Nauseating Notation
Very Much in Draft

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Notation exists to be abused[1]

> the abuses of language without which any mathematical text threatens to become pedantic and even unreadable. [3]

but some abuse is more harmful than others.

1 Intervals

We raise this old chestnut first because it illustrates some of the problems. How do we represent $\{x : 0 < x \leq 1\}$? Semantically, there is no problem.

```
<OMA>
  <OMS name="interval_oc" cd="interval1''/>
  <OMI>0</OMI>
  <OMI>1</OMI>
<\OMA>
```

Presentationally, there are two well-kown routes: the "anglo-saxon" way $(0, 1]$ and the "french" way $]0, 1]$. The purpose of this paper is not to argue that one is "better" than the other: merely that there are two competing ones, and the use of the unfamiliar one may well baffle.

2 Plus or Minus

This is familiar to us all from the solution to the quadratic:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{1}$$

[1] On 10.6.2007, a quick Google demonstrated 783 uses of "abus de notation", roughly 10% of which were in english-language papers.

which can be seen as shorthand for

$$\left\{\frac{-b-\sqrt{b^2-4ac}}{2a}, \frac{-b+\sqrt{b^2-4ac}}{2a}\right\}. \tag{2}$$

We are prepared to accept it in formulae such as [1, Equation 4.3.38]

$$\tan z_1 \pm \tan z_2 = \frac{\sin(z_1 \pm z_2)}{\cos z_1 \cos z_2}, \tag{3}$$

which we read as shorthand for two equations:

$$\begin{aligned} \tan z_1 - \tan z_2 &= \frac{\sin(z_1 - z_2)}{\cos z_1 \cos z_2} \\ \tan z_1 + \tan z_2 &= \frac{\sin(z_1 + z_2)}{\cos z_1 \cos z_2}. \end{aligned}$$

But what of [1, Equations 4.6.26,27]

$$\begin{aligned} \operatorname{Arcsinh} z_1 \pm \operatorname{Arcsinh} z_2 &= \operatorname{Arcsinh}\left(z_1\sqrt{1-z_2^2} \pm z_2\sqrt{1-z_1^2}\right) \\ \operatorname{Arccosh} z_1 \pm \operatorname{Arccosh} z_2 &= \operatorname{Arccosh}\left(z_1 z_2 \pm \sqrt{(z_1^2-1)(z_2^2-1)}\right)? \end{aligned}$$

3 Alphabetical Order?

The author recently heard a speaker [13] state that, in Arabic Mathematics, the alphabetic order used in formulae is different from the usual one. “How perverse”, the author thought, as probably does the reader. But is it that perverse? Any text in ideal theory will normally call the variables $x_1 \ldots, x_n$ for generic theorems and definitions. However, explicit examples will usually make use of particular letters, e.g. x and y in two dimensions, or x, y and z in three. In four, the variable w (or possibly t) is pressed into service, but the lexicographic order is then normally taken to be $x > y > z > w$. Maybe it *is* perverse, but it’s a common perversion.

There are other instances of non-obvious order: the theory of elliptic functions (see section 6) tends to order the relevant letters as s, c, n, d, though this doesn’t have any mathematical significance, merely the order in which one finds things in tables.

4 Iterated Functions

5 To Capitalise or not to Capitalise

Half of the 26 standard transcendental elementary functions (log, the six inverse trigonometric functions and the six inverse hyperbolic) are one–many functions, at least on the complex plane, since the functions whose inverses they are (exp, trigonometric and hyperbolic) are many–one. However, it is normal to restrict them, by means of "branch cuts" [4][2] to be one–one, at the price of being discontinuous.

This means that, if f is a many–one function $\mathbf{C} \to \mathbf{C}$, its inverse[3], which will be denoted g, has two possible definitions: the one–one discontinuous one, and the one–many continuous one. It is usual in anglo-saxon cultures to denote a[4] one–one function with a lower-case initial letter, as g, and the one–many one with an upper-case initial letter, as G. Regrettably, in France the convention is apparently reversed[5]. Here the situation is worse than in section 1: here the notations are not merely baffling but contradictory, and any attempt at understanding them will need to know the (linguistic, in this case) context.

6 Pq

[1, equation 16.25.1] defines

$$\mathrm{Pq}(u) = \int_0^u \mathrm{pq}^2(t)\mathrm{d}t \tag{4}$$

(where $\mathrm{pq}^2(t)$ means $\mathrm{pq}(t)^2$: see section 4). This is, of course, in defiance of the conventions of section 5, but we are dealing with elliptic functions, not elementary ones. However, the joker here is that equation (4) applies whenever p and q are any of the letters s,c,n,d (note the order, which is traditional, and see section 3). Hence this equation is in fact shorthand for twelve equations of the form

$$\mathrm{Sn}(u) = \int_0^u \mathrm{sn}^2(t)\mathrm{d}t, \tag{5}$$

except that, when q is s, equation (4) should be read as

$$\mathrm{Pq}(u) = \int_0^u \left(\mathrm{pq}^2(t) - \frac{1}{t^2}\right)\mathrm{d}t - \frac{1}{u}, \tag{6}$$

where the changes are to remove the removable singularity at $t = 0$.

[2]Where the branch cuts *are* is largely irrelevant to this discussion, though there is no standard notation for distinguishing between functions which differ only in their branch cuts.

[3]We use a different letter, to avoid the problem in section 4.

[4]It would be tempting, but wrong, to write "the one-one function". Since it is 'obvious' that the correct inverse of $x \mapsto x^2$ as $\mathbf{R} \to \mathbf{R}$ is the positive square root, we may be tempted to think there is an obvious inverse in other circumstances. While it is normal these days to define log to have imaginary part in $(-\pi, \pi]$, the author was initially taught to have the imaginary part in $[0, 2\pi)$. [1] changed the branch cut of arctan between printings, and systems have been known to be internally inconsistent [4].

[5]Various mathematical textbooks seem to indicate this. However [2, Arcsin] gives capitals to Arcsin, Arccos and Arctan, but not to the others. There is clearly an inconsistency here, as [2, Arctan] describes arctan as the inverse function, and makes no mention of Arctan. The other inverse functions seem to have no entries in [2].

A similar equation, but this time with explanation, can be seen as

$$\mathrm{pq}(u) = \frac{\mathrm{pr}(u)}{\mathrm{qr}(u)} \qquad \text{([1, Equation 16.3.4])}$$

To quote [1, coda to section 16.27]

> There is a bewildering variety of notations ... so that in consulting books caution should be used.

As an example of this, or showing that not all apparent misprints are such, we can see [1, Equation 17.2.8–10]

$$E(u|m) = \int_0^x (1-t^2)^{-1/2}(1-mt^2)^{1/t}\mathrm{d}t = \int_0^u \mathrm{dn}^2(w)\mathrm{d}w. \tag{7}$$

Does this tell us what $\mathrm{Dn}(u)$ is — indeed [1, Equation 16.26.3] has $\mathrm{Dn}(u) = E(u)$. However, the 'x' in equation (7) is not a misprint, and in fact [1, Equation 17.2.2] $x = \mathrm{sn}\, u$. So in Maple-speak

```
EllipticE(JacobiSN(u,m),m)=int(JacobiDN(t)^2,t=0..u).
```

Quite how this is to be reconciled with [11, Equation 5.138(3)] —

$$\int \mathrm{dn}^2(u) = E(\mathrm{am}\, u, k)$$

— is not clear ($m = k^2$ here).

7 While we're on the subject ...

The 'help' for Maple 10 under `JacobiSN` helpfully states that

```
In A&S, these functions are expressed in terms of a parameter m,
representing the square of the modulus k entering the definition of
these functions in Maple or G&R. So, for example, the formula
JacobiDN(z,k)^2 = 1 - k^2 * JacobiSN(z,k)^2 appears in A&S as
dn(z,m)^2 = 1 - m * sn(z,m)^2.
```

However, the corresponding warning is missing from the help on `EllipticE`, but can be deduced from the fact that the example

```
EllipticE(0.3);
                          1.534833465
```

in the help corresponds to the entry for $E(0.09)$ [1, p. 609], noting, however, that both this and Maple's `EllipticE` are $E(x)$, not $E(u)$.

8 O and friends

We have written elsewhere [7] as follows.

> Every student is taught that $O(f(n))$ is really a set, and that when we write "$g(n) = O(f(n))$", we really mean "$g(n) \in O(f(n))$". Almost all[6] textbooks then use '=', having apparently placated the gods of confusion. However, actual uses of O as a set are rare: the author has never[7] seen "$O(f) \cap O(g)$", and, while a textbook might[8] write "$O(n^2) \subset O(n^3)$", this would only be for pedagogy of the O-notation.

That paper proposes an OpenMath symbol `Landauin`, whose semantics would be that of set membership, but whose notation *might be* (OpenMath does not prescribe notation) that of '='.

Another notation that has come into use[9] is the so-called "soft O", generally written $\tilde{O}$ but also O^*, but which has two fundamentally differing definitions.

1. 'where the "soft O" $\tilde{O}$ indicates an implicit factor of $(\log n)^{O(1)}$' [16], attributed by [17] to [18].

1' 'where $f = \tilde{O}(g)$ if and only if there exists a constant $k \geq 0$ such that $f = O(g \cdot (\log g)^k)$' [10].

2. 'we write $O(n^{3+\epsilon})$ for $O(n^{3+o(1)})$, which is also sometimes written $\tilde{O}(n^3)$' [8, footnote 1].

2' 'We write[10] $\tilde{O}(f)$, or $O(f^{1+\epsilon})$, for $O(f^{1+o(1)})$'.

Of these, 1 and 1' agree for $\tilde{O}(n^d)$, often the intended domain of application, but disagree otherwise. In the sense of 1, $\tilde{O}(\log n) = \tilde{O}(1)$, whereas for 1', $\tilde{O}(\log n) = O(\log n (\log\log n)^k)$ for some k, so 1' clearly makes more sense than 1.

The difference between 1' and 2' is more fundamental. 1' means "g times something polylogarithmic in g", whereas 2' means "g times someting slower than any power of g". Hence when $g = e^n$, 1' means $\exists k O(e^n n^k)$, while 2' means $\forall \epsilon o(e^{(1+\epsilon)n})$. So 2' would include $e^n L_{1/2}(n)$ (where $L_{1/2}(n) = \exp(O(\sqrt{\log n \log\log n}))$), whereas 1' would not. A second, more subtle, point is that it is not clear in sense 1' whether k is explicitly calculable (just as it is not always clear in standard O notation).

While the difference is moot in most current uses, we believe that 2' *should be* the correct usage of $\tilde{O}$.

---

[6] [14] is an honourable exception.

[7] Not even in the one context where it *would* be useful: $\Theta(f) = O(f) \cap \Omega(f)$.

[8] [5, p. 41] write $\Theta(n) \subset O(n)$.

[9] A quick Google on 14.6.2007 showed 846 uses, though not all were mathematical.

[10] Not actually seen anywhere, but related to 2 as 1' is to 1.

9 The sins of $\TeX$

We have seen the following.

> Then the functor $T \mapsto \{$generically smooth T-morphisms $T \times_S \mathcal{C}' \to \mathcal{T} \times_S \mathcal{C}\}$ from ((S-schemes)) to ((sets)) is

However, the input $\LaTeX$ was[11]

```
Then the functor $T\mapsto\{$generically smooth $T$-morphisms
$T\times_S\Cal
C'\to T\times_S\Cal C\}$ from $((S$-schemes)) to ((sets)) is
```

10 The sins of Notation

We have seen the following.

$$2\pi\phi = \left\{ \int_\delta^{2\pi-\delta} + \int_{-\delta}^{0} + \int_0^\delta \right\} \frac{a^2 - r^2}{a^2 - 2ar\cos\vartheta + r^2} f(\theta+\vartheta)d\vartheta$$

[12, (8) p. 435]. Presumably $d\vartheta$ is meant to close all three integrals. The mixture of θ and ϑ might also be considered challenging.

11 Abuse of declarations

The author recently encountered the following abstract.

```
Let k,x,x' be nonzero natural numbers.  Let M be a tropical
matrix with tropical rank k.  We show that Kapranov rank is
k too if x and x' are not too big; namely if we are in one
of the following cases:  a) k>=6 and x, x' <=2 b) k=4,5, x<=2
and x'<=3 (or obviously the converse) c) k=3 and either x,x'<=3
or x<=2 and x'<=4 (or the converse).
```

This is nonsense as it stands. Further research found the following version (our re-typesetting).

> Let M be a tropical matrix $(k + x) \times (k + x')$ for some k, x, $x' \in \mathbf{N} \setminus \{0\}$ with tropical rank k. We show that Kapranov rank is k too if x and x' are not too big; namely if we are in one of the following cases:
>
> a) $k \geq 6$ and $x, x' \leq 2$;
>
> b) $k = 4, 5$, $x \leq 2$ and $x' \leq 3$ (or obviously the converse, that is $x \leq 3$ and $x' \leq 2$)

[11] We did `\def\Cal{\cal}` to make it $\LaTeX$. `http://arXiv.org/abs/math/0701407`

c) $k = 3$ and either $x, x' \leq 3$ or $x \leq 2$ and $x' \leq 4$ (or obviously the converse).

Omitting the declaration of the dimensions of M has made x and x' into free variables, making nonsense[12] of the whole statement.

12 Other notation we have seen

12.1 `\overline`

We have seen [?] the expression $i = \overline{0, n}$, and in other places we have seen $i = \overline{0; n}$ (semi-colon rather than comma). In context, it was relatively clear that this meant $i \in \{0, 1, \ldots, n\}$, but the usage was new to this author. The use of $=$ here is at least as egregious as its use in section 8, and is not hallowed by time. This author sees no case for $=$ over $\in$, as in $i \in \overline{0, n}$.

12.2 "Suggestive Notation"

We have seen [15] the following (our typesetting, attempting to preserve the original).

> We use suggestive notation like $\mathbf{R}[\overline{X}]^2 := \{p^2 \mid p \in \mathbf{R}[\overline{X}]\}$ for the set of squares and $\sum \mathbf{R}[\overline{X}]^2$ for the set of sums of squares of polynomials in $\mathbf{R}[\overline{X}]$.

While it cannot be denied that these are indeed suggestive, they would probably cause 'presentation to content' converters a great deal of difficulty. Consider the following [15, (2)].

$$T(\overline{g}) = \sum_{\delta \in \{0,1\}^m} \sum \mathbf{R}[\overline{X}]^2 \overline{g}^\delta := \left\{ \sum_{\delta \in \{0,1\}^m} \sigma_\delta \overline{g}^\delta \mid \sigma_\delta \in \sum \mathbf{R}[\overline{X}]^2 \right\}, \tag{2}$$

where $\sum \mathbf{R}[\overline{X}]^2$ has to be read as a compound symbol, and the usual precedence rules for $\sum$, *viz.* that it binds everything to its right, do not apply to it.

12.3 Abuse of weights/fonts

It is normal to say that juxtaposition indicates multiplication (MathML's symbol `InvisibleTimes`) or function application (MathML's `⁡`) [6], but in fact the general rules are more complex, and highly context-sensitive. In general, we can state the observed properties of juxtaposition as being those in table 1. Hence the font, or even the weight, of characters carries quite detailed semantic information. However, this can be abused to make differences of font or weight carry undue importance. This is shown in [9], where B denotes an

[12] The author, admittedly not an expert in this area of mathematics, had to retrieve the second version before it made any sense to him.

Table 1: Properties of juxtaposition

left weight	right weight	meaning	example
normal	normal	lexical	sin
normal	italic	application	$\sin x$
italic	italic	multiplication	xy
italic	normal	multiplication	$a \sin x$
digit	digit	lexical	42
digit	italic	multiplication	$2x$
digit	normal	multiplication	$2 \sin x$

(arbitrary) category, as does A and other italic capitals, but $\mathbf{B}$ denotes a specific object, the groupoid of finite sets and bijections. This leads to us considering [9, p. 206] "(A, B)-species of structures either as functors $\mathbf{B}A \to \hat{B}$". The review[13] of this paper even refers to "the category of colimit-preserving, symmetric, strong monoidal functors from $\mathrm{Set}^{\mathbf{B}B}$ to $\mathrm{Set}^{\mathbf{B}A}$", which to the author's mind requires keen eyes to read correctly.

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