

Nauseating Notation Very Much in Draft

James H. Davenport
J.H.Davenport@bath.ac.uk

August 30, 2013

Notation exists to be abused¹

the abuses of language without which any mathematical text threatens to become pedantic and even unreadable. [4]

but some abuses are more harmful than others.

An interesting set of observations on the pedagogic problems of notational ambiguity is at [35], himself quoting

To illustrate this, I often ask teachers to write $4x$ and $4\frac{1}{2}$. I then ask them what the mathematical operation is between the 4 and the x , which most realize is multiplication. I then ask what the mathematical operation is between the 4 and the $\frac{1}{2}$, which is, of course, addition ... [36, p. 53].

1 Intervals

We raise this old chestnut first because it illustrates some of the problems. How do we represent $\{x : 0 < x \leq 1\}$? Semantically, there is no problem.

```
<OMA>  
  <OMS name="interval_oc" cd="interval1" />  
  <OMI>0</OMI>  
  <OMI>1</OMI>  
<\OMA>
```

Presentationally, there are two well-known routes: the “anglo-saxon” way $(0, 1]$ and the “french” way $]0, 1]$. The purpose of this paper is not to argue that one is “better” than the other: merely that there are two competing ones, and the use of the unfamiliar one may well baffle.

¹On 10.6.2007, a quick Google search demonstrated 783 uses of “abus de notation”, roughly 10% of which were in english-language papers.

1.1 Signed Intervals

The construction $\int_a^b f(x)dx$ is clear when $a \leq b$. Once we have learned about contour integrals, we realise that this is $\int_{\mathcal{C}} f(x)dx$, where \mathcal{C} is the contour running from a to b along the real axis. If \mathcal{C}' is the contour running from b to a along the real axis, it is immediate that $\int_{\mathcal{C}} f(x)dx = -\int_{\mathcal{C}'} f(x)dx$, and hence we are tempted to write

$$\int_a^b f(x)dx = -\int_b^a f(x)dx. \quad (1)$$

We are somewhat more surprised to see a similar convention [24, Definition 3]

$$\sum_{m \leq i < n} f(i) \triangleq -\sum_{n \leq i < m} f(i) \quad \text{Where } m > n. \quad (2)$$

That author helpfully points out ‘This abuse of notation means that “ $m \leq i < n$,” when written under a \sum , does not imply that $m < n$ ’. The notation is genuinely useful, as the author points out.

Proposition 1 ([24, Proposition 2]) (a) $\Delta g = f \Rightarrow \sum_{m \leq i < n} f(i) = g(n) - g(m)$ regardless of the ordering of m, n ;

(B) $\sum_{l \leq i < n} f(i) = \sum_{l \leq i < m} f(i) + \sum_{m \leq i < n} f(i)$ regardless of the ordering of l, m, n .

2 Plus or Minus

This is familiar to us all from the solution to the quadratic:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad (3)$$

which can be seen as shorthand for

$$\left\{ \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right\}. \quad (4)$$

We are prepared to accept it in formulae such as [1, Equation 4.3.38]

$$\tan z_1 \pm \tan z_2 = \frac{\sin(z_1 \pm z_2)}{\cos z_1 \cos z_2}, \quad (5)$$

which we read as shorthand for two equations:

$$\begin{aligned} \tan z_1 - \tan z_2 &= \frac{\sin(z_1 - z_2)}{\cos z_1 \cos z_2} \\ \tan z_1 + \tan z_2 &= \frac{\sin(z_1 + z_2)}{\cos z_1 \cos z_2}. \end{aligned}$$

But what of [1, Equations 4.6.26,27] (note that capital letters denote set-valued inverse functions — section 5)

$$\begin{aligned} \operatorname{Arcsinh} z_1 \pm \operatorname{Arcsinh} z_2 &= \operatorname{Arcsinh} \left(z_1 \sqrt{1 + z_2^2} \pm z_2 \sqrt{1 + z_1^2} \right) \\ \operatorname{Arccosh} z_1 \pm \operatorname{Arccosh} z_2 &= \operatorname{Arccosh} \left(z_1 z_2 \pm \sqrt{(z_1^2 - 1)(z_2^2 - 1)} \right)? \end{aligned}$$

These are reproduced, after α -conversion, as [29, 4.38.15,16]. This latter text gives this useful note.

The above equations are interpreted in the sense that every value of the left-hand side is a value of the right-hand side and vice-versa. All square roots have either possible value.

3 Alphabetical Order?

The author recently heard a speaker [27] state that, in Arabic Mathematics, the alphabetic order used in formulae is different from the usual one. “How perverse”, the author thought, as probably does the reader. But is it that perverse? Any text in ideal theory will normally call the variables $x_1 \dots, x_n$ for generic theorems and definitions. However, explicit examples will usually make use of particular letters, e.g. x and y in two dimensions, or x , y and z in three. In four, the variable w (or possibly t) is pressed into service, but the lexicographic order is then normally taken to be $x > y > z > w$. Maybe it is perverse, but it’s a common perversion.

There are other instances of non-obvious order: the theory of elliptic functions (see section 6) tends to order the relevant letters as s, c, n, d , though this doesn’t have any mathematical significance, merely the order in which one finds things in tables.

4 Iterated Functions

There is a curious anomaly in mathematical notation when it comes to iterated functions.

- If G is a permutation group, and $\pi \in G$, then π^2 is clearly the iterated permutation: $(\pi^2)(a) = \pi(\pi(a))$. Similarly π^{-1} is the inverse permutation: $\pi^{-1}(a) = b : \pi(b) = a$.
- If f is a function $\mathbf{R} \rightarrow \mathbf{R}$, then $f^2(x)$ is an alternative² way of writing $(f(x))^2$: one need merely consider the much-quoted

$$\sin^2 \theta + \cos^2 \theta = 1. \tag{6}$$

²Many, including the author would say “regrettable”.

- While the preceding works for positive powers, it is nevertheless the case that $\sin^{-1}(x)$ does *not* mean $1/\sin(x)$, but rather the inverse function: $\sin^{-1}(a) = b : \sin(b) = a$.



This means that $\sin^{-2}(x)$ is ambiguous to the point of uselessness: $\sin^{-2}(a)$ could mean any of:

- b : $\sin \sin b = a$ (as if \sin were the permutation π);
 - c : $(\sin c)^2 = a$ (the inverse of the traditional \sin^2);
 - d^2 : $\sin(d) = a$ (the square of the traditional \sin^{-1}).
- A consequence of the above is that we have no really good notation for iterated functions, and have to write expressions such as $\frac{n \log n \log \log n}{\log \log \log n}$.

5 To Capitalise or not to Capitalise

Half of the 26 standard transcendental elementary functions (\log , the six inverse trigonometric functions and the six inverse hyperbolic) are one–many functions, at least on the complex plane, since the functions whose inverses they are (\exp , trigonometric and hyperbolic) are many–one. However, it is normal to restrict them, by means of “branch cuts” [7]³ to be one–one, at the price of being discontinuous.

This means that, if f is a many–one function $\mathbf{C} \rightarrow \mathbf{C}$, its inverse⁴, which will be denoted g , has two possible definitions: the one–one discontinuous one, and the one–many continuous one. It is usual in anglo-saxon cultures to denote a⁵ one–one function with a lower-case initial letter, as g , and the one–many one with an upper-case initial letter, as G . Regrettably, in France the convention is apparently reversed⁶, and some Anglosaxon texts adopt this convention, as in [19, p. 294]. This situation is worse than in section 1: here the notations are not merely baffling but contradictory, and any attempt at understanding them will need to know the (linguistic, in this case) context.

³Where the branch cuts *are* is largely irrelevant to this discussion, though there is no standard notation for distinguishing between functions which differ only in their branch cuts.

⁴We use a different letter, to avoid the problem in section 4.

⁵It would be tempting, but wrong, to write “the one-one function”. Since it is ‘obvious’ that the correct inverse of $x \mapsto x^2$ as $\mathbf{R} \rightarrow \mathbf{R}$ is the positive square root, we may be tempted to think there is an obvious inverse in other circumstances. While it is normal these days to define \log to have imaginary part in $(-\pi, \pi]$, the author was initially taught to have the imaginary part in $[0, 2\pi)$. [1] changed the branch cut of arccot between printings, and systems have been known to be internally inconsistent [7].

⁶Various mathematical textbooks seem to indicate this. However [2, Arcsin] gives capitals to Arcsin, Arccos and Arctan, but not to the others. There is clearly an inconsistency here, as [2, Arctan] describes arctan as the inverse function, and makes no mention of Arctan. The other inverse functions seem to have no entries in [2].

6 Pq

[1, equation 16.25.1] defines

$$Pq(u) = \int_0^u pq^2(t)dt \quad (7)$$

(where $pq^2(t)$ means $pq(t)^2$: see section 4). This is, of course, in defiance of the conventions of section 5, but we are dealing with elliptic functions, not elementary ones. However, the joker here is that equation (7) applies whenever p and q are any of the letters s,c,n,d (note the order, which is traditional, and see section 3). Hence this equation is in fact shorthand for twelve equations of the form

$$Sn(u) = \int_0^u sn^2(t)dt, \quad (8)$$

except that, when q is s , equation (7) should be read as

$$Pq(u) = \int_0^u \left(pq^2(t) - \frac{1}{t^2} \right) dt - \frac{1}{u}, \quad (9)$$

where the changes are to remove the removable singularity at $t = 0$.

A similar equation, but this time with explanation, can be seen as

$$pq(u) = \frac{pr(u)}{qr(u)} \quad ([1, \text{Equation 16.3.4}])$$

To quote [1, coda to section 16.27]

There is a bewildering variety of notations . . . so that in consulting books caution should be used.

As an example of this, or showing that not all apparent misprints are such, we can see [1, Equation 17.2.8–10]

$$E(u|m) = \int_0^x (1-t^2)^{-1/2}(1-mt^2)^{1/t} dt = \int_0^u dn^2(w)dw. \quad (10)$$

Does this tell us what $Dn(u)$ is — indeed [1, Equation 16.26.3] has $Dn(u) = E(u)$. However, the ‘ x ’ in equation (10) is not a misprint, and in fact [1, Equation 17.2.2] $x = sn u$. So in Maple-speak

`EllipticE(JacobiSN(u,m),m)=int(JacobiDN(t)^2,t=0..u).`

Quite how this is to be reconciled with [17, Equation 5.138(3)] —

$$\int dn^2(u) = E(am u, k)$$

— is not clear ($m = k^2$ here).

7 While we're on the subject ...

The 'help' for Maple 10 under `JacobiSN` helpfully states that

In A&S, these functions are expressed in terms of a parameter m , representing the square of the modulus k entering the definition of these functions in Maple or G&R. So, for example, the formula $\text{JacobiDN}(z,k)^2 = 1 - k^2 * \text{JacobiSN}(z,k)^2$ appears in A&S as $\text{dn}(z,m)^2 = 1 - m * \text{sn}(z,m)^2$.

However, the corresponding warning is missing from the help on `EllipticE`, but can be deduced from the fact that the example

```
EllipticE(0.3);  
1.534833465
```

in the help corresponds to the entry for $E(0.09)$ [1, p. 609], noting, however, that both this and Maple's `EllipticE` are $E(x)$, not $E(u)$.

8 O and friends

We have written elsewhere [10] as follows.

Every student is taught that $O(f(n))$ is really a set, and that when we write “ $g(n) = O(f(n))$ ”, we really mean “ $g(n) \in O(f(n))$ ”. Almost all⁷ textbooks then use ‘=’, having apparently placated the gods of confusion. However, actual uses of O as a set are rare: the author has never⁸ seen “ $O(f) \cap O(g)$ ”, and, while a textbook might⁹ write “ $O(n^2) \subset O(n^3)$ ”, this would only be for pedagogy of the O -notation.

That paper proposes an OpenMath symbol `Landauin`, whose semantics would be that of set membership, but whose notation *might be* (OpenMath does not prescribe notation) that of ‘=’.

Another notation that has come into use¹⁰ is the so-called “soft O ”, generally written \tilde{O} but also O^* , but which has two fundamentally differing definitions.

1. ‘where the “soft O ” \tilde{O} indicates an implicit factor of $(\log n)^{O(1)}$ ’ [32], attributed by [33] to [34].
- 1’ ‘where $f = \tilde{O}(g)$ if and only if there exists a constant $k \geq 0$ such that $f = O(g \cdot (\log g)^k)$ ’ [16].
2. ‘we write $O(n^{3+\epsilon})$ for $O(n^{3+o(1)})$, which is also sometimes written $\tilde{O}(n^3)$ ’ [12, footnote 1].

⁷[28] is an honourable exception.

⁸Not even in the one context where it *would* be useful: $\Theta(f) = O(f) \cap \Omega(f)$.

⁹[8, p. 41] write $\Theta(n) \subset O(n)$.

¹⁰A quick Google on 14.6.2007 showed 846 uses, though not all were mathematical.

2' 'We write¹¹ $\tilde{O}(f)$, or $O(f^{1+\epsilon})$, for $O(f^{1+o(1)})$ '.

Of these, 1 and 1' agree for $\tilde{O}(n^d)$, often the intended domain of application, but disagree otherwise. In the sense of 1, $\tilde{O}(\log n) = \tilde{O}(1)$, whereas for 1', $\tilde{O}(\log n) = O(\log n(\log \log n)^k)$ for some k , so 1' clearly makes more sense than 1.

The difference between 1' and 2' is more fundamental. 1' means “ g times something polylogarithmic in g ”, whereas 2' means “ g times something slower than any power of g ”. Hence when $g = e^n$, 1' means $\exists k O(e^n n^k)$, while 2' means $\forall \epsilon O(e^{(1+\epsilon)n})$. So 2' would include $e^n L_{1/2}(n)$ (where $L_{1/2}(n) = \exp(O(\sqrt{\log n \log \log n}))$), whereas 1' would not. A second, more subtle, point is that it is not clear in sense 1' whether k is explicitly calculable (just as it is not always clear in standard O notation).

While the difference is moot in most current uses, we believe that 2' *should* be the correct usage of \tilde{O} .

9 Other asymmetric notations

[26] writes, quoting [21] (we use Langer's notation and equation numbers), as follows.

$$(19a) \quad \frac{1}{|\phi|^{1/2}} e^{-|\xi|} \longleftrightarrow \frac{2}{\phi^{1/2}} \cos\left(\xi - \frac{\pi}{4}\right),$$

$$(19b) \quad \frac{1}{|\phi|^{1/2}} e^{|\xi|} \longleftrightarrow \frac{2}{\phi^{1/2}} \cos\left(\xi - \frac{\pi}{12}\right),$$

where, to quote [Jeffreys]: “the sign \longleftrightarrow is used to indicate that the functions it connects are asymptotic approximations to the same function in different circumstances.”

He then goes on to criticise the notation here.

Concerning the form of the statement of results (19) I would object to the use of the symbol \longleftrightarrow , on the ground that it invites misconceptions which have not failed to show themselves in the literature. I find it not unnatural to read into a pair of forms connected by an arrow the thought that in the direction of the arrow the one form implies the other. In the case of the relations (19) this inference would certainly be incorrect. [...]

Firstly, *the relation (19a) is one in which the left hand member implies and necessarily leads to the right hand member. The converse, however, is not true, ...*

¹¹Not actually seen anywhere, but related to 2 as 1' is to 1.

Secondly, the relation (19b) is one in which the right hand member implies and necessarily leads to the left hand member, but not conversely.

It seems to the current author that we have here an analogous case to the previous one: a symbol implying symmetry (here \longleftrightarrow , there $=$) is used in an asymmetric sense. The fact that the senses are reversed between (19a) and (19b) merely adds to the confusion.

10 The sins of T_EX

A trivial example [14, abstract] is $\text{Zar}(K|A)$, which is admittedly easier to type than $\mathcal{Z}\text{ar}(K|A)$, but which

- a) suggests, falsely, that the paper is about $(K|A)$ (possibly a conditional probability?);
- b) loses (at the formula level) the fact that it's about Zar.

We have also seen the following piece of apparently good mathematics.

Then the functor $T \mapsto \{\text{generically smooth } T\text{-morphisms } T \times_S C' \rightarrow T \times_S C\}$ from $((S\text{-schemes}))$ to $((\text{sets}))$ is

However, the input L^AT_EX was¹²

```
Then the functor  $T \mapsto \{\text{generically smooth } T\text{-morphisms } T \times_S C' \rightarrow T \times_S C\}$  from  $((S\text{-schemes}))$  to  $((\text{sets}))$  is
```

with the $\{\dots\}$ and $((\dots))$ not properly nested with respect to $\$$.

See also [18].

11 The sins of MathML

We have seen the following.

```
<mml:msup><mml:mn>10</mml:mn><mml:mrow><mml:mn>10</mml:mn>
<mml:mspace width="0.2em"></mml:mspace><mml:mn>000</mml:mn>
</mml:mrow></mml:msup></mml:math>
```

12 The sins of Notation

We have seen the following.

$$2\pi\phi = \left\{ \int_{\delta}^{2\pi-\delta} + \int_{-\delta}^0 + \int_0^{\delta} \right\} \frac{a^2 - r^2}{a^2 - 2ar \cos \vartheta + r^2} f(\theta + \vartheta) d\vartheta$$

¹²We did $\text{\def\Cal{\cal}}$ to make it L^AT_EX. <http://arXiv.org/abs/math/0701407>

[22, (8) p. 435]. Presumably $d\vartheta$ is meant to close all three integrals. The mixture of θ and ϑ might also be considered challenging.

We were surprised to see the following [13].

... the use of XOR operators, denoted $(q_1 \oplus q_2 \oplus \dots \oplus q_n)$ meaning that exactly one q_i holds for $1 \leq i \leq n$.

We note that this interpretation of \oplus is *not* associative.

13 The sins of Language

13.1 Translation

[6] has the following abstracts in `arxiv.org`.

Let a be a nonzero integer. If a is not congruent to 4 or 5 modulo 9 then there is no Brauer

Soit a un entier non nul. Si a n'est pas congru à 4 ou 5 modulo 9, il n'y a pas d'obstruction

So far, so good, and the translation is better than reasonable. However, the abstract in the paper itself is (our typesetting) as follows.

Soit a un entier non nul. Si a n'est pas de la forme $9n \pm 4$ pour un $n \in \mathbf{Z}$, il n'y a pas d'obstruction de Brauer–Manin à l'existence d'une solution de l'équation $x^3 + y^3 + z^3 = a$ en entiers $x, y, z \in \mathbf{Z}$. D'autre part, il n'y a pas d'obstruction de Brauer–Manin à l'existence d'une solution de l'équation $x^3 + y^3 + 2z^3 = a$ en entiers $x, y, z \in \mathbf{Z}$.

While the deep meaning is the same, we note how a congruence condition has become an “of the form” condition.

13.2 (Almost) How not to begin a paper

[5] begins as follows.

Recall that a simplicial complex Δ is called shellable if there exists a linear ordering F_1, F_2, \dots, F_k of the facets of Δ in such a way that, for each $j = 2, \dots, k$, the intersection of the sub-complex of F_j with the union of all sub-complexes of previous facets F_1, \dots, F_{j-1} is a pure sub-complex of Δ of dimension $\dim F_j - 1$.

This obscure start is rescued by the next paragraph, which begins “Although its definition is not illuminating, the notion of shellability has remarkable topological consequences.”

14 Abuse of declarations

The author recently encountered the following abstract.

Let k, x, x' be nonzero natural numbers. Let M be a tropical matrix with tropical rank k . We show that Kapranov rank is k too if x and x' are not too big; namely if we are in one of the following cases: a) $k \geq 6$ and $x, x' \leq 2$ b) $k = 4, 5$, $x \leq 2$ and $x' \leq 3$ (or obviously the converse) c) $k = 3$ and either $x, x' \leq 3$ or $x \leq 2$ and $x' \leq 4$ (or the converse).

This is nonsense as it stands. Further research found the following version (our re-typesetting).

Let M be a tropical matrix $(k + x) \times (k + x')$ for some $k, x, x' \in \mathbf{N} \setminus \{0\}$ with tropical rank k . We show that Kapranov rank is k too if x and x' are not too big; namely if we are in one of the following cases:

- a) $k \geq 6$ and $x, x' \leq 2$;
- b) $k = 4, 5$, $x \leq 2$ and $x' \leq 3$ (or obviously the converse, that is $x \leq 3$ and $x' \leq 2$)
- c) $k = 3$ and either $x, x' \leq 3$ or $x \leq 2$ and $x' \leq 4$ (or obviously the converse).

Omitting the declaration of the dimensions of M has made x and x' into free variables, making nonsense¹³ of the whole statement.

15 Other notation we have seen

15.1 $\overline{}$

We have seen [31] the expression $i = \overline{0, n}$, and in other places we have seen $i = \overline{0; n}$ (semi-colon rather than comma). In context, it was relatively clear that this meant $i \in \{0, 1, \dots, n\}$, but the usage was new to this author. The use of $\overline{}$ here is at least as egregious as its use in section 8, and is not hallowed by time. This author sees no case for $\overline{}$ over \in , as in $i \in \overline{0, n}$.

15.2 “Suggestive Notation”

We have seen [30] the following (our typesetting, attempting to preserve the original).

We use suggestive notation like $\mathbf{R}[\overline{X}]^2 := \{p^2 \mid p \in \mathbf{R}[\overline{X}]\}$ for the set of squares and $\sum \mathbf{R}[\overline{X}]^2$ for the set of sums of squares of polynomials in $\mathbf{R}[\overline{X}]$.

¹³The author, admittedly not an expert in this area of mathematics, had to retrieve the second version before it made any sense to him.

Table 1: Properties of juxtaposition			
left	right	meaning	example
weight	weight		
normal	normal	lexical	\sin
normal	italic	application	$\sin x$
italic	italic	multiplication	xy
italic	normal	multiplication	$a \sin x$
digit	digit	lexical	42
digit	italic	multiplication	$2x$
digit	normal	multiplication	$2 \sin x$
normal	digit	application	$\sin 2$
(but note the precedence in $2 \sin 3x$)			
italic	digit	error	$x2$
		(but reconsider)	x^2 or x_2 ?
digit	fraction	addition	$4\frac{1}{2}$
italic	greek	application ⁻¹	$a\phi$
		(as in group theory)	i.e. $\phi(a)$

While it cannot be denied that these are indeed suggestive, they would probably cause ‘presentation to content’ converters a great deal of difficulty. Consider the following [30, (2)].

$$(2) \quad T(\bar{g}) = \sum_{\delta \in \{0,1\}^m} \sum \mathbf{R}[\bar{X}]^2 \bar{g}^\delta := \left\{ \sum_{\delta \in \{0,1\}^m} \sigma_\delta \bar{g}^\delta \mid \sigma_\delta \in \sum \mathbf{R}[\bar{X}]^2 \right\},$$

where $\sum \mathbf{R}[\bar{X}]^2$ has to be read as a compound symbol, and the usual precedence rules for \sum , *viz.* that it binds everything to its right, do not apply to it.

15.3 Abuse of weights/fonts

It is normal to say that juxtaposition indicates multiplication (MathML’s symbol `InvisibleTimes`) or function application (MathML’s `⁡`) [9], but in fact the general rules are more complex, and highly context-sensitive. In general, we can state the observed properties of juxtaposition as being those in Table 1. Hence the font, or even the weight, of characters carries quite detailed semantic information. However, this can be abused to make differences of font or weight carry undue importance. This is shown in [15], where B denotes an (arbitrary) category, as does A and other italic capitals, but \mathbf{B} denotes a specific object, the groupoid of finite sets and bijections. This leads to us considering [15, p. 206] “ (A, B) -species of structures either as functors $\mathbf{B}A \rightarrow \hat{B}$ ”. The review¹⁴ of this paper even refers to “the category of colimit-preserving, symmetric, strong monoidal functors from $\text{Set}^{\mathbf{B}B}$ to $\text{Set}^{\mathbf{B}A}$ ”, which to the author’s mind requires keen eyes to read correctly.

¹⁴Mathematical Reviews 2389925.

We have noted in Table 1 the issue of precedence. This is very subtle in mathematics: can any-one explain satisfactorily why $2 \sin 3x \cos 4x$ means $2 \cdot (\sin(3 \cdot x)) \cdot (\cos(4 \cdot x))$, and not, say, $2 \cdot (\sin(3 \cdot x \cdot \cos(4 \cdot x)))$ or $2 \cdot (\sin 3) \cdot (x \cos 4x)$.

15.4 Units

Though initially hard to read, the notation

$$A = \{A\}_{XX}[A]_{XX},$$

[25, (1)]¹⁵ “where $\{A\}_{XX}$ is the numerical value (a pure number) of quantity A and $[A]_{XX}$ is the corresponding unit” has certain advantages.

15.5 The uses of |

We have seen [3, p. 4] the equation

$$M = M[1]|| \cdots ||M[|M|]. \quad (11)$$

where the outfix $|\cdot|$ means “size of”, but the infix $||$ (note that this is single operator) indicates concatenation. The fact that “[” and “|” are hard to tell apart doesn’t help. Incidentally, the meaning is that a bit string is composed of the string of bits.

15.6 The uses of \times

[11, (6.8)] is

$$\begin{aligned} N_{\text{loc}}(P) &= \frac{1}{4} \sum_{\iota \in \{0,1\}} \sum_{\epsilon \in \{\pm\}^3 \setminus (+,-,-)} \sum_{\substack{m,n \in \mathcal{B} \\ \gcd(m,n)=1}} \sum_{\substack{\ell \in L(m,n) \\ \neg(6.7)}} \\ &\times \sum_{\substack{m'',n'' \in \mathcal{A} \\ \gcd(m'',n'')=1 \\ \gcd(m''n'',2mn)=1}} \sum_{(\beta,\gamma,\delta) \in L_2^{(\iota)}} N_{\text{loc}} + O(P^{2+\epsilon}) \end{aligned}$$

This in fact raises various queries.

1. What is the rôle of \times here? It seems to be some form of continuation marker, even though no real product is taking place.
2. We are used to seeing \pm as a shorthand for $+, -$, but to read $\{\pm\}^3$ as shorthand for $\{+, -\}^3$ seems to be taking this a little far.
3. The $O(P^{2+\epsilon})$ is presumably outside the summation.
4. The ϵ being summed over in the second summation presumably is not the same (and indeed a strict reading of variable bindings assuming the previous point would concur) as the ϵ in $O(P^{2+\epsilon})$.

¹⁵The notation itself is attributed to [20].

15.7 Symbol Reuse

We note [23, p. 3] the use of (infix) \vee to mean (prefix) lcm.

References

- [1] M. Abramowitz and I. Stegun. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. *US Government Printing Office*, 1964.
- [2] Anonymous. Wikipedia, Français. <http://fr.wikipedia.org>, 2007.
- [3] M. Bellare and T. Ristenpart. Hash Functions in the Dedicated-Key Setting: Design Choices and MPP Transforms (full version). <http://eprint.iacr.org/2007/271.pdf>, 2007.
- [4] N. Bourbaki. Théorie des Ensembles. *Diffusion C.C.L.S.*, 1970.
- [5] M.B. Can and T. Twelbeck. Lexicographic Shellability of Partial Involutions. <http://arxiv.org/abs/1205.0062>, 2012.
- [6] J.L. Colliot-Thélène and O. Wittenberg. Groupe de Brauer et points entiers de deux surfaces cubiques affines. <http://arxiv.org/abs/0911.3539>, 2009.
- [7] R.M. Corless, J.H. Davenport, D.J. Jeffrey, and S.M. Watt. According to Abramowitz and Stegun, or arccoth needn't be uncouth. *SIGSAM Bulletin* 2, 34:58–65, 2000.
- [8] T.H. Cormen, C.E. Leiserson, R.L. Rivest, and C. Stein. Introduction to Algorithms, 2nd. ed. *M.I.T. Press*, 2001.
- [9] J.H. Davenport. Artificial Intelligence Meets Natural Typography. In S. Autexier *et al.*, editor, *Proceedings AISC/Calcuemus/MKM 2008*, pages 53–60, 2008.
- [10] J.H. Davenport and P. Libbrecht. The Freedom to Extend OpenMath and its Utility. *Mathematics in Computer Science 2(2008/9)*, pages 379–398, 2008.
- [11] R. de la Bretèche and T.D. Browning. Density of Châtelet surfaces failing the Hasse principle. <http://arxiv.org/abs/1210.4010>, 2012.
- [12] A. Denise and P. Zimmermann. Uniform Random Generation of Decomposable Structures Using Floating-Point Arithmetic. *INRIA RR 3242*, 1997.
- [13] C. Dixon, M. Fisher, and B. Konev. Tractable Temporal Reasoning. In *Proceedings IJCAI '07*, pages 318–323, 2007.
- [14] C. Finocchiaro, M. Fontana, and K.A. Loper. The constructible topology on spaces of valuation domains. <http://arxiv.org/abs/1206.3521>, 2012.

- [15] M. Fiore, N. Gambino, M. Hyland, and G. Winskel. The cartesian closed bicategory of generalised species of structures. *J. London Math. Soc. (2)*, 77:203–220, 2008.
- [16] M. Giesbrecht. Fast algorithms for matrix normal forms. In *Proceedings 33 Symp. FOCS*, pages 121–130, 1992.
- [17] I.S. Gradshteyn and I.M. Ryzhik. Table of Integrals, Series and Products (ed A. Jeffrey). *5th ed.*, 1994.
- [18] E. Gregorio. Horrors in L^AT_EX: How to misuse L^AT_EX and make a copy editor unhappy. *TUGBoat*, 26:273–279, 2005.
- [19] P. Henrici. Applied and Computational Complex Analysis I. *Wiley*, 1974.
- [20] ISO. International standard ISO 31-0: Quantities and units — Part 0: General principles. *International Organization for Standardization*, 2002.
- [21] H. Jeffreys. On certain approximate solutions of linear differential equations of the second order. *Proc. L.M.S. (2)*, 23:428–436, 1923.
- [22] H. Jeffreys and B. Jeffreys. Methods of Mathematical Physics (3rd edition). *Cambridge University Press*, 1956.
- [23] Antoine Joux and Vanessa Vitse. A variant of the f4 algorithm. In Aggelos Kiayias, editor, *Topics in Cryptology . CT-RSA 2011*, volume 6558 of *Lecture Notes in Computer Science*, pages 356–375. Springer Berlin / Heidelberg, 2011.
- [24] M. Karr. Summation in Finite Terms. *J. ACM*, 28:305–350, 1981.
- [25] S.A. Klioner. Relativistic scaling of astronomical quantities and the system of astronomical units. <http://arxiv.org/abs/astro-ph/0508292v3>, 2007.
- [26] R.E. Langer. The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to the Stokes phenomenon. *Bull. A.M.S.*, 40:545–582, 1934.
- [27] A. Lazrek. Multilingual Mathematical e-Document Processing. <http://www.ima.umn.edu/2006-2007/SW12.8-9.06/activities/Lazrek-Azzeddine/MathArabIMAE.pdf>, 2006.
- [28] A. Levitin. Introduction to the design and analysis of algorithms. *Pearson Addison-Wesley*, 2007.
- [29] National Institute for Standards and Technology. The NIST Digital Library of Mathematical Functions. <http://dlmf.nist.gov>, 2010.
- [30] J. Nie and M. Schweighofer. On the complexity of Putinar’s Positivstellensatz. <http://arxiv.org/abs/0812.2657>, 2008.

- [31] E. Shmerling. Algorithm for Defining the Distribution of Zeros of Random Polynomials. In *Proceedings 11th WSEAS International Conference on Computers*, pages 659–662, 2007.
- [32] V. Shoup. Fast construction of irreducible polynomials over finite fields. In *Proceedings 4th ACM-SIAM Symposium on Discrete algorithms*, pages 484–492, 1993.
- [33] I.E. Shparlinski. A deterministic test for permutation polynomials. *Computational Complexity*, 2:129–132, 1992.
- [34] J. von zur Gathen. Irreducibility of multivariate polynomials. *J. Computer Syst. Sci.*, 31:225–264, 1985.
- [35] D. Wees. Ambiguity in mathematical notation. <http://davidwees.com/content/ambiguity-mathematical-notation>, 2013.
- [36] D. William. Embedded Formative Assessment. *Solution Tree*, 2011.