Nauseating Notation
Very Much in Draft

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Notation exists to be abused[1]

> the abuses of language without which any mathematical text threatens to become pedantic and even unreadable. [4]

but some abuses are more harmful than others.

An interesting set of observations on the pedagogic problems of notational ambiguity (discussed in more detail in section 16.3) is at [40], himself quoting

> To illustrate this, I often ask teachers to write $4x$ and $4\frac{1}{2}$. I then ask them what the mathematical operation is between the 4 and the x, which most realize is multiplication. I then ask what the mathematical operation is between the 4 and the $\frac{1}{2}$, which is, of course, addition . . . [41, p. 53].

1 Intervals

We raise this old chestnut first because it illustrates some of the problems.

1.1 Presentation of Intervals

How do we represent $\{x : 0 < x \leq 1\}$? Semantically, there is no problem.

```
<OMA>
  <OMS name="interval_oc" cd="interval1''/>
  <OMI>0</OMI>
  <OMI>1</OMI>
<\OMA>
```

[1]On 10.6.2007, a quick Google search demonstrated 783 uses of "abus de notation", roughly 10% of which were in english-language papers.

Presentationally, there are two well-kown routes: the “anglo-saxon” way $(0, 1]$ and the “french” way $]0, 1]$. The purpose of this paper is not to argue that one is “better” than the other: merely that there are two competing ones, and the use of the unfamiliar one may well baffle. The “anglo-saxon” is officially recommended by the international standards community [5, 2–6.10], but doubtless the other usage will continue.

1.2 Signed Intervals

The construction $\int_a^b f(x)\mathrm{d}x$ is clear when $a \le b$. Once we have learned about contour integrals, we realise that this is $\int_{\mathcal{C}} f(x)\mathrm{d}x$, where $\mathcal{C}$ is the contour running from a to b along the real axis. If $\mathcal{C}'$ is the contour running from b to a along the real axis, it is immediate that $\int_{\mathcal{C}} f(x)\mathrm{d}x = -\int_{\mathcal{C}'} f(x)\mathrm{d}x$, and hence we are tempted to write

$$\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x. \tag{1}$$

We are somewhat more surprised to see a similar convention [26, Definition 3]

$$\sum_{m \le i < n} f(i) \stackrel{\Delta}{=} -\sum_{n \le i < m} f(i) \qquad \text{Where } m > n. \tag{2}$$

That author helpfully points out ‘This abuse of notation means that “$m \le i < n$,” when written under a $\sum$, does not imply that $m < n$”. He might also have added (which the current author finds curious) that “$m \le i < n$,” when written under a $\sum$, does not necessarily mean that the case $i = m$ is included: if $m > n$, the case $i = m$ is excluded but $i = n$ is included”.

The notation is genuinely useful, as the author points out in the following result.

Proposition 1 ([26, Proposition 2])

(a) $\Delta g = f \Rightarrow \sum_{m \le i < n} f(i) = g(n) - g(m)$ *regardless of the ordering of* m, n*;*

(b) $\displaystyle\sum_{l \le i < n} f(i) = \sum_{l \le i < m} f(i) + \sum_{m \le i < n} f(i)$ *regardless of the ordering of* l, m, n.

The current author would find “$i \in [m, n]$” less confusing than “$m \le i < n$”

2 To Capitalise or not to Capitalise

Half of the 26 standard transcendental elementary functions (log, the six inverse trigonometric functions and the six inverse hyperbolic) are one–many functions, at least on the complex plane, since the functions whose inverses they are (exp, trigonometric and hyperbolic) are many–one. However, it is normal to restrict them, by means of "branch cuts" [8][2] to be one–one, at the price of being discontinuous.

This means that, if f is a many–one function $\mathbf{C} \to \mathbf{C}$, its inverse[3], which will be denoted g, has two possible definitions: the one–one discontinuous one, and the one–many continuous one. It is usual in anglo-saxon cultures to denote a[4] one–one function with a lower-case initial letter, as g, and the one–many one with an upper-case initial letter, as G. Regrettably, in France the convention is apparently reversed[5], and some Anglosaxon texts adopt this convention, as in [21, p. 294]. This situation is worse than in section 1.1: here the notations are not merely baffling but contradictory, and any attempt at understanding them will need to know the (linguistic, in this case) context.

In the world of international standardisation[6], the Anglosaxon convention for lower case has won [5, 2–13.8 *et seq.*], but the multivalued functions are not defined.

3 Plus or Minus

This is familiar to us all from the solution to the quadratic:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{3}$$

which can be seen as shorthand for

$$\left\{ \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right\}. \tag{4}$$

We are prepared to accept it in formulae such as [1, Equation 4.3.38]

$$\tan z_1 \pm \tan z_2 = \frac{\sin(z_1 \pm z_2)}{\cos z_1 \cos z_2}, \tag{5}$$

[2]Where the branch cuts *are* is largely irrelevant to this discussion, though there is no standard notation for distinguishing between functions which differ only in their branch cuts.

[3]We use a different letter, to avoid the problem discussed in section 5.

[4]It would be tempting, but wrong, to write "the one-one function". Since it is 'obvious' that the correct inverse of $x \mapsto x^2$ as $\mathbf{R} \to \mathbf{R}$ is the positive square root, we may be tempted to think there is an obvious inverse in other circumstances. While it is normal these days to define log to have imaginary part in $(-\pi, \pi]$, the author was initially taught to have the imaginary part in $[0, 2\pi)$. [1] changed the branch cut of arccot between printings, and systems have been known to be internally inconsistent [8].

[5]Various mathematical textbooks seem to indicate this. However [2, Arcsin] gives capitals to Arcsin, Arccos and Arctan, but not to the others. There is clearly an inconsistency here, as [2, Arctan] describes arctan as the inverse function, and makes no mention of Arctan. The other inverse functions seem to have no entries in [2]. [**?**], however, says "Arc sin ou Asin en notation franaise, $\sin^{-1}$, asin ou asn en notation anglo-saxonne)"

[6]Which has also settled the longstanding controversy over $\mathbf{N}$: [5, 2-6.1] states that $\mathbf{N}$ includes 0 and $\mathbf{N}^*$ does not.

which we read as shorthand for two equations:

$$\begin{aligned}
\tan z_1 - \tan z_2 &= \frac{\sin(z_1 - z_2)}{\cos z_1 \cos z_2} \\
\tan z_1 + \tan z_2 &= \frac{\sin(z_1 + z_2)}{\cos z_1 \cos z_2}.
\end{aligned}$$

But what of [1, Equations 4.6.26,27] (note that capital letters denote set-valued inverse functions — section 2)

$$\begin{aligned}
\operatorname{Arcsinh} z_1 \pm \operatorname{Arcsinh} z_2 &= \operatorname{Arcsinh}\left(z_1\sqrt{1+z_2^2} \pm z_2\sqrt{1+z_1^2}\right) \\
\operatorname{Arccosh} z_1 \pm \operatorname{Arccosh} z_2 &= \operatorname{Arccosh}\left(z_1 z_2 \pm \sqrt{(z_1^2-1)(z_2^2-1)}\right)?
\end{aligned}$$

These are reproduced, after α-conversion, as [32, 4.38.15,16]. This latter text gives this useful note.

> The above equations are interpreted in the sense that every value of the left-hand side is a value of the right-hand side and vice-versa. All square roots have either possible value.

4 Alphabetical Order?

The author recently heard a speaker [30] state that, in Arabic Mathematics, the alphabetic order used in formulae is different from the usual one. "How perverse", the author thought, as probably does the reader. But is it that perverse? Any text in ideal theory will normally call the variables $x_1 \ldots, x_n$ for generic theorems and definitions. However, explicit examples will usually make use of particular letters, e.g. x and y in two dimensions, or x, y and z in three. In four, the variable w (or possibly t) is pressed into service, but the lexicographic order is then normally taken to be $x > y > z > w$. Maybe it *is* perverse, but it's a common perversion.

There are other instances of non-obvious order: the theory of elliptic functions (see section 6) tends to order the relevant letters as s, c, n, d, though this doesn't have any mathematical significance, merely the order in which one finds things in tables.

5 Iterated Functions

There is a curious anomaly in mathematical notation when it comes to iterated functions.

- If G is a permutation group, and $\pi \in G$, then π^2 is clearly the iterated permutation: $(\pi^2)(a) = \pi(\pi(a))$. Similarly π^{-1} is the inverse permutation: $\pi^{-1}(a) = b \Leftrightarrow \pi(b) = a$.

- If f is a function $\mathbf{R} \to \mathbf{R}$, then $f^2(x)$ is an alternative[7] way of writing $(f(x))^2$: one need merely consider the much-quoted

$$\sin^2\theta + \cos^2\theta = 1. \tag{6}$$

- While the preceding works for positive powers, it is nevertheless the case that $\sin^{-1}(x)$ does *not* mean $1/\sin(x)$, but rather the inverse function: $\sin^{-1}(a) = b : \sin(b) = a$.

This means that $\sin^{-2}(x)$ is ambiguous to the point of uselessness: $\sin^{-2}(a)$ could mean any of:

  - b: $\sin\sin b = a$ (as if sin were the permutation π);
  - c: $(\sin c)^2 = a$ (the inverse of the traditional $\sin^2$);
  - d^2: $\sin(d) = a$ (the square of the traditional $\sin^{-1}$).

- A consequence of the above is that we have no really good notation for iterated functions, and have to write expressions such as $\frac{n \log n \log\log n}{\log\log\log n}$. Some authors write $\frac{n \log n \log_2 n}{\log_3 n}$, but the current author finds this too readily confused with subscripts indicating the base of the logarithm.

- The ISO community [5, 2–13.8 *et seq.*] uses arcsin rather than $\sin^{-1}$: the current author regrets that [5] does not repeat the explicit disapprobation of $\sin^{-1}$ etc. that was in its predecessor [22, 11-9.8].

6 Pq

[1, equation 16.25.1] defines

$$\mathrm{Pq}(u) = \int_0^u \mathrm{pq}^2(t)\mathrm{d}t \tag{7}$$

(where $\mathrm{pq}^2(t)$ means $\mathrm{pq}(t)^2$: see section 5). This is, of course, in defiance of the conventions of section 2, but we are dealing with elliptic functions, not elementary ones. However, the joker here is that equation (7) applies whenever p and q are any of the letters s,c,n,d (note the order, which is traditional, and see section 4). Hence this equation is in fact shorthand for twelve equations of the form

$$\mathrm{Sn}(u) = \int_0^u \mathrm{sn}^2(t)\mathrm{d}t, \tag{8}$$

except that, when q is s, equation (7) should be read as

$$\mathrm{Pq}(u) = \int_0^u \left(\mathrm{pq}^2(t) - \frac{1}{t^2}\right)\mathrm{d}t - \frac{1}{u}, \tag{9}$$

[7] Many, including the author would say “regrettable”.

where the changes are to remove the removable singularity at $t = 0$.

A similar equation, but this time with explanation, can be seen as

$$\mathrm{pq}(u) = \frac{\mathrm{pr}(u)}{\mathrm{qr}(u)} \qquad ([1, \text{Equation } 16.3.4])$$

To quote [1, coda to section 16.27]

> There is a bewildering variety of notations ... so that in consulting books caution should be used.

As an example of this, or showing that not all apparent misprints are such, we can see [1, Equation 17.2.8–10]

$$E(u|m) = \int_0^x (1-t^2)^{-1/2}(1-mt^2)^{1/t}\mathrm{d}t = \int_0^u \mathrm{dn}^2(w)\mathrm{d}w. \tag{10}$$

Does this tell us what $\mathrm{Dn}(u)$ is — indeed [1, Equation 16.26.3] has $\mathrm{Dn}(u) = E(u)$. However, the 'x' in equation (10) is not a misprint, and in fact [1, Equation 17.2.2] $x = \mathrm{sn}\, u$. So in Maple-speak

```
EllipticE(JacobiSN(u,m),m)=int(JacobiDN(t)^2,t=0..u).
```

Quite how this is to be reconciled with [19, Equation 5.138(3)] —

$$\int \mathrm{dn}^2(u) = E(\mathrm{am}\, u, k)$$

— is not clear ($m = k^2$ here).

7 While we're on the subject ...

The 'help' for Maple 10 under `JacobiSN` helpfully states that

```
In A&S, these functions are expressed in terms of a parameter m,
representing the square of the modulus k entering the definition of
these functions in Maple or G&R. So, for example, the formula
JacobiDN(z,k)^2 = 1 - k^2 * JacobiSN(z,k)^2 appears in A&S as
dn(z,m)^2 = 1 - m * sn(z,m)^2.
```

However, the corresponding warning is missing from the help on `EllipticE`, but can be deduced from the fact that the example

```
EllipticE(0.3);
                              1.534833465
```

in the help corresponds to the entry for $E(0.09)$ [1, p. 609], noting, however, that both this and Maple's `EllipticE` are $E(x)$, not $E(u)$.

8 O and friends

We have written elsewhere [11] as follows.

> Every student is taught that $O(f(n))$ is really a set, and that when we write "$g(n) = O(f(n))$", we really mean "$g(n) \in O(f(n))$". Almost all[8] textbooks then use '=', having apparently placated the gods of confusion. However, actual uses of O as a set are rare: the author has never[9] seen "$O(f) \cap O(g)$", and, while a textbook might[10] write "$O(n^2) \subset O(n^3)$", this would only be for pedagogy of the O-notation.

That paper proposes an OpenMath symbol `Landauin`, whose semantics would be that of set membership, but whose notation *might be* (OpenMath does not prescribe notation) that of '='.

Another notation that has come into use[11] is the so-called "soft O", generally written $\tilde{O}$ but also O^*, but which has two fundamentally differing definitions.

1. 'where the "soft O" $\tilde{O}$ indicates an implicit factor of $(\log n)^{O(1)}$' [36], attributed by [37] to [39].

1' 'where $f = \tilde{O}(g)$ if and only if there exists a constant $k \geq 0$ such that $f = O(g \cdot (\log g)^k)$' [17].

2. 'we write $O(n^{3+\epsilon})$ for $O(n^{3+o(1)})$, which is also sometimes written $\tilde{O}(n^3)$' [13, footnote 1].

2' 'We write[12] $\tilde{O}(f)$, or $O(f^{1+\epsilon})$, for $O(f^{1+o(1)})$'.

Of these, 1 and 1' agree for $\tilde{O}(n^d)$, often the intended domain of application, but disagree otherwise. In the sense of 1, $\tilde{O}(\log n) = \tilde{O}(1)$, whereas for 1', $\tilde{O}(\log n) = O(\log n(\log\log n)^k)$ for some k, so 1' clearly makes more sense than 1.

The difference between 1' and 2' is more fundamental. 1' means "g times something polylogarithmic in g", whereas 2' means "g times someting slower than any power of g". Hence when $g = e^n$, 1' means $\exists k O(e^n n^k)$, while 2' means $\forall \epsilon o(e^{(1+\epsilon)n})$. So 2' would include $e^n L_{1/2}(n)$ (where $L_{1/2}(n) = \exp(O(\sqrt{\log n \log\log n}))$), whereas 1' would not. A second, more subtle, point is that it is not clear in sense 1' whether k is explicitly calculable (just as it is not always clear in standard O notation).

While the difference is moot in most current uses, we believe that 2' *should be* the correct usage of $\tilde{O}$.

[8] [31] is an honourable exception.
[9] Not even in the one context where it *would* be useful: $\Theta(f) = O(f) \cap \Omega(f)$.
[10] [9, p. 41] write $\Theta(n) \subset O(n)$.
[11] A quick Google on 14.6.2007 showed 846 uses, though not all were mathematical.
[12] Not actually seen anywhere, but related to 2 as 1' is to 1.

9 Other asymmetric notations

[29] writes, quoting [23] (we use Langer's notation and equation numbers), as follows.

> $$\text{(19}a\text{)} \qquad \frac{1}{|\phi|^{1/2}}e^{-|\xi|} \longleftrightarrow \frac{2}{\phi^{1/2}}\cos\left(\xi - \frac{\pi}{4}\right),$$
>
> $$\text{(19}b\text{)} \qquad \frac{1}{|\phi|^{1/2}}e^{|\xi|} \longleftrightarrow \frac{2}{\phi^{1/2}}\cos\left(\xi - \frac{\pi}{12}\right),$$
>
> where, to quote [Jeffreys]: "the sign $\longleftrightarrow$ is used to indicate that the functions it connects are asymptotic approximations to the same function in different circumstances."

He then goes on to criticise the notation here.

> Concerning the form of the statement of results (19) I would object to the use of the symbol $\longleftrightarrow$, on the ground that it invites misconceptions which have not failed to show themselves in the literature. I find it not unnatural to read into a pair of forms connected by an arrow the thought that in the direction of the arrow the one form implies the other. In the case of the relations (19) this inference would certainly be incorrect. [...]
>
> Firstly, *the relation (19a) is one in which the left hand member implies and necessarily leads to the right hand member. The converse, however, is not true, ...*
>
> Secondly, *the relation (19b) is one in which the right hand member implies and necessarily leads to the left hand member, but not conversely.*

It seems to the current author that we have here an aalogous case to the previous one: a symbol implying symmetry (here $\longleftrightarrow$, there $=$) is used in an asymmetric sense. The fact that the senses are reversed between (19a) and (19b) merely adds to the confusion.

10 The sins of TeX

A trivial example [15, abstract] is `Zar$(K|A)$`, which is admittedly easier to type than `$\Zar(K|A)$`, but which

a) suggests, falsely, that the paper is about $(K|A)$ (possibly a conditional probability?);

b) loses (at the formula level) the fact that it's about Zar.

We have also seen the following piece of apparently good mathematics.

> Then the functor $T \mapsto \{$generically smooth T-morphisms $T \times_S \mathcal{C}' \to \mathcal{T} \times_S \mathcal{C}\}$ from ((S-schemes)) to ((sets)) is

However, the input LaTeX was[13]

```
Then the functor $T\mapsto\{$generically smooth $T$-morphisms
$T\times_S\Cal
C'\to T\times_S\Cal C\}$ from $((S$-schemes)) to ((sets)) is
```

with the `\{...\}` and `((` not properly nested with respect to `$`.

See also [20].

11 The sins of MathML

We have seen the following.

```
<mml:msup><mml:mn>10</mml:mn><mml:mrow><mml:mn>10</mml:mn>
<mml:mspace width="0.2em"></mml:mspace><mml:mn>000</mml:mn>
</mml:mrow></mml:msup></mml:math>
```

12 The sins of Notation

We have seen the following.

$$2\pi\phi = \left\{ \int_{\delta}^{2\pi-\delta} + \int_{-\delta}^{0} + \int_{0}^{\delta} \right\} \frac{a^2 - r^2}{a^2 - 2ar\cos\vartheta + r^2} f(\theta + \vartheta) d\vartheta$$

[24, (8) p. 435]. Presumably $d\vartheta$ is meant to close all three integrals. The mixture of θ and ϑ might also be considered challenging.

We were surprised to see the following [14].

> ...the use of XOR operators, denoted $(q_1 \oplus q_2 \oplus \ldots \oplus q_n)$ meaning that exactly one q_i holds for $1 \le i \le n$.

13 The sins of Layout/Size

Ieuan Evans drew our attention to the tables in [38], which are practically unreadable if printed.

[13] We did `\def\Cal{\cal}` to make it LaTeX. `http://arXiv.org/abs/math/0701407`

14 The sins of Language

14.1 Translation

[7] has the following abstracts in `arxiv.org`.

```
Let a be a nonzero integer. If a is not congruent to 4 or 5
modulo 9 then there is no Brauer-Manin obstruction to the
existence of integers x, y, z such that x^3+y^3+z^3=a. In addition,
there is no Brauer-Manin obstruction to the existence of integers
x, y, z such that x^3+y^3+2z^3=a.
-----
Soit a un entier non nul. Si a n'est pas congru \'a 4 ou 5 modulo
9, il n'y a pas d'obstruction de Brauer-Manin \'a l'existence
d'entiers x, y, z tels que x^3+y^3+z^3=a. D'autre part, il n'y a
pas d'obstruction de Brauer-Manin \'a l'existence d'entiers
x, y, z tels que x^3+y^3+2z^3=a.
```

So far, so good, and the translation is better than reasonable. However, the abstract in the paper itself is (our typesetting) as follows.

> Soit a un entier non nul. Si a n'est pas de la forme $9n \pm 4$ pour un $n \in \mathbf{Z}$, il n'y a pas d'obstruction de Brauer–Manin à l'existence d'une solution de l'équation $x^3 + y^3 + z^3 = a$ en entiers $x, y, z \in \mathbf{Z}$. D'autre part, il n'y a pas d'obstruction de Brauer–Manin à l'existence d'une solution de l'équation $x^3 + y^3 + 2z^3 = a$ en entiers $x, y, z \in \mathbf{Z}$.

While the deep meaning is the same, we note how a congruence condition has become an "of the form" condition.

14.2 (Almost) How not to begin a paper

[6] begins as follows.

> Recall that a simplicial complex Δ is called shellable if there exits a linear ordering $F_1, F_2, \ldots, F_k$ of the facets of Δ in such a way that, for each $j = 2, \ldots, k$, the intersection of the sub-complex of F_j with the union of all sub-complexes of previous facets $F_1, \ldots, F_{j-1}$ is a pure sub-complex of Δ of dimension $\dim F_j - 1$.

This obscure start is rescued by the next paragraph, whih begins "Although its definition is not illuminating, the notion of shellability has remarkable topological consequences."

15 Abuse of declarations

The author recently encountered the following abstract.

> Let k,x,x' be nonzero natural numbers. Let M be a tropical matrix with tropical rank k. We show that Kapranov rank is k too if x and x' are not too big; namely if we are in one of the following cases: a) k>=6 and x, x' <=2 b) k=4,5, x<=2 and x'<=3 (or obviously the converse) c) k=3 and either x,x'<=3 or x<=2 and x'<=4 (or the converse).

This is nonsense as it stands. Further research found the following version (our re-typesetting).

> Let M be a tropical matrix $(k+x)\times(k+x')$ for some k, x, $x' \in \mathbf{N}\setminus\{0\}$ with tropical rank k. We show that Kapranov rank is k too if x and x' are not too big; namely if we are in one of the following cases:
>
> a) $k \geq 6$ and $x, x' \leq 2$;
>
> b) $k = 4, 5$, $x \leq 2$ and $x' \leq 3$ (or obviously the converse, that is $x \leq 3$ and $x' \leq 2$)
>
> c) $k = 3$ and either $x, x' \leq 3$ or $x \leq 2$ and $x' \leq 4$ (or obviously the converse).

Omitting the declaration of the dimensions of M has made x and x' into free variables, making nonsense[14] of the whole statement.

16 Other notation we have seen

16.1 `\overline`

We have seen [35] the expression $i = \overline{0, n}$, and in other places we have seen $i = \overline{0; n}$ (semi-colon rather than comma). In context, it was relatively clear that this meant $i \in \{0, 1, \ldots, n\}$, but the usage was new to this author. The use of $=$ here is at least as egregious as its use in section 8, and is not hallowed by time. This author sees no case for $=$ over $\in$, as in $i \in \overline{0, n}$.

16.2 "Suggestive Notation"

We have seen [33] the following (our typesetting, attempting to preserve the original).

> We use suggestive notation like $\mathbf{R}[\overline{X}]^2 := \{p^2 \mid p \in \mathbf{R}[\overline{X}]\}$ for the set of squares and $\sum \mathbf{R}[\overline{X}]^2$ for the set of sums of squares of polynomials in $\mathbf{R}[\overline{X}]$.

[14]The author, admittedly not an expert in this area of mathematics, had to retrieve the second version before it made any sense to him.

Table 1: Properties of juxtaposition

left weight	right weight	meaning	example
normal	normal	lexical	sin
normal	italic	application	$\sin x$
italic	italic	multiplication	xy
		(or `⁣`)	M_{ij}
italic	normal	multiplication	$a\sin x$
digit	digit	lexical	42
		(or `⁣`)	M_{12}
digit	italic	multiplication	$2x$
digit	normal	multiplication	$2\sin x$
normal	digit	application	$\sin 2$
(but note the precedence in $2\sin 3x$)			
italic	digit	error	$x2$
		(but reconsider)	x^2 or x_2?
digit	fraction	addition	$4\frac{1}{2}$
		`&InvisiblePlus;`	
italic	greek	application^{-1}	$a\phi$
		(as in group theory)	i.e. $\phi(a)$
italic	(	unclear	$f(y+z)$ or $x(y+z)$
		what is $f(g+h)$?	

While it cannot be denied that these are indeed suggestive, they would probably cause 'presentation to content' converters a great deal of difficulty. Consider the following [33, (2)].

$$T(\overline{g}) = \sum_{\delta\in\{0,1\}^m} \sum \mathbf{R}[\overline{X}]^2\overline{g}^\delta := \left\{ \sum_{\delta\in\{0,1\}^m} \sigma_\delta\overline{g}^\delta \mid \sigma_\delta \in \sum \mathbf{R}[\overline{X}]^2 \right\}, \tag{2}$$

where $\sum \mathbf{R}[\overline{X}]^2$ has to be read as a compound symbol, and the usual precedence rules for $\sum$, *viz.* that it binds everything to its right, do not apply to it.

16.3 Precedence and weights/fonts

It is normal to say that juxtaposition indicates multiplication (MathML's symbol `InvisibleTimes`) or function application (MathML's `⁡`) [10], but in fact the general rules are more complex, and highly context-sensitive. In general, we can state the observed properties of juxtaposition as being those in Table 1.

We have noted in Table 1 the issue of precedence. This is very subtle in mathematics: can any-one explain satisfactorily why $2\sin 3x\cos 4x$ means $2\cdot(\sin(3\cdot x))\cdot(\cos(4\cdot x))$, and not, say, $2\cdot(\sin(3\cdot x\cdot\cos(4\cdot x)))$ or $2\cdot(\sin 3)\cdot(x\cdot\cos(4\cdot x))$?

A further problem is that, at least in linear algebra, the rules are often different in subscripts, so that M_{ij} is really $M_{i,j}$, and M_{12} is not the twelth element of a vector[15]. It is the author's *personal* opinion that this is lazy, and writers *should* insert the comma.

16.4 Abuse of weights/fonts

Hence the font, or even the weight, of characters carries quite detailed semantic information. However, this can be abused to make differences of font or weight carry undue importance. This is shown in [16], where B denotes an (arbitrary) category, as does A and other italic capitals, but $\mathbf{B}$ denotes a specific object, the groupoid of finite sets and bijections. This leads to us considering [16, p. 206] "(A, B)-species of structures either as functors $\mathbf{B}A \to \hat{B}$". The review[16] of this paper even refers to "the category of colimit-preserving, symmetric, strong monoidal functors from $\mathrm{Set}^{\mathbf{B}B}$ to $\mathrm{Set}^{\mathbf{B}A}$", which to the author's mind requires keen eyes to read correctly.

But there is a fine line between use and abuse. A PhD thesis the author was reading has the following statement "So $\mathcal{X} = (X, \sqsubseteq_X)$ for X equal to T, S, V". This is a shorthand way of defining $\mathcal{T}$, $\mathcal{S}$, $\mathcal{V}$ in one equation, though the author would have preferred it had the thesis said "$X/\mathcal{X}$ equal to $T/\mathcal{T}$, $S/\mathcal{S}$, $V/\mathcal{V}$, making all the symbols explicit. He is still scarred from a Galois Theory lecturer who began by saying "upper case letters define fields, lower case letters denote elements of the corresponding fields, upper case Gothic letters denote corresponding Galois groups, and lower case Gothic letters denote elements of the corresponding groups, so" and then wrote on the board "$f\mathfrak{f} = f'$ is an example of $\mathfrak{F}$ acting on F". There was a stampede for copies of the Gothic alphabet, as simply copying down funny symbols was no longer sufficient: one had to understand their significance.

We also note this use of fonts [27, (1)].

$$I(T) = \frac{\mathcal{N}(T-2) + \mathcal{N}(T-1)}{N(T-2) + N(T-1)}. \tag{11}$$

It would probably be hard for a text-to-speech renderer to convey this subtlety to the hearer.

16.5 An obscure text on Units

Though initially hard to read, the notation

$$A = \{A\}_{XX}[A]_{XX},$$

[28, (1)][17] "where $\{A\}_{XX}$ is the numerical value (a pure number) of quantity A and $[A]_{XX}$ is the corresponding unit" has certain advantages.

[15] It may, of course, be a Mathieu group in Group Theory, just to confuse further.

[16] Mathematical Reviews 2389925.

[17] The notation itself is attributed to [**?**].

16.6 The uses of |

We have seen [3, p. 4] the equation

$$M = M[1]||\cdots||M[|M|]. \tag{12}$$

where the outfix $|\cdot|$ means "size of", but the infix $||$ (note that this is single operator) indicates concatenation. The fact that "[" and "|" are hard to tell apart doesn't help. Incidentally, the meaning is that a bit string is composed of the string of bits.

16.7 The uses of ×

[12, (6.8)] is

$$\begin{aligned} N_{\rm loc}(P) &= \frac{1}{4}\sum_{\iota\in\{0,1\}}\ \sum_{\epsilon\in\{\pm\}^3\setminus(+,-,-)}\ \sum_{\substack{m,n\in\mathcal{B}\\ \gcd(m,n)=1}}\ \sum_{\substack{\ell\in L(m,n)\\ \neg(6.7)}} \\ &\quad\times \sum_{\substack{m'',n''\in\mathcal{A}\\ \gcd(m'',n'')=1\\ \gcd(m''n'',2mn)=1}}\ \sum_{(\beta,\gamma,\delta)\in L_2^{(\iota)}} N_{\rm loc} + O(P^{2+\epsilon}) \end{aligned}$$

This in fact raises various queries.

1. What is the rôle of $\times$ here? It seems to be some form of continuation marker, even though no real product is taking place.
2. We are used to seeing $\pm$ as a shorthand for $+,-$, but to read $\{\pm\}^3$ as shorthand for $\{+,-\}^3$ seems to be taking this a little far.
3. The $O(P^{2+\epsilon})$ is presumably outside the summation.
4. The ϵ being summed over in the second summation presumably is not the same (and indeed a strict reading of variable bindings assuming the previous point would concur) as the ϵ in $O(P^{2+\epsilon})$.

16.8 Symbol Reuse

We note [25, p. 3] the use of (infix) $\vee$ to mean (prefix) lcm.

16.9 Abuse of ≠

The author has seen this more than once, but the immediate example is in [18]:

$$BC(v) := \sum_{s\neq t\neq v\in V}\frac{\sigma_{s,t}(v)}{\sigma_{s,t}}.$$

What is not implied at all by this notation is that $s \neq v$, and it takes fairly careful reading to see that the sum is over s, t pairs, but *not* over v. The author would write

$$BC(v) := \sum_{s,t \in V : |\{s,t,v\}|=3} \frac{\sigma_{s,t}(v)}{\sigma_{s,t}},$$

which is admittedly slightly longer, but, the author claims, clearer.

17 "What is a number"

There has been a significant discussion in October 2013 on the LinkedIn group "Math, Math Education, Math Culture", with the title "Math, Math Education, Math Culture".[18] This started out discussion `4^3^2` and the fact that Excel gives 4096, i.e. $(4^3)^2$ whereas other languages[19] give 262144, i.e. $4^{(3^2)}$. This discussion rapidly morphed into one of `-3^2`, with, essentially, two camps for the result, but three arguments.

-9: **Fuchun Huang etc.** This can be summarised as "`-` is an operator, and the usual rules of precedence mean that `-` has lower precedence than `^`.

9: **Art DiVito[20] etc.** This can be summarised as "`-3` is a number, and therefore `-3^2` is the square of -3. No-one would consider `43^2` to be $4(3^2) = 36$".

9: **David Radcliffe** "It's an Excel design decision that unary operators have higher precedence".

The discussion also touched on, via Karen Wanless, on `3i^2`. She observed "I would not consider squaring the 3". This caused JHD to look again at MatLab, which *does* allow the syntax `3i`, which appears to be an alternative to `3*i`, but is not, since the variable `i` can be re-assigned, but the lexical `i` in `3i` is always $\sqrt{-1}$. MatLab here is consistent, but hard to explain.

`3i^2` This is -9, the square of the "number" `3i`, exactly as `43^2` is the square of the number `43`.

18 It's not all bad

Consider this use of colour to make a pedagogic point [34]:

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{i 2\pi k \frac{n}{N}} \tag{13}$$

[18] `http://www.linkedin.com/groupAnswers?viewQuestionAndAnswers=&discussionID=5795493826755837952&gid=33207&commentID=5799367054851067904&trk=view_disc&fromEmail=&ut=0HeG-`.LjWNfEBY1

[19] JHD corrected the author of the post. MatLab is in fact in the Excel camp here.

To find the energy at a particular frequency, spin your signal around a circle at that frequency, and average a bunch of points along that path.

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