, e^{-i\arg(\tanh(z))} = \left(\frac{\cosh(2x)-\cos(2y)}{\cosh(2x)+\cos(2y)}\right)^{1/2}.$$

Again, "absolute value" is not an analytic operator, and hence cannot be brought within the framework of [14].

6. NON-ELEMENTARY FUNCTIONS

Many[8] non-elementary special functions can be defined in terms of ordinary algebraic-differential equations — one can think of Bessel and error functions, for example. Does this method extend to them? The abstract method of [14] does indeed so generalise. However, there are two main stumbling blocks to generalising the method of this paper to non-elementary special functions.

- Many non-elementary special functions have infinitely many branch cuts (e.g. the Lambert W function[8]), or the branch cuts may not be algebraic, and hence in either case do not fall within the remit of cylindrical algebraic decomposition to solve step (b). In some cases the non-algebraic branch cuts may fall within the sub-Pfaffian extension of cylindrical decomposition [11].
- The "constant problem" is in practice less well studied than it is for elementary functions.

However, we still have a partial "equal/unequal/cannot decide" procedure in these cases. Note that, to get a genuine "unequal/cannot decide" split, the "constant problem" solver must implement an "unequal/cannot decide" split, instead of, as is all too often the case, returning "unequal" where what is meant is "unable to prove equal".

7. CONCLUSION

7.1 Comparison of methods

A simplified comparison of the two methods of [5] and this paper is given in the table below.

We should note that both methods require specification of the branch cuts, since they are essentially human-imposed conventions, rather than mathematically deducible (their existence is deducible, but where they are is purely conventional, and sometimes controversial — see the discussion of the branch cut for arccot in [7]).

7.2 Future Research

A major goal clearly has to be to implement this algorithm, and see how it compares with that of [5] in practice on examples open to both. It would also be nice to produce some examples with non-elementary special functions which were amenable to this method, though so far this has not proved trivial — see section 5.2 for one stumbling block we encountered.

On the theoretical side, much of what we said about step (b) in [5] remains valid.

> Cylindrical algebraic decomposition [6]. Here the difficulty is that the semi-algebraic sets we are using to partition the complex plane, as in equation (3), cannot be given directly to cylindrical algebraic decomposition, and the transformation of them may lead to far larger decompositions than are desirable. A clustering algorithm [2, 3] would help but in general it would need to be aware of the *original* set of semi-algebraic sets, rather than the transformed set of constraints.

Multivalued Functions [5]	Power Series: section 3 and [14]
Can handle multivariate expressions	Currently restricted to one variable
Does not allow inverses in denominator	Permits inverses in denominator
Does not allow nesting of functions	Allows nesting of functions
(except for square roots inside other functions)	(provided branch cuts finite and algebraic)
Needs multivalued simplification rules [4]	Only needs differential equation definitions
	but needs power series and differential algebra technology
Restricted to elementary functions	Not necessarily so restricted (see section 6)
Relies on numerical evaluation	Relies on symbolic evaluation of constants
Therefore unstable on lower-dimensional cuts [4]	Cannot always short-cut via numerics
	(but can to prove inequality — see c-2.5 in section 4.2)
Allows complex conjugation	Does not allow complex conjugation

[8]But not all: for example the Γ function cannot be so defined [13].

At the moment, the method of [14] is limited to one independent variable, so cannot consider the example of

$$\arctan(x) + \arctan(y) \stackrel{?}{=} \arctan\left(\frac{x+y}{1-xy}\right) \qquad (12)$$

from [5]. [14] does say

> on the longer run, the algorithm might generalize to the multivariate setting of partial differential equations with initial conditions on a subspace of dimension > 0.

This would definitely widen the applicability of this method.

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