Asymptotic Analysis of American-Style Options

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Summary

We investigate the behaviour of both European and American-style options on dividend paying underlyings under the assumptions used in the Black-Scholes-Merton model [17, 84]. For European options we show explicitly how the small-time, large-time and small-volatility limiting behaviour may be derived in the context of matched asymptotic expansions. In particular, exponentially small terms are obtained for the small-time and small-volatility problems using WKBJ expansions. This work is consistent with that of Addison et al. [2] in relation to the Stefan problem and the terms carry over into the American option problem. The large-time work results in a similarity solution which has not previously been mentioned in the context of the European option problem and which is accurate for surprisingly small times. In extending the asymptotic analysis to the standard American option and American barrier option problems we demonstrate that reverse barriers give rise to different small-time asymptotic behaviour versus the standard problem. The large-time behaviour for both American-style problems has not previously been derived explicitly, but is identified in the course of this work.

In an attempt to bridge the gap between the small- and large-time asymptotic work for standard American options, we extend the popular uniformly valid approximation due to MacMillan [79] and Barone-Adesi & Whaley [10] (the *MBAW* approximation). By posing the approximation as the leading order term in a homotopic series, following an approach adopted recently by Zhu [106], we are able to derive three-term analytic expressions for the optimal exercise boundary and price of the standard American option. The resulting expression for the optimal exercise boundary improves greatly on the accuracy of the leading order term, but maintains the advantage of being fast and easy to determine, requiring only an accurate method for the calculation of the cumulative normal distribution function. The approach generalises attempts by previous authors such as Ju & Zhong [69] to find correction terms to the MBAW approximation. We demonstrate the potential to extend this approach to other American-style options by deriving a two-term analytical approximation for the American up-and-out put option, where the leading order term has previously been identified by AitSahlia & Lai [3], with similarly promising results.

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Chapter 1

Introduction

1.1 Financial Derivatives and Option Pricing

A derivative financial instrument is one whose value is dependent on, or *derived* from, the price of an underlying asset S. Examples of typical types of underlying are equities, interest rate products such as bonds, currency rates, commodities (precious metals, energy products etc.) and even other derivatives since these can also be classed as assets.

An option is a particular instance of a derivative, which conveys upon the purchaser (the *holder*) the right, but not the obligation, to buy (a *call* option) or sell (a *put* option) the underlying at a future time T (the *expiry*) and at a pre-specified price K (the *strike*). This optionality comes at a price which the option holder pays to the seller of the option (the *writer*) at time t, which we denote $V_e(S,t)$. At expiry a rational holder will take up this right (*exercise* the option) if there is a profit to be made, which leads to the payoff functions at expiry

$$\text{Call option payoff:} \qquad C_e(S,T) = \max\left(S-K,0\right), \tag{1.1a}$$

$$\text{Put option payoff:} \qquad P_e(S,T) = \max\left(K-S,0\right). \tag{1.1b}$$

The max notation represents the holder's choice to exercise the option at expiry if it is profitable to do so, or leave it to expire unexercised if not. For the holder, the maximum loss is the cost of the option, while the potential profit is unlimited for a call and capped at K for a put. The position of the option writer is therefore inherently more risky as his best outcome is that the option remains unexercised, whereas his worst-case losses are potentially unlimited if he has written a call option.

The option described above, where the holder may only exercise at expiry, is known as a *European* style option. Such options are often given the name *vanilla* as they were among the earliest exchange-traded options, with the Chicago Board Options Exchange (CBOE) first trading call options in 1973 and put options in 1977 [62].

In the intervening years many non-vanilla, or *exotic*, option contracts have been conceived with increasingly more complex payoffs, such as options on extreme and average values of the price of the underlying, path dependent options with barriers which affect the state of the option and options which allow the holder to exercise prior to expiry at a time of their choosing. Options exhibiting the latter early-exercise feature are known as *American* style options.

Though the increasing complexity of derivative products allows the mitigation of increasingly more complex risks, the valuation of the instruments involved, together with the quantification of the risks around the positions they create, leads to ever more challenging mathematical problems.

1.2 European Options and the Black-Scholes-Merton Equation

Black & Scholes [17] showed that, under a certain set of assumptions, the European option problem has a unique solution for a given set of parameters. The assumptions of the classical Black-Scholes model are:

- The risk-free interest rate, r, is deterministic and constant;
- Instantaneous price returns for the underlying, defined as $\frac{dS}{S}$, are normally distributed;
- The underlying pays no dividends;
- There are no transaction costs; and
- The underlying is infinitely divisible and short selling, or holding a negative position in the stock, is permitted.

The second assumption is equivalent to the existence of a non-negative price process for the underlying which evolves according to the stochastic differential equation

$$dS = \mu S dt + \sigma S dW, \tag{1.2}$$

where μ is the mean (or *drift*), σ is the standard deviation (or *volatility*) of the price returns and dW is the increment of a Brownian motion which is normally distributed with zero mean and variance dt. An underlying which follows such a price process is said to exhibit geometric Brownian motion (or *GBM*) and was first proposed as a model for stock prices by Samuelson [95].

The solution derived in [17] uses a combination of the available assets (or *portfolio*) composed of a positive (or *long*) position in the stock and a fractional negative (or *short*) position in the option. It is shown, through continuously adjusting the short position (or *hedging*), that the risky stochastic component of the portfolio return can be eliminated, resulting in a deterministic payoff for the portfolio at expiry.

The central concept of *no-arbitrage*, which prohibits the existence of riskless profits, forces such a deterministic portfolio to have a return equal to the risk-free interest rate. The construction of a riskless hedging portfolio and no-arbitrage lead to the Black-Scholes PDE

$$S \in (0, \infty), t \in (0, T) \qquad \frac{\partial V_e}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V_e}{\partial S^2} + rS \frac{\partial V_e}{\partial S} - rV_e = 0, \tag{1.3a}$$

which has the following properties:

- Linear,
- First-order in time,
- Second-order in space (except at $S = 0$),
- Homogeneous in S,
- Backwards parabolic and
- Degenerate at $S = 0$.

The approach of Black & Scholes relies on the option being a continuously tradeable asset in order to perfectly hedge the risky stock price movements. However, this can be circumvented as long as there exists a bank account which pays or charges the risk-free rate on long or short cash positions. In this case a continuously adjusted portfolio can be formed which perfectly reproduces (or *replicates*) the option payoff at expiry. A further consequence of no-arbitrage is that a portfolio which does not require a net flow of funds from an external source, called a *self financing* portfolio, and which replicates the option payoff at expiry must have the same cost to construct at all earlier times. A more detailed discussion of replication, no-arbitrage and hedging can be found in Bjork [16].

We note that the form of the Black-Scholes PDE is independent of the type of option which is defined through the payoff function and the boundary conditions. For a European put option these consist of

$$\text{as } S \to \infty \qquad P_e(S,t) \to 0, \tag{1.3b}$$

$$\text{as } S \to 0 \qquad P_e(S,t) \to e^{-r(T-t)}K, \tag{1.3c}$$

$$\text{at } t = T \qquad P_e(S,T) = \max(K-S,0). \tag{1.3d}$$

The large S condition (1.3b) specifies that the price of a put option on an underlying which is far above the strike ($S \gg K$, or deeply *out-of-the-money*) approaches zero. The economic intuition for this is that as S becomes very large, there is a vanishingly small probability of the option having a positive payoff ($S < K$, or being *in-the-money*) at expiry, and is therefore likely to expire unexercised without generating a cash flow.

The small S condition (1.3c) specifies the value of a put option on an underlying with zero value. To motivate this condition, we can see from the form of GBM (1.2) that an underlying which has reached zero is deterministically zero at all subsequent times. Thus our option has a known payoff K at expiry and no-arbitrage enforces the price to be the discounted value of the payoff, or $Ke^{-r(T-t)}$ under a deterministic interest rate. The final condition (1.3d) provides the final condition at expiry which is simply the put option payoff (1.1b).

This initial boundary value problem can be solved using a number of methods, for example by transformation to the heat equation and solving using a Green's function [103], to give the Black-Scholes equation for a put option on a non-dividend paying underlying

$$P_e(S,t) = \frac{1}{2}\left[Ke^{-r(T-t)}\mathrm{erfc}\left(\frac{d_2}{\sqrt{2}}\right) - S\mathrm{erfc}\left(\frac{d_1}{\sqrt{2}}\right)\right], \tag{1.4}$$

where $\mathrm{erfc}(\zeta)$ is the complementary error function

$$\mathrm{erfc}(\zeta) = \frac{2}{\sqrt{\pi}}\int_{\zeta}^{\infty} e^{-\frac{s^2}{2}}\,ds \tag{1.5}$$

and

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}}, \tag{1.6a}$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}}. \tag{1.6b}$$

Although this thesis centres around the PDE approach to option pricing, any introduction to the subject would not be complete without reference to the *risk neutral valuation* approach, which is a body of work initiated and formalised in the papers of Cox & Ross [33], Cox, Ross & Rubinstein [34], Harrison & Kreps [52] and Harrison & Pliska [53]. This approach leads to the concept of pricing options as the expected value of the final payoff under a particular probability measure $\mathbb{Q}$ (the *risk-neutral measure*) discounted at the risk free rate. Under $\mathbb{Q}$, the expected return of all assets is the risk-free rate, with the dynamics of S given by

$$dS = rSdt + \sigma SdW \tag{1.7}$$

and the value of the option written as

$$V_e(S_t, t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[f(S_T)|S_t\right], \tag{1.8}$$

where S_t and S_T represent the price of the underlying at times t and T respectively and $f(S_T)$ represents the payoff at expiry. It transpires that results which hold in the risk-neutral world are valid using the real world (or *objective*) measure $\mathbb{P}$. We mention that the formulation of the problem as the expectation (1.8) can be shown to be equivalent to the PDE formulation (1.3a) the using the Feynman-Kac theorem, a discussion of which can be found in Joshi [67]. The book by Baxter & Rennie [11] contains an informal but illuminating description of this approach to derivative pricing, while a more rigorous discussion can be found in Karatzas and Shreve [71].

1.2.1 The Inclusion of Dividends

The limitations of the Black-Scholes assumptions when applied to real-world situations are widely discussed in the literature, with broad discussions of the issues involved and possible extensions found in Hull [62] and Wilmott [103].

For the purpose of our work, the most relevant extension is the generalisation of the price process (1.2) to include underlyings which pay continuous dividends, which was introduced by Merton [84]. The risk-neutral price process for an underlying yielding a continuous dividend D, defined as a percentage of the prevailing price of the underlying, is

$$dS = (r - D)\,Sdt + \sigma SdW, \tag{1.9}$$

where the quantity $r - D$ is sometimes referred to as the *cost-of-carry* and represents the cost of financing the purchase of a long position in the underlying, with the cost of borrowing offset by any dividends received through holding the underlying. This leads to the problem for a put option on a dividend paying underlying

$$S \in (0,\infty), t \in (0,T) \qquad \frac{\partial P_e}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 P_e}{\partial S^2} + (r-D)S\frac{\partial P_e}{\partial S} - rP_e = 0, \tag{1.10a}$$

subject to

$$\text{as } S \to 0 \qquad P_e(S,t) \to e^{-r(T-t)}K, \tag{1.10b}$$

$$\text{as } S \to \infty \qquad P_e(S,t) \to 0, \tag{1.10c}$$

$$\text{at } t = T \qquad P_e(S,T) = \max\left(K - S, 0\right). \tag{1.10d}$$

The solution of this initial boundary value problem is

$$P_e(S,t) = \frac{1}{2}\left[Ke^{-r(T-t)}\text{erfc}\left(\frac{d_2}{\sqrt{2}}\right) - Se^{-D(T-t)}\text{erfc}\left(\frac{d_1}{\sqrt{2}}\right)\right], \tag{1.11}$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left((r-D) + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}}, \tag{1.12a}$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left((r-D) - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}}. \tag{1.12b}$$

In what follows, we shall work under the Black-Scholes-Merton model assumptions for a dividend paying underlying which follows GBM, unless otherwise specified.

1.2.2 Put-Call Parity for European Options

Although the relevant results presented in this thesis relate specifically to European put options, the equivalent results for a European call option may be obtained through the put-call parity relationship

$$P_e(S,t) + e^{-D(T-t)}S = C_e(S,t) + e^{-r(T-t)}K. \tag{1.13}$$

This result can be obtained simply by demonstrating that the European put option plus a stock position, with any dividends received reinvested in the stock, is a self-financing replicating portfolio for the European call option plus a position in the bank account. The reader is directed towards Hull [62] for the details.

1.3 Early Exercise and American Options

American options differ from European style options in that the holder is able to exercise at any time prior to expiry. The American option remains an important problem in finance as a significant proportion of options transacted today have early exercise features. In the discrete case, where early exercise is only permitted at one or more specified times before expiry, the option is said to be *Bermudan*.

The extra optionality afforded to the holder through early exercise translates into a higher option value (or *premium*) compared to the European option price. The European option price therefore forms a trivial lower bound on the American option price.

From an economic viewpoint, the holder's decision over whether or not to exercise is influenced by three competing factors:

1. Interest income foregone (call) or gained (put) from a change in cash position from early exercise,
2. Dividend income gained (call) or foregone (put) from holding or giving up the underlying,
3. Insurance value from holding an unexercised option.

Allowing the holder of a put option the right to exercise at any point in time, receiving the intrinsic value, gives rise to the formulation of an optimal stopping problem [13, 70, 100] where the value of the option at time t is given by the maximum value taken by considering all possible stopping times t^*, or

$$V_a(S_t, t) = \sup_{t^*} \left(e^{-r(T-t^*)} \mathbb{E}_{\mathbb{Q}} \left[f(S_{t^*}) | S_t \right] \right), \tag{1.14}$$

where $f(S_{t^*})$ represents the payoff achieved by exercising the option at t^*.

McKean [83] showed that the optimal stopping problem for an American call could be posed as a free boundary problem with two conditions applied at an unknown *optimal exercise boundary* $S^*(t)$. These conditions separate the problem into a region in which the option is active and a region in which the option is exercised. The properties of $S^*(t)$ were investigated by van Moerbeke [100], who derived an integral equation for the optimal exercise boundary. Proof of the existence and uniqueness of the optimal exercise boundary is provided by Peskir [89].

A summary of this body of work, together with proofs for the American put option in the absence of dividends, is contained in the review paper by Myneni [86] who derives the following conditions for the American put option

$$\text{as } S \to \infty \qquad P_a(S,t) \to 0, \tag{1.15a}$$

$$\text{at } t = T \qquad P_a(S,T) = \max(K - S, 0), \tag{1.15b}$$

$$\text{at } S = S^*(t) \qquad P_a(S^*(t),t) = K - S^*(t), \tag{1.15c}$$

$$\left.\frac{\partial P_a}{\partial S}\right|_{S=S^*(t)} = -1. \tag{1.15d}$$

Conditions (1.15a) and (1.15b) are the same as for the European option, with the former representing the vanishingly small chance of the option finishing in-the-money as S becomes very large, while the latter is simply the final payoff. Condition (1.15c) is termed the *value matching* condition and represents the continuity of the option price on the optimal exercise boundary. Condition (1.15d) is termed the *high contact* condition, and together with (1.15a-1.15c) provides sufficient conditions to specify the location of $S^*(t)$. We term the value matching and high contact conditions together, the *early exercise* conditions. One further condition on the option price is the requirement

$$P_a(S,t) \geq \max(K - S, 0), \tag{1.16}$$

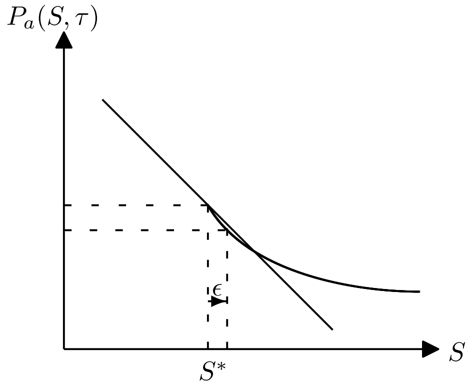
which states that the intrinsic value forms a lower bound on the American option price.

Somewhat heuristic motivations of the high contact condition are contained in Dewynne et al. [38] and also in Joshi [67]. The approach contained in the latter is illustrated in Figures 1.1(a) & 1.1(b). Consider first the case illustrated in Figure 1.1(a) where the unexercised price meets the intrinsic value line such that $\frac{\partial P}{\partial S} < -1$. We can find $0 < \epsilon \ll 1$ such that $\frac{\partial P}{\partial S} < -1 - \delta$ on the interval $(S^*, S^* + \epsilon)$, for some $0 < \delta \ll 1$. The option price at $S = S^* + \epsilon$ is then

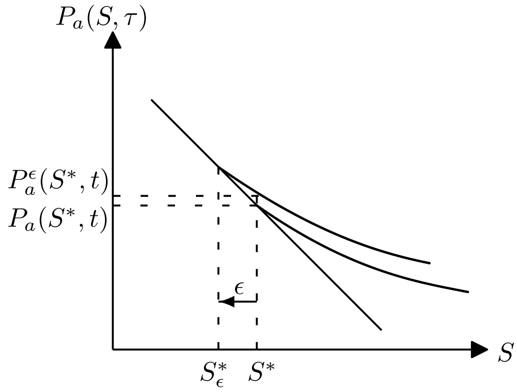
$$\begin{aligned} P(S^* + \epsilon, t) &= P(S^*, t) + \epsilon \left.\frac{\partial P}{\partial S}\right|_{S^*} + O(\epsilon^2) \\ &= (K - S^*) + \epsilon(-1 - \delta) + O(\epsilon^2) \\ &= (K - (S^* + \epsilon)) - \epsilon\delta + O(\epsilon^2), \end{aligned} \tag{1.17}$$

which is less than the intrinsic value of the option and is a violation of no-arbitrage. This leads us to conclude $\frac{\partial P}{\partial S}\big|_{S^*(\tau)} \geq -1$.

Next, consider the case illustrated in Figure 1.1(b) where the unexercised price meets the



(a) $\frac{\partial P}{\partial S}\big|_{S^*(\tau)} < -1$



(b) $\frac{\partial P}{\partial S}\big|_{S^*(\tau)} > -1$

Figure 1-1: Motivation by Joshi [67] of the high contact condition at the optimal exercise boundary $\left(\frac{\partial P_a}{\partial S}\big|_{S^*(t)} = -1\right)$. If the gradient were less than -1 (Figure 1.1(a)) then an option price below the intrinsic value would exist in a region local to the boundary. If the gradient were greater than -1 (Figure 1.1(a)) then a different boundary would exist for which the option price is higher, and therefore the assumed boundary is suboptimal.

intrinsic value line such that $\frac{\partial P}{\partial S}\big|_{S^*(\tau)} > -1$. We perturb the position of the optimal exercise boundary according to $S^*_\epsilon = S^* - \epsilon$ for $0 < \epsilon \ll 1$ such that $\frac{\partial P}{\partial S} > -1$ on the interval (S^*_ϵ, S^*). Thus we can find some $0 < \delta \ll 1$ such that $\frac{\partial P}{\partial S}\big|_{S^*_\epsilon(\tau)} > -1 + \delta$. The option price at $S = S^*_\epsilon$ is then

$$\begin{aligned} P_a^\epsilon(S^*,t) &= P_a^\epsilon(S^*_\epsilon,t) + \epsilon \frac{\partial P_a^\epsilon}{\partial S}\bigg|_{S^*_\epsilon} + O(\epsilon^2) \\ &> (K - S^*_\epsilon) + \epsilon(-1+\delta) + O(\epsilon^2) \\ &> (K - (S^*_\epsilon + \epsilon)) + \epsilon\delta + O(\epsilon^2) \\ &> (K - S^*) + \epsilon\delta + O(\epsilon^2), \end{aligned} \tag{1.18}$$

which is greater than the value attainable by exercising at S^* and therefore S^* is a suboptimal exercise strategy. Therefore, combined with the previous scenario, we conclude that the option price at the optimal exercise boundary must satisfy the condition $\frac{\partial P}{\partial S}\big|_{S^*(\tau)} = -1$.

1.3.1 The Inclusion of Dividends

Since the American option obeys the same dynamics as the European option in the active region, we should expect the inclusion of dividends to produce the same governing equation as in the European case. There is an additional impact however regarding the initial position of the optimal exercise boundary $S^*(0)$ which is illustrated using an approach due to Dewynne et al. [38]. Using the transformation

$$P_a(S,t) = (K - S) + e^{-r(T-t)}p(S,t), \tag{1.19}$$

gives the governing equation

$$S \in (S^*(t), \infty), t \in (0,T) \qquad \frac{\partial p}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 p}{\partial S^2} + (r-D)S\frac{\partial p}{\partial S} = e^{r(T-t)}\left(rK - DS\right), \tag{1.20a}$$

subject to

$$\text{as } S \to \infty \qquad p(S,t) \sim e^{r(T-t)}(S-K), \tag{1.20b}$$

$$\text{at } t = T \qquad p(S,T) = \max\left(S-K, 0\right), \tag{1.20c}$$

$$\text{at } S = S^*(t) \qquad p(S^*(t),t) = 0, \tag{1.20d}$$

$$\left.\frac{\partial p}{\partial S}\right|_{S=S^*(t)} = 0. \tag{1.20e}$$

The source term in (1.20a) is negative for $S < \frac{rK}{D}$ and positive for $S > \frac{rK}{D}$. Since the constraint $P_a(S,t) \geq 0$ also implies $p(S,t) \geq 0$, this forbids $S^*(T) < \frac{rK}{D}$ for $r < D$ as it would give rise to $p(S,t) < 0$ instantaneously for $S \in \left[S^*(T), \frac{rK}{D}\right)$ with $S < K$. The positive source term for $S > \frac{rK}{D}$ indicates that $p(S,t) > 0$ instantaneously on the interval $S \in \left(\frac{rK}{D}, K\right]$, which does not satisfy condition (1.20d). Considering the case $r < D$, this indicates the optimal exercise boundary starts at $S^*(T) = \frac{rK}{D}$. For $r \geq D$ the source term is instantaneously negative for $S < K$ and therefore $S^*(T) = K$. We therefore conclude that the initial condition of the optimal exercise boundary for a put option on a dividend paying underlying is

$$S^*(0) = \min\left(\frac{rK}{D}, K\right). \tag{1.21}$$

1.3.2 Formulation of the American Put Option Problem

The results above lead us to the free boundary problem for the American put option in the presence of dividends

$$S \in (S^*(t), \infty), t \in (0,T) \qquad \frac{\partial P_a}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 P_a}{\partial S^2} + (r-D)S\frac{\partial P_a}{\partial S} - rP_a = 0, \tag{1.22a}$$

subject to

$$\text{as } S \to \infty \qquad P_a(S,t) \to 0, \tag{1.22b}$$

$$\text{at } t = T \qquad P_a(S,T) = \max(K - S, 0), \tag{1.22c}$$

$$S^*(T) = \min(rK/D, K), \tag{1.22d}$$

$$\text{at } S = S^*(t) \qquad P_a(S^*(t),t) = K - S^*(t), \tag{1.22e}$$

$$\left.\frac{\partial P_a}{\partial S}\right|_{S=S^*(t)} = -1. \tag{1.22f}$$

Unlike the European option problem, the presence of the optimal exercise boundary makes the American option problem nonlinear. We note that the Landau transformation $X = \ln(S/S^*(t))$ transforms the problem onto a fixed semi-infinite domain, and would reveal the nonlinearity in the governing equation via the time dependence of the optimal exercise boundary.

Existence and uniqueness results for the problem (1.22a-1.22f) in the absence of dividends ($\beta = \alpha$) are shown by Chen et al. [30], with reference to the texts by Friedman [43, 44]. The optimal exercise boundary is shown to be convex by Ekström [40], a result also obtained in [31]. Similar results in the presence of dividends have not been found to date. Regularity of the optimal exercise boundary for both American calls and puts on non-dividend paying underlyings is demonstrated by Blanchet [18].

In the strictest sense, no general analytic closed-form solution to the American option problem has been found to date, though closed-form solutions have been derived in particular cases discussed later.

Although the standard American problem can be solved via a range of numerical schemes, much effort has been placed in finding analytical approximations which are accurate in certain limits, while easy to implement uniform approximations continue to be popular among practitioners though their accuracy tends to be reduced for intermediate times to expiry. Any improvement in the accuracy of such approximations would be valuable to practitioners as a basis for valuation, while such approximations are also used as seed inputs for other numerical routines and therefore a more accurate starting point would improve their efficiency.

1.3.3 Put Call Parity for American Options

An equivalent American put-call parity result to that found for European options was derived by Bjerksund & Stensland [14] and MacDonald & Schroder [82]

$$P_a(S,t;r,D,\sigma,K) = C_a(K,t;D,r,\sigma,S), \tag{1.23}$$

while the optimal exercise boundary of a put option S_p^* and the corresponding American call option S_c^* are related via

$$S_p^*(t;r,D,\sigma,K)S_c^*(t;D,r,\sigma,K) = K^2. \tag{1.24}$$

From (1.23) & (1.24), results obtained for American put options within this document can be applied to the corresponding American call option.

1.3.4 Relationship to Other Free Boundary Problems

As discussed by Dewynne et al [38] in relation to the American call option, the introduction of the non-dimensional interest rate α, cost-of-carry β and time τ

$$\alpha = \frac{2r}{\sigma^2}, \qquad \beta = \frac{2(r-D)}{\sigma^2}, \qquad \tau = \frac{\sigma^2(T-t)}{2}, \tag{1.25}$$

together with

$$\begin{aligned} &\eta = \ln\left(\frac{S}{K}\right) + (\beta-1)\tau, \qquad \eta^*(\tau) = \ln\left(\frac{S^*(t)}{K}\right) + (\beta-1)\tau, \\ &P_a(S,t) = (1 - e^{\eta-(\beta-1)\tau}) + e^{-\alpha\tau}u(\eta,\tau), \end{aligned} \tag{1.26}$$

transforms the problem (1.22a-1.22f) into

$$\bar{\eta} \in (\eta^*,\infty), \tau \in (0,T) \qquad \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \eta^2} + e^{\alpha\tau}\left[(\alpha-\beta)e^{-(\beta-1)\tau}e^{\eta} - \alpha\right], \tag{1.27a}$$

subject to

$$\text{as } \eta \to \infty \qquad u(\eta,\tau) \sim e^{\alpha\tau}\left[e^{-(\beta-1)\tau}e^{\eta}-1\right], \tag{1.27b}$$

$$\text{at } \tau = 0 \qquad u(\eta,0) = \max\left(e^{\eta}-1,0\right), \tag{1.27c}$$

$$\eta^*(0) = \min\left(\ln\left(\alpha/(\alpha-\beta)\right),0\right), \tag{1.27d}$$

$$\text{at } \eta = \eta^*(\tau) \qquad u\left(\eta^*,\tau\right) = 0, \tag{1.27e}$$

$$\left.\frac{\partial u}{\partial \eta}\right|_{\eta^*} = 0. \tag{1.27f}$$

which is analogous to the problem for oxygen diffusion in an absorbing medium developed by Crank and Gupta [35], but with a time and space-varying oxygen source.

Ockendon et al. [87] show the oxygen consumption problem can be related to the most widely studied parabolic free boundary problem, the Stefan problem for melting or solidification of a material by heat conduction, by defining it as the time derivative of the oxygen consumption problem $v(\eta,\tau) = \frac{\partial u}{\partial \tau}$. Howison [59] discusses the move from the Stefan problem to the oxygen consumption problem using the Baiocchi transformed Stefan problem $u(\eta,\tau) = \int_{\eta^{*-1}(\eta)}^{\tau} v(\eta,s)$ where $\eta^{*-1}(\eta)$ is the inverse of $\eta^*(\tau)$.

The classification of the Stefan problem within a class of free boundary problems satisfying the heat equation with the boundary condition $v\left(\eta^*,\tau\right) = 0$, but with a general dynamic interface equation can be found in the book by Galaktionov [45]. This class of equations contains the Florin equation which has the first spatial derivative condition $\left.\frac{\partial v}{\partial \eta}\right|_{\eta^*} = 1$ but, like the Stefan problem, the derivative is not smooth at the interface which is a requirement of the American option problem. References for both the Stefan and Florin problems can be found in [45].

1.4 Summary of Previous Relevant Work

We classify previous work relevant to this thesis into five broad categories: closed form solutions available in specific cases, uniformly valid approximations, work based around an integral formulation, asymptotic analysis, the identification of bounds on the option price or boundary and schemes developed for the numerical solution of the problem.

1.4.1 Closed-Form Solutions

To date, no closed-form analytical solution has been found to the general problem. However, solutions have been proposed which are valid in certain situations.

Merton [84] demonstrated that early exercise is never optimal for an American call option in the absence of dividends and hence the price must be the same as for the corresponding European call option. In the same paper, a closed-form solution for the value of a perpetual American put was derived on the basis that the problem is the steady state solution to (1.22a-1.22f).

Closed-form solutions have also been derived in the case of an underlying which pays discrete dividends prior to expiry. This work is usually known as the Roll-Geske-Whaley model. The origin of the work is the observation by Roll [93] that the holder of an American call option on an underlying paying a single discrete dividend faces a choice at an instant before the stock goes ex-dividend; exercise, in which case we receive the underlying and the dividend at the expense of the strike; hold, in which case we effectively have the equivalent European option since it is never optimal to exercise a call in the absence of dividends [84].

The important observation by Roll is that the outcomes of this decision can be replicated via the use of vanilla European options and a European compound option which is an option on an underlying, which is itself an option. By the usual no arbitrage arguments, this portfolio then prices the American option. Since the compound option was priced in closed-form by Geske [48], a price for the American option on an underlying paying a single dividend may be derived. In a subsequent note, Geske [47] observes that a simpler replicating portfolio exists by considering only the ex-dividend portfolios at t with the critical value of the stock price separating the two regions. Essentially the holder then has a chooser option with expiry at t with one payoff consisting of the stock plus a cash amount, and the other is an option itself (hence the compound characteristic). The same approach employed in [48] is used to form, then evaluate, the resulting risk neutral expectation. Geske shows how this approach may be extended to underlyings with more than one dividend at the expense of calculating high dimensional multivariate normal distributions.

Whaley [101] identifies discrepancies in the arguments of both the previous authors and provides corrected formulae. He points out that the options Roll uses in his replicating portfolio are not a unique solution and in fact, the strike price for one of the options is misspecified leading to incorrect cash flows at the ex-dividend date. He further identifies that Geske has an incorrect correlation coefficient in his formula. Correction of both these errors leads to an equivalent valuation formula.

1.4.2 Uniformly Valid Approximations

Analyic approximations to the PDE have been investigated by a number of authors [79, 10, 9, 4, 69], who have produced approximations for the price and the optimal exercise boundary which are quick to determine, but which are only typically accurate in certain limits. A summary of this work is included in the review paper by Barone-Adesi [8].

MacMillan [79] was the first to approach this problem for American put options on non-dividend paying assets. The option price is decomposed into a European option plus a premium, representing the optimal exercise feature, both of which satisfy the Black-Scholes PDE subject to different boundary conditions. Through an appropriate time transformation, MacMillan's formulation contains a term which is small at either small or large times to expiry. Neglecting this term leads to simple free boundary ODE with a closed-form solution. The free boundary is obtained as the solution to a transcendental equation which can be simply solved via bisection. Barone-Adesi & Whaley [10] use the same approach and extend the model to cover both puts and calls on dividend paying underlyings.

Barone-Adesi & Elliot [9] and later Allegretto et al. [4] assume the premium has the form derived in [10, 79] and attempt to solve the PDE exactly by choice of the parameters and through the transcendental equation for the optimal exercise boundary. In similarity with the approach of MacMillan, finding a tractable solution requires neglecting a term which is small near the boundary and/or far from expiry. The resulting expression for the boundary is more accurate than in [79], but the expression for the price deteriorates rapidly in regions away from the boundary. Allegretto et al. also note that the form for the boundary is not monotone far from expiry which violates no-arbitrage and introduce an ad-hoc relaxation constant to adjust for this behaviour.

Ju & Zhong [69] extend the approach in [10, 79] by adding a correction term into the form for the exercise premium which leads to more accurate prices, particularly for intermediate times to expiry. However the approach does not include a corresponding correction term in the calculation of the boundary and requires some apparently ad-hoc assumptions to produce a tractable form for the correction term.

Aitsahlia & Lai [3] apply the same approach adopted in [10, 79] to the American barrier option problem, and the resulting approximation is used as a seed input for another numerical routine. Other expressions have been derived which represent solutions to the full problem if an infinite number of terms can be determined, but in practice require the numerical computation of a truncated series.

Geske & Johnson [49] consider the American put option problem in the absence of dividends as a discrete series of exercise decisions, or a Bermudan option, which converges to the continuous solution as the time between decisions goes to zero and the series becomes infinite. Geske's technique [48] is used to evaluate the series of decisions working back from expiry using the discounted expected value under the risk-neutral measure. The convergence of the series is improved using Richardson extrapolation on a sequence of Bermudan options with an increasing number of exercise dates. An approach to allow for a dividend paying underlying is suggested based on interpolating between the price in the absence of dividends and the price at the critical dividend value above which exercise is not optimal. A more efficient scheme for evaluating the series is proposed by Bunch & Johnson [27] while the model is extended to include stochastic interest rates by Ho et al. [58].

More recently, Zhu [106] derives a series solution to the American put option problem in the absence of dividends based on homotopy analysis. As with the Geske & Johnson approach, exact evaluation of the resulting expression requires the sum of an infinite series. Zhu claims that the series is convergent after 30 terms if the European option price is taken as a starting point but it is not clear to us why Zhu's solution is any more closed-form than Geske and Johnson's.

1.4.3 Integral Formulation

In a further thread of work, Kim [72] formulates the American option problem as the limiting case of a series of Bermudan options in the same vein as Geske & Johnson [49]. Rather than the approach taken in [49], Kim forms an integral equation by taking the expected value and working backwards recursively. The resulting expression decomposes the American put option price into the equivalent European option plus the American put option premium

$$\begin{aligned} P_a(S,t) = P_e(S,t) + \frac{1}{2}\int_t^T \Bigg[& rKe^{-r(T-s)}\mathrm{erfc}\left(\frac{d_2(S,S^*(s),s)}{\sqrt{2}}\right), \\ & - DSe^{-D(T-s)}\mathrm{erfc}\left(\frac{d_1(S,S^*(s),s)}{\sqrt{2}}\right)\Bigg] ds, \end{aligned} \tag{1.28a}$$

where

$$d_1(S,S^*(s),s) = \frac{\ln\left(\frac{S}{S^*(s)}\right) + \left((r-D) + \frac{\sigma^2}{2}\right)(T-s)}{\sigma\sqrt{T-s}}, \tag{1.28b}$$

$$d_2(S,S^*(s),s) = \frac{\ln\left(\frac{S}{S^*(s)}\right) + \left((r-D) - \frac{\sigma^2}{2}\right)(T-s)}{\sigma\sqrt{T-s}} \tag{1.28c}$$

and the boundary $S^*(s)$ is the solution to

$$K - S^*(s) = P_e(S^*(s), s) + \frac{1}{2}\int_s^T \left[rKe^{-r(T-\xi)}\mathrm{erfc}\left(\frac{d_2(S, S^*(\xi), \xi)}{\sqrt{2}}\right), \right.$$
$$\left. - DSe^{-D(T-\xi)}\mathrm{erfc}\left(\frac{d_1(S, S^*(\xi), \xi)}{\sqrt{2}}\right)\right] d\xi. \qquad (1.28\text{d})$$

Uniqueness and regularity results for the optimal exercise boundary $S^*(t)$ given in Kim's integral equation (1.28d) are derived by Peskir [89].

The same integral formulation is also derived by Jacka [65] using an optimal stopping problem, and Carr et al. [28] in the absence of dividends by forming an integral expression for the discounted price process for an American option and taking expectations under the risk-neutral measure. The inclusion of dividends is trivial and results in identical expressions to (1.28a) and (1.28d). Carr et al. also show that the option price can be decomposed into its intrinsic value plus a time value component.

An advantage of the integral formulation is that, with a known boundary form or approximation, evaluation of the option price becomes a simple quadrature exercise. If the boundary is not known, it can be determined at discrete points by solving (1.28d) which is a nonlinear Volterra equation of the second kind and can be solved recursively. However, as Kim mentions, this procedure is not necessarily more efficient than finite difference methods.

A number of attempts have been made to optimise the solution of the integral decomposition. Huang et al. [105] use Richardson extrapolation in the same vein as [49] to approximate the price in terms of the equivalent Bermudan options. The advantage computationally is that the boundary only needs to be determined at a discrete set of points determined by the exercise dates of the composite Bermudan options. Further, the scheme avoids the use of time-intensive multivariate normal integrals which hamper the approach of Geske & Johnson. The disadvantage of this approach is the underlying assumption that boundary is piecewise linear on each interval. Ju [68] assumes a multipiece exponential form for the optimal exercise boundary which allows the integral equations for both the boundary and the option price to be solved in closed-form. The resulting unknown coefficients specifying the boundary on each subinterval are determined by application of the value matching and high contact conditions and the use of multidimensional Newton-Raphson routine. Richardson extrapolation is used to approximate the American option price based on options with different numbers of sub-intervals. Ibáñez [64] summarises the work in this area, highlighting the strengths and weaknesses of the various approaches. The non-monotonicity of the approximate option value with increasing terms in the Richardson extrapolation is demonstrated for typical parameters which impacts negatively on the work of Huang et al. [105]. By discounting the optimal exercise premium, Ibáñez forms a monotonic series which is more amenable to Richardson extrapolation and allows more efficient determination of the option price and the boundary.

1.4.4 Asymptotic Analysis

The asymptotic behaviour of the optimal exercise boundary near expiry has been investigated using a variety of methods and incorporating both the PDE and integral equation formulations, with the type of behaviour dependent on the relationship between the interest rate and the dividend yield.

Dewynne et al. [38] identify that the boundary for a call option on a dividend paying underlying with $r > D$ exhibits a jump as we move away from expiry and has parabolic leading order behaviour thereafter. Alobaidi & Mallier [5] use the same approach as in [38] but include higher order terms in their asymptotic expansion. Knessl [74] derives an integral equation for the optimal exercise boundary, using a Laplace transform, and investigates its small-time asymptotic properties for both $r > D$, where the behaviour is parabolic, and $r \leq D$, where the behaviour is parabolic-logarithmic.

Barles at al [7] construct super- and sub-solutions for the boundary of an American put option on a non-dividend paying underlying and show that the two bounds tend to the same asymptotic limit near expiry. A number of approaches are used to derive an integral equation for the optimal exercise boundary of the American put option. Kuske & Keller [75], Goodman & Ostrov [51] and Chen et al. [29] use Green's functions, Stamicar et al. [97] use a Fourier transform, and Knessl [73] uses a Laplace transform. All of these approaches produce the same parabolic-logarithmic small-time behaviour as in [7] up to a constant term. Kuske & Keller's initial paper contains an error which is corrected in a later paper [41] and brings their result into line with other authors. Chen et al. derive higher order terms in the expansion, but these are only accurate for very small times. As in his paper on call options [74], Knessl also looks at the behaviour in a range of parameter limits.

Evans et al. [41] are the first to derive the small-time asymptotic behaviour of the American put option for both $r \geq D$ and $r < D$, using integral equations and via the use of matched asymptotic expansions. Lamberton & Villeneuve [76] independently derive the same behaviour for $r = D$ and, up to a multiplicative constant, for $r > D$. Alobaidi & Maillier [6] assume a logarithmic series expansion to derive higher order behaviour for $r \geq D$. The expression for $r > D$ shows the same parabolic-logarithmic behaviour as in [41], while the behaviour for $r = D$ is expressed in terms of the Lambert W function.

In a separate paper, Mallier [80] prices American options using Monte Carlo simulation with an exercise strategy given by the boundary approximations discussed above. This provides a measure of the sensitivity of the price to errors in the boundary.

1.4.5 Option Price Bounds

An alternative approach to that of finding an approximation for either the option price or the exercise boundary, is to look for bounds on the price. The first attempt to price an American option through the use of option bounds was performed for a put option, in the absence of dividends, by Johnson [66]. Johnson uses a result from Margrabe's [81] work on exchange options, which suggests an American option with a strike replaced by the underlying process will never be exercised before expiry. This leads to an equivalence between the price of an American option with a strike which grows at the risk-free rate and the corresponding European option with the same strike. A weighted average price is formed based on these bounds, with the form for the weighting function determined by regression analysis on option price data from Parkinson [88]. Blomeyer [19] modifies Johnson's work to include underlyings which pay a single discrete and known dividend before expiry.

Broadie & Detemple [24] use a constant exercise boundary strategy to derive bounds on both the price and on the optimal exercise boundary of American options on dividend paying underlyings. A lower price bound is determined by optimising the option price for all constant exercise boundary strategies; since constant exercise boundary strategies are suboptimal, the true price must be greater than any price obtained using such a strategy. The lower bound on the optimal exercise boundary is found as the limiting case of a scheme formed by iterating the lower price bound routine. The latter is used in conjunction with the integral formulation developed by Kim [72] and Carr et al. [28] to form an upper bound on the option price.

Although the work in [24] refers to previous work on capped calls [23, 20], the constant exercise boundary is more easily understood in the framework of barrier options: since the cap provides the maximum payoff, exercise is always optimal at the cap and the option behaves like an up-and-out barrier option with a rebate equal to the payoff at the cap. The valuation of barrier options due to Rubenstein & Renier [94] and is correctly identified as the solution to the low dividend problem for capped calls in [23].

Bjerksund & Stensland [15] independently consider the same constant exercise boundary approach to the lower bound as Broadie and Detemple, though without reference to capped options. The result can be shown to be identical to those given in [94] and [24]. In setting the level of the constant boundary, two approaches are adopted. The first considers an approximate form based on the closed forms for the perpetual boundary and the boundary at expiry. The second considers the same optimisation problem as in [24].

In a subsequent paper [15], the same authors consider an extension to this approach in the form of a stepped boundary over two time intervals. The boundary levels are based on a modified form from the earlier paper while the time intervals are chosen based on a "golden rule". The method is shown to increase the accuracy of the lower bound.

1.4.6 Numerical Schemes

Many numerical schemes have been proposed for the American option problem. Some of these schemes, such as the formulation of analytical approximations and the methods for solution of the integral formulation of the American option problem, have been discussed in the relevant sections. What we discuss here are the most popular remaining schemes which we broadly classify into (i) binomial methods, (ii) finite difference methods and (iii) Monte Carlo simulation.

Binomial Methods

The use of binomial models in option pricing is attributable to Cox, Ross & Rubenstein [34] (or the *CRR* model) who model GBM as a discrete random walk with the parameters chosen such that its distributional properties converge to that of the continuous model as the step size tends to zero. When applied to the American option and its early exercise feature, the CRR model becomes a dynamic programming problem with the price at each step determined as the greater of the backward induced value and the intrinsic value. Many extensions have been proposed to the CRR model, particularly in relation to the use of different lattices, and the reader is directed towards [25] for a discussion.

The ease of implementation means tree-based numerical routines are often developed as an initial numerical approach to new valuation problems. The intuitive feel of the CRR model also means it is prevalent as a teaching aid throughout the literature, with almost every option pricing text containing a sizeable reference to the material [67, 62, 63, 103]. Relevant to American options however, a weakness arises when attempting to pin down the location of the optimal exercise boundary, which typically occurs between lattice points.

Finite Difference Methods

Finite difference methods have been similarly popular among practitioners and academics due to the wide range of literature available from numerical studies of other physical problems. The earliest use of finite difference methods in solving the American option problem is attributed to

Brennan & Schwartz [21, 22], who use an implicit time-stepping scheme to solve the variational inequality formulation of the American option problem, with the intrinsic value condition (1.16) imposed at each grid point in the solution and the optimal exercise boundary identified as the first point at which the solution meets this lower bound.

Dewynne at al. [38] pose the American call option as a linear complementarity problem and solve the system iteratively using a projected successive over relaxation (or *PSOR*) algorithm. This approach, typically employing a Crank-Nicolson [36] stepping scheme, has become highly popular as a first approach to modelling the American option problem using finite differences. For a discussion of this approach, the reader is directed to the book by Duffy [39].

To avoid the iterative overhead required by the PSOR method, Wu & Kwok [104] use a front-fixing transformation due to Landau [77]

$$\hat{X} = \ln\left(\frac{S}{S^*(t)}\right), \tag{1.29}$$

for the American put option problem in the absence of dividends, which places the problem on a fixed domain at the expense of a nonlinear term, derived from the rate of movement of the boundary, appearing in the governing equation

$$\hat{X} \in (0,\infty), t \in (0,T) \qquad \frac{\partial P_a}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 P_a}{\partial \hat{X}^2} + \left[r - \frac{\sigma^2}{2} + \frac{1}{S^*}\frac{dS^*}{dt}\right]\frac{\partial P_a}{\partial \hat{X}} - rP_a = 0, \tag{1.30a}$$

subject to

$$\text{as } \hat{X} \to \infty \qquad P_a(\hat{X},t) \to 0, \tag{1.30b}$$

$$\text{at } t = T \qquad P_a(\hat{X},T) = 0, \tag{1.30c}$$

$$S^*(T) = K, \tag{1.30d}$$

$$\text{at } \hat{X} = 0 \qquad P_a(0,t) = K - S^*(t), \tag{1.30e}$$

$$\left.\frac{\partial P_a}{\partial \hat{X}}\right|_{\hat{X}=0} = -S^*(t). \tag{1.30f}$$

As an aside, we note that the Landau transformation is used by authors in other areas of work on the American option problem [69, 106].

A numerical scheme seemingly less popular in the available literature is the method of lines (or *MOL*) approach due to Meyer et al. [85]. This is a well tested technique which has been applied to a number of problems and has the advantage of only employing temporal discretisation of the PDE, resulting in an ODE at each time-step which can be solved using a range of available numerical routines such as those contained in [92].

Monte Carlo Simulation

Though we do not consider the numerical issues relating to the expectation pricing of American options, which has traditionally been viewed as a less suitable way of tackling the problem than grid-based methods, we mention the popular least-squares approach of Longstaff and Schwartz [78] and refer the reader to the comprehensive text by Glasserman [50] for a detailed discussion and further references.

1.5 Parameter Considerations

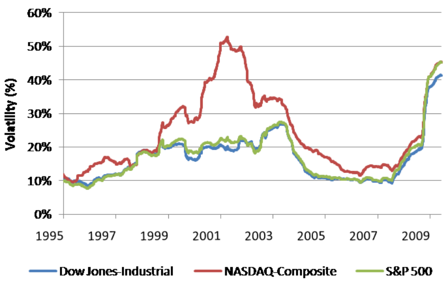
Before selection of a numerical scheme, we first present a brief discussion of the considerations used in selecting the parameters used in this thesis. This is intended to ensure the work is relevant to the prevailing market conditions, but takes into account some historic variations around this position to produce a reasonable parameter set. Since we have restricted ourselves to the Black-Scholes-Merton model, under which the underlying follows GBM, we use data from global stock market indices for the underlying parameters. GBM is typically seen as a reasonable model in the case of indices and is often assumed in broad texts on the subject such as Hull [62].

For the estimation of the volatility and risk-free rate we have looked at historical data since the start of 1995. This takes in a short period prior to the dot-com boom through to the current global financial crisis. For the estimation of the dividend yield we have used data going back to the turn of the last century as yields have remained below the long-run average since 1995, though they have risen sharply with the recent collapse in asset prices.

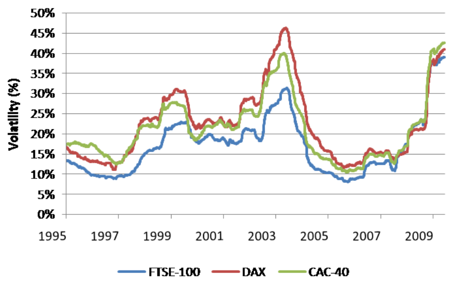
Volatility

In selecting a volatility parameter, we have looked at a 250-day moving average of annualised daily stock market returns for the major US, European and Asian indices. Figure 1-2 shows that the volatility of Asian index returns are typically higher than that of Continental Europe which are in turn, typically higher than those of the UK and US. The moving average volatility has broadly ranged from 10-50% over the period, though this is weighted towards the lower end of the range with a simple average of the volatility of returns over the period giving figures in the range 20-25%. Against this, the volatility has been rising since the end of 2007, with the moving average currently in the range 40-50%.

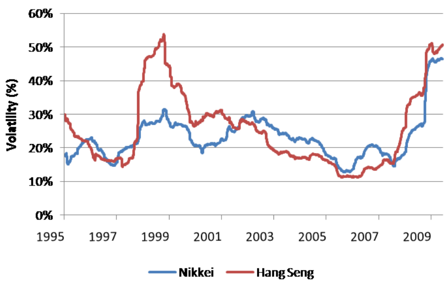
As we do not wish to make a prediction as to how long the current global financial crisis may last, and therefore whether the current level of market volatility will persist, we feel it is prudent to take a value slightly above the average observed since 1995, and we will therefore use a volatility of 30%.



(a) US Indices



(b) European Indices



(c) Asian Indices

Figure 1-2: Evolution of the volatility of major global stock indices since 1995.

Risk-Free Rate

When considering which of the wide range of available interest rates is the most relevant as a proxy for the risk-free rate in the Black-Scholes-Merton model, we remind ourselves that the derivation of the pricing equation assumes the formation of a risk-free portfolio via continuous hedging, and therefore a continuously changing cash position. We therefore require the risk-free rate to be a short-term rate of borrowing and have a high degree of liquidity.

On this basis, we have selected the 3-Month LIBOR rate as the risk-free proxy as this is typically considered a short-term maturity, while it offers more liquidity than alternative short-term rates such as the 3-Month US T-Bill. We highlight that this may not be the most applicable rate for all writers and holders of options, particularly where access to cheaper collateralised short-term lending via overnight repo rates are available.

Figure 1-3 shows 3-Month LIBOR to be at its lowest level over our period of interest, which is a result of the attempt by central banks to stimulate interbank and retail lending which collapsed as the credit crunch worsened through the third quarter of 2008. The simple average over the period provides a figure of 5.5%, but as with the choice of volatility parameter we take a steer from the prevailing level and therefore choose a risk-free rate slightly below the average at 5.0%.

Dividend Yield

Quarterly dividend yield for US stocks going back to before the start of the 1900s has been aggregated by Schiller [96]. The evolution of the dividend yield is shown in Figure 1-4 and, as with other financial metrics, shows a wide degree of variation over time. Typically high dividend yields are associated with times of economic difficulty such as the great depression of the 1930s. This occurs due to declining asset prices, while companies will typically seek to maintain their divided policy where possible.

Since the early 1990s, dividend yields have been at historically low levels with the dot-com bubble driving yields well below 2% as investors increasingly looked to share price appreciation as a source of capital growth rather than income received via dividends. The existing global financial crisis has seen a dramatic deflation in asset prices resulting in a return of yields to the level seen at the start of the 1990s, though dividend yield is a forward looking measure and the impact of the crisis on dividend policy is likely to have some influence as we move forward.

Based on the period studied in Figure 1-4, we consider a reasonable range for the dividend yield parameter to be 2-8%.

1.6 Numerical Scheme Considerations

From the approaches investigated, our preferred scheme for determining benchmark numerics is the MOL [85]. This approach satisfies the requirements of providing a smooth boundary with the capability to run out to very large times to expiry ($T = 200$) in reasonable computation time. Finally we are mindful of the requirement to be able to easily extend the scheme to model barrier options if required. In comparison, we found the finite difference approach employing PSOR [38] did not provide a smooth enough boundary unless the time discretisation was reduced to the point where large expiry runs took a prohibitively large amount of time. Though interpolation schemes were investigated, none were reliably able to provide a suitably smooth boundary.

Our MOL scheme is implemented in Microsoft Visual C++.NET 2003 using a second-order backward time approximation and the solution to the ODE at each step is performed using a fourth order Runge-Kutta routine [92] with a variable spatial mesh.

The extension to allow investigation of the American barrier option problem simply involves the replacement of the condition at the truncated boundary in the standard problem, with the barrier condition and any assumed rebate as discussed in Chapter 4.

1.6.1 Asymptotic Behaviour of Benchmark Numerics

A comparison of the asymptotic behaviour of our MOL boundary with the small-time asymptotics of the optimal exercise boundary derived by Evans et al. [41], together with the large-time approach to the perpetual boundary, is shown in Figure 1-5. We note that the boundary in these numerics has been normalised by the strike, while the time transformation $h(t) = 1 - e^{-r(T-t)}$ has also been used.

The small-time behaviour of the optimal exercise boundary is observed to be consistent with the relevant asymptotic results, while for large times, the boundary approaches the steady-state solution, though we mention that a large number of time-steps is required to avoid the propagation of errors from small times significantly affecting the large-time behaviour.

All of the numerics in this thesis were performed on a laptop with a 1.66GHz Intel® Core2™ CPU and 1GB of RAM. The generation of the small-time boundary in Figures (1.5(a),1.5(c),1.5(e)) took 200 minutes each, while the generation of the large-time boundary in Figures (1.5(b),1.5(d),1.5(f)) took 18 hours each. Corresponding results for the American option MOL scheme adapted for the inclusion of a barrier are provided in Chapter 4.

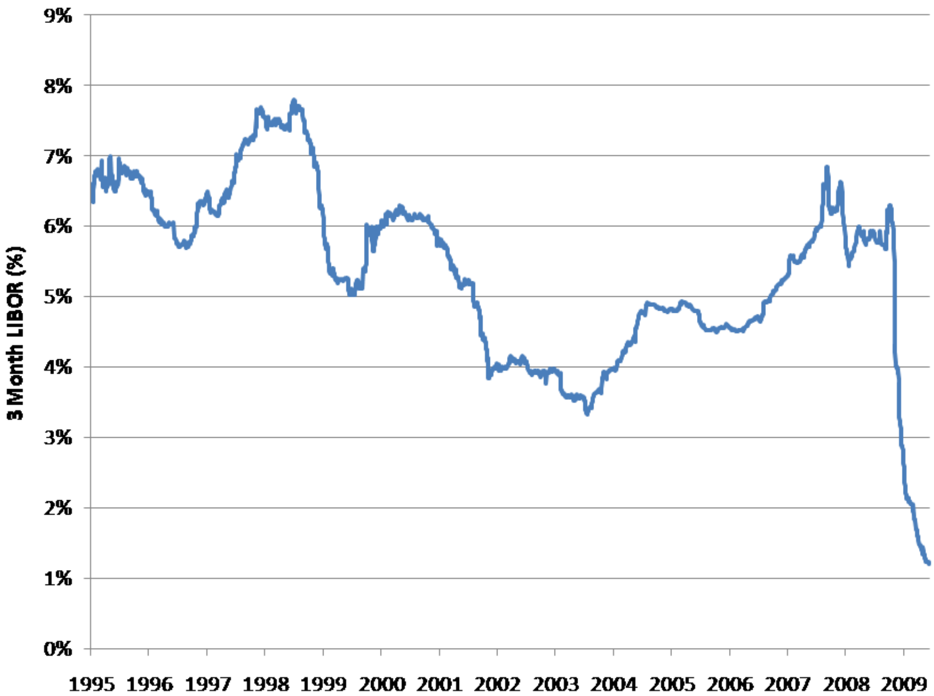


Figure 1-3: Evolution of the 3-Month LIBOR rate since 1995.

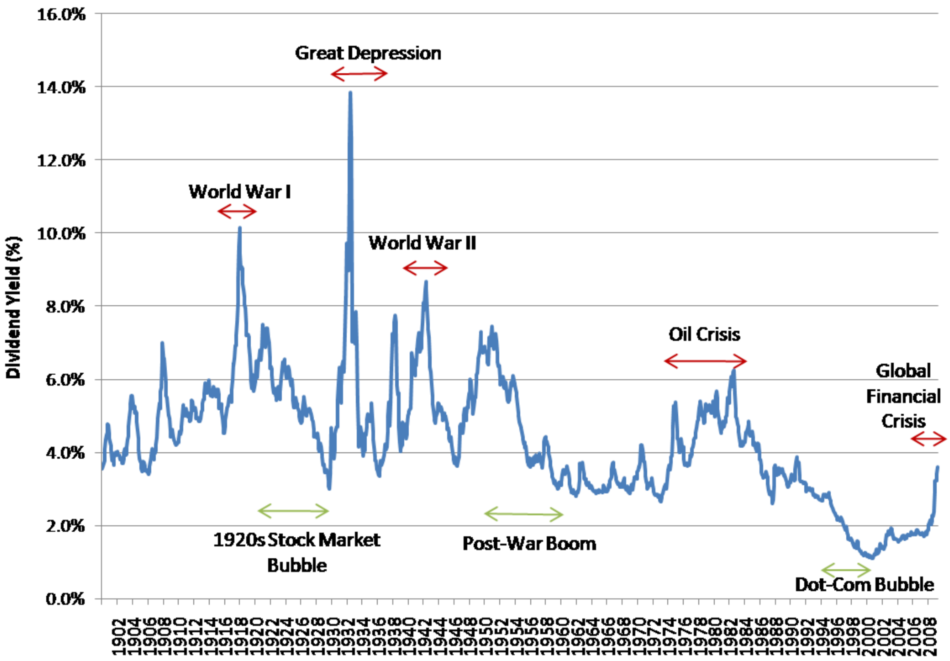
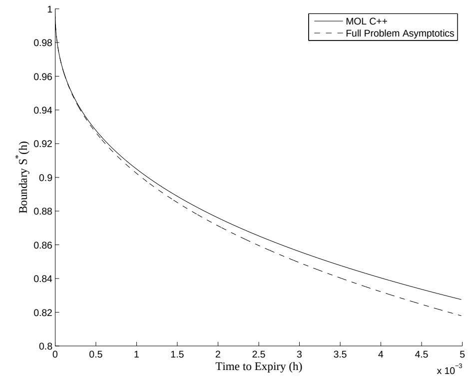
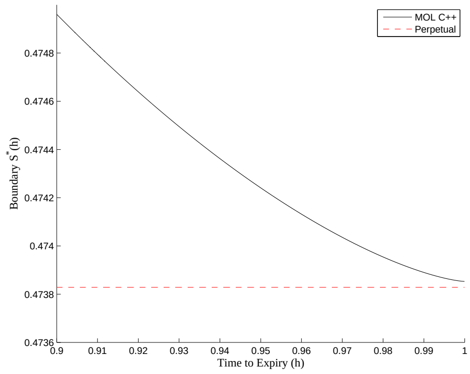


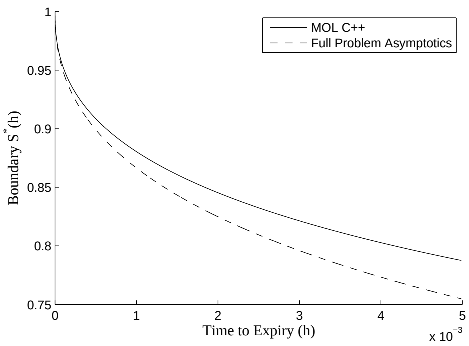
Figure 1-4: Evolution of the dividend yield of US stocks since 1900 as reported by Schiller [96]. The magnitude of the dividend yield is typically inversely correlated with the strength of the global economy.



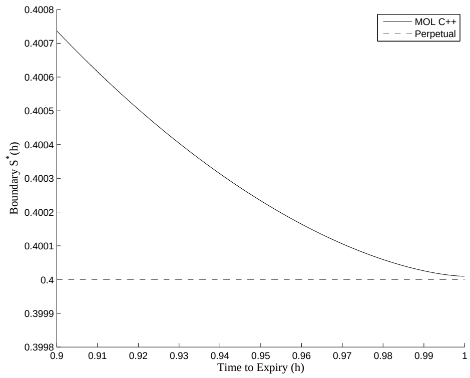
(a) Small-time asymptotics ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$)



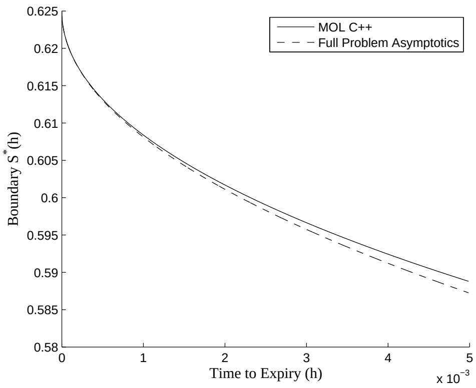
(b) Perpetual approach ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$)



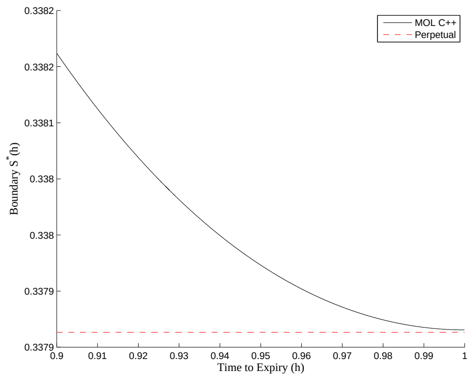
(c) Small-time asymptotics ($\alpha = \frac{10}{9}$, $\beta = 0$)



(d) Perpetual approach ($\alpha = \frac{10}{9}$, $\beta = 0$)



(e) Small-time asymptotics ($\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$)



(f) Perpetual approach ($\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$)

Figure 1-5: A comparison of the benchmark MOL boundary with the asymptotic results from Evans et al. [41] for the small-time plots, and with the steady state boundary for the large-time plots. For the small-time numerics 20000 time-steps and 25000 spatial points were used with 15000 space-steps on the interval $[\bar{S}^*_\infty, 1]$, and 10000 on the interval $[1, 5]$. For the large-time numerics 200000 time-steps and 25000 spatial points were used with 15000 space-steps on the interval $[\bar{S}^\infty, 1]$, and 10000 on the interval $[1, 50]$

Chapter 2

The European Option Problem

In preparation for the investigation of the asymptotic properties of the American option problem, we first consider the case of the European put option problem under the Black-Scholes-Merton model (1.10a-1.10d). This will illustrate the main principles of the technique of matched asymptotic expansions [12, 57] on a well-known problem with a closed-form solution. We look first at the small-time asymptotics of the European put option problem and subsequently at the problem when the volatility is small relative to the risk-free rate and the cost-of-carry. The European option problem has recently attracted interest in terms of the application of asymptotic techniques, with Howison [60] looking at the small-time behaviour of the European call option in the absence of dividends, while Firth et al. [42] and Widdicks et al. [102] look at the small-volatility behaviour of the European call option with and without dividends respectively. Non-dimensionalisation of the problem (1.10a-1.10d) through the introduction of the scalings

$$\alpha = \frac{2r}{\sigma^2}, \qquad \beta = \frac{2(r-D)}{\sigma^2}, \qquad \tau = \frac{\sigma^2(T-t)}{2}, \tag{2.1}$$
$$S = K\bar{S}, \qquad P_e(S,t) = \bar{P}_e(\bar{S},\tau),$$

gives

$$\bar{S} \in (0,\infty), \tau \in (0,T) \qquad \frac{\partial \bar{P}_e}{\partial \tau} = \bar{S}^2 \frac{\partial^2 \bar{P}_e}{\partial \bar{S}^2} + \beta \bar{S} \frac{\partial \bar{P}_e}{\partial \bar{S}} - \alpha \bar{P}_e, \tag{2.2a}$$

subject to the boundary conditions

$$\text{as } \bar{S} \to 0 \qquad \bar{P}_e(\bar{S}, \tau) \to e^{-\alpha\tau}, \tag{2.2b}$$

$$\text{as } \bar{S} \to \infty \qquad \bar{P}_e(\bar{S}, \tau) \to 0, \tag{2.2c}$$

$$\text{at } \tau = 0 \qquad \bar{P}_e(\bar{S}, 0) = \max\big(1 - \bar{S}, 0\big). \tag{2.2d}$$

The problem (2.2a-2.2d) has solution

$$\bar{P}_e(\bar{S}, \tau) = \frac{e^{-\alpha\tau}}{2}\left[\operatorname{erfc}\left(\frac{\bar{d}_2}{\sqrt{2}}\right) - e^{\beta\tau}\bar{S}\operatorname{erfc}\left(\frac{\bar{d}_1}{\sqrt{2}}\right)\right], \tag{2.3}$$

where

$$\frac{\bar{d}_1}{\sqrt{2}} = \frac{1}{2}\left[\frac{\ln(\bar{S})}{\tau^{\frac{1}{2}}} + (\beta + 1)\tau^{\frac{1}{2}}\right], \tag{2.4a}$$

$$\frac{\bar{d}_2}{\sqrt{2}} = \frac{1}{2}\left[\frac{\ln(\bar{S})}{\tau^{\frac{1}{2}}} + (\beta - 1)\tau^{\frac{1}{2}}\right]. \tag{2.4b}$$

We note that we may also choose to transform the governing equation (2.2a) into one with constant coefficients through the use of the transformation $\bar{X} = \ln(\bar{S})$.

2.1 Small-Time Behaviour

The small-time European put option problem presents a simple situation with which to begin our asymptotic analysis. We introduce the time scaling $\tau = \epsilon^2 \hat{T}$ into (2.2a-2.2d), where ϵ is a small artificial parameter ($0 < \epsilon \ll 1$) and $\hat{T} = O(1)$. This gives the small-time problem

$$\bar{S} \in (0, \infty), \hat{T} \in (0, \infty) \qquad \frac{1}{\epsilon^2}\frac{\partial \bar{P}_e}{\partial \hat{T}} = \bar{S}^2 \frac{\partial^2 \bar{P}_e}{\partial \bar{S}^2} + \beta \bar{S} \frac{\partial \bar{P}_e}{\partial \bar{S}} - \alpha \bar{P}_e, \tag{2.5a}$$

subject to

$$\text{as } \bar{S} \to 0 \qquad \bar{P}_e(\bar{S}, \hat{T}) \to e^{-\epsilon^2 \alpha \hat{T}}, \tag{2.5b}$$

$$\text{as } \bar{S} \to \infty \qquad \bar{P}_e(\bar{S}, \hat{T}) \to 0, \tag{2.5c}$$

$$\text{at } \hat{T} = 0 \qquad \bar{P}_e(\bar{S}, 0) = (1 - \bar{S})^+. \tag{2.5d}$$

The presence of the small scaling factor leads to a global problem with reduced, easier to solve sub-problems in certain regions. For instance, we can observe immediately that for $\bar{S} = O(1)$ the dominant term is the time derivative but that for some small region about some judiciously chosen point in $\bar{S}$, we can recover the diffusion term. Intuition tells us that this is the behaviour we expect for the small-time European option: for small times the majority of the change will occur in a small region about the discontinuity in the option payoff, while we expect little change far from this region. All of this points towards a three-region structure with two *outer* regions for $\bar{S} < 1$ and $\bar{S} > 1$ and an *inner* region for $\bar{S} = 1 + \delta(\epsilon)\hat{S}$, where $\delta(\epsilon)$ is a scaling factor to be determined, and $\hat{S} = O(1)$.

Outer Regions

For $\bar{S} < 1$ we write $\bar{P}_e(\bar{S}, \hat{T}) = \bar{P}_e^{Out1}(\bar{S}, \hat{T})$ and we have the outer 1 problem

$$\bar{S} \in (0,1), \hat{T} \in (0,\infty) \qquad \frac{1}{\epsilon^2}\frac{\partial \bar{P}_e^{Out1}}{\partial \hat{T}} = \bar{S}^2 \frac{\partial^2 \bar{P}_e^{Out1}}{\partial \bar{S}^2} + \beta \bar{S}\frac{\partial \bar{P}_e^{Out1}}{\partial \bar{S}} - \alpha \bar{P}_e^{Out1}, \tag{2.6a}$$

subject to

$$\text{as } \bar{S} \to 0 \qquad \bar{P}_e^{Out1}(\bar{S}, \hat{T}) \to e^{-\alpha\epsilon^2\hat{T}}, \tag{2.6b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{P}_e^{Out1}(\bar{S}, 0) = 1 - \bar{S}. \tag{2.6c}$$

Posing a regular expansion in even powers of ϵ

$$\bar{P}_e^{Out1}(\bar{S}, \hat{T}; \epsilon) = \bar{P}_0^{Out1}(\bar{S}, \hat{T}) + \epsilon^2 \bar{P}_1^{Out1}(\bar{S}, \hat{T}) + \epsilon^4 \bar{P}_2^{Out1}(\bar{S}, \hat{T}) + O(\epsilon^6) \qquad \text{as } \epsilon \to 0, \tag{2.7}$$

gives the following subproblems: for $\bar{P}_0^{Out1}(\bar{S}, \hat{T})$

$$\bar{S} \in (0,1), \hat{T} \in (0,\infty) \qquad \frac{\partial \bar{P}_0^{Out1}}{\partial \hat{T}} = 0, \tag{2.8a}$$

subject to

$$\text{as } \bar{S} \to 0 \qquad \bar{P}_0^{Out1}(\bar{S}, \hat{T}) \to 1, \tag{2.8b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{P}_0^{Out1}(\bar{S}, 0) = 1 - \bar{S}; \tag{2.8c}$$

for $\bar{P}_1^{Out1}(\bar{S}, \hat{T})$

$$\bar{S} \in (0,1), \hat{T} \in (0,\infty) \qquad \frac{\partial \bar{P}_1^{Out1}}{\partial \hat{T}} = \bar{S}^2 \frac{\partial^2 \bar{P}_0^{Out1}}{\partial \bar{S}^2} + \beta \bar{S}\frac{\partial \bar{P}_0^{Out1}}{\partial \bar{S}} - \alpha \bar{P}_0^{Out1}, \tag{2.9a}$$

subject to

$$\text{as } \bar{S} \to 0 \qquad \bar{P}_1^{Out1}(\bar{S},\hat{T}) \sim -\alpha\hat{T}, \tag{2.9b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{P}_1^{Out2}(\bar{S},0) = 0; \tag{2.9c}$$

and for $\bar{P}_2^{Out1}(\bar{S},\hat{T})$

$$\bar{S} \in (0,1), \hat{T} \in (0,\infty) \qquad \frac{\partial \bar{P}_2^{Out1}}{\partial \hat{T}} = \bar{S}^2 \frac{\partial^2 \bar{P}_1^{Out1}}{\partial \bar{S}^2} + \beta\bar{S}\frac{\partial \bar{P}_1^{Out1}}{\partial \bar{S}} - \alpha\bar{P}_1^{Out1}, \tag{2.10a}$$

subject to

$$\text{as } \bar{S} \to 0 \qquad \bar{P}_2^{Out1}(\bar{S},\hat{T}) \sim \frac{1}{2}\alpha^2\hat{T}^2, \tag{2.10b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{P}_2^{Out2}(\bar{S},0) = 0. \tag{2.10c}$$

The outer 1 expansion up to algebraic terms is therefore

$$\bar{P}_e^{Out1}(\bar{S},\hat{T};\epsilon) \sim (1-\bar{S}) + \epsilon^2\hat{T}\left((\alpha-\beta)\bar{S}-\alpha\right) + \frac{\epsilon^4}{2}\hat{T}^2\left(\alpha^2-(\alpha-\beta)^2\bar{S}\right) + O(\epsilon^6) \quad \text{as } \epsilon \to 0, \tag{2.11}$$

though we note that we could construct a simple inductive proof which would demonstrate that the remaining terms of the algebraic expansion generate the solution

$$\bar{P}_e^{Out1}(\bar{S},\hat{T};\epsilon) \sim e^{-\alpha\epsilon^2\hat{T}} - \bar{S}e^{-(\alpha-\beta)\epsilon^2\hat{T}} \quad \text{as } \epsilon \to 0, \tag{2.12}$$

which can be shown to be the trivial lower bound on the European put option price ensured by no-arbitrage.

For $\bar{S} > 1$ we write $\bar{P}_e(\bar{S},\hat{T}) = \bar{P}_e^{Out2}(\bar{S},\hat{T})$ and we have the outer 2 problem

$$\bar{S} \in (1,\infty), \hat{T} \in (0,\infty) \qquad \frac{1}{\epsilon^2}\frac{\partial \bar{P}_e^{Out2}}{\partial \hat{T}} = \bar{S}^2 \frac{\partial^2 \bar{P}_e^{Out2}}{\partial \bar{S}^2} + \beta\bar{S}\frac{\partial \bar{P}_e^{Out2}}{\partial \bar{S}} - \alpha\bar{P}_e^{Out2}, \tag{2.13a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{P}_e^{Out2}(\bar{S},\hat{T}) \to 0, \tag{2.13b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{P}_e^{Out2}(\bar{S},0) = 0. \tag{2.13c}$$

Posing a regular expansion in even powers of ϵ

$$\bar{P}_e^{Out2}(\bar{S},\hat{T};\epsilon) = \bar{P}_0^{Out2}(\bar{S},\hat{T}) + \epsilon^2 \bar{P}_1^{Out2}(\bar{S},\hat{T}) + O(\epsilon^4) \qquad \text{as } \epsilon \to 0, \tag{2.14}$$

gives the following subproblems: for $\bar{P}_0^{Out2}(\bar{S},\hat{T})$

$$\bar{S} \in (1,\infty), \hat{T} \in (0,\infty) \qquad \frac{\partial \bar{P}_0^{Out2}}{\partial \hat{T}} = 0, \tag{2.15a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{P}_0^{Out2}(\bar{S},\hat{T}) \to 0, \tag{2.15b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{P}_0^{Out2}(\bar{S},0) = 0; \tag{2.15c}$$

and for $\bar{P}_1^{Out2}(\bar{S},\hat{T})$

$$\bar{S} \in (1,\infty), \hat{T} \in (0,\infty) \qquad \frac{\partial \bar{P}_1^{Out2}}{\partial \hat{T}} = \bar{S}^2 \frac{\partial^2 \bar{P}_0^{Out2}}{\partial \bar{S}^2} + \beta \bar{S} \frac{\partial \bar{P}_0^{Out2}}{\partial \bar{S}} - \alpha \bar{P}_0^{Out2}, \tag{2.16a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{P}_1^{Out2}(\bar{S},\hat{T}) \to 0, \tag{2.16b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{P}_1^{Out2}(\bar{S},0) = 0. \tag{2.16c}$$

These problems have the trivial solutions $\bar{P}_0^{Out2}(\bar{S},\hat{T}) = \bar{P}_1^{Out2}(\bar{S},\hat{T}) = 0$ with the same true for all algebraic terms giving the outer 2 expansion

$$\bar{P}_e^{Out2}(\bar{S},\hat{T};\epsilon) = 0 \qquad \text{as } \epsilon \to 0, \tag{2.17}$$

to all powers of ϵ. We highlight that the algebraic expansions only capture the trivial lower option price bound in both outer regions. To find the non-trivial perturbative terms to the lower option price bound in the outer 2 region, we use an approach adopted by Addison et al. [2] in relation to the Stefan problem and pose a WKBJ-type expansion [12] of the form

$$\bar{P}_e^{Out2}(\bar{S},\hat{T};\epsilon) = \epsilon^q \bar{A}_0^{Out2}(\bar{S},\hat{T}) \left(1 + \epsilon^2 \bar{A}_1^{Out2}(\bar{S},\hat{T}) + O(\epsilon^4)\right) e^{-\frac{a(\bar{S},\hat{T})}{\epsilon^2}} \qquad \text{as } \epsilon \to 0, \tag{2.18}$$

where the scaling in the exponential follows from the governing equation and the index q will be determined via matching. This leads to the following subproblems: for $a(\bar{S}, \hat{T})$

$$\bar{S} \in (1, \infty), \hat{T} \in (0, \infty) \qquad \frac{\partial a}{\partial \hat{T}} + \bar{S}^2 \left(\frac{\partial a}{\partial \bar{S}} \right)^2 = 0, \tag{2.19a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad a(\bar{S}, \hat{T}) \to \infty, \tag{2.19b}$$

$$\text{as } \hat{T} \to 0 \qquad a(\bar{S}, 0) \to \infty; \tag{2.19c}$$

for $\bar{A}_0^{Out2}(\bar{S}, \hat{T})$

$$\bar{S} \in (1, \infty), \hat{T} \in (0, \infty) \qquad \frac{\partial \bar{A}_0^{Out2}}{\partial \hat{T}} = -2\bar{S}^2 \frac{\partial a}{\partial \bar{S}} \frac{\partial \bar{A}_0^{Out2}}{\partial \bar{S}} - \left(\beta \bar{S} \frac{\partial a}{\partial \bar{S}} + \bar{S}^2 \frac{\partial^2 a}{\partial \bar{S}^2} \right) \bar{A}_0^{Out2}, \tag{2.20a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{A}_0^{Out2}(\bar{S}, \hat{T}) \to 0, \tag{2.20b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{A}_0^{Out2}(\bar{S}, 0) = 0; \tag{2.20c}$$

and for $\bar{A}_1^{Out2}(\bar{S}, \hat{T})$

$$\begin{aligned} \bar{S} \in (1, \infty), \hat{T} \in (0, \infty) \qquad \frac{\partial \bar{A}_1^{Out2}}{\partial \hat{T}} =& -2\bar{S}^2 \frac{\partial a}{\partial \bar{S}} \frac{\partial \bar{A}_1^{Out2}}{\partial \bar{S}} - \left(\beta \bar{S} \frac{\partial a}{\partial \bar{S}} + \bar{S}^2 \frac{\partial^2 a}{\partial \bar{S}^2} \right) \bar{A}_1^{Out2} \\ &+ \bar{S}^2 \frac{\partial^2 \bar{A}_0^{Out2}}{\partial \bar{S}^2} + \beta \bar{S} \frac{\partial \bar{A}_0^{Out2}}{\partial \bar{S}} - \alpha \bar{A}_0^{Out2}, \end{aligned} \tag{2.21a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{A}_1^{Out2}(\bar{S}, \hat{T}) \to 0, \tag{2.21b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{A}_1^{Out2}(\bar{S}, 0) = 0. \tag{2.21c}$$

The problem for $a(\bar{S}, \hat{T})$ is a first order nonlinear PDE with complete integral

$$a(\bar{S}, \hat{T}) = C_1 \ln \bar{S} - C_1^2 \hat{T} + C_2, \tag{2.22}$$

which can be derived using Charpit's method [99, 87] or identified directly in a reference such as [91] where C_1 and C_2 are arbitrary constants. The complete integral (2.22) however fails to satisfy the boundary conditions (2.19b,2.19c). A further solution exists which is not captured by the references above and which is singular in $\hat{T}$ with no arbitrary constants. The form of this solution can be determined by considering the value of a for which the discriminant of the quadratic formed by the arbitrary constant C_1 is zero. Details of singular solutions and their identification can be found in Piaggio [90]. The solution in this case is given by

$$a(\bar{S},\hat{T}) = \frac{\left(\ln(\bar{S})\right)^2}{4\hat{T}} \tag{2.23}$$

and we observe that this solution satisfies both conditions (2.19b) and (2.19c). We mention in passing that we could have reasoned the form of (2.23) from the scaling of the controlling factor. The scaling ϵ^{-2} suggests we look for a solution $a(\bar{S},\hat{T}) = \frac{\tilde{a}(\bar{S})}{\hat{T}}$ which, upon substitution into (2.19a) yields $\tilde{a}(\bar{S}) = \frac{(\ln(\bar{S}))^2}{4}$ after matching with the leading order Inner expression derived later and it is reassuring that we can obtain the same solution using both arguments.

Having obtained the solution for a, we can now find $\bar{A}_0^{Out2}(\bar{S},\hat{T})$ as the solution to the linear first order PDE

$$\bar{S} \in (1,\infty), \hat{T} \in (0,\infty) \qquad \frac{\partial \bar{A}_0^{Out2}}{\partial \hat{T}} = -\frac{\bar{S}}{\hat{T}} \ln(\bar{S}) \frac{\partial \bar{A}_0^{Out2}}{\partial \bar{S}} - \frac{1}{2\hat{T}} \left((\beta-1)\ln(\bar{S})+1\right) \bar{A}_0^{Out2}, \tag{2.24a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{A}_0^{Out2}(\bar{S},\hat{T}) \to 0, \tag{2.24b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{A}_0^{Out2}(\bar{S},0) = 0, \tag{2.24c}$$

which is amenable to the method of characteristics [87], or the solution may be found by direct reference [91] as

$$\bar{A}_0^{Out2}(\bar{S},\hat{T}) = \Phi_0\left(\frac{\hat{T}}{\ln(\bar{S})}\right) \frac{\bar{S}^{\frac{1-\beta}{2}}}{\left(\ln(\bar{S})\right)^{\frac{1}{2}}}, \tag{2.25}$$

for some arbitrary function $\Phi\left(\frac{\hat{T}}{\ln(\bar{S})}\right)$. From (2.23) and (2.25) the asymptotic behaviour of the outer 2 expansion is therefore

$$\bar{P}_e^{Out2}(\bar{S},\hat{T};\epsilon) \sim \Phi_0\left(\frac{\epsilon^2\hat{T}}{\ln(\bar{S})}\right) \frac{\bar{S}^{\frac{1-\beta}{2}}}{\left(\ln(\bar{S})\right)^{\frac{1}{2}}} e^{-\frac{(\ln(\bar{S}))^2}{4\epsilon^2\hat{T}}} \left(1+\epsilon^2\bar{A}_1^{Out2}(\bar{S},\hat{T})+O(\epsilon^4)\right) \qquad \text{as } \epsilon \to 0, \tag{2.26}$$

where we have written $\epsilon^q \Phi_0\left(\frac{\hat{T}}{\ln(\bar{S})}\right) = \Phi_0\left(\frac{\epsilon^2 \hat{T}}{\ln(\bar{S})}\right)$ without loss of generality and satisfying the correct scaling of the problem.

We can use the same approach to identify the non-trivial asymptotic behaviour of outer 1 problem with a WKBJ-type expansion of the form

$$\begin{aligned}\bar{P}_e^{Out1}(\bar{S},\hat{T};\epsilon) =& e^{-\alpha\epsilon^2\hat{T}} - \bar{S}e^{-(\alpha-\beta)\epsilon^2\hat{T}} \\ &+ \epsilon^q \bar{A}_0^{Out1}(\bar{S},\hat{T})\left(1+\epsilon^2\bar{A}_1^{Out1}(\bar{S},\hat{T})+O(\epsilon^4)\right)e^{-\frac{a(\bar{S},\hat{T})}{\epsilon^2}} \qquad \text{as } \epsilon \to 0,\end{aligned} \tag{2.27}$$

which leads to the following subproblems: for $a(\bar{S},\hat{T})$

$$\bar{S} \in (1,\infty), \hat{T} \in (0,\infty) \qquad \frac{\partial a}{\partial \hat{T}} + \bar{S}^2\left(\frac{\partial a}{\partial \bar{S}}\right)^2 = 0, \tag{2.28a}$$

subject to

$$\text{as } \bar{S} \to 0 \qquad a(\bar{S},\hat{T}) \to \infty, \tag{2.28b}$$

$$\text{at } \hat{T} = 0 \qquad a(\bar{S},0) = \infty; \tag{2.28c}$$

for $\bar{A}_0^{Out1}(\bar{S},\hat{T})$

$$\bar{S} \in (1,\infty), \hat{T} \in (0,\infty) \qquad \frac{\partial \bar{A}_0^{Out1}}{\partial \hat{T}} = -2\bar{S}^2\frac{\partial a}{\partial \bar{S}}\frac{\partial \bar{A}_0^{Out1}}{\partial \bar{S}} - \left(\beta\bar{S}\frac{\partial a}{\partial \bar{S}} + \bar{S}^2\frac{\partial^2 a}{\partial \bar{S}^2}\right)\bar{A}_0^{Out1}, \tag{2.29a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{A}_0^{Out1}(\bar{S},\hat{T}) \to 0, \tag{2.29b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{A}_0^{Out1}(\bar{S},0) = 0; \tag{2.29c}$$

and for $\bar{A}_1^{Out1}(\bar{S},\hat{T})$

$$\begin{aligned}\bar{S} \in (0,1), \hat{T} \in (0,\infty) \qquad \frac{\partial \bar{A}_1^{Out1}}{\partial \hat{T}} =& -2\bar{S}^2\frac{\partial a}{\partial \bar{S}}\frac{\partial \bar{A}_1^{Out1}}{\partial \bar{S}} - \left(\beta\bar{S}\frac{\partial a}{\partial \bar{S}} + \bar{S}^2\frac{\partial^2 a}{\partial \bar{S}^2}\right)\bar{A}_1^{Out1} \\ &+ \bar{S}^2\frac{\partial^2 \bar{A}_0^{Out1}}{\partial \bar{S}^2} + \beta\bar{S}\frac{\partial \bar{A}_0^{Out1}}{\partial \bar{S}} - \alpha\bar{A}_0^{Out1},\end{aligned} \tag{2.30a}$$

subject to

$$\text{as } \bar{S} \to 0 \qquad \bar{A}_1^{Out1}(\bar{S}, \hat{T}) \to 0, \tag{2.30b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{A}_1^{Out1}(\bar{S}, 0) = 0. \tag{2.30c}$$

The solutions to these problems are the same as those for the corresponding outer 2 problems and lead to the outer 1 expansion

$$\begin{aligned}\bar{P}_e^{Out1}(\bar{S}, \hat{T}; \epsilon) =& e^{-\alpha\epsilon^2\hat{T}} - \bar{S}e^{-(\alpha-\beta)\epsilon^2\hat{T}} \\ &+ \Phi_0\left(\frac{\epsilon^2\hat{T}}{\ln(\bar{S})}\right)\frac{\bar{S}^{\frac{1-\beta}{2}}}{\left(\ln(\bar{S})\right)^{\frac{1}{2}}}e^{-\frac{(\ln(\bar{S}))^2}{4\epsilon^2\hat{T}}}\left(1 + \epsilon^2\bar{A}_1^{Out1}(\bar{S}, \hat{T}) + O(\epsilon^4)\right) \quad \text{as } \epsilon \to 0,\end{aligned} \tag{2.31}$$

where the unknown function Φ_0 will again be determined during matching.

Inner Region

In order to smooth the discontinuity in the derivative of the final payoff at the strike for small times, we look in a small inner region about $\bar{S} = 1$, where intuition tells us that most of the price change will occur. We introduce the length scaling $\bar{S} = 1 + \delta(\epsilon)\hat{S}$, where the required scaling $\delta(\epsilon) = \epsilon$ is motivated by the dominant balance in the governing equation and $\hat{S} = O(1)$. Writing the outer 1 expansion in terms of the inner variable $\hat{S}$ suggests an inner expansion with the scaling $\bar{P}_e(\bar{S}, \hat{T}; \epsilon) = \epsilon\hat{P}_e^{In}(\hat{S}, \hat{T}; \epsilon)$, which gives the inner problem

$$\hat{S} \in (-\infty, \infty), \hat{T} \in (0, \infty) \qquad \frac{1}{\epsilon^2}\frac{\partial \hat{P}_e^{In}}{\partial \hat{T}} = \left(1 + \epsilon\hat{S}\right)^2\frac{1}{\epsilon^2}\frac{\partial^2 \hat{P}_e^{In}}{\partial \hat{S}^2} + \beta\left(1 + \epsilon\hat{S}\right)\frac{1}{\epsilon}\frac{\partial \hat{P}_e^{In}}{\partial \hat{S}} - \alpha\hat{P}_e^{In}, \tag{2.32a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_e^{In}(\hat{S}, 0) = \left(-\hat{S}\right)^+. \tag{2.32b}$$

Posing a regular expansion in powers of ϵ

$$\hat{P}_e^{In}(\hat{S}, \hat{T}; \epsilon) = \hat{P}_0^{In}(\hat{S}, \hat{T}) + \epsilon\hat{P}_1^{In}(\hat{S}, \hat{T}) + \epsilon^2\hat{P}_2^{In}(\hat{S}, \hat{T}) + O(\epsilon^3) \qquad \text{as } \epsilon \to 0, \tag{2.33}$$

gives the following subproblems: for $\hat{P}_0^{In}(\hat{S},\hat{T})$

$$\hat{S} \in (-\infty,\infty), \hat{T} \in (0,\infty) \qquad \frac{\partial \hat{P}_0^{In}}{\partial \hat{T}} = \frac{\partial^2 \hat{P}_0^{In}}{\partial \hat{S}^2}, \tag{2.34a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_0^{In}(\hat{S},0) = \left(-\hat{S}\right)^+; \tag{2.34b}$$

for $\hat{P}_1^{In}(\hat{S},\hat{T})$

$$\hat{S} \in (-\infty,\infty), \hat{T} \in (0,\infty) \qquad \frac{\partial \hat{P}_1^{In}}{\partial \hat{T}} = \frac{\partial^2 \hat{P}_1^{In}}{\partial \hat{S}^2} + 2\hat{S}\frac{\partial^2 \hat{P}_0^{In}}{\partial \hat{S}^2} + \beta\frac{\partial \hat{P}_0^{In}}{\partial \hat{S}}, \tag{2.35a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_1^{In}(\hat{S},0) = 0; \tag{2.35b}$$

and for $\hat{P}_2^{In}(\hat{S},\hat{T})$

$$\hat{S} \in (-\infty,\infty), \hat{T} \in (0,\infty) \qquad \frac{\partial \hat{P}_2^{In}}{\partial \hat{T}} = \frac{\partial^2 \hat{P}_2^{In}}{\partial \hat{S}^2} + 2\hat{S}\frac{\partial^2 \hat{P}_1^{In}}{\partial \hat{S}^2} + \beta\frac{\partial \hat{P}_1^{In}}{\partial \hat{S}} + \hat{S}^2\frac{\partial^2 \hat{P}_0^{In}}{\partial \hat{S}^2} + \beta\hat{S}\frac{\partial \hat{P}_0^{In}}{\partial \hat{S}} - \alpha\hat{P}_0, \tag{2.36a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_2^{In}(\hat{S},0) = 0. \tag{2.36b}$$

The problem (2.34a-2.34b) has the similarity solution $\hat{P}_0^{In}(\hat{S},\hat{T}) = \sqrt{\hat{T}}h_0(\zeta)$ where $\zeta = \frac{\hat{S}}{2\sqrt{\hat{T}}}$ and where $h_0(\zeta)$ is the solution to

$$h_0^{''} + 2\zeta h_0^{'} - 2h_0 = 0, \tag{2.37a}$$

subject to

$$\text{as } \zeta \to \infty \qquad \hat{T}^{\frac{1}{2}}h_0(\zeta) \to 0, \tag{2.37b}$$

$$\text{as } \zeta \to -\infty \qquad h_0(\zeta) \sim -2\zeta. \tag{2.37c}$$

The ODE (2.37a) has the general solution

$$h_0(\zeta) = 2C_{01}\zeta + C_{02}\mathrm{ierfc}(\zeta), \tag{2.38}$$

for arbitrary constants C_{01} and C_{02}, where the expression

$$\mathrm{ierfc}(\zeta) = \frac{1}{\sqrt{\pi}}e^{-\zeta^2} - \zeta\mathrm{erfc}(\zeta) \tag{2.39}$$

is the first repeated integral of the complementary error function which can be determined via the recurrence relation [1] for $n \geq 1$

$$\mathrm{i}^n\mathrm{erfc}(\zeta) = -\frac{\zeta}{n}\mathrm{i}^{n-1}\mathrm{erfc}(\zeta) + \frac{1}{2n}\mathrm{i}^{n-2}\mathrm{erfc}(\zeta), \tag{2.40}$$

where

$$\mathrm{i}^0\mathrm{erfc}(\zeta) = \mathrm{erfc}(\zeta), \tag{2.41}$$

$$\mathrm{i}^{-1}\mathrm{erfc}(\zeta) = \frac{2}{\sqrt{\pi}}e^{-\zeta^2}. \tag{2.42}$$

For the purpose of determining C_{01} and C_{02}, we need the asymptotic behaviours

$$\mathrm{erfc}(\zeta) \sim \begin{cases} \frac{e^{-\zeta^2}}{\sqrt{\pi}\zeta}\left(1 - \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + O(\zeta^{-6})\right) & \text{as } \zeta \to \infty, \\ 2 + \frac{e^{-\zeta^2}}{\sqrt{\pi}\zeta}\left(1 - \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + O(\zeta^{-6})\right) & \text{as } \zeta \to -\infty, \end{cases} \tag{2.43}$$

$$\mathrm{ierfc}(\zeta) \sim \begin{cases} \frac{e^{-\zeta^2}}{2\sqrt{\pi}}\left(\frac{1}{\zeta^2} - \frac{3}{2\zeta^4} + O(\zeta^{-6})\right) & \text{as } \zeta \to \infty, \\ -2\zeta + \frac{e^{-\zeta^2}}{2\sqrt{\pi}}\left(\frac{1}{\zeta^2} - \frac{3}{2\zeta^4} + O(\zeta^{-6})\right) & \text{as } \zeta \to -\infty, \end{cases} \tag{2.44}$$

from which we can write

$$h_0(\zeta) \sim \begin{cases} 2C_{01}\zeta + O\left(\zeta^{-2}e^{-\zeta^2}\right) & \text{as } \zeta \to \infty, \\ 2(C_{01} - C_{02})\zeta + O\left(\zeta^{-2}e^{-\zeta^2}\right) & \text{as } \zeta \to -\infty. \end{cases} \tag{2.45}$$

The asymptotic behaviour (2.45) together with conditions (2.37b) and (2.37c) requires

$$C_{01} = 0, \qquad C_{02} = 1 \tag{2.46}$$

and therefore

$$h_0(\zeta) = \frac{1}{\sqrt{\pi}} e^{-\zeta^2} - \zeta \operatorname{erfc}(\zeta). \tag{2.47}$$

The problem (2.35a-2.35b) has the similarity solution $\hat{P}_1^{In}(\hat{S}, \hat{T}) = \hat{T} h_1(\zeta)$ where $h_1(\zeta)$ is the solution to

$$h_1^{''} + 2\zeta h_1^{'} - 4h_1 = -4\zeta h_0^{''} - 2\beta h_0^{'}, \tag{2.48a}$$

subject to

$$\text{as } \zeta \to \infty \qquad \hat{T} h_1(\zeta) \to 0, \tag{2.48b}$$

$$\text{as } \zeta \to -\infty \qquad \hat{T} h_1(\zeta) \to 0. \tag{2.48c}$$

This has the general solution [1]

$$h_1(\zeta) = C_{11}(2\zeta^2 + 1) + 2C_{12}\mathrm{i}^2\operatorname{erfc}(\zeta) + \frac{\zeta}{\sqrt{\pi}} e^{-\zeta^2} - \frac{\beta}{2}\operatorname{erfc}(\zeta), \tag{2.49}$$

for arbitrary constants C_{11} and C_{12}, where (2.40) determines

$$\mathrm{i}^2\operatorname{erfc}(\zeta) = \frac{1}{4}\left[(2\zeta^2 + 1)\operatorname{erfc}(\zeta) - \frac{2}{\sqrt{\pi}}\zeta e^{-\zeta^2}\right], \tag{2.50}$$

which has the asymptotic behaviour

$$\mathrm{i}^2\operatorname{erfc}(\zeta) \sim \begin{cases} \frac{1}{4}\frac{e^{-\zeta^2}}{\sqrt{\pi}\zeta^3}\left(1 - \frac{3}{\zeta^2} + O(\zeta^{-4})\right) & \text{as } \zeta \to \infty, \\ \zeta^2 + \frac{1}{2} + \frac{1}{4}\frac{e^{-\zeta^2}}{\sqrt{\pi}\zeta^3}\left(1 - \frac{3}{\zeta^2} + O(\zeta^{-4})\right) & \text{as } \zeta \to -\infty. \end{cases} \tag{2.51}$$

To specify C_{11} and C_{12} we use (2.49) and (2.51) to write

$$h_1(\zeta) \sim \begin{cases} C_{21}\left(2\zeta^2 + 1\right) + O\left(\zeta e^{-\zeta^2}\right) & \text{as } \zeta \to \infty, \\ (C_{11} + C_{12})\left(2\zeta^2 + 1\right) - \beta + O\left(\zeta e^{-\zeta^2}\right) & \text{as } \zeta \to -\infty, \end{cases} \tag{2.52}$$

which along with conditions (2.48b) and (2.48c) leads to

$$C_{11} = C_{12} = 0 \tag{2.53}$$

and therefore

$$h_1(\zeta) = \frac{\zeta}{\sqrt{\pi}} e^{-\zeta^2} - \frac{\beta}{2} \mathrm{erfc}(\zeta). \tag{2.54}$$

The problem (2.36a-2.36b) has the similarity solution $\hat{P}_2^{In}(\hat{S}, \hat{T}) = \hat{T}^{\frac{3}{2}} h_2(\zeta)$ where $h_2(\zeta)$ is the solution to

$$h_2^{''} + 2\zeta h_2^{'} - 6h_1 = -4\zeta h_1^{''} - 2\beta h_1^{'} - 4\zeta^2 h_0^{''} - 4\beta\zeta h_0^{'} + 4\alpha h_0, \tag{2.55a}$$

subject to

$$\text{as } \zeta \to \infty \qquad \hat{T}^{\frac{3}{2}} h_2(\zeta) \to 0, \tag{2.55b}$$

$$\text{as } \zeta \to -\infty \qquad \hat{T}^{\frac{3}{2}} h_2(\zeta) \to 0, \tag{2.55c}$$

which has the general solution [1]

$$\begin{aligned} h_2(\zeta) =& C_{21} \frac{1}{3} \left(\zeta^3 + \frac{3}{2}\zeta \right) + C_{22} \mathrm{i}^3\mathrm{erfc}(\zeta) + \frac{e^{-\zeta^2}}{\sqrt{\pi}} \left(\zeta^4 - \frac{1}{3}(1+3\beta)\zeta^2 + \frac{1}{4} \left(\beta(\beta-1) - \frac{1}{3} \right) \right) \\ & - (\alpha - \beta)\mathrm{ierfc}(\zeta), \end{aligned} \tag{2.56}$$

for arbitrary constants C_{21} and C_{22} and where, from (2.40),

$$\mathrm{i}^3\mathrm{erfc}(\zeta) = \frac{1}{6} \left[\frac{1}{\sqrt{\pi}} (1+\zeta^2) e^{-\zeta^2} - \left(\frac{3\zeta}{2} + \zeta^3 \right) \mathrm{erfc}(\zeta) \right], \tag{2.57}$$

which has the asymptotic behaviour

$$\mathrm{i}^3\mathrm{erfc}(\zeta) \sim \begin{cases} \frac{e^{-\zeta^2}}{6\sqrt{\pi}} \left(\frac{3}{4\zeta^6} - \frac{4}{7\zeta^4} + O(\zeta^{-6}) \right) & \text{as } \zeta \to \infty, \\ -\frac{1}{3} \left(\zeta^3 + \frac{3\zeta}{2} \right) + \frac{e^{-\zeta^2}}{6\sqrt{\pi}} \left(\frac{3}{4\zeta^6} - \frac{4}{7\zeta^4} O(\zeta^{-6}) \right) & \text{as } \zeta \to -\infty. \end{cases} \tag{2.58}$$

To specify C_{21} and C_{22} we use (2.56) and (2.58) to write

$$h_2(\zeta) \sim \begin{cases} C_{21} \frac{1}{3} \left(\zeta^3 + \frac{3}{2}\zeta \right) + O\left(\zeta^4 e^{-\zeta^2} \right) & \text{as } \zeta \to \infty, \\ (C_{21} - C_{22}) \frac{1}{3} \left(\zeta^3 + \frac{3}{2}\zeta \right) + 2(\alpha - \beta)\zeta + O\left(\zeta^4 e^{-\zeta^2} \right) & \text{as } \zeta \to -\infty, \end{cases} \tag{2.59}$$

which along with the conditions (2.55b) and (2.55c) leads to

$$C_{21} = C_{22} = 0 \tag{2.60}$$

and therefore

$$h_2(\zeta) = \frac{e^{-\zeta^2}}{\sqrt{\pi}}\left(\zeta^4 - \frac{1}{3}(1+3\beta)\zeta^2 + \frac{1}{4}\left(\beta(\beta-1) - \frac{1}{3}\right)\right) - (\alpha-\beta)\mathrm{ierfc}(\zeta). \tag{2.61}$$

Hence we can write the Inner expansion in terms of the Inner spatial variable $\hat{S}$ as

$$\hat{P}_e^{In}(\hat{S},\hat{T};\epsilon) = \epsilon\sqrt{\hat{T}}h_0\left(\frac{\hat{S}}{2\sqrt{\hat{T}}}\right) + \epsilon^2\hat{T}h_1\left(\frac{\hat{S}}{2\sqrt{\hat{T}}}\right) + \epsilon^3\hat{T}^{\frac{3}{2}}h_2\left(\frac{\hat{S}}{2\sqrt{\hat{T}}}\right) + O(\epsilon^4) \quad \text{as } \epsilon \to 0, \tag{2.62}$$

where

$$h_0(\zeta) = \frac{1}{\sqrt{\pi}}e^{-\zeta^2} - \zeta\mathrm{erfc}(\zeta), \tag{2.63a}$$

$$h_1(\zeta) = \frac{\zeta}{\sqrt{\pi}}e^{-\zeta^2} - \frac{\beta}{2}\mathrm{erfc}(\zeta), \tag{2.63b}$$

$$h_2(\zeta) = \frac{e^{-\zeta^2}}{\sqrt{\pi}}\left(\zeta^4 - \frac{1}{3}(1+3\beta)\zeta^2 + \frac{1}{4}\left(\beta(\beta-1) - \frac{1}{3}\right)\right) - (\alpha-\beta)\mathrm{ierfc}(\zeta). \tag{2.63c}$$

Matching Region

For a discussion of the possible approaches to matching inner and outer expansions the reader is directed to Hinch [57]. In this piece of work we define a *matching* region where $\epsilon \ll \bar{S} - 1 = \epsilon\hat{S} \ll 1$ through the introduction of an intermediate variable $\tilde{S}$ which is related to the outer spatial variable through $\bar{S} - 1 = \epsilon^n\tilde{S}$ and the inner spatial variable through $\hat{S} = \epsilon^{n-1}\tilde{S}$ where $0 < n < 1$. In the matching region we take the limit $\epsilon \to 0$ with $\tilde{S}$ fixed which requires $\bar{S} \to 1^-$ and $\hat{S} \to -\infty$ in the outer 1 region, and $\bar{S} \to 1^+$ and $\hat{S} \to \infty$ for the outer 2 region. By looking at the asymptotic behaviour of the outer (2.26,2.31) and inner (2.62) expressions in the appropriate limits, together with the unknown function in the outer expansion, we will seek to form a matching expression by forcing the expressions to be the same in the relevant matching region. A suitable matching expression allows the formation of a *uniform* expansion, valid for all $\bar{S}$, by the summation of the outer and inner expansion and the subtraction of the matching expression. For a discussion of the formation of uniform expansions the reader is directed to Bender & Orszag [12].

In the matching region we define the contribution of the inner expansion in terms of the intermediate variable through $\hat{P}_e^{In}(\hat{S},\hat{T};\epsilon) = \tilde{P}_e^{In}(\tilde{S},\hat{T};\epsilon)$ and the contribution of the terms of the inner expansion in the matching region as $h_0(\zeta) = \tilde{h}_0\left(\frac{\tilde{S}}{2\sqrt{\hat{T}}};\epsilon\right)$, $h_1(\zeta) = \tilde{h}_1\left(\frac{\tilde{S}}{2\sqrt{\hat{T}}};\epsilon\right)$ and $h_2(\zeta) = \tilde{h}_2\left(\frac{\tilde{S}}{2\sqrt{\hat{T}}};\epsilon\right)$.

Using (2.63a-2.63c) together with (2.43) and (2.44), the asymptotic behaviours of the terms of the inner expansion are

$$h_0(\zeta) \sim \begin{cases} \frac{e^{-\zeta^2}}{2\sqrt{\pi}}\left(\frac{1}{\zeta^2} - \frac{3}{2\zeta^4} + O(\zeta^{-6})\right) & \text{as } \zeta \to \infty, \\ -2\zeta + \frac{e^{-\zeta^2}}{2\sqrt{\pi}}\left(\frac{1}{\zeta^2} - \frac{3}{2\zeta^4} + O(\zeta^{-6})\right) & \text{as } \zeta \to -\infty, \end{cases} \tag{2.64a}$$

$$h_1(\zeta) \sim \begin{cases} \frac{e^{-\zeta^2}}{2\sqrt{\pi}}\left(2\zeta - \frac{\beta}{\zeta} + \frac{\beta}{2\zeta^3} + O(\zeta^{-4})\right) & \text{as } \zeta \to \infty, \\ -\beta + \frac{e^{-\zeta^2}}{2\sqrt{\pi}}\left(2\zeta - \frac{\beta}{2\zeta} + \frac{\beta}{4\zeta^3} + O(\zeta^{-4})\right) & \text{as } \zeta \to -\infty, \end{cases} \tag{2.64b}$$

$$h_2(\zeta) \sim \begin{cases} \frac{e^{-\zeta^2}}{2\sqrt{\pi}}\left(2\zeta^4 - \frac{2}{3}(1+3\beta)^2\zeta^2 + O(1)\right) & \text{as } \zeta \to \infty, \\ 2(\alpha-\beta)\zeta + \frac{e^{-\zeta^2}}{2\sqrt{\pi}}\left(2\zeta^4 - \frac{2}{3}(1+3\beta)^2\zeta^2 + O(1)\right) & \text{as } \zeta \to -\infty \end{cases} \tag{2.64c}$$

and therefore the contribution of the terms of the inner expansion in the matching region have the following forms: for $\hat{P}_0^{In}(\hat{S},\hat{T})$

$$\epsilon\sqrt{\hat{T}}\tilde{h}_0\left(\frac{\tilde{S}}{2\sqrt{\hat{T}}};\epsilon\right) \sim \begin{cases} \frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\tilde{\psi}_0(\tilde{S},\hat{T};\epsilon) & \text{as } \epsilon \to 0, \tilde{S} > 0, \\ -\epsilon^n\tilde{S} + \frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\tilde{\psi}_0(\tilde{S},\hat{T};\epsilon) & \text{as } \epsilon \to 0, \tilde{S} < 0, \end{cases} \tag{2.65a}$$

where

$$\tilde{\psi}_0(\tilde{S},\hat{T};\epsilon) = \epsilon^{3-2n}\frac{2\hat{T}^{\frac{3}{2}}}{\tilde{S}^2} - \epsilon^{5-4n}\frac{12\hat{T}^{\frac{5}{2}}}{\tilde{S}^4} + \epsilon^{7-6n}\frac{120\hat{T}^{\frac{7}{2}}}{\tilde{S}^6} + O\left(\epsilon^{9-8n}\right); \tag{2.65b}$$

for $\hat{P}_1^{In}(\hat{S},\hat{T})$

$$\epsilon^2\hat{T}\tilde{h}_1\left(\frac{\tilde{S}}{2\sqrt{\hat{T}}};\epsilon\right) \sim \begin{cases} \frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\tilde{\psi}_1(\tilde{S},\hat{T};\epsilon) & \text{as } \epsilon \to 0, \tilde{S} > 0, \\ -\beta\epsilon^2\hat{T} + \frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\tilde{\psi}_1(\tilde{S},\hat{T};\epsilon) & \text{as } \epsilon \to 0, \tilde{S} < 0, \end{cases} \tag{2.66a}$$

where

$$\tilde{\psi}_1(\tilde{S},\hat{T};\epsilon) = \epsilon^{n+1}\frac{\hat{T}^{\frac{1}{2}}\tilde{S}}{2} - \epsilon^{3-n}\frac{\beta\hat{T}^{\frac{3}{2}}}{\tilde{S}} + \epsilon^{5-3n}\frac{2\beta\hat{T}^{\frac{5}{2}}}{\tilde{S}^3} - O\left(\epsilon^{5-3n}\right); \tag{2.66b}$$

and for $\hat{P}_2^{In}(\hat{S},\hat{T})$

$$\epsilon^3\hat{T}^{\frac{3}{2}}\tilde{h}_2\left(\frac{\tilde{S}}{2\sqrt{\hat{T}}};\epsilon\right)\sim\begin{cases}\frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\tilde{\psi}_2(\tilde{S},\hat{T};\epsilon) & \text{as } \epsilon\to 0, \tilde{S}>0,\\ (\alpha-\beta)\epsilon^{n+2}\tilde{S}\hat{T}+\frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\tilde{\psi}_2(\tilde{S},\hat{T};\epsilon) & \text{as } \epsilon\to 0, \tilde{S}<0,\end{cases} \tag{2.67a}$$

where

$$\tilde{\psi}_2(\tilde{S},\hat{T};\epsilon)=\epsilon^{4n-1}\frac{\tilde{S}^4}{16\hat{T}^{\frac{1}{2}}}+O\left(\epsilon^{2n+1}\right). \tag{2.67b}$$

We note from the asymptotic expressions (2.65b,2.66b,2.67b) that the ordering of terms in the expansion (2.62) breaks down for $n\le\frac{2}{3}$ and we therefore look for a matching region such that $n>\frac{2}{3}$. However, leaving n unrestricted on the interval $n\in\left(\frac{2}{3},1\right)$ does not allow us to determine where the first term of (2.66b) at $O(\epsilon^{n+1})$ contributes to the matching expression. Indeed for $n=1$, all orders of the expansion (2.65b) are dominant.

We also observe that the matching expression formed by the inner is singular at $\tilde{S}=0$ ($\bar{S}=1$). This is to be expected as the WKBJ terms in the outer expansions are singular at $\bar{S}=1$. We would like the singular terms of the matching and outer expressions to cancel in the limit as $\tilde{S}\to 0$ ($\bar{S}\to 1$) otherwise our uniform expansion will also be singular in this limit. To this end we note that the $O(\epsilon^{3-n})$ term from (2.66b), which contains $\tilde{S}^{-1}$, can only dominate the $O(\epsilon^{5-4n})$ term from (2.65b), which contains $\tilde{S}^{-4}$, for $n<\frac{2}{3}$. In fact for some general integer $k>0$, terms of $O(\epsilon^{k-(k-1)n})$ from (2.66b), which contain $\tilde{S}^{(1-k)}$, can only dominate terms of $O(\epsilon^{(k-2)-(k-4)n})$ from (2.65b), which contain $\tilde{S}^{(4-k)}$, for $n<\frac{2}{3}$. Therefore, the restriction $n>\frac{2}{3}$ prohibits the formation of a truncated matching expression containing all negative integral powers of $\tilde{S}$ up to some finite integer.

We now look for the contribution of the outer expansions (2.26) and (2.31) to the matching regions which we previously related to the intermediate variable through $\bar{S}-1=\epsilon^n\tilde{S}$. Writing the terms of the outer expansion in terms of the intermediate variable $\tilde{S}$ and function Φ_0 as a Taylor series

$$\Phi_0\left(\frac{\epsilon^2\hat{T}}{\ln\left(1+\epsilon^n\tilde{S}\right)}\right)\simeq\Phi_0\left(\epsilon^{2-n}\frac{\hat{T}}{\tilde{S}}\right)-\left(\epsilon^2\hat{T}+\dots\right)\Phi_0'\left(\epsilon^{2-n}\frac{\hat{T}}{\tilde{S}}\right)\qquad\text{as } \epsilon\to 0, \tag{2.68}$$

produces the leading order contribution of the outer 2 expansion in the matching region for

$\tilde{S} > 0$

$$\tilde{P}_e^{Out2}(\tilde{S}, \hat{T}; \epsilon) \sim \frac{1}{(\epsilon\tilde{S})^{\frac{n}{2}}} \Phi_0 \left(\epsilon^{2-n} \frac{\hat{T}}{\tilde{S}} \right) e^{-\epsilon^{2n-2} \frac{\tilde{S}^2}{4\hat{T}}} e^{-\epsilon^{3n-2} \frac{\tilde{S}^3}{4\hat{T}}} \qquad \text{as } \epsilon \to 0 \tag{2.69}$$

and the outer 1 expansion in the matching region for $\tilde{S} < 0$

$$\begin{aligned} \tilde{P}_e^{Out1}(\tilde{S}, \hat{T}; \epsilon) \sim & \epsilon^n \tilde{S} - \beta\epsilon^2 \hat{T} + (\alpha - \beta)\epsilon^{n+2} \tilde{S}\hat{T} + O(\epsilon^4) \\ & + \frac{1}{(\epsilon\tilde{S})^{\frac{n}{2}}} \Phi_0 \left(\epsilon^{2-n} \frac{\hat{T}}{\tilde{S}} \right) e^{-\epsilon^{2n-2} \frac{\tilde{S}^2}{4\hat{T}}} e^{-\epsilon^{3n-2} \frac{\tilde{S}^3}{4\hat{T}}} \qquad \text{as } \epsilon \to 0. \end{aligned} \tag{2.70}$$

Matching the forms of (2.69) and (2.70) with the relevant leading order term of the inner terms in the matching region for $\tilde{S} > 0$ and $\tilde{S} < 0$ from (2.65b) identifies that matching of the exponential terms is only possible for $n > \frac{2}{3}$, which is consistent with our observation from the ordering of terms in the inner expansion in the matching region, and further that the unknown function Φ_0 has the form

$$\Phi_0 \left(\frac{\epsilon^2 \hat{T}}{\ln(\bar{S})} \right) = \frac{2}{\sqrt{\pi}} \left(\frac{\epsilon^2 \hat{T}}{\ln(\bar{S})} \right)^{\frac{3}{2}} \qquad \text{as } \epsilon \to 0 \tag{2.71}$$

and therefore we can rewrite (2.25) as

$$\bar{A}_0^{Out2}(\bar{S}, \hat{T}; \epsilon) = \bar{A}_0^{Out1}(\bar{S}, \hat{T}; \epsilon) \sim \frac{2\epsilon^3 \hat{T}^{\frac{3}{2}}}{\sqrt{\pi}} \frac{\bar{S}^{\frac{1-\beta}{2}}}{\left(\ln(\bar{S})\right)^2} \qquad \text{as } \epsilon \to 0. \tag{2.72}$$

Defining the contribution of the leading order terms of the outer expansion to the matching region, for $n \in \left(\frac{2}{3}, 1\right)$, through $\bar{A}_0^{Out2}(\bar{S}, \hat{T}) = \tilde{A}_0^{Out2}(\tilde{S}, \hat{T}; \epsilon)$ and $\bar{A}_0^{Out1}(\bar{S}, \hat{T}) = \tilde{A}_0^{Out1}(\tilde{S}, \hat{T}; \epsilon)$ gives

$$\tilde{A}_0^{Out2}(\tilde{S}, \hat{T}; \epsilon) = \tilde{A}_0^{Out1}(\tilde{S}, \hat{T}; \epsilon) \sim \frac{1}{\sqrt{\pi}} \left(\epsilon^{3-2n} \frac{2\hat{T}^{\frac{3}{2}}}{\tilde{S}^2} + \epsilon^{3-n}(3-\beta) \frac{\hat{T}^{\frac{3}{2}}}{\tilde{S}} + O(\epsilon^3) \right) \qquad \text{as } \epsilon \to 0 \tag{2.73}$$

and therefore for $\tilde{S} > 0$

$$\begin{aligned} \tilde{P}^{Out2}(\tilde{S}, \hat{T}; \epsilon) \sim & \frac{1}{\sqrt{\pi}} e^{-\epsilon^{2(n-1)} \frac{\tilde{S}^2}{4\hat{T}}} \left(\epsilon^{3-2n} \frac{2\hat{T}^{\frac{3}{2}}}{\tilde{S}^2} + \epsilon^{n+1} \frac{\hat{T}^{\frac{1}{2}} \tilde{S}}{2} + \epsilon^{4n-1} \frac{\tilde{S}^4}{16\hat{T}^{\frac{1}{2}}} \right. \\ & \left. + \epsilon^{3-n}(3-\beta) \frac{\hat{T}^{\frac{3}{2}}}{\tilde{S}} + O(\epsilon^{2n+1}) \right) \text{ as } \epsilon \to 0, \end{aligned} \tag{2.74}$$

while for $\tilde{S} < 0$

$$\begin{aligned}\tilde{P}^{Out1}(\tilde{S},\hat{T};\epsilon) \sim & \epsilon^n\tilde{S} - \beta\epsilon^2\hat{T} + (\alpha-\beta)\epsilon^{n+2}\tilde{S}\hat{T} + O(\epsilon^4) \\ & + \frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\Bigg(\epsilon^{3-2n}\frac{2\hat{T}^{\frac{3}{2}}}{\tilde{S}^2} + \epsilon^{n+1}\frac{\hat{T}^{\frac{1}{2}}\tilde{S}}{2} + \epsilon^{4n-1}\frac{\tilde{S}^4}{16\hat{T}^{\frac{1}{2}}} + \epsilon^{3-n}(3-\beta)\frac{\hat{T}^{\frac{3}{2}}}{\tilde{S}} \\ & + O(\epsilon^{2n+1})\Bigg) \quad \text{as } \epsilon \to 0. \end{aligned} \tag{2.75}$$

Comparison of (2.74) and (2.75) with the inner matching conditions from (2.65b) reveals a term of $O\left(\epsilon^{5-4n}\right)$ missing from the outer expressions which dominates the $O\left(\epsilon^{3-n}\right)$ term for $n > \frac{2}{3}$, but it is dominated by the $O\left(\epsilon^{4n-1}\right)$ term for $n < \frac{3}{4}$. We could form the three-term matching expression

$$\frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{2\tilde{S}^2}{4\hat{T}}}\left(\epsilon^{3-2n}\frac{\hat{T}^{\frac{3}{2}}}{\tilde{S}^2} + \epsilon^{n+1}\frac{\hat{T}^{\frac{1}{2}}\tilde{S}}{2} - \epsilon^{4n-1}\frac{\tilde{S}^4}{16\hat{T}^{\frac{1}{2}}}\right) \quad \text{as } \epsilon \to 0, \tilde{S} > 0, \tag{2.76}$$

which is valid for $n \in (\frac{2}{3}, \frac{3}{4})$, but does not cancel with the singular terms at $O(\epsilon^{3-n})$ in (2.74) and (2.75) and therefore the corresponding uniform expansion will be singular as $\bar{S} \to 1$. We also observe that the coefficient $(3-\beta)$ at $O(\epsilon^{3-n})$ in (2.74,2.75) does not match with the corresponding coefficient β from (2.66b). We therefore look to the contribution in the matching region of the next order term in the outer expansions, $\bar{A}_1^{Out2}(\bar{S},\hat{T})$ and $\bar{A}_1^{Out1}(\bar{S},\hat{T})$.
Using the behaviour of Φ_0 (2.71) determined during matching at leading order, we can rewrite the problem for $A_1^{Out2}(\bar{S},\hat{T})$ (2.21a-2.21c) which becomes

$$\bar{S} \in (1,\infty), \hat{T} \in (0,\infty) \qquad \frac{\partial \bar{A}_1^{Out2}}{\partial \hat{T}} = -\frac{\bar{S}}{\hat{T}}\ln(\bar{S})\frac{\partial \bar{A}_1^{Out2}}{\partial \bar{S}} - \left(\frac{(\beta-1)^2}{4} + \alpha - \frac{6}{\left(\ln(\bar{S})\right)^2}\right), \tag{2.77a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{A}_1^{Out2}(\bar{S},\hat{T}) \to 0, \tag{2.77b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{A}_1^{Out2}(\bar{S},0) = 0. \tag{2.77c}$$

As in the case of the $\bar{A}_0$ problem, (2.77a-2.77c) is a linear first order PDE which is amenable to method of characteristics, giving the solution

$$\bar{A}_1^{Out2}(\bar{S},\hat{T}) = -\hat{T}\left[\frac{6}{\left(\ln(\bar{S})\right)^2} + \frac{C_1^{Out2}\left(\ln(\bar{S})/\hat{T}\right)}{\ln(\bar{S})} + \frac{(\beta-1)^2}{4} + \alpha\right], \tag{2.78}$$

for some arbitrary function $C_1^{Out2}\left(\ln(\bar{S})/\hat{T}\right)$.

The problem for $\bar{A}_1^{Out1}(\bar{S},\hat{T})$ (2.30a-2.30c) becomes

$$\bar{S}\in(0,1), \hat{T}\in(0,\infty) \qquad \frac{\partial \bar{A}_1^{Out1}}{\partial \hat{T}} = -\frac{\bar{S}}{\hat{T}}\ln(\bar{S})\frac{\partial \bar{A}_1^{Out1}}{\partial \bar{S}} - \left(\frac{(\beta-1)^2}{4} + \alpha - \frac{6}{\left(\ln(\bar{S})\right)^2}\right), \tag{2.79a}$$

subject to

$$\text{as } \bar{S}\to 0 \qquad \bar{A}_1^{Out1}(\bar{S},\hat{T}) \to 0, \tag{2.79b}$$

$$\text{at } \hat{T}=0 \qquad \bar{A}_1^{Out1}(\bar{S},0) = 0. \tag{2.79c}$$

which has the same solution as the outer 2 problem

$$\bar{A}_1^{Out1}(\bar{S},\hat{T}) = -\hat{T}\left[\frac{6}{\left(\ln(\bar{S})\right)^2} + \frac{C_1^{Out1}\left(\ln(\bar{S})/\hat{T}\right)}{\ln(\bar{S})} + \frac{(\beta-1)^2}{4} + \alpha\right], \tag{2.80}$$

for some arbitrary function $C_1^{Out1}\left(\ln(\bar{S})/\hat{T}\right)$.

We can now use (2.71) together with (2.78) and (2.80) to write the outer 2 expansion (2.26) as

$$\begin{aligned}\bar{P}_e^{Out2}(\bar{S},\hat{T};\epsilon) \sim{}& \frac{2\epsilon^3}{\sqrt{\pi}}\frac{\hat{T}^{\frac{3}{2}}\bar{S}^{\frac{1-\beta}{2}}}{\left(\ln(\bar{S})\right)^2}e^{-\frac{(\ln(\bar{S}))^2}{4\epsilon^2\hat{T}}}\times\\ &\left(1-\epsilon^2\hat{T}\left[\frac{6}{\left(\ln(\bar{S})\right)^2} + \frac{C_1^{Out2}}{\ln(\bar{S})} + \frac{(\beta-1)^2}{4} + \alpha\right] + O(\epsilon^4)\right) \quad \text{as } \epsilon\to 0\end{aligned} \tag{2.81}$$

and the outer 1 expansion (2.31) as

$$\begin{aligned}\bar{P}_e^{Out1}(\bar{S},\hat{T};\epsilon) \sim{}& e^{-\alpha\epsilon^2\hat{T}} - \bar{S}e^{-(\alpha-\beta)\epsilon^2\hat{T}} + \frac{2\epsilon^3}{\sqrt{\pi}}\frac{\hat{T}^{\frac{3}{2}}\bar{S}^{\frac{1-\beta}{2}}}{\left(\ln(\bar{S})\right)^2}e^{-\frac{(\ln(\bar{S}))^2}{4\epsilon^2\hat{T}}}\times\\ &\left(1-\epsilon^2\hat{T}\left[\frac{6}{\left(\ln(\bar{S})\right)^2} + \frac{C_1^{Out1}}{\ln(\bar{S})} + \frac{(\beta-1)^2}{4} + \alpha\right] + O(\epsilon^4)\right) \quad \text{as } \epsilon\to 0.\end{aligned} \tag{2.82}$$

In terms of the intermediate variable, the contributions of $\bar{A}_1^{Out1}(\bar{S},\hat{T})$ and $\bar{A}_1^{Out2}(\bar{S},\hat{T})$ to the matching region expressions (2.74) and (2.75) are

$$\tilde{A}_1^{Out1}(\tilde{S},\hat{T};\epsilon)=\tilde{A}_1^{Out2}(\tilde{S},\hat{T};\epsilon)\sim\frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\left(-\epsilon^{5-4n}\frac{12\hat{T}^{\frac{5}{2}}}{\tilde{S}^4}-\epsilon^{3-n}\frac{3\hat{T}^{\frac{5}{2}}}{\tilde{S}}+O(\epsilon^{5-3n})\right)\quad\text{as }\epsilon\to 0 \tag{2.83}$$

and we observe that this contributes both the terms required to match with the inner matching expressions at $O(\epsilon^{5-4n})$ and $O(\epsilon^{3-n})$. However, in order to match these terms we have introduced a $\frac{1}{\left(\ln(\bar{S})\right)^4}$ term into the outer expression which will contribute a term of the form $\bar{S}^{-3}$. As discussed previously, we are unable to form a matching expression which contains the relevant term without going to further orders in the outer series. This process prevents the cancellation of singular terms as $\bar{S}\to 1$ and therefore the creation of a non-singular uniform expression is not possible using a finite number of terms. Finally, we note that determination of the arbitrary functions C_1^{Out2}, C_1^{Out1} in (2.78) and (2.80) requires matching at $O(\epsilon^{5-3n})$ in the intermediate variable. This term is dominated by the $O(\epsilon^{7-6n})$ term from the Inner matching expression (2.65b) which contains the term $\bar{S}^{-6}$ suggesting we need to go to additional terms in the outer to match to this order. We observe however that the inner matching expression (2.66b) suggests we should find $C_1^{Out1}=C_1^{Out2}=2\beta$.

We therefore define an outer 2 matching expression for $\tilde{S}>0$ which is valid for $n\in(\frac{2}{3},\frac{4}{5})$ as

$$\tilde{P}_e^{Match2}(\tilde{S},\hat{T};\epsilon)\sim\frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\tilde{\psi}(\tilde{S},\hat{T};\epsilon)\qquad\text{as }\epsilon\to 0 \tag{2.84a}$$

and the corresponding outer 1 matching expression for $\tilde{S}<0$ as

$$\tilde{P}_e^{Match1}(\tilde{S},\hat{T};\epsilon)\sim e^{-\alpha\epsilon^2\hat{T}}-\bar{S}e^{-(\alpha-\beta)\epsilon^2\hat{T}}+\frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\tilde{\psi}(\tilde{S},\hat{T};\epsilon)\quad\text{as }\epsilon\to 0, \tag{2.84b}$$

where

$$\tilde{\psi}(\tilde{S},\hat{T};\epsilon)=\epsilon^{3-2n}\frac{2\hat{T}^{\frac{3}{2}}}{\tilde{S}^2}+\epsilon^{n+1}\frac{\hat{T}^{\frac{1}{2}}\tilde{S}}{2}+\epsilon^{4n-1}\frac{\tilde{S}^4}{16\hat{T}^{\frac{1}{2}}}-\epsilon^{5-4n}\frac{12\hat{T}^{\frac{5}{2}}}{\tilde{S}^4}-\epsilon^{3-n}\beta\frac{\hat{T}^{\frac{3}{2}}}{\tilde{S}}, \tag{2.84c}$$

with the relative magnitude of the $O(\epsilon^{4n-1})$ and $O(\epsilon^{5n-4})$ terms depending on whether n is greater than or less than $\frac{3}{4}$.

For consistency, we define our final outer 1 and outer 2 expansions as

$$\bar{P}_e^{Out2}(\bar{S},\hat{T};\epsilon)\sim\frac{2\epsilon^3}{\sqrt{\pi}}\frac{\hat{T}^{\frac{3}{2}}\bar{S}^{\frac{1-\beta}{2}}}{\left(\ln(\bar{S})\right)^2}e^{-\frac{(\ln(\bar{S}))^2}{4\epsilon^2\hat{T}}}\left(1-\epsilon^2\hat{T}\frac{6}{\left(\ln(\bar{S})\right)^2}\right)\quad\text{as }\epsilon\to 0 \tag{2.85}$$

and

$$\bar{P}_e^{Out1}(\bar{S},\hat{T};\epsilon) \sim e^{-\alpha\epsilon^2\hat{T}} - \bar{S}e^{-(\alpha-\beta)\epsilon^2\hat{T}} + \frac{2\epsilon^3}{\sqrt{\pi}}\frac{\hat{T}^{\frac{3}{2}}\bar{S}^{\frac{1-\beta}{2}}}{\left(\ln(\bar{S})\right)^2}e^{-\frac{(\ln(\bar{S}))^2}{4\epsilon^2\hat{T}}}\left(1-\epsilon^2\hat{T}\frac{6}{\left(\ln(\bar{S})\right)^2}\right) \text{ as } \epsilon \to 0, \tag{2.86}$$

which contain the terms successfully matched with the inner expansion.

Comparison with the Closed-Form Solution

We can verify that the WKBJ expansion has captured the correct behaviour by looking at the small-time asymptotic behaviour of the European option price in the inner and outer regions. We can represent the non-dimensional small-time European put option price in terms of the complementary error function as

$$\bar{P}_e(\bar{S},\hat{T};\epsilon) = \frac{1}{2}\left(e^{-\alpha\epsilon^2\hat{T}}\mathrm{erfc}\left(\frac{\bar{d}_2}{\sqrt{2}}\right) - \bar{S}e^{-(\alpha-\beta)\epsilon^2\hat{T}}\mathrm{erfc}\left(\frac{\bar{d}_1}{\sqrt{2}}\right)\right), \tag{2.87}$$

where

$$\frac{\bar{d}_1}{\sqrt{2}} = \frac{\ln(\bar{S})}{2\epsilon\sqrt{\hat{T}}} + \frac{(\beta+1)}{2}\epsilon\sqrt{\hat{T}}, \tag{2.88a}$$

$$\frac{\bar{d}_2}{\sqrt{2}} = \frac{\ln(\bar{S})}{2\epsilon\sqrt{\hat{T}}} + \frac{(\beta-1)}{2}\epsilon\sqrt{\hat{T}}. \tag{2.88b}$$

In the outer regions, $\frac{\bar{d}_1}{\sqrt{2}}, \frac{\bar{d}_2}{\sqrt{2}} \to -\infty$ as $\epsilon \to 0$, for $\bar{S} \gg 1$ and $\frac{\bar{d}_1}{\sqrt{2}}, \frac{\bar{d}_2}{\sqrt{2}} \to \infty$ as $\epsilon \to 0$, for $\bar{S} \ll 1$. Using (2.43) together with (2.87) we derive the small-time behaviour in the outer 2 region ($\bar{S} \gg 1$) as

$$\begin{aligned}\bar{P}_e^{Out2}(\bar{S},\hat{T};\epsilon) \sim{}& \frac{2\epsilon^3}{\sqrt{\pi}}\frac{\hat{T}^{\frac{3}{2}}}{(\ln(\bar{S}))^2}\frac{e^{-\frac{(\ln(\bar{S}))^2}{4\epsilon^2\hat{T}}}}{\bar{S}^{\frac{\beta-1}{2}}}\times \\ & \left(1-\epsilon^2\hat{T}\left[\alpha+\frac{(\beta-1)^2}{4}+\frac{2\beta}{\ln(\bar{S})}+\frac{6}{(\ln(\bar{S}))^2}\right]+O\left(\epsilon^4\right)\right) \qquad \text{as } \epsilon \to 0,\end{aligned} \tag{2.89}$$

and in the outer 1 region ($\bar{S} \ll 1$) as

$$\bar{P}_e^{Out1}(\bar{S}, \hat{T}; \epsilon) \sim e^{-\alpha\epsilon^2\hat{T}} - \bar{S}e^{-(\alpha-\beta)\epsilon^2\hat{T}} + \frac{2\epsilon^3}{\sqrt{\pi}} \frac{\hat{T}^{\frac{3}{2}}}{(\ln(\bar{S}))^2} \frac{e^{-\frac{(\ln(\bar{S}))^2}{4\epsilon^2\hat{T}}}}{\bar{S}^{\frac{\beta-1}{2}}} \times \left(1 - \epsilon^2\hat{T}\left[\alpha + \frac{(\beta-1)^2}{4} + \frac{2\beta}{\ln(\bar{S})} + \frac{6}{(\ln(\bar{S}))^2}\right] + O\left(\epsilon^4\right)\right) \qquad \text{as } \epsilon \to 0, \tag{2.90}$$

which we observe are consistent with the outer expansions, and further confirms that we expect $C_1 = 2\beta$.

For the inner region, defined by $\bar{S} = 1 + \epsilon\hat{S}$, as $\epsilon \to 0$ we have

$$\frac{\hat{d}_1}{\sqrt{2}} \simeq \frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}} + \frac{\epsilon}{2}\left((\beta+1)\hat{T}^{\frac{1}{2}} - \frac{\hat{S}^2}{2\hat{T}^{\frac{1}{2}}}\right) + \frac{\epsilon^2}{6}\frac{\hat{S}}{\hat{T}^{\frac{1}{2}}} + O(\epsilon^3), \tag{2.91a}$$

$$\frac{\hat{d}_2}{\sqrt{2}} \simeq \frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}} + \frac{\epsilon}{2}\left((\beta-1)\hat{T}^{\frac{1}{2}} - \frac{\hat{S}^2}{2\hat{T}^{\frac{1}{2}}}\right) + \frac{\epsilon^2}{6}\frac{\hat{S}}{\hat{T}^{\frac{1}{2}}} + O(\epsilon^3), \tag{2.91b}$$

and we use the Taylor expansions

$$\operatorname{erfc}\left(\frac{\hat{d}_1}{\sqrt{2}}\right) \simeq \operatorname{erfc}\left(\frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}\right) + \left(\frac{\epsilon}{2}\left((\beta+1)\hat{T}^{\frac{1}{2}} - \frac{\hat{S}^2}{2\hat{T}^{\frac{1}{2}}}\right) + \frac{\epsilon^2}{6}\frac{\hat{S}}{\hat{T}^{\frac{1}{2}}} + \dots\right)\operatorname{erfc}'\left(\frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}\right), \tag{2.92a}$$

$$\operatorname{erfc}\left(\frac{\hat{d}_2}{\sqrt{2}}\right) \simeq \operatorname{erfc}\left(\frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}\right) + \left(\frac{\epsilon}{2}\left((\beta-1)\hat{T}^{\frac{1}{2}} - \frac{\hat{S}^2}{2\hat{T}^{\frac{1}{2}}}\right) + \frac{\epsilon^2}{6}\frac{\hat{S}}{\hat{T}^{\frac{1}{2}}} + \dots\right)\operatorname{erfc}'\left(\frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}\right), \tag{2.92b}$$

to derive the inner behaviour

$$\hat{P}_e^{In}(\bar{S}, \hat{T}; \epsilon) \sim \epsilon\left(\frac{\hat{T}^{\frac{1}{2}}}{\sqrt{\pi}}e^{-\frac{\hat{S}^2}{4\hat{T}}} - \frac{\hat{S}}{2}\operatorname{erfc}\left(\frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}\right)\right) + \epsilon^2\left(\frac{\hat{S}\hat{T}^{\frac{1}{2}}}{2\sqrt{\pi}}e^{-\frac{\hat{S}^2}{4\hat{T}}} - \beta\hat{T}\operatorname{erfc}\left(\frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}\right)\right) + O(\epsilon^3) \tag{2.93}$$

which can be shown to be equivalent to (2.62) using the substitution $\zeta = \frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}$, and the results (2.63a) and (2.63b).

The three region structure for the small-time problem is illustrated in Figure 2-1, while a numerical comparison of our inner (2.63a-2.63c), outer (2.81-2.82) and matching (2.84a-2.84c) expansions with the closed-form solution (2.87) is shown in Figure 2-2. Although graphically it appears that the outer and inner expressions are indistinguishable for $\bar{S} \gg 1$, the error graphs show that the inner expression goes to zero much faster than the closed-form solution, whereas the outer expression captures the correct asymptotic behaviour in this limit. This demonstrates that the validity of the inner series as an approximation for the outer behaviour of the European option, as suggested by previous authors [42, 60, 102] is valid only up to algebraic terms in the out-of-the-money outer region. The same effect would be observed for the exponentially small terms for $\bar{S} \ll 1$, but the effect is not leading order and is dominated by the size of the intrinsic value for small times. The put-call parity result (1.13) ensures the behaviour in (2.89) will be observed in the small-time European call option problem for $\bar{S} \ll 1$, whereas the use of the first term of the inner expansion of the corresponding call option, as provided in [60], can be shown to give to the small-time asymptotic behaviour as $\bar{S} \to 0$

$$\bar{C}_e^{Out1}(\bar{S}, \hat{T}; \epsilon) \sim \frac{2}{\sqrt{\pi}} \frac{\epsilon^3 \hat{T}^{\frac{3}{2}}}{(\bar{S}-1)^2} e^{-\frac{(\bar{S}-1)^2}{4\epsilon^2 \hat{T}}} \qquad \text{as } \epsilon \to 0, \tag{2.94}$$

which does not capture the leading order exponentially small behaviour (2.89) as $\bar{S} \to 0$ and which contains a residual non-zero term at $\bar{S} = 0$.

Finally, the breakdown of the outer expansion near $\bar{S} = 1$ is clear. This is caused by the presence of logarithmic terms in the outer expansion which require an increasing number of terms in order to match in the inner region.

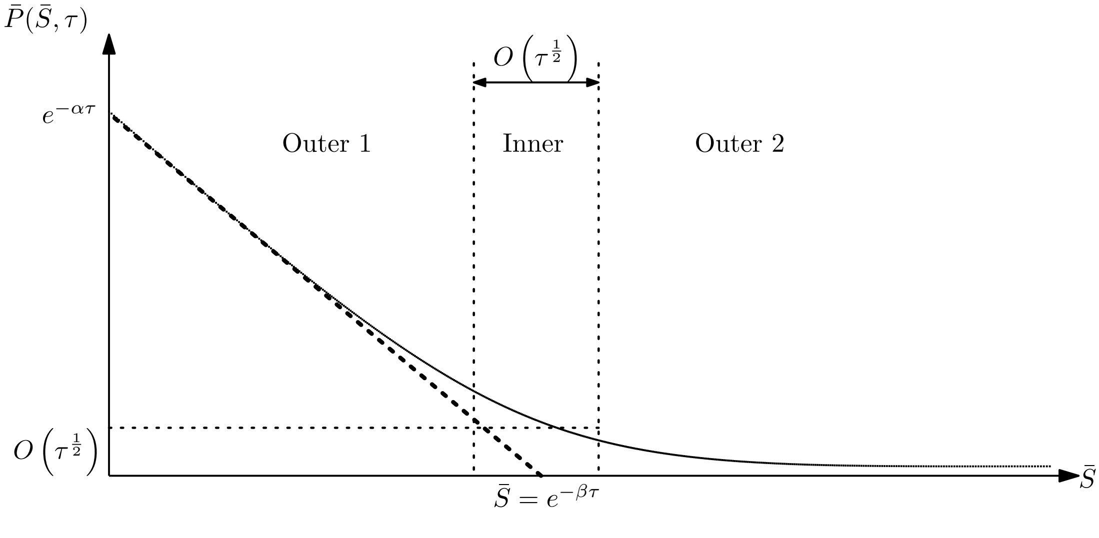
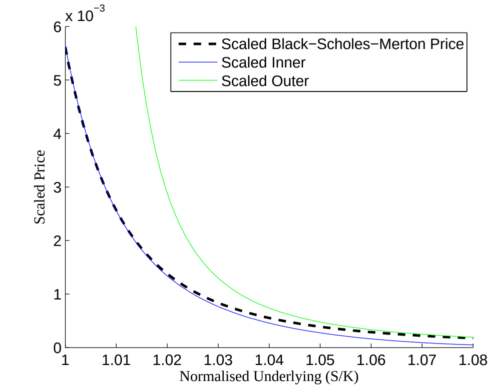
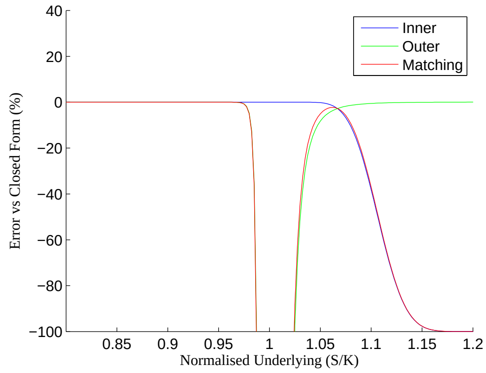


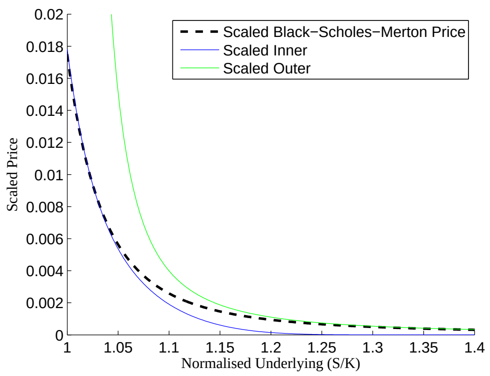
Figure 2-1: A schematic showing the small-time asymptotic structure of the European put option. The structure uses an $O(\sqrt{\tau})$ inner layer about $\bar{S} = 1$ in which the price is $O(\sqrt{\tau})$. For $\bar{S} < 1$ an outer region exists in which the price can be represented by an algebraic series in powers of τ. For $\bar{S} > 1$ an second outer region exists in which the price is exponentially small and can be shown to have the leading order asymptotic behaviour $\bar{P}_e \sim \frac{2e^{-\alpha\tau}}{\sqrt{\pi}} \frac{\tau^{\frac{3}{2}} \bar{S}^{\frac{1-\beta}{2}}}{\left(\ln(\bar{S})\right)^2} e^{-\frac{(\ln(\bar{S}))^2}{4\tau}}$.



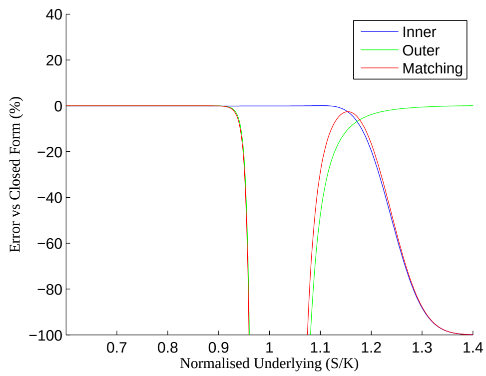
(a) Option Price $\tau = 0.0001$



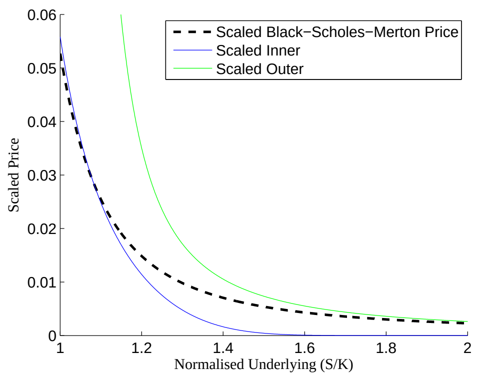
(b) Error $\tau = 0.0001$



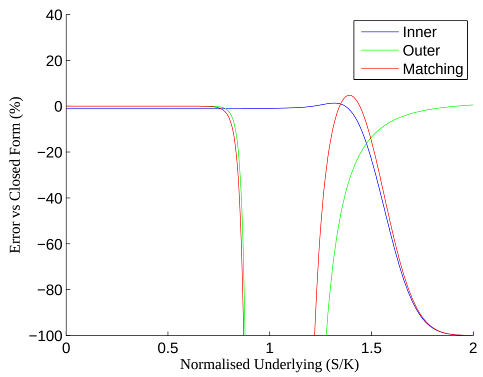
(c) Option Price $\tau = 0.001$



(d) Error $\tau = 0.001$



(e) Option Price $\tau = 0.01$



(f) Error $\tau = 0.01$

Figure 2-2: Comparison of the small-time behaviour of the Black-Scholes-Merton price for the European put option with the derived inner (2.62), outer (2.85-2.86) and matching (2.84a-2.84b) expressions for a range of values of τ and with $\alpha = \frac{10}{9}$ and $\beta = \frac{2}{3}$. The prices in Figures 2.2(a), 2.2(c) & 2.2(e) are scaled by the exponential factor $\exp\left(\left(\ln\left(\bar{S}\right)\right)^2/4\tau\right)$ and a change in the asymptotic behaviour of the Black-Scholes-Merton price from the inner expression to the outer expression for large $\bar{S}$ can be observed. The error plotted in Figures 2.2(b), 2.2(d) & 2.2(f) is defined as the difference between the value of the relevant expression and the Black-Scholes-Merton price, expressed as a percentage of the Black-Scholes-Merton price.

2.2 Small-Volatility Behaviour

The scenario in which the variance of returns is small compared to the risk-free interest rate and the cost-of-carry is investigated by Widdicks et al [102], though in the absence of dividends and without the investigation of outer terms beyond the trivial lower bound. We can represent this case through the scalings of our non-dimensional variables (2.1) $\tau = \epsilon^2 \hat{T}$, $\alpha = \frac{\hat{\alpha}}{\epsilon^2}$ and $\beta = \frac{\hat{\beta}}{\epsilon^2}$, where ϵ is a small artificial parameter ($0 < \epsilon \ll 1$) and $\hat{\alpha}, \hat{\beta} = O(1)$. Much of the approach used in this section is similar to that used in the previous section and we therefore omit some of the detail and refer to the previous section where relevant.

The non-dimensional European put option problem (2.2a-2.2d) under our small volatility scalings is

$$\bar{S} \in (0,\infty), \hat{T} \in (0,\infty) \qquad \frac{1}{\epsilon^2}\frac{\partial \bar{P}_e}{\partial \hat{T}} = \bar{S}^2 \frac{\partial^2 \bar{P}_e}{\partial \bar{S}^2} + \frac{\hat{\beta}}{\epsilon^2}\bar{S}\frac{\partial \bar{P}_e}{\partial \bar{S}} - \frac{\hat{\alpha}}{\epsilon^2}\bar{P}_e, \tag{2.95a}$$

subject to

$$\text{as } \bar{S} \to 0 \qquad \bar{P}_e(\bar{S},\hat{T}) \to e^{-\hat{\alpha}\hat{T}}, \tag{2.95b}$$

$$\text{as } \bar{S} \to \infty \qquad \bar{P}_e(\bar{S},\hat{T}) \to 0, \tag{2.95c}$$

$$\text{at } \hat{T} = 0 \qquad \bar{P}_e(\bar{S},0) = (1-\bar{S})^+. \tag{2.95d}$$

This problem has a three-region structure with two outer regions for $e^{\hat{\beta}\hat{T}}\bar{S} < 1$ and $e^{\hat{\beta}\hat{T}}\bar{S} > 1$ and an inner region for $e^{\hat{\beta}\hat{T}}\bar{S} = 1 + \epsilon\hat{S}$, where $\hat{S} = O(1)$.

Outer Regions

For $e^{\hat{\beta}\hat{T}}\bar{S} < 1$ we write $\bar{P}_e(\bar{S},\hat{T}) = \bar{P}_e^{Out1}(\bar{S},\hat{T})$ and we have the outer 1 problem

$$\bar{S} \in (0,1), \hat{T} \in (0,\infty) \qquad \frac{1}{\epsilon^2}\frac{\partial \bar{P}_e^{Out1}}{\partial \hat{T}} = \bar{S}^2 \frac{\partial^2 \bar{P}_e^{Out1}}{\partial \bar{S}^2} + \frac{\hat{\beta}}{\epsilon^2}\bar{S}\frac{\partial \bar{P}_e^{Out1}}{\partial \bar{S}} - \frac{\hat{\alpha}}{\epsilon^2}\bar{P}_e^{Out1}, \tag{2.96a}$$

subject to

$$\text{as } \bar{S} \to 0 \qquad \bar{P}_e^{Out1}(\bar{S},\hat{T}) \to e^{-\hat{\alpha}\hat{T}}, \tag{2.96b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{P}_e^{Out1}(\bar{S},0) = 1-\bar{S}. \tag{2.96c}$$

Posing a regular expansion in even powers of ϵ yields the outer 1 expansion

$$\bar{P}_e^{Out1}(\bar{S},\hat{T};\epsilon) = e^{-\hat{\alpha}\hat{T}} - \bar{S}e^{-(\hat{\alpha}-\hat{\beta})\hat{T}} \qquad \text{as } \epsilon \to 0. \tag{2.97}$$

For $e^{\hat{\beta}\hat{T}}\bar{S} > 1$ we write $\bar{P}_e(\bar{S},\hat{T}) = \bar{P}_e^{Out2}(\bar{S},\hat{T})$ and we have the outer 2 problem

$$\bar{S} \in (1,\infty), \hat{T} \in (0,\infty) \qquad \frac{1}{\epsilon^2}\frac{\partial \bar{P}_e^{Out2}}{\partial \hat{T}} = \bar{S}^2\frac{\partial^2 \bar{P}_e^{Out2}}{\partial \bar{S}^2} + \frac{\hat{\beta}}{\epsilon^2}\bar{S}\frac{\partial \bar{P}_e^{Out2}}{\partial \bar{S}} - \frac{\hat{\alpha}}{\epsilon^2}\bar{P}_e^{Out2}, \tag{2.98a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{P}_e^{Out2}(\bar{S},\hat{T}) \to 0, \tag{2.98b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{P}_e^{Out2}(\bar{S},0) = 0. \tag{2.98c}$$

Posing a regular expansion in even powers of ϵ yields the trivial the trivial outer 2 expansion

$$\bar{P}_e^{Out2}(\bar{S},\hat{T};\epsilon) = 0 \qquad \text{as } \epsilon \to 0, \tag{2.99}$$

to all orders of ϵ. The results (2.97) and (2.99) are given by Widdicks et al [102] as the asymptotic behaviour of the small-volatility European put option price in the outer regions. This is a trivial result which can be obtained as a trivial lower bound from no-arbitrage arguments. Our small-time work suggests we can do better, and we look to obtain the non-trivial leading order behaviour in the outer 2 region using a WKBJ type expansion of the form

$$\bar{P}_e(\bar{S},\hat{T};\epsilon) = \epsilon^q \bar{A}_0^{Out2}(\bar{S},\hat{T})\left(1 + \epsilon^2 \bar{A}_1^{Out2}(\bar{S},\hat{T}) + O(\epsilon^4)\right)e^{-\frac{a(\bar{S},\hat{T})}{\epsilon^2}} \qquad \text{as } \epsilon \to 0, \tag{2.100}$$

which gives the following subproblems: for $a(\bar{S},\hat{T})$

$$\bar{S} \in (1,\infty), \hat{T} \in (0,\infty) \qquad \frac{\partial a}{\partial \hat{T}} + \bar{S}^2\left(\frac{\partial a}{\partial \bar{S}}\right)^2 - \hat{\beta}\bar{S}\frac{\partial a}{\partial \bar{S}} = 0, \tag{2.101a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad a \to \infty, \tag{2.101b}$$

$$\text{at } \hat{T} = 0 \qquad a = \infty; \tag{2.101c}$$

for $\bar{A}_0^{Out2}(\bar{S},\hat{T})$

$$\bar{S}\in(1,\infty),\hat{T}\in(0,\infty) \qquad \frac{\partial \bar{A}_0^{Out2}}{\partial \hat{T}} = \bar{S}\left[\hat{\beta}-2\bar{S}\frac{\partial a}{\partial \bar{S}}\right]\frac{\partial \bar{A}_0^{Out2}}{\partial \bar{S}} - \left[\hat{\alpha}+\bar{S}^2\frac{\partial^2 a}{\partial \bar{S}^2}\right]\bar{A}_0^{Out2}, \tag{2.102a}$$

subject to

$$\text{as } \bar{S}\to\infty \qquad \bar{A}_0^{Out2}(\bar{S},\hat{T})\to 0, \tag{2.102b}$$

$$\text{at } \hat{T}=0 \qquad \bar{A}_0^{Out2}(\bar{S},0)=0; \tag{2.102c}$$

and for $\bar{A}_1^{Out2}(\bar{S},\hat{T})$

$$\bar{S}\in(1,\infty),\hat{T}\in(0,\infty) \qquad \frac{\partial \bar{A}_1^{Out2}}{\partial \hat{T}} = \bar{S}\left[\hat{\beta}-2\bar{S}\frac{\partial a}{\partial \bar{S}}\right]\frac{\partial \bar{A}_1^{Out2}}{\partial \bar{S}} + \frac{\bar{S}^2}{\bar{A}_0^{Out2}}\frac{\partial^2 \bar{A}_0^{Out2}}{\partial \bar{S}^2}, \tag{2.103a}$$

subject to

$$\text{as } \bar{S}\to\infty \qquad \bar{A}_1^{Out2}(\bar{S},\hat{T})\to 0, \tag{2.103b}$$

$$\text{at } \hat{T}=0 \qquad \bar{A}_1^{Out1}(\bar{S},0)=0. \tag{2.103c}$$

We can again use Charpit's method or look for the relevant first order nonlinear PDE solution in [91] to show that (2.101a) has the complete integral

$$a(\bar{S},\hat{T}) = C_1\ln(e^{\hat{\beta}\hat{T}}\bar{S}) - C_1^2\hat{T} + C_2, \tag{2.104}$$

where C_1 and C_2 are arbitrary constants. Again, the complete integral (2.104) fails to satisfy the boundary conditions and we find a singular solution with no arbitrary constants which has the form

$$a(\bar{S},\hat{T}) = \frac{\left(\ln\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)\right)^2}{4\hat{T}} \tag{2.105}$$

and we observe that this satisfies the boundary conditions. $\bar{A}_0^{Out2}(\bar{S},\hat{T})$ is then the solution to

$$\bar{S}\in(1,\infty),\hat{T}\in(0,\infty) \qquad \frac{\partial \bar{A}_0^{Out2}}{\partial \hat{T}} = \left(\hat{\beta}\bar{S} - \frac{\bar{S}}{\hat{T}}\ln(e^{\hat{\beta}\hat{T}}\bar{S})\right)\frac{\partial \bar{A}_0^{Out2}}{\partial \bar{S}} - \left(\hat{\alpha}+\frac{1}{2\hat{T}}\left(1-\ln(e^{\hat{\beta}\hat{T}}\bar{S})\right)\right)\bar{A}_0^{Out2}, \tag{2.106a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{A}_0^{Out2}(\bar{S}, \hat{T}) \to 0, \tag{2.106b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{A}_0^{Out2}(\bar{S}, 0) = 0, \tag{2.106c}$$

which is a first order linear PDE with general solution found via the method of characteristics to be

$$\bar{A}_0^{Out2}(\bar{S}, \hat{T}) = \Phi_0\left(\frac{\hat{T}}{\ln\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)}\right) e^{-\hat{\alpha}\hat{T}} \frac{\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)^{\frac{1}{2}}}{\left(\ln\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)\right)^{\frac{1}{2}}}, \tag{2.107}$$

for some arbitrary function $\Phi_0\left(\frac{\hat{T}}{\ln\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)}\right)$. The asymptotic behaviour of the outer 2 solution is therefore of the form

$$\bar{P}_e^{Out2}(\bar{S}, \hat{T}; \epsilon) \sim \Phi_0\left(\frac{\epsilon^2\hat{T}}{\ln\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)}\right) \frac{e^{-\hat{\alpha}\hat{T}}\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)^{\frac{1}{2}}}{\left(\ln\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)\right)^{\frac{1}{2}}} e^{-\frac{\left(\ln\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)\right)^2}{4\epsilon^2\hat{T}}} \left(1 + \epsilon^2\bar{A}_1^{Out2}(\bar{S}, \hat{T}) + O(\epsilon^4)\right) \text{ as } \epsilon \to 0, \tag{2.108}$$

where the problem for the second term in the WKBJ expansion again depends on an unknown function Φ_0.
Though we omit the details here, the same approach can be used to identify the asymptotic behaviour to the outer 1 problem as

$$\begin{aligned} \bar{P}_e^{Out1}(\bar{S}, \hat{T}; \epsilon) \sim & e^{-\hat{\alpha}\hat{T}} - \bar{S}e^{-(\hat{\alpha}-\hat{\beta})\hat{T}} + \\ & \Phi_0\left(\frac{\epsilon^2\hat{T}}{\ln\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)}\right) \frac{e^{-\hat{\alpha}\hat{T}}\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)^{\frac{1}{2}}}{\left(\ln\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)\right)^{\frac{1}{2}}} e^{-\frac{\left(\ln\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)\right)^2}{4\epsilon^2\hat{T}}} \left(1 + \epsilon^2\bar{A}_1^{Out1}(\bar{S}, \hat{T}) + O(\epsilon^4)\right) \text{ as } \epsilon \to 0, \end{aligned} \tag{2.109}$$

where the governing equation for $\bar{A}_1^{Out1}(\bar{S}, \hat{T})$ will be the same as for $\bar{A}_1^{Out2}(\bar{S}, \hat{T})$ but subject to the relevant boundary conditions in the outer 1 region.

Inner Region

For the small volatility problem, we define an inner region through the scaling $e^{\hat{\beta}\hat{T}}\bar{S} = 1 + \epsilon\hat{S}$, $\hat{S} = O(1)$. Letting $\bar{P}_e(\bar{S}, \hat{T}; \epsilon) = \epsilon\hat{P}_e^{In}(\hat{S}, \hat{T}; \epsilon)$

gives the inner problem

$$\hat{S} \in (-\infty, \infty), \hat{T} \in (0, \infty) \qquad \frac{\partial \hat{P}_e^{In}}{\partial \hat{T}} = \left(1 + \epsilon \hat{S}\right)^2 \frac{\partial^2 \hat{P}_e^{In}}{\partial \hat{S}^2} - \hat{\alpha} \hat{P}_e^{In}, \tag{2.110a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_e^{In}(\hat{S}, 0) = \left(-\hat{S}\right)^+. \tag{2.110b}$$

Posing a regular expansion in powers of ϵ

$$\hat{P}_e^{In}(\hat{S}, \hat{T}; \epsilon) = e^{-\hat{\alpha}\hat{T}} \left(\hat{P}_0^{In}(\hat{S}, \hat{T}) + \epsilon \hat{P}_1^{In}(\hat{S}, \hat{T}) + \epsilon^2 \hat{P}_2^{In}(\hat{S}, \hat{T}) + O(\epsilon^3)\right) \qquad \text{as } \epsilon \to 0, \tag{2.111}$$

gives the following subproblems: for $\hat{P}_0^{In}(\hat{S}, \hat{T})$

$$\hat{S} \in (-\infty, \infty), \hat{T} \in (0, \infty) \qquad \frac{\partial \hat{P}_0^{In}}{\partial \hat{T}} = \frac{\partial^2 \hat{P}_0^{In}}{\partial \hat{S}^2}, \tag{2.112a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_0^{In}(\hat{S}, 0) = \left(-\hat{S}\right)^+; \tag{2.112b}$$

for $\hat{P}_1^{In}(\hat{S}, \hat{T})$

$$\hat{S} \in (-\infty, \infty), \hat{T} \in (0, \infty) \qquad \frac{\partial \hat{P}_1^{In}}{\partial \hat{T}} = \frac{\partial^2 \hat{P}_1^{In}}{\partial \hat{S}^2} + 2\hat{S} \frac{\partial^2 \hat{P}_0^{In}}{\partial \hat{S}^2}, \tag{2.113a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_1^{In}(\hat{S}, 0) = 0; \tag{2.113b}$$

and for $\hat{P}_2^{In}(\hat{S}, \hat{T})$

$$\hat{S} \in (-\infty, \infty), \hat{T} \in (0, \infty) \qquad \frac{\partial \hat{P}_2^{In}}{\partial \hat{T}} = \frac{\partial^2 \hat{P}_2^{In}}{\partial \hat{S}^2} + 2\hat{S} \frac{\partial^2 \hat{P}_1^{In}}{\partial \hat{S}^2} + \hat{S}^2 \frac{\partial^2 \hat{P}_0^{In}}{\partial \hat{S}^2}, \tag{2.114a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_2^{In}(\hat{S}, 0) = 0. \tag{2.114b}$$

As in the small-time case, though subject to a different spatial scaling, the problem (2.34a-2.34b) has the similarity solution $\hat{P}_0^{In}(\hat{S},\hat{T}) = \sqrt{\hat{T}}h_0(\zeta)$ where $\zeta = \frac{\hat{S}}{2\sqrt{\hat{T}}}$ and where $h_0(\zeta)$ is the solution to

$$h_0^{''} + 2\zeta h_0^{'} - 2h_0 = 0, \tag{2.115a}$$

subject to

$$\text{as } \zeta \to \infty \qquad \hat{T}^{\frac{1}{2}}h_0(\zeta) \to 0, \tag{2.115b}$$

$$\text{as } \zeta \to -\infty \qquad h_0(\zeta) \sim -2\zeta. \tag{2.115c}$$

This has the general solution

$$h_0(\zeta) = 2C_{01}\zeta + C_{02}\text{ierfc}(\zeta), \tag{2.116}$$

where $\text{ierfc}(\zeta)$ is given by (2.39) and C_{01} and C_{02} are arbitrary constants determined using the asymptotic behaviour

$$h_0(\zeta) \sim \begin{cases} 2C_{01}\zeta + O\left(\zeta^{-2}e^{-\zeta^2}\right) & \text{as } \zeta \to \infty, \\ 2(C_{01} - C_{02})\zeta + O\left(\zeta^{-2}e^{-\zeta^2}\right) & \text{as } \zeta \to -\infty, \end{cases} \tag{2.117}$$

which along with conditions (2.115b) and (2.115c) leads to

$$C_{01} = 0, \qquad C_{02} = 1 \tag{2.118}$$

and therefore

$$h_0(\zeta) = \frac{1}{\sqrt{\pi}}e^{-\zeta^2} - \zeta\text{erfc}(\zeta). \tag{2.119}$$

The problem (2.113a-2.113b) has the similarity solution $\hat{P}_1^{In}(\hat{S},\hat{T}) = \hat{T}h_1(\zeta)$ where $h_1(\zeta)$ is the solution to

$$h_1^{''} + 2\zeta h_1^{'} - 4h_1 = -4\zeta h_0^{''}, \tag{2.120a}$$

subject to

$$\text{as } \zeta \to \infty \qquad \hat{T} h_1(\zeta) \to 0, \tag{2.120b}$$

$$\text{as } \zeta \to -\infty \qquad \hat{T} h_1(\zeta) \to 0. \tag{2.120c}$$

This has the general solution [1]

$$h_1(\zeta) = C_{11}(2\zeta^2 + 1) + 2C_{12}\mathrm{i}^2\mathrm{erfc}(\zeta) + \frac{\zeta}{\sqrt{\pi}} e^{-\zeta^2}, \tag{2.121}$$

where $\mathrm{i}^2\mathrm{erfc}(\zeta)$ is given by (2.50) and C_{11} and C_{12} are arbitrary constants determined using the asymptotic behaviour

$$h_1(\zeta) \sim \begin{cases} C_{21}\left(2\zeta^2 + 1\right) + O\left(\zeta e^{-\zeta^2}\right) & \text{as } \zeta \to \infty, \\ \left(C_{11} + C_{12}\right)\left(2\zeta^2 + 1\right) + O\left(\zeta e^{-\zeta^2}\right) & \text{as } \zeta \to -\infty, \end{cases} \tag{2.122}$$

which, along with conditions (2.120b) and (2.120c), leads to

$$C_{11} = C_{12} = 0 \tag{2.123}$$

and therefore

$$h_1(\zeta) = \frac{\zeta}{\sqrt{\pi}} e^{-\zeta^2}. \tag{2.124}$$

The problem (2.114a-2.114b) has the similarity solution $\hat{P}_2^{In}(\hat{S}, \hat{T}) = \hat{T}^{\frac{3}{2}} h_2(\zeta)$ where $h_2(\zeta)$ is the solution to

$$h_2^{''} + 2\zeta h_2^{'} - 6h_1 = -4\zeta h_1^{''} - 4\zeta^2 h_0^{''}, \tag{2.125a}$$

subject to

$$\text{as } \zeta \to \infty \qquad \hat{T}^{\frac{3}{2}} h_2(\zeta) \to 0, \tag{2.125b}$$

$$\text{as } \zeta \to -\infty \qquad \hat{T}^{\frac{3}{2}} h_2(\zeta) \to 0. \tag{2.125c}$$

This has the general solution [1]

$$h_2(\zeta) = C_{21}\frac{1}{3}\left(\zeta^3 + \frac{3}{2}\zeta\right) + C_{22}\mathrm{i}^3\mathrm{erfc}(\zeta) + \frac{e^{-\zeta^2}}{\sqrt{\pi}}\left(\zeta^4 - \frac{\zeta^2}{3} + \frac{1}{12}\right), \tag{2.126}$$

where $\mathrm{i}^3\mathrm{erfc}(\zeta)$ is given by (2.57) and C_{21} and C_{22} are arbitrary constants which are determined using the asymptotic behaviour

$$h_2(\zeta) \sim \begin{cases} C_{21}\frac{1}{3}\left(\zeta^3 + \frac{3}{2}\zeta\right) + O\left(\zeta^4 e^{-\zeta^2}\right) & \text{as } \zeta \to \infty, \\ (C_{21} - C_{22})\frac{1}{3}\left(\zeta^3 + \frac{3}{2}\zeta\right) + O\left(\zeta^4 e^{-\zeta^2}\right) & \text{as } \zeta \to -\infty, \end{cases} \tag{2.127}$$

which, along with the conditions (2.125b) and (2.125c), leads to

$$C_{21} = C_{22} = 0 \tag{2.128}$$

and therefore

$$h_2(\zeta) = \frac{e^{-\zeta^2}}{\sqrt{\pi}}\left(\zeta^4 - \frac{\zeta^2}{3} + \frac{1}{12}\right). \tag{2.129}$$

Hence we can write the small volatility inner series in terms of the inner variable $\hat{S}$ as

$$\hat{P}^{In}(\hat{S}, \hat{T}; \epsilon) = \epsilon\sqrt{\hat{T}}h_0\left(\frac{\hat{S}}{2\sqrt{\hat{T}}}\right) + \epsilon^2\hat{T}h_1\left(\frac{\hat{S}}{2\sqrt{\hat{T}}}\right) + \epsilon^3\hat{T}^{\frac{3}{2}}h_2\left(\frac{\hat{S}}{2\sqrt{\hat{T}}}\right) + O(\epsilon^4) \qquad \text{as } \epsilon \to 0, \tag{2.130}$$

where

$$h_0(\zeta) = \frac{1}{\sqrt{\pi}}e^{-\zeta^2} - \zeta\mathrm{erfc}(\zeta), \tag{2.131a}$$

$$h_1(\zeta) = \frac{\zeta}{\sqrt{\pi}}e^{-\zeta^2}, \tag{2.131b}$$

$$h_2(\zeta) = \frac{e^{-\zeta^2}}{\sqrt{\pi}}\left(\zeta^4 - \frac{\zeta^2}{3} + \frac{1}{12}\right). \tag{2.131c}$$

We note here that our choice of inner scaling ($e^{\hat{\beta}\hat{T}}\bar{S} = 1 + \epsilon\hat{S}$) differs from the equivalent expression of Widdicks et al. [102] if dividends were included ($\bar{S} = e^{-\hat{\beta}\hat{T}} + \epsilon\hat{S}$). This results in a series of problems which have simple similarity solutions which do not require numerical schemes such as the Crank-Nicolson scheme used therein.

Matching Region

For the purpose of matching, we define a matching region through the introduction of an intermediate variable $\tilde{S}$ such that $e^{\hat{\beta}\hat{T}}\bar{S} = 1 + \epsilon^n \tilde{S}$ and $\hat{S} = \epsilon^{n-1}\tilde{S}$ where $0 < n < 1$. We define the contribution of the inner series in the matching region as $\hat{P}_e^{In}(\hat{S}, \hat{T}; \epsilon) = \tilde{P}_e^{In}(\tilde{S}, \hat{T}; \epsilon)$.
Using (2.43) we can write the first three terms of the inner series in terms of the intermediate variable $\tilde{S}$ as

$$\epsilon\sqrt{\hat{T}}h_0\left(\frac{\epsilon^{n-1}\tilde{S}}{2\sqrt{\hat{T}}}\right) \sim \begin{cases} \frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\tilde{\psi}_0(\tilde{S},\hat{T}) & \text{as } \epsilon \to 0, \tilde{S} > 0, \\ -\epsilon^n\tilde{S} + \frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\tilde{\psi}_0(\tilde{S},\hat{T}) & \text{as } \epsilon \to 0, \tilde{S} < 0, \end{cases} \tag{2.132a}$$

where

$$\tilde{\psi}_0(\tilde{S},\hat{T}) = \epsilon^{3-2n}\frac{2\hat{T}^{\frac{3}{2}}}{\tilde{S}^2} - \epsilon^{5-4n}\frac{12\hat{T}^{\frac{5}{2}}}{\tilde{S}^4} + \epsilon^{7-6n}\frac{120\hat{T}^{\frac{7}{2}}}{\tilde{S}^6} + O(\epsilon^{9-8n}), \tag{2.132b}$$

and

$$\epsilon^2\hat{T}h_1\left(\frac{\epsilon^{n-1}\tilde{S}}{2\sqrt{\hat{T}}}\right) = \frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\left(\epsilon^{n+1}\frac{\tilde{S}\hat{T}^{\frac{1}{2}}}{2}\right) \qquad \text{as } \epsilon \to 0, \tag{2.133}$$

$$\epsilon^3\hat{T}^{\frac{3}{2}}h_2\left(\frac{\epsilon^{n-1}\tilde{S}}{2\sqrt{\hat{T}}}\right) = \frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\left(\epsilon^{4n-1}\frac{\tilde{S}^4}{16\hat{T}^{\frac{1}{2}}} - \epsilon^{2n+1}\frac{\tilde{S}^2\hat{T}^{\frac{1}{2}}}{12} + \epsilon^3\frac{\hat{T}^{\frac{3}{2}}}{12}\right) \qquad \text{as } \epsilon \to 0. \tag{2.134}$$

As with the small-time problem, the series order breaks down for $n \leq \frac{2}{3}$ and is singular as $\tilde{S} \to 0$ ($\bar{S} \to 1$).
To find the outer expressions in terms of the intermediate variable we expand the terms of (2.108) and we write the unknown function Φ_0 as a Taylor series

$$\Phi_0\left(\frac{\epsilon^2\hat{T}}{\ln(1+\epsilon^n\tilde{S})}\right) \simeq \Phi_0\left(\epsilon^{2-n}\frac{\hat{T}}{\tilde{S}}\right) - \left(\epsilon^2\hat{T} + \dots\right)\Phi_0'\left(\epsilon^{2-n}\frac{\hat{T}}{\tilde{S}}\right) \qquad \text{as } \epsilon \to 0, \tag{2.135}$$

which produces the leading order behaviour in the matching region for $\tilde{S} > 0$

$$\tilde{P}_e^{Out2}(\tilde{S},\hat{T}) \sim \frac{1}{(\epsilon\tilde{S})^{\frac{n}{2}}}\Phi_0\left(\epsilon^{2-n}\frac{\hat{T}}{\tilde{S}}\right)e^{-\epsilon^{2n-2}\frac{\tilde{S}^2}{4\hat{T}}}e^{-\epsilon^{3n-2}\frac{\tilde{S}^3}{4\hat{T}}} \qquad \text{as } \epsilon \to 0 \tag{2.136}$$

and for $\tilde{S} < 0$

$$\tilde{P}_e^{Out1}(\tilde{S},\hat{T}) \sim e^{-\hat{\alpha}\hat{T}} - \bar{S}e^{-(\hat{\alpha}-\hat{\beta})\hat{T}} + \frac{1}{(\epsilon\tilde{S})^{\frac{n}{2}}}\Phi_0\left(\epsilon^{2-n}\frac{\hat{T}}{\tilde{S}}\right)e^{-\epsilon^{2n-2}\frac{\tilde{S}^2}{4\hat{T}}}e^{-\epsilon^{3n-2}\frac{\tilde{S}^3}{4\hat{T}}} \quad \text{as } \epsilon \to 0. \tag{2.137}$$

Matching the form of (2.136) and (2.137) with the leading order inner matching condition from (2.132a-2.132b) identifies that matching of the exponential terms is only possible for $n > \frac{2}{3}$, which is consistent with our observation from the inner series, and suggests the unknown function Φ has the form

$$\Phi_0\left(\frac{\epsilon^2\hat{T}}{\ln\left(e^{b\hat{T}}\bar{S}\right)}\right) = \frac{2}{\sqrt{\pi}}\left(\frac{\epsilon^2\hat{T}}{\ln\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)}\right)^{\frac{3}{2}}. \tag{2.138}$$

We can therefore rewrite (2.107) as

$$\bar{A}_0^{Out2}(\bar{S},\hat{T}) = \bar{A}_0^{Out1}(\bar{S},\hat{T}) = e^{-\hat{\alpha}\hat{T}}\frac{2\epsilon^3\hat{T}^{\frac{3}{2}}}{\sqrt{\pi}}\frac{\left(e^{\hat{\beta}\hat{T}}\bar{S}\right)^{\frac{1}{2}}}{(\ln\left(e^{\hat{\beta}\hat{T}}\bar{S}\right))^2}, \tag{2.139}$$

with the behaviour of $\bar{A}_0^{Out2}(\bar{S},\hat{T})$ and $\bar{A}_0^{Out1}(\bar{S},\hat{T})$ for $n \in (\frac{2}{3}, 1)$ given by

$$\tilde{A}_0^{Out2}(\tilde{S},\hat{T}) = \tilde{A}_0^{Out1}(\tilde{S},\hat{T}) \sim \frac{e^{-\hat{\alpha}\hat{T}}}{\sqrt{\pi}}\left(\epsilon^{3-2n}\frac{2\hat{T}^{\frac{3}{2}}}{\tilde{S}^2} + \epsilon^{3-n}\frac{3\hat{T}^{\frac{3}{2}}}{\tilde{S}} + O(\epsilon^3)\right) \quad \text{as } \epsilon \to 0. \tag{2.140}$$

Therefore the contribution of the outer expressions to the matching region become, for $\tilde{S} > 0$

$$\tilde{P}_e^{Out2}(\tilde{S},\hat{T}) \sim \frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\left(\epsilon^{3-2n}\frac{2\hat{T}^{\frac{3}{2}}}{\tilde{S}^2} + \epsilon^{n+1}\frac{\hat{T}^{\frac{1}{2}}\tilde{S}}{2} + \epsilon^{4n-1}\frac{\tilde{S}^4}{16\hat{T}^{\frac{1}{2}}} + \epsilon^{3-n}\frac{3\hat{T}^{\frac{3}{2}}}{\tilde{S}}\right.$$
$$\left. + O(\epsilon^{2n+1})\right) \quad \text{as } \epsilon \to 0 \tag{2.141}$$

and for $\tilde{S} < 0$

$$\tilde{P}_e^{Out1}(\tilde{S},\hat{T}) \sim e^{-\hat{\alpha}\hat{T}} - \bar{S}e^{-(\hat{\alpha}-\hat{\beta})\hat{T}} + \frac{1}{\sqrt{\pi}}e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}}\left(\epsilon^{3-2n}\frac{2\hat{T}^{\frac{3}{2}}}{\tilde{S}^2} + \epsilon^{n+1}\frac{\hat{T}^{\frac{1}{2}}\tilde{S}}{2} + \epsilon^{4n-1}\frac{\tilde{S}^4}{16\hat{T}^{\frac{1}{2}}}\right.$$
$$\left. + \epsilon^{3-n}\frac{3\hat{T}^{\frac{3}{2}}}{\tilde{S}} + O(\epsilon^{2n+1})\right) \quad \text{as } \epsilon \to 0, \tag{2.142}$$

with the relative magnitude of the $O(\epsilon^{4n-1})$ and $O(\epsilon^{3-n})$ terms dependent whether n is greater or less than $\frac{4}{5}$.

A comparison of the outer (2.141-2.142) and inner (2.132a-2.132b,2.133-2.134) matching conditions reveals a term of $O\left(\epsilon^{5-4n}\right)$ missing from the outer expressions. For $n > \frac{2}{3}$ this term dominates the $O\left(\epsilon^{3-n}\right)$ term and therefore these singular terms cannot be matched without finding the next term in the outer expansions, $\bar{A}_1^{Out1}(\bar{S},\hat{T})$ and $\bar{A}_1^{Out2}(\bar{S},\hat{T})$.

Using the behaviour of Φ determined during matching at leading order (2.138), we can rewrite problem for $\bar{A}_1^{Out2}(\bar{S},\hat{T})$ (2.103a-2.103c) as

$$\bar{S} \in (1,\infty), \hat{T} \in (0,\infty) \qquad \frac{\partial \bar{A}_1^{Out2}}{\partial \hat{T}} = \bar{S}\left[\hat{\beta} - 2\bar{S}\frac{\partial a}{\partial \bar{S}}\right]\frac{\partial \bar{A}_1^{Out2}}{\partial \bar{S}} + \left[\frac{6}{(\ln(e^{\hat{\beta}\hat{T}}\bar{S}))^2} - \frac{1}{4}\right], \tag{2.143a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{A}_1^{Out2}(\bar{S},\hat{T}) \to 0, \tag{2.143b}$$

$$\text{at } \hat{T} = 0 \qquad \bar{A}_1^{Out1}(\bar{S},0) = 0. \tag{2.143c}$$

The problem (2.143a-2.143c) is a linear first order PDE which is amenable to the method of characteristics, giving the solution

$$\bar{A}_1^{Out2}(\bar{S},\hat{T}) = -\hat{T}\left[\frac{6}{(\ln(e^{\hat{\beta}\hat{T}}\bar{S}))^2} + \frac{C_1^{Out2}\left(\ln(e^{\hat{\beta}\hat{T}}\bar{S})/\hat{T}\right)}{\ln(e^{\hat{\beta}\hat{T}}\bar{S})} + \frac{1}{4}\right], \tag{2.144}$$

for some arbitrary function $C_1^{Out2}\left(e^{\hat{\beta}\hat{T}}\ln(\bar{S})/\hat{T}\right)$. We note that the problem for $\bar{A}_1^{Out1}(\bar{S},\hat{T})$ yields the same solution as (2.144) up to an arbitrary function $C_1^{Out1}\left(e^{\hat{\beta}\hat{T}}\ln(\bar{S})/\hat{T}\right)$.

We use (2.138) and (2.144) and the fact that $\bar{A}_1^{Out1}(\bar{S},\hat{T}) = \bar{A}_1^{Out2}(\bar{S},\hat{T})$ to write the outer expressions for $\bar{S} > 1$ as

$$\bar{P}_e^{Out2}(\bar{S},\hat{T}) \sim \frac{2\epsilon^3}{\sqrt{\pi}}\frac{e^{-\hat{\alpha}\hat{T}}\hat{T}^{\frac{3}{2}}}{(\ln(e^{\hat{\beta}\hat{T}}\bar{S}))^2}e^{-\frac{\left(\ln(e^{\hat{\beta}\hat{T}}\bar{S})\right)^2}{4\epsilon^2\hat{T}}} \times \left(1 - \epsilon^2\hat{T}\left[\frac{6}{(\ln(e^{\hat{\beta}\hat{T}}\bar{S}))^2} + \frac{C_1^{Out2}}{\ln(e^{\hat{\beta}\hat{T}}\bar{S})} + \frac{1}{4}\right] + O(\epsilon^4)\right) \quad \text{as } \epsilon \to 0 \tag{2.145}$$

and for $\bar{S} < 1$ as

$$\bar{P}_e^{Out1}(\bar{S},\hat{T}) \sim e^{-\hat{\alpha}\hat{T}} - \bar{S}e^{-(\hat{\alpha}-\hat{\beta})\hat{T}} + \frac{2\epsilon^3}{\sqrt{\pi}}\frac{e^{-\hat{\alpha}\hat{T}}\hat{T}^{\frac{3}{2}}}{(\ln(e^{\hat{\beta}\hat{T}}\bar{S}))^2}e^{-\frac{\left(\ln(e^{\hat{\beta}\hat{T}}\bar{S})\right)^2}{4\epsilon^2\hat{T}}} \times \left(1 - \epsilon^2\hat{T}\left[\frac{6}{(\ln(e^{\hat{\beta}\hat{T}}\bar{S}))^2} + \frac{C_1^{Out1}}{\ln(e^{\hat{\beta}\hat{T}}\bar{S})} + \frac{1}{4}\right] + O(\epsilon^4)\right) \quad \text{as } \epsilon \to 0. \tag{2.146}$$

In terms of the intermediate variable, the contribution in the matching region of $\bar{A}_1^{Out1}(\bar{S},\hat{T})$ and $\bar{A}_1^{Out2}(\bar{S},\hat{T})$, is

$$\tilde{A}_1^{Out1} = \tilde{A}_1^{Out2} \sim \frac{1}{\sqrt{\pi}} e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}} \left(-\epsilon^{5-4n}\frac{12\hat{T}^{\frac{5}{2}}}{\tilde{S}^4} + \epsilon^{3-n}\frac{3\hat{T}^{\frac{3}{2}}}{\tilde{S}} + O(\epsilon^{5-3n},\epsilon^{2n+1}) \right) \text{ as } \epsilon \to 0. \tag{2.147}$$

We observe that this contributes the term required to match with the inner matching expression (2.132b) while determination of the arbitrary functions C_1^{Out2} and C_1^{Out1} requires matching at $O(\epsilon^{5-3n})$ in the intermediate variable which is dominated by the $O(\epsilon^{7-6n})$ term in (2.132b) which requires additional terms in the outer expansion to match and restricts the validity of our matching expression to $n \in (\frac{2}{3}, \frac{4}{5})$. As with the small-time case, the inability to match the singular terms using a truncated series prohibits the formation of a non-singular uniform expression.

We therefore define matching expressions for $\tilde{S} > 0$ as

$$\tilde{P}_e^{Match2}(\tilde{S},\hat{T}) \sim \frac{1}{\sqrt{\pi}} e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}} \tilde{\psi}(\tilde{S},\hat{T}) \qquad \text{as } \epsilon \to 0 \tag{2.148}$$

and for $\tilde{S} < 0$ as

$$\tilde{P}_e^{Match1}(\tilde{S},\hat{T}) \sim e^{-\hat{\alpha}\hat{T}} - \bar{S}e^{-(\hat{\alpha}-\hat{\beta})\hat{T}} + \frac{1}{\sqrt{\pi}} e^{-\epsilon^{2(n-1)}\frac{\tilde{S}^2}{4\hat{T}}} \tilde{\psi}(\tilde{S},\hat{T}) \qquad \text{as } \epsilon \to 0, \tag{2.149}$$

where

$$\tilde{\psi}(\tilde{S},\hat{T}) \sim \epsilon^{3-2n}\frac{2\hat{T}^{\frac{3}{2}}}{\tilde{S}^2} + \epsilon^{n+1}\frac{\hat{T}^{\frac{1}{2}}\tilde{S}}{2} + \epsilon^{4n-1}\frac{\tilde{S}^4}{16\hat{T}^{\frac{1}{2}}} - 12\epsilon^{5-4n}\frac{\hat{T}^{\frac{5}{2}}}{\tilde{S}^4} \qquad \text{as } \epsilon \to 0, \tag{2.150}$$

with the relative magnitude of the $O(\epsilon^{4n-1})$ and $O(\epsilon^{5n-4})$ terms depending on whether n is greater than or less than $\frac{3}{4}$. We note that the transformation used for the small volatility problem has removed the term at $O(\epsilon^{3-n})$ seen in the small-time problem.

For consistency, we define our final outer 1 and outer 2 expressions as

$$\bar{P}_e^{Out2}(\bar{S},\hat{T}) \sim \frac{2\epsilon^3}{\sqrt{\pi}} \frac{e^{-\hat{\alpha}\hat{T}}\hat{T}^{\frac{3}{2}}}{(\ln{(e^{\hat{\beta}\hat{T}}\bar{S})})^2} e^{-\frac{\left(\ln{(e^{\hat{\beta}\hat{T}}\bar{S})}\right)^2}{4\epsilon^2\hat{T}}} \left(1 - \epsilon^2\hat{T}\frac{6}{(\ln{(e^{\hat{\beta}\hat{T}}\bar{S})})^2}\right) \text{ as } \epsilon \to 0 \tag{2.151}$$

and for $\bar{S} < 1$ as

$$\bar{P}_e^{Out1}(\bar{S},\hat{T}) \sim e^{-\hat{\alpha}\hat{T}} - \bar{S}e^{-(\hat{\alpha}-\hat{\beta})\hat{T}} + \frac{2\epsilon^3}{\sqrt{\pi}}\frac{e^{-\hat{\alpha}\hat{T}}\hat{T}^{\frac{3}{2}}}{(\ln(e^{\hat{\beta}\hat{T}}\bar{S}))^2}e^{-\frac{\left(\ln(e^{\hat{\beta}\hat{T}}\bar{S})\right)^2}{4\epsilon^2\hat{T}}}\left(1-\epsilon^2\hat{T}\frac{6}{(\ln(e^{\hat{\beta}\hat{T}}\bar{S}))^2}\right) \text{ as } \epsilon \to 0, \quad (2.152)$$

which contain the terms successfully matched with the inner region.

Comparison with the Closed-Form Solution

We can verify that the WKBJ expansion has captured the correct behaviour by looking at the asymptotic behaviour of the European option price in the inner and outer regions. We can represent European put option price under our small-volatility scalings as

$$\bar{P}_e(\bar{S},\tau) = \frac{e^{-\hat{\alpha}\hat{T}}}{2}\left(\text{erfc}\left(\frac{\bar{d}_2}{\sqrt{2}}\right) - (e^{\hat{\beta}\hat{T}}\bar{S})\text{erfc}\left(\frac{\bar{d}_1}{\sqrt{2}}\right)\right), \quad (2.153)$$

where

$$\frac{\bar{d}_1}{\sqrt{2}} = \frac{\ln(e^{\hat{\beta}\hat{T}}\bar{S})}{2\epsilon\sqrt{\hat{T}}} + \frac{\epsilon\sqrt{\hat{T}}}{2}, \quad (2.154a)$$

$$\frac{\bar{d}_2}{\sqrt{2}} = \frac{\ln(e^{\hat{\beta}\hat{T}}\bar{S})}{2\epsilon\sqrt{\hat{T}}} - \frac{\epsilon\sqrt{\hat{T}}}{2}. \quad (2.154b)$$

In the outer regions, $\frac{\bar{d}_1}{\sqrt{2}}, \frac{\bar{d}_2}{\sqrt{2}} \to -\infty$ as $\epsilon \to 0$, for $\bar{S} > 1$, and $\frac{\bar{d}_1}{\sqrt{2}}, \frac{\bar{d}_2}{\sqrt{2}} \to \infty$ as $\epsilon \to 0$, for $\bar{S} < 1$. Using (2.43) together with (2.153) we derive the small-time behaviour in the outer regions for $\bar{S} > 1$ as

$$\bar{P}_e^{Out2}(\bar{S},\hat{T}) \sim \frac{2}{\sqrt{\pi}}\frac{e^{-\hat{\alpha}\hat{T}}\epsilon^3\hat{T}^{\frac{3}{2}}}{(\ln(e^{\hat{\beta}\hat{T}}\bar{S}))^2}e^{-\frac{\left(\ln(e^{\hat{\beta}\hat{T}}\bar{S})\right)^2}{4\epsilon^2\hat{T}}} \left(1-\epsilon^2\hat{T}\left[\frac{6}{(\ln(e^{\hat{\beta}\hat{T}}\bar{S}))^2}+\frac{1}{\ln(e^{\hat{\beta}\hat{T}}\bar{S})}+\frac{1}{4}\right]+O(\epsilon^4)\right) \quad \text{as } \epsilon \to 0 \quad (2.155)$$

and for $\bar{S} < 1$ as

$$\bar{P}_e^{Out1}(\bar{S},\hat{T}) \sim e^{-\hat{\alpha}\hat{T}} - \bar{S}e^{-(\hat{\alpha}-\hat{\beta})\hat{T}} + \frac{2}{\sqrt{\pi}}\frac{e^{-\hat{\alpha}\hat{T}}\epsilon^3\hat{T}^{\frac{3}{2}}}{(\ln(e^{\hat{\beta}\hat{T}}\bar{S}))^2}e^{-\frac{\left(\ln(e^{\hat{\beta}\hat{T}}\bar{S})\right)^2}{4\epsilon^2\hat{T}}} \times \left(1-\epsilon^2\hat{T}\left[\frac{6}{(\ln(e^{\hat{\beta}\hat{T}}\bar{S}))^2}+\frac{1}{\ln(e^{\hat{\beta}\hat{T}}\bar{S})}+\frac{1}{4}\right]+O(\epsilon^4)\right) \quad \text{as } \epsilon \to 0, \quad (2.156)$$

which we observe are consistent with the outer expansions, and further suggests that we expect $C_1^{Out1} = C_1^{Out2} = 1$.

For the inner region, defined by $e^{\hat{\beta}\hat{T}}\bar{S} = 1 + \epsilon\hat{S}$, we have

$$\frac{\hat{d}_1}{\sqrt{2}} \simeq \frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}} + \frac{\epsilon}{2}\left(\frac{\hat{S}^2}{2\hat{T}^{\frac{1}{2}}} + \hat{T}^{\frac{1}{2}}\right) + \frac{\epsilon^2}{6}\frac{\hat{S}}{\hat{T}^{\frac{1}{2}}} \qquad \text{as } \epsilon \to 0, \tag{2.157a}$$

$$\frac{\hat{d}_2}{\sqrt{2}} \simeq \frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}} + \frac{\epsilon}{2}\left(\frac{\hat{S}^2}{2\hat{T}^{\frac{1}{2}}} - \hat{T}^{\frac{1}{2}}\right) + \frac{\epsilon^2}{6}\frac{\hat{S}}{\hat{T}^{\frac{1}{2}}} \qquad \text{as } \epsilon \to 0 \tag{2.157b}$$

and we use the Taylor expansions

$$\operatorname{erfc}\left(\frac{\hat{d}_1}{\sqrt{2}}\right) \simeq \operatorname{erfc}\left(\frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}\right) + \left(\frac{\epsilon}{2}\left(\frac{\hat{S}^2}{2\hat{T}^{\frac{1}{2}}} + \hat{T}^{\frac{1}{2}}\right) + \frac{\epsilon^2}{6}\frac{\hat{S}}{\hat{T}^{\frac{1}{2}}} + \ldots\right)\operatorname{erfc}'\left(\frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}\right) \qquad \text{as } \epsilon \to 0, \tag{2.158}$$

$$\operatorname{erfc}\left(\frac{\hat{d}_2}{\sqrt{2}}\right) \simeq \operatorname{erfc}\left(\frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}\right) + \left(\frac{\epsilon}{2}\left(\frac{\hat{S}^2}{2\hat{T}^{\frac{1}{2}}} - \hat{T}^{\frac{1}{2}}\right) + \frac{\epsilon^2}{6}\frac{\hat{S}}{\hat{T}^{\frac{1}{2}}} + \ldots\right)\operatorname{erfc}'\left(\frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}\right) \qquad \text{as } \epsilon \to 0, \tag{2.159}$$

to derive the inner behaviour

$$\hat{P}_e^{In}(\bar{S},\tau) \sim \frac{\epsilon}{2}\left(-\hat{S}\operatorname{erfc}\left(\frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}\right) - \frac{\hat{T}^{\frac{1}{2}}}{\sqrt{\pi}}e^{-\frac{\hat{S}^2}{4\hat{T}}}\right) - \frac{\epsilon^2}{2}\frac{\hat{S}\hat{T}^{\frac{1}{2}}}{\sqrt{\pi}}e^{-\frac{\hat{S}^2}{4\hat{T}}} + O(\epsilon^4) \qquad \text{as } \epsilon \to 0, \tag{2.160}$$

which can be shown to be equivalent to (2.130) using the substitution $\zeta = \frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}$ and the results (2.131a) and (2.131b).

The three term structure for the small-volatility problem is illustrated in Figure 2-3 and a numerical comparison of the outer (2.145-2.146), inner (2.130) and matching (2.148-2.149) expressions with the closed-form solution (2.153) are given in Figure 2-4. As with the small-time case, the Error graphs show that the inner expression goes to zero much faster than the closed-form solution, whereas the outer expression captures the asymptotic behaviour in this limit. We could also demonstrate that the assumption made by Widdicks et al. [102] and Firth et al. [42] that the inner expansion for the European call option captures the correct outer region behavior, does not hold in the out-of-the-money outer region $e^{\hat{\beta}\hat{T}}\bar{S} < 1$. The breakdown of the outer expansion near $\bar{S} = e^{-\hat{\beta}\hat{T}}$ is also seen.

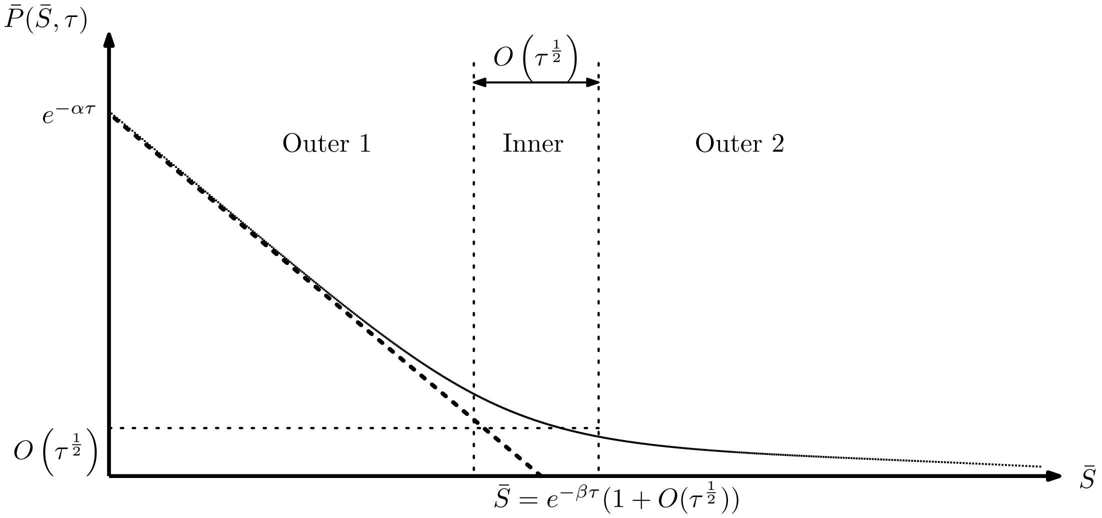


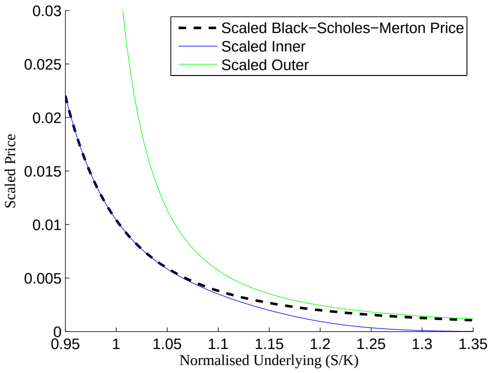
Figure 2-3: A schematic showing the small-volatility asymptotic structure of the European put option. The structure uses an $O(\sqrt{\tau})$ inner region about $e^{\beta\tau}\bar{S} = 1$ in which the price is $O(\sqrt{\tau})$. For $e^{\beta\tau}\bar{S} < 1$ an outer region exists in which the price can be represented by an algebraic series in powers of τ. For $e^{\beta\tau}\bar{S} > 1$ an second outer region exists in which the price is exponentially small and can be shown to have the leading order asymptotic behaviour $\bar{P}_e \sim \frac{2e^{-\alpha\tau}}{\sqrt{\pi}} \frac{\tau^{\frac{3}{2}}(e^{\beta\tau}\bar{S})^{\frac{1}{2}}}{\left(\ln(e^{\beta\tau}\bar{S})\right)^2} e^{-\frac{\left(\ln\left(e^{\beta\tau}\bar{S}\right)\right)^2}{4\tau}}$.

2.3 Large-Time Behaviour

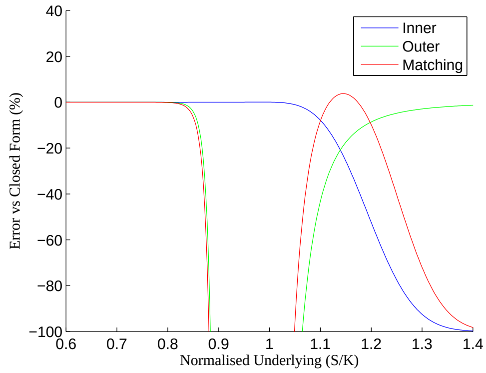
The large-time behaviour for the European put option is non-trivial with the condition $\bar{P}_e(\bar{S},\tau) \to e^{-\alpha\tau}$ as $\bar{S} \to 0$, together with the convexity of the option price, indicating we should expect exponentially small terms throughout the space domain. Further, in the perpetual limit the European put option has zero value. This is obvious if we consider that maximum payoff of the European put option is the strike, the discounted value of which is zero if it occurs at perpetuity. Nonetheless, for large but finite times, the option will have a positive value and we may use the approach of the previous section to identify the large-time asymptotic structure.

Introducing the artificial small parameter ϵ into the non-dimensional European put option problem (2.2a-2.2d) through the time scaling $\tau = \frac{\hat{T}}{\epsilon^2}$ where $\hat{T} = O(1)$ gives the large-time problem

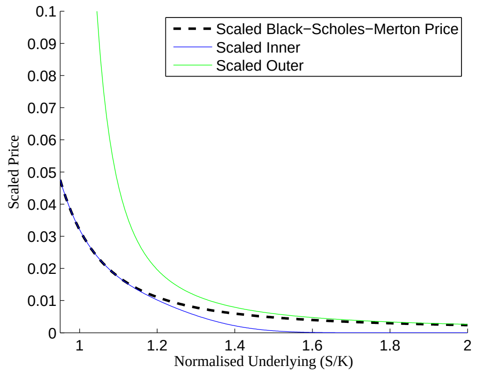
$$\bar{S} \in (0,\infty), \hat{T} \in (0,\infty) \qquad \epsilon^2 \frac{\partial \bar{P}_e}{\partial \hat{T}} = \bar{S}^2 \frac{\partial^2 \bar{P}_e}{\partial \bar{S}^2} + \beta \bar{S} \frac{\partial \bar{P}_e}{\partial \bar{S}} - \alpha \bar{P}_e, \tag{2.161a}$$



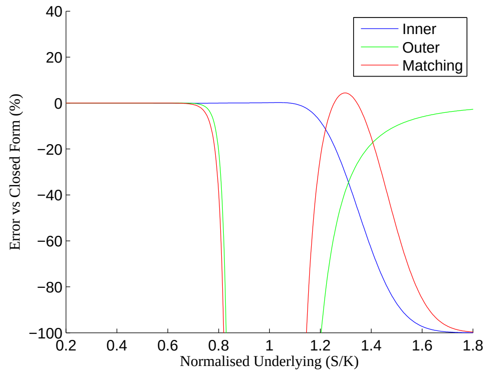
(a) Option Price $\alpha = 40$, $\beta = 24$, $\tau = 0.0025$



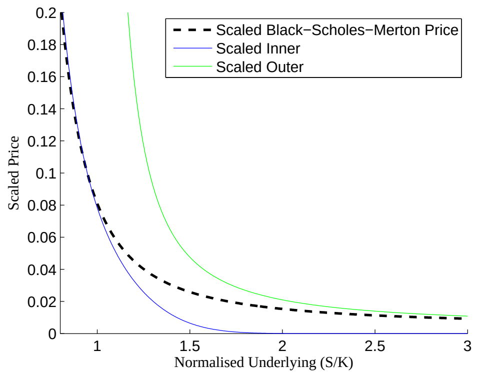
(b) Error $\alpha = 40$, $\beta = 24$, $\tau = 0.0025$



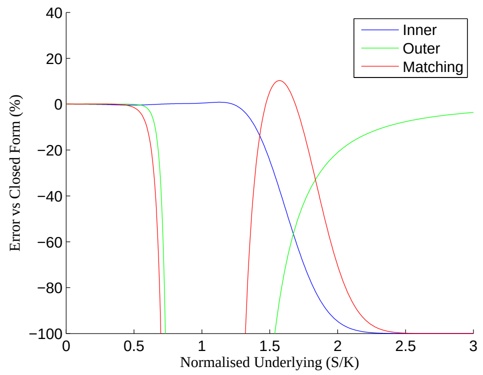
(c) Option Price $\alpha = 10$, $\beta = 6$, $\tau = 0.01$



(d) Error $\alpha = 10$, $\beta = 6$, $\tau = 0.01$



(e) Option Price $\alpha = 2.5$, $\beta = 1.5$, $\tau = 0.04$



(f) Error $\alpha = 2.5$, $\beta = 1.5$, $\tau = 0.04$

Figure 2-4: Comparison of the small-volatility behaviour of the Black-Scholes-Merton price for the European put option, with the derived inner (2.130), outer (2.151-2.152) and matching (2.148-2.149) expressions for a range of values of α, β and τ. The prices in Figures 2.4(a),2.4(c) & 2.4(e) are scaled by the exponential factor $\exp\left(\left(\ln\left(e^{\beta\tau}\bar{S}\right)\right)^2/4\tau\right)$ and a change in the asymptotic behaviour of the Black-Scholes-Merton price from the inner expression to the outer expression for large $\bar{S}$ can be observed. The error plotted in Figures 2.4(b),2.4(d) & 2.4(f) is defined as the difference between the value of the relevant expression and the Black-Scholes-Merton price, expressed as a percentage of the Black-Scholes-Merton price.

subject to

$$\text{as } \bar{S} \to 0 \qquad \bar{P}_e(\bar{S}, \hat{T}) \to e^{-\alpha \frac{\hat{T}}{\epsilon^2}}, \tag{2.161b}$$

$$\text{as } \bar{S} \to \infty \qquad \bar{P}_e(\bar{S}, \hat{T}) \to 0. \tag{2.161c}$$

In comparison to the small-time problem, there seems little intuition over any regions involved in the large-time solution. We know the solution approaches an exponentially small value in the limiting case as the stock price approaches zero. We also require the problem to support decaying solutions to satisfy the boundary condition as $\bar{S} \to \infty$. A transformation of the form $\bar{P}_e(\bar{S}, \hat{T}) = e^{-\alpha \frac{\hat{T}}{\epsilon^2}} \hat{P}_e(\bar{X}, \hat{T})$ where $\bar{X} = \ln\left(e^{(\beta-1)\frac{\hat{T}}{\epsilon^2}} \bar{S}\right)$, gives the large-time problem

$$\bar{X} \in (-\infty, \infty), \hat{T} \in (0, \infty) \qquad \epsilon^2 \frac{\partial \hat{P}_e}{\partial \hat{T}} = \frac{\partial^2 \hat{P}_e}{\partial \bar{X}^2}, \tag{2.162a}$$

subject to

$$\text{as } \bar{X} \to -\infty \qquad \hat{P}_e(\bar{X}, \hat{T}) \to 1, \tag{2.162b}$$

$$\text{as } \bar{X} \to \infty \qquad \hat{P}_e(\bar{X}, \hat{T}) \to 0. \tag{2.162c}$$

We further define the variable $\zeta = \frac{\epsilon \hat{X}}{2\sqrt{\hat{T}}}$, and let $\hat{P}_e(\bar{X}, \hat{T}) = h(\zeta, \hat{T})$, which gives the problem

$$\zeta \in (-\infty, \infty), \hat{T} \in (0, \infty) \qquad 4\hat{T} \frac{\partial h}{\partial \hat{T}} = \frac{\partial^2 h}{\partial \zeta^2} + 2\zeta \frac{\partial h}{\partial \zeta}, \tag{2.163a}$$

subject to

$$\text{as } \zeta \to -\infty \qquad h(\zeta, \hat{T}) \to 1, \tag{2.163b}$$

$$\text{as } \zeta \to \infty \qquad h(\zeta, \hat{T}) \to 0. \tag{2.163c}$$

We note that, to this point, no approximations have been made. The steady state solution to (2.163a-2.163c) is the similarity solution $h(\zeta, \hat{T}) = h_s(\zeta)$ where

$$\zeta \in (-\infty, \infty), \hat{T} \in (0, \infty) \qquad \frac{\partial^2 h_s}{\partial \zeta^2} + 2\zeta \frac{\partial h_s}{\partial \zeta} = 0, \tag{2.164a}$$

subject to

$$\text{as } \zeta \to -\infty \qquad h_s(\zeta) \to 1, \tag{2.164b}$$

$$\text{as } \zeta \to \infty \qquad h_s(\zeta) \to 0, \tag{2.164c}$$

which has the solution

$$h_s(\zeta) = \frac{1}{2}\text{erfc}(\zeta). \tag{2.165}$$

We can therefore write the large-time behaviour of European option price in terms ζ as $\bar{P}_e(\bar{S}, \hat{T}) = \tilde{P}_e(\zeta, \hat{T}; \epsilon)$ where

$$\tilde{P}_e(\zeta, \hat{T}; \epsilon) = \frac{1}{2}e^{-\alpha\frac{\hat{T}}{\epsilon^2}}\text{erfc}(\zeta). \tag{2.166}$$

Comparison with the Closed-Form Solution

We again have reference to the closed-form solution to the European put option to validate our expression. In non-dimensional variables, the large-time European put option price in terms of the complementary error function is

$$\bar{P}_e(\bar{S}, \hat{T}; \epsilon) = \frac{1}{2}e^{-\frac{\alpha\hat{T}}{\epsilon^2}}\left[\text{erfc}\left(\frac{\bar{d}_2}{\sqrt{2}}\right) - \bar{S}e^{\frac{\beta\hat{T}}{\epsilon^2}}\text{erfc}\left(\frac{\bar{d}_1}{\sqrt{2}}\right)\right], \tag{2.167a}$$

where

$$\frac{\bar{d}_1}{\sqrt{2}} = \frac{1}{2}\left(\frac{\epsilon\ln(\bar{S})}{\hat{T}^{\frac{1}{2}}} + (\beta+1)\epsilon^{-1}\hat{T}^{\frac{1}{2}}\right), \tag{2.167b}$$

$$\frac{\bar{d}_2}{\sqrt{2}} = \frac{1}{2}\left(\frac{\epsilon\ln(\bar{S})}{\hat{T}^{\frac{1}{2}}} + (\beta-1)\epsilon^{-1}\hat{T}^{\frac{1}{2}}\right), \tag{2.167c}$$

Making the substitution $\bar{X} = \ln\left(e^{(\beta-1)\frac{\hat{T}}{\epsilon^2}}\bar{S}\right)$ and rearranging (2.88a,2.88b) gives

$$\hat{P}_e(\bar{X}, \hat{T}; \epsilon) = \frac{1}{2}e^{-\frac{\alpha\hat{T}}{\epsilon^2}}\left[\text{erfc}\left(\frac{\hat{d}_2}{\sqrt{2}}\right) - e^{\bar{X}}e^{\frac{\hat{T}}{\epsilon^2}}\text{erfc}\left(\frac{\hat{d}_1}{\sqrt{2}}\right)\right], \tag{2.168a}$$

where

$$\frac{\hat{d}_1}{\sqrt{2}} = \frac{\epsilon \bar{X}}{2\hat{T}^{\frac{1}{2}}} + \epsilon^{-1}\hat{T}^{\frac{1}{2}}, \tag{2.168b}$$

$$\frac{\hat{d}_2}{\sqrt{2}} = \frac{\epsilon \bar{X}}{2\hat{T}^{\frac{1}{2}}}. \tag{2.168c}$$

Finally we let $\zeta = \frac{\epsilon \bar{X}}{2\hat{T}^{\frac{1}{2}}}$ to give

$$\hat{P}_e(\zeta, \hat{T}; \epsilon) = \frac{1}{2} e^{-\frac{\alpha \hat{T}}{\epsilon^2}} \left[\operatorname{erfc}(\zeta) - e^{2\zeta\epsilon^{-1}\hat{T}^{\frac{1}{2}}} e^{\frac{\hat{T}}{\epsilon^2}} \operatorname{erfc}\left(\zeta + \epsilon^{-1}\hat{T}^{\frac{1}{2}}\right) \right]. \tag{2.169}$$

Using the asymptotic behaviour of the complementary error function given in (2.43), we can see for $-\infty < \zeta \ll -O(\epsilon^{-1})$ that $\operatorname{erfc}(\zeta)$ and $\operatorname{erfc}\left(\zeta + \epsilon^{-1}\hat{T}^{\frac{1}{2}}\right)$ are both $O(1)$, but the term $e^{2\zeta\epsilon^{-1}\hat{T}^{\frac{1}{2}}} e^{\frac{\hat{T}}{\epsilon^2}}$ is exponentially small and therefore the first term dominates. For $0 < \zeta \ll O(\epsilon^{-1})$ $\operatorname{erfc}(\zeta)$ remains $O(1)$ and the product $e^{2\zeta\epsilon^{-1}\hat{T}^{\frac{1}{2}}} e^{\frac{\hat{T}}{\epsilon^2}} \operatorname{erfc}\left(\zeta + \epsilon^{-1}\hat{T}^{\frac{1}{2}}\right)$ is exponentially small, so again the first term dominates. For $\zeta = O(\epsilon^{-1})$, both $\operatorname{erfc}(\zeta)$ and $\operatorname{erfc}\left(\zeta + \epsilon^{-1}\hat{T}^{\frac{1}{2}}\right)$ are exponentially small, giving the leading order behaviour

$$\hat{P}_e(\zeta, \hat{T}; \epsilon) \sim \frac{1}{2} e^{-\frac{\alpha \hat{T}}{\epsilon^2}} \frac{e^{-\zeta^2}}{\sqrt{\pi}} \left(\frac{1}{\zeta} - \frac{1}{\zeta + \epsilon^{-1}\hat{T}^{\frac{1}{2}}} \right), \qquad \text{for } \epsilon \to 0,\ \zeta \gg 0. \tag{2.170}$$

As $\zeta \to O\left(\epsilon^{-1}\right)$ it is clear that we can no longer ignore the second term and the rate at which the expression approaches infinity is modified in this far-field region. We therefore have the leading order behaviour of the large-time European put option as $\epsilon \to 0$

$$\hat{P}_e(\zeta, \hat{T}; \epsilon) \sim \begin{cases} \frac{1}{2} e^{-\frac{\alpha \hat{T}}{\epsilon^2}} \operatorname{erfc}(\zeta), & \text{for } \zeta \ll O(\epsilon^{-1}), \\ \frac{1}{2} e^{-\frac{\alpha \hat{T}}{\epsilon^2}} \frac{e^{-\zeta^2}}{\sqrt{\pi}} \frac{\epsilon^{-1}\hat{T}^{\frac{1}{2}}}{\zeta(\zeta + \epsilon^{-1}\hat{T}^{\frac{1}{2}})}, & \text{for } \zeta = O\left(\epsilon^{-1}\right), \\ \frac{1}{2} e^{-\frac{\alpha \hat{T}}{\epsilon^2}} \frac{e^{-\zeta^2}}{\sqrt{\pi}} \frac{\epsilon^{-1}\hat{T}^{\frac{1}{2}}}{\zeta^2}, & \text{for } \zeta \gg O(\epsilon^{-1}). \end{cases} \tag{2.171}$$

Comparison with (2.165) indicates that our large-time asymptotic expression breaks down when $\zeta = O(\epsilon^{-1})$ and that a modification is required in the far-field which changes the rate at which the solution approaches zero. Furthermore, we may no longer expect the solution to be self-similar in this region.

In the far field, $h_s(\zeta)$ has the asymptotic behaviour

$$h_s(\zeta) \sim \frac{e^{-\zeta^2}}{\sqrt{\pi}} \left(\frac{1}{\zeta} + \dots \right), \quad \text{as } \zeta \to \infty \tag{2.172}$$

we therefore look for the far-field behaviour $h(\zeta,\hat{T}) = \tilde{h}(\zeta,\hat{T})$ to have the form

$$\tilde{h}(\zeta,\hat{T}) = \frac{2}{\sqrt{\pi}} e^{-\zeta^2} \left(a_0(\zeta) - a_1(\zeta,\hat{T}) \right) \tag{2.173}$$

where a_0 is the solution to

$$\frac{d^2 a_0}{d\zeta^2} - 2\zeta \frac{da_0}{d\zeta} - 2a_0 = 0, \tag{2.174}$$

which yields

$$a_0 \sim \frac{1}{\zeta} - \frac{1}{2\zeta^3} + \ldots \quad \text{as } \zeta \to \infty \tag{2.175}$$

and a_1 is the solution to

$$\frac{\partial^2 a_1}{\partial \zeta^2} - 2\zeta \frac{\partial a_1}{\partial \zeta} - 2a_1 = 4\hat{T} \frac{\partial a_1}{\partial \hat{T}}. \tag{2.176}$$

We expect the location of the far-field to occur for increasingly large $\bar{S}$ as we approach perpetuity. We therefore look for a transition region, in which the leading order is modified, through the scaling $\Gamma = \zeta + \frac{f(\hat{T})}{\delta(\epsilon)}$ where $\delta(\epsilon) \ll 1$ and $f(\hat{T})$ is a function to be determined which provides the location of the transition region. Letting $a_1(\zeta,\hat{T}) = a_1(\Gamma)$ gives the expression

$$\frac{\partial^2 a_1}{\partial \Gamma^2} - 2\zeta \frac{\partial a_1}{\partial \Gamma} - 2a_1 = \frac{2}{\delta(\epsilon)} \left(2\hat{T} \frac{\partial f}{\partial \hat{T}} - f \right) \frac{\partial a_1}{\partial \Gamma}, \tag{2.177}$$

which has non-trivial solutions for

$$\frac{1}{f(\hat{T})} \frac{\partial f}{\partial \hat{T}} = \frac{1}{2\hat{T}}, \tag{2.178}$$

or

$$f(\hat{T}) = \hat{C} \hat{T}^{\frac{1}{2}}, \tag{2.179}$$

for some arbitrary constant $\tilde{C}$, with the scaling of $f(\hat{T})$ allowing us to specify $\delta(\epsilon) = \epsilon$. The governing equation for $a_1(\Gamma)$ is now the same as for $a_0(\zeta)$ and we therefore expect it to have the solution

$$a_1 \sim \frac{1}{\Gamma} - \frac{1}{2\Gamma^3} + \ldots \tag{2.180}$$

and therefore in the transition region

$$\tilde{h}(\zeta,\hat{T}) = \frac{2}{\sqrt{\pi}}e^{-\zeta^2}\left(\left(\frac{1}{\zeta}-\frac{1}{2\zeta^3}+\dots\right)-\left(\frac{1}{\Gamma}-\frac{1}{2\Gamma^3}+\dots\right)\right), \tag{2.181}$$

or to leading order, in terms of ζ

$$\tilde{h}(\zeta,\hat{T}) \sim \frac{2}{\sqrt{\pi}}e^{-\zeta^2}\left(\frac{1}{\zeta}-\frac{1}{\zeta+\hat{C}\epsilon^{-1}\hat{T}^{\frac{1}{2}}}\right), \tag{2.182}$$

and therefore the leading order expression has the behaviour

$$\tilde{h}(\zeta,\hat{T}) \sim \begin{cases} \frac{2}{\sqrt{\pi}}e^{-\zeta^2}\frac{1}{\zeta}, & \text{for } \zeta \ll \epsilon^{-1}, \\ \frac{2}{\sqrt{\pi}}e^{-\zeta^2}\frac{\tilde{C}\epsilon^{-1}\hat{T}^{\frac{1}{2}}}{\zeta(\zeta+\tilde{C}\epsilon^{-1}\hat{T}^{\frac{1}{2}})}, & \text{for } \zeta = O(\epsilon^{-1}), \\ \frac{2}{\sqrt{\pi}}e^{-\zeta^2}\frac{\tilde{C}\epsilon^{-1}\hat{T}^{\frac{1}{2}}}{\zeta^2}, & \text{for } \zeta \gg \epsilon^{-1}, \end{cases} \tag{2.183}$$

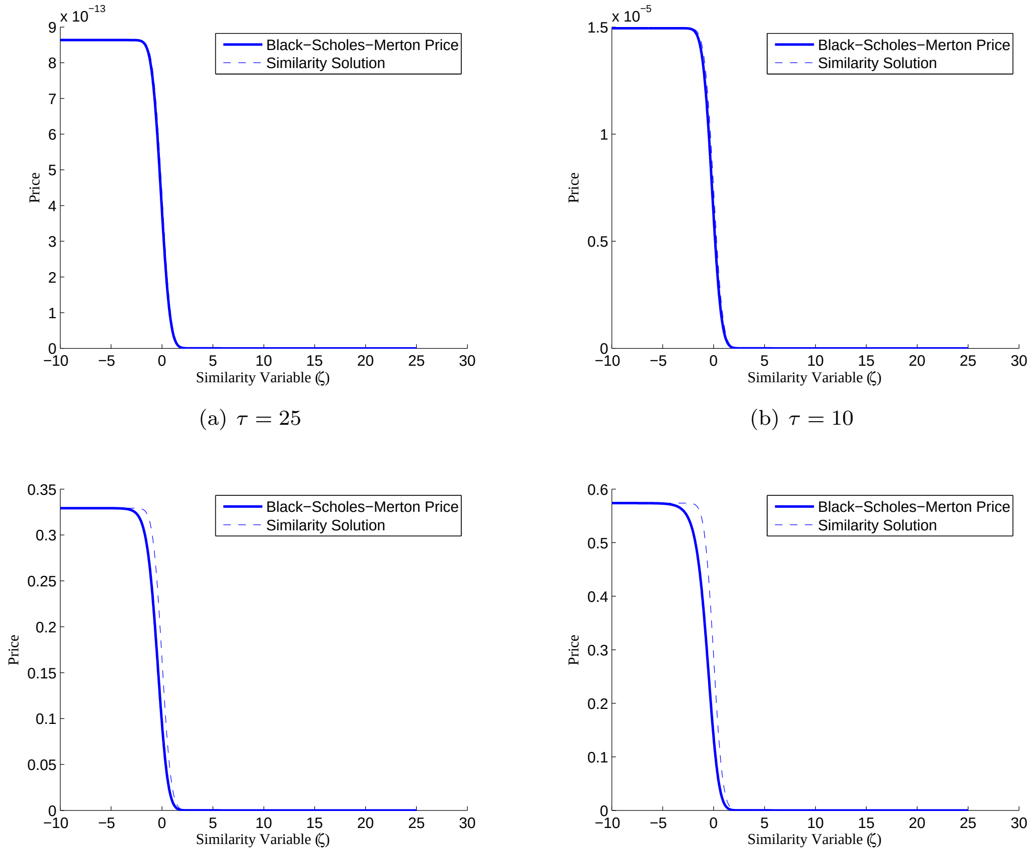
which gives the large-time asymptotic structure of the European put option problem in terms of our non-dimensional time variable τ and similarity variable ζ

$$\tilde{P}_e(\zeta,\tau) \sim \begin{cases} \frac{1}{2}e^{-\alpha\tau}\operatorname{erfc}(\zeta), & \text{for } \zeta \ll O(\sqrt{\tau}), \\ \frac{1}{2}e^{-\alpha\tau}\frac{e^{-\zeta^2}}{\sqrt{\pi}}\frac{\tilde{C}\sqrt{\tau}}{\zeta(\zeta+\tilde{C}\sqrt{\tau})}, & \text{for } \zeta = O(\sqrt{\tau}), \\ \frac{1}{2}e^{-\alpha\tau}\frac{e^{-\zeta^2}}{\sqrt{\pi}}\frac{\tilde{C}\sqrt{\tau}}{\zeta^2}, & \text{for } \zeta \gg O(\sqrt{\tau}). \end{cases} \tag{2.184}$$

We note that this expression captures the required behaviour in the far-field seen in (2.171) providing $\tilde{C} = 1$.

A natural question is whether we could have predicted the existence of this far-field behaviour without recourse to the closed-form expression. A clue to the answer lies in the form of the solution in the far-field for $\zeta \gg O(\sqrt{\tau})$, which we observe has the same form as the asymptotic behaviour of the solution to the leading order small-time inner problem (2.63a). The far-field behaviour in the large-time problem is the remnant effect of the smoothing of the option payoff near expiry and this is the only way in which information from the small-time solution enters the large-time problem.

A comparison of the large-time approximation with the Black-Scholes-Merton price is shown in Figure 2-5. We observe that the approximation is indistinguishable from the closed-form solution for surprisingly small values of τ and even for $\tau = 5$ the approximation is reasonably accurate. The existence of the transition region for large ζ is shown in Figure 2-6. We observe that the transition expression in (2.183) captures the correct asymptotic behaviour of the closed-form solution (2.169) for large ζ.



(a) $\tau = 25$ (b) $\tau = 10$

(c) $\tau = 5$ (d) $\tau = 1$

Figure 2-5: Comparison of the large-time behaviour of the Black-Scholes-Merton price for the European put option (2.167a) with the similarity solution (2.165) for a range of values of τ and for $\alpha = \frac{10}{9}$.

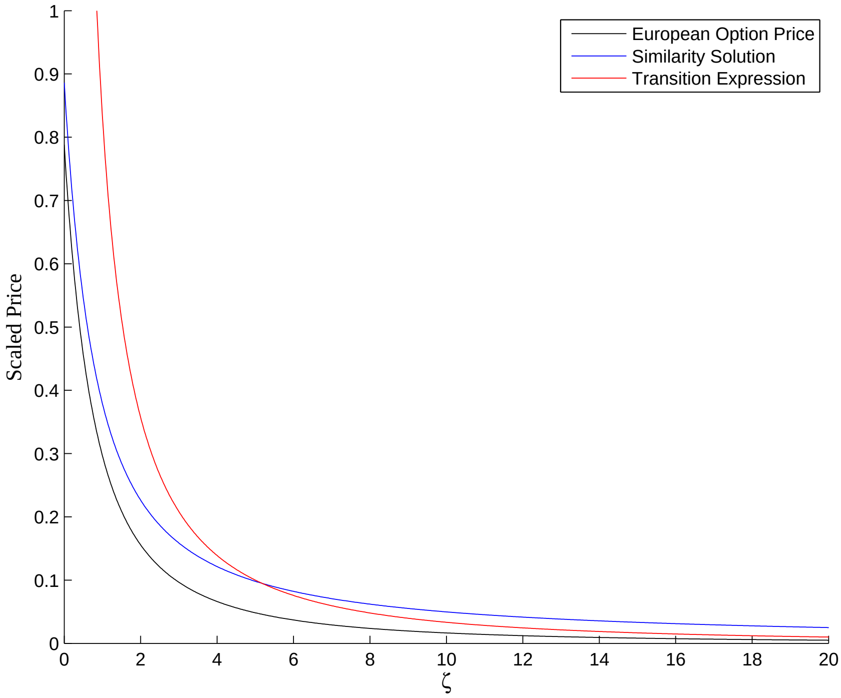


Figure 2-6: Comparison of the large-time behaviour of the Black-Scholes-Merton price for the European put option (2.167a) with the similarity solution (2.165) and transition region expression (2.182) for large values of the similarity variable $\zeta\ \left(= \ln\left(e^{\beta\tau}\bar{S}\right)/2\sqrt{\tau}\right)$ and for the parameters $\tau = 25$, $\alpha = \frac{10}{9}$. Due to the exponentially small magnitude of the terms involved, prices have been scaled by the exponential factor $\sqrt{\pi}e^{\zeta^2}$. We observe a change in the asymptotic behaviour of the Black-Scholes-Merton price from the similarity solution to the transition region expression, beginning at $\zeta = O(\tau^{\frac{1}{2}})$.

Chapter 3

The American Option Problem

Using the framework developed in Chapter 2, we now investigate the asymptotic behaviour of the American put option problem for small and large times to expiry. Several authors have investigated the small-time behaviour of the optimal exercise boundary of the American put option [6, 7, 30, 51, 73, 75, 76, 97], using a variety of techniques, but only Evans et al. [41] look at all possible configurations of the risk-free rate and dividend yield. The large-time behaviour has been discussed by Knessl [73], but not derived explicitly.

We move on to extend an analytic approximation developed to MacMillan [79] and Barone-Adesi & Whaley [10] by posing the problem as a leading order term in a homotopic series, an approach used recently by Zhu [106] in relation to the full American option problem. This generalises the work of other authors such as Ju & Zhong [69] in looking for correction terms to the MBAW approximation.

Non-dimensionalising the problem (1.22a-1.22f) using the same scalings used in Chapter 2 (2.1), but with the addition of a non-dimensional American option price $\bar{P}_a(\bar{S},\tau)$ and optimal exercise boundary $\bar{S}^*(\tau)$ defined through the scalings

$$S^*(t) = K\bar{S}^*(\tau), \qquad P_a(S,t) = K\bar{P}_a(\bar{S},\tau), \tag{3.1}$$

gives the non-dimensional American put option problem

$$\bar{S} \in (\bar{S}^*(\tau),\infty), \tau \in (0,T) \qquad \frac{\partial \bar{P}_a}{\partial \tau} = \bar{S}^2\frac{\partial^2 \bar{P}_a}{\partial \bar{S}^2} + \beta\bar{S}\frac{\partial \bar{P}_a}{\partial \bar{S}} - \alpha\bar{P}_a, \tag{3.2a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{P}_a(\bar{S}, \tau) \to 0, \tag{3.2b}$$

$$\text{at } \tau = 0 \qquad \bar{P}_a(\bar{S}, 0) = \max\big(1 - \bar{S}, 0\big), \tag{3.2c}$$

$$\bar{S}^*(0) = \min\left(\alpha/(\alpha - \beta), 1\right), \tag{3.2d}$$

$$\text{at } \bar{S} = \bar{S}^*(\tau) \qquad \bar{P}_a\left(\bar{S}^*, \tau\right) = \left(1 - \bar{S}^*\right), \tag{3.2e}$$

$$\left.\frac{\partial \bar{P}_a}{\partial \bar{S}}\right|_{\bar{S}^*} = -1. \tag{3.2f}$$

We note that we may also choose to transform the problem into one with constant coefficients through the use of the transformation $\bar{X} = \ln(\bar{S})$, or alternatively transform the problem on a fixed semi-infinite domain using a Landau transformation of the form $\tilde{X} = \ln\big(\bar{S}/\bar{S}^*(\tau)\big)$.

3.1 Small-Time Behaviour

The small-time asymptotic behaviour of the American option has been studied by a number of authors, in particular Evans et al. [41] who derive the behaviour in the cases $\beta < 0$, $\beta = 0$ and $\beta > 0$ in the presence of dividends using both matched asymptotic expansions and via the use of integral equations. Also of interest is the work of Knessl who looks at the behaviour in various parameter limits of the American put in the absence of dividends [73] and the American call in the presence of dividends [74]. Here we pose the small-time American option problem as an extension to our work on the European put option in Chapter 2 which allows us to capture the WKBJ terms in the relevant region. As much of this work is covered in previous papers, though using different formulations of the problem, we omit some of the detail in the small-time work.

Using the non-dimensional parameters given in (3.1) and introducing an artificially small parameter ϵ through the time scaling $\tau = \epsilon^2 \hat{T}$, where $\hat{T} = O(1)$, gives the small-time problem

$$\bar{S} \in (\bar{S}^*(\hat{T}), \infty), \hat{T} \in (0, T) \qquad \frac{1}{\epsilon^2}\frac{\partial \bar{P}_a}{\partial \hat{T}} = \bar{S}^2 \frac{\partial^2 \bar{P}_a}{\partial \bar{S}^2} + \beta \bar{S} \frac{\partial \bar{P}_a}{\partial \bar{S}} - \alpha \bar{P}_a, \tag{3.3a}$$

subject to

$$\begin{aligned}
&\text{as } \bar{S} \to \infty && \bar{P}_a(\bar{S}, \hat{T}) \to 0, && (3.3\text{b})\\
&\text{at } \hat{T} = 0 && \bar{P}_a(\bar{S}, 0) = \max\left(1 - \bar{S}, 0\right), && (3.3\text{c})\\
& && \bar{S}^*(0) = \min\left(\alpha/(\alpha - \beta), 1\right), && (3.3\text{d})\\
&\text{at } \bar{S} = \bar{S}^*(\hat{T}) && \bar{P}_a\left(\bar{S}^*, \hat{T}\right) = \left(1 - \bar{S}^*\right), && (3.3\text{e})\\
& && \left.\frac{\partial \bar{P}_a}{\partial \bar{S}}\right|_{\bar{S}^*} = -1. && (3.3\text{f})
\end{aligned}$$

The addition of the optimal exercise boundary does not affect the inner (2.32a-2.32b) or outer 2 (2.13a-2.13c) problems derived for the European put option and therefore, for all values of β, an outer 2 region exists for $\bar{S} > 1$ in which the American put option has the leading order small-time behaviour

$$\bar{P}_a^{Out2}(\bar{S}, \tau) \sim \frac{2\epsilon^3}{\sqrt{\pi}} \frac{\hat{T}^{\frac{3}{2}} \bar{S}^{\frac{1-\beta}{2}}}{\left(\ln(\bar{S})\right)^2} e^{-\frac{(\ln(\bar{S}))^2}{4\epsilon^2 \hat{T}}} \qquad \text{as } \epsilon \to 0. \tag{3.4}$$

This has potential relevance in relation to numerical routines for American-style options, which are typically performed on a truncated domain and required the specification of some behaviour at the truncated boundary.

Also in common with the small-time European put option problem, an inner region exists for $\bar{S} - 1 = O(\epsilon)$ in terms of a similarity variable $\zeta = \frac{(\bar{S}-1)^2}{2\epsilon \hat{T}^{\frac{1}{2}}}$ which can be written as an expansion of the form

$$\hat{P}_a^{In}(\zeta, \hat{T}) = \epsilon \hat{T}^{\frac{1}{2}} h_0(\zeta) + \epsilon^2 \hat{T} h_1(\zeta) + \epsilon^3 \hat{T}^{\frac{3}{2}} h_2(\zeta) + O(\epsilon^4) \qquad \text{as } \epsilon \to 0, \tag{3.5}$$

where

$$h_0(\zeta) = \frac{1}{\sqrt{\pi}} e^{-\zeta^2} - \zeta \operatorname{erfc}(\zeta), \tag{3.6a}$$

$$h_1(\zeta) = \frac{\zeta}{\sqrt{\pi}} e^{-\zeta^2} - \frac{\beta}{2} \operatorname{erfc}(\zeta), \tag{3.6b}$$

$$h_2(\zeta) = \frac{e^{-\zeta^2}}{\sqrt{\pi}} \left(\zeta^4 - \frac{1}{3}(1 + 3\beta)\zeta^2 + \frac{1}{4}\left(\beta(\beta - 1) - \frac{1}{3}\right)\right) - (\alpha - \beta) \operatorname{ierfc}(\zeta). \tag{3.6c}$$

The region structure for $\bar{S} < 1$ depends on the value of the non-dimensional cost of carry β, which determines the initial position of the optimal exercise boundary as shown in Chapter 1.

Additional Regions ($\beta \geq 0$)

Other authors have observed that the optimal exercise boundary for the case $\beta \geq 0$ is outside the inner region for small times. In order to apply the boundary conditions, we look in a small region about the boundary, which we call the *boundary inner* region, through the scaling

$$\bar{S} = \bar{S}^*(\hat{T}) + \delta_1(\epsilon)\check{S} \tag{3.7}$$

and we pose the boundary expansion in the boundary inner region

$$\bar{S}^*(\hat{T}) = 1 + \delta_0(\epsilon)\check{S}^*(\hat{T}) + o\left(\delta_0(\epsilon)\right), \tag{3.8}$$

where $\delta_0, \delta_1 \ll 1$ are to be determined and $\check{S}, \check{S}^*(\hat{T}) = O(1)$. We note that knowledge that the boundary lies outside of the inner region requires $\delta_0 \gg \epsilon$.

The boundary inner problem, $\bar{P}_a(\bar{S}, \hat{T}) = \check{P}_a^{BIn}(\check{S}, \hat{T})$ becomes

$$\check{S} \in (0,\infty), \hat{T} \in (0,T) \qquad \frac{1}{\epsilon^2}\left[\frac{\partial \check{P}_a^{BIn}}{\partial \hat{T}} - \frac{\delta_0}{\delta_1}\frac{d\bar{S}^*}{d\hat{T}}\frac{\partial \check{P}_a^{BIn}}{\partial \check{S}}\right] = (1+\delta_0\check{S}^*(\hat{T})+\delta_1\check{S})^2\frac{1}{\delta_1^2}\frac{\partial^2 \check{P}_a^{BIn}}{\partial \check{S}^2} + \beta(1+\delta_0\check{S}^*(\hat{T})+\delta_1\check{S})\frac{1}{\delta_1}\frac{\partial \check{P}_a^{BIn}}{\partial \check{S}} - \alpha\check{P}_a^{BIn}, \tag{3.9a}$$

subject to

$$\text{at } \check{S} = 0 \qquad \check{P}_a^{BIn}\left(0,\hat{T}\right) = -\delta_0\check{S}^*, \tag{3.9b}$$

$$\left.\frac{\partial \check{P}_a^{BIn}}{\partial \check{S}}\right|_{\check{S}=0} = -\delta_1. \tag{3.9c}$$

The leading order balance in the governing equation (3.9a) determines $\delta_0\delta_1 = \epsilon^2$ and, along with $\delta_0 \gg \epsilon$, gives the restriction $\epsilon^2 \ll \delta_1 \ll \epsilon$. Combined with the boundary conditions (3.9b-3.9c), this suggests a boundary inner expansion of the form

$$\check{P}_a^{BIn}(\check{S},\hat{T};\epsilon) = \frac{\epsilon^2}{\delta_1(\epsilon)}\check{P}_0^{BIn}(\check{S},\hat{T}) + \delta_1(\epsilon)\check{P}_1^{BIn}(\check{S},\hat{T}) + \delta_2(\epsilon)\check{P}_2^{BIn}(\check{S},\hat{T}) + o(\delta_2) \tag{3.10}$$

where $\delta_2 \ll \delta_1$, which gives a series of subproblems with the first two solutions

$$\check{P}_0^{BIn}(\check{S},\hat{T}) = -\check{S}^*, \tag{3.11}$$

$$\check{P}_1^{BIn}(\check{S},\hat{T}) = -\check{S}, \tag{3.12}$$

while the next non-zero problem for $\beta > 0$ occurs when $\delta_2 = \delta_1^2 \ll \epsilon^2$ and the solution to the subproblem is

$$\check{P}_2^{BIn}(\check{S},\hat{T}) = \beta\left(\frac{\partial\check{S}^*}{\partial\hat{T}}\right)^{-1}\left(\check{S} - \left(\frac{\partial\check{S}^*}{\partial\hat{T}}\right)^{-1}\left(1 - e^{-\frac{\partial\check{S}^*}{\partial\hat{T}}\check{S}}\right)\right). \tag{3.13}$$

From our work on the small-time European problem in Chapter 2, the asymptotic behaviour of the inner expression as $\epsilon \to 0$ for $\bar{S} < 1$ is

$$\begin{aligned}\bar{P}_a^{In}(\bar{S},\hat{T}) \sim & 1 - \bar{S} - \beta\epsilon^2\hat{T} + (\alpha-\beta)\epsilon^3\hat{T}(\bar{S}-1) + O(\epsilon^4)\\ &+ \frac{1}{2\sqrt{\pi}}\frac{\epsilon^3\hat{T}^{\frac{3}{2}}}{(\bar{S}-1)^2}e^{-\frac{(\bar{S}-1)^2}{4\epsilon^2\hat{T}}} \qquad \text{as } \epsilon \to 0.\end{aligned} \tag{3.14}$$

Writing (3.14) in terms of the boundary inner variable $\check{S}$ and the scaled optimal exercise boundary correction term $\check{S}^*$ gives

$$\begin{aligned}\check{P}_a^{In}(\check{S},\hat{T}) \sim & -\frac{\epsilon^2}{\delta_1}\check{S}^* - \delta_1\check{S} - \beta\epsilon^2\hat{T} + \frac{\epsilon^4}{\delta_1}\hat{T}(\alpha-\beta)\check{S}^* + O(\delta_1\epsilon^4)\\ &+ \frac{2}{\sqrt{\pi}}\frac{\delta_1^2}{\epsilon}\frac{\hat{T}^{\frac{3}{2}}}{(\check{S}^*)^2}e^{-\frac{\left(\frac{\epsilon^2}{\delta_1}\bar{S}^* + \delta_1\check{S}\right)^2}{4\epsilon^2\hat{T}}} \qquad \text{as } \epsilon \to 0.\end{aligned} \tag{3.15}$$

The first two terms in (3.18) match with the first two terms in the boundary inner series (3.11-3.12), however we have identified that there are no further terms in the boundary inner larger than $O(\delta_1^2)$, where we have previously determined that $\epsilon^2 \ll \delta_1 \ll \epsilon$. We therefore require terms in (3.18) larger than $O(\epsilon^4)$ to cancel, which determines that

$$\frac{2}{\sqrt{\pi}}\frac{\delta_1^2}{\epsilon}\frac{\hat{T}^{\frac{3}{2}}}{(\check{S}^*)^2}e^{-\frac{\left(\frac{\epsilon^2}{\delta_1}\bar{S}^* + \delta_1\check{S}\right)^2}{4\epsilon^2\hat{T}}} = \beta\epsilon^2\hat{T} \qquad \text{as } \epsilon \to 0, \tag{3.16}$$

and by taking logs and using $\frac{\left(\frac{\epsilon^2}{\delta_1}\bar{S}^* + \delta_1\check{S}\right)^2}{4\epsilon^2\hat{T}} \to \infty$ as $\epsilon \to 0$ we find the dominant behaviour

$$\delta_0\bar{S}^*(\hat{T}) \sim -\sqrt{-4\epsilon^2\hat{T}\ln\left(2\sqrt{\pi}\beta\epsilon\hat{T}^{\frac{1}{2}}\right)} \qquad \text{as } \epsilon \to 0. \tag{3.17}$$

For $\beta = 0$ the third term in the boundary inner series occurs at $O(\delta_1\epsilon^2)$ while the inner series in the boundary inner variables is

$$\check{P}_a^{In}(\check{S},\hat{T}) \sim -\frac{\epsilon^2}{\delta_1}\check{S}^* - \delta_1\check{S} + \frac{\epsilon^4}{\delta_1}\hat{T}\alpha\check{S}^* + O(\delta_1\epsilon^4)$$
$$+\frac{2}{\sqrt{\pi}}\frac{\delta_1^2}{\epsilon}\frac{\hat{T}^{\frac{3}{2}}}{(\check{S}^*)^2}e^{-\frac{\left(\frac{\epsilon^2}{\delta_1}\check{S}^*+\delta_1\check{S}\right)^2}{4\epsilon^2\hat{T}}} \qquad \text{as } \epsilon \to 0. \tag{3.18}$$

Comparison of terms in this case requires

$$\frac{2}{\sqrt{\pi}}\frac{\delta_1^2}{\epsilon}\frac{\hat{T}^{\frac{3}{2}}}{(\check{S}^*)^2}e^{-\frac{\left(\frac{\epsilon^2}{\delta_1}\check{S}^*+\delta_1\check{S}\right)^2}{4\epsilon^2\hat{T}}} = -\frac{\epsilon^4}{\delta_1}\hat{T}\alpha\check{S}^* \qquad \text{as } \epsilon \to 0 \tag{3.19}$$

and again taking logs and using $\frac{\left(\frac{\epsilon^2}{\delta_1}\check{S}^*+\delta_1\check{S}\right)^2}{4\epsilon^2\hat{T}} \to \infty$ as $\epsilon \to 0$ gives the dominant behaviour

$$\delta_0\check{S}^*(\hat{T}) \sim -\sqrt{-4\epsilon^2\hat{T}\ln\left(4\sqrt{\pi}\alpha\epsilon^2\hat{T}\right)} \qquad \text{as } \epsilon \to 0. \tag{3.20}$$

The structure of the problems for $\beta \geq 0$ is shown in Figure 3-1.

Additional Regions ($\beta < 0$)

For $\beta < 0$ the starting point of the optimal exercise boundary is $\bar{S}^*(0) = \frac{\alpha}{\alpha-\beta}$ and we expect an outer 1 region which bridges from the inner region to a region near the boundary. Posing an expansion in powers of ϵ^2

$$\bar{P}_a^{Out1}(\bar{S},\hat{T};\epsilon) = \bar{P}_0^{Out1}(\bar{S},\hat{T}) + \epsilon^2\bar{P}_1^{Out1}(\bar{S},\hat{T}) + \epsilon^4\bar{P}_2^{Out1}(\bar{S},\hat{T}) + O(\epsilon^6), \tag{3.21}$$

gives a series of problems with solutions

$$\bar{P}_0^{Out1}(\bar{S},\hat{T}) = 1 - \bar{S}, \tag{3.22}$$
$$\bar{P}_1^{Out1}(\bar{S},\hat{T}) = \left[(\alpha-\beta)\bar{S} - \alpha\right]\hat{T}, \tag{3.23}$$
$$\bar{P}_2^{Out1}(\bar{S},\hat{T}) = \left[\alpha^2 - (\alpha-\beta)^2\bar{S}\right]\frac{\hat{T}^2}{2}. \tag{3.24}$$

We note that $\bar{P}_0^{Out1}(\bar{S},\hat{T})$ is the intrinsic value of the American option in the outer 1 region obtainable via early exercise, while the first correction term $\bar{P}_1^{Out1}(\bar{S},\hat{T})$ is positive for the case $\beta < 0$ provided $\bar{S} > \frac{\alpha}{\alpha-\beta}$ which is consistent with the definition of the outer 1 region.

Though we could investigate the existence of WKBJ terms in a similar vein to our work on the European problem, these would be dominated by the algebraic terms in the outer 1 region and

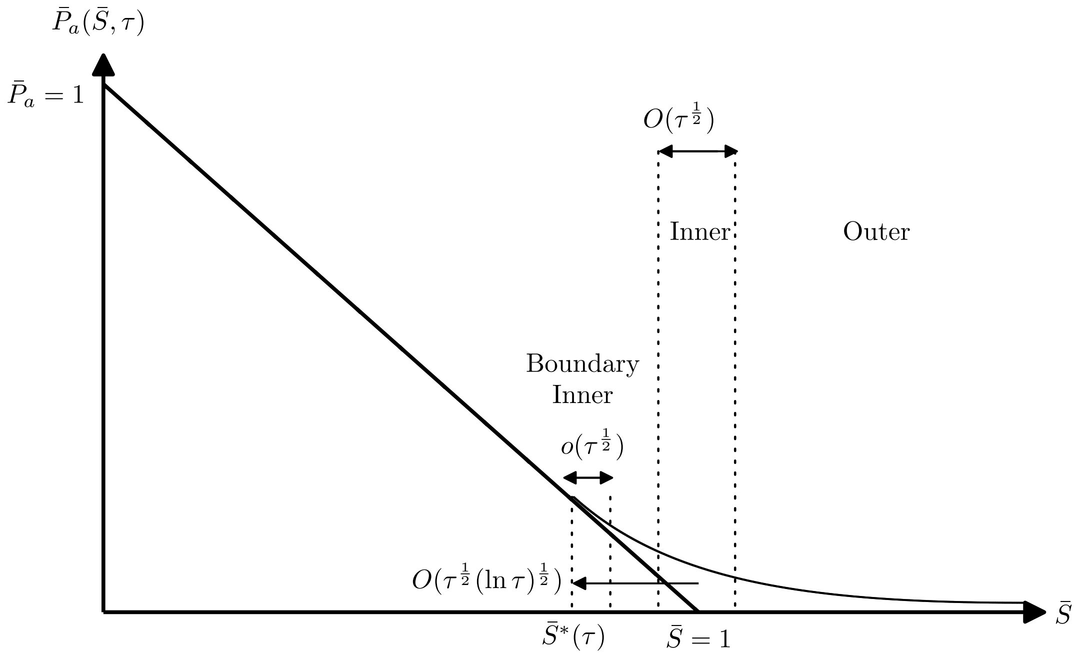


Figure 3-1: A schematic showing the small-time asymptotic structure of the American put option for $\beta \geq 0$. The asymptotic structure uses an $O(\sqrt{\tau})$ inner layer about $\bar{S} = 1$ in which the price is $O(\sqrt{\tau})$. The optimal exercise boundary lies at $O(\sqrt{\tau \ln(\tau)})$ from the initial starting point, $\bar{S}^*(0) = 1$, which is outside of the inner region. For $\bar{S} > 1$ a second outer region exists in which the price is exponentially small and can be shown to have the leading order asymptotic behaviour $\bar{P}_a \sim \frac{2e^{-\alpha\tau}}{\sqrt{\pi}} \frac{\tau^{\frac{3}{2}} \bar{S}^{\frac{1-\beta}{2}}}{\left(\ln(\bar{S})\right)^2} e^{-\frac{(\ln(\bar{S}))^2}{4\tau}}$

are not required for the determination of the leading order behaviour of the optimal exercise boundary. Since we do not perform the same matching exercise as in Chapter 2, we omit the WKBJ terms in this region.

To apply the boundary conditions, we consider a boundary inner region local to the initial location of the optimal exercise boundary $\bar{S}^*(0)$, via the introduction of a boundary inner variable $\check{S}$ such that $\check{P}^{BIn}(\check{S},\hat{T}) = \bar{P}(\bar{S},\hat{T})$ with the scalings

$$\bar{S} = \bar{S}^*(0)\left(1 + \epsilon\check{S}\right), \tag{3.25}$$

$$\bar{S}^*(\hat{T}) = \bar{S}^*(0)\left(1 + \epsilon\check{S}^*(\hat{T})\right), \tag{3.26}$$

which leads to the problem

$$\check{S} \in (\check{S}^*(\hat{T}), \infty), \hat{T} \in (0, T) \qquad \frac{\partial \check{P}_a^{BIn}}{\partial \hat{T}} = (1+\epsilon\check{S})^2 \frac{\partial^2 \check{P}_a^{BIn}}{\partial \check{S}^2} + \epsilon\beta(1+\epsilon\check{S}) \frac{\partial \check{P}_a^{BIn}}{\partial \check{S}} - \epsilon^2 \alpha \check{P}_a^{BIn}, \tag{3.27a}$$

subject to

$$\text{at } \check{S} = \check{S}^* \qquad \check{P}_a^{BIn}\left(\check{S}^*, \hat{T}\right) = 1 - \bar{S}^*(0) - \epsilon\bar{S}^*(0)\check{S}^*, \tag{3.27b}$$

$$\left.\frac{\partial \check{P}_a^{BIn}}{\partial \check{S}}\right|_{\check{S}^*} = -\epsilon\bar{S}^*(0), \tag{3.27c}$$

$$\text{as } \check{S} \to \infty \qquad \check{P}_a^{BIn}\left(\check{S}, \hat{T}\right) \sim 1 - \bar{S}^*(0) - \epsilon\bar{S}^*(0)\check{S} + \epsilon^3\alpha\check{S}\hat{T}, \tag{3.27d}$$

where the matching condition (3.27d) is derived by writing the terms outer 1 expressions (3.22-3.24) in terms of the boundary inner variable $\check{S}$.
Posing an expansion in powers of ϵ

$$\check{P}_a^{BIn}(\check{S}, \hat{T}; \epsilon) = \check{P}_0^{BIn}(\check{S}, \hat{T}) + \epsilon\check{P}_1^{BIn}(\check{S}, \hat{T}) + \epsilon^2\check{P}_2^{BIn}(\check{S}, \hat{T}) + \epsilon^3\check{P}_3^{BIn}(\check{S}, \hat{T}) + O(\epsilon^4), \tag{3.28}$$

gives a series of subproblems with trivial solutions for the first three terms

$$\check{P}_0^{BIn}(\check{S}, \hat{T}) = 1 - \bar{S}^*(0), \tag{3.29}$$

$$\check{P}_1^{BIn}(\check{S}, \hat{T}) = -\bar{S}^*(0)\check{S}, \tag{3.30}$$

$$\check{P}_2^{BIn}(\check{S}, \hat{T}) = 0, \tag{3.31}$$

while $\check{P}_3^{BIn}(\check{S}, \hat{T})$ is the solution to

$$\check{S} \in (\check{S}^*(\hat{T}), \infty), \hat{T} \in (0, T) \qquad \frac{\partial \check{P}_3^{BIn}}{\partial \hat{T}} = \frac{\partial^2 \check{P}_3^{BIn}}{\partial \check{S}^2} + \alpha\check{S}, \tag{3.32a}$$

subject to

$$\text{at } \check{S} = \check{S}^* \qquad \check{P}_3^{BIn}\left(\check{S}^*, \hat{T}\right) = 0, \tag{3.32b}$$

$$\frac{\partial \check{P}_3^{BIn}}{\partial \check{S}} = 0, \tag{3.32c}$$

$$\text{as } \check{S} \to \infty \qquad \check{P}_3^{BIn}\left(\check{S}, \hat{T}\right) \sim \alpha\check{S}\hat{T}. \tag{3.32d}$$

The problem (3.32a-3.32d) has the similarity solution

$$\check{P}_3^{BIn} = \hat{T}^{\frac{3}{2}} h_3(\zeta), \qquad \zeta = \frac{\check{S}}{2\hat{T}^{\frac{1}{2}}}, \tag{3.33}$$

where

$$h_3(\zeta) = \check{C}_{30}\frac{1}{3}\left(\zeta^3 + \frac{3\zeta}{2}\right) + \check{C}_{31}\mathrm{i}^3\mathrm{erfc}(\zeta) + 2\alpha\zeta \tag{3.34}$$

and conditions (3.32b) and (3.32d) require

$$\check{C}_{30} = 0, \qquad \check{C}_{31} = -\frac{2\alpha\zeta^*}{\mathrm{i}^3\mathrm{erfc}(\zeta^*)}, \tag{3.35}$$

where $\zeta^* = \frac{\bar{S}_1^*}{2}$, and condition (3.32c) leads to the transcendental expression for ζ^*

$$2\zeta^* - 1 = 2\sqrt{\pi}(\zeta^*)^3 e^{(\zeta^*)^2}\mathrm{erfc}(\zeta^*), \tag{3.36}$$

which has the numerical solution $\zeta^* = -0.45172$.

The small-time asymptotic behaviour of the optimal exercise boundary for $\beta < 0$ is therefore

$$\bar{S}^*(\hat{T}) \sim \frac{\alpha}{\alpha - \beta}\left(1 - 0.90344\epsilon\hat{T}^{\frac{1}{2}}\right) \qquad \text{as } \epsilon \to 0. \tag{3.37}$$

The structure of the problem for $\beta < 0$ is shown in Figure 3-2.

Summary of Results

The small-time asymptotics of the American option problem are derived in scaled variables in (3.17), (3.20) & (3.37) and are summarised below using the non-dimensional variables (3.1) as $\tau \to 0$

$$\bar{S}^*(\tau) \sim \begin{cases} 1 - \sqrt{-4\tau \ln\left(2\sqrt{\pi}\beta\tau^{\frac{1}{2}}\right)} & \text{for } \beta > 0, \\ 1 - \sqrt{-4\tau \ln\left(4\sqrt{\pi}(\alpha-\beta)\tau\right)} & \text{for } \beta = 0, \\ \frac{\alpha}{\alpha-\beta}\left(1 - 0.90344\tau^{\frac{1}{2}}\right) & \text{for } \beta < 0, \end{cases} \tag{3.38}$$

which are consistent with the work of previous authors.

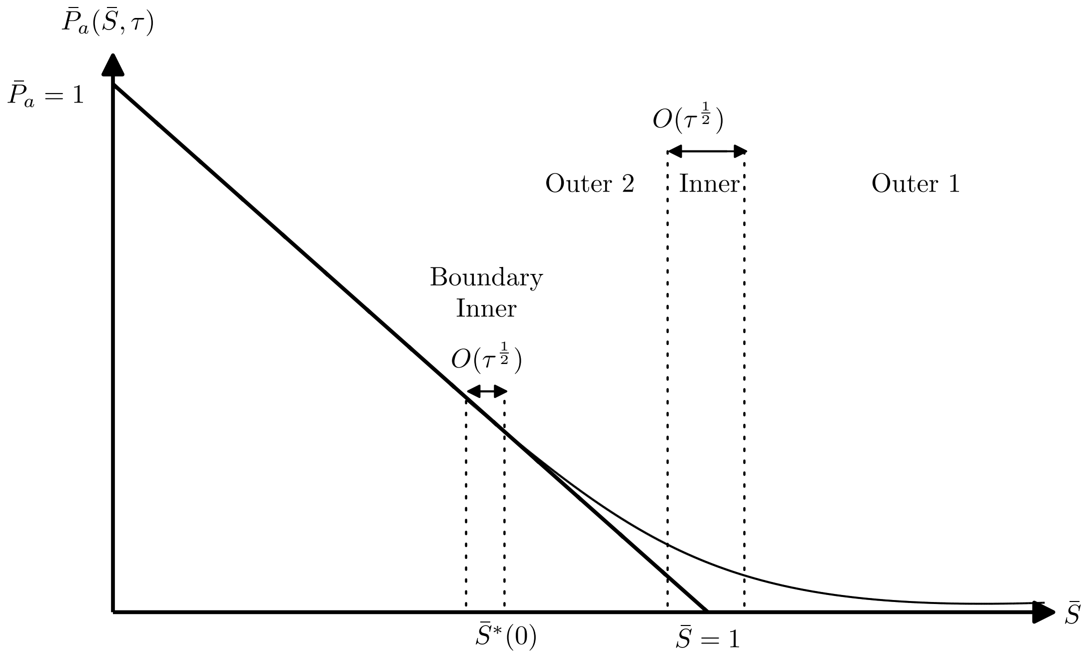


Figure 3-2: A schematic showing the small-time asymptotic structure of the American put option for $\beta < 0$. The asymptotic structure uses an $O(\sqrt{\tau})$ inner layer about $\bar{S} = 1$ in which the price is $O(\sqrt{\tau})$. The optimal exercise boundary lies at $O(\sqrt{\tau})$ from the initial starting point, $\bar{S}^*(0) = \frac{\alpha}{\alpha-\beta}$. For $\bar{S} < 1$ an outer region exists which bridges from the inner region to a region $O(\sqrt{\tau})$ about the boundary, in which the price can be represented as an algebraic series in powers of τ. For $\bar{S} > 1$ a second outer region exists in which the price is exponentially small and can be shown to have the leading order asymptotic behaviour $\bar{P}_a \sim \frac{2e^{-\alpha\tau}}{\sqrt{\pi}} \frac{\tau^{\frac{3}{2}} \bar{S}^{\frac{1-\beta}{2}}}{\left(\ln(\bar{S})\right)^2} e^{-\frac{(\ln(\bar{S}))^2}{4\tau}}$

3.2 Large-Time Behaviour

Unlike the European option problem, the steady-state solution to the American option is non-trivial, and is identified by Merton [84] as one of the situations in which American options have a closed-form solution. To our knowledge, the asymptotic behaviour of the American put option problem in the approach to perpetuity is not derived in the literature, though Knessl [73] does show that the difference between the large-time optimal exercise boundary and the perpetual boundary is at least exponentially small.

3.2.1 The Perpetual American Put Option

The perpetual American put option problem is the steady-state solution to the problem (3.2a-3.2f), or

$$\bar{S} \in (\bar{S}^*_\infty, \infty) \qquad \bar{S}^2 \frac{\partial^2 \bar{P}^\infty_a}{\partial \bar{S}^2} + \beta \bar{S} \frac{\partial \bar{P}^\infty_a}{\partial \bar{S}} - \alpha \bar{P}^\infty_a = 0, \tag{3.39a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{P}^\infty_a(\bar{S}) \to 0, \tag{3.39b}$$

$$\text{at } \bar{S} = \bar{S}^*_\infty \qquad \bar{P}^\infty_a\left(\bar{S}^*\right) = \left(1 - \bar{S}^*_\infty\right), \tag{3.39c}$$

$$\left.\frac{\partial \bar{P}^\infty_a}{\partial \bar{S}}\right|_{\bar{S}^*_\infty} = -1. \tag{3.39d}$$

The large $\bar{S}$ condition (3.39b) requires the decaying solution to the ODE (3.39a)

$$\bar{P}^\infty_a(\bar{S}) = \bar{A}^\infty_0 \bar{S}^{\lambda^\infty_-}, \tag{3.40}$$

where

$$\lambda^\infty_- = \frac{-(\beta - 1) - \sqrt{(\beta - 1)^2 + 4\alpha}}{2}, \tag{3.41}$$

while the early exercise conditions (3.39c) and (3.39d) can be used to find the expressions for $\bar{A}^\infty_0$ and the perpetual optimal exercise boundary $\bar{S}^*_\infty$

$$\bar{A}^\infty_0 = \frac{\left(1 - \bar{S}^*_\infty\right)}{(\bar{S}^*_\infty)^{\lambda^\infty_-}}, \tag{3.42}$$

$$\bar{S}^*_\infty = \frac{\lambda^\infty_-}{\lambda^\infty_- - 1}, \tag{3.43}$$

which leads to the non-dimensional price for a perpetual American option

$$\bar{P}^\infty_a(\bar{S}) = \left(1 - \bar{S}^*_\infty\right) \left(\frac{\bar{S}}{\bar{S}^*_\infty}\right)^{\lambda^\infty_-}. \tag{3.44}$$

No-arbitrage requires that $\bar{P}_\infty(\bar{S})$ forms an upper bound on $\bar{P}_a(\bar{S}, \hat{T})$, while $\bar{S}^*_\infty$ forms a lower bound on $\bar{S}^*(\tau)$.

3.2.2 Perturbative Behaviour to the Perpetual Problem

To investigate the perturbative behaviour to the steady-state solution, we introduce the small parameter $0 < \epsilon^2 \ll 1$ through the time scaling

$$\tau = \frac{\hat{T}}{\epsilon^2}, \tag{3.45}$$

where $\hat{T} = O(1)$. Using the spatial transformation $\bar{X} = \ln(\bar{S})$ and defining $\bar{P}_a(\bar{S}, \hat{T}) = \hat{P}_a(\bar{X}, \hat{T})$, leads to the large-time problem

$$\bar{X} \in (\ln(\bar{S}^*(\hat{T})), \infty), \hat{T} \in (0, \infty) \qquad \epsilon^2 \frac{\partial \hat{P}_a}{\partial \hat{T}} = \frac{\partial^2 \hat{P}_a}{\partial \bar{X}^2} + (\beta - 1)\frac{\partial \hat{P}_a}{\partial \bar{X}} - \alpha \hat{P}_a, \tag{3.46a}$$

subject to

$$\text{as } \bar{X} \to \infty \qquad \hat{P}_a\left(\bar{X}, \hat{T}\right) \to 0, \tag{3.46b}$$

$$\text{at } \bar{X} = \ln(\bar{S}^*(\hat{T})) \qquad \hat{P}_a\left(\ln(\bar{S}^*(\hat{T})), \hat{T}\right) = 1 - \bar{S}^*(\hat{T}), \tag{3.46c}$$

$$\left.\frac{\partial \hat{P}_a}{\partial \bar{X}}\right|_{\ln(\bar{S}^*(\hat{T}))} = -\bar{S}^*(\hat{T}). \tag{3.46d}$$

Under $\bar{X}$, the perpetual option solution (3.44-3.43) is $\bar{P}_a^{\infty}(\bar{S}, \hat{T}) = \hat{P}_a^{\infty}(\bar{X}, \hat{T})$ where

$$\hat{P}_a^{\infty}(\bar{X}) = \left(1 - \bar{S}_{\infty}^*\right) e^{\lambda_-^{\infty}\left(\bar{X} - \ln(\bar{S}_{\infty}^*)\right)}. \tag{3.47}$$

Outer Region

In the outer region, which we define by $\bar{X} - \ln(\bar{S}_{\infty}^*) = O(1)$ and $\hat{P}_a(\bar{X}, \hat{T}) = \hat{P}_a^{Out}(\bar{X}, \hat{T}; \epsilon)$, a regular expansion in powers of ϵ fails to capture the perturbative behaviour to the perpetual problem, which is consistent with the findings of Knessl [73]. Instead we consider a WKBJ expansion of the form

$$\hat{P}_a^{Out}(\bar{X}, \hat{T}; \epsilon) = \hat{P}_{\infty}(\bar{X}) + \hat{P}_0^{Out}(\bar{X}) e^{-\frac{\lambda \hat{T}}{\epsilon^2}} + o\left(e^{-\frac{\lambda \hat{T}}{\epsilon^2}}\right) \qquad \text{as } \epsilon \to 0, \tag{3.48a}$$

$$\bar{S}^*(\hat{T}) = \bar{S}_{\infty}^* + \bar{S}_0^* e^{-\frac{\lambda \hat{T}}{\epsilon^2}} + o\left(e^{-\frac{\lambda \hat{T}}{\epsilon^2}}\right) \qquad \text{as } \epsilon \to 0, \tag{3.48b}$$

with the form of the controlling factor obvious from (3.46a). Expanding the boundary conditions about the perpetual option price and optimal exercise boundary gives the problem for the correction terms $\left(\bar{P}_0, \bar{S}_0^*\right)$

$$\bar{X} \in \left(\ln(\bar{S}_\infty^*), \infty\right), \hat{T} \in (0,\infty) \qquad \frac{\partial^2 \hat{P}_0^{Out}}{\partial \bar{X}^2} + (\beta - 1)\frac{\partial \hat{P}_0^{Out}}{\partial \bar{X}} - (\alpha - \lambda)\,\hat{P}_0^{Out} = 0, \tag{3.49a}$$

subject to

$$\text{as } \bar{X} \to \infty \qquad \hat{P}_0^{Out}\left(\bar{X}\right) \to 0, \tag{3.49b}$$

$$\text{at } \bar{X} = \ln(\bar{S}_\infty^*) \qquad \hat{P}_0^{Out}\left(\ln(\bar{S}_\infty^*)\right) = 0, \tag{3.49c}$$

$$\left.\frac{\partial \hat{P}_0^{Out}}{\partial \bar{X}}\right|_{\ln(\bar{S}_\infty^*)} = (\lambda_-^\infty - 1)\bar{S}_0^*. \tag{3.49d}$$

The form of the solution to the ODE (3.49a) depends on the magnitude of λ, which determines the nature of the roots of the characteristic equation. The boundary conditions, together with the restriction that the correction term should be strictly negative, suggest we look for the repeated root solution given by

$$\lambda = \frac{(\beta - 1)^2}{4} + \alpha, \tag{3.50}$$

which leads to

$$\hat{P}_0^{Out}(\bar{X}) = (\lambda_-^\infty - 1)\bar{S}_0^* e^{-\frac{\beta-1}{2}\left(\bar{X} - \ln(\bar{S}_\infty^*)\right)}\left(\bar{X} - \ln(\bar{S}_\infty^*)\right). \tag{3.51}$$

Thus we have the large-time behaviour in the outer region

$$\begin{aligned}\hat{P}_a^{Out}(\bar{X}, \hat{T}) \sim{}& \left(1 - \bar{S}_\infty^*\right) e^{\lambda_-^\infty\left(\bar{X} - \ln(\bar{S}_\infty^*)\right)} \\ &+ (\lambda_-^\infty - 1)\bar{S}_0^* e^{-\frac{\lambda\hat{T}}{\epsilon^2}} e^{-\frac{\beta-1}{2}\left(\bar{X} - \ln(\bar{S}_\infty^*)\right)}\left(\bar{X} - \ln(\bar{S}_\infty^*)\right) \quad \text{as } \epsilon \to 0,\end{aligned} \tag{3.52}$$

where the value of $\bar{S}_0^*$ is determined by matching back to the optimal exercise boundary at earlier times, with the constraint $\bar{S}_0^* > 0$ required as the steady-state boundary forms a lower bound on the finite time boundary. This implies $(\lambda_-^\infty - 1)\bar{S}_0^* < 0$, which is consistent with the requirement that the perpetual option price is an upper bound on the finite time price.

We highlight two areas of concern in relation to the expression derived in (3.52). The first is that it only satisfies the large-$\bar{X}$ condition (3.49b) for $\beta > 1$. Secondly, the restriction that the option price must be strictly positive indicates our expansion breaks down if the correction term is of comparable magnitude to the perpetual option prices. We therefore look to see if a region containing such behaviour exists, and whether a modification of the problem in this region leads to a solution which satisfies (3.49b).

Transition Region

To identify whether the correction term can become of similar magnitude to the perpetual option price we look for some value $\bar{Z}(\hat{T}) = \bar{X} - \ln(\bar{S}^*_\infty)$ which satisfies the transcendental expression

$$e^{\lambda_0 \bar{Z}(\hat{T})} = -k\hat{S}^*_0 e^{-\frac{\lambda\hat{T}}{\epsilon^2}} \bar{Z}(\hat{T}) \quad \text{as } \epsilon \to 0, \tag{3.53}$$

with k as yet undetermined, but restricted to $k > 1$ in order that $\hat{P}^{Out}_a\left(\bar{Z}(\hat{T}), \hat{T}\right)$ is positive, and where

$$\lambda_0 = \lambda^\infty_- + \frac{\beta - 1}{2}, \tag{3.54}$$

$$\hat{S}^*_0 = (\lambda^\infty_- - 1)^2 \bar{S}^*_0. \tag{3.55}$$

Noting that $\lambda_0 < 0$, the location of $\bar{Z}(\hat{T})$ has the asymptotic behaviour

$$\bar{Z}(\hat{T}) \sim -\frac{\lambda\hat{T}}{\lambda_0\epsilon^2} + \frac{1}{\lambda_0}\ln\left(k\hat{S}^*_0\frac{\lambda\hat{T}}{\lambda_0\epsilon^2}\right) - \frac{\epsilon^2}{\lambda_0\lambda\hat{T}}\ln\left(k\hat{S}^*_0\frac{\lambda\hat{T}}{\lambda_0\epsilon^2}\right) + O\left(\epsilon^4\left(\ln\left(\frac{1}{\epsilon^2}\right)\right)^2\right) \quad \text{as } \epsilon \to 0. \tag{3.56}$$

Locally to $\bar{Z}(\hat{T})$ the expansions (3.48a) and (3.48b) break down and we look for a *transition region* by introducing the small parameter $0 < \delta(\epsilon) \ll 1$ and transition region variable $\bar{Y}$ through the scaling

$$\bar{X} - \ln(\bar{S}^*_\infty) = \bar{Z}(\hat{T}) + \delta(\epsilon)\bar{Y}, \tag{3.57}$$

where $\bar{Y} = O(1)$.

In terms of the transition region variable, the outer expression is

$$\begin{aligned}\check{P}^{Out}_a(\bar{Y}, \hat{T}; \epsilon) = &\frac{\hat{S}^*_0}{\lambda_0(\lambda^\infty_- - 1)} e^{-\frac{\lambda\lambda^\infty_-\hat{T}}{\lambda_0\epsilon^2}} \left(k\hat{S}^*_0\frac{\lambda\hat{T}}{\lambda_0\epsilon^2}\right)^{-\frac{\beta-1}{2\lambda_0}} \left[(k-1)\left(\frac{\lambda\hat{T}}{\epsilon^2}\right) - (k-1)\lambda^\infty_-\ln\left(k\hat{S}^*_0\frac{\lambda\hat{T}}{\lambda_0\epsilon^2}\right)\right. \\ &\left. + \lambda\left(k\lambda^\infty_- + \frac{\beta-1}{2}\right)\bar{Y}\hat{T}\left(\frac{\delta}{\epsilon^2}\right) + O\left(\epsilon^2\left(\ln\left(\frac{1}{\epsilon^2}\right)\right)^2\right)\right] \quad \text{as } \epsilon \to 0,\end{aligned} \tag{3.58}$$

where the constraint $k > 1$ ensures that the price is positive in the transition region. We note that the first two terms in the series for $\bar{Z}(\hat{T})$, together with the restriction on k, are required to produce a positive price in the transition region. This suggests a transition region expression with the form

$$\bar{P}_a(\bar{X},\hat{T}) = \frac{\hat{S}_0^*}{\lambda_0(\lambda_-^\infty - 1)} e^{-\frac{\lambda\lambda_-^\infty \hat{T}}{\lambda_0 \epsilon^2}} \left(k\hat{S}_0^* \frac{\lambda \hat{T}}{\lambda_0 \epsilon^2} \right)^{-\frac{\beta-1}{2\lambda_0}} \check{P}_a^{Tran}(\bar{Y},\hat{T};\epsilon) \quad \text{as } \epsilon \to 0, \tag{3.59}$$

which, noting that

$$\frac{\partial \bar{Z}}{\partial \hat{T}} \sim -\frac{\lambda}{\lambda_0 \epsilon^2} + \frac{1}{\lambda_0 \hat{T}} + \frac{1}{\lambda_0} \frac{\epsilon^2}{\lambda \hat{T}^2} \ln \left(k\hat{S}_0^* \frac{\lambda \hat{T}}{\lambda_0 \epsilon^2} \right) + O\left(\epsilon^2\right) \quad \text{as } \epsilon \to 0, \tag{3.60}$$

gives the transition region problem

$$\bar{Y} \in (-\infty,\infty), \hat{T} \in (0,\infty) \qquad \epsilon^2 \frac{\partial \check{P}_a^{Tran}}{\partial \hat{T}} = \frac{1}{\delta^2} \frac{\partial^2 \check{P}_a^{Tran}}{\partial \bar{Y}^2} + \frac{1}{\delta} \left(\frac{(\beta-1)}{2} - \lambda_-^\infty + O(\epsilon^2) \right) \frac{\partial \check{P}_a^{Tran}}{\partial \bar{Y}}$$
$$- \frac{\beta-1}{2} \left(\lambda_-^\infty - \frac{\epsilon^2}{\lambda_0 \hat{T}} \right) \check{P}_a^{Tran}, \tag{3.61a}$$

subject to

$$\bar{Y} \to -\infty \qquad \check{P}_a^{Tran} \sim \hat{P}_a^{Out}. \tag{3.61b}$$

Expression (3.58) suggests $\check{P}_a^{Tran}(\bar{Y},\hat{T};\epsilon)$ has the form

$$\check{P}_a^{Tran}(\bar{Y},\hat{T};\epsilon) = \frac{1}{\epsilon^2} \check{P}_0^{Tran}(\bar{Y},\hat{T}) + \ln \left(k\hat{S}_0^* \frac{\lambda \hat{T}}{\lambda_0 \epsilon^2} \right) \check{P}_1^{Tran}(\bar{Y},\hat{T}) + \left(\frac{\delta}{\epsilon^2} \right) \check{P}_2^{Trans}(\bar{Y},\hat{T})$$
$$+ O\left(\epsilon^2 \left(\ln \left(\frac{1}{\epsilon^2} \right) \right)^2 \right) \quad \text{as } \epsilon \to 0, \tag{3.62}$$

which leads to subproblems with solutions

$$\check{P}_0^{Tran}(\bar{Y},\hat{T}) = (k-1)\lambda \hat{T}, \tag{3.63a}$$

$$\check{P}_1^{Tran}(\bar{Y},\hat{T}) = -(k-1)\lambda_-^\infty, \tag{3.63b}$$

$$\check{P}_2^{Tran}(\bar{Y},\hat{T}) = \lambda \left(k\lambda_-^\infty + \frac{\beta-1}{2} \right) \bar{Y}\hat{T}, \tag{3.63c}$$

where the solution to the problem for $\check{P}_2^{Trans}$ requires $\delta \ll \epsilon$.

Far-Field

To complete the structure, we look for a *far-field* region beyond the transition region, defined through the scaling $\bar{X} - \bar{X}^*_\infty = \bar{Z}(\hat{T}) + \bar{Y}_2$, where $\bar{Y}_2 = O(1)$, in which the price approaches zero as $\bar{Y}_2 \to \infty$. Motivated by the transition region expression, we write the far-field expression in the form

$$\bar{P}_a(\bar{X}, \hat{T}) = \frac{\hat{S}^*_0}{\lambda_0(\lambda^\infty_- - 1)} e^{-\frac{\lambda\lambda^\infty_- \hat{T}}{\lambda_0 \epsilon^2}} \left(k\hat{S}^*_0 \frac{\lambda \hat{T}}{\lambda_0 \epsilon^2} \right)^{-\frac{\beta-1}{2\lambda_0}} \tilde{P}^{Far}_a(\bar{Y}_2, \hat{T}; \epsilon) \quad \text{as } \epsilon \to 0, \tag{3.64}$$

for some general series in ϵ, which gives the far-field problem

$$\bar{Y}_2 \in (0,\infty), \hat{T} \in (0,\infty) \qquad \epsilon^2 \frac{\partial \tilde{P}^{Far}_a}{\partial \hat{T}} = \frac{\partial^2 \tilde{P}^{Far}_a}{\partial \bar{Y}_2^2} + \left(\frac{(\beta-1)}{2} - \lambda^\infty_- + O(\epsilon^2) \right) \frac{\partial \tilde{P}^{Far}_a}{\partial \bar{Y}_2} - \frac{\beta-1}{2} \left(\lambda^\infty_- - \frac{\epsilon^2}{\lambda_0 \hat{T}} \right) \tilde{P}^{Far}_a, \tag{3.65a}$$

subject to

$$\bar{Y}_2 \to \infty \qquad \tilde{P}^{Far}_a\left(\bar{Y}_2, \hat{T}\right) \to 0. \tag{3.65b}$$

The leading-order terms in the transition region expression suggest posing the series

$$\tilde{P}^{Far}_a(\bar{Y}_2, \hat{T}; \epsilon) = \frac{1}{\epsilon^2} \tilde{P}^{Far}_0(\bar{Y}_2, \hat{T}) + \ln\left(k\hat{S}^*_0 \frac{\lambda \hat{T}}{\lambda_0 \epsilon^2} \right) \tilde{P}^{Far}_1(\bar{Y}_2, \hat{T}) + O\left(\epsilon^2 \ln\left(\frac{1}{\epsilon^2} \right) \right) \text{ as } \epsilon \to 0, \tag{3.66}$$

which leads to the following subproblems: for $\tilde{P}^{Far}_0(\bar{Y}_2, \hat{T})$

$$\bar{Y}_2 \in (0,\infty), \hat{T} \in (0,\infty) \qquad \frac{\partial^2 \tilde{P}^{Far}_0}{\partial \bar{Y}_2^2} + \left(\frac{(\beta-1)}{2} - \lambda^\infty_- \right) \frac{\partial \tilde{P}^{Far}_0}{\partial \bar{Y}_2} - \frac{\beta-1}{2} \lambda^\infty_- \tilde{P}^{Far}_0 = 0, \tag{3.67a}$$

subject to

$$\bar{Y}_2 \to \infty \qquad \tilde{P}^{Far}_0\left(\bar{Y}_2, \hat{T}\right) \to 0; \tag{3.67b}$$

and for $\tilde{P}_1^{Far}(\bar{Y}_2, \hat{T})$

$$\bar{Y}_2 \in (0,\infty), \hat{T} \in (0,\infty) \qquad \frac{\partial^2 \tilde{P}_1^{Far}}{\partial \bar{Y}_2^2} + \left(\frac{(\beta-1)}{2} - \lambda_-^{\infty}\right)\frac{\partial \tilde{P}_1^{Far}}{\partial \bar{Y}_2} - \frac{\beta-1}{2}\lambda_-^{\infty}\tilde{P}_1^{Far} = 0, \tag{3.68a}$$

subject to

$$\bar{Y}_2 \to \infty \qquad \tilde{P}_1^{Far}\left(\bar{Y}_2, \hat{T}\right) \to 0. \tag{3.68b}$$

These problems have the general solutions

$$\tilde{P}_0^{Far}(\bar{Y}_2, \hat{T}) = \tilde{C}_{00}^{Far}(\hat{T})e^{-\frac{\beta-1}{2}\bar{Y}_2} + \left(\tilde{C}_{01}^{Far}(\hat{T}) - \tilde{C}_{00}^{Far}(\hat{T})\right)e^{\lambda_-^{\infty}\bar{Y}_2}, \tag{3.69a}$$

$$\tilde{P}_1^{Far}(\bar{Y}_2, \hat{T}) = \tilde{C}_{10}^{Far}(\hat{T})e^{-\frac{\beta-1}{2}\bar{Y}_2} + \left(\tilde{C}_{11}^{Far}(\hat{T}) - \tilde{C}_{10}^{Far}(\hat{T})\right)e^{\lambda_-^{\infty}\bar{Y}_2}, \tag{3.69b}$$

for undetermined coefficients $\tilde{C}_{00}^{Far}$, $\tilde{C}_{01}^{Far}$, $\tilde{C}_{10}^{Far}$, $\tilde{C}_{11}^{Far}$. The large $\bar{Y}_2$ conditions (3.67b), (3.68b) do not specify any constraint on the coefficients for $\beta > 1$, however for $\beta \leq 1$ we have

$$\tilde{C}_{00}^{Far}(\hat{T}) = \tilde{C}_{10}^{Far}(\hat{T}) = 0. \tag{3.70}$$

The remaining coefficients are determined by matching back to the transition region, which we do by writing $\bar{Y}_2 = \delta\bar{Y}$ in the far-field expression, that is $\tilde{P}_a^{Far}(\bar{Y}_2, \hat{T}) = \hat{P}_a^{Far}(\bar{Y}, \hat{T})$ where for $\beta > 1$

$$\begin{aligned}\hat{P}_a^{Far}(\bar{Y}, \hat{T}; \epsilon) =& \frac{1}{\epsilon^2}\left(\tilde{C}_{01}^{far} + \delta\left(\lambda_-^{\infty}\tilde{C}_{01}^{Far} - \lambda_0\tilde{C}_{00}^{Far}\right)\bar{Y} + \dots\right) \\ &+ \ln\left(k\hat{S}_0^*\frac{\lambda\hat{T}}{\lambda_0\epsilon^2}\right)\left(\tilde{C}_{11}^{Far} + \delta\left(\lambda_-^{\infty}\tilde{C}_{11}^{Far} - \lambda_0\tilde{C}_{10}^{Far}\right)\bar{Y} + \dots\right) + O\left(\frac{1}{\epsilon^2}\ln\left(\epsilon^2\right)\right)\end{aligned} \tag{3.71}$$

and for $\beta \leq 1$

$$\begin{aligned}\hat{P}_a^{Far}(\bar{Y}, \hat{T}) =& \frac{1}{\epsilon^2}\left(\tilde{C}_{01}^{Far} + \delta\lambda_-^{\infty}\tilde{C}_{01}^{Far}\bar{Y} + \dots\right) \\ &+ \ln\left(k\hat{S}_0^*\frac{\lambda\hat{T}}{\lambda_0\epsilon^2}\right)\left(\tilde{C}_{11}^{Far} + \delta\lambda_-^{\infty}\tilde{C}_{11}^{Far}\bar{Y} + \dots\right) + O\left(\epsilon^2\ln\left(\frac{1}{\epsilon^2}\right)\right).\end{aligned} \tag{3.72}$$

Matching (3.71) with (3.58) for $\beta > 1$ occurs for any $0 < \delta \ll \epsilon$ with

$$\tilde{C}_{01}^{Far}(\hat{T}) = (k-1)\lambda\hat{T}, \tag{3.73a}$$

$$\tilde{C}_{00}^{Far}(\hat{T}) = -\lambda\hat{T}, \tag{3.73b}$$

$$\tilde{C}_{11}^{Far}(\hat{T}) = -(k-1)\lambda_{-}^{\infty}, \tag{3.73c}$$

while the matching of $\tilde{C}_{10}^{Far}(\hat{T})$ requires additional terms in the transition region expansions, but can be shown to be $\tilde{C}_{10}^{Far}(\hat{T}) = \lambda_{-}^{\infty}$.

Matching (3.72) with (3.58) for $\beta \leq 1$ only occurs for exponentially small δ with

$$\tilde{C}_{01}^{Far}(\hat{T}) = (k-1)\lambda\hat{T}, \tag{3.74a}$$

$$\tilde{C}_{00}^{Far}(\hat{T}) = 0, \tag{3.74b}$$

$$\tilde{C}_{11}^{Far}(\hat{T}) = -(k-1)\lambda_{-}^{\infty}, \tag{3.74c}$$

while the matching of $\tilde{C}_{10}^{Far}(\hat{T})$ requires additional terms in the transition region expansions, but can be shown to be $\tilde{C}_{10}^{Far}(\hat{T}) = 0$.

Specification of the parameter k, which translates the location of the transition region, is determined by the behaviour of the option price as it approaches from an earlier time.

The large-time asymptotic structure of the American put option problem is shown in Figure 3-3, while a comparison of the large-time asymptotic expressions for the optimal exercise boundary with the benchmark MOL numerics are shown in Figure 3-4.

3.3 An Analytic Approximation for the American Put Option

Having derived the small- and large-time asymptotic behaviour of the American put option we now look to bridge the gap to include intermediate cases. Our starting point is the uniformly valid approximation to the full American option problem derived by MacMillan [79] in the absence of dividends, which was extended to include a general cost-of-carry by Barone-Adesi & Whaley [10]. We shall call this approach the MBAW approximation, with the associated boundary and price approximations termed the MBAW boundary and MBAW price.

Attempts have previously been made to extend this work by looking for correction terms to the MBAW boundary or price, notably Ju & Zhong [69]. As yet no attempt has been made to look for series solutions to both the price and the boundary using the MBAW boundary and price as

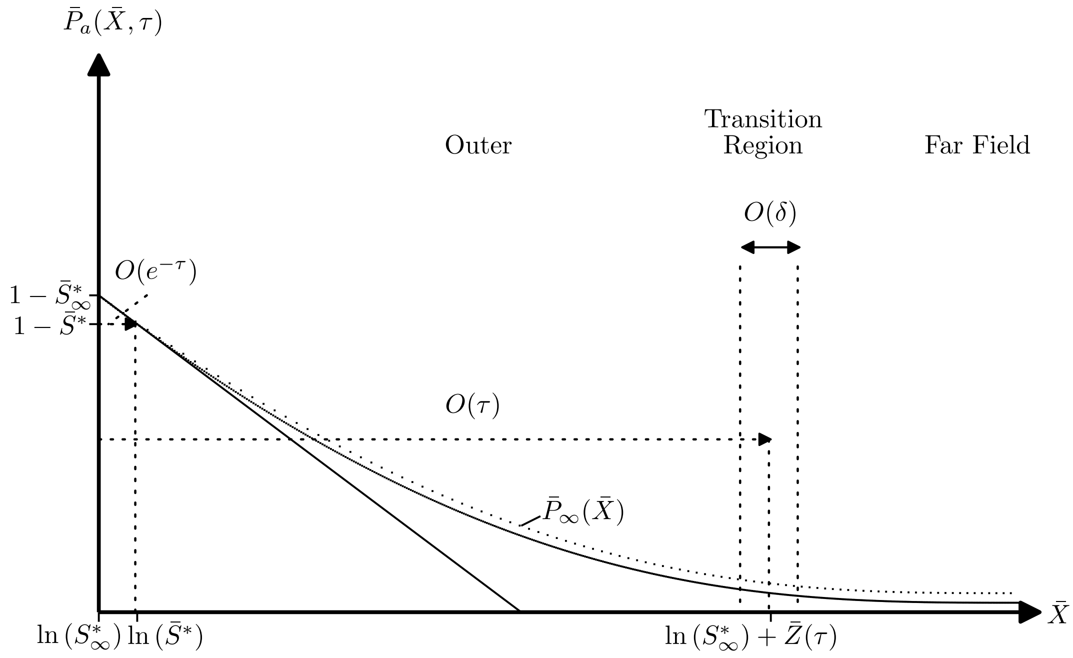
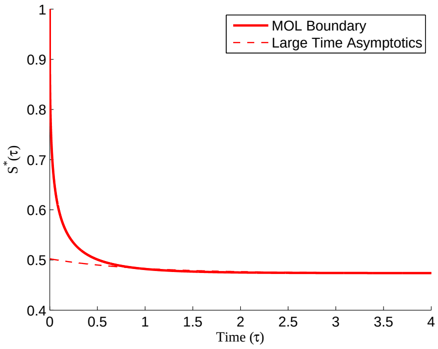


Figure 3-3: A schematic showing the large-time asymptotic structure of the American put option. The asymptotic structure has an outer expression which is valid up to $\bar{X} - \ln(\bar{S}^*_\infty) = O(\tau)$, where a transition region is required to match into a decaying solution in the far-field. In order to match with the far-field expression, the width of the transition region is determined to be $o(\sqrt{\tau})$ for $\beta > 1$ and is at least exponentially small in the case $\beta \leq 1$. The perturbative terms to the perpetual option price and optimal exercise boundary are both exponentially small.

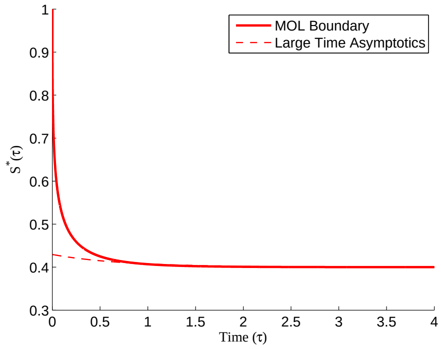
a basis, and this is the aim of the work in this section. To derive the subproblems for the terms in our series, we utilise an approach used by Zhu [106] in developing what he terms a closed-form solution to the American put option problem, but in practice is an infinite series which requires truncation to be determined numerically. We first decompose the American put option price into the equivalent European option price plus a premium which reflects the value of the holder's right to exercise early, as discussed in Chapter 1. Using the same time transformation and assuming the same form for the premium as that used in the MBAW approximation, we pose the premium as a homotopic series in an artificial parameter p, with the MBAW problem corresponding to the case $p \to 0$, and the full American option premium problem corresponding to the case $p \to 1$.

3.3.1 The American Option Premium

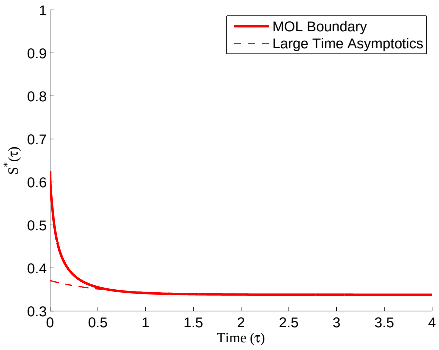
The difference between the European and American put option price can be thought of as a premium a holder is prepared to pay for the right to exercise early and receive the intrinsic value.



(a) $\alpha = \frac{10}{9}, \beta = \frac{2}{3}$



(b) $\alpha = \frac{10}{9}, \beta = 0$



(c) $\alpha = \frac{10}{9}, \beta = -\frac{2}{3}$

Figure 3-4: The large-time asymptotic behavior of the optimal exercise boundary of the American put option obtained from (3.48b). The constant $\bar{S}_0^*$ is determined by matching back into the optimal exercise boundary at $\tau = O(1)$.

Since the European option forms a lower bound on the American option price, the premium is always positive for $\tau > 0$. Mathematically, because both the European and American option prices satisfy the Black-Scholes-Merton PDE in the unexercised region, the American option premium must also satisfy the same PDE, though subject to different boundary conditions. This result is adopted in the body of work looking at approximations to the PDE [10, 69, 79] and also in the integral formulation of this decomposition developed by Kim [72], Jacka [65] and Carr et al. [28].

We define the non-dimensional American put option premium $\bar{p}_a(\bar{S}, \tau)$ through

$$\bar{P}_a(\bar{S}, \tau) = \bar{P}_e(\bar{S}, \tau) + \bar{p}_a(\bar{S}, \tau), \tag{3.75}$$

where the premium solves

$$\bar{S} \in (\bar{S}^*(\tau), \infty), \tau \in (0, T) \qquad \frac{\partial \bar{p}_a}{\partial \tau} = \bar{S}^2 \frac{\partial^2 \bar{p}_a}{\partial \bar{S}^2} + \beta \bar{S} \frac{\partial \bar{p}_a}{\partial \bar{S}} - \alpha \bar{p}_a, \tag{3.76a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{p}_a(\bar{S}, \tau) \to 0, \tag{3.76b}$$

$$\text{at } \tau = 0 \qquad \bar{p}_a(\bar{S}, 0) = 0, \tag{3.76c}$$

$$\bar{S}^*(0) = \min(\alpha/(\alpha - \beta), 1), \tag{3.76d}$$

$$\text{at } \bar{S} = \bar{S}^*(\tau) \qquad \bar{p}_a\left(\bar{S}^*, \tau\right) = \left(1 - \bar{S}^*\right) - \bar{P}_e\left(\bar{S}^*, \tau\right), \tag{3.76e}$$

$$\left.\frac{\partial \bar{p}_a}{\partial \bar{S}}\right|_{\bar{S}^*} = -1 - \left.\frac{\partial \bar{P}_e}{\partial \bar{S}}\right|_{\bar{S}^*}. \tag{3.76f}$$

3.3.2 The Approximation of MacMillan and Barone-Adesi & Whaley

To obtain a uniformly valid approximation, MacMillan [79] proposed decomposing the American option price into the European option price plus an early exercise premium (3.75). A time transformation $h(\tau)$ is introduced, where

$$h(\tau) = 1 - e^{-\alpha \tau}, \tag{3.77}$$

while the premium is assumed to have the form

$$\bar{p}_a(\bar{S}, \tau) = h(\tau) \bar{g}(\bar{S}, h), \tag{3.78}$$

which transforms the problem described in (3.76a-3.76d) into

$$\bar{S} \in (\bar{S}^*(h), \infty), h \in (0, 1) \qquad \bar{S}^2 \frac{\partial^2 \bar{g}}{\partial \bar{S}^2} + \beta \bar{S} \frac{\partial \bar{g}}{\partial \bar{S}} - \frac{\alpha}{h}\left(\bar{g} + h(1-h)\frac{\partial \bar{g}}{\partial h}\right) = 0, \tag{3.79a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{g}(\bar{S}, h) \to 0, \tag{3.79b}$$

$$\text{at } h = 0 \qquad h\bar{g}(\bar{S}, 0) = 0, \tag{3.79c}$$

$$\bar{S}^*(0) = \min\left(\alpha/(\alpha - \beta), 1\right), \tag{3.79d}$$

$$\text{at } \bar{S} = \bar{S}^*(h) \qquad h\bar{g}\left(\bar{S}^*, h\right) = \left(1 - \bar{S}^*\right) - \bar{P}_e(\bar{S}^*, h), \tag{3.79e}$$

$$h\frac{\partial \bar{g}}{\partial \bar{S}}\bigg|_{\bar{S}^*} = -1 - \frac{\partial \bar{P}_e}{\partial \bar{S}}\bigg|_{\bar{S}^*}. \tag{3.79f}$$

The key assumption made by MacMillan is that the term $\bar{g} \ll h(1-h)\frac{\partial \bar{g}}{\partial h}$, which is motivated by h being small near expiry, $1-h$ being small far from expiry and the product $h(1-h)$ having a maximum value of $\frac{1}{4}$. Neglecting this term, and denoting the price approximation under this assumption as $\bar{g}_0(\bar{S}, h)$ and the boundary approximation as $\bar{S}_0^*(h)$, results in the problem

$$\bar{S} \in (\bar{S}_0^*(h), \infty), h \in (0,1) \qquad \bar{S}^2\frac{\partial^2 \bar{g}_0}{\partial \bar{S}^2} + \beta\bar{S}\frac{\partial \bar{g}_0}{\partial \bar{S}} - \frac{\alpha}{h}\bar{g}_0 = 0, \tag{3.80a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{g}_0(\bar{S}, h) \to 0, \tag{3.80b}$$

$$\text{at } h = 0 \qquad h\bar{g}_0(\bar{S}, 0) = 0, \tag{3.80c}$$

$$\bar{S}_0^*(0) = \min\left(\alpha/(\alpha - \beta), 1\right), \tag{3.80d}$$

$$\text{at } \bar{S} = \bar{S}_0^*(h) \qquad h\bar{g}_0(\bar{S}_0^*, h) = \left(1 - \bar{S}_0^*\right) - \bar{P}_e(\bar{S}_0^*, h), \tag{3.80e}$$

$$h\frac{\partial \bar{g}_0}{\partial \bar{S}}\bigg|_{\bar{S}_0^*} = -1 - \frac{\partial \bar{P}_e}{\partial \bar{S}}\bigg|_{\bar{S}_0^*}. \tag{3.80f}$$

The assumption used to generate this approximation has the following implications:

- the governing equation is transformed from a PDE into a second order ODE, with the time variable h entering the approximation as a parameter in the ODE and via the early exercise conditions;
- together with the optimal exercise boundary this admits three conditions to be imposed to fully specify the problem, and therefore the problem for the approximation is overspecified; and
- the large $\bar{S}$ (3.80b) and early exercise conditions (3.80e) & (3.80f) may therefore be imposed upon the solution, but the asymptotic behaviour of the solution will need to be

checked to confirm whether or not it satisfies the conditions (3.80c) & (3.80d).

The problem (3.80a-3.80f) has solution

$$\bar{g} = A_{00}(h)\left(\frac{\bar{S}}{\bar{S}_0^*}\right)^{\lambda_-}, \tag{3.81}$$

where

$$\lambda_- = \frac{-(\beta-1) - \sqrt{(\beta-1)^2 + \frac{4\alpha}{h}}}{2}, \tag{3.82}$$

$$A_{00}(h) = \frac{\left(1-\bar{S}_0^*\right) - \bar{P}_e(\bar{S}_0^*, h)}{h} \tag{3.83}$$

and where the MBAW boundary $\bar{S}_0^*(h)$ is the solution to the transcendental expression

$$\lambda_-(h)\left(1 - \bar{S}_0^* - \bar{P}_e(\bar{S}_0^*, h)\right) = -\bar{S}_0^* - \bar{S}_0^* \left.\frac{\partial \bar{P}_e}{\partial \bar{S}}\right|_{\bar{S}_0^*}. \tag{3.84}$$

To investigate whether the asymptotic properties of the boundary approximation are consistent with those of the full problem (3.38) we look at the corresponding behaviour of (3.84) but, for ease of comparison, we perform this analysis in the time variable τ.

Small-Time Asymptotics of the MBAW Boundary

To look at the small-time asymptotic behaviour of the MBAW boundary, we again introduce a small parameter $\epsilon \ll 1$ via the time scaling $\tau = \epsilon^2 \hat{T}$ where $\hat{T} = O(1)$, under which the small-time behaviour of $\lambda_-(\hat{T})$ is simply

$$\lambda_-(\hat{T}) \sim -\frac{1}{\epsilon \hat{T}^{\frac{1}{2}}} - \frac{\beta-1}{2} - \frac{\epsilon \hat{T}^{\frac{1}{2}}}{4}\left(\frac{(\beta-1)^2}{2} - \alpha\right) + O\left(\epsilon^3\right) \qquad \text{as } \epsilon \to 0, \tag{3.85}$$

and the European put option evaluated at $\bar{S}_0^*$, which we define as $\bar{P}_e(\bar{S}_0^*, \hat{T}; \epsilon) = \bar{P}_e^*$, is

$$\bar{P}_e^* = \frac{1}{2}\left(e^{-\alpha\epsilon^2\hat{T}}\operatorname{erfc}\left(\frac{\bar{d}_2^*}{\sqrt{2}}\right) - \bar{S}_0^* e^{-(\alpha-\beta)\epsilon^2\hat{T}}\operatorname{erfc}\left(\frac{\bar{d}_1^*}{\sqrt{2}}\right)\right), \tag{3.86a}$$

where

$$\frac{\bar{d}_1^*}{\sqrt{2}} = \frac{\ln\left(\bar{S}_0^*\right)}{2\epsilon\hat{T}^{\frac{1}{2}}} + \frac{(\beta+1)}{2}\epsilon\hat{T}^{\frac{1}{2}}, \tag{3.86b}$$

$$\frac{\bar{d}_2^*}{\sqrt{2}} = \frac{\ln\left(\bar{S}_0^*\right)}{2\epsilon\hat{T}^{\frac{1}{2}}} + \frac{(\beta-1)}{2}\epsilon\hat{T}^{\frac{1}{2}} \tag{3.86c}$$

and from our work on the small-time asymptotics of the European option problem (2.90), if $\frac{\bar{d}_1}{\sqrt{2}}, \frac{\bar{d}_2}{\sqrt{2}} \to -\infty$ as $\epsilon \to 0$ then (3.86a) has the behaviour

$$\begin{aligned} \bar{P}_e^* \sim &(1-\bar{S}_0^*) + \epsilon^2 \hat{T}\left((\alpha-\beta)\bar{S}_0^* - \alpha\right) + \frac{\epsilon^4 \hat{T}^2}{2}\left(\alpha^2 - (\alpha-\beta)^2 \bar{S}_0^*\right) + O(\epsilon^6) \\ &+ \frac{2}{\sqrt{\pi}} \frac{\epsilon^3 \hat{T}^{\frac{3}{2}}}{(\ln(\bar{S}_0^*))^2} \frac{e^{-\frac{(\ln(\bar{S}_0^*))^2}{4\epsilon^2 \hat{T}}}}{(\bar{S}_0^*)^{\frac{\beta-1}{2}}} \left(1 + O\left(\epsilon^2\right)\right) \text{ as } \epsilon \to 0 \end{aligned} \tag{3.87}$$

and therefore

$$\begin{aligned} 1 - \bar{S}_0^* - \bar{P}_e^* \sim &-\epsilon^2 \hat{T}\left((\alpha-\beta)\bar{S}_0^* - \alpha\right) - \frac{\epsilon^4 \hat{T}^2}{2}\left(\alpha^2 - (\alpha-\beta)^2 \bar{S}_0^*\right) - O(\epsilon^6) \\ &- \frac{2}{\sqrt{\pi}} \frac{\epsilon^3 \hat{T}^{\frac{3}{2}}}{(\ln(\bar{S}_0^*))^2} \frac{e^{-\frac{(\ln(\bar{S}_0^*))^2}{4\epsilon^2 \hat{T}}}}{(\bar{S}_0^*)^{\frac{\beta-1}{2}}} \left(1 + O\left(\epsilon^2\right)\right) \text{ as } \epsilon \to 0. \end{aligned} \tag{3.88}$$

We define the European put option evaluated at $\bar{S}_0^*$ as $\frac{\partial \bar{P}_e}{\partial \bar{S}}\Big|_{\bar{S}_0^*} = \frac{\partial \bar{P}_e^*}{\partial \bar{S}}$, so

$$\frac{\partial \bar{P}_e^*}{\partial \bar{S}} = -\frac{1}{2} e^{-(\alpha-\beta)\epsilon^2 \hat{T}} \operatorname{erfc}\left(\frac{\bar{d}_1^*}{\sqrt{2}}\right) \tag{3.89a}$$

and therefore

$$-\bar{S}_0^*\left(1 + \frac{\partial \bar{P}_e^*}{\partial \bar{S}}\right) \sim -(\alpha-\beta)\bar{S}_0^* \epsilon \hat{T} - O\left(\epsilon^4\right) \qquad \text{as } \epsilon \to 0,\ \ln(\bar{S}_0^*) < 0. \tag{3.89b}$$

From (3.85), (3.88) and (3.89b), the most obvious leading order behaviour comes at $O(\epsilon)$ on the LHS of (3.84) which requires

$$\bar{S}_0^* = \frac{\alpha}{\alpha - \beta}. \tag{3.90}$$

To find the perturbative behaviour we look for a new solution of the form $\bar{S}_0^* = \frac{\alpha}{\alpha-\beta} + \delta_1(\epsilon)\bar{S}_1^*$, where $\delta_1(\epsilon) \ll \frac{\alpha}{\alpha-\beta}$ and $\bar{S}_1^* = O(1)$, giving

$$-\bar{S}_0^*\left(1 + \frac{\partial \bar{P}_e^*}{\partial \bar{S}}\right) \sim -\alpha\epsilon^2 \hat{T} + O\left(\epsilon^4, \delta_1 \epsilon^2\right) \qquad \text{as } \epsilon \to 0 \tag{3.91}$$

and

$$1 - \bar{S}_0^* - \bar{P}_e^* \sim (\alpha-\beta)\delta_1 \epsilon^2 \bar{S}_1^* \hat{T} + O(\epsilon^4) \qquad \text{as } \epsilon \to 0, \tag{3.92}$$

from which we identify that the dominant balance occurs at $O(\epsilon^2)$ provided $\delta_1 = \epsilon$ and

$$\bar{S}_1^* = -\frac{\alpha}{\alpha - \beta}\hat{T}^{\frac{1}{2}}. \tag{3.93}$$

We can therefore identify a small-time asymptotic solution to (3.84) for $\ln(\bar{S}_0^*) < 0$ as

$$\bar{S}_0^* = \frac{\alpha}{\alpha - \beta}\left(1 - \epsilon\hat{T}^{\frac{1}{2}}\right) \qquad \text{as } \epsilon \to 0, \tag{3.94}$$

which holds providing $\beta < 0$, which we note is consistent with the corresponding asymptotic behaviour of the full problem.

As $\beta \to 0$, $\frac{\alpha}{\alpha-\beta} \to 1$ and for $\beta \geq 0$ we therefore look for an asymptotic expression of the form $\bar{S}_0^* = 1 + \delta_1(\epsilon)\bar{S}_1^*$, where $\delta_1(\epsilon) \ll 1$ and $\bar{S}_1^* = O(1)$. We note that, although $\ln(\bar{S}_0^*) \to 0$ as $\epsilon \to 0$, our work on the full problem suggests we should expect the ratio $\frac{\ln(\bar{S}_0^*)}{4\epsilon^2\hat{T}} \to -\infty$ as $\epsilon \to 0$. The components of our transcendental expression now become

$$1 - \bar{S}_0^* - \bar{P}_e^* \sim \beta\epsilon^2\hat{T} + O(\epsilon^4, \epsilon^2\delta_1) - \frac{2}{\sqrt{\pi}}\frac{\epsilon^3\hat{T}^{\frac{3}{2}}}{(\delta_1\bar{S}_1^*)^2}e^{-\frac{(\delta_1\bar{S}_1^*)^2}{4\epsilon^2\hat{T}}} \quad \text{as } \epsilon \to 0 \tag{3.95}$$

and

$$-\bar{S}_0^*\left(1 + \frac{\partial\bar{P}_e^*}{\partial\bar{S}}\right) \sim -(\alpha - \beta)\epsilon^2\hat{T} + O(\epsilon^4, \epsilon^2\delta_1) \qquad \text{as } \epsilon \to 0. \tag{3.96}$$

For $\beta > 0$, the dominant balance for $\bar{S}_1^*$ comes at $O(\epsilon)$ from (3.95) and (3.85), requiring

$$\beta\epsilon^2\hat{T} = \frac{2}{\sqrt{\pi}}\frac{\epsilon^3\hat{T}^{\frac{3}{2}}}{(\delta_1\bar{S}_1^*)^2}e^{-\frac{(\delta_1\bar{S}_1^*)^2}{4\epsilon^2\hat{T}}}, \tag{3.97}$$

or

$$\delta_1\bar{S}_1^* = -\sqrt{-4\epsilon^2\hat{T}\ln\left(2\sqrt{\pi}\beta\epsilon\hat{T}^{\frac{1}{2}}\right)} \tag{3.98}$$

and therefore the small-time asymptotic behaviour of the MBAW boundary is

$$\bar{S}_0^* \sim 1 - \sqrt{-4\epsilon^2\hat{T}\ln\left(2\sqrt{\pi}\beta\epsilon\hat{T}^{\frac{1}{2}}\right)} \qquad \text{as } \epsilon \to 0, \tag{3.99}$$

which we observe is the same as the asymptotic behaviour of the full problem (3.38).

For $\beta = 0$, the dominant balance for $\bar{S}_1^*$ comes at $O(\epsilon)$ from (3.96) with the algebraic terms all zero. We therefore require

$$\alpha\epsilon^2\hat{T} = \frac{2}{\sqrt{\pi}}\frac{\epsilon^2\hat{T}}{(\delta_1\bar{S}_1^*)^2}e^{-\frac{(\delta_1\bar{S}_1^*)^2}{4\epsilon^2\hat{T}}} \qquad \text{as } \epsilon \to 0, \tag{3.100}$$

or

$$\delta_1\bar{S}_1^* = -\sqrt{-4\epsilon^2\hat{T}\ln\left(2\sqrt{\pi}\beta\epsilon\hat{T}^{\frac{1}{2}}\right)} \tag{3.101}$$

and therefore the small-time asymptotic behaviour of the MBAW boundary is

$$\bar{S}_0^* \sim 1 - \sqrt{-4\epsilon^2\hat{T}\ln\left(2\sqrt{\pi}\alpha\epsilon^2\hat{T}\right)} \qquad \text{as } \epsilon \to 0, \tag{3.102}$$

which we observe is consistent with the asymptotic behaviour of the full problem (3.38) up to a constant in the logarithmic term. A comparison of the small-time asymptotic behaviour of the MBAW boundary with that of the full problem is shown in Figure 3-5.

We have determined that the MBAW boundary has the same asymptotic form as that of the full problem in the small-time limit, and therefore satisfies the condition (3.80d). For the condition at expiry (3.80c), we observe from the solution (3.81) that $hg(\bar{S}, h) \to 0$ as $h \to 0$, provided $hA_{00}(h) \to 0$. We observe from the form of $A_{00}(h)$ (3.83) that this is satisfied by the condition (3.88).

Large-Time Asymptotics of the MBAW Boundary

To investigate the large-time asymptotic behaviour of (3.84), we introduce the small parameter $H = e^{-\alpha\tau}$ under which $\lambda_-(H)$ is

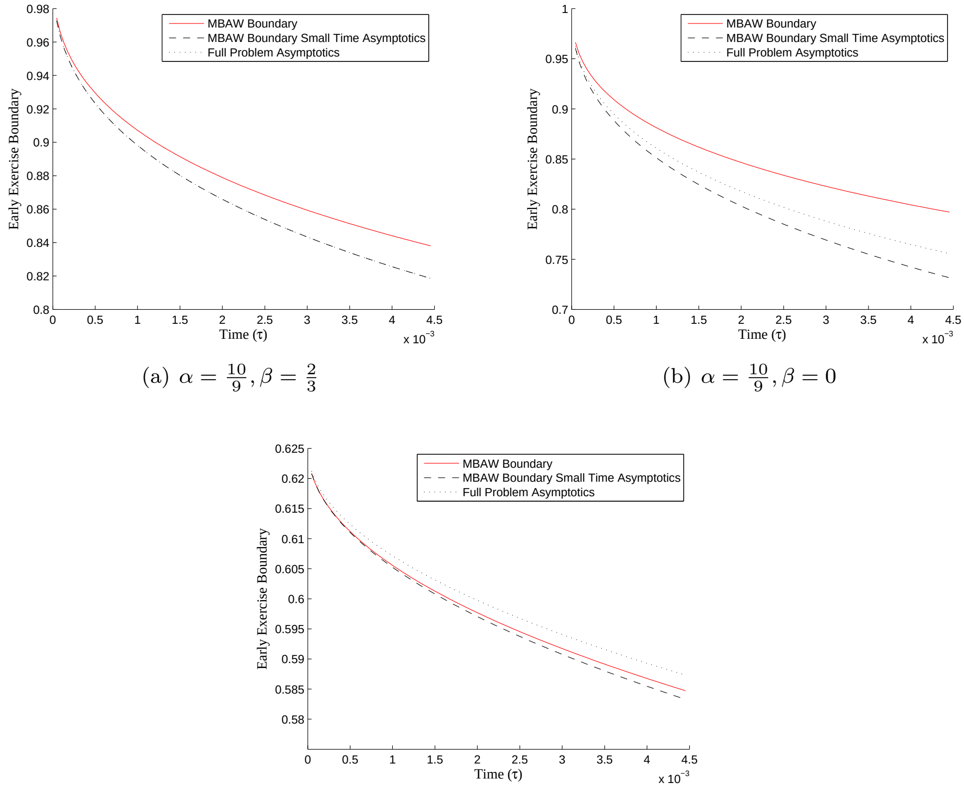
$$\lambda(H) \sim \lambda_\infty - \lambda_1 H + O\left(H^2\right) \qquad \text{as } H \to 0, \tag{3.103}$$

where

$$\lambda_1 = \frac{\alpha}{\sqrt{\alpha + \frac{(\beta-1)^2}{4}}} \tag{3.104}$$

and the European put option evaluated at $\bar{S}_0^*$ is

$$\bar{P}_e^* = \frac{H}{2}\left[\text{erfc}\left(\frac{\bar{d}_2^*}{\sqrt{2}}\right) - H^{-\frac{\beta}{\alpha}}\bar{S}_0^*\text{erfc}\left(\frac{\bar{d}_1^*}{\sqrt{2}}\right)\right], \tag{3.105a}$$



(a) $\alpha = \frac{10}{9}, \beta = \frac{2}{3}$

(b) $\alpha = \frac{10}{9}, \beta = 0$

(c) $\alpha = \frac{10}{9}, \beta = -\frac{2}{3}$

Figure 3-5: Small-time behaviour of the MBAW Boundary $\bar{S}_0^*(\tau)$. The relevant full problem asymptotics (3.38) are included together with the small-time asymptotics of $\bar{S}_0^*(\tau)$ derived in (3.94),(3.99) & (3.102).

where

$$\frac{\bar{d}_1^*}{\sqrt{2}} = \frac{\sqrt{\alpha}}{2} \frac{\ln\left(\bar{S}_0^*\right)}{\left(\ln\left(\frac{1}{H}\right)\right)^{\frac{1}{2}}} + \frac{(\beta+1)}{2\sqrt{\alpha}} \left(\ln\left(\frac{1}{H}\right)\right)^{\frac{1}{2}}, \tag{3.105b}$$

$$\frac{\bar{d}_2^*}{\sqrt{2}} = \frac{\sqrt{\alpha}}{2} \frac{\ln\left(\bar{S}_0^*\right)}{\left(\ln\left(\frac{1}{H}\right)\right)^{\frac{1}{2}}} + \frac{(\beta-1)}{2\sqrt{\alpha}} \left(\ln\left(\frac{1}{H}\right)\right)^{\frac{1}{2}} \tag{3.105c}$$

and the European put option delta evaluated at $\bar{S}_0^*$ is

$$\frac{\partial \bar{P}_e^*}{\partial \bar{S}} = -\frac{1}{2} H^{\frac{\alpha-\beta}{\alpha}} \operatorname{erfc}\left(\frac{\bar{d}_1^*}{\sqrt{2}}\right). \tag{3.106}$$

Collecting terms in H, the large-time expression for (3.84) becomes

$$\begin{aligned}\left(\lambda_\infty - (\lambda_\infty - 1)\bar{S}_0^*\right) + \left(\lambda_1(1-\bar{S}_0^*) - \frac{\lambda_\infty}{2}\text{erfc}\left(\frac{\bar{d}_2^*}{\sqrt{2}}\right)\right)H \\ -(\lambda_\infty - 1)\frac{\bar{S}_0^*}{2}\text{erfc}\left(\frac{\bar{d}_1^*}{\sqrt{2}}\right)H^{\frac{\alpha-\beta}{\alpha}} + O\left(H^2, HH^{\frac{\alpha-\beta}{\alpha}}\right) = 0 \qquad \text{as } H \to 0. \end{aligned} \tag{3.107}$$

We note that, since $\beta \in (-\infty, \alpha]$, $H^{\frac{\alpha-\beta}{\alpha}}$ dominates H for $\beta > 0$ and is $O(1)$ when $\beta = \alpha$. Conversely, H dominates $H^{\frac{\alpha-\beta}{\alpha}}$ for $\beta < 0$. In looking for the dominant balance however, one must consider the effect of β on the coefficient at the relevant order. For example, although on initial inspection it appears that the term at $O\left(H^{\frac{\alpha-\beta}{\alpha}}\right)$ contributes at $O(1)$ for $\beta = \alpha$, $\text{erfc}\left(\frac{\bar{d}_1^*}{\sqrt{2}}\right)$ is exponentially small and therefore the leading order behaviour is

$$\bar{S}_0^* = \frac{\lambda_\infty}{\lambda_\infty - 1} \qquad \text{as } H \to 0, \tag{3.108}$$

for all values of β, which we observe, as expected, to be the perpetual boundary $\bar{S}_\infty^*$ (3.43). Posing $\bar{S}_0^* = \bar{S}_\infty^* + \bar{S}_1^*(H)$ and expanding the complementary error functions as

$$\text{erfc}\left(\frac{\bar{d}_1^*}{\sqrt{2}}\right) = \text{erfc}\left(\frac{\bar{d}_{1\infty}^*}{\sqrt{2}}\right) + O\left(\frac{\bar{S}_1^*}{\left(\ln\left(\frac{1}{H}\right)\right)^{\frac{1}{2}}}\right), \tag{3.109}$$

$$\text{erfc}\left(\frac{\bar{d}_2^*}{\sqrt{2}}\right) = \text{erfc}\left(\frac{\bar{d}_{2\infty}^*}{\sqrt{2}}\right) + O\left(\frac{\bar{S}_1^*}{\left(\ln\left(\frac{1}{H}\right)\right)^{\frac{1}{2}}}\right), \tag{3.110}$$

where

$$\frac{\bar{d}_{1\infty}^*}{\sqrt{2}} = \left(\frac{\sqrt{\alpha}}{2}\frac{\ln\left(\bar{S}_\infty^*\right)}{\left(\ln\left(\frac{1}{H}\right)\right)^{\frac{1}{2}}} + \frac{(\beta+1)}{2\sqrt{\alpha}}\left(\ln\left(\frac{1}{H}\right)\right)^{\frac{1}{2}}\right), \tag{3.111}$$

$$\frac{\bar{d}_{2\infty}^*}{\sqrt{2}} = \left(\frac{\sqrt{\alpha}}{2}\frac{\ln\left(\bar{S}_\infty^*\right)}{\left(\ln\left(\frac{1}{H}\right)\right)^{\frac{1}{2}}} + \frac{(\beta-1)}{2\sqrt{\alpha}}\left(\ln\left(\frac{1}{H}\right)\right)^{\frac{1}{2}}\right), \tag{3.112}$$

gives

$$\begin{aligned} -(\lambda_\infty - 1)\bar{S}_1^* + \left(\frac{\lambda_1}{\lambda_\infty - 1} - \frac{\lambda_\infty}{2}\text{erfc}\left(\frac{\bar{d}_{2\infty}^*}{\sqrt{2}}\right)\right)H \\ -\frac{\lambda_\infty}{2}\text{erfc}\left(\frac{\bar{d}_{1\infty}^*}{\sqrt{2}}\right)H^{\frac{\alpha-\beta}{\alpha}} + O\left(H^2, HH^{\frac{\alpha-\beta}{\alpha}}, \bar{S}_1^*H, \bar{S}_1^*H^{\frac{\alpha-\beta}{\alpha}}\right) = 0 \qquad \text{as } H \to 0. \end{aligned} \tag{3.113}$$

As discussed previously, it would appear that dominant balance occurs at $O(H)$ for $\beta \leq 0$ or $O\left(H^{\frac{\alpha-\beta}{\beta}}\right)$ for $\beta > 0$. However, we observe from (3.105b) that $\frac{\bar{d}_1^*}{\sqrt{2}} \to \infty$ for $\beta \gg -1$ and therefore $\mathrm{erfc}\left(\frac{\bar{d}_1^*}{\sqrt{2}}\right)$ is exponentially small for $\beta > 0$. Thus the first correction term to $\bar{S}_\infty^*$ $\forall\beta$ is

$$\bar{S}_1^* = \frac{1}{\lambda_\infty - 1}\left(\frac{\lambda_1}{\lambda_\infty - 1} - \frac{\lambda_\infty}{2}\mathrm{erfc}\left(\frac{\bar{d}_{2\infty}^*}{\sqrt{2}}\right)\right) H \qquad \text{as } H \to 0 \tag{3.114}$$

and therefore to one correction term, the large-time asymptotic behaviour of $\bar{S}_0^*$ is

$$\bar{S}_0^* \sim \bar{S}_\infty^* - \frac{1}{\lambda_\infty - 1}\left(\frac{\lambda_1}{\lambda_\infty - 1} - \frac{\lambda_\infty}{2}\mathrm{erfc}\left(\frac{\bar{d}_{2\infty}^*}{\sqrt{2}}\right)\right) H \qquad \text{as } H \to 0. \tag{3.115}$$

We observe from Figure (3-6) that the correction term is negative as we approach the perpetual limit which is a violation of no-arbitrage for a put option as it drives the optimal exercise boundary below the perpetual boundary. Also included in this figure is the asymptotic approximation formed by including the term of $O\left(H^{\frac{\alpha-\beta}{\beta}}\right)$ in (3.113) which we observe captures the asymptotic behaviour more accurately as we move away from perpetuity.

3.3.3 Ju & Zhong's Extension

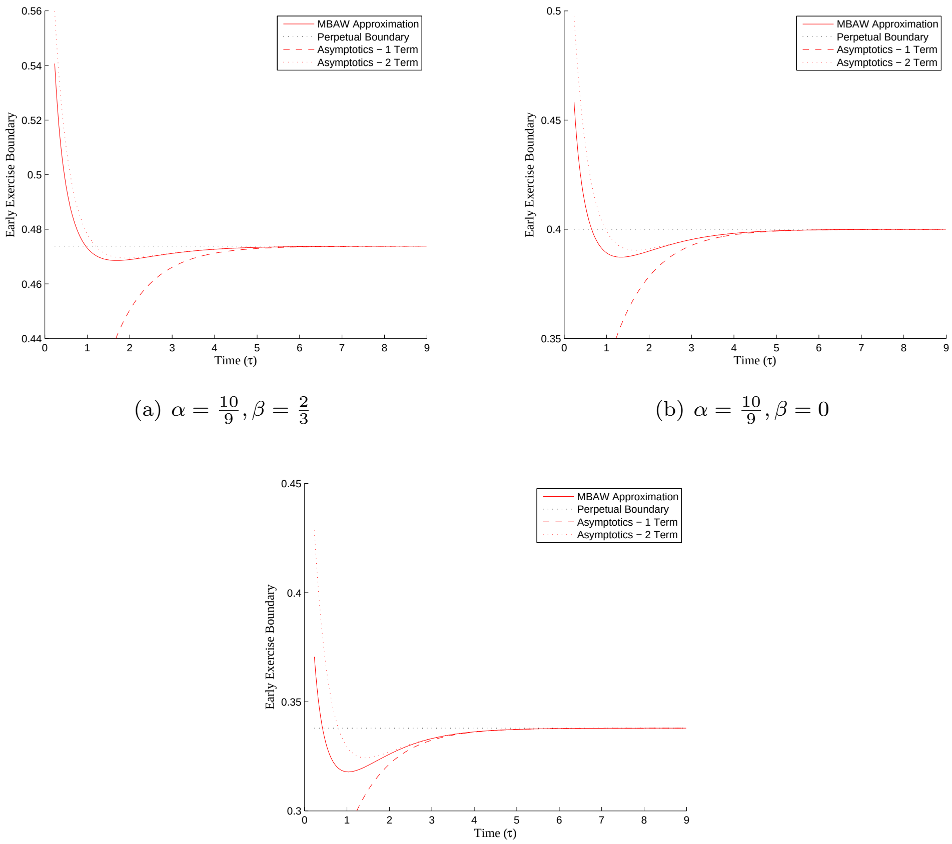
In deriving the MBAW approximation, an assumption is made that a term in the governing equation is small and can be ignored at leading order. This results in a second order ODE with a simple closed-form solution. A natural extension seems to be to pose a series in an appropriate parameter and to look for higher order terms based around the MBAW approximation.

An attempt towards this end was made by Ju & Zhong [69] although they only look for a single correction term to the MBAW price, with no attempt made to look for a correction to the MBAW boundary. The mis-specification of the problem is hidden in the boundary conditions, which are not provided in the paper, and the resulting solution fails to satisfy the high contact condition of the American option problem. Although the price approximation is typically more accurate than the MBAW price, particularly for intermediate maturities, it exhibits singular behaviour at small times as shown in Figure 3-7.

In the following section, we use a homotopic series approach adopted by Zhu [106], to correctly formulate the correction terms to the MBAW approximation.

3.3.4 An Analytic Approximation for the American Put Option

To extend the MBAW approximation, we look for a suitable expansion of both option price and optimal exercise boundary, with the MBAW price and MBAW boundary as the leading order term in the series. One possible method of generating terms in the series is adopted by



(a) $\alpha = \frac{10}{9}, \beta = \frac{2}{3}$

(b) $\alpha = \frac{10}{9}, \beta = 0$

(c) $\alpha = \frac{10}{9}, \beta = -\frac{2}{3}$

Figure 3-6: Large-time asymptotic behaviour of the MBAW Boundary $\bar{S}_0^*$. The 1 and 2-Term asymptotic results are obtained in (3.115) while the perpetual boundary is derived in (3.43). We observe that the MBAW Boundary is not monotonic and crosses the perpetual boundary which is a violation of no-arbitrage.

Zhu [106] who poses the full American option problem as a homotopic series in an artificial parameter p, which corresponds to the full American option problem when $p = 1$. Successive terms in the homotopic series are generated by differentiating the problem the requisite number of times with respect to p. Zhu claims this leads to a closed-form solution to the American option problem, though since the exact solution requires the computation of an infinite number of terms, it is not clear to us that the solution is any more a closed-form than the work of Geske & Johnson [49].

We are aware of some criticism by practitioners of the ease of numerical implementation of Zhu's solution and the time taken to achieve a convergent solution, which Zhu demonstrates numerically in his paper. However the technique does provide a framework for generating a

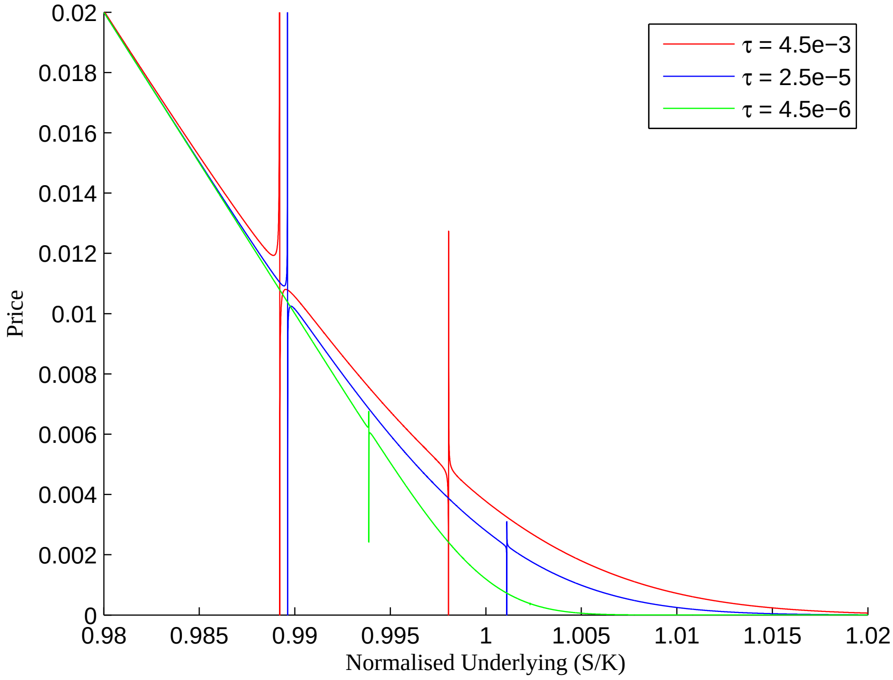


Figure 3-7: Singular behaviour occuring in Ju & Zhong's extension to the MBAW price approximation ($r = 0.05, D = 0.02, \sigma = 0.3$).

series of terms based around the MBAW approximation, and we adopt that approach here. Starting with the problem (3.80a-3.80f) we use a Landau transformation of the form

$$\tilde{X} = \ln\left(\frac{\bar{S}}{\bar{S}^*(h)}\right), \tag{3.116}$$

under which the decomposition of the American option price is

$$\tilde{P}_a(\tilde{X}, h) = \bar{P}_e(\bar{S}, h) + h\tilde{g}(\tilde{X}, h), \tag{3.117}$$

where the European option price is written in terms of the spatial variable $\bar{S}$ as it will afford some convenience in determining the European option Greeks in our numerical work. Using Zhu's approach, we introduce an artificial parameter p into the forcing term of the governing equation, such that we have the premium corresponding to the full American option problem for $p = 1$. The problem becomes

$$\tilde{X} \in (0,\infty), h \in (0,1) \qquad \frac{\partial^2 \tilde{g}}{\partial \tilde{X}^2} + (\beta - 1)\frac{\partial \tilde{g}}{\partial \tilde{X}} - \frac{\alpha}{h}\tilde{g} = p\alpha(1-h)\left[\frac{\partial \tilde{g}}{\partial h} - \frac{1}{\bar{S}^*}\frac{d\bar{S}^*}{dh}\frac{\partial \tilde{g}}{\partial \tilde{X}}\right], \tag{3.118a}$$

subject to

$$\text{as } \tilde{X} \to \infty \qquad \tilde{g}(\tilde{X}, h; p) \to 0, \tag{3.118b}$$

$$\text{at } h = 0 \qquad h\tilde{g}(\tilde{X}, 0; p) = 0, \tag{3.118c}$$

$$\bar{S}^*(0;p) = \min\left(\alpha/(\alpha-\beta), 1\right), \tag{3.118d}$$

$$\text{at } \tilde{X} = 0 \qquad h\tilde{g}(\tilde{X}, h; p) = \left(1 - \bar{S}^*(h;p)\right) - \bar{P}_e(\bar{S}^*(h;p), h), \tag{3.118e}$$

$$h\frac{\partial \tilde{g}(\tilde{X}, h; p)}{\partial \tilde{X}}\bigg|_{\tilde{X}=0} = -\bar{S}^*(h;p)\left(1 + \frac{\partial \bar{P}_e}{\partial \bar{S}}\bigg|_{\bar{S}^*(h;p)}\right). \tag{3.118f}$$

We propose the analytic expansions for $\tilde{g}(\tilde{X}, h; p)$ and $\bar{S}^*(h;p)$

$$\tilde{g}(\tilde{X}, h; p) = \sum_{n=0}^{\infty} \frac{p^n}{n!}\tilde{g}_n(\tilde{X}, h), \tag{3.119a}$$

$$\bar{S}^*(h;p) = \sum_{n=0}^{\infty} \frac{p^n}{n!}\bar{S}_n^*(h), \tag{3.119b}$$

which allows us to derive a sequence for the pairs of problems $(\tilde{g}_n(\tilde{X}, h), \bar{S}_n^*(h))$ where

$$\tilde{g}_n(\tilde{X}, h) = \frac{\partial^n \tilde{g}}{\partial p^n}\bigg|_{p=0}, \tag{3.120a}$$

$$\bar{S}_n^*(h) = \frac{\partial^n \bar{S}^*}{\partial p^n}\bigg|_{p=0}. \tag{3.120b}$$

We also introduce a function $\hat{F}(\tilde{X}, h; p)$ given by

$$\tilde{F}(\tilde{X}, h; p) = \frac{\partial \tilde{g}}{\partial h} - \frac{1}{\bar{S}^*}\frac{d\bar{S}^*}{dh}\frac{\partial \tilde{g}}{\partial \tilde{X}} \tag{3.121}$$

so that $p\alpha(1-h)\tilde{F}(\tilde{X}, h; p)$ is the forcing term for the base problem.

As with the assumption which gave rise to the MBAW approximation, the problem changes nature when $p = 0$, with the time dependence of the solution no longer contained as a variable with the governing equation, but entering the problem as a parameter in the ODE and the boundary conditions. After using the conditions (3.118b), (3.118e) & (3.118f) to fully specify the leading order problem, we are unable to impose the conditions (3.118c) & (3.118d). We are limited therefore to verifying that the solution satisfies these conditions after determination, either analytically or numerically. In the following problems we omit these conditions and mention them where relevant.

The $(\tilde{g}_n, \bar{S}^*_n)$ Problem

The problem for the general n^{th} term in the series can be found by differentiating the problem (3.118a-3.118f) n times and evaluating at $p = 0$ to give

$$\tilde{X} \in (0,\infty), h \in (0,1) \qquad \frac{\partial^2 \tilde{g}_n}{\partial \tilde{X}^2} + (\beta - 1)\frac{\partial \tilde{g}_n}{\partial \tilde{X}} - \frac{\alpha}{h}\tilde{g}_n = n\alpha(1-h)\hat{F}_{n-1}(\tilde{X}, h; 0), \quad (3.122a)$$

subject to

$$\text{as } \tilde{X} \to \infty \qquad \tilde{g}_n(\tilde{X}, h) \to 0, \quad (3.122b)$$

$$\text{at } \tilde{X} = 0 \qquad h\tilde{g}_n(\tilde{X}, h) = \big(1 - \bar{S}^*_n(h)\big) - \left.\frac{\partial^n \bar{P}^*_e}{\partial p^n}\right|_{p=0}, \quad (3.122c)$$

$$h\left.\frac{\partial \tilde{g}_n}{\partial \tilde{X}}\right|_{\tilde{X}=0} = -\bar{S}^*_n\left(1 + \left.\frac{\partial \bar{P}_e}{\partial \bar{S}}\right|_{\bar{S}^*_0}\right) - \sum_{i=1}^{n}\binom{n}{i}\bar{S}^*_{n-i}\frac{\partial^i}{\partial p^i}\left(\left.\frac{\partial \bar{P}_e}{\partial \bar{S}}\right|_{\bar{S}^*}\right)\Bigg|_{p=0}, \quad (3.122d)$$

where for convenience of notation, $\bar{P}^*_e$ represents the European option price evaluated at $\bar{S}^*(h)$ and the subscript in $\hat{F}_{n-1}$ represents the number of times the derivative of this term is taken with respect to p before evaluation at $p = 0$.

Ideally we would like to be able to solve the general expression for the n^{th} term and iterate between successive terms in our series and investigate the convergence of the series, either analytically or numerically. The difficulty in this respect is that the number of terms generated by differentiation of the forcing term and boundary conditions with respect to p increases exponentially and we have not found a general iterative scheme to cope with this. The steps we have made so far are included in the Discussion, but this remains further work to be done. Nonetheless, we can derive the first three terms in our series explicitly and investigate the performance of the resulting uniform approximation.

For convenience, a summary of the formulae required to calculate each term is included in Table 3.1 at the end of this Chapter, while the relevant Greeks are derived in Appendix A.

The $(\tilde{g}_0, \bar{S}_0^*)$ Problem

Setting $p = 0$ in (3.118a-3.118f) gives the problem

$$\tilde{X} \in (0,\infty), h \in (0,1) \qquad \frac{\partial^2 \tilde{g}_0}{\partial \tilde{X}^2} + (\beta - 1)\frac{\partial \tilde{g}_0}{\partial \tilde{X}} - \frac{\alpha}{h}\tilde{g}_0 = 0, \tag{3.123a}$$

subject to

$$\text{as } \tilde{X} \to \infty \qquad \tilde{g}_0(\tilde{X}, h) \to 0, \tag{3.123b}$$

$$\text{at } \tilde{X} = 0 \qquad h\tilde{g}_0(0,h) = \left(1 - \bar{S}_0^*\right) - \bar{P}_e(\bar{S}_0^*, h), \tag{3.123c}$$

$$h\frac{\partial \tilde{g}_0}{\partial \tilde{X}}\bigg|_{\tilde{X}=0} = -\bar{S}_0^* - \bar{S}_0^* \frac{\partial \bar{P}_e}{\partial \bar{S}}\bigg|_{\bar{S}_0^*}. \tag{3.123d}$$

This problem is identical to that leading to the MBAW approximation and has the solution

$$\tilde{g}_0 = A_{00}(h)e^{\lambda_- \tilde{X}}, \tag{3.124}$$

where $\lambda_-(h)$ and $A_{00}(h)$ are given by (3.82) and (3.83) respectively and where the leading order boundary approximation is the solution to the transcendental equation (3.84). Our previous work in this Chapter has demonstrated that the leading order boundary satisfies the condition (3.118d) and also exhibits the same small-time asymptotic behaviour of the full problem, though it is not monotonic and breaches the perpetual option boundary for large times. The leading order price approximation meanwhile satisfies the condition (3.118c).

The $(\tilde{g}_1, \bar{S}_1^*)$ Problem

The problem for the first correction term is

$$\tilde{X} \in (0,\infty), h \in (0,1) \qquad \frac{\partial^2 \tilde{g}_1}{\partial \tilde{X}^2} + (\beta - 1)\frac{\partial \tilde{g}_1}{\partial \tilde{X}} - \frac{\alpha}{h}\tilde{g}_1 = \alpha(1-h)\tilde{F}_0(\tilde{X}, h), \tag{3.125a}$$

subject to

$$\text{as } \tilde{X} \to \infty \qquad \tilde{g}_1(\tilde{X}, h) \to 0, \tag{3.125b}$$

$$\text{at } \tilde{X} = 0 \qquad h\tilde{g}_1(0,h) = -\bar{S}_1^* - \frac{\partial \bar{P}_e^*}{\partial p}\bigg|_{p=0}, \tag{3.125c}$$

$$h\frac{\partial \tilde{g}_1}{\partial \tilde{X}}\bigg|_{\tilde{X}=0} = -\bar{S}_1^* - \frac{\partial}{\partial p}\left(\bar{S}^* \frac{\partial \bar{P}_e}{\partial \bar{S}}\bigg|_{\bar{S}^*}\right)\bigg|_{p=0}. \tag{3.125d}$$

The function $\tilde{F}_0(\tilde{X}, h)$ in the forcing term of (3.125a) is given by

$$\tilde{F}_0(\tilde{X}, h) = \frac{\partial \tilde{g}_0}{\partial \hat{h}} - \frac{1}{\bar{S}_0^*}\frac{\partial \bar{S}_0^*}{\partial h}\frac{\partial \tilde{g}_0}{\partial \tilde{X}} = e^{\lambda_- \tilde{X}}\left(B_{01}(h)\tilde{X} + B_{00}(h)\right) \tag{3.126}$$

and using (3.124) and (3.126) gives the forms for the coefficients of the forcing term

$$B_{01} = \alpha(1-h)\frac{\partial \lambda_-}{\partial h}A_0, \tag{3.127a}$$

$$B_{00} = \alpha(1-h)\left[\frac{\partial A_0}{\partial h} - \frac{1}{\bar{S}_0^*}\frac{\partial \bar{S}_0^*}{\partial h}\lambda_- A_0\right]. \tag{3.127b}$$

The solution to (3.125a-3.125d) is given by

$$\tilde{g}_1(\tilde{X}, h) = e^{\lambda_- \tilde{X}}\sum_{j=0}^{2} A_{1j}(h)\tilde{X}^j, \tag{3.128}$$

where the coefficients $A_{1j}(h)$ for $j \neq 0$ can be found by substitution into (3.125a), giving

$$A_{12} = \frac{B_{01}}{2\left(2\lambda_- + (\beta - 1)\right)}, \tag{3.129}$$

$$A_{11} = \frac{1}{(2\lambda_- + (\beta-1))}\left[B_{00} - 2A_{12}\right]. \tag{3.130}$$

The coefficient A_{10} and boundary correction term $\bar{S}_1^*$ are found through application of the boundary conditions (3.125c) and (3.125d)

$$\bar{S}_1^* = \frac{hA_{11}}{(\lambda - 1)(1 + \bar{\Delta}^*) - \bar{S}_0^*\bar{\Gamma}^*}, \tag{3.131}$$

$$A_{10} = \frac{-\bar{S}_1^* - \bar{S}_1^*\bar{\Delta}^*}{h}, \tag{3.132}$$

where the non-dimensional European Greeks $\bar{\Delta}^*$ and $\bar{\Gamma}^*$ are derived in Appendix A.

The $(\tilde{g}_2, \bar{S}_2^*)$ Problem

The problem for the second correction term is

$$\tilde{X} \in (0, \infty), h \in (0,1) \qquad \frac{\partial^2 \tilde{g}_2}{\partial \tilde{X}^2} + (\beta - 1)\frac{\partial \tilde{g}_2}{\partial \tilde{X}} - \frac{\alpha}{h}\tilde{g}_2 = 2\alpha(1-h)\tilde{F}_1(\tilde{X}, h), \tag{3.133a}$$

subject to

$$\text{as } \tilde{X} \to \infty \qquad \tilde{g}_2(\tilde{X}, h) \to 0, \tag{3.133b}$$

$$\text{at } \tilde{X} = 0 \qquad h\tilde{g}_1(0, h) = -\bar{S}_2^* - \left.\frac{\partial^2 \bar{P}_e^*}{\partial p^2}\right|_{p=0}, \tag{3.133c}$$

$$\left.\frac{\partial \tilde{g}_2}{\partial \tilde{X}}\right|_{\tilde{X}=0} = -\bar{S}_2^* - \left.\frac{\partial^2}{\partial p^2}\left(\bar{S}^* \left.\frac{\partial \hat{P}_e}{\partial \bar{S}}\right|_{\bar{S}^*}\right)\right|_{p=0}. \tag{3.133d}$$

The function $\tilde{F}_1(\tilde{X}, h)$ in the forcing term of (3.125a) is given by

$$\tilde{F}_1(\tilde{X}, h) = \frac{\partial \tilde{g}_1}{\partial \hat{h}} - \left(\frac{1}{\bar{S}_0^*}\frac{\partial \bar{S}_1^*}{\partial h} - \frac{\bar{S}_1^*}{(\bar{S}_0^*)^2}\frac{\partial \bar{S}_0^*}{\partial h}\right)\frac{\partial \tilde{g}_0}{\partial \tilde{X}} - \frac{1}{\bar{S}_0^*}\frac{\partial \bar{S}_0^*}{\partial h}\frac{\partial \tilde{g}_1}{\partial \tilde{X}} = e^{\lambda_- \tilde{X}} \sum_{j=0}^{j=3} B_{1j}(h)\tilde{X}^j \tag{3.134}$$

and using (3.124), (3.126) and (3.128) gives the forms for the coefficients of the forcing term

$$B_{13} = \alpha(1-h)\left[\frac{\partial \lambda_-}{\partial h} A_{12}\right], \tag{3.135}$$

$$B_{12} = \alpha(1-h)\left[\frac{\partial \lambda_-}{\partial h} A_{11} + \frac{\partial A_{12}}{\partial h} - \frac{1}{\bar{S}_0^*}\frac{\partial \bar{S}_0^*}{\partial h}\lambda_- A_{12}\right], \tag{3.136}$$

$$B_{11} = \alpha(1-h)\left[\frac{\partial \lambda_-}{\partial h} A_{10} + \frac{\partial A_{11}}{\partial h} - \frac{1}{\bar{S}_0^*}\frac{\partial \bar{S}_0^*}{\partial h}(\lambda_- A_{11} + 2A_{12})\right], \tag{3.137}$$

$$B_{10} = \alpha(1-h)\left[\frac{\partial A_{10}}{\partial h} - \frac{1}{\bar{S}_0^*}\frac{\partial \bar{S}_0^*}{\partial h}(\lambda_- A_{10} + A_{11}) - \left(\frac{1}{\bar{S}_0^*}\frac{\partial \bar{S}_1^*}{\partial h} - \frac{\bar{S}_1^*}{(\bar{S}_0^*)^2}\frac{\partial \bar{S}_0^*}{\partial h}\right)\lambda_- A_0\right]. \tag{3.138}$$

The solution to (3.133a-3.133d) is given by

$$\tilde{g}_2(\tilde{X}, h) = e^{\lambda_- \tilde{X}} \sum_{j=0}^{4} A_{2j}(h)\tilde{X}^j, \tag{3.139}$$

where the coefficients $A_{2j}(h)$ for $j \neq 0$ can be found by substitution into (3.133a), giving

$$A_{24} = \frac{B_{13}}{4\left(2\lambda_- + (\beta - 1)\right)}, \tag{3.140}$$

$$A_{23} = \frac{1}{3\left(2\lambda_- + (\beta - 1)\right)}\left[B_{12} - 12A_{24}\right], \tag{3.141}$$

$$A_{22} = \frac{1}{2\left(2\lambda_- + (\beta - 1)\right)}\left[B_{11} - 6A_{23}\right], \tag{3.142}$$

$$A_{21} = \frac{1}{1\left(2\lambda_- + (\beta - 1)\right)}\left[B_{10} - 2A_{22}\right]. \tag{3.143}$$

The coefficient A_{20} and boundary correction term $\bar{S}_2^*$ are found through application of the boundary conditions (3.133c) and (3.133d)

$$\bar{S}_2^* = \frac{hA_{21} - \left[(\lambda_- - 2)\bar{\Gamma}^* - \bar{S}_0^*\bar{\Gamma}_\Delta^*\right](\bar{S}_1^*)^2}{(\lambda_- - 1)(1 + \bar{\Delta}^*) - \bar{S}_0^*\bar{\Gamma}^*}, \tag{3.144}$$

$$A_{20} = \frac{-\bar{S}_2^* - \bar{S}_2^*\bar{\Delta}^* - (\bar{S}_1^*)^2\bar{\Gamma}^*}{h}, \tag{3.145}$$

where the non-dimensional European Greeks $\bar{\Delta}^*$, $\bar{\Gamma}^*$ and $\bar{\Gamma}_\Delta^*$ are derived in Appendix A.

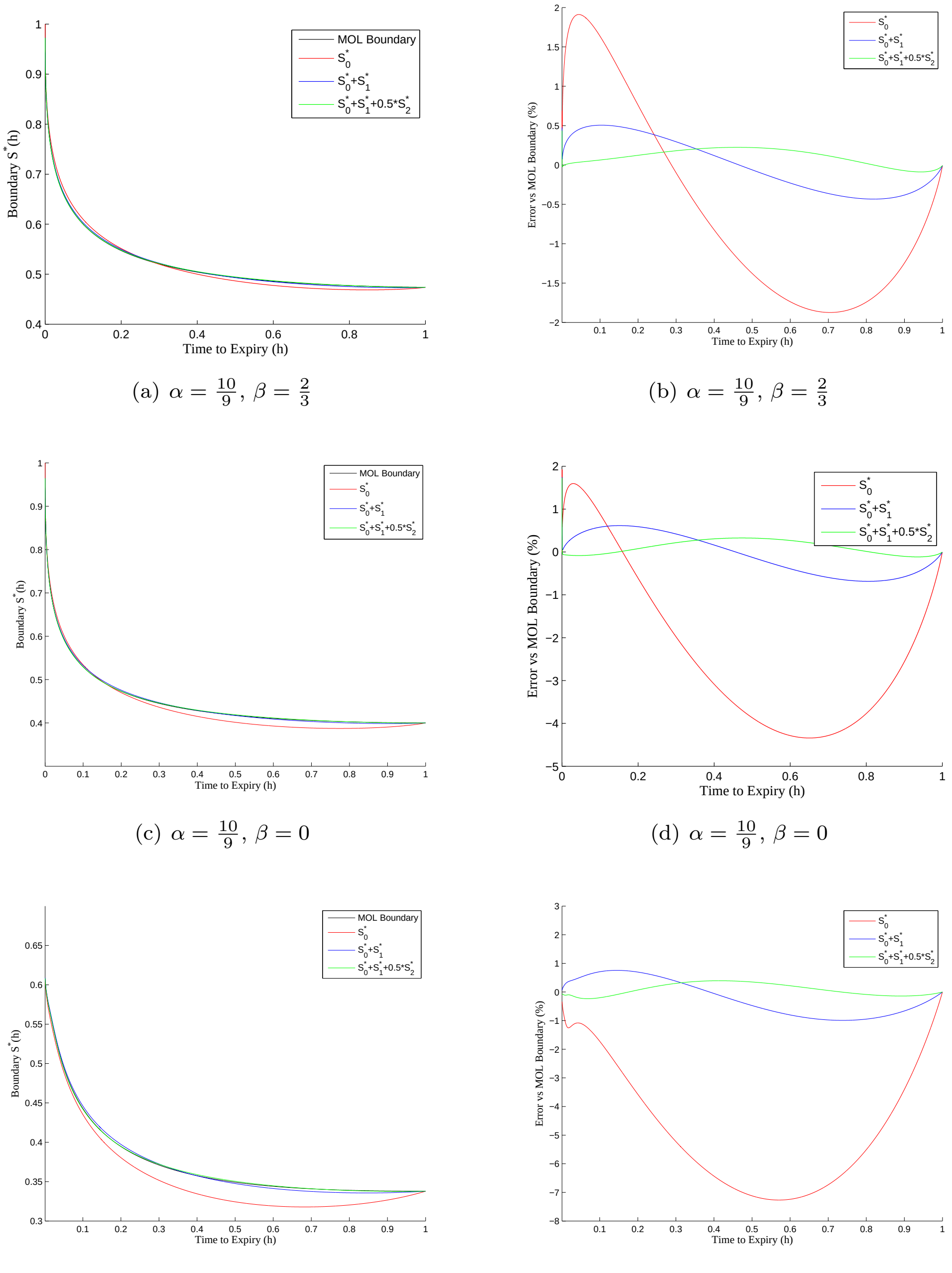
Numerical Results

A comparison of the terms of our analytic approximation for the optimal exercise boundary with the MOL numerics are shown in Figure 3-8. We observe from the relative error Figures 3.8(b), 3.8(d) & 3.8(f) that the two- and three-term boundary approximations provide a significant improvement over the MBAW boundary with the three-term series better than 1% accurate at all times.

The improvement in the large- and small-time asymptotic behaviour can be observed in Figures 3.9(a) & 3.9(f). For the approach to expiry, our MOL numerics were performed on a truncated domain using $\tilde{X}_{max} = 5$ and with $25,000$ space-steps and $20,000$ time-steps. For the approach to perpetuity, we ran the MOL numerics out to $T - t = 200$, with $200,000$ time-steps. We mention the existence of small differences between the convergence of the large-time MOL benchmark and the perpetual boundary, which result from discretisation errors in the calculation of the boundary at small times propagating into the large-time numerics. A significant improvement over the MBAW boundary is observed. For the approach to expiry, the MOL boundary is almost indistinguishable from our three-term expression. For the approach to perpetuity we note that the boundary remains non-monotonic and breaches the perpetual boundary, but the effect reduces with successive terms.

The performance of the price approximation in Figures 3-10, 3-11 & 3-12 is less impressive, typically doing well near the boundary, but with increasingly large percentage errors appearing as we move deeply out-of-the-money. This is not an unexpected result given the structure of the additional terms in the series which add increasingly high order powers of $\tilde{X}$, though these are dominated by an exponentially small term which makes the errors small in absolute terms. We make the additional observation, that in $\tilde{X}$ it is not clear that the Ju & Zhong approximation performs much better than even the MBAW approximation.

In summary, our three-term series provides an accurate approximation to the location of the American option boundary. The procedure is easy to implement, requiring only a single spreadsheet and access to an accurate approximation to the cumulative normal distribution function such as that due to Marsaglia, a routine for which can be found in [50]. As an indication of the speed of the routine, generating a curve for $h \in (0, 1)$ using 1,000 time-steps takes 14 seconds in Matlab using the built-in normal distribution approximation. The price approximation is typically only accurate near the boundary, and therefore use of the boundary approximation together with, for example, the integral formulation of the problem due to Kim [72] and an appropriate quadrature routine may be preferred if more consistently accurate prices are desired.

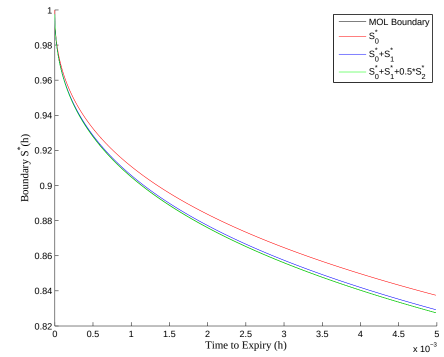


(a) $\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$ (b) $\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$

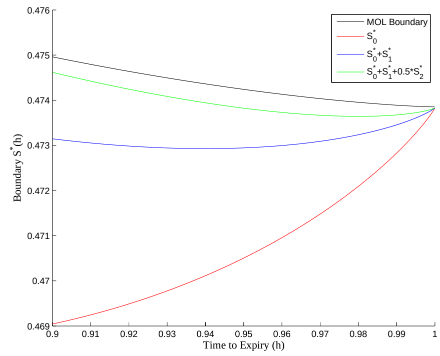
(c) $\alpha = \frac{10}{9}$, $\beta = 0$ (d) $\alpha = \frac{10}{9}$, $\beta = 0$

(e) $\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$ (f) $\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$

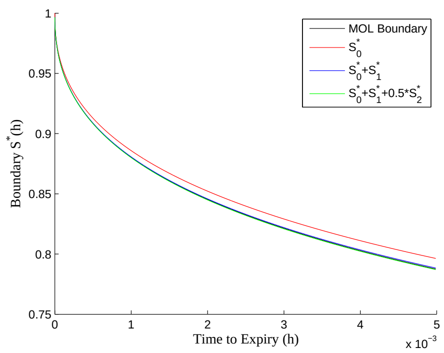
Figure 3-8: Comparison of the benchmark MOL boundary with our analytic approximation. For the MOL Boundary, 200000 time-steps and 25000 spatial points were used with 15000 on the interval $[\bar{S}^{\infty}, 1]$, and 10000 on the interval $[1, 50]$. The improvement of the 2-term $(\bar{S}_0^* + \bar{S}_1^*)$ and 3-term $(\bar{S}_0^* + \bar{S}_1^* + 0.5\bar{S}_2^*)$ boundary approximations versus the MBAW approximation $(\bar{S}_0^*)$ is most clearly shown in the error Figures 3.8(b), 3.8(d) & 3.8(f) where the error is defined as the difference between the relevant boundary approximation and the MOL Boundary, divided by the MOL boundary and expressed as a percentage.



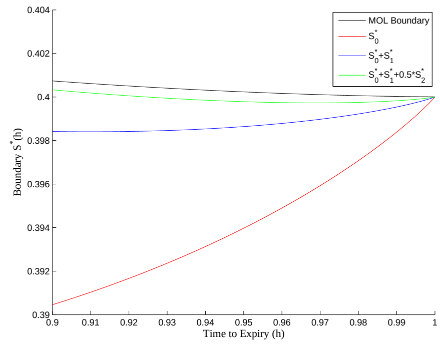
(a) Approach to Expiry $\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$



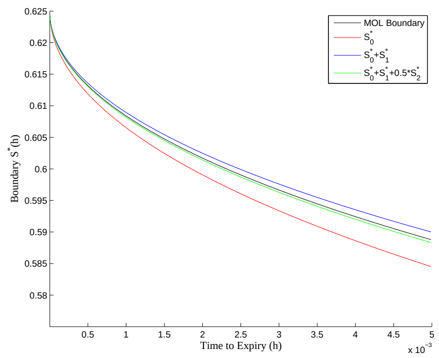
(b) Approach to Perpetuity $\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$



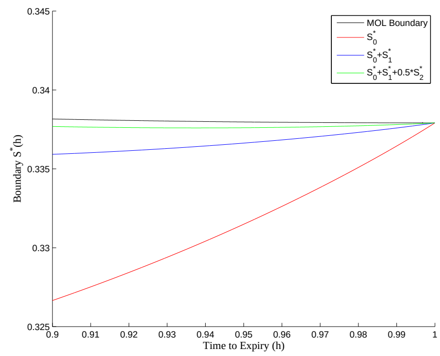
(c) Approach to Expiry $\alpha = \frac{10}{9}$, $\beta = 0$



(d) Approach to Perpetuity $\alpha = \frac{10}{9}$, $\beta = 0$

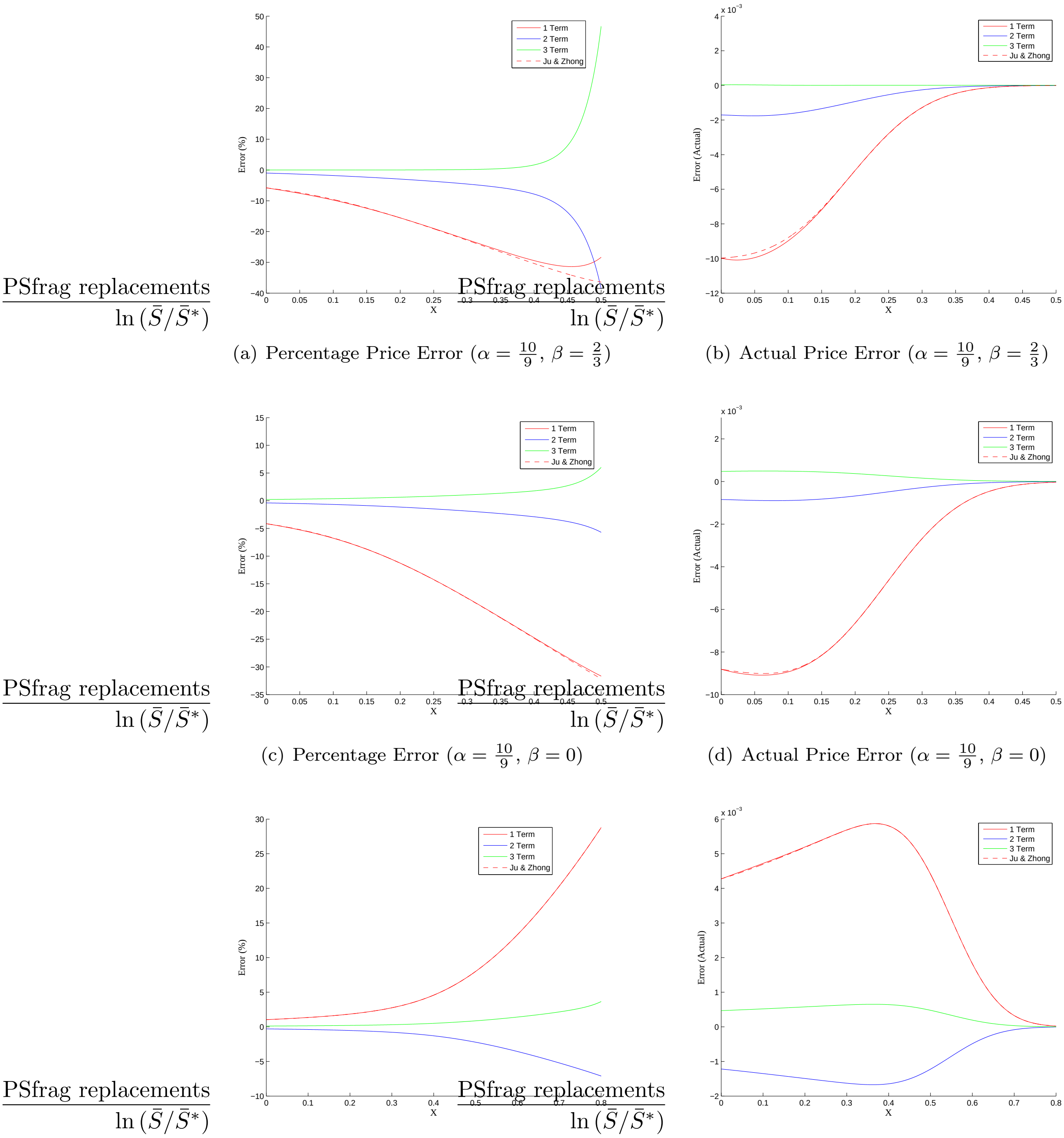


(e) Approach to Expiry $\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$



(f) Approach to Perpetuity $\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$

Figure 3-9: Asymptotic comparison of the benchmark MOL boundary with the homotopic series. For the approach to expiry, 20000 time-steps and 25000 spatial points were used with 15000 on the interval $[\bar{S}^{\infty}, 1]$, and 10000 on the interval $[1, 5]$. For the approach to perpetuity, 200000 time steps and 25000 spatial points were used with 15000 on the interval $[\bar{S}^{\infty}, 1]$, and 10000 on the interval $[1, 50]$.



(a) Percentage Price Error ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$)

(b) Actual Price Error ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$)

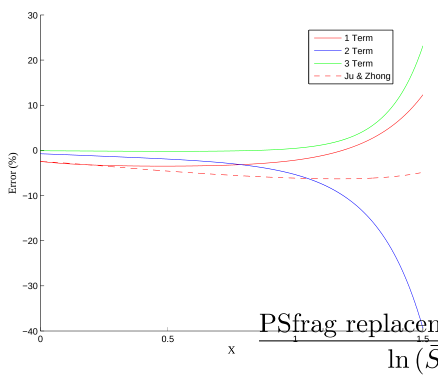
(c) Percentage Error ($\alpha = \frac{10}{9}$, $\beta = 0$)

(d) Actual Price Error ($\alpha = \frac{10}{9}$, $\beta = 0$)

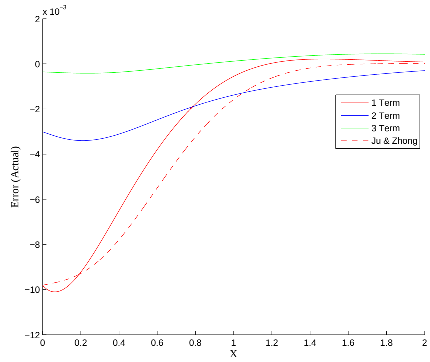
(e) Percentage Price Error ($\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$)

(f) Actual Price Error ($\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$)

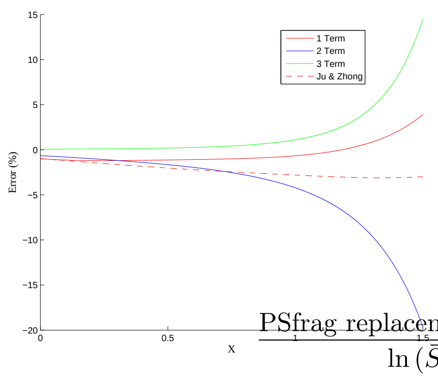
Figure 3-10: A comparison of the MBAW (1-term) and Ju & Zhong price approximations with the 2-term and 3-term price approximations from the homotopic series for $h = 0.005$. The errors are defined as the difference between the relevant approximation and the MOL benchmark price and expressed as both the actual difference (3.10(b),3.10(d),3.11(f)) or as a percentage of the MOL benchmark price (3.10(a),3.10(c),3.11(e)). The benchmark was determined using 20000 time-steps and 25000 spatial points with 15000 on the interval $[\bar{S}^\infty, 1]$, and 10000 on the interval $[1, 5]$, and transformed onto the fixed domain $\tilde{X} = \ln\left(\bar{S}/\bar{S}^*(h)\right)$ using the corresponding MOL boundary at $h = 0.005$.



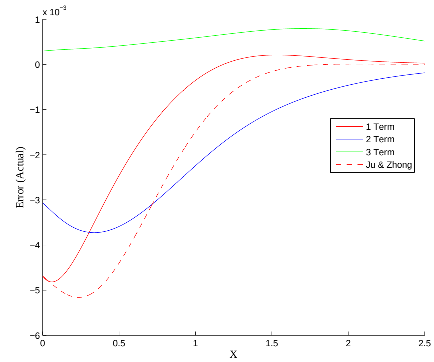
(a) Percentage Price Error ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$)



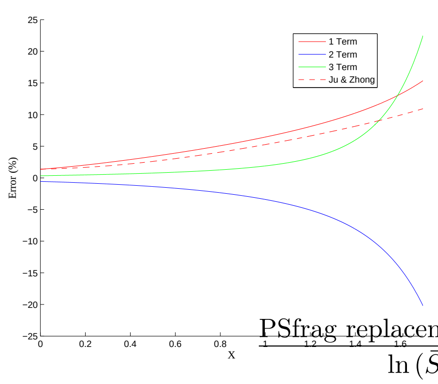
(b) Actual Price Error ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$)



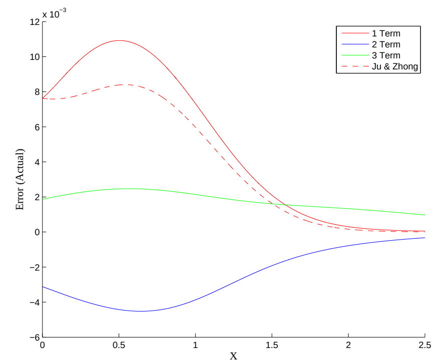
(c) Percentage Price Error ($\alpha = \frac{10}{9}$, $\beta = 0$)



(d) Actual Price Error ($\alpha = \frac{10}{9}$, $\beta = 0$)

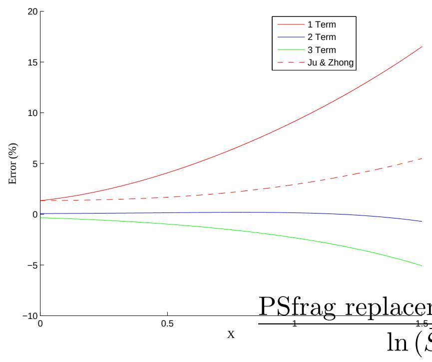


(e) Percentage Price Error ($\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$)

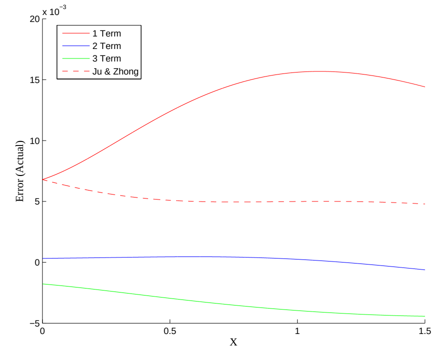


(f) Actual Price Error ($\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$)

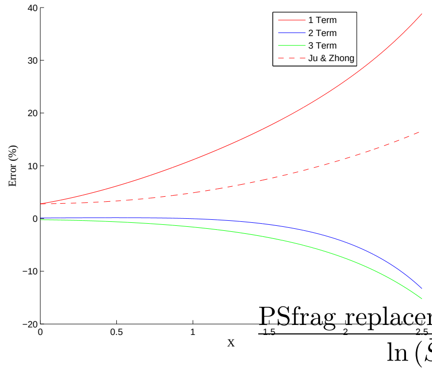
Figure 3-11: A comparison of the MBAW (1-term) and Ju & Zhong price approximations with the 2-term and 3-term price approximations from the homotopic series for $h = 0.1$. The errors are defined as the difference between the relevant approximation and the MOL benchmark price and expressed as both the actual difference (3.10(b),3.10(d),3.11(f)) or as a percentage of the MOL benchmark price (3.10(a),3.10(c),3.11(e)). The benchmark was determined using 20000 time-steps and 25000 spatial points with 15000 on the interval $[\bar{S}^\infty, 1]$, and 10000 on the interval $[1, 10]$, and transformed onto the fixed domain $\tilde{X} = \ln\left(\bar{S}/\bar{S}^*(h)\right)$ using the corresponding MOL boundary at $h = 0.1$.



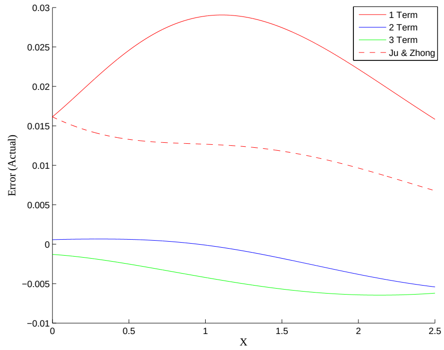
(a) Percentage Price Error ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$)



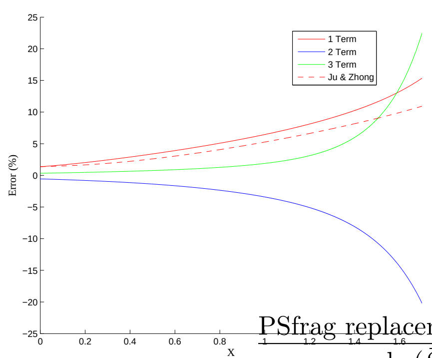
(b) Actual Price Error ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$)



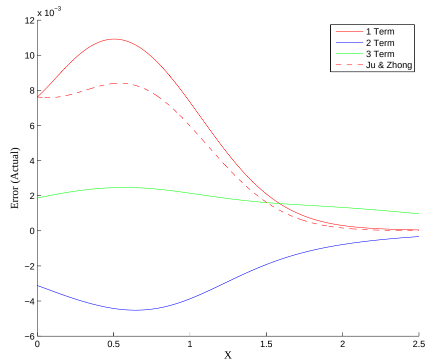
(c) Percentage Price Error ($\alpha = \frac{10}{9}$, $\beta = 0$)



(d) Actual Price Error ($\alpha = \frac{10}{9}$, $\beta = 0$)



(e) Percentage Price Error ($\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$)



(f) Actual Price Error ($\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$)

Figure 3-12: A comparison of the MBAW (1-term) and Ju & Zhong price approximations with the 2-term and 3-term price approximations from the homotopic series for $h = 0.5$. The errors are defined as the difference between the relevant approximation and the MOL benchmark price and expressed as both the actual difference (3.10(b),3.10(d),3.11(f)) or as a percentage of the MOL benchmark price (3.10(a),3.10(c),3.11(e)). The benchmark was determined using 50000 time-steps and 25000 spatial points with 15000 on the interval $[\bar{S}^\infty, 1]$, and 10000 on the interval $[1, 25]$, and transformed onto the fixed domain $\tilde{X} = \ln\left(\bar{S}/\bar{S}^*(h)\right)$ using the corresponding MOL boundary at $h = 0.5$.

Term	Formula	Greeks Required
$(\bar{g}_0, \bar{S}_0^*)$ **Problem**		
$\bar{g}_0(\tilde{X}, h)$	$e^{\lambda_- \tilde{X}} A_{00}(h)$	$\bar{P}_e$ (2.3)
λ_-	$\frac{-(\beta-1)-\sqrt{(\beta-1)^2+\frac{4\alpha}{h}}}{2}$	$\bar{\Delta}$ (A.7)
$A_{00}(h)$	$\frac{1-\bar{S}_0^*-\bar{P}_e^*}{h}$	
$S_0^*(h)$	$\lambda_- \left(1-\bar{S}_0^*-\bar{P}_e^*\right) = -\bar{S}_0^* - \bar{S}_0^*\bar{\Delta}^*$	
$(\bar{g}_1, \bar{S}_1^*)$ **Problem**		
$\bar{g}_1(\tilde{X}, h)$	$e^{\lambda_- \tilde{X}} \left(A_{12}(h)\tilde{X}^2 + A_{11}(h)\tilde{X} + A_{10}(h)\right)$	$\bar{\Gamma}$ (A.8)
$\frac{\partial \lambda_-}{\partial h}$	$\frac{\alpha}{h^2(2\lambda_-+(\beta-1))}$	$\bar{\Theta}$ (A.10)
$\frac{\partial \bar{S}_0^*}{\partial h}$	$\frac{h\frac{\partial \lambda_-}{\partial h}A_{00}+\bar{S}_0^*\bar{\Delta}_\Theta^*-\lambda_-\bar{\Theta}^*}{(\lambda_--1)(1+\Delta^*)-\bar{S}_0^*\bar{\Gamma}^*}$	$\bar{\Delta}_\Theta$ (A.11)
$\frac{\partial \bar{A}_{00}}{\partial h}$	$-\frac{A_{00}+\frac{\partial \bar{S}_0^*}{\partial h}+\bar{\Theta}^*}{h}$	
$B_{00}(h)$	$\alpha(1-h)\left[\frac{\partial A_{00}}{\partial h} - \lambda_- \frac{A_{00}}{\bar{S}_0^*}\frac{\partial \bar{S}_0^*}{\partial h}\right]$	
$B_{01}(h)$	$\alpha(1-h)A_{00}\frac{\partial \lambda_-}{\partial h}$	
$A_{12}(h)$	$\frac{B_{01}}{2(2\lambda_-+(\beta-1))}$	
$A_{11}(h)$	$\frac{1}{(2\lambda_-+(\beta-1))}\left[B_{00}-2A_{12}\right]$	
$A_{10}(h)$	$\frac{-\bar{S}_1^*-\bar{S}_1^*\bar{\Delta}_e^*}{h}$	
$\bar{S}_1^*(h)$	$\frac{hA_{11}}{(\lambda_--1)\bar{\Delta}^*-\bar{S}_0^*\bar{\Gamma}_e^*}$	

Table 3.1: Formulae used in the derivation of the first two terms of the analytic approximation for the American put option. The required Greeks are derived in Appendix A. The star superscript denotes that the Greeks are evaluated at the leading order boundary approximation $\bar{S}_0^*$.

Term	Formula	Greeks Required
$(\bar{g}_2, \bar{S}_2^*)$ **Problem**		
$\bar{g}_2$	$e^{\lambda_- \tilde{X}} \left(A_{24}(h)\tilde{X}^4 + A_{23}(h)\tilde{X}^3 + A_{22}(h)\tilde{X}^2 + A_{21}(h)\tilde{X} + A_{20}(h) \right)$	$\bar{\Gamma}_\Delta$ (A.9)
$\frac{\partial^2 \lambda_-}{\partial h^2}$	$-\frac{\partial \lambda_-}{\partial h} \left(\frac{2}{h} + \frac{1}{2\lambda_- + (\beta-1)} \frac{\partial \lambda_-}{\partial h} \right)$	$\bar{\Gamma}_\Theta$ (A.12)
$\frac{\partial^2 \bar{S}_0^*}{\partial h^2}$	$\frac{\frac{\partial \lambda_-}{\partial h} A_{00} + h \frac{\partial^2 \lambda_-}{\partial h^2} A_{00} + h \frac{\partial \lambda_-}{\partial h} \frac{\partial A_{00}}{\partial h} + \frac{\partial \bar{S}_0^*}{\partial h} \bar{\Delta}_\Theta^* + \bar{S}_0^* \left(\bar{\Theta}_{\Theta\Delta}^* + \frac{\partial \bar{S}_0^*}{\partial h} \bar{\Gamma}_\Theta^* \right) - \frac{\partial \lambda_-}{\partial h} \bar{\Theta}^* - \lambda_- \left(\bar{\Theta}_\Theta^* + \frac{\partial \bar{S}_0^*}{\partial h} \bar{\Delta}_\Theta^* \right)}{(\lambda_- - 1)(1 + \Delta^*) - \bar{S}_0^* \bar{\Gamma}^*}$	$\bar{\Gamma}_\Gamma$ (A.13)
	$- \frac{\frac{\partial \lambda_-}{\partial h} (1 + \bar{\Delta}^*) + (\lambda_- - 1) \left(\bar{\Delta}_\Theta^* + \frac{\partial \bar{S}_0^*}{\partial h} \bar{\Gamma}^* \right) - \frac{\partial \bar{S}_0^*}{\partial h} \bar{\Gamma}^* - \bar{S}_0^* \left(\bar{\Gamma}_\Theta^* + \frac{\partial \bar{S}_0^*}{\partial h} \bar{\Gamma}_\Delta^* \right)}{(\lambda_- - 1)(1 + \Delta^*) - \bar{S}_0^* \bar{\Gamma}^*} \frac{\partial \bar{S}_0^*}{\partial h}$	$\bar{\Theta}_\Theta$ (A.14)
$\frac{\partial \bar{S}_1^*}{\partial h}$	$\frac{\bar{S}_1^*}{h} + \frac{\bar{S}_1^*}{A_{11}} \frac{\partial A_{11}}{\partial h} - \bar{S}_1^* \frac{\frac{\partial \lambda_-}{\partial h} (1 + \bar{\Delta}^*) + (\lambda_- - 1) \left(\bar{\Delta}_\Theta^* + \frac{\partial \bar{S}_0^*}{\partial h} \bar{\Gamma}^* \right) - \frac{\partial \bar{S}_0^*}{\partial h} - \bar{S}_0^* \left(\bar{\Gamma}_\Theta^* + \frac{\partial \bar{S}_0^*}{\partial h} \bar{\Gamma}_\Delta^* \right)}{(\lambda_- - 1)(1 + \bar{\Delta}^*) - \bar{S}_0^* \bar{\Gamma}^*}$	$\bar{\Delta}_{\Theta\Theta}$ (A.15)
$\frac{\partial^2 A_{00}}{\partial h^2}$	$-\frac{1}{h} \left(2 \frac{\partial A_{00}}{\partial h} + \frac{\partial^2 \bar{S}_0^*}{\partial h^2} + \bar{\Theta}_\Theta^* + \frac{\partial \bar{S}_0^*}{\partial h} \bar{\Delta}_\Theta^* \right)$	$\bar{\Gamma}_{\Delta\Theta}$ (A.16)
$\frac{\partial B_{00}}{\partial h}$	$-\frac{A_{11}}{1-h} + \alpha(1-h) \left[\frac{\partial^2 \bar{A}_{00}}{\partial h^2} + \frac{\left(\frac{\partial \bar{S}_0^*}{\partial h} \right)^2 - \bar{S}_0^* \frac{\partial^2 \bar{S}_0^*}{\partial h^2}}{(\bar{S}_0^*)^2} \lambda_- A_{00} - \frac{A_{00}}{\bar{S}_0^*} \frac{\partial \bar{S}_0^*}{\partial h} \frac{\partial \lambda_-}{\partial h} - \frac{\lambda_-}{\bar{S}_0^*} \frac{\partial \bar{S}_0^*}{\partial h} \frac{\partial A_{00}}{\partial h} \right]$	
$\frac{\partial B_{01}}{\partial h}$	$-\frac{B_{01}}{1-h} + \alpha(1-h) \frac{\partial^2 \lambda_-}{\partial h^2} A_{00} + \alpha(1-h) \frac{\partial \lambda_-}{\partial h} \frac{\partial A_{00}}{\partial h}$	
$\frac{\partial A_{12}}{\partial h}$	$\frac{1}{2\lambda_- + (\beta-1)} \left(\frac{1}{2} \frac{\partial B_{01}}{\partial h} - 2A_{12} \frac{\partial \lambda_-}{\partial h} \right)$	
$\frac{\partial A_{11}}{\partial h}$	$\frac{1}{2\lambda_- + (\beta-1)} \left(\frac{\partial B_{00}}{\partial h} - 2 \frac{\partial A_{12}}{\partial h} - 2A_{11} \frac{\partial \lambda_-}{\partial h} \right)$	
$\frac{\partial A_{10}}{\partial h}$	$-\frac{A_{10}}{h} + \frac{1}{\bar{S}_1^*} \frac{\partial \bar{S}_1^*}{\partial h} A_{10} - \frac{\bar{S}_1^*}{h} \left(\bar{\Delta}_\Theta^* + \frac{\partial \bar{S}_0^*}{\partial h} \bar{\Gamma}^* \right)$	
$B_{10}(h)$	$\alpha(1-h) \left[\frac{\partial A_{10}}{\partial h} - \frac{1}{\bar{S}_1^*} \frac{\partial \bar{S}_0^*}{\partial h} (\lambda_- A_{10} + A_{11}) - \left(\frac{1}{\bar{S}_0^*} \frac{\partial \bar{S}_1^*}{\partial h} - \frac{\bar{S}_0^*}{\bar{S}_0^*} \frac{\partial \bar{S}_0^*}{\partial h} \right) \lambda_- A_{00} \right]$	
$B_{11}(h)$	$\alpha(1-h) \left[\frac{\partial \lambda_-}{\partial h} A_{10} + \frac{\partial A_{11}}{\partial h} - \frac{1}{\bar{S}_0^*} \frac{\partial \bar{S}_0^*}{\partial h} (\lambda_- A_{11} + 2A_{12}) \right]$	
$B_{12}(h)$	$\alpha(1-h) \left[\frac{\partial \lambda_-}{\partial h} A_{11} + \frac{\partial A_{12}}{\partial h} - \frac{1}{\bar{S}_0^*} \frac{\partial \bar{S}_0^*}{\partial h} \lambda_- A_{12} \right]$	
$B_{13}(h)$	$\alpha(1-h) \left[\frac{\partial \lambda_-}{\partial h} A_{12} \right]$	
$A_{24}(h)$	$\frac{1}{4(2\lambda_- + (\beta-1))} B_{13}$	
$A_{23}(h)$	$\frac{1}{3(2\lambda_- + (\beta-1))} [B_{12} - 12A_{24}]$	
$A_{22}(h)$	$\frac{1}{2(2\lambda_- + (\beta-1))} [B_{11} - 6A_{23}]$	
$A_{21}(h)$	$\frac{1}{(2\lambda_- + (\beta-1))} [B_{10} - 2A_{22}]$	
$A_{20}(h)$	$\frac{-\bar{S}_2^* - \bar{S}_2^* \bar{\Delta}^* - (\bar{S}_1^*)^2 \bar{\Gamma}^*}{h}$	
$\bar{S}_2^*(h)$	$\frac{h A_{21} - \left[(\lambda_- - 2)\bar{\Gamma}^* - \bar{S}_0^* \bar{\Gamma}_\Delta^* \right] (\bar{S}_1^*)^2}{(\lambda_- - 1)(1 + \bar{\Delta}^*) - \bar{S}_0^* \bar{\Gamma}^*}$	

Table 3.2: Formulae used in the derivation of the third term of the analytic approximation for the American put option. The required Greeks are derived in Appendix A. The star superscript denotes that the Greeks are evaluated at the leading order boundary approximation $\bar{S}_0^*$.

Chapter 4

The American Barrier Option Problem

Barrier options are path dependent options with a payoff dependent on whether the price of the underlying reaches a defined level B (the *barrier*) before expiry. They are among the most commonly traded non-standard, or *exotic*, options. A barrier option which is inactive at inception, but becomes active if the barrier is reached is termed a *knock-in* option. Conversely, a barrier option which is active at inception, but becomes inactive if the barrier is reached is termed a *knock-out* option. A further level of classification specifies whether the price of the underlying at inception is above (a *down* option) or below (an *up* option) the barrier. Finally, a specific classification is given to options where the barrier is in-the-money ($B > K$ for a call option or $B < K$ for a put option) with respect to the strike, in which case we have a *reverse* barrier option. As an example, a put option which is active at inception, with a barrier which is above the strike is termed an *up-and-out* put option. If the barrier were below the strike, it would be termed a *reverse up-and-out* put option.

In theory we therefore have 16 distinct mathematical problems, though in the absence of rebates two of these problems lead to zero payoff everywhere (the up-and-out call and the down-and-out put) and two other pairs of problems are the same whether $B > K$ or $B < K$ (the down-and-in call and the up-and-in put), all of which results in 12 barrier option problems of mathematical interest. The contracts, along with their payoff functions at expiry are given in Table 4.1.

Barrier options are potentially attractive from an investor's point of view because they are cheaper than the corresponding vanilla option. This is true because the existence of the barrier restricts the optionality of the holder who may find their option *knocked-out* or never *knocked-in*

Option	Payoff at Expiry		Value at Barrier
	$B < K$	$B > K$	
Up-and-Out Call	0	$\mathbb{1}_{(S_{max}<B)}\left(S_T - K\right)^+$	0
Up-and-In Call	$\left(S_T - K\right)^+$	$\mathbb{1}_{(S_{max}>B)}\left(S_T - K\right)^+$	$C_e(B,t)$
Down-and-Out Call	$\mathbb{1}_{(S_{min}>B)}\left(S_T - K\right)^+$	$\mathbb{1}_{(S_{min}>B)}\left(S_T - K\right)$	0
Down-and-In Call	$\mathbb{1}_{(S_{min}<B)}\left(S_T - K\right)^+$	$\mathbb{1}_{(S_{min}<B)}\left(S_T - K\right)^+$	$C_e(B,t)$
Up-and-Out Put	$\mathbb{1}_{(S_{max}<B)}\left(K - S_T\right)$	$\mathbb{1}_{(S_{max}<B)}\left(K - S_T\right)^+$	0
Up-and-In Put	$\mathbb{1}_{(S_{max}>B)}\left(K - S_T\right)^+$	$\mathbb{1}_{(S_{max}>B)}\left(K - S_T\right)^+$	$P_e(B,t)$
Down-and-Out Put	$\mathbb{1}_{(S_{min}>B)}\left(K - S_T\right)^+$	0	0
Down-and-In Put	$\mathbb{1}_{(S_{min}>B)}\left(K - S_T\right)^+$	$\left(K - S_T\right)^+$	$P_e(B,t)$

Table 4.1: Payoff functions for European barrier options.

while a vanilla option would continue to exist and potentially provide a non-zero payoff. Thus an investor with a firm belief over the directionality of the underlying can participate fully in price movements if they are correct, without paying for the part of the price distribution they believe will not outturn.

The use of barrier options for risk management is typically restricted to the hedging of exotic options as they present a more complicated hedging problem than standard options, particularly in the case of reverse options. Though we do not discuss the reality of hedging of barrier options, for which the reader is directed to Taleb [98], we do mention the difficulty of identifying the point at which the underlying reaches the barrier in the presence of discontinuous price movements. Mathematically, the presence of a barrier separates the pricing problem into two regions, or more in the case of options with multiple barriers. At inception, one of these regions is active with the option price obeying the usual Black-Scholes-Merton governing equation whilst the other is inactive. Thus the PDE formulation of the problem for an European up-and-out put option with $B > K$ is

$$S \in (0,B), t \in (0,T) \qquad \frac{\partial P_e^{UO}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_e^{UO}}{\partial S^2} + (r-D)S\frac{\partial P_e^{UO}}{\partial S} - rP_e^{UO} = 0, \quad (4.1a)$$

subject to

$$\text{as } S \to 0 \qquad P_e^{UO}(S,t) \to Ke^{-r(T-t)}, \quad (4.1b)$$

$$\text{at } S = B \qquad P_e^{UO}(B,t) = 0, \quad (4.1c)$$

$$\text{at } t = T \qquad P_e^{UO}(S,T) = (K-S)^+ . \quad (4.1d)$$

Solving the European barrier option problem in closed-form is performed using several techniques in the literature: by transformation of the PDE into the heat equation and solving the resulting semi-infinite problem using the method of images, or directly with the relevant Green's function; by expectation pricing [67]; and by use of certain symmetric properties of the solutions to the Black-Scholes-Merton PDE [26].

Summaries of the solutions to these equations are presented in the literature using either a consolidated approach based on indicator variables [54] or explicitly in terms of the relevant cumulative normal distributions [94]. Our preferred representation however is adopted by Howison & Steinberg [61] where the barrier option price is represented as the sum of a European option price less a term which is a function of the barrier. This provides a more intuitive feel to the sources of value and has the added advantage that the barrier option Greeks can be written in terms of the equivalent vanilla European Greeks plus a correction term.

Defining the non-dimensional barrier as $B = K\bar{B}$ gives the non-dimensional form for the up-and-out put option

$$\bar{P}_e^{UO}(\bar{S},\tau;\bar{B}) = \begin{cases} \bar{P}_e(\bar{S},\tau) - \left(\frac{\bar{S}}{\bar{B}}\right)^{2a} \bar{P}_e\left(\frac{\bar{B}^2}{\bar{S}},\tau\right) & \text{for } \bar{B} > 1, \\ \bar{B}\left[\bar{P}_e\left(\frac{\bar{S}}{\bar{B}},\tau\right) - \left(\frac{\bar{S}}{\bar{B}}\right)^{2a} \bar{P}_e\left(\frac{\bar{B}}{\bar{S}},\tau\right)\right] & \\ -(1-\bar{B})\left[\bar{P}_d\left(\frac{\bar{S}}{\bar{B}},\tau\right) - \left(\frac{\bar{S}}{\bar{B}}\right)^{2a} \bar{P}_d\left(\frac{\bar{B}}{\bar{S}},\tau\right)\right] & \text{for } \bar{B} \leq 1, \end{cases} \tag{4.2}$$

where $2a = -(\beta - 1)$ and $\bar{P}_d(\bar{S},\tau)$ is a non-dimensional European digital put option, which pays unity if $\bar{S} < 1$ at expiry and zero otherwise and has value

$$\bar{P}_d(\bar{S},\tau) = \frac{e^{-\alpha\tau}}{2}\text{erfc}\left(\frac{\bar{d}_2}{\sqrt{2}}\right). \tag{4.3}$$

Although more thinly traded, some barrier option contracts contain a *rebate* which is paid if the option is knocked-out or fails to be knocked-in. Since the Black-Scholes-Merton PDE is linear, the rebate valuation problem can be formulated and solved separately. In substance, a rebate is simply a pre-agreed value $R = K\bar{R}$ which is paid if the price of the underlying reaches the barrier $\bar{B}$. The valuation problem is a scaling by $\bar{R}$ of an American digital option which pays unity if the exercise condition is met. If the barrier is above the underlying at inception, the contract is an American digital call $C_d^{Am}(S,t;B)$ and if the barrier is below the underlying at inception, the contract is an American digital put $P_d^{Am}(S,t;B)$. Such options are often termed *one-touch* digital options as they would be immediately exercised through no-arbitrage. The timing of the rebate's payment has an effect on the valuation problem and for the work described here we assume the rebate is paid immediately at the time the barrier is reached.

The non-dimensional American digital option prices are then

$$\bar{C}_d^{Am}(\bar{S},\tau;\bar{B}) = \frac{1}{2}\left[\left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^+}\operatorname{erfc}\left(-\frac{\bar{d}^+}{\sqrt{2}}\right) + \left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^-}\operatorname{erfc}\left(-\frac{\bar{d}^-}{\sqrt{2}}\right)\right], \tag{4.4a}$$

$$\bar{P}_d^{Am}(\bar{S},\tau;\bar{B}) = \frac{1}{2}\left[\left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^+}\operatorname{erfc}\left(\frac{\bar{d}^+}{\sqrt{2}}\right) + \left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^-}\operatorname{erfc}\left(\frac{\bar{d}^-}{\sqrt{2}}\right)\right], \tag{4.4b}$$

where

$$\lambda_\infty^\pm = \frac{-(\beta-1)\pm\sqrt{(\beta-1)^2+4\alpha}}{2}, \tag{4.4c}$$

$$\bar{d}^\pm = \frac{\ln\left(\frac{\bar{S}}{\bar{B}}\right)\pm(\lambda_\infty^+-\lambda_\infty^-)\,\tau}{\sqrt{2\tau}}. \tag{4.4d}$$

Table 4.2 contains digital American option contract corresponding to the relevant barrier option which leads to the non-dimensional rebate valuation for an up-and-out put option

$$\bar{R}^{UO}(\bar{S},\tau;\bar{B},\bar{R}) = \bar{R}\left[\left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^+}\operatorname{erfc}\left(-\frac{\bar{d}^+}{\sqrt{2}}\right) + \left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^-}\operatorname{erfc}\left(-\frac{\bar{d}^-}{\sqrt{2}}\right)\right]. \tag{4.5}$$

Option	Rebate Value
Up-and-Out	$RC_d^{Am}(S,t;B)$
Up-and-In	$RC_d^{Am}(S,t;B)$
Down-and-Out	$RP_d^{Am}(S,t;B)$
Down-and-In	$RP_d^{Am}(S,t;B)$

Table 4.2: Rebate payoffs for European barrier options. Payment of the rebate is assumed to occurs at the instant the barrier is reached.

The Greeks for the European up-and-out barrier option (4.2) together with the relevant rebate term (4.5), which will be required in subsequent numerical work, are derived in Appendix A.

4.1 The American Up-and-Out Put Option Problem

In the following work, we consider an American up-and-out put option $\bar{P}_a^{UO}(\bar{S},\tau)$ which knocks out and pays a rebate $R = K\bar{R}$ if the stock price reaches the barrier $B = K\bar{B}$. The impact on the non-dimensional American option problem discussed in Chapter 3 (3.2a-3.2f) is to change the domain from semi-infinite to one that is finite, though not fixed. This PDE formulation of the American up-and-out put option problem is

$$\bar{S} \in (\bar{S}^*(\tau), \bar{B}), \tau \in (0, T) \qquad \frac{\partial \bar{P}_a^{UO}}{\partial \tau} = \bar{S}^2 \frac{\partial^2 \bar{P}_a^{UO}}{\partial \bar{S}^2} + \beta \bar{S} \frac{\partial \bar{P}_a^{UO}}{\partial \bar{S}} - \alpha \bar{P}_a^{UO}, \tag{4.6a}$$

subject to

$$\text{at } \tau = 0 \qquad \bar{P}_a^{UO}(\bar{S}, 0) = \max\big(1 - \bar{S}, 0\big), \tag{4.6b}$$

$$\bar{S}^*(0) = \min\left(\alpha/(\alpha - \beta), 1\right), \tag{4.6c}$$

$$\text{at } \bar{S} = \bar{S}^*(\tau) \qquad \bar{P}_a^{UO}\left(\bar{S}^*, \tau\right) = \left(1 - \bar{S}^*\right), \tag{4.6d}$$

$$\left.\frac{\partial \bar{P}_a^{UO}}{\partial \bar{S}}\right|_{\bar{S}^*} = -1, \tag{4.6e}$$

$$\text{at } \bar{S} = \bar{B} \qquad \bar{P}_a^{UO}(\bar{B}, 0) = \bar{R}. \tag{4.6f}$$

4.2 Small-Time Behaviour

As in previous chapters, to investigate the small-time behaviour we introduce the time scaling $\tau = \epsilon^2 \hat{T}$ ($0 < \epsilon \ll 1$, $\hat{T} = O(1)$) into the non-dimensional American up-and-out put problem to give

$$\bar{S} \in (\bar{S}^*(\hat{T}), \bar{B}), \hat{T} \in (0, T) \qquad \frac{1}{\epsilon^2}\frac{\partial \bar{P}_a^{UO}}{\partial \hat{T}} = \bar{S}^2 \frac{\partial^2 \bar{P}_a^{UO}}{\partial \bar{S}^2} + \beta \bar{S} \frac{\partial \bar{P}_a^{UO}}{\partial \bar{S}} - \alpha \bar{P}_a^{UO}, \tag{4.7a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \bar{P}_a^{UO}(\bar{S}, 0) = \max\big(1 - \bar{S}, 0\big), \tag{4.7b}$$

$$\bar{S}^*(0) = \min\left(\alpha/(\alpha - \beta), 1\right), \tag{4.7c}$$

$$\text{at } \bar{S} = \bar{S}^*(\hat{T}) \qquad \bar{P}_a^{UO}\left(\bar{S}^*, \hat{T}\right) = \left(1 - \bar{S}^*\right), \tag{4.7d}$$

$$\left.\frac{\partial \bar{P}_a^{UO}}{\partial \bar{S}}\right|_{\bar{S}^*} = -1, \tag{4.7e}$$

$$\text{at } \bar{S} = \bar{B} \qquad \bar{P}_a^{UO}(\bar{B}, 0) = \bar{R}. \tag{4.7f}$$

Out-of-the-Money Barrier ($\bar{B} > 1$)

For out-of-the-money barriers, the small-time behaviour of the American up-and-out put option in the region of the boundary is indistinguishable from the regular American option problem. The only additional feature is a *barrier inner* region near $\bar{S} = \bar{B}$ which matches into the outer

2 region seen in the regular American option problem.

We look in a small region $\bar{S} = \bar{B}(1+\epsilon\hat{S})$ about the barrier where $\bar{P}_a^{UO}(\bar{S},\hat{T}) = \bar{R}\hat{P}_a^{Bar.In}(\hat{S},\hat{T})$, which gives to the barrier inner problem

$$\hat{S} \in (-\infty,0), \hat{T} \in (0,T) \qquad \frac{1}{\epsilon^2}\frac{\partial \hat{P}_a^{Bar.In}}{\partial \hat{T}} = (1+\epsilon\hat{S})^2\frac{1}{\epsilon^2}\frac{\partial^2 \bar{P}_a^{Bar.In}}{\partial \hat{S}^2} + \beta(1+\epsilon\hat{S})\frac{1}{\epsilon}\frac{\partial \bar{P}_a^{Bar.In}}{\partial \hat{S}} - \alpha\bar{P}_a^{Bar.In}, \tag{4.8a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_a^{Bar.In}(\hat{S},0) = 0, \tag{4.8b}$$

$$\text{at } \hat{S} = 0 \qquad \hat{P}_a^{Bar.In}(0,\hat{T}) = 1. \tag{4.8c}$$

Posing a barrier inner expansion of the form

$$\check{P}_a^{Bar.In}(\hat{S},\hat{T};\epsilon) = \hat{P}_0^{Bar.In}(\hat{S},\hat{T}) + \epsilon\hat{P}_1^{Bar.In}(\hat{S},\hat{T}) + O(\epsilon^2) \qquad \text{as } \epsilon \to 0, \tag{4.9}$$

gives the following subproblems: for $\hat{P}_0^{Bar.In}$

$$\hat{S} \in (-\infty,0), \hat{T} \in (0,T) \qquad \frac{\partial \hat{P}_0^{Bar.In}}{\partial \hat{T}} = \frac{\partial^2 \bar{P}_0^{Bar.In}}{\partial \hat{S}^2}, \tag{4.10a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_0^{Bar.In}(\hat{S},0) = 0, \tag{4.10b}$$

$$\text{at } \hat{S} = 0 \qquad \hat{P}_0^{Bar.In}(0,\hat{T}) = 1; \tag{4.10c}$$

and for $\hat{P}_1^{Bar.In}$

$$\hat{S} \in (-\infty,0), \hat{T} \in (0,T) \qquad \frac{\partial \hat{P}_1^{Bar.In}}{\partial \hat{T}} = \frac{\partial^2 \bar{P}_1^{Bar.In}}{\partial \hat{S}^2} + 2\hat{S}\frac{\partial^2 \bar{P}_0^{Bar.In}}{\partial \hat{S}^2} + \beta\frac{\partial \hat{P}_0^{Bar.In}}{\partial \hat{S}}, \tag{4.11a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_1^{Bar.In}(\hat{S},0) = 0, \tag{4.11b}$$

$$\text{at } \hat{S} = 0 \qquad \hat{P}_1^{Bar.In}(0,\hat{T}) = 0. \tag{4.11c}$$

The problem (4.10a-4.10c) has the similarity solution

$$\hat{P}_0^{BIn} = h_0(\zeta), \qquad \zeta = \frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}, \tag{4.12}$$

where $h_0(\zeta)$ solves

$$h_0^{''} + 2\zeta h_0^{'} = 0. \tag{4.13}$$

This has general solution

$$h_0(\zeta) = 2\hat{C}_{00} + \hat{C}_{01}\mathrm{erfc}(\zeta) \tag{4.14}$$

and conditions (4.10b,4.10c) require

$$\hat{C}_{00} = 1, \qquad \hat{C}_{01} = -1, \tag{4.15}$$

giving

$$h_0(\zeta) = 2 - \mathrm{erfc}(\zeta). \tag{4.16}$$

The problem (4.11a-4.11c) becomes

$$\hat{S} \in (-\infty, 0), \hat{T} \in (0, T) \qquad \frac{\partial \hat{P}_1^{Bar.In}}{\partial \hat{T}} = \frac{\partial^2 \hat{P}_1^{Bar.In}}{\partial \hat{S}^2} + 2\hat{S}\frac{\partial^2 \hat{P}_0^{Bar.In}}{\partial \hat{S}^2} + \beta\frac{\partial \hat{P}_0^{Bar.In}}{\partial \hat{S}}, \tag{4.17a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_1^{Bar.In}(\hat{S}, 0) = 0, \tag{4.17b}$$

$$\text{at } \hat{S} = 0 \qquad \hat{P}_1^{Bar.In}(0, \hat{T}) = 0, \tag{4.17c}$$

which has the similarity solution

$$\hat{P}_1^{BIn} = \hat{T}^{\frac{1}{2}} h_1(\zeta), \tag{4.18}$$

where

$$h_1^{''} + 2\zeta h_1^{'} - 2h_1 = -4\zeta h_0^{''} - 2\beta h_0^{'}. \tag{4.19}$$

This has general solution

$$h_1(\zeta) = 2\hat{C}_{10}\zeta + \hat{C}_{11}\mathrm{ierfc}(\zeta) - \frac{2e^{-\zeta^2}}{\sqrt{\pi}}\left(\zeta^2 + \frac{1}{2}(\beta+1)\right) \tag{4.20}$$

and conditions (4.17b,4.17c) require

$$\hat{C}_{10} = \hat{C}_{11} = \beta + 1, \tag{4.21}$$

giving

$$h_1(\zeta) = (\beta+1)\left(2\zeta + \mathrm{ierfc}(\zeta)\right) - \frac{2e^{-\zeta^2}}{\sqrt{\pi}}\left(\zeta^2 + \frac{1}{2}(\beta+1)\right). \tag{4.22}$$

Apart from the barrier region, the remainder of this problem, including the transcendental expression for free boundary, is identical to the standard American option problem. For small times, the boundary does not feel the effect of the out-of-the-money barrier. The small-time asymptotic structure of the barrier option problem for out-of-the-money barriers is given in Figure 4-1.

Reverse Barrier ($\bar{B} < 1$)

As the barrier moves near-the-money ($\bar{B} \to 1^+$), the outer 2 region disappears and we may investigate the possibility that the local presence of the barrier affects the small-time asymptotic behaviour of the optimal exercise boundary.

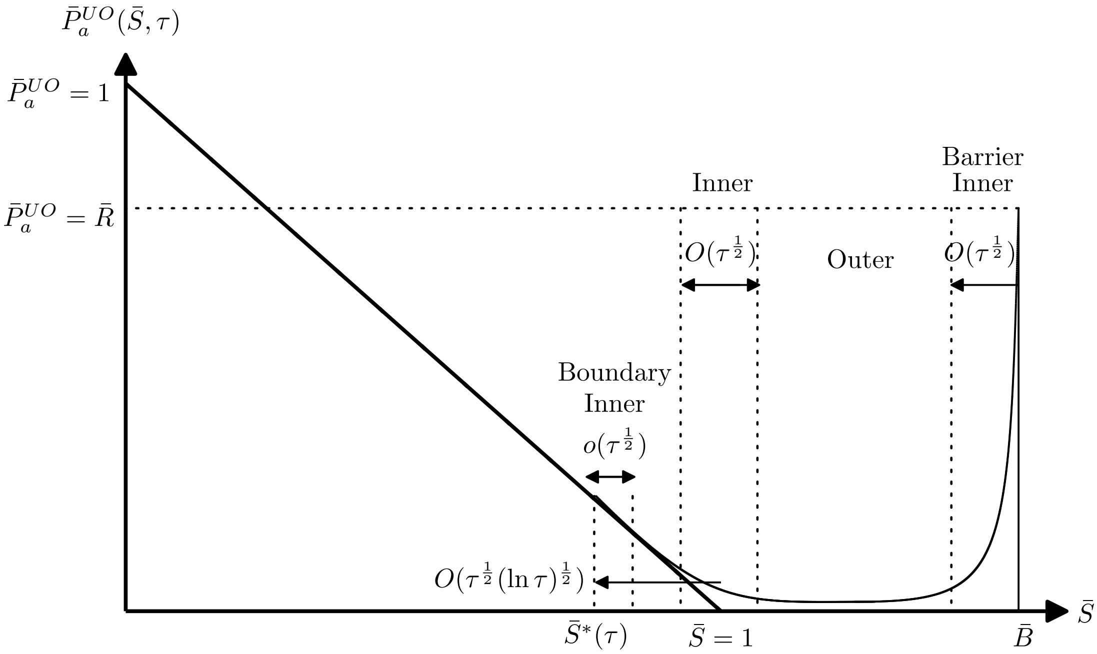
We consider a reverse barrier ($\bar{B} < 1$) by looking in a small region $\bar{S} = \bar{B}(1 + \epsilon\hat{S})$ about the barrier where $\bar{P}_a^{UO}(\bar{S}, \hat{T}) = \hat{P}_a^{Bar.In}(\hat{S}, \hat{T})$, which gives to the barrier inner problem

$$\hat{S} \in (-\infty, 0), \hat{T} \in (0, \infty) \qquad \frac{\partial \hat{P}_a^{Bar.In}}{\partial \hat{T}} = (1+\epsilon\hat{S})^2 \frac{\partial^2 \bar{P}_a^{Bar.In}}{\partial \hat{S}^2} + \beta\epsilon(1+\epsilon\hat{S})\frac{\partial \bar{P}_a^{Bar.In}}{\partial \hat{S}} - \epsilon^2\alpha\bar{P}_a^{Bar.In}, \tag{4.23a}$$

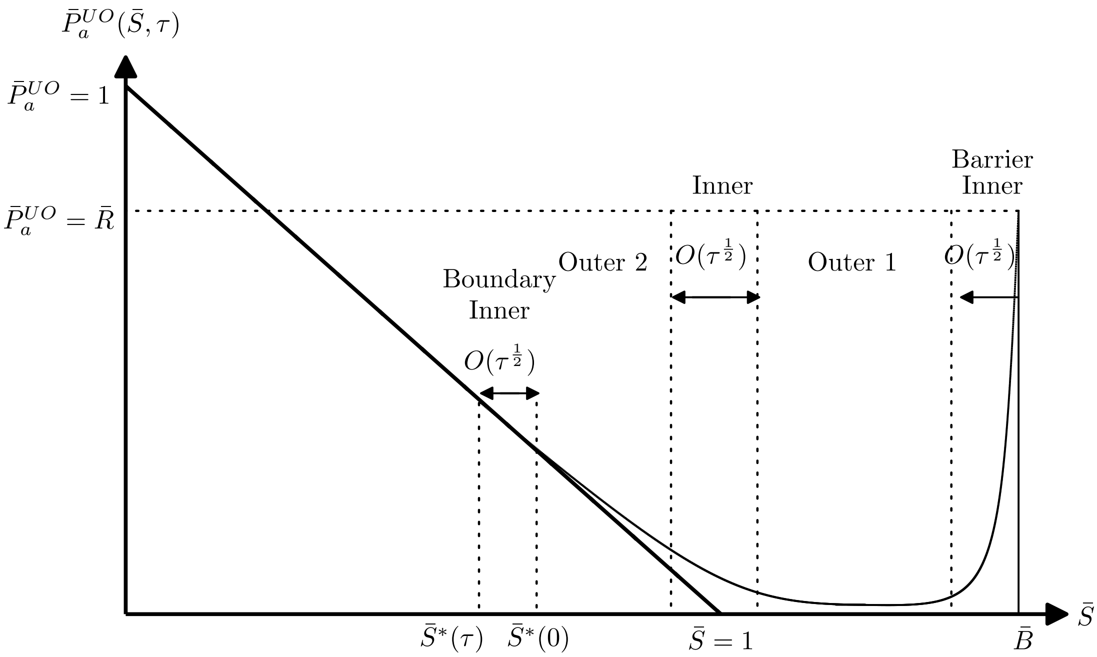
subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_a^{Bar.In}(\hat{S}, 0) = 1 - \bar{B} - \epsilon\bar{B}\hat{S}, \tag{4.23b}$$

$$\text{at } \hat{S} = 0 \qquad \hat{P}_a^{Bar.In}(0, \hat{T}) = \bar{R}. \tag{4.23c}$$



(a) Small-time structure ($\beta \geq 0$)



(b) Small-time structure ($\beta < 0$)

Figure 4-1: A schematic showing the small-time structure for the American up-and-out put with out-of-the-money barrier ($\bar{B} > 1$). The structure differs from the standard American option problem (Figures 3-1 & 3-2) only via an $O(\sqrt{\tau})$ region near the barrier.

Posing a barrier inner expansion of the form

$$\check{P}_a^{Bar.In}(\hat{S},\hat{T};\epsilon) = \hat{P}_0^{Bar.In}(\hat{S},\hat{T}) + \epsilon\hat{P}_1^{Bar.In}(\hat{S},\hat{T}) + O(\epsilon^2) \qquad \text{as } \epsilon \to 0, \tag{4.24}$$

gives the following subproblems: for $\hat{P}_0^{Bar.In}$

$$\hat{S} \in (-\infty, 0), \hat{T} \in (0,\infty) \qquad \frac{\partial \hat{P}_0^{Bar.In}}{\partial \hat{T}} = \frac{\partial^2 \bar{P}_0^{Bar.In}}{\partial \hat{S}^2}, \tag{4.25a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_0^{Bar.In}(\hat{S}, 0) = 1 - \bar{B}, \tag{4.25b}$$

$$\text{at } \hat{S} = 0 \qquad \hat{P}_0^{Bar.In}(0, \hat{T}) = \bar{R}; \tag{4.25c}$$

for $\hat{P}_1^{Bar.In}$

$$\hat{S} \in (-\infty, 0), \hat{T} \in (0,\infty) \qquad \frac{\partial \hat{P}_1^{Bar.In}}{\partial \hat{T}} = \frac{\partial^2 \hat{P}_1^{Bar.In}}{\partial \hat{S}^2} + 2\hat{S}\frac{\partial^2 \hat{P}_0^{Bar.In}}{\partial \hat{S}^2} + \beta\frac{\partial \hat{P}_0^{Bar.In}}{\partial \hat{S}}, \tag{4.26a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_1^{Bar.In}(\hat{S}, 0) = -\bar{B}\hat{S}, \tag{4.26b}$$

$$\text{at } \hat{S} = 0 \qquad \hat{P}_1^{Bar.In}(0, \hat{T}) = 0; \tag{4.26c}$$

and for $\hat{P}_2^{Bar.In}$

$$\begin{aligned}\hat{S} \in (-\infty, 0), \hat{T} \in (0,\infty) \qquad \frac{\partial \hat{P}_2^{Bar.In}}{\partial \hat{T}} &= \frac{\partial^2 \hat{P}_2^{Bar.In}}{\partial \hat{S}^2} + 2\hat{S}\frac{\partial^2 \hat{P}_1^{Bar.In}}{\partial \hat{S}^2} + \beta\frac{\partial \hat{P}_1^{Bar.In}}{\partial \hat{S}} \\ &\quad \hat{S}^2\frac{\partial^2 \hat{P}_0^{Bar.In}}{\partial \hat{S}^2} + \beta\hat{S}\frac{\partial \hat{P}_0^{Bar.In}}{\partial \hat{S}} - \alpha\hat{P}_0^{Bar.In},\end{aligned} \tag{4.27a}$$

subject to

$$\text{at } \hat{T} = 0 \qquad \hat{P}_2^{Bar.In}(\hat{S}, 0) = 0, \tag{4.27b}$$

$$\text{at } \hat{S} = 0 \qquad \hat{P}_2^{Bar.In}(0, \hat{T}) = 0. \tag{4.27c}$$

As in previous work, we define the similarity variable $\zeta = \frac{\hat{S}}{2\hat{T}^{\frac{1}{2}}}$ and let

$$\hat{P}_0^{Bar.In} = h_0(\zeta), \tag{4.28}$$

which transforms the problem (4.25a-4.25c) into

$$h_0^{''} + 2\zeta h_0^{'} = 0, \tag{4.29a}$$

subject to

$$\text{as } \zeta \to -\infty \qquad h_0 \sim 1 - \bar{B}, \tag{4.29b}$$

$$\text{at } \zeta = 0 \qquad h_0 = \bar{R}, \tag{4.29c}$$

which has the solution

$$h_0(\zeta) = (1 - \bar{B}) + \left(\bar{R} - (1 - \bar{B})\right)(2 - \operatorname{erfc}(\zeta)). \tag{4.30}$$

The problem (4.26a-4.26c) has the similarity solution

$$\hat{P}_1^{Bar.In} = \hat{T}^{\frac{1}{2}} h_1(\zeta), \tag{4.31}$$

where

$$h_1^{''} + 2\zeta h_1^{'} - 2h_1 = -4\zeta h_0^{''} - 2\beta h_0^{'}, \tag{4.32a}$$

subject to

$$\text{as } \zeta \to -\infty \qquad h_1 \sim -2\bar{B}\zeta, \tag{4.32b}$$

$$\text{at } \zeta = 0 \qquad h_1 = 0, \tag{4.32c}$$

which has the solution

$$h_1(\zeta) = -2\bar{B}\zeta - (\beta - 1)\left(\bar{R} - (1 - \bar{B})\right)\zeta\,(2 - \operatorname{erfc}(\zeta)) - \left(\bar{R} - (1 - \bar{B})\right)\zeta^2 \frac{e^{-\zeta^2}}{\sqrt{\pi}}. \tag{4.33}$$

The problem (4.27a-4.27c) has the similarity solution

$$\hat{P}_2^{Bar.In} = \hat{T} h_2(\zeta), \tag{4.34}$$

where

$$h_2'' + 2\zeta h_2' - 2h_2 = -4\zeta h_1'' - 2\beta h_1' - 4\zeta^2 h_0'' - 4\beta\zeta h_0' + 4\alpha h_0, \tag{4.35a}$$

subject to

$$\text{as } \zeta \to -\infty \qquad h_2 \sim 0, \tag{4.35b}$$

$$\text{at } \zeta = 0 \qquad h_2 = 0, \tag{4.35c}$$

which has the solution

$$h_2(\zeta) = -2\bar{B}\zeta - (\beta - 1)\left(\bar{R} - (1 - \bar{B})\right)\zeta\left(2 - \operatorname{erfc}(\zeta)\right) - \left(\bar{R} - (1 - \bar{B})\right)\zeta^2 \frac{e^{-\zeta^2}}{\sqrt{\pi}}. \tag{4.36}$$

The asymptotic behaviours of the barrier inner terms (4.30,4.33,4.36) as $\zeta \to -\infty$ are

$$h_0(\zeta) \sim (1 - \bar{B}) - \left(\bar{R} - (1 - \bar{B})\right)\frac{e^{-\zeta^2}}{\sqrt{\pi}}\left(\frac{1}{\zeta} + O\left(\zeta^{-3}\right)\right), \tag{4.37}$$

$$h_1(\zeta) \sim -2\bar{B}\zeta - \left(\bar{R} - (1 - \bar{B})\right)\frac{e^{-\zeta^2}}{\sqrt{\pi}}\left(2\zeta^2 + O(1)\right), \tag{4.38}$$

$$h_2(\zeta) \sim \left((\alpha - \beta)\bar{B} - \alpha\right) - \left(\bar{R} - (1 - \bar{B})\right)\frac{e^{-\zeta^2}}{\sqrt{\pi}}\left(2\zeta^5 + O(\zeta^3)\right). \tag{4.39}$$

Following the method used in the American option case for $\beta \geq 0$, we look for a small boundary inner region about the optimal exercise boundary through the scaling $\bar{S} = \bar{S}^*(\hat{T}) + \delta_1(\epsilon)\check{S}$ and expand the boundary about the initial starting point using $\bar{S}^*(\hat{T}) = \bar{B} + \delta_0(\epsilon)\check{S}^*(\hat{T}) + o(\delta_0)$. In the boundary inner region, the price has the expansion

$$\check{P}_a^{B.In}(\check{S}, \hat{T}; \epsilon) = 1 - \bar{B} - \delta_0\check{S}^*(\hat{T}) - \delta_1\check{S} + o(\epsilon^2) \quad \text{as } \epsilon \to 0, \tag{4.40}$$

where the scalings are restricted to $\delta_0\delta_1 = \epsilon^2$ and $\epsilon^2 \ll \delta_1 \ll \epsilon$, while writing the inner variable in terms of the boundary inner variable, $\zeta = \frac{\delta_0\check{S}^* + \delta_1\check{S}}{2\epsilon\bar{B}\hat{T}^{\frac{1}{2}}}$, leads to the asymptotic behaviour of the terms of the inner series

$$\begin{aligned}\check{P}_a^{B.In}(\check{S}, \hat{T}; \epsilon) =& 1 - \bar{B} - \delta_0\check{S}^*(\hat{T}) - \delta_1\check{S} + \epsilon^2\hat{T}\left(\alpha - (\alpha - \beta)\bar{B}\right) \\ &- \left(\bar{R} - (1 - \bar{B})\right)\frac{e^{-\frac{(\delta_0\check{S}^* + \delta_1\check{S})^2}{4\epsilon^2\bar{B}^2\hat{T}}}}{\sqrt{\pi}}\left(\frac{2\epsilon\bar{B}\hat{T}^{\frac{1}{2}}}{\delta_0\check{S}^* + \delta_1\check{S}} + \dots\right) + o(\epsilon^2) \quad \text{as } \epsilon \to 0.\end{aligned} \tag{4.41}$$

Comparison of the boundary inner expression (4.40) and the asymptotic behaviour of the inner expression in the boundary inner variable (4.41) requires

$$\left(\bar{R}-(1-\bar{B})\right)\frac{e^{-\frac{(\delta_0\check{S}^*+\delta_1\check{S})^2}{4\epsilon^2\bar{B}^2\hat{T}}}}{\sqrt{\pi}}\frac{2\epsilon\bar{B}\hat{T}^{\frac{1}{2}}}{\delta_0\check{S}^*+\delta_1\check{S}}=\epsilon^2\hat{T}\left(\alpha-(\alpha-\beta)\bar{B}\right)\quad\text{as }\epsilon\to 0. \tag{4.42}$$

Taking logs and remembering $\frac{\delta_0\check{S}^*+\delta_1\check{S}}{2\epsilon\bar{B}\hat{T}^{\frac{1}{2}}}\to-\infty$ as $\epsilon\to 0$ gives

$$\delta_0\check{S}^*(\hat{T})\sim\bar{B}-\sqrt{4\epsilon^2\hat{T}\bar{B}^2\ln\left(\epsilon^2\hat{T}\sqrt{\pi}\frac{((\alpha-\beta)\bar{B}-\alpha)}{(1-\bar{B})-\bar{R}}\right)} \tag{4.43}$$

and therefore the small-time asymptotic behaviour of the reverse American up-and-out option boundary for $\beta\geq 0$ is

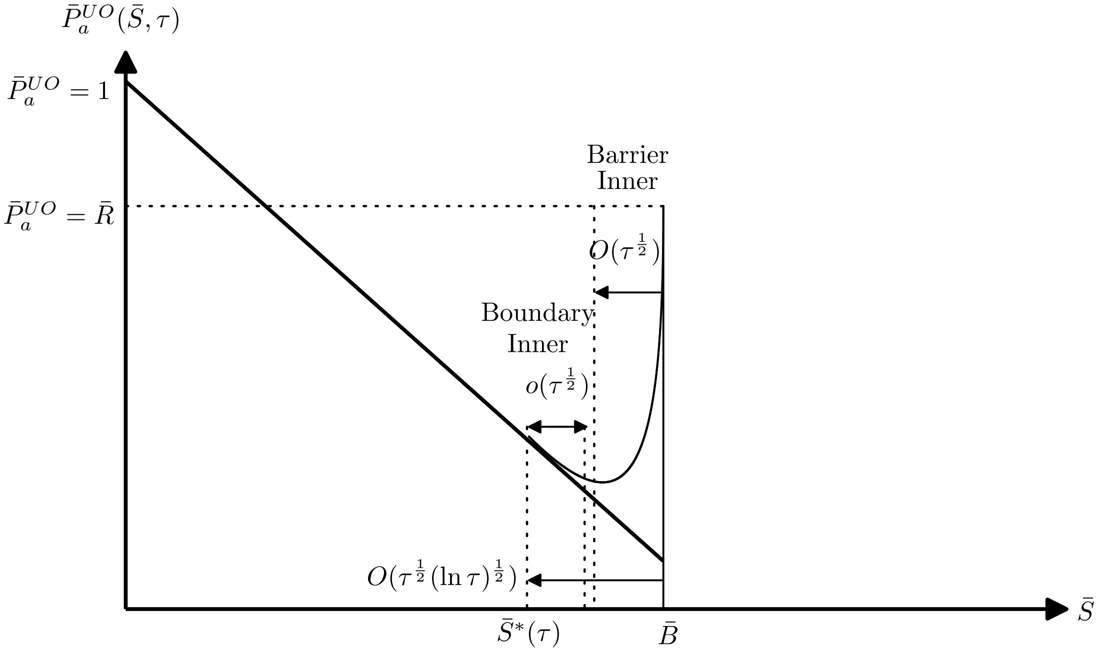
$$\bar{S}^*(\hat{T})\sim\bar{B}-\sqrt{4\epsilon^2\hat{T}\bar{B}^2\ln\left(\epsilon^2\hat{T}\sqrt{\pi}\frac{((\alpha-\beta)\bar{B}-\alpha)}{(1-\bar{B})-\bar{R}}\right)}. \tag{4.44}$$

We note that for $\beta\in(0,\alpha)$, $\alpha-\beta>0$ and therefore $\left(\alpha-(\alpha-\beta)\bar{B}\right)>0$ since $\bar{B}<1$, which ensures the RHS is positive for the purpose of taking logarithms. Further, (4.44) is only valid when the rebate is greater than the intrinsic value of the option at the barrier ($\bar{R}>1-\bar{B}$).
We note that the problem for $\beta<0$ will only feel the effect of the barrier when it is local to $\frac{\alpha}{\alpha-\beta}$ when the problem becomes identical to the problem for $\beta\geq 0$. Transition regions will exist for both the $\beta\geq 0$ problem as $\bar{B}\to 1^+$ and the $\beta<0$ problem as $\bar{B}\to\frac{\alpha}{\alpha-\beta}^+$ but these are not derived in this thesis.
The small-time asymptotic structure of the barrier option problem for out-of-the-money barriers is given in Figure 4-2, while a comparison of the small-time asymptotic expression for the optimal exercise boundary of reverse American up-and-out put options (4.44) with the barrier MOL numerics is shown in Figure 4-3.

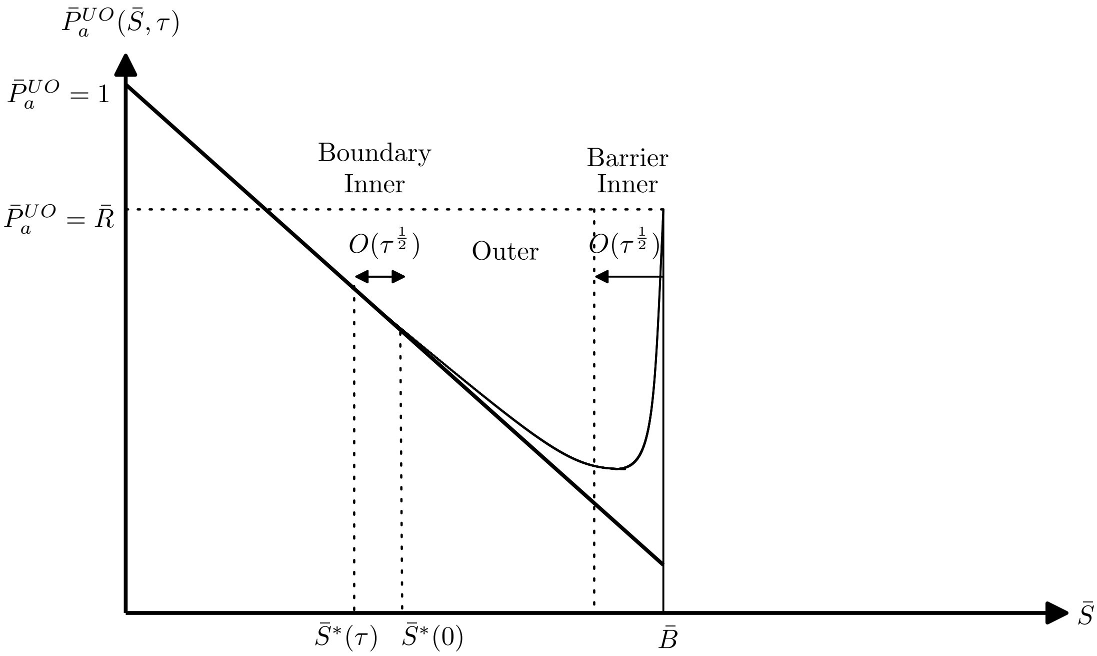
4.3 Large-Time Behaviour

To our knowledge the large-time asymptotic behaviour of the American barrier option problem has not been discussed in the literature. The behaviour may be obtained by posing a suitable perturbation problem. Introducing the small parameter $0<\epsilon\ll 1$ through the time scaling

$$\tau=\frac{\hat{T}}{\epsilon^2}, \tag{4.45}$$

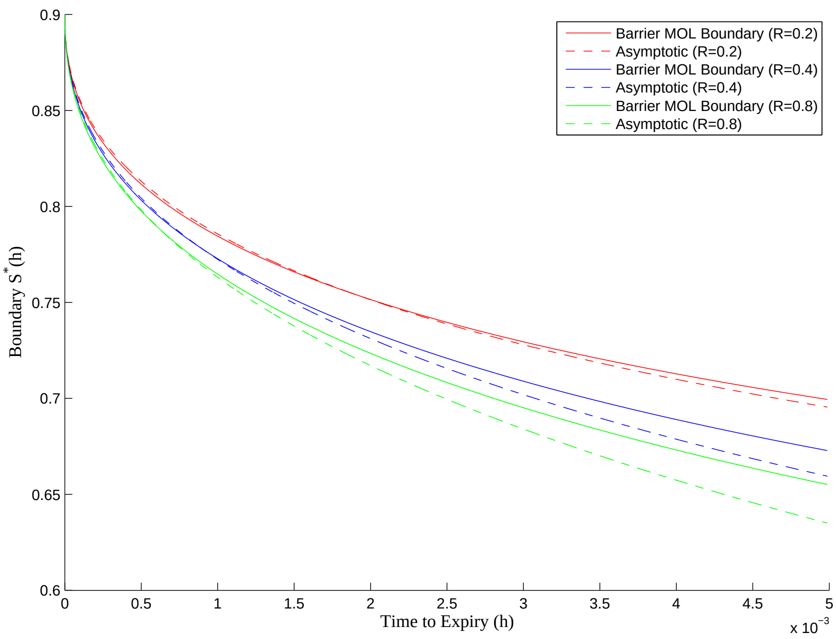


(a) Small-time structure ($\beta \geq 0$)

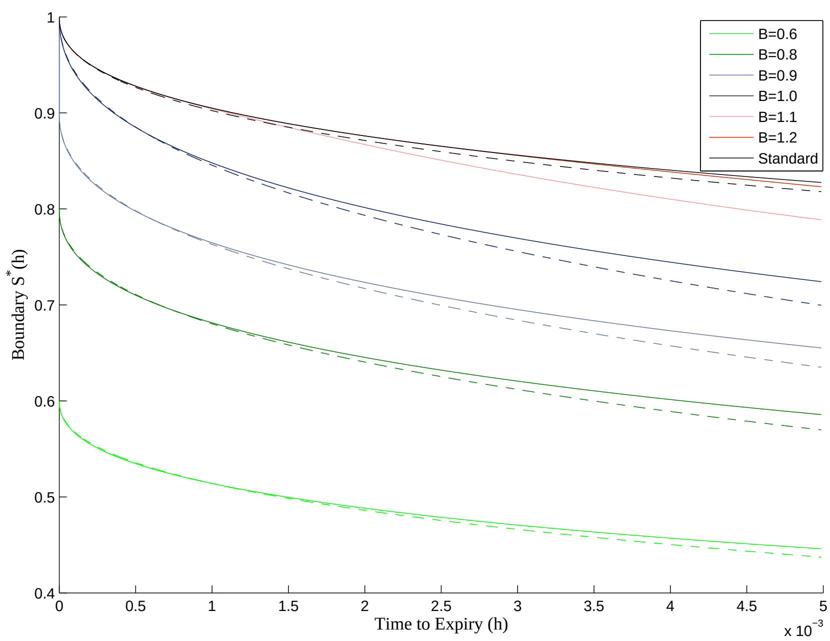


(b) Small-time structure ($\beta < 0$)

Figure 4-2: A schematic showing the small-time structure of the American up-and-out put with a reverse barrier ($\bar{B} < 1$). For $\beta \geq 0$, the asymptotic structure uses an $O(\sqrt{\tau})$ layer near the barrier while the optimal exercise boundary lies at $O(\sqrt{\tau \ln(\tau)})$ from the barrier, though with different constant terms to the standard American option problem. For $\beta > 0$ the presence of the barrier does not affect the behaviour of the boundary unless $\bar{B} \leq \bar{S}^*(0)$ in which case we expect the same structure as for the $\beta \geq 0$ problem.



(a) Small-time asymptotics ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$, $\bar{B} = 0.9$)



(b) Small-time asymptotics ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$, $\bar{R} = 0.8$)

Figure 4-3: A comparison of the benchmark barrier MOL boundary with the full problem asymptotics (4.44). Figure 4.3(a) shows a comparison for fixed barrier ($\bar{B} = 0.9$) and a range of rebates. Figure 4.3(b) shows a comparison for fixed rebate ($\bar{R} = 0.8$) and a range of barriers. The equivalent MOL boundary and asymptotics for the standard American option are also shown. The MOL scheme uses 20000 time steps and 40000 spatial points on the interval $[\bar{S}^*_\infty, \bar{B}]$.

where $\hat{T} = O(1)$, leads to the large-time problem for the up-and-out put option which pays a rebate $\bar{R}$ at the barrier

$$\bar{S} \in (\bar{S}^*(\hat{T}), \bar{B}), \hat{T} \in (0, \infty) \qquad \epsilon^2 \frac{\partial \bar{P}^{UO}}{\partial \hat{T}} = \bar{S}^2 \frac{\partial^2 \bar{P}^{UO}}{\partial \bar{S}^2} + \beta \bar{S} \frac{\partial \bar{P}^{UO}}{\partial \bar{S}} - \alpha \bar{P}^{UO}, \tag{4.46a}$$

subject to

$$\text{at } \bar{S} = \bar{B} \qquad \bar{P}^{UO}\left(\bar{B}, \hat{T}\right) = \bar{R}, \tag{4.46b}$$

$$\text{at } \bar{S} = \bar{S}^*(\hat{T}) \qquad \bar{P}^{UO}\left(\bar{S}^*, \hat{T}\right) = (1 - S^*), \tag{4.46c}$$

$$\left.\frac{\partial \bar{P}^{UO}}{\partial \bar{S}}\right|_{\bar{S}^*} = -1. \tag{4.46d}$$

The Perpetual American Barrier Option

As in the case of the standard American option, the American barrier option problem can be solved in closed form in the perpetual limit as $\epsilon \to 0$. In the non-dimensional setting, the perpetual up-and-out put price and boundary are the solutions to

$$\bar{S} \in \left(\bar{S}^*_\infty, \bar{B}\right) \qquad \bar{S}^2 \frac{\partial^2 \bar{P}^{UO}_\infty}{\partial \bar{S}^2} + \beta \bar{S} \frac{\partial \bar{P}^{UO}_\infty}{\partial \bar{S}} - \alpha \bar{P}^{UO}_\infty = 0, \tag{4.47a}$$

subject to

$$\text{at } \bar{S} = \bar{B} \qquad \bar{P}^{UO}_\infty\left(\bar{B}\right) = \bar{R}, \tag{4.47b}$$

$$\text{at } \bar{S} = \bar{S}^*_\infty \qquad \bar{P}^{UO}_\infty\left(\bar{S}^*_\infty\right) = (1 - S^*_\infty), \tag{4.47c}$$

$$\left.\frac{\partial \bar{P}^{UO}_\infty}{\partial \bar{S}}\right|_{\bar{S}^*_\infty} = -1. \tag{4.47d}$$

This second order ODE problem has the solution

$$\bar{P}^{UO}_\infty(\bar{S}) = A^+_\infty \left(\frac{\bar{S}}{\bar{S}^*_\infty}\right)^{\lambda^\infty_+} + A^-_\infty \left(\frac{\bar{S}}{\bar{S}^*_\infty}\right)^{\lambda^\infty_-}, \tag{4.48a}$$

where

$$\lambda_{\pm}^{\infty} = \frac{-(\beta-1) \pm \sqrt{(\beta-1)^2 + 4\alpha}}{2}, \tag{4.48b}$$

$$A_{\infty}^{+} = \frac{\bar{S}_{\infty}^{*} + \lambda_{-}^{\infty}\left(1 - \bar{S}_{\infty}^{*}\right)}{\lambda_{-}^{\infty} - \lambda_{+}^{\infty}}, \tag{4.48c}$$

$$A_{\infty}^{-} = \frac{\bar{S}_{\infty}^{*} + \lambda_{+}^{\infty}\left(1 - \bar{S}_{\infty}^{*}\right)}{\lambda_{+}^{\infty} - \lambda_{-}^{\infty}} \tag{4.48d}$$

and $\bar{S}_{\infty}^{*}$ is the solution to the transcendental expression

$$A_{\infty}^{+}\left(\frac{\bar{B}}{\bar{S}_{\infty}^{*}}\right)^{\lambda_{+}^{\infty}} + A_{\infty}^{-}\left(\frac{\bar{B}}{\bar{S}_{\infty}^{*}}\right)^{\lambda_{-}^{\infty}} = \bar{R}. \tag{4.48e}$$

No-arbitrage requires that $\bar{P}_{\infty}^{UO}(\bar{S})$ forms an upper bound on $\bar{P}_{a}^{UO}(\bar{S},\hat{T})$, while $\bar{S}_{\infty}^{*}$ forms a lower bound on $\bar{S}^{*}(\hat{T})$ and that the option price is strictly positive.

We note that in the large barrier limit $\bar{B} \to \infty$ the transcendental expression is dominated by the term $A_{\infty}^{+}\left(\frac{\bar{B}}{\bar{S}_{\infty}^{*}}\right)^{\lambda_{+}^{\infty}}$ which therefore requires $A_{\infty}^{+} \to 0$ as $\bar{B} \to \infty$, or from (4.48c)

$$\bar{S}_{\infty}^{*} \sim \frac{\lambda_{-}^{\infty}}{\lambda_{-}^{\infty} - 1}, \qquad A_{\infty}^{-} \sim \frac{1}{1 - \lambda_{-}^{\infty}}, \qquad \text{as } \bar{B} \to \infty \tag{4.49}$$

and the limiting perpetual barrier option price is

$$\bar{P}_{\infty}^{UO}(\bar{S}) \sim A_{\infty}^{-}\left(\frac{\bar{S}}{\bar{S}_{\infty}^{*}}\right)^{\lambda_{-}^{\infty}} \qquad \text{as } \bar{B} \to \infty, \tag{4.50}$$

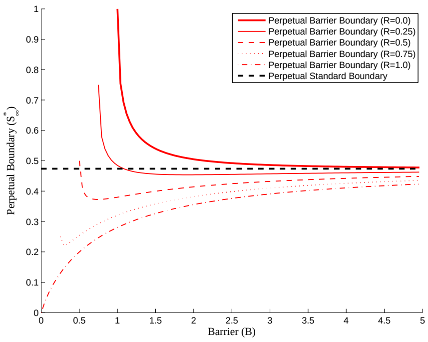
all of which is consistent with the standard perpetual American put option. The approach of the perpetual boundary of the American up-and-out put option to the corresponding boundary of the standard problem is shown in Figure 4-4.

Perturbative Terms

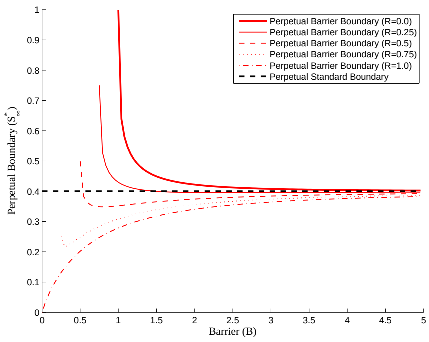
To capture the perturbative behaviour to the perpetual barrier problem we consider a WKBJ expansion of the form

$$\bar{P}^{UO}(\bar{S},\hat{T}) = \bar{P}_{\infty}^{UO}(\bar{S}) + \bar{P}_{0}^{UO}(\bar{S})e^{-\frac{\lambda\hat{T}}{\epsilon^2}} + o\left(e^{-\frac{\lambda\hat{T}}{\epsilon^2}}\right) \qquad \text{as } \epsilon \to 0, \tag{4.51a}$$

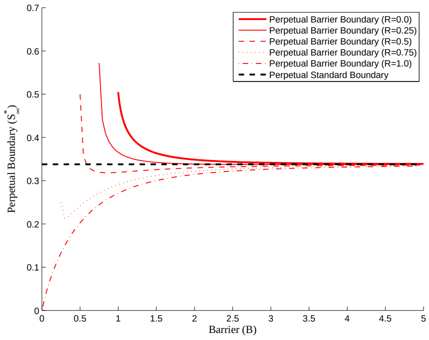
$$\bar{S}^{*}(\hat{T}) = \bar{S}_{\infty}^{*} + \bar{S}_{0}^{*}e^{-\frac{\lambda\hat{T}}{\epsilon^2}} + o\left(e^{-\frac{\lambda\hat{T}}{\epsilon^2}}\right) \qquad \text{as } \epsilon \to 0. \tag{4.51b}$$



(a) $\alpha = \frac{10}{9}, \beta = \frac{2}{3}$



(b) $\alpha = \frac{10}{9}, \beta = 0$



(c) $\alpha = \frac{10}{9}, \beta = -\frac{2}{3}$

Figure 4-4: The perpetual American up-and-out put boundary $\bar{S}^*_\infty$ obtained from (4.48e). Both standard ($\bar{B} \geq 1$) and reverse ($\bar{B} < 1$) barriers are considered together with a range of rebates ($\bar{R} = 0, 0.25, 0.5, 0.75, 1.0$). Convergence towards the standard American perpetual boundary is observed for $\bar{B} \gg 1$, with convergence strongest for $\beta < 0$. Roots of (4.48e) only exist for $\bar{B} > 1 - \bar{R}$ which is consistent with our small-time asymptotic findings.

Expanding the boundary conditions about the perpetual solution gives the problem for $\left(\bar{P}_0^{UO}, \bar{S}_0^*\right)$

$$\bar{S} \in \left(\bar{S}^*, \bar{B}\right), \hat{T} \in (0, \infty) \qquad \bar{S}^2 \frac{\partial^2 \bar{P}_0^{UO}}{\partial \bar{S}^2} + \beta \bar{S} \frac{\partial \bar{P}_0^{UO}}{\partial \bar{S}} - (\alpha - \lambda) \bar{P}_0^{UO} = 0, \tag{4.52a}$$

subject to

$$\text{at } \bar{S} = \bar{B} \qquad \bar{P}_0^{UO}\left(\bar{B}, \hat{T}\right) = 0, \tag{4.52b}$$

$$\text{at } \bar{S} = \bar{S}^*_\infty \qquad \bar{P}_0^{UO}\left(\bar{S}^*_\infty, \hat{T}\right) = 0, \tag{4.52c}$$

$$\left.\frac{\partial \bar{P}_0^{UO}}{\partial \bar{S}}\right|_{\bar{S}^*_\infty} = -\bar{S}_0^* \left.\frac{\partial^2 \bar{P}_\infty^{UO}}{\partial \bar{S}^2}\right|_{\bar{S}^*_\infty}. \tag{4.52d}$$

Conditions (4.52b) and (4.52c) are satisfied by the solution corresponding to complex roots of the characteristic equation. After application of the boundary conditions (4.52c) & (4.52d), this is

$$\bar{P}_0^{UO}(\bar{S}) = -\frac{\bar{S}_0^*\bar{S}_\infty^*}{k}\frac{\partial^2 \bar{P}_\infty^{UO}}{\partial \bar{S}^2}\bigg|_{\bar{S}_\infty^*}\left(\frac{\bar{S}}{\bar{S}_\infty^*}\right)^{-\frac{\beta-1}{2}}\sin\left(k\ln\left(\frac{\bar{S}}{\bar{S}_\infty^*}\right)\right), \tag{4.53}$$

where

$$k = \sqrt{(\lambda-\alpha) - \frac{(\beta-1)^2}{4}} \tag{4.54}$$

and λ is constrained by

$$\lambda > \frac{(\beta-1)^2}{4} + \alpha. \tag{4.55}$$

Application of the barrier condition (4.52b) specifies

$$k\ln\left(\bar{B}/\bar{S}_\infty^*\right) = n\pi, \tag{4.56}$$

but the no-arbitrage condition which requires $\bar{P}_\infty^{UO}$ to be an upper bound on $\bar{P}_a^{UO}$, restricts $\bar{P}_0^{UO}$ to be strictly negative which is only true if $n = 1$. This pins λ to be

$$\lambda = \frac{(\beta-1)^2}{4} + \alpha + \left(\frac{\pi}{\ln\left(\bar{B}/\bar{S}_\infty^*\right)}\right)^2 \tag{4.57}$$

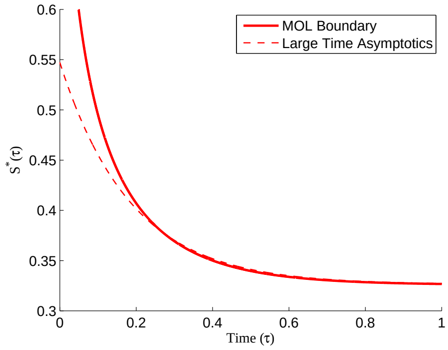
and we therefore have the large-time asymptotic behaviour of the American up-and-out put option

$$\bar{P}_a^{UO}(\bar{S},\hat{T}) \sim \bar{P}_\infty^{UO}(\bar{S}) - \frac{\bar{S}_0^*\bar{S}_\infty^*}{k}\frac{\partial^2 \bar{P}_\infty^{UO}}{\partial \bar{S}^2}\bigg|_{\bar{S}_\infty^*} e^{-\frac{\lambda\hat{T}}{\epsilon^2}}\left(\frac{\bar{S}}{\bar{S}_\infty^*}\right)^{-\frac{\beta-1}{2}}\sin\left(k\ln\left(\frac{\bar{S}}{\bar{S}_\infty^*}\right)\right) \quad \text{as } \epsilon \to 0, \tag{4.58}$$

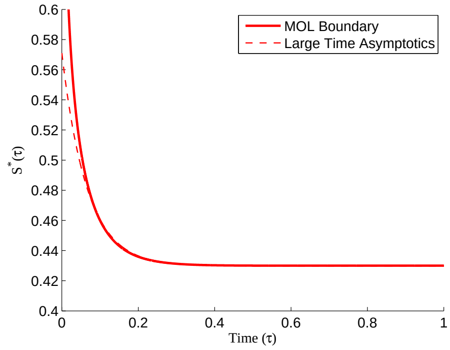
with the optimal exercise boundary $\bar{S}^*(\hat{T})$

$$\bar{S}^*(\hat{T}) \sim \bar{S}_\infty^* + \bar{S}_0^* e^{-\frac{\lambda\hat{T}}{\epsilon^2}} \qquad \text{as } \epsilon \to 0. \tag{4.59}$$

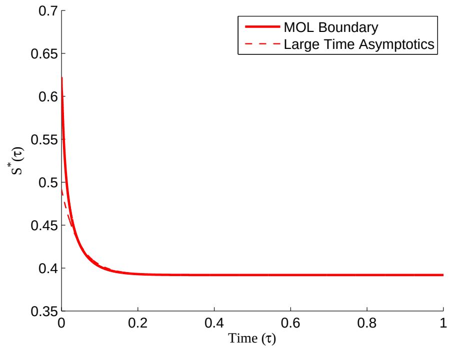
We note that the value of $\bar{S}_0^*$ is determined by matching back to the solution in $\tau = O(1)$ and that we expect $\bar{S}_0^* > 0$ in order to satisfy the no-arbitrage requirement that $\bar{S}_\infty^*$ forms a lower bound on $\hat{S}^*(\hat{T})$. Further, we mention that in the limiting case of an American barrier option with a barrier approaching infinity, we expect to recover the standard American option behaviour. We observe that, in this limit, the controlling factor λ approaches that value which gives rise to the coincident root solution which is consistent with our findings in Chapter 3.



(a) $\alpha = \frac{10}{9}, \beta = \frac{2}{3}, \bar{B} = 1.5, \bar{R} = 1.0$



(b) $\alpha = \frac{10}{9}, \beta = 0, \bar{B} = 1.0, \bar{R} = 0.25$



(c) $\alpha = \frac{10}{9}, \beta = -\frac{2}{3}, \bar{B} = 0.8, \bar{R} = 0.3$

Figure 4-5: Comparison of the large-time asymptotic behaviour of the optimal exercise boundary of the American up-and-out put option obtained from (4.51b) with the benchmark barrier MOL numerics. The constant $\bar{S}_0^*$ is determined by matching back into the optimal exercise boundary at $\tau = O(1)$.

4.4 An Analytic Approximation for the American Up-and-Out Put Option

Following the methodology used in Chapter 3, we look for a decomposition of the American up-and-out put option of the form

$$\bar{P}_a^{UO}(\bar{S}, h) = \bar{P}_e^{UO}(\bar{S}, h) + h\tilde{g}^{UO}(\tilde{X}, h), \tag{4.60}$$

where $\bar{P}_e^{UO}(\bar{S}, h)$ is the equivalent European up-and-out put option and $h\tilde{g}^{UO}(\tilde{X}, h)$ represents the American barrier option premium. This leads to the problem

$$\tilde{X} \in (0, \ln\left(\bar{B}/\bar{S}^*\right)), h \in (0,1) \qquad \begin{aligned} &\frac{\partial^2 \tilde{g}^{UO}}{\partial \tilde{X}^2} + (\beta - 1)\frac{\partial \tilde{g}^{UO}}{\partial \tilde{X}} - \frac{\alpha}{h}\tilde{g}^{UO} \\ &= p\alpha(1-h)\left[\frac{\partial \tilde{g}^{UO}}{\partial h} - \frac{1}{\bar{S}^*}\frac{d\bar{S}^*}{dh}\frac{\partial \tilde{g}^{UO}}{\partial \tilde{X}}\right], \end{aligned} \tag{4.61a}$$

subject to the boundary conditions

$$\text{at } \tilde{X} = \ln\left(\frac{\bar{B}}{\bar{S}^*}\right) \qquad h\tilde{g}^{UO}(\tilde{X}, h; p) = \bar{R} - \bar{P}_e^{UO}(\bar{B}, h) = 0, \tag{4.61b}$$

$$\text{at } h = 0 \qquad h\tilde{g}^{UO}(\tilde{X}, 0; p) = 0, \tag{4.61c}$$

$$\bar{S}^*(0; p) = \min\left(\alpha/(\alpha - \beta), 1, \bar{B}\right), \tag{4.61d}$$

$$\text{at } \tilde{X} = 0 \qquad h\tilde{g}^{UO}(\tilde{X}, h; p) = \left(1 - \bar{S}^*(h; p)\right) - \bar{P}_e^{UO}(\bar{S}^*(h; p), h), \tag{4.61e}$$

$$h\frac{\partial \tilde{g}^{UO}(\tilde{X}, h; p)}{\partial \tilde{X}}\bigg|_{\tilde{X}=0} = -\bar{S}^*(h; p)\left(1 + \frac{\partial \bar{P}_e^{UO}}{\partial \bar{S}}\bigg|_{\bar{S}^*(h;p)}\right). \tag{4.61f}$$

We propose the analytical expansions for $\tilde{g}^{UO}(\tilde{X}, h; p)$ and $\bar{S}^*(h; p)$

$$\tilde{g}^{UO}(\tilde{X}, h; p) = \sum_{n=0}^{\infty} \frac{p^n}{n!}\tilde{g}_n(\tilde{X}, h), \tag{4.62a}$$

$$\bar{S}^*(h; p) = \sum_{n=0}^{\infty} \frac{p^n}{n!}\bar{S}_n^*(h), \tag{4.62b}$$

which allows us to derive a sequence for the pairs of problems $(\tilde{g}_n^{UO}(\tilde{X}, h), \bar{S}_n^*(h))$ where

$$\tilde{g}_n^{UO}(\tilde{X}, h) = \frac{\partial^n \tilde{g}^{UO}}{\partial p^n}\bigg|_{p=0}, \tag{4.63a}$$

$$\bar{S}_n^*(h) = \frac{\partial^n \bar{S}^*}{\partial p^n}\bigg|_{p=0}. \tag{4.63b}$$

We also introduce a function $\hat{F}^{UO}(\tilde{X}, h; p)$ given by

$$\hat{F}^{UO}(\tilde{X}, h; p) = \frac{\partial \tilde{g}^{UO}}{\partial h} - \frac{1}{\bar{S}^*}\frac{d\bar{S}^*}{dh}\frac{\partial \tilde{g}^{UO}}{\partial \tilde{X}}, \tag{4.64}$$

so that $p\alpha(1-h)\hat{F}(\tilde{X}, h; p)$ is the forcing term for the base problem.

Note that unlike the vanilla American option problem, the use of the Landau transformation does not result in a fixed domain problem due to the finite domain imposed by the barrier. Although we could choose to normalise the stock price by the barrier it would make taking the limiting case of the infinite barrier, which allows us to recover the behaviour of the vanilla American option, more onerous.

The $\left(\tilde{g}_0^{UO}, \bar{S}_0^*\right)$ Problem

Setting $p = 0$ in (4.61a-4.61f) gives the problem

$$\tilde{X} \in (0, \ln\left(\bar{B}/\bar{S}_0^*\right)), h \in (0,1) \qquad \frac{\partial^2 \tilde{g}_0^{UO}}{\partial \tilde{X}^2} + (\beta - 1)\frac{\partial \tilde{g}_0^{UO}}{\partial \tilde{X}} - \frac{\alpha}{h}\tilde{g}_0^{UO} = 0, \tag{4.65a}$$

subject to the boundary conditions

$$\text{at } \tilde{X} = \ln\left(\bar{B}/\bar{S}_0^*\right) \qquad h\tilde{g}_0^{UO}(\tilde{X}, h) = \bar{R} - \bar{P}_e^{UO}(\bar{R}, h), \tag{4.65b}$$

$$\text{at } h = 0 \qquad h\tilde{g}_0^{UO}(\tilde{X}, 0) = 0, \tag{4.65c}$$

$$\bar{S}_0^*(0) = \min\left(\alpha/(\alpha - \beta), 1, \bar{B}\right), \tag{4.65d}$$

$$\text{at } \tilde{X} = 0 \qquad h\tilde{g}_0^{UO}(\tilde{X}, h) = \left(1 - \bar{S}_0^*(h)\right) - \bar{P}_e^{UO}(\bar{S}_0^*, h), \tag{4.65e}$$

$$h\left.\frac{\partial \tilde{g}_0^{UO}(\tilde{X}, h)}{\partial \tilde{X}}\right|_{\tilde{X}=0} = -\bar{S}_0^*\left(1 + \left.\frac{\partial \bar{P}_e^{UO}}{\partial \bar{S}}\right|_{\bar{S}_0^*}\right). \tag{4.65f}$$

This problem has been posed by Aitsahlia & Lai [3] as an extension to the MBAW price for the standard American option problem, with solution

$$\tilde{g}_0^{UO}(\tilde{X}, h) = A_{00}^+(h)e^{\lambda_+ \tilde{X}} + A_{00}^-(h)e^{\lambda_- \tilde{X}}, \tag{4.66a}$$

where

$$\lambda_\pm(h) = \frac{-(\beta - 1) \pm \sqrt{(\beta - 1)^2 + \frac{4\alpha}{h}}}{2}, \tag{4.66b}$$

$$A_{00}^+(h) = \frac{\lambda_-\left(1 - \bar{S}_0^* - \bar{P}_e^{UO}(\bar{S}_0^*, h)\right) + \bar{S}_0^*\left(1 + \frac{\partial \bar{P}_e^{UO}}{\partial \bar{S}}\Big|_{\bar{S}_0^*}\right)}{h\left(\lambda_- - \lambda_+\right)}, \tag{4.66c}$$

$$A_{00}^-(h) = \frac{\lambda_+\left(1 - \bar{S}_0^* - \bar{P}_e^{UO}(\bar{S}_0^*, h)\right) + \bar{S}_0^*\left(1 + \frac{\partial \bar{P}_e^{UO}}{\partial \bar{S}}\Big|_{\bar{S}_0^*}\right)}{h\left(\lambda_+ - \lambda_-\right)}, \tag{4.66d}$$

where $\bar{P}_e^{UO}(\bar{S}_0^*, h)$ and $\left.\frac{\partial \bar{P}_e^{UO}}{\partial \bar{S}}\right|_{\bar{S}_0^*}$ are the non-dimensional European up-and-out put option price and its delta, evaluated at the leading order boundary, $\bar{S}_0^*$, which is given by the solution to the transcendental expression

$$h\left(A_{00}^{+}(h)\left(\frac{\bar{B}}{\bar{S}_0^*}\right)^{\lambda_+}+A_{00}^{-}(h)\left(\frac{\bar{B}}{\bar{S}_0^*}\right)^{\lambda_-}\right)=\bar{R}-\bar{P}_e^{UO}(\bar{B},h). \tag{4.67}$$

We observe that in the limit $\bar{B}\to\infty$, with $\bar{R}\to 0$, the dominant term is the $A_{00}^{+}(h)\left(\frac{\bar{B}}{\bar{S}_0^*}\right)^{\lambda_+}$ term on the LHS. Thus in the limit we require $A_{00}^{+}(h)=0$ which, from (4.66c), leads to

$$\lambda_-\left(1-\bar{S}_0^*-\bar{P}_e^{UO}(\bar{S}_0^*,h)\right)=-\bar{S}_0^*\left(1+\frac{\partial\bar{P}_e^{UO}}{\partial\bar{S}}|_{\bar{S}_0^*}\right), \tag{4.68}$$

which we note is the equation for the MBAW approximation to the standard American option boundary as expected.

The $(\tilde{g}_1^{UO},\bar{S}_1^*)$ Problem

The problem for the first correction term is

$$\tilde{X}\in(0,\ln\left(\bar{B}/\bar{S}_0^*\right)),h\in(0,1)\qquad \frac{\partial^2\tilde{g}_1^{UO}}{\partial\tilde{X}^2}+(\beta-1)\frac{\partial\tilde{g}_1^{UO}}{\partial\tilde{X}}-\frac{\alpha}{h}\tilde{g}_1^{UO}=\alpha(1-h)\hat{F}_0^{UO}(\tilde{X},h), \tag{4.69a}$$

subject to the boundary conditions

$$\text{at } \tilde{X}=\ln\left(\bar{B}/\bar{S}_0^*\right)\qquad \tilde{g}_1^{UO}(\tilde{X},h)=\frac{\bar{S}_1^*}{\bar{S}_0^*}\frac{\partial\tilde{g}_0^{UO}}{\partial\tilde{X}}, \tag{4.69b}$$

$$\text{at } \tilde{X}=0\qquad h\tilde{g}_1^{UO}(0,h)=-\bar{S}_1^*-\left.\frac{\partial\bar{P}_e^{UO*}}{\partial p}\right|_{p=0}, \tag{4.69c}$$

$$h\left.\frac{\partial\tilde{g}_1^{UO}}{\partial\tilde{X}}\right|_{\tilde{X}=0}=-\bar{S}_1^*-\frac{\partial}{\partial p}\left(\bar{S}^*\left.\frac{\partial\bar{P}_e^{UO}}{\partial\bar{S}}\right|_{\bar{S}^*}\right)\bigg|_{p=0}. \tag{4.69d}$$

The function $\hat{F}_0^{UO}(\tilde{X},h)$ in the forcing term of (4.69a) is given by

$$\begin{aligned}\hat{F}_0(\tilde{X},h)=\frac{\partial\tilde{g}_0^{UO}}{\partial\hat{h}}-\frac{1}{\bar{S}_0^*}\frac{\partial\bar{S}_0^*}{\partial h}\frac{\partial\tilde{g}_0^{UO}}{\partial\tilde{X}}=&e^{\lambda_+\tilde{X}}\left(B_{01}^{+}\tilde{X}+B_{00}^{+}(h)\right)\\&+e^{\lambda_-\tilde{X}}\left(B_{01}^{-}(h)\tilde{X}+B_{00}^{-}(h)\right)\end{aligned} \tag{4.70}$$

and using (4.66a) and (4.70) gives the forms for the coefficients of the forcing term

$$B_{01}^{+} = \alpha(1-h)\frac{\partial \lambda_{+}}{\partial h}A_{0}^{+}, \tag{4.71a}$$

$$B_{00}^{+} = \alpha(1-h)\left[\frac{\partial A_{0}^{+}}{\partial h} - \frac{1}{\bar{S}_{0}^{*}}\frac{\partial \bar{S}_{0}^{*}}{\partial h}\lambda_{+}A_{0}^{+}\right], \tag{4.71b}$$

$$B_{01}^{-} = \alpha(1-h)\frac{\partial \lambda_{-}}{\partial h}A_{0}^{-}, \tag{4.71c}$$

$$B_{00}^{-} = \alpha(1-h)\left[\frac{\partial A_{0}^{-}}{\partial h} - \frac{1}{\bar{S}_{0}^{*}}\frac{\partial \bar{S}_{0}^{*}}{\partial h}\lambda_{-}A_{0}^{-}\right]. \tag{4.71d}$$

The solution to (4.69a-4.69d) is given by

$$\tilde{g}_{1}^{UO}(\tilde{X},h) = e^{\lambda_{+}\tilde{X}}\sum_{j=0}^{2}A_{1j}^{+}(h)\tilde{X}^{j} + e^{\lambda_{-}\tilde{X}}\sum_{j=0}^{2}A_{1j}^{-}(h)\tilde{X}^{j}, \tag{4.72}$$

where the coefficients $A_{1j}^{+}(h)$,$A_{1j}^{-}(h)$ for $j \neq 0$ can be found by substitution into (4.69a), giving

$$A_{12}^{+} = \frac{B_{01}^{+}}{2\left(2\lambda_{+} + (\beta-1)\right)}, \tag{4.73}$$

$$A_{11}^{+} = \frac{1}{\left(2\lambda_{+} + (\beta-1)\right)}\left[B_{00}^{+} - 2A_{12}^{+}\right], \tag{4.74}$$

$$A_{12}^{-} = \frac{B_{01}^{-}}{2\left(2\lambda_{-} + (\beta-1)\right)}, \tag{4.75}$$

$$A_{11}^{-} = \frac{1}{\left(2\lambda_{-} + (\beta-1)\right)}\left[B_{00}^{-} - 2A_{12}^{-}\right]. \tag{4.76}$$

The coefficients A_{10}^{-},A_{10}^{+} are found through application of the boundary conditions (4.69c) and (4.69d)

$$A_{10}^{+} = \bar{S}_{1}^{*}\frac{(\lambda_{+}-1)(1+\bar{\Delta}^{UO*}) - \bar{S}_{0}^{*}\bar{\Gamma}^{UO*}}{h(\lambda_{-}-\lambda_{+})} - \frac{A_{11}^{+}+A_{11}^{-}}{\lambda_{-}-\lambda_{+}}, \tag{4.77}$$

$$A_{10}^{-} = \bar{S}_{1}^{*}\frac{(\lambda_{-}-1)(1+\bar{\Delta}^{UO*}) - \bar{S}_{0}^{*}\bar{\Gamma}^{UO*}}{h(\lambda_{+}-\lambda_{-})} - \frac{A_{11}^{+}+A_{11}^{-}}{\lambda_{+}-\lambda_{-}} \tag{4.78}$$

and the boundary correction term $\bar{S}_{1}^{*}$ is found using (4.69b)

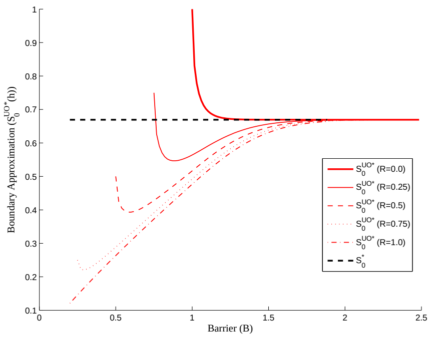
$$\bar{S}_{1}^{*} = \bar{S}_{0}^{*}\frac{\left(\frac{\bar{B}}{\bar{S}_{0}^{*}}\right)^{\lambda_{+}}\left(A_{12}^{+}\left(\ln\left(\frac{\bar{B}}{\bar{S}_{0}^{*}}\right)^{2}\right) + A_{11}^{+}\ln\left(\frac{\bar{B}}{\bar{S}_{0}^{*}}\right)\right) + \left(\frac{\bar{B}}{\bar{S}_{0}^{*}}\right)^{\lambda_{-}}\left(A_{12}^{-}\left(\ln\left(\frac{\bar{B}}{\bar{S}_{0}^{*}}\right)^{2}\right) + A_{11}^{-}\ln\left(\frac{\bar{B}}{\bar{S}_{0}^{*}}\right)\right)}{(\lambda_{-}-1)(1+\bar{\Delta}^{UO*}) - \bar{S}_{0}^{*}\bar{\Gamma}^{UO*}}, \tag{4.79}$$

where the European Greeks $\bar{\Delta}^{UO*}$ and $\bar{\Gamma}^{UO*}$ are derived in Appendix A. A summary of the formulae required to determine the analytic approximation is provided in Table 4.3.

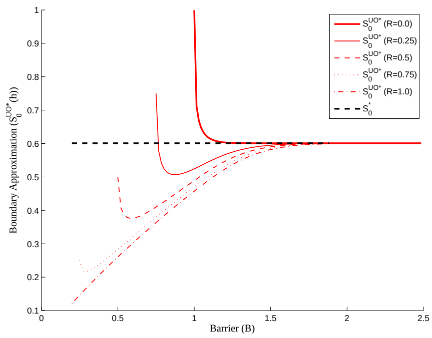
Numerical Results

The behaviour of the leading order term of the analytic approximation $\bar{S}_0^{UO*}$ due to AitSahlia and Lai [3] is shown in Figure 4-6 for a range of rebates and barriers. We observe that $\bar{S}_0^{UO*} \to \bar{S}_0^*$ as $\bar{B} \to \infty$ as expected.

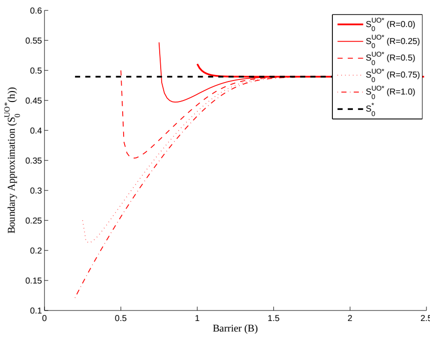
A comparison of the two-term analytic approximation for the optimal exercise boundary of the American up-and-out put option with the barrier MOL numerics is shown in Figure 4-7. We observe from the relative error Figures 4.7(b), 4.7(d) & 4.7(f) that the two-term boundary approximations provide a significant improvement over the AitSahlia & Lai boundary. An improvement in the large- and small-time asymptotic behaviour can be observed in Figure 4-8. Moreover, unlike our standard American option approximation, the performance of our two-term price approximation for the up-and-out barrier improves greatly on the leading order term, as can be seen in Figure 4-9. It appears that the presence of barriers which are near-the-money minimises the impact of higher order terms in $\ln(\bar{S}/\bar{S}_0^*)$ and therefore we may expect the barrier option approximation to perform less well in cases where the barrier is deeply out-of-the-money. In summary, our two-term series provides an accurate approximation to the location of the American option boundary. The procedure benefits from the same ease to determine as the standard American option approximation, however the increased complexity of the calculation of European up-and-out put option together with the rebate and the corresponding Greeks leads to an additional computational overhead with a curve for $h \in [0, 1]$ using 1000 time-steps taking 48 seconds in Matlab using the built-in normal distribution approximation.



(a) $\alpha = \frac{10}{9}, \beta = \frac{2}{3}$

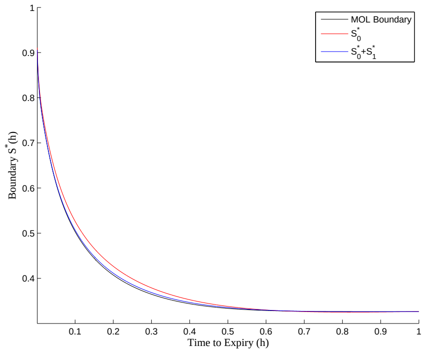


(b) $\alpha = \frac{10}{9}, \beta = 0$

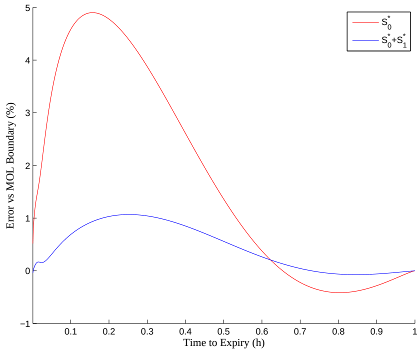


(c) $\alpha = \frac{10}{9}, \beta = -\frac{2}{3}$

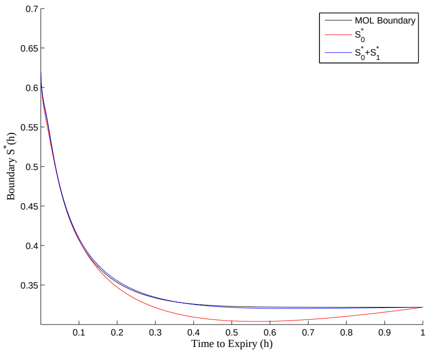
Figure 4-6: The leading order term in the American up-and-out put boundary approximation due to AitSahlia & Lai [3] (4.67) for $h = 0.05$. Both standard ($\bar{B} \geq 1$) and reverse ($\bar{B} < 1$) barriers are considered together with a range of rebates ($\bar{R} = 0, 0.25, 0.5, 0.75, 1.0, 2.0$). Convergence towards the MBAW approximation $\bar{S}_0^*$ is observed for $\bar{B} \gg 1$. Consistent with work in the previous sections, the location of $\bar{S}_0^{UO*}$ is only defined when $\bar{B} > 1 - \bar{R}$.



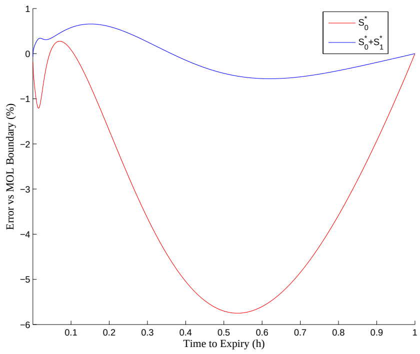
(a) Boundary Approximation vs MOL ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$, $\bar{B} = 1.5$, $\bar{R} = 1.0$)



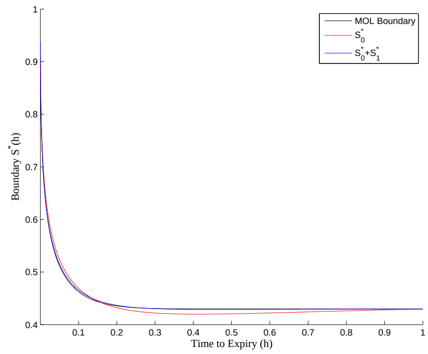
(b) Boundary Error vs MOL ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$, $\bar{B} = 1.5$, $\bar{R} = 1.0$)



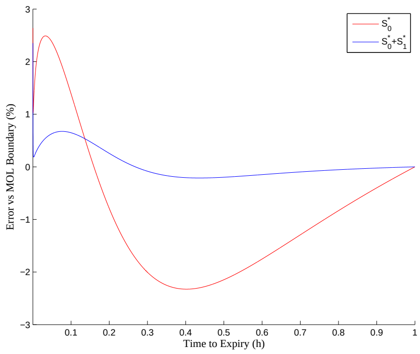
(c) Boundary Approximation vs MOL ($\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$, $\bar{B} = 1.2$, $\bar{R} = 0.5$)



(d) Boundary Error vs MOL ($\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$, $\bar{B} = 1.2$, $\bar{R} = 0.5$)

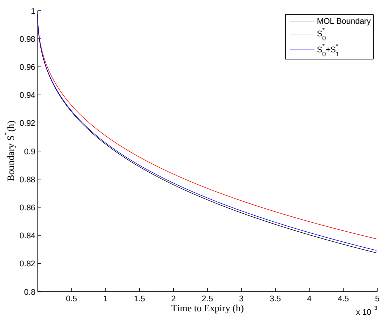


(e) Boundary Approximation vs MOL ($\alpha = \frac{10}{9}$, $\beta = 0$, $\bar{B} = 1.0$, $\bar{R} = 0.25$)

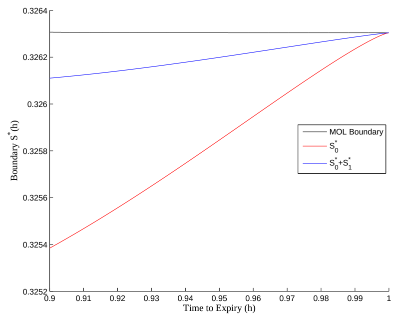


(f) Boundary Error vs MOL ($\alpha = \frac{10}{9}$, $\beta = 0$, $\bar{B} = 1.0$, $\bar{R} = 0.25$)

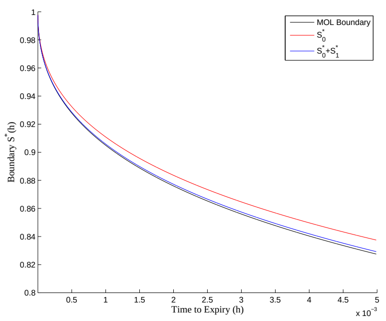
Figure 4-7: A comparison of the benchmark barrier MOL boundary with the two-term analytic approximation. For the MOL Boundary, 200000 time-steps and 50000 spatial points. The improvement over the leading order ($\bar{S}_0^* + \bar{S}_1^*$) approximation due to AitSahlia & Lai [3] ($\bar{S}_0^*$) is most clearly shown in the error Figures 4.7(b), 4.7(d) & 4.7(f) where the error is defined as the difference between the relevant boundary approximation and the barrier MOL boundary, divided by the barrier MOL boundary and expressed as a percentage.



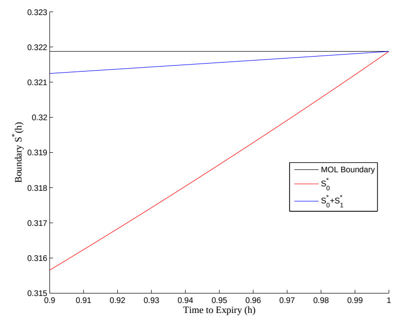
(a) Approach to Expiry ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$, $\bar{B} = 1.5$, $\bar{R} = 1.0$)



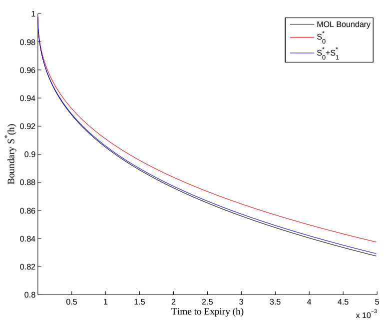
(b) Approach to Perpetuity ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$, $\bar{B} = 1.5$, $\bar{R} = 1.0$)



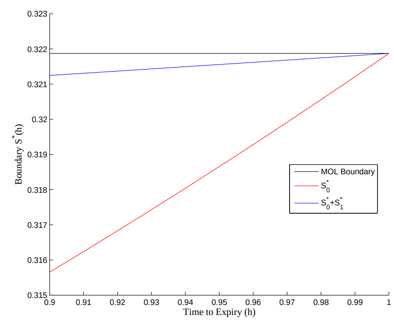
(c) Approach to Expiry ($\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$, $\bar{B} = 1.2$, $\bar{R} = 0.5$)



(d) Approach to Perpetuity ($\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$, $\bar{B} = 1.2$, $\bar{R} = 0.5$)

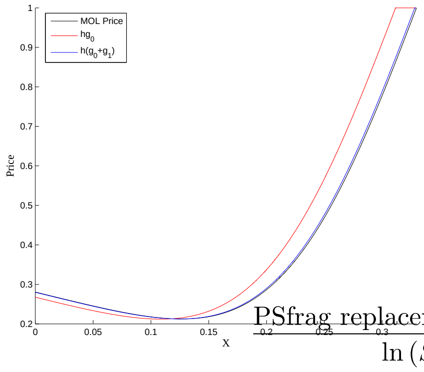


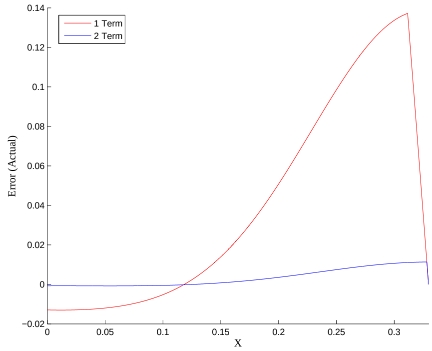
(e) Approach to Expiry ($\alpha = \frac{10}{9}$, $\beta = 0$, $\bar{B} = 1.0$, $\bar{R} = 0.25$)



(f) Approach to Perpetuity ($\alpha = \frac{10}{9}$, $\beta = 0$, $\bar{B} = 1.0$, $\bar{R} = 0.25$)

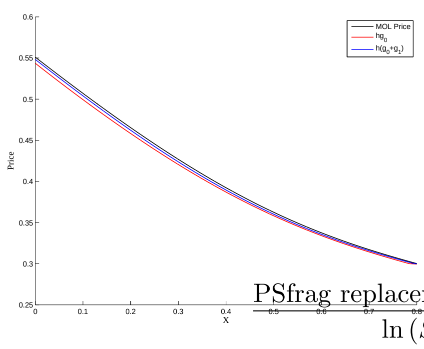
Figure 4-8: A comparison of asymptotic behaviour of the benchmark barrier MOL boundary with the two-term analytic approximation. For the approach to expiry, 20000 time-steps and 25000 spatial points were used. For the approach to perpetuity, 200000 time-steps and 50000 spatial points were used.

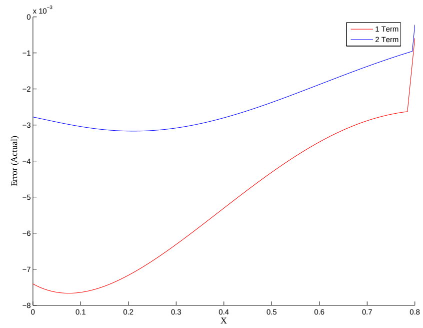




(a) Price Approximations vs MOL numerics ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$, $\bar{B} = 1.5$, $\bar{R} = 1.0$, $h = 0.05$)

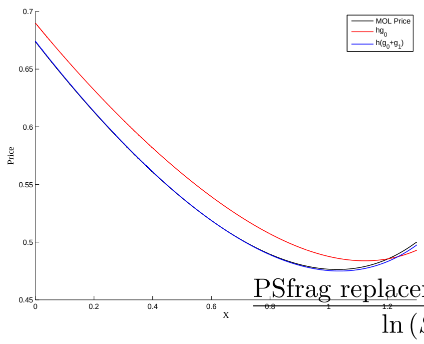
(b) Error vs MOL numerics ($\alpha = \frac{10}{9}$, $\beta = \frac{2}{3}$, $\bar{B} = 1.5$, $\bar{R} = 1.0$, $h = 0.05$)

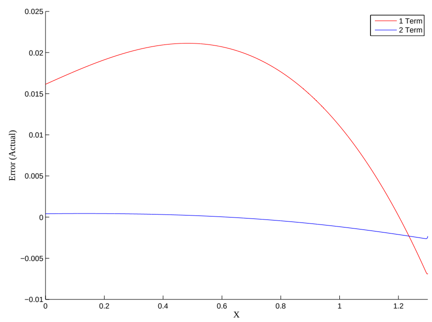




(c) Price Approximations vs MOL numerics ($\alpha = \frac{10}{9}$, $\beta = 0$, $\bar{B} = 1.0$, $\bar{R} = 0.3$, $h = 0.1$)

(d) Error vs MOL numerics ($\alpha = \frac{10}{9}$, $\beta = 0$, $\bar{B} = 1.0$, $\bar{R} = 0.3$, $h = 0.1$)





(e) Price Approximations vs MOL numerics ($\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$, $\bar{B} = 1.2$, $\bar{R} = 0.5$, $h = 0.5$)

(f) Error vs MOL numerics ($\alpha = \frac{10}{9}$, $\beta = -\frac{2}{3}$, $\bar{B} = 1.2$, $\bar{R} = 0.5$, $h = 0.5$)

Figure 4-9: A comparison of the leading order approximation due to Aitsahlia & Lai [3] the two-term analytic approximations for a selection of parameters and times to expiry. The error is defined as the difference between the relevant approximation and the MOL benchmark price and expressed as both the actual difference (4.9(a), 4.9(c) & 4.9(e)) or as a percentage of the MOL benchmark price (4.9(b), 4.9(d) & 4.9(f)). The benchmark was determined using 20000 time-steps and 25000 spatial points, and transformed onto the fixed domain $\tilde{X} = \ln\left(\bar{S}/\bar{S}^*(h)\right)$ using the corresponding MOL boundary.

Term	Formula	Greeks Required
$(\bar{g}_0, \bar{S}_0^*)$ **Problem**		
$\bar{g}_0(\tilde{X}, h)$	$e^{\lambda_- \tilde{X}} A_{00}^-(h) + e^{\lambda_+ \tilde{X}} A_{00}^+(h)$	$\bar{P}_e^{UO}$ (4.2)
$\lambda_\pm$	$\frac{-(\beta-1)\pm\sqrt{(\beta-1)^2+\frac{4\alpha}{h}}}{2}$	$\bar{\Delta}^{UO}$ (A.22,A.26)
$A_{00}^\pm(h)$	$\frac{\lambda_\mp\left(1-\bar{S}_0^*-\bar{P}_e^{UO*}\right)+\bar{S}_0^*\left(1+\bar{\Delta}^{UO*}\right)}{h(\lambda_\mp-\lambda_\pm)}$	
$S_0^*(h)$	$h\left(A_{00}^+(h)\left(\frac{\bar{B}}{\bar{S}_0^*}\right)^{\lambda_+} + A_{00}^-(h)\left(\frac{\bar{B}}{\bar{S}_0^*}\right)^{\lambda_-}\right) = \bar{R} - \bar{P}_e(\bar{B}, h)$	
$(\bar{g}_1, \bar{S}_1^*)$ **Problem**		
$\bar{g}_1(\tilde{X}, h)$	$e^{\lambda_- \tilde{X}}\left(A_{12}^-(h)\tilde{X}^2 + A_{11}^-(h)\tilde{X} + A_{10}^-(h)\right) + e^{\lambda_+ \tilde{X}}\left(A_{12}^+(h)\tilde{X}^2 + A_{11}^+(h)\tilde{X} + A_{10}^+(h)\right)$	$\bar{\Gamma}^{UO}$ (A.23,A.27)
$\frac{\partial\lambda_\pm}{\partial h}$	$\mp\frac{\alpha}{h^2(2\lambda_\pm+(\beta-1))}$	$\bar{\Theta}^{UO}$ (A.24,A.28)
$\frac{\partial\bar{S}_0^*}{\partial h}$	$\frac{h\frac{\partial\lambda_-}{\partial h}A_{00}+\bar{S}_0^*\bar{\Delta}_\Theta^*-\lambda_-\bar{\Theta}^*}{(\lambda_--1)(1+\Delta^*)-\bar{S}_0^*\bar{\Gamma}^*}$	$\bar{\Delta}_\Theta^{UO}$ (A.25,A.29)
$\frac{\partial\bar{A}_{00}}{\partial h}$	$-A_{00}^\pm\left(\frac{1}{\lambda_\pm-\lambda_\mp}\left(\frac{\partial\lambda_\pm}{\partial h}-\frac{\partial\lambda_\mp}{\partial h}\right)+\frac{1}{h}\right) - \frac{\frac{\partial\lambda_\mp}{\partial h}(1-\bar{S}_0^*-\bar{P}_e^{UO*})-\lambda_\mp\bar{\Theta}_e^{UO*}-(\lambda_\mp-1)(1+\bar{\Delta}_e^{UO*})\frac{\partial\bar{S}_0^*}{\partial h}+\bar{S}_0^*\left(\bar{\Delta}_\Theta^{UO*}+\frac{\partial\bar{S}_0^*}{\partial h}\bar{\Gamma}^{UO*}\right)}{h(\lambda_\pm-\lambda_\mp)}$	
$B_{00}^\pm(h)$	$\alpha(1-h)\left[\frac{\partial A_{00}^\pm}{\partial h} - \lambda_\pm\frac{A_{00}^\pm}{\bar{S}_0^*}\frac{\partial\bar{S}_0^*}{\partial h}\right]$	
$B_{01}^\pm(h)$	$\alpha(1-h)A_{00}^\pm\frac{\partial\lambda_\pm}{\partial h}$	
$A_{12}^\pm(h)$	$\frac{B_{01}^\pm}{2(2\lambda_\pm+(\beta-1))}$	
$A_{11}^\pm(h)$	$\frac{1}{(2\lambda_\pm+(\beta-1))}\left[B_{00}^\pm - 2A_{12}^\pm\right]$	
$A_{10}^\pm(h)$	$\bar{S}_1^*\frac{(\lambda_\mp-1)(1+\bar{\Delta}^{UO*})-\bar{S}_0^*\bar{\Gamma}^{UO*}}{h(\lambda_\mp-\lambda_\pm)} + \frac{A_{11}^\pm+A_{11}^\mp}{\lambda_\pm-\lambda_\mp}$	
$\bar{S}_1^*(h)$	$\bar{S}_0^*\frac{\left(\frac{\bar{B}}{\bar{S}_0^*}\right)^{\lambda_+}\left(A_{12}^+\left(\ln\left(\frac{\bar{B}}{\bar{S}_0^*}\right)^2\right)+A_{11}^+\ln\left(\frac{\bar{B}}{\bar{S}_0^*}\right)\right)+\left(\frac{\bar{B}}{\bar{S}_0^*}\right)^{\lambda_-}\left(A_{12}^-\left(\ln\left(\frac{\bar{B}}{\bar{S}_0^*}\right)^2\right)+A_{11}^-\ln\left(\frac{\bar{B}}{\bar{S}_0^*}\right)\right)}{(\lambda_--1)(1+\bar{\Delta}^{UO*})-\bar{S}_0^*\bar{\Gamma}^{UO*}}$	

Table 4.3: Formulae used in the derivation of the analytic approximation for the American up-and-out put option. The required Greeks are derived in Appendix A. The star superscript denotes that the Greeks are evaluated at the leading order boundary approximation $\bar{S}_0^*$.

Chapter 5

Discussion

The main aims of this thesis were twofold. The first was to add to the body of work looking at the asymptotic behaviour of option pricing problems under the Black-Scholes-Merton assumptions. The second was to develop an analytic approximation to the American option problem based around the popular work of MacMillan [79] and Barone-Adesi & Whaley [10], in order to bridge the gap between the small- and large-time asymptotic limits. This work is of relevance in cases where there is no known closed-form solution to a particular option pricing problem, such as the American option problem. In this case, analytic expressions which are asymptotically correct in certain limits, or provide accurate and easy to determine approximations, are of use as a basis for valuation or to validate more computationally intensive numerical approaches. The extent to which these aims have been achieved and where further work may be focussed, are discussed in the following sections.

5.1 Conclusions and Further Work

5.1.1 Asymptotic Analysis

In our asymptotic analysis of the small-time European put option problem we have derived exponentially small leading order behaviour in the out-of-the-money outer region. Despite these terms having been identified by Addison et al. [2] in relation to the Stefan problem, authors have attributed the behaviour in this region to the inner series; however, this only captures the behaviour seen in the closed form up to algebraic terms. This result is also applicable in the corresponding region of the American put option problem. The solution is derived using a WKBJ-type expansion with the resulting leading order problem requiring knowledge of singular solutions to differential equations. Despite displaying the correct asymptotic behaviour as $\bar{S} \to \infty$, the outer expression is singular at $\bar{S} = 1$ and we have be unable to perform matching with the inner expansion using an intermediate variable approach and a finite number of terms. This prevents the formation of a uniform expression and this remains an open issue to be resolved.

To complete the temporal asymptotic analysis of the European put option problem, we derive the large-time asymptotic behaviour not previously discussed in the literature. This results in a similarity solution which is accurate surprisingly far from perpetuity, while a far-field region is also identified where we observe the remnant effect of smoothing the payoff function near expiry.

The standard American put option and American up-and-out put option problems are posed as an extension to the European option framework and the small and large-time asymptotic behaviour of the price and optimal exercise boundary are derived. Notwithstanding our identification of the additional out-of-the-money outer region behaviour, the small-time work of the standard American option is shown to be consistent with that of previous authors. Knowledge of the correct asymptotic form as $\bar{S} \to \infty$ may be of interest in the implementation of grid-based numerical routines, where an assumption has to be made over how to specify the problem at the edge of the truncated domain. The extension to include the small-time behaviour of the American barrier option is new and of particular interest in the case of reverse barriers, where the presence of the barrier changes the small-time asymptotic behaviour of the optimal exercise boundary. Though transition region behaviour will exist as the normalised barrier approaches unity for $\beta \geq 0$ and as the barrier approaches the initial position of the boundary for $\beta < 0$, this has not been derived in this thesis. The large-time perturbative behaviour to the well-known perpetual solutions of both of these American-style problems are also derived. Knowledge of this behaviour may be of value as numerical routines require a large amount of time to accurately determine the large-time behaviour, with errors resulting from earlier times propagating into the large-time problem.

5.1.2 Analytic Approximations

The application of the homotopic series approach adopted by Zhu [106] has allowed us to extend the popular analytic approximation of MacMillan [79] and Barone-Adesi & Whaley [10] to derive a three-term analytic approximation to the price and optimal exercise boundary of the American put option problem. The terms in the series can be determined in a single spreadsheet if desired, requiring only access to an accurate method for the calculation of the cumulative normal distribution. The results for a single point in time are almost instantaneous and the resulting boundary is better than 1% accurate across the whole time domain for our parameters when compared to our benchmark MOL scheme, representing a significant improvement on the equivalent MBAW boundary with negligible additional overhead. The price approximation improves upon the corresponding MBAW approximation near the boundary, but breaks down as the underlying moves out-of-the-money due to the introduction of increasingly large powers of $\ln(\bar{S}/\bar{S}^*)$ by higher order terms in the series. This effect is of reduced importance if one considers that the expression is dominated by an exponentially small multiplicative term as $\bar{S}$ becomes large, and that the solution tends to zero in the limit $\bar{S} \to \infty$. Nevertheless, we expect the boundary approximation is more likely to be of practical use in driving a numerical scheme, such as a quadrature scheme based around the integral equation due to Kim [72], however the accuracy of such an approach has not been investigated here.

The approach adopted lends itself to extension to other American-style option problems. The approach uses the following general steps:

1. Decomposition of the American-style option into an equivalent European option plus an expression capturing the premium attributable to the right to exercise early;

2. Use of MacMillan's time transformation and identification of a term in the governing equation which can be assumed to be negligible to give a closed-form leading order approximation;

3. Formation of a homotopic series in a parameter p, with the leading order approximation corresponding to $p = 0$ and the full solution corresponding to $p = 1$. The problems for successive terms in the series are derived via repeated differentiation of the problem with respect to p; and

4. The solution of the resulting problems, which depend upon previously derived terms along with the Greeks of the equivalent European-style option.

The ease to which the final step may be implemented for successive problems depends upon the complexity of the leading order term and its derivatives, along with the existence of a closed-form solution for the European-style option and the capability to determine the corresponding Greeks. This limitation is demonstrated by application of the approach to the American up-and-out put option problem, for which a fully analytical approximation beyond two terms is onerous due to the increased complexity of the leading order approximation, originally derived by AitSahlia & Lai [3]. However, the significant improvement in accuracy of the two-term series versus the leading order term highlights the potential gain to be made.

Further potential complexity issues are highlighted through the formulation of the American option problem under the CEV model for the underlying, where the determination of the leading order term requires the availability of a routine to determine the confluent hypergeometric function, while determination of additional terms will almost surely require a numerical scheme. We furnish some of the details of this possible direction of work in subsection 5.2.2.

As a further limitation on our analysis, we have not found a suitable way of iterating between successive terms in the series, either analytically or numerically, and therefore we have been unable to investigate the convergence properties of this series, even in the standard American option case. Our work towards this end is included in subsection 5.2.1.

5.2 Further Work

Here we add further detail to two of the possible directions in which we would propose to take the work contained in this thesis, but which remain unfinished as of the time of writing.

5.2.1 Towards a General Analytic Expression

Finding a general analytic solution to the problem for the general $n+1^{th}$ term in the series (3.122a-3.122d) requires the derivation of a general expression for the forcing term, the solution of the resulting governing equation and the specification and application of the boundary conditions.

The forcing term for the $n+1^{th}$ problem, is found via differentiation with respect to p of the term (3.121) n times, with evaluation at $p=0$. This yields the expression for the forcing term of the problem $(\tilde{g}_{n+1}, \bar{S}^*_{n+1})$

$$(n+1)\alpha(1-h)\tilde{F}_n(\tilde{X},h) = (n+1)\alpha(1-h)\left[\frac{\partial \tilde{g}_n}{\partial h} + \sum_{m=0}^{n}\binom{n}{m}\frac{\partial^{n-m}}{\partial p^{n-m}}\left(\frac{1}{\bar{S}^*}\frac{\partial \bar{S}^*}{\partial h}\right)\bigg|_{p=0}\frac{\partial \tilde{g}_m}{\partial \tilde{X}}\right]. \tag{5.1}$$

In order to determine a general form for the forcing term, we need an expression for the general derivative with respect to p of the boundary term, which entered the governing equation by virtue of the Landau transformation, or

$$\frac{\partial^{n-m}}{\partial p^{n-m}}\left(\frac{1}{\bar{S}^*}\frac{\partial \bar{S}^*}{\partial h}\right)\bigg|_{p=0} = \sum_{j=0}^{j=n-m}\binom{n-m}{j}Y_j\frac{\partial \bar{S}^*_{n-m-j}}{\partial h}, \tag{5.2}$$

where $Y_j(h) = \frac{\partial^j}{\partial p^j}\left(\frac{1}{\bar{S}^*}\right)\Big|_{p=0}$. This can be written recursively as

$$Y_j(h) = \begin{cases} \frac{1}{\bar{S}_0^*} & j = 0, \\ -\sum_{k=0}^{k=j-1}\binom{j-1}{k}\left(\sum_{l=0}^{l=k}\binom{k}{l}Y_l Y_{k-l}\right)\bar{S}_{j-k}^* & \text{if } j = 1, 2, \dots m, \end{cases} \tag{5.3}$$

and therefore the forcing term becomes

$$(n+1)\alpha(1-h)\tilde{F}_{n+1}(\tilde{X},h) = (n+1)\alpha(1-h)\left[\frac{\partial \tilde{g}_n}{\partial h} + \sum_{m=0}^{n}\binom{n}{m}\frac{\partial \tilde{g}_m}{\partial \tilde{X}}\sum_{j=0}^{j=n-m}\binom{n-m}{j}Y_j\frac{\partial \bar{S}_{n-m-j}^*}{\partial h}\right]. \tag{5.4}$$

Further, we can show that if $\tilde{g}_n$ has the form

$$\tilde{g}_n(\tilde{X},h) = e^{\lambda_- \tilde{X}}\sum_{j=0}^{j=\Phi(n)} A_{nj}(h)\tilde{X}^j, \tag{5.5}$$

for some integer $\Phi(n)$, then the forcing term for the $(\tilde{g}_{n+1}, \tilde{S}_{n+1}^*)$ problem has the form $e^{\lambda_- \tilde{X}}\sum_{j=0}^{j=\Phi(n)+1} B_{nj}(h)\tilde{X}^j$. Given this form for the forcing term, we can show that general solution to the governing equation for $\hat{g}_{n+1}$ has the form

$$\tilde{g}_{n+1}(\tilde{X},h) = e^{\lambda_- \tilde{X}}\sum_{j=0}^{j=\Phi(n)+2} A_{(n+1)j}(h)\tilde{X}^j. \tag{5.6}$$

We know that the leading order expression has the form $\tilde{g}_0(\tilde{X},h) = A_{00}(h)e^{\lambda_- \tilde{X}}$ and therefore $\Phi(0) = 0$ which requires $\tilde{g}_{n+1}(\tilde{X},h)$ to have the general form

$$\hat{g}_{n+1}(\tilde{X},h) = e^{\lambda_- \tilde{X}}\sum_{j=0}^{2n+2} A_{(n+1)j}(h)\tilde{X}^j. \tag{5.7}$$

The development of an analytical expression therefore reduces to finding a recurrence relation between the coefficients $A_{(n)j}$ and $A_{(n+1)j}$ which will determine $A_{(n+1)(2n+2)}, \dots, A_{(n+1)1}$, while the boundary conditions will determine $A_{(n+1)0}$ and $\bar{S}_{n+1}^*$.

Determination of the coefficients $B_{nj}(h)$ can be achieved through the relationships

$$\frac{\partial \tilde{g}_m}{\partial \tilde{X}} = \begin{cases} e^{\lambda_- \tilde{X}}\lambda_- A_{00} & \text{if } m = 0, \\ e^{\lambda_- \tilde{X}}\left[\sum_{j=0}^{2m-1}\left((j+1)A_{m(j+1)} + \lambda_- A_{(mj)}\right)\tilde{X}^j + \lambda_- A_{m(2m)}\tilde{X}^{2m}\right] & \text{if } m > 0 \end{cases} \tag{5.8}$$

and

$$\frac{\partial \tilde{g}_n}{\partial h} = \begin{cases} e^{\lambda_- \tilde{X}} \left[\frac{\partial A_{00}}{\partial h} + \frac{\partial \lambda_-}{\partial h} A_{00} \tilde{X} \right] & \text{if } n = 0, \\ e^{\lambda_- \tilde{X}} \left[\frac{\partial A_{n0}}{\partial h} + \sum_{j=1}^{2n} \left(\frac{\partial \lambda_-}{\partial h} A_{n(j-1)} + \frac{\partial A_{nj}}{\partial h} \right) \tilde{X}^j + \frac{\partial \lambda_-}{\partial h} A_{n(2n)} \tilde{X}^{2n+1} \right] & \text{if } n > 0 \end{cases} \tag{5.9}$$

and these can be related to the coefficients $A_{(n+1)j}$ for $j \neq 0$ by substituting the general form (5.7) in the governing equation (3.122a) and comparing powers of $\tilde{X}$.

Expressions for $A_{(n+1)0}$ and $\bar{S}^*_{n+1}$ are found through application of the boundary conditions (3.122c-3.122d) which yield

$$hA_{(n+1)0} = -\bar{S}^*_{n+1} - \left. \frac{\partial^{n+1} \hat{P}^*_e}{\partial p^{n+1}} \right|_{p=0}, \tag{5.10}$$

where

$$\bar{S}^*_{n+1} = \frac{hA_{(n+1)1} + \sum_{k=0}^{n} \bar{S}^*_k \frac{\partial^{n+1-k}}{\partial p^{n+1-k}} \left(\frac{\partial \hat{P}^*_e}{\partial \bar{S}} \right) \Big|_{p=0}}{(\lambda_- - 1) - \frac{\partial \hat{P}_e}{\partial \bar{S}} \Big|_{\bar{S}^*_0}}. \tag{5.11}$$

Though the expressions relating successive terms in the series look relatively benign, there are some difficulties moving along the series in practice. The h-derivatives in forcing term (5.4) together with the p-derivatives of the put option price and its delta (5.10-5.11) produce ever larger numbers of terms which are onerous to determine analytically and we have not found an iterative method for their determination. The development of a numerical routine which may facilitate the investigation of convergence of the optimal exercise boundary approximation is therefore an area of further work.

5.2.2 An Analytical Approximation Under the CEV Model

One possible extension of our anaytical approximation for American-style options is application to option pricing under a general constant elasticity of variance or *CEV*, process which was first proposed by Cox [32]. This model aims to reproduce the observation that implied option volatility varies with the strike price, in an effect typically referred to as the *volatility smile*. Under the CEV model, a dividend paying underlying satisfies the SDE

$$dS = (r - D)Sdt + \sigma S^{n+1} dW, \tag{5.12}$$

under the risk neutral measure, where n represents the sensitivity of the variance to the value of the underlying.

For $n = 1$, we have geometric Brownian motion and return to the Black-Scholes-Merton model and the results derived therein. For $n < 1$ the volatility decreases/increases relative to GBM as the value of the underlying increases/decreases. This shifts the weight of probability distribution to the left, and makes large positive returns less easily attainable than large negative returns. This behaviour is typically seen in the equity market. For $n > 1$ the volatility increases/decreases relative to GBM as the value of the underlying increases/decreases. This shifts the weight of probability distribution to the right, and makes higher returns more easily attainable. This behaviour is typically seen in the market for options on futures. It is worth mentioning that none of these cases represent a symmetrical smile as observed in the foreign exchange market.

The American put option problem under the CEV model is given by

$$S \in (S^*(t), \infty), t \in (0,T) \qquad \frac{\partial P_a^{CEV}}{\partial t} + \frac{1}{2}\sigma^2 S^2 S^{2n} \frac{\partial^2 P_a^{CEV}}{\partial S^2} + (r-D)S\frac{\partial P_a^{CEV}}{\partial S} - rP_a^{CEV} = 0, \tag{5.13a}$$

subject to

$$\text{as } S \to \infty \qquad P_a^{CEV}(S,t) \to 0, \tag{5.13b}$$

$$\text{at } t = T \qquad P_a^{CEV} = (K-S)^+, \tag{5.13c}$$

$$\text{at } S = S^*(t) \qquad P_a^{CEV}(S^*(t),t) = 1 - S^*(t), \tag{5.13d}$$

$$\frac{\partial P_a^{CEV}}{\partial S} = -1. \tag{5.13e}$$

Introducing the scalings

$$S = K\bar{S}, \qquad P_a^{CEV} = K\bar{P}_a^{CEV}, \qquad \tilde{\sigma}^2 = K^{2n}\sigma^2, \qquad \tilde{\alpha} = \frac{2r}{\tilde{\sigma}^2},$$
$$\tilde{\beta} = \frac{2(r-D)}{\tilde{\sigma}^2}, \qquad \tilde{\tau} = \frac{\tilde{\sigma}^2(T-t)}{2}, \tag{5.14}$$

together with the MacMillan transformation $\tilde{h}(\tilde{\tau}) = 1 - e^{-\tilde{\alpha}\tilde{\tau}}$, the decomposition $\bar{P}_a^{CEV}(\bar{S},\tilde{\tau}) = \bar{P}_e^{CEV}(\bar{S},\tilde{\tau}) + \tilde{h}g(\bar{S},h)$ and the analytic expansions in a parameter $p \in [0,1]$

$$\bar{g}^{CEV}(\bar{S},\tilde{h};p) = \sum_{i=0}^{\infty} \frac{p^i}{i!}\bar{g}_i^{CEV}(\bar{S},\tilde{h}), \tag{5.15}$$

$$\bar{S}^*(\tilde{h};p) = \sum_{i=0}^{\infty} \frac{p^i}{i!}\bar{S}_i^*(\tilde{h}), \tag{5.16}$$

gives the problem

$$\bar{S} \in (\bar{S}^*(\tilde{h}), \infty), \tilde{h} \in (0,1) \qquad \bar{S}^2 \bar{S}^{2n} \frac{\partial^2 \bar{g}^{CEV}}{\partial \bar{S}^2} + \tilde{\beta} \bar{S} \frac{\partial \bar{g}^{CEV}}{\partial \bar{S}} - \frac{\tilde{\alpha}}{\tilde{h}} \bar{g}^{CEV} = p\tilde{\alpha}(1-\tilde{h}) \frac{\partial \bar{g}^{CEV}}{\partial \tilde{h}}, \tag{5.17a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{g}^{CEV}(\bar{S}, \tilde{h}) \to 0, \tag{5.17b}$$

$$\text{at } \tilde{h} = 0 \qquad \tilde{h}\bar{g}^{CEV}(\bar{S}, 0) = 0, \tag{5.17c}$$

$$\text{at } \bar{S} = \bar{S}^*(\tilde{h}) \qquad \tilde{h}\bar{g}^{CEV}(\bar{S}^*(\tilde{h}), \tilde{h}) = 1 - \bar{S}^*(\tilde{h}) - \bar{P}_e^{CEV}(\bar{S}^*(\tilde{h}), \tilde{h}), \tag{5.17d}$$

$$\tilde{h} \frac{\partial \bar{g}^{CEV}}{\partial \bar{S}} = -1 - \left. \frac{\partial \bar{P}_e^{CEV}}{\partial \bar{S}} \right|_{\bar{S}^*}, \tag{5.17e}$$

where the parameter $p \in [0,1]$ is also introduced into the forcing term of the governing equation. Following the same approach as in Chapters 3 & 4, we differentiate the problem (5.17a-5.17e) wrt p and evaluate at $p = 0$ to give the following problems for the first two terms in the series: for $(\bar{g}_0^{CEV}, \bar{S}_0^*)$

$$\bar{S} \in (\bar{S}_0^*(\tilde{h}), \infty), \tilde{h} \in (0,1) \qquad \bar{S}^2 \bar{S}^{2n} \frac{\partial^2 \bar{g}_0^{CEV}}{\partial \bar{S}^2} + \tilde{\beta} \bar{S} \frac{\partial \bar{g}_0^{CEV}}{\partial \bar{S}} - \frac{\tilde{\alpha}}{\tilde{h}} \bar{g}_0^{CEV} = 0, \tag{5.18a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{g}_0^{CEV}(\bar{S}, \tilde{h}) \to 0, \tag{5.18b}$$

$$\text{at } \tilde{h} = 0 \qquad \tilde{h}\bar{g}_0^{CEV}(\bar{S}, 0) = 0, \tag{5.18c}$$

$$\text{at } \bar{S} = \bar{S}_0^* \qquad \tilde{h}\bar{g}_0^{CEV} = 1 - \bar{S}_0^* - \bar{P}_e^{CEV}(\bar{S}_0^*(\tilde{h}), \tilde{h}), \tag{5.18d}$$

$$\tilde{h} \frac{\partial \bar{g}_0^{CEV}}{\partial \bar{S}} = -1 - \frac{\partial \bar{P}_e^{CEV}}{\partial \bar{S}}, \tag{5.18e}$$

and for $(\bar{g}_1^{CEV}, \bar{S}_1^*)$

$$\bar{S} \in (\bar{S}_0^*(\tilde{h}), \infty), \tilde{h} \in (0,1) \qquad \bar{S}^2 \bar{S}^{2n} \frac{\partial^2 \bar{g}_1^{CEV}}{\partial \bar{S}^2} + \tilde{\beta} \bar{S} \frac{\partial \bar{g}_1^{CEV}}{\partial \bar{S}} - \frac{\tilde{\alpha}}{\tilde{h}} \bar{g}_1^{CEV} = \tilde{\alpha}(1-\tilde{h}) \frac{\partial \bar{g}_0^{CEV}}{\partial \tilde{h}}, \tag{5.19a}$$

subject to

$$\text{as } \bar{S} \to \infty \qquad \bar{g}_1^{CEV}(\bar{S}, \tilde{h}) \to 0, \tag{5.19b}$$

$$\text{at } \tilde{h} = 0 \qquad \tilde{h}\bar{g}_1^{CEV}(\bar{S}, 0) = 0, \tag{5.19c}$$

$$\text{at } \bar{S} = \bar{S}_0^*(\tilde{h}) \qquad \tilde{h}\bar{g}_1^{CEV} = 0, \tag{5.19d}$$

$$\tilde{h}\frac{\partial \bar{g}_1^{CEV}}{\partial \bar{S}} = -\bar{S}_1^* \left[\frac{\partial^2 \bar{P}_e^{CEV}}{\partial \bar{S}^2} + \tilde{h}\frac{\partial^2 \bar{g}_0^{CEV}}{\partial \bar{S}^2} \right]. \tag{5.19e}$$

Davydov and Linetsky [37] give the following general solutions to the ODE (5.18a), where in each case $C_0^+(\tilde{h})$ is the unknown coefficient of an increasing function in $\bar{S}$ and $C_0^-(\tilde{h})$ is the unknown coefficient of a decreasing function in $\bar{S}$.
For $n > 0$

$$\bar{g}_0^{CEV}(\bar{S}, \tilde{h}) = \begin{cases} \bar{S}^{n+\frac{1}{2}} e^{\frac{\tilde{\beta}}{2n}\bar{S}^{-2n}} \left[C_0^+(\tilde{h}) W_{k,m}\left(\frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right) + C_0^-(\tilde{h}) M_{k,m}\left(\frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right)\right], & \text{for } \tilde{\beta} > 0 \\ \bar{S}^{n+\frac{1}{2}} e^{\frac{\tilde{\beta}}{2n}\bar{S}^{-2n}} \left[C_0^+(\tilde{h}) W_{-k,m}\left(-\frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right) + C_0^-(\tilde{h}) M_{-k,m}\left(-\frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right)\right], & \text{for } \tilde{\beta} < 0 \\ \bar{S}^{\frac{1}{2}} \left[C_0^+(\tilde{h}) K_\nu\left(\sqrt{2\tilde{\alpha}}\frac{\bar{S}^{-n}}{n}\right) + C_0^-(\tilde{h}) I_\nu\left(\sqrt{2\tilde{\alpha}}\frac{\bar{S}^{-n}}{n}\right)\right], & \text{for } \tilde{\beta} = 0, \end{cases} \tag{5.20}$$

and for $n < 0$

$$\bar{g}_0^{CEV}(\bar{S}, \tilde{h}) = \begin{cases} \bar{S}^{n+\frac{1}{2}} e^{\frac{\tilde{\beta}}{2n}\bar{S}^{-2n}} \left[C_0^-(\tilde{h}) W_{-k,-m}\left(-\frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right) + C_0^+(\tilde{h}) M_{-k,-m}\left(-\frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right)\right], & \text{for } \tilde{\beta} > 0 \\ \bar{S}^{n+\frac{1}{2}} e^{\frac{\tilde{\beta}}{2n}\bar{S}^{-2n}} \left[C_0^-(\tilde{h}) W_{k,-m}\left(\frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right) + C_0^+(\tilde{h}) M_{k,-m}\left(\frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right)\right], & \text{for } \tilde{\beta} < 0 \\ \bar{S}^{\frac{1}{2}} \left[C_0^-(\tilde{h}) K_{-\nu}\left(-\sqrt{2\tilde{\alpha}}\frac{\bar{S}^{-n}}{n}\right) + C_0^+(\tilde{h}) I_{-\nu}\left(-\sqrt{2\tilde{\alpha}}\frac{\bar{S}^{-n}}{n}\right)\right], & \text{for } \tilde{\beta} = 0. \end{cases} \tag{5.21}$$

where $W_{k,m}(x)$ and $M_{k,m}(x)$ are the Whittaker functions and $K_\nu(x)$ and $I_\nu(x)$ are the modified Bessel functions, which can all be written in terms of the confluent hypergeometric functions $U(a, b; x)$ and $M(a, b; x)$ [1]

$$W_{k,m}(x) = e^{-\frac{x}{2}} x^{m+\frac{1}{2}} U\left(\frac{1}{2} + m - k, 1 + 2m; x\right), \tag{5.22}$$

$$M_{k,m}(x) = e^{-\frac{x}{2}} x^{m+\frac{1}{2}} M\left(\frac{1}{2} + m - k, 1 + 2m; x\right), \tag{5.23}$$

$$K_\nu(x) = \left(\frac{x}{2}\right)^\nu \frac{e^{-x}}{\Gamma(\nu + 1)} M\left(\frac{1}{2} + \nu, 1 + 2\nu; 2x\right), \tag{5.24}$$

$$I_\nu(x) = \pi^{\frac{1}{2}} (2x)^\nu e^{-x} U\left(\frac{1}{2} + \nu, 1 + 2\nu; 2x\right), \tag{5.25}$$

where $U(a,b;x)$ and $M(a,b;x)$ have the integral representations

$$U(a,b;x) = \frac{e^x}{\Gamma(a)} \int_1^\infty e^{-xt} t^{b-a-1} (1-t)^{a-1} \, dt, \tag{5.26}$$

$$M(a,b;x) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt. \tag{5.27}$$

Writing them in this form has the advantage that derivatives with respect to the independent variable x have well known forms [1]

$$\frac{\partial U}{\partial x}(a,b;x) = -aU(a+1,b+1;x), \tag{5.28}$$

$$\frac{\partial M}{\partial x}(a,b;x) = \frac{a}{b} M(a+1,b+1;x), \tag{5.29}$$

which makes the representation of derivatives of $\bar{g}_0$ more compact.

For the American put option problem, we need decreasing solutions in order to satisfy the large $\bar{S}$ constraint and therefore we require $C_0^+ = 0$ in (5.20,5.21), leading to the form for $\bar{g}_0^{CEV}$ for $n > 0$

$$\bar{g}_0^{CEV}(\bar{S},\tilde{h}) = \begin{cases} C_0^-(\tilde{h}) \left(\frac{\tilde{\beta}}{n}\right)^{\frac{1}{2}\left(1+\frac{1}{2n}\right)} M\left(\frac{\tilde{\alpha}}{2\tilde{\beta}n}, 1+\frac{1}{2n}; \frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right), & \text{for } \tilde{\beta} > 0, \\ C_0^-(\tilde{h}) \left(-\frac{\tilde{\beta}}{n}\right)^{\frac{1}{2}\left(1+\frac{1}{2n}\right)} e^{\frac{\tilde{\beta}}{2n}\bar{S}^{-2n}} M\left(1+\frac{1}{2n}-\frac{\tilde{\alpha}}{2\tilde{\beta}n}, 1+\frac{1}{2n}; -\frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right), & \text{for } \tilde{\beta} < 0, \\ C_0^-(\tilde{h}) \left(\frac{\sqrt{2\tilde{\alpha}}}{2n}\right)^{\frac{1}{2n}} e^{-\frac{\sqrt{2\tilde{\alpha}}}{n}\bar{S}^{-n}} M\left(\frac{1}{2}+\frac{1}{2n}, 1+\frac{1}{n}; \frac{2\sqrt{2\tilde{\alpha}}}{n}\bar{S}^{-n}\right), & \text{for } \tilde{\beta} = 0, \end{cases} \tag{5.30}$$

and for $n < 0$

$$\bar{g}_0^{CEV}(\bar{S},\tilde{h}) = \begin{cases} C_0^-(\tilde{h}) \left(-\frac{\tilde{\beta}}{n}\right)^{\frac{1}{2}\left(1-\frac{1}{2n}\right)} \bar{S} e^{\frac{\tilde{\beta}}{2n}\bar{S}^{-2n}} U\left(1-\frac{\tilde{\alpha}}{2\tilde{\beta}n}, 1-\frac{1}{2n}; -\frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right), & \text{for } \tilde{\beta} > 0, \\ C_0^-(\tilde{h}) \left(\frac{\tilde{\beta}}{n}\right)^{\frac{1}{2}\left(1-\frac{1}{2n}\right)} \bar{S} U\left(\frac{\tilde{\alpha}}{2\tilde{\beta}n}-\frac{1}{2n}, 1-\frac{1}{2n}; \frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right), & \text{for } \tilde{\beta} < 0, \\ C_0^-(\tilde{h}) \pi^{\frac{1}{2}} \left(-\frac{2\sqrt{2\tilde{\alpha}}}{n}\right)^{-\frac{1}{2n}} \bar{S} e^{\frac{\sqrt{2\tilde{\alpha}}}{n}\bar{S}^{-n}} U\left(\frac{1}{2}-\frac{1}{2n}, 1-\frac{1}{n}; -\frac{2\sqrt{2\tilde{\alpha}}}{n}\bar{S}^{-n}\right), & \text{for } \tilde{\beta} = 0. \end{cases} \tag{5.31}$$

Determination of $\bar{S}_0^*(\tilde{h})$

From conditions (5.18d-5.18e), $\bar{S}_0^*$ solves the transcendental expression

$$\bar{g}_0^{CEV}(\bar{S},\tilde{h}) = -\left.\frac{\partial \bar{g}_0^{CEV}}{\partial \bar{S}}\right|_{\bar{S}_0^*} \frac{1-\bar{S}_0^* - \bar{P}_e^{CEV*}}{1+\bar{\Delta}^{CEV*}}, \tag{5.32}$$

where $\bar{P}_e^{CEV*}$ and $\bar{\Delta}^{CEV*}$ respectively represent the European put option price and its Delta under the CEV model evaluated at $\bar{S}_0^*$. To solve this expression we need the form for $\frac{\partial \bar{g}_0^{CEV}}{\partial \bar{S}}$ which is for $n > 0$

$$\frac{\partial \bar{g}_0^{CEV}}{\partial \bar{S}} = \begin{cases} \frac{\tilde{\alpha} C_0^-}{2n+1} \left(\frac{\tilde{\beta}}{n}\right)^{\frac{1}{2}\left(1+\frac{1}{2n}\right)} \bar{S}^{-2n-1} M\left(1 + \frac{\tilde{\alpha}}{2\tilde{\beta}n}, 2 + \frac{1}{2n}; \frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right), & \text{for } \tilde{\beta} > 0, \\ \tilde{\beta} C_0^- \left(-\frac{\tilde{\beta}}{n}\right)^{\frac{1}{2}\left(1+\frac{1}{2n}\right)} \bar{S}^{-2n-1} e^{\frac{\tilde{\beta}}{2n}\bar{S}^{-2n}} \times \dots & \\ \left[\left(1 - \frac{\tilde{\alpha}}{\tilde{\beta}(2n+1)}\right) M\left(2 + \frac{1}{2n}\frac{\tilde{\alpha}}{2\tilde{\beta}n}, 2 + \frac{1}{2n} - \frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right)\right. & \\ \left.\dots - M\left(1 + \frac{1}{2n} - \frac{\tilde{\alpha}}{2\tilde{\beta}n}, 1 + \frac{1}{2n}; -\frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right)\right], & \text{for } \tilde{\beta} < 0, \\ \sqrt{2\tilde{\alpha}} C_0^- \left(\frac{\sqrt{2\tilde{\alpha}}}{2n}\right)^{\frac{1}{2n}} \bar{S}^{-n-1} e^{-\frac{\sqrt{2\tilde{\alpha}}}{n}\bar{S}^{-n}} \left[M\left(\frac{1}{2} + \frac{1}{2n}, 1 + \frac{1}{n}; \frac{2\sqrt{2\tilde{\alpha}}}{n}\bar{S}^{-n}\right)\right. & \\ \left.\dots - M\left(\frac{3}{2} + \frac{1}{2n}, 2 + \frac{1}{n}; \frac{2\sqrt{2\tilde{\alpha}}}{n}\bar{S}^{-n}\right)\right], & \text{for } \tilde{\beta} = 0 \end{cases} \tag{5.33}$$

and for $n < 0$

$$\frac{\partial \bar{g}_0^{CEV}}{\partial \bar{S}} = \begin{cases} C_0^- \left(-\frac{\tilde{\beta}}{n}\right)^{\frac{1}{2}\left(1-\frac{1}{2n}\right)} e^{\frac{\tilde{\beta}}{2n}\bar{S}^{-2n}} \left[\left(1 - \tilde{\beta}\bar{S}^{-2n}\right) U\left(1 - \frac{\tilde{\alpha}}{2\tilde{\beta}n}, 1 - \frac{1}{2n}; -\frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right)\right. & \\ \left.\dots + \left(\frac{\tilde{\alpha}}{n} - 2\tilde{\beta}\right) \bar{S}^{-2n} U\left(2 - \frac{\tilde{\alpha}}{2\tilde{\beta}n}, 2 - \frac{1}{2n}; -\frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right)\right], & \text{for } \tilde{\beta} > 0, \\ C_0^- \left(\frac{\tilde{\beta}}{n}\right)^{\frac{1}{2}\left(1-\frac{1}{2n}\right)} \left[U\left(\frac{\tilde{\alpha}}{2\tilde{\beta}n} = \frac{1}{2n}, 1 - \frac{1}{2n}; \frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right)\right. & \\ \left.\dots + \left(\tilde{\alpha} - \tilde{\beta}\right) \frac{\bar{S}^{-2n}}{n} U\left(1 + \frac{\tilde{\alpha}}{2\tilde{\beta}n} - \frac{1}{2n}, 2 - \frac{1}{2n}; \frac{\tilde{\beta}}{n}\bar{S}^{-2n}\right)\right], & \text{for } \tilde{\beta} > 0, \\ C_0^- \pi^{\frac{1}{2}} \left(\frac{\sqrt{2\tilde{\alpha}}}{2n}\right)^{\frac{1}{2n}} e^{-\frac{\sqrt{2\tilde{\alpha}}}{n}\bar{S}^{-n}} \left[\left(1 - \sqrt{2\tilde{\alpha}}\bar{S}^{-n}\right) U\left(\frac{1}{2} - \frac{1}{2n}, 1 - \frac{1}{n}; -\frac{2\sqrt{2\tilde{\alpha}}}{n}\bar{S}^{-n}\right)\right. & \\ \left.\dots - \sqrt{2\tilde{\alpha}}\left(1 - \frac{1}{n}\right) \bar{S}^{-n} U\left(\frac{3}{2} - \frac{1}{2n}, 2 - \frac{1}{n}; -\frac{2\sqrt{2\tilde{\alpha}}}{n}\bar{S}^{-n}\right)\right], & \text{for } \tilde{\beta} = 0, \end{cases} \tag{5.34}$$

while the European put option under the CEV model [33] is given by

$$\bar{P}_e^{CEV}(\bar{S}, t) = \begin{cases} e^{-\tilde{\alpha}\tilde{\tau}}\left(1 - \chi^2(c, b, a)\right) - \bar{S}e^{-\tilde{\beta}\tilde{\tau}}\chi^2(a, b+2, c) & \text{for } n < 0, \\ e^{-\tilde{\alpha}\tilde{\tau}}\left(1 - \chi^2(a, 2-b, c)\right) - \bar{S}e^{-\tilde{\beta}\tilde{\tau}}\chi^2(c, -b, a) & \text{for } n > 0, \end{cases} \tag{5.35}$$

where

$$a = \frac{\left(e^{-\tilde{\beta}\tilde{\tau}}\right)^{-2n}}{n^2 m}, \qquad b = -\frac{1}{n}, \qquad c = \frac{\bar{S}^{-2n}}{n^2 m}, \qquad m = -\frac{1}{2\tilde{\beta}n} e^{-2n\tilde{\beta}\tilde{\tau} - 1} \tag{5.36a}$$

and $\chi^2(z, k, w)$ is the cumulative distribution function of the non-central chi-squared distribution with k degrees of freedom and non-centrality parameter w. We note that only c depends on the underlying and therefore will be the only contribution to the delta from the chi-squared distribution.

$$\bar{\Delta}_e^{CEV}(\bar{S},\tilde{\tau}) = \begin{cases} -e^{-\tilde{\beta}\tilde{\tau}}\chi^2(a,b+2,c) - \frac{\partial c}{\partial \bar{S}}\left[e^{-\tilde{\alpha}\tilde{\tau}}\frac{\partial}{\partial c}\chi^2(c,b,a) + \bar{S}e^{-\tilde{\beta}\tilde{\tau}}\frac{\partial}{\partial c}\chi^2(a,b+2,c)\right] & \text{for } n<0, \\ -e^{-\tilde{\beta}\tilde{\tau}}\chi^2(c,-b,a) - \frac{\partial c}{\partial \bar{S}}\left[e^{-\tilde{\alpha}\tilde{\tau}}\frac{\partial}{\partial c}\chi^2(a,2-b,c) + \bar{S}e^{-\tilde{\beta}\tilde{\tau}}\frac{\partial}{\partial c}\chi^2(c,-b,a)\right] & \text{for } n>0. \end{cases} \tag{5.37}$$

The numerical solution to the transcendental expression for $\bar{S}_0^*$ for the general CEV model thus requires evaluation of the non-centralised chi-squared distribution together with its derivatives with respect to its argument and the non-centrality parameter. A further requirement is a robust routine for the evaluation of the confluent hypergeometric functions. This remains work to be completed.

An analytical solution to the problem for $(g_1^{CEV}, \bar{S}_1^*)$ (5.19a-5.19e) requires an expression for the forcing term in the ODE, which depends on the term $\frac{\partial C_0^-}{\partial h}$ and in turn on the term $\frac{\partial \bar{S}_0^*}{\partial h}$. Deriving analytical forms for these terms will probably be onerous and one may therefore prefer to use a numerical approximation for the forcing term and seek to solve the ODE problem for $\bar{g}_1$ numerically, though naturally at the likely expense of computational time versus an analytical expression.

Appendix A

Greeks

In order to calculate the analytical approximations derived for the standard American put option and American up-and-out put option in Chapter's 3 & 4, we require derivatives of the equivalent European-style options with respect to model variables. In finance these sensitivities are termed *Greeks*.

Discussions regarding some of the lower order vanilla European Greeks can be found in most financial mathematics texts (eg. Hull [62]) while some higher order vanilla European Greeks are discussed by Garman [46]. Good technical discussions of the role of Greeks in finance from the point of view of practitioners can be found in Haug [55, 56] and Taleb [98].

We shall derive the European Greeks for the non-dimensional prices $\bar{P}_e(\bar{S},\tau)$ (2.3) and $\bar{P}_e^{UO}(\bar{S},\tau)$ (4.2) based on the non-dimensional parameters α & β (2.1), with the Greeks for vanilla European options notated using a bar (ie $\bar{\Delta}$,$\bar{\Gamma}$,$\bar{\Theta}$,...) and a similar notation used for European barrier option Greeks but with a superscript denoting the type of barrier (ie $\bar{\Delta}^{UO}$,$\bar{\Gamma}^{UO}$,$\bar{\Theta}^{UO}$,...).

The Greeks evaluated at the leading or order boundary term in the analytic approximations, are notated by a superscript star (ie $\bar{\Delta}^*$, $\bar{\Gamma}^*$,$\bar{\Theta}^*$,... or $\bar{\Delta}^{UO*}$, $\bar{\Gamma}^{UO*}$,$\bar{\Theta}^{UO*}$,...).

Since we use MacMillan's time-transformation in deriving our series approximations, we derive the time related Greeks under $h = 1 - e^{-\alpha\tau}$, which can be recovered for the τ-derivatives using

$$\frac{\partial}{\partial h} = \frac{1}{\alpha(1-h)}\frac{\partial}{\partial \tau}. \tag{A.1}$$

A.1 European Put Option Greeks

The non-dimensional Black-Scholes-Merton equation for a European put option on a dividend paying asset was given in (2.3) as

$$\bar{P}_e(\bar{S},\tau) = \frac{e^{-\alpha\tau}}{2}\left[\mathrm{erfc}\left(\frac{\bar{d}_2}{\sqrt{2}}\right) - e^{\beta\tau}\bar{S}\mathrm{erfc}\left(\frac{\bar{d}_1}{\sqrt{2}}\right)\right],$$

where

$$\begin{aligned}\frac{\bar{d}_1}{\sqrt{2}} &= \frac{1}{2}\left[\frac{\ln(\bar{S})}{\tau^{\frac{1}{2}}} + (\beta+1)\tau^{\frac{1}{2}}\right],\\ \frac{\bar{d}_2}{\sqrt{2}} &= \frac{1}{2}\left[\frac{\ln(\bar{S})}{\tau^{\frac{1}{2}}} + (\beta-1)\tau^{\frac{1}{2}}\right]\end{aligned}$$

and $\mathrm{erfc}(\zeta)$ is the complementary error function

$$\mathrm{erfc}(\zeta) = \frac{2}{\sqrt{\pi}}\int_{\zeta}^{\infty} e^{-s^2}ds. \tag{A.2}$$

In deriving the Greeks, we use the relationships

$$\mathrm{erfc}(-\zeta) = 2 - \mathrm{erfc}(\zeta), \tag{A.3}$$

$$\mathrm{erfc}'(\zeta) = \frac{d\,(\mathrm{erfc}(\zeta))}{d\zeta} = -\frac{2}{\sqrt{\pi}}e^{-\zeta^2}, \tag{A.4}$$

the derivatives of (2.4a) and (2.4b) with respect to $\bar{S}$ and τ

$$\frac{\partial}{\partial\bar{S}}\left(\frac{\bar{d}_1}{\sqrt{2}}\right) = \frac{\partial}{\partial\bar{S}}\left(\frac{\bar{d}_2}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}\bar{S}(\bar{d}_1-\bar{d}_2)}, \tag{A.5a}$$

$$\frac{\partial}{\partial\tau}\left(\frac{\bar{d}_1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}(\bar{d}_1-\bar{d}_2)}\left[(\beta+1) - \frac{\bar{d}_1}{\bar{d}_1-\bar{d}_2}\right], \tag{A.5b}$$

$$\frac{\partial}{\partial\tau}\left(\frac{\bar{d}_2}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}(\bar{d}_1-\bar{d}_2)}\left[(\beta-1) - \frac{\bar{d}_2}{\bar{d}_1-\bar{d}_2}\right] \tag{A.5c}$$

and the relation

$$\mathrm{erfc}'\left(\frac{\bar{d}_2}{\sqrt{2}}\right) - e^{\beta\tau}\bar{S}\mathrm{erfc}'\left(\frac{\bar{d}_1}{\sqrt{2}}\right) = 0. \tag{A.6}$$

Delta

The *Delta* of the European put option is the first derivative with respect to the option price

$$\begin{aligned}\bar{\Delta} = \frac{\partial \bar{P}_e}{\partial \bar{S}} &= \frac{e^{-\alpha\tau}}{2}\left[\text{erfc}'\left(\frac{\bar{d}_2}{\sqrt{2}}\right)\frac{\partial}{\partial \bar{S}}\left(\frac{\bar{d}_2}{\sqrt{2}}\right) - e^{\beta\tau}\bar{S}\text{erfc}'\left(\frac{\bar{d}_1}{\sqrt{2}}\right)\frac{\partial}{\partial \bar{S}}\left(\frac{\bar{d}_1}{\sqrt{2}}\right) - e^{\beta\tau}\text{erfc}\left(\frac{\bar{d}_1}{\sqrt{2}}\right)\right] \\ &= -\frac{e^{-(\alpha-\beta)\tau}}{2}\text{erfc}\left(\frac{\bar{d}_1}{\sqrt{2}}\right) = -\frac{(1-h)^{\frac{\alpha-\beta}{\alpha}}}{2}\text{erfc}\left(\frac{\bar{d}_1}{\sqrt{2}}\right) \qquad \text{(A.7)}\end{aligned}$$

,

using (A.6) and (A.5a).

Gamma

The *Gamma* of the European put option is the second derivative with respect to the option price

$$\bar{\Gamma} = \frac{\partial^2 \bar{P}_e}{\partial \bar{S}^2} = -\frac{(1-h)^{\frac{\alpha-\beta}{\alpha}}}{2}\text{erfc}'\left(\frac{\bar{d}_1}{\sqrt{2}}\right)\frac{\partial}{\partial \bar{S}}\left(\frac{\bar{d}_2}{\sqrt{2}}\right) = (1-h)^{\frac{\alpha-\beta}{\alpha}}\frac{e^{-\frac{\bar{d}_1^2}{2}}}{\sqrt{2\pi}}\frac{1}{\bar{S}(\bar{d}_1-\bar{d}_2)}. \qquad \text{(A.8)}$$

Speed

The third derivative of the option price with respect to the underlying is given the term *Speed* by Garman [46] and we notate it as $\bar{\Gamma}_\Delta$. Differentiating (A.8) with respect to $\bar{S}$ gives

$$\bar{\Gamma}_{\bar{\Delta}} = \frac{\partial^3 \bar{P}_e}{\partial \bar{S}^3} = -\frac{\bar{\Gamma}}{\bar{S}}\left[\frac{\bar{d}_1}{\bar{d}_1-\bar{d}_2}+1\right]. \qquad \text{(A.9)}$$

Theta

The *Theta* of an option is its first derivative with respect time, or for the purpose of analytic approximation the transformed time variable h. We use the Black-Scholes PDE to write Theta in terms of Delta (A.7), Gamma (A.8) and the European put option price (2.3)

$$\bar{\Theta} = \frac{1}{\alpha(1-h)}\left[\bar{S}^2\bar{\Gamma} + \beta\bar{S}\bar{\Delta} - \alpha\bar{P}_e\right]. \qquad \text{(A.10)}$$

Charm/Delta Bleed

The decay in Delta with time is given the term *Charm* by Garman and *Delta Bleed* by Taleb and we shall notate it as $\bar{\Delta}_{\Theta}$. Differentiating (A.10) with respect to $\bar{S}$ gives

$$\bar{\Delta}_{\bar{\Theta}} = \frac{1}{\alpha(1-h)}\left[\bar{S}^2\bar{\Gamma}_{\Delta} + (\beta+2)\bar{S}\bar{\Gamma} - (\alpha-\beta)\bar{\Delta}\right]. \tag{A.11}$$

Colour/Gamma Bleed

The decay in Gamma with time is given the term *Colour* by Garman and *Gamma Bleed* by Taleb and we notate it as $\bar{\Gamma}_{\Theta}$. Differentiating (A.11) with respect to $\bar{S}$ gives

$$\bar{\Gamma}_{\bar{\Theta}} = \frac{1}{\alpha(1-h)}\left[\bar{S}^2\bar{\Gamma}_{\Gamma} + (\beta+4)\bar{S}\bar{\Gamma}_{\Delta} - (\alpha-2\beta-2)\bar{\Gamma}\right]. \tag{A.12}$$

Higher Order Greeks

The Greeks of higher order than previously derived are not typically discussed in the literature. We derive the results needed for the purpose of determining the terms in our analytic approximation here.

Differentiating (A.9) with respect to $\bar{S}$ gives

$$\bar{\Gamma}_{\bar{\Gamma}} = \frac{\partial^4 \bar{P}_e}{\partial \hat{S}^4} = -\frac{\bar{\Gamma}_{\Delta}}{\bar{S}}\left[2 + \frac{\bar{d}_1}{\bar{d}_1 - \bar{d}_2}\right] - \frac{\bar{\Gamma}}{\bar{S}^2}\frac{1}{(\bar{d}_1 - \bar{d}_2)^2}. \tag{A.13}$$

Differentiating (A.11) with respect to h gives

$$\begin{aligned}\bar{\Theta}_{\bar{\Theta}} = \frac{\partial^2 \bar{P}_e}{\partial h^2} &= \frac{\Theta}{(1-h)} + \frac{1}{\alpha(1-h)}\left[\bar{S}^2\bar{\Gamma}_{\Theta} + \beta\bar{S}\bar{\Delta}_{\Theta} - \alpha\Theta\right] \\ &= \frac{1}{\alpha(1-h)}\left[\bar{S}^2\bar{\Gamma}_{\Theta} + \beta\bar{S}\bar{\Delta}_{\Theta}\right]. \end{aligned} \tag{A.14}$$

Differentiating (A.14) with respect to $\bar{S}$ gives

$$\bar{\Delta}_{\Theta\Theta} = \frac{\partial^3 \bar{P}_e}{\partial \bar{S}\partial h^2} = \frac{1}{\alpha(1-h)}\left[2\bar{S}\bar{\Gamma}_{\Theta} + \bar{S}^2\bar{\Gamma}_{\Delta\Theta} + \beta\bar{\Delta}_{\Theta} + \beta\bar{S}\bar{\Gamma}_{\Theta}\right]. \tag{A.15}$$

Differentiating (A.9) with respect to h gives

$$\bar{\Gamma}_{\Delta\Theta} = \frac{\partial^4 \bar{P}_e}{\partial \bar{S}^3\partial h} = -\bar{\Gamma}_{\Theta}\left[\bar{d}_1 + \frac{1}{\bar{S}}\right] - \frac{\bar{\Gamma}}{\alpha(1-h)(\bar{d}_1 - \bar{d}_2)}\left[(\beta+1) - \frac{\bar{d}_1}{\bar{d}_1 - \bar{d}_2}\right]. \tag{A.16}$$

A.2 European Up-and-Out Put Option Greeks

Choosing to represent the European barrier option price in terms of vanilla European option greatly simplifies the calculation of the equivalent Greeks. From (4.2) the price of a European up-and-out put option is given by

$$\bar{P}^{UO}(\bar{S},\tau;\bar{B}) = \begin{cases} \bar{P}_e(\bar{S}) - \left(\frac{\bar{S}}{\bar{B}}\right)^{2a} \bar{P}_e\left(\frac{\bar{B}^2}{\bar{S}}\right) & \text{for } \bar{B} > 1, \\ \bar{B}\left[\bar{P}_e\left(\frac{\bar{S}}{\bar{B}}\right) - \left(\frac{\bar{S}}{\bar{B}}\right)^{2a} \bar{P}_e\left(\frac{\bar{B}}{\bar{S}}\right)\right] - (1-\bar{B})\left[\bar{P}_d\left(\frac{\bar{S}}{\bar{B}}\right) - \left(\frac{\bar{S}}{\bar{B}}\right)^{2a} \bar{P}_d\left(\frac{\bar{B}}{\bar{S}}\right)\right] & \text{for } \bar{B} \leq 1, \end{cases} \tag{A.17}$$

where $2a = -(\beta - 1)$ and the time dependence of the European put option terms is omitted for clarity. $\bar{P}_d$ is a European digital put option given by

$$\bar{P}_d(\bar{S},\tau) = \frac{e^{-\alpha\tau}}{2}\text{erfc}\left(\frac{\bar{d}_2}{\sqrt{2}}\right). \tag{A.18}$$

For an up-and-out put option, the rebate is given by an American digital call option struck at the barrier and scaled by the rebate, or

$$\bar{R}\bar{C}_d^{Am}(\bar{S},\tau;\bar{B}) = \frac{\bar{R}}{2}\left[\left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^+} \text{erfc}\left(-\frac{\bar{d}^+}{\sqrt{2}}\right) + \left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^-} \text{erfc}\left(-\frac{\bar{d}^-}{\sqrt{2}}\right)\right], \tag{A.19}$$

where

$$\lambda_\pm^\infty = \frac{-(\beta-1) \pm \sqrt{(\beta-1)^2 + 4\alpha}}{2}, \tag{A.20}$$

$$\bar{d}^\pm = \frac{\ln\left(\frac{\bar{S}}{\bar{B}}\right) \pm \left(\lambda_+^\infty - \lambda_-^\infty\right)\tau}{\sqrt{2\tau}}. \tag{A.21}$$

We will therefore also require the Greeks of European and American digital options for our analytical approximation for the American barrier option. These are derives in subsections (A.2.3) and (A.2.4) respectively.

A.2.1 Greeks for Regular Barrier Options ($\bar{B} \geq 1$)

Using (4.2) the following Greeks can be derived for the regular European up-an-out put option, with the European digital put option Greeks derived in subsection (A.2.3)

Delta

$$\bar{\Delta}_e^{UO} = \bar{\Delta}(\bar{S}) - \frac{2a}{\bar{S}} \left(\frac{\bar{S}}{\bar{B}}\right)^{2a} \bar{P}_e \left(\frac{\bar{B}^2}{\bar{S}}\right) + \left(\frac{\bar{S}}{\bar{B}}\right)^{2a-2} \bar{\Delta} \left(\frac{\bar{B}^2}{\bar{S}}\right). \tag{A.22}$$

Gamma

$$\bar{\Gamma}_e^{UO} = \bar{\Gamma}(\bar{S}) + \frac{2a - 4a^2}{\bar{S}^2} \left(\frac{\bar{S}}{\bar{B}}\right)^{2a} \bar{P}_e \left(\frac{\bar{B}^2}{\bar{S}}\right) + \frac{4a-2}{\bar{S}} \left(\frac{\bar{S}}{\bar{B}}\right)^{2a-2} \bar{\Delta} \left(\frac{\bar{B}^2}{\bar{S}}\right) - \left(\frac{\bar{S}}{\bar{B}}\right)^{2a-4} \bar{\Gamma} \left(\frac{\bar{B}^2}{\bar{S}}\right). \tag{A.23}$$

Theta

$$\bar{\Theta}_e^{UO} = \frac{1}{\alpha(1-h)} \left(\bar{S}^2 \bar{\Gamma}_e^{UO} + \beta \bar{S} \bar{\Delta}_e^{UO} - \alpha \bar{P}_e^{UO}\right). \tag{A.24}$$

Delta Bleed

$$\bar{\Delta}_{\Theta}^{UO} = \bar{\Delta}_{\Theta}(\bar{S}) - \frac{2a}{\bar{S}} \left(\frac{\bar{S}}{\bar{B}}\right)^{2a} \bar{\Theta} \left(\frac{\bar{B}^2}{\bar{S}}\right) + \left(\frac{\bar{S}}{\bar{B}}\right)^{2a-2} \bar{\Delta}_{\Theta} \left(\frac{\bar{B}^2}{\bar{S}}\right). \tag{A.25}$$

A.2.2 Greeks for Reverse Barrier Options ($\bar{B} < 1$)

Using (4.2) the following Greeks can be derived for the regular European up-an-out put option, with the European digital put option Greeks derived in subsection (A.2.3).

Delta

$$\begin{aligned} \bar{\Delta}_e^{UOR} = &\left[\bar{\Delta}\left(\frac{\bar{S}}{\bar{B}}\right) - 2a \left(\frac{\bar{S}}{\bar{B}}\right)^{2a-1} \bar{P}_e \left(\frac{\bar{B}}{\bar{S}}\right) + \left(\frac{\bar{S}}{\bar{B}}\right)^{2a-2} \bar{\Delta}\left(\frac{\bar{B}}{\bar{S}}\right)\right] \\ &- \frac{(1-\bar{B})}{\bar{B}} \left[\bar{\Delta}_d\left(\frac{\bar{S}}{\bar{B}}\right) - 2a \left(\frac{\bar{S}}{\bar{B}}\right)^{2a-1} \bar{P}_d \left(\frac{\bar{B}}{\bar{S}}\right) + \left(\frac{\bar{S}}{\bar{B}}\right)^{2a-2} \bar{\Delta}_d\left(\frac{\bar{B}}{\bar{S}}\right)\right]. \end{aligned} \tag{A.26}$$

Gamma

$$
\begin{aligned}
\bar{\Gamma}_e^{UOR} =& \frac{1}{\bar{B}}\left[\bar{\Gamma}\left(\frac{\bar{S}}{\bar{B}}\right) - 2a(2a-1)\left(\frac{\bar{S}}{\bar{B}}\right)^{2a-2}\bar{P}_e\left(\frac{\bar{B}}{\bar{S}}\right) + (4a-2)\left(\frac{\bar{S}}{\bar{B}}\right)^{2a-3}\bar{\Delta}\left(\frac{\bar{B}}{\bar{S}}\right)\right. \\
&\left. - \left(\frac{\bar{S}}{\bar{B}}\right)^{2a-4}\bar{\Gamma}\left(\frac{\bar{B}}{\bar{S}}\right)\right] - \frac{(1-\bar{B})}{\bar{B}^2}\left[\bar{\Gamma}_d\left(\frac{\bar{S}}{\bar{B}}\right) - 2a(2a-1)\left(\frac{\bar{S}}{\bar{B}}\right)^{2a-2}\bar{P}_d\left(\frac{\bar{B}}{\bar{S}}\right)\right. \\
&\left. + (4a-2)\left(\frac{\bar{S}}{\bar{B}}\right)^{2a-3}\bar{\Delta}_d\left(\frac{\bar{B}}{\bar{S}}\right) - \left(\frac{\bar{S}}{\bar{B}}\right)^{2a-4}\bar{\Gamma}_d\left(\frac{\bar{B}}{\bar{S}}\right)\right].
\end{aligned}
\tag{A.27}
$$

Theta

$$
\bar{\Theta}_e^{UOR} = \frac{1}{\alpha(1-h)}\left(\bar{S}^2\bar{\Gamma}_e^{UOR} + \beta\bar{S}\bar{\Delta}_e^{UOR} - \alpha\bar{P}_e^{UOR}\right). \tag{A.28}
$$

Delta Bleed

$$
\begin{aligned}
\bar{\Delta}_{\Theta e}^{UOR} =& \left[\bar{\Delta}_\Theta\left(\frac{\bar{S}}{\bar{B}}\right) - 2a\left(\frac{\bar{S}}{\bar{B}}\right)^{2a-1}\bar{\Theta}\left(\frac{\bar{B}}{\bar{S}}\right) + \left(\frac{\bar{S}}{\bar{B}}\right)^{2a-2}\bar{\Delta}_\Theta\left(\frac{\bar{B}}{\bar{S}}\right)\right] \\
&- \frac{(1-\bar{B})}{\bar{B}}\left[\bar{\Delta}_{\Theta d}\left(\frac{\bar{S}}{\bar{B}}\right) - 2a\left(\frac{\bar{S}}{\bar{B}}\right)^{2a-1}\bar{\Theta}_d\left(\frac{\bar{B}}{\bar{S}}\right) + \left(\frac{\bar{S}}{\bar{B}}\right)^{2a-2}\bar{\Delta}_{\Theta d}\left(\frac{\bar{B}}{\bar{S}}\right)\right].
\end{aligned}
\tag{A.29}
$$

A.2.3 European Digital Put Option Greeks

For the purpose of determining the European digital put option Greeks, we can rewrite (A.18) using (2.3) and (A.7) as

$$
\bar{P}_d(\bar{S},\tau) = \bar{P}_e(\bar{S},\tau) - \bar{S}\bar{\Delta}_e(\bar{S},\tau), \tag{A.30}
$$

which allows us to trivially derive

$$
\bar{\Delta}_d = -\bar{S}\bar{\Gamma}, \tag{A.31}
$$

$$
\bar{\Gamma}_d = -\bar{\Gamma} - \bar{S}\bar{\Gamma}_\Delta, \tag{A.32}
$$

$$
\bar{\Theta}_d = \frac{1}{\alpha(1-h)}\left(\bar{S}^2\bar{\Gamma}_d + \beta\bar{S}\bar{\Delta}_d - \alpha\bar{P}_d\right), \tag{A.33}
$$

$$
\bar{\Delta}_{\Theta d} = -\bar{S}\bar{\Gamma}_\Theta. \tag{A.34}
$$

A.2.4 American Digital Call Option Greeks

To determine the Greeks of the rebate term (A.19), we need the Greeks of the American digital call option, struck at the barrier.

Delta

$$\begin{aligned}\bar{\Delta}_d^{Am} =& \frac{1}{2}\Bigg[\frac{\lambda_\infty^+}{\bar{S}}\left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^+}\operatorname{erfc}\left(-\frac{\bar{d}^+}{\sqrt{2}}\right)+\frac{\lambda_\infty^-}{\bar{S}}\left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^-}\operatorname{erfc}\left(-\frac{\bar{d}^-}{\sqrt{2}}\right)\\ &+\frac{1}{\sqrt{\pi}\bar{S}\tau^{\frac{1}{2}}}\left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^+}e^{-\frac{(\bar{d}^+)^2}{2}}+\frac{1}{\sqrt{\pi}\bar{S}\tau^{\frac{1}{2}}}\left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^-}e^{-\frac{(\bar{d}^-)^2}{2}}\Bigg].\end{aligned} \tag{A.35}$$

Gamma

$$\begin{aligned}\bar{\Gamma}_d^{Am} =& \frac{1}{2}\Bigg[\frac{\lambda_\infty^+(\lambda_\infty^+-1)}{\bar{S}^2}\left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^+}\operatorname{erfc}\left(-\frac{\bar{d}^+}{\sqrt{2}}\right)+\frac{\lambda_\infty^-(\lambda_\infty^--1)}{\bar{S}^2}\left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^-}\operatorname{erfc}\left(-\frac{\bar{d}^-}{\sqrt{2}}\right)\\ &+\frac{1}{\sqrt{\pi}\bar{S}^2\tau^{\frac{1}{2}}}\left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^+}e^{-\frac{(\bar{d}^+)^2}{2}}\left(2\lambda_\infty^+-1+\frac{\bar{d}^+}{\sqrt{2}\tau^{\frac{1}{2}}}\right)\\ &+\frac{1}{\sqrt{\pi}\bar{S}^2\tau^{\frac{1}{2}}}\left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^-}e^{-\frac{(\bar{d}^-)^2}{2}}\left(2\lambda_\infty^--1+\frac{\bar{d}^-}{\sqrt{2}\tau^{\frac{1}{2}}}\right)\Bigg].\end{aligned} \tag{A.36}$$

Theta

$$\bar{\Theta}_d^{Am} = \frac{1}{\alpha(1-h)}\left(\bar{S}^2\bar{\Gamma}_d^{Am}+\beta\bar{S}\bar{\Delta}_d^{Am}-\alpha\bar{P}_d^{Am}\right). \tag{A.37}$$

Delta Bleed

$$\begin{aligned}\bar{\Delta}_{\Theta d}^{Am} =& \frac{1}{2\alpha(1-h)}\Bigg[\frac{1}{\sqrt{\pi}\bar{S}}\left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^+}e^{-\frac{(\bar{d}^+)^2}{2}}\left(\left(\frac{\bar{d}^+}{2\tau^{\frac{1}{2}}}-\lambda_\infty^+\right)\left(\frac{\bar{d}^+}{\sqrt{2}\tau}-\frac{\lambda_+^\infty-\lambda_-^\infty}{\tau^{\frac{1}{2}}}\right)-\frac{1}{2\tau^{\frac{3}{2}}}\right)\\ &+\frac{1}{\sqrt{\pi}\bar{S}}\left(\frac{\bar{S}}{\bar{B}}\right)^{\lambda_\infty^-}e^{-\frac{(\bar{d}^-)^2}{2}}\left(\left(\frac{\bar{d}^-}{\tau^{\frac{1}{2}}}-\lambda_\infty^-\right)\left(\frac{\bar{d}^-}{\sqrt{2}\tau}+\frac{\lambda_+^\infty-\lambda_-^\infty}{\tau^{\frac{1}{2}}}\right)-\frac{1}{2\tau^{\frac{3}{2}}}\right)\Bigg].\end{aligned} \tag{A.38}$$

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, 1972.

[2] J. A. Addison, S. D. Howison, and J. R. King, *Ray methods for free boundary problems*, Quarterly of Applied Mathematics, 64 (2006), pp. 41–59.

[3] F. AitSahlia, L. Imhof, and T. L. Lai, *Fast and accurate valuation of American barrier options*, Journal of Computational Finance, 7 (2003), pp. 129–145.

[4] W. Allegretto, G. Barone-Adesi, and R. J. Elliot, *Numerical evaluation of the critical price and American options*, The European Journal of Finance, 1 (1995), pp. 69–78.

[5] G. Alobaidi and R. Mallier, *Asymptotic analysis of American call options*, Journal of Mathematics and Mathematical Sciences, 27 (2001), pp. 177–188.

[6] ——, *The American put option close to expiry*, Acta Mathematica Universitatis Comenianae, 2 (2004), pp. 161–174.

[7] G. Barles, *Critical stock price near expiration*, Mathematical Finance, 5 (1995), pp. 77–95.

[8] G. Barone-Adesi, *The saga of the American put*, Journal of Banking and Finance, 29 (2005), pp. 2909–2918.

[9] G. Barone-Adesi and R. J. Elliot, *Approximations for the values of American options*, Stochastic Analysis and Applications, 9 (1991), pp. 115–131.

[10] G. Barone-Adesi and R. E. Whaley, *Efficient analytical approximation of American option values*, Journal of Finance, 42 (1987), pp. 301–320.

[11] M. Baxter and A. Rennie, *Financial Calculus: An Introduction to Derivative Pricing*, Cambridge University Press, 2003.

[12] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, Springer, 1999.

[13] A. Bensoussan, *On the theory of option pricing*, Acta Applicandae Mathematicae, 2 (1984), pp. 139–158.

[14] P. Bjerksund and G. Stensland, *American exchange options and a put-call transformation: A note*, Journal of Business Finance and Accounting, 20 (1993), pp. 761–764.

[15] ——, *Closed form approximations of American options*, Scandinavian Journal of Management, 9 (1993), pp. 87–99.

[16] T. Bjork, *Arbitrage Theory in Continuous Time*, Oxford Univeristy Press, 1998.

[17] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy, 81 (1973), pp. 637–659.

[18] A. Blanchet, *On the regularity of the free boundary in the parabolic obstacle problem. application to American options*, Nonlinear Analysis, 65 (2006), pp. 1362–1378.

[19] E. C. Blomeyer, *An analytical approximation for the American put price for options on stocks with dividends*, Journal of Financial and Quantitative Analysis, 21 (1986), pp. 229–233.

[20] P. Boyle and S. Turnbull, *Pricing and hedging capped options*, Journal of Futures Markets, 9 (1989), pp. 41–54.

[21] M. J. Brennan and E. S. Schwartz, *The valuation of American put options*, Journal of Finance, 32 (1977), pp. 449–462.

[22] ——, *Finite difference methods and jump processes arising in the pricing of contingent clains: A synthesis*, Journal of Financial Quantitative Analysis, 13 (1978), pp. 461–474.

[23] M. Broadie and J. Detemple, *American capped call options on dividend paying assets*, Review of Financial Studies, 8 (1995), pp. 161–190.

[24] ——, *American option valuation: New bounds, approximations, and a comparison of existing methods*, Review of Financial Studies, 9 (1996), pp. 1211–1250.

[25] ——, *Option pricing: Valuation models and applications*, Management Science, 50 (2004), pp. 1145–1177.

[26] P. Buchan, *Pricing European barrier options*, Preprint.

[27] D. S. Bunch and H. E. Johnson, *A simple and numerically efficient valuation method for American puts using a modified Geske-Johnson approach*, Journal of Finance, 47 (1992), pp. 809–816.

[28] P. Carr, R. Jarrow, and R. Myeni, *Alternative characteristics of American put options*, Mathematical Finance, 2 (1992), pp. 87–106.

[29] X. Chen and J. Chadam, *Analytical and numerical approximations for the early exercise boundary for American options*, Dynamics of Continuous, Discrete and Impulsive Systems, 10 (2003), pp. 649–660.

[30] ——, *A mathematical analysis of the optimal exercise boundary for American put options*, SIAM Journal of Mathematical Analysis, 38 (2006), pp. 1613–1641.

[31] X. Chen, J. Chadam, L. Jiang, and W. Zheng, *Convexity of the exercise boundary of the American put option on a zero dividend asset*, Mathematical Finance, 18 (2008), pp. 185–197.

[32] J. Cox, *Notes on option pricing I: Constant elasticity of variance diffusions.* Working Paper, Stanford University (reprinted in the *Journal of Computational Finance*, 22:15-17,1996).

[33] J. Cox and S. Ross, *The valuation of options for alternative stochastic processes*, Journal of Financial Economics, 3 (1976), pp. 145–166.

[34] J. Cox, S. Ross, and M. Rubinstein, *Option pricing: A simplified approach*, Journal of Financial Economics, 7 (1979), pp. 229–263.

[35] J. Crank and R. S. Gupta, *A moving boundary problem arising from the diffusion of oxygen in absorbing tissue*, Journal of the Institute of Mathematics and its Applications, 10 (1972), pp. 19–33.

[36] J. Crank and P. Nicolson, *A practical method for numerical evaluation of solutions of partial differential equations of the heat conduction type*, Proceedings of the Cambridge Philosophical Society, 43 (1947), pp. 50–64.

[37] D. Davidov and V. Linetsky, *Pricing and hedging path-dependent options under the CEV process*, Management Science, 47 (2001), pp. 949–965.

[38] J. N. Dewynne, S. D. Howison, I. Rupf, and P. Wilmott, *Some mathematical results in the pricing of American options*, Journal of Applied Mathematics, 4 (1993), pp. 381–398.

[39] D. J. Duffy, *Financial Instrument Pricing Using C++*, Wiley, 2004.

[40] E. Ekström, *Convexity of the optimal stopping boundary for the american put option*, Journal of Mathematical Analysis and Applications, 299 (2004), pp. 147–156.

[41] J. D. Evans, R. A. Kuske, and J. B. Keller, *American options on assets with dividends near expiry*, Mathematical Finance, 12 (2002), pp. 219–237.

[42] N. P. Firth, J. N. Dewynne, and S. J. Chapman, *An asymptotic analysis of an American call option with small volatility.* OFRC Working Papers Series 2004mf03. Available at http://ideas.repec.org/p/sbs/wpsefe/2004mf03.html.

[43] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hill, 1964.

[44] ——, *Variational Principles and Free Boundary Problems*, Wiley, 1982.

[45] V. A. Galaktionov, *Geometric Sturmian Theory of Nonlinear Parabolic Equations and Applications*, Chapman and Hall/CRC, 2004.

[46] M. Garman, *Charm school*, Risk Magazine, 5 (1992), pp. 53–56.

[47] R. Geske, *A note on an analytical valuation formula for unprotected American options on stocks with known dividends*, Journal of Financial Economics, 7 (1979), pp. 375–380.

[48] ——, *The valuation of compound options*, Journal of Financial Economics, 7 (1979), pp. 63–81.

[49] R. Geske and H. Johnson, *The American put option valued analytically*, Journal of Finance, 39 (1984), pp. 1511–1524.

[50] P. Glasserman, *Monte Carlo Methods in Financial Engineering*, Springer, 2000.

[51] J. Goodman and D. N. Ostrov, *Early exercise boundary of the American put option*, Journal of Applied Mathematics, 2 (2002), pp. 71–92.

[52] J. M. Harrison and D. M. Kreps, *Martingales and arbitrage in multiperiod security markets*, Journal of Economic Theory, 20 (1979), pp. 381–408.

[53] J. M. Harrison and S. R. Pliska, *Martingales and stochastic integrals in the theory of continuous trading*, Stochastic Processes and their Applications, 11 (1981), pp. 215–260.

[54] E. Haug, *Complete Guide to Option Pricing Formulas*, McGraw and Hill, 1998.

[55] ———, *Know your weapon part I*, Wilmott, May (2003), pp. 49–57.

[56] ———, *Know your weapon part II*, Wilmott, (2003), pp. 50–56.

[57] E. J. Hinch, *Perturbation Methods*, Cambridge University Press, 1991.

[58] T. S. Ho, R. C. Stapleton, and M. G. Subrahmanyam, *The valuation of American options with stochastic interest rates: A generalisation of the Geske-Johnson formula*, Journal of Finance, 52 (1997), pp. 827–840.

[59] S. D. Howison, *Applied maths and finance*, Philosophical Transactions of the Royal Society A, 347 (1994), pp. 465–470.

[60] ———, *Matched asymptotic expansions in financial engineering*, Journal of Engineering Mathematics, 53 (2005), pp. 385–406.

[61] S. D. Howison and M. Steinberg, *A matched asymptotic expansions approach to continuity corrections for discretely sampled options. part 1: Barrier options*, Applied Mathematical Finance, 14 (2007), pp. 63–89.

[62] J. Hull, *Options, Futures, and Other Derivatives*, Prentice Hall, 2003.

[63] P. J. Hunt and J. E. Kennedy, *Financial Derivatives in Theory and Practice*, Wiley, 2000.

[64] A. Ibáñez, *Robust pricing of the American put option: A note on Richardson extrapolation and the early exercise premium*, Management Science, 49 (2003), pp. 1210–1228.

[65] S. D. Jacka, *Optimal stopping and the American put*, Mathematical Finance, 1 (1991), pp. 1–14.

[66] H. E. Johnson, *An analytical approximation for the American put price*, Journal of Financial and Quantitative Analysis, 18 (1983), pp. 141–148.

[67] M. Joshi, *The Concepts and Practices of Mathematical Finance*, Cambridge University Press, 2003.

[68] N. Ju, *Pricing an American option by approximating its early exercise boundary as a multipiece exponential function*, The Review of Financial Studies, 11 (1998), pp. 627–646.

[69] N. Ju and R. Zhong, *An approximate formula for pricing American options*, Journal of Derivatives, 42 (1999), pp. 301–320.

[70] I. Karatzas, *On the pricing of American options*, Applied Mathematics and Optimization, 17 (1988), pp. 37–60.

[71] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer Verlag, 1997.

[72] I. J. Kim, *The analytical valuation of American options*, Review of Financial Studies, 3 (1990), pp. 547–572.

[73] C. Knessl, *A note on the moving boundary problem arising in the American put option*, Studies in Applied Mathematics, 107 (2001), pp. 157–183.

[74] ———, *Asymptotic analysis of the American call option with dividends*, European Journal of Applied Mathematics, 13 (2002), pp. 587–616.

[75] R. A. Kuske and J. B. Keller, *Optimal exercise boundary for an American put option*, Applied Mathematical Finance, 5 (1998), pp. 107–116.

[76] D. Lamberton and S. Villeneuve, *Critical price near maturity for an American option on a dividend paying stock*, The Annals of Applied Probability, 13 (2003), pp. 800–815.

[77] H. G. Landau, *Heat conduction in a melting solid*, Quarterly of Applied Mathematics, 8 (1950), pp. 81–94.

[78] F. A. Longstaff and E. S. Schwartz, *Valuing American options by simulation: A simple least squares approach*, The Review of Financial Studies, 14 (2001), pp. 113–147.

[79] L. W. MacMillan, *Analytical approximation for the American put option*, Advances in Futures and Options Research, 1 (1986), pp. 119–139.

[80] R. Mallier, *Approximations for American options*, Journal of Applied Mathematics, 2 (2002), pp. 71–92.

[81] W. Margrabe, *The value of an option to exchange one asset for another*, Journal of Finance, 33 (1978), pp. 177–186.

[82] R. D. McDonald and M. D. Schroder, *A parity result for American options*, Journal of Computational Finance, 1 (1998), pp. 5–13.

[83] H. P. McKean, *Appendix: A free boundary problem for the heat equation arising from a problem in mathematical finance*, Industrial Management Review, 6 (1965), pp. 32–39.

[84] R. C. Merton, *Theory of rational option pricing*, Bell Journal of Economics and Management Science, 4 (1973), pp. 141–183.

[85] G. H. Meyer and J. van der Hoek, *The evaluation of American options with the method of lines*, Advances in Futures and Options Research, 9 (1997), pp. 265–285.

[86] R. Myneni, *The pricing of the American option*, The Annals of Applied Probability, 2 (1992), pp. 1–23.

[87] J. Ockendon, S. Howison, A. Lacey, and A. Movchan, *Applied Partial Differential Equations*, Oxford University Press, 1999.

[88] M. Parkinson, *Option pricing: The American put*, Journal of Business, 50 (1977), pp. 21–36.

[89] G. Peskir, *On the American option problem*, Mathematical Finance, 15 (2005), pp. 169–181.

[90] H. T. H. Piaggio, *An Elementary Treatise on Differential Equations and their Applications*, Bell, 1971.

[91] A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux, *Handbook of First Order Partial Differential Equations*, Taylor and Francis, 2002.

[92] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C++: The Art of Scientific Computing*, Cambridge University Press, 2002.

[93] R. Roll, *Call option values for stocks with known dividends*, Journal of Financial Economics, 5 (1977), pp. 251–258.

[94] M. Rubenstein and Reiner, *Breaking down the barriers*, Risk, 4 (1991), pp. 28–35.

[95] P. Samuelson, *Rational theory of warrant pricing*, Industrial Management Review, 6 (1965), pp. 13–32.

[96] R. J. Schiller, *http://www.econ.yale.edu/ shiller/*.

[97] R. Stamicar, D. Ševčovič, and J. Chadam, *The early exercise boundary for the American put option near expiry: Numerical approximation*, Canadian Applied Mathematics Quarterly, 7 (1999), pp. 427–444.

[98] N. N. Taleb, *Dynamic Hedging*, Wiley, 1997.

[99] A. B. Tayler, *Mathematical Models in Applied Mechanics*, Oxford Univeristy Press, 1996.

[100] P. van Moerbeke, *On optimal stopping and free boundary problems*, Archive for Rational Mechanics and Analysis, 60 (1976), pp. 101–148.

[101] R. E. Whaley, *On the valuation of American call options with known dividends*, Journal of Financial Economics, 9 (1981), pp. 207–211.

[102] M. Widdicks, P. Duck, A. Andricopoulos, and D. P. Newton, *The Black-Scholes equation revisited: Asymptotic expansions and singular perturbations*, Mathematical Finance, 15 (2005), pp. 373–391.

[103] P. Wilmott, *Paul Wilmott on Quantitative Finance*, Wiley, 2003.

[104] L. Wu and Y. K. Kwok, *A front fixing finite difference method for the valuation of American options*, Journal of Financial Engineering, 6 (1997), pp. 83–97.

[105] J. zhi Huang, M. G. Subrahmanyam, and G. Yu, *Pricing and hedging American options: A recursive integration method*, Review of Financial Studies, 9 (1996), pp. 277–300.

[106] S. Zhu, *An exact and explicit solution for the valuation of American put options*, Quantitative Finance, 6 (2006), pp. 229–242.