The Elastic Limit of Interpolated Maxwell Models for Fluids with Memory

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1 Introduction

We begin by looking at the governing equations for fluid motion. These are derived from the balance laws of conservation of mass and conservation of momentum. Conservation of mass gives rise to the continuity equation which reads:

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0 \tag{1}$$

Where ρ is the density, $\mathbf{v}$ is the velocity and $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)$. For an incompressible fluid, which is the case we shall study, $\frac{D\rho}{Dt} = 0$ and so the continuity equation reduces to:

$$\nabla\cdot\mathbf{v} = 0 \tag{2}$$

Further, by the conservation of linear momentum we recover the equation:

$$\rho\frac{D\mathbf{v}}{Dt} = \nabla\cdot\sigma + \rho\mathbf{F} \tag{3}$$

Where $\mathbf{F}$ is the body force acting on the fluid and σ is the stress tensor. Everything else is defined as before. We are going to assume the absence of a body force ($\mathbf{F} = \mathbf{0}$). σ is given by:

$$\sigma = -p\mathbf{I} + \mathbf{T} \tag{4}$$

Where p is the pressure, $I_{ij} = \delta_{ij}$ and $\mathbf{T}$ is the extra stress tensor. Conservation of angular momentum does not give us another equation, but we do recover the condition that the stress tensor is symmetric, i.e.

$$\sigma^T = \sigma \tag{5}$$

It then follows that the extra stress tensor $\mathbf{T}$ is also symmetric. In component form, the equation reduces to:

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{x_j}\right) = \frac{\partial\sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_j}\delta_{ij} + \frac{\partial T_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial T_{ij}}{\partial x_j} \quad i = 1,2,3 \tag{6}$$

Where the summation convention has been used. In vector form this reads:

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \nabla\cdot\mathbf{T} \tag{7}$$

In order to solve these equations we need additional constitutive relations that relate the extra stress tensor to the motion. In the interpolated Maxwell models this relation takes the following form:

$$\mathbf{T} = \mathbf{T}^P + \mathbf{T}^S \tag{8}$$

Here $\mathbf{T}^P$ is polymer stress and $\mathbf{T}^S$ is solvent stress which is newtonian like. The maxwell models treat the fluid as strings of polymers lubricated by solvent. Further information may be gained from the relations for $\mathbf{T}^P$ and $\mathbf{T}^S$ which read[1]:

$$\mathbf{T}^S = 2\eta_S\mathbf{D} \tag{9}$$

$$\mathbf{T}^P + \lambda_P \overset{\square}{\mathbf{T}^{\mathbf{P}}} = 2\eta_P\mathbf{D} \tag{10}$$

Where $\mathbf{D} = \frac{1}{2}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right)$ is the rate of strain tensor (the symmetric part of the velocity gradient), η_S and η_P are the solvent and polymer viscosities respectively and λ_P is the relaxation time and can be interpreted as a measure of the strength of the fluid's memory. $\overset{\square}{\mathbf{T}^{\mathbf{P}}}$ is given by:

$$\overset{\square}{\mathbf{T}^{\mathbf{P}}} = \frac{\partial\mathbf{T}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{T}^P - (\mathbf{W} + a\mathbf{D})\mathbf{T}^P + \mathbf{T}^P(\mathbf{W} - a\mathbf{D}) \tag{11}$$

Where $\mathbf{W} = \frac{1}{2}\left(\nabla\mathbf{v} - (\nabla\mathbf{v})^T\right)$ (the anti-symmetric part of the velocity gradient), and $-1 \leq a \leq 1$ is known as the 'slip parameter' and accounts for non-affine deformation of the fluid. $a = 1$ represents affine motion, and deviation from this value measures deviation from affine motion[11].

1.1 Typical Parameter Values

The most commonly used values for a are $-1, 0, 1$. These give rise to the Lower Convected, Co-rotational and Upper Convected Maxwell models respectively. The other parameters naturally will vary from fluid to fluid. Below is a list of actual viscoelastic fluids and their parameter values[1].

Fluid	Density (kg/m^3)	Viscosity ($Pa \cdot s$)	Relaxation Time ($10^{-3}s$)
Glycerol	1255	0.69	0.12
Olive Oil	914	0.06	0.65
Soybean Oil	922	0.046	1.26
SAE 30 Motor Oil	886	0.098	0.19
Honey	1400	249	0.97
Cherry Jam (Pillsbury)	1090	474	0.32

1.2 Non-dimensionalisation

We seek to non-dimensionalise, we have the governing equations:

$$\nabla \cdot \mathbf{v} = 0 \qquad \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nabla \cdot \mathbf{T} \qquad \mathbf{T} = \mathbf{T}^P + \mathbf{T}^S$$

$$\mathbf{T}^S = 2\eta_S \mathbf{D} \qquad \mathbf{T}^P + \lambda_P \overset{\square}{\mathbf{T}^P} = 2\eta_P \mathbf{D}$$

The density ρ, solvent and polymer viscosities η_S, η_P and relaxation time λ_P are assumed to be constant. We can now nondimensionalise as follows[5]:

$$\mathbf{x} = L\bar{\mathbf{x}}, \qquad \mathbf{v} = U\bar{\mathbf{v}}, \qquad p = \tfrac{\eta U}{L}\bar{p}, \qquad \mathbf{T} = \tfrac{\eta U}{L}\bar{\mathbf{T}}, \qquad \mathbf{T}^S = \tfrac{\eta U}{L}\bar{\mathbf{T}}^S, \qquad \mathbf{T}^P = \tfrac{\eta U}{L}\bar{\mathbf{T}}^P$$

Where L and U are characteristic length and flow speed resp. and $\eta = \eta_S + \eta_P$. We thus obtain:

$$\bar{\nabla} \cdot \bar{\mathbf{v}} = 0, \qquad \mathrm{Re}\bar{\mathbf{v}} \cdot \bar{\nabla}\bar{\mathbf{v}} = -\bar{\nabla}\bar{p} + \bar{\nabla} \cdot \bar{\mathbf{T}}, \qquad \bar{\mathbf{T}} = \bar{\mathbf{T}}^S + \bar{\mathbf{T}}^P$$

$$\bar{\mathbf{T}}^S = 2\beta\bar{\mathbf{D}}, \qquad \bar{\mathbf{T}}^P + \mathrm{We}\,\overset{\square}{\bar{\mathbf{T}}^P} = 2(1-\beta)\bar{\mathbf{D}}$$

Where $\mathrm{Re} = \frac{\rho U L}{\eta}$ is the Reynolds number, $\mathrm{We} = \frac{\lambda_P U}{L}$ is the Weissenberg number and $\beta = \frac{\eta_S}{\eta_S + \eta_P}$. The solutions to the non-dimensional equations depend on these three dimensionless parameters, the original equations have four dimensional parameters and each solution of these corresponds to a solution of the non-dimensional equations, and different cases in the dimensional case correspond to a single solution in the non-dimensionalised case. This means it is easier to investigate the effects of the parameters and also easier to design experiments to verify the predictions of the model. For the rest of the report we will use the non-dimensionalised equations and drop the bars. In component form the equations become:

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{12}$$

$$\mathrm{Re} v_k \frac{\partial v_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \frac{\partial T_{ik}}{\partial x_k} \tag{13}$$

$$T_{ij} = T_{ij}^P + T_{ij}^S \tag{14}$$

$$T_{ij}^S = 2\beta D_{ij} \tag{15}$$

$$T_{ij}^P + \mathrm{We}\left(\frac{\partial T_{ij}^P}{\partial t} + v_k \frac{\partial}{\partial x_k} T_{ij}^P - \left(\frac{a+1}{2}\frac{\partial v_i}{\partial x_k} + \frac{a-1}{2}\frac{\partial v_k}{\partial x_i}\right) T_{kj}^P - T_{ik}^P \left(\frac{a-1}{2}\frac{\partial v_k}{\partial x_j} + \frac{a+1}{2}\frac{\partial v_j}{\partial x_k}\right)\right) = (1-\beta)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{16}$$

2 Cartesian and Natural Stress Formulations

We will now consider steady planar flow. That is, there is no time dependence and we work in two dimensions. The momentum and constitutive equations may be written in the form:

$$\mathrm{Re}\,\mathbf{v}\cdot\nabla u = -\frac{\partial p}{\partial x} + \frac{\partial T_{11}}{\partial x} + \frac{\partial T_{12}}{\partial y} \tag{17}$$

$$\mathrm{Re}\,\mathbf{v}\cdot\nabla v = -\frac{\partial p}{\partial y} + \frac{\partial T_{12}}{\partial x} + \frac{\partial T_{22}}{\partial y} \tag{18}$$

and

$$T_{11} + \mathrm{We}\left(u\frac{\partial T_{11}}{\partial x} + v\frac{\partial T_{11}}{\partial y} - 2a\frac{\partial u}{\partial x}T_{11} - (a-1)\frac{\partial v}{\partial x}T_{12} - (a+1)\frac{\partial u}{\partial y}T_{12}\right) = 2(1-\beta)\frac{\partial u}{\partial x} \tag{19}$$

$$T_{22} + \mathrm{We}\left(u\frac{\partial T_{22}}{\partial x} + v\frac{\partial T_{22}}{\partial y} - 2a\frac{\partial v}{\partial y}T_{22} - (a-1)\frac{\partial u}{\partial y}T_{12} - (a+1)\frac{\partial v}{\partial x}T_{12}\right) = 2(1-\beta)\frac{\partial v}{\partial y} \tag{20}$$

$$T_{12} + \mathrm{We}\left(u\frac{\partial T_{12}}{\partial x} + v\frac{\partial T_{12}}{\partial y} - \left(\frac{a+1}{2}\frac{\partial u}{\partial y} + \frac{a-1}{2}\frac{\partial v}{\partial x}\right)T_{22} - \left(\frac{a-1}{2}\frac{\partial u}{\partial y} + \frac{a+1}{2}\frac{\partial v}{\partial x}\right)T_{11}\right) = (1-\beta)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{21}$$

The velocity field is given by:

$$\mathbf{v} = (u, v) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right) \tag{22}$$

where ψ is the stream function. We can also express the stress tensor with respect to a natural stress basis which is spanned by the velocity field and its orthogonal counterpart. The vector $\mathbf{w}$ given by:

$$\mathbf{w} = (w_1, w_2) = \left(-\frac{v}{u^2+v^2}, \frac{u}{u^2+v^2}\right) \tag{23}$$

is orthogonal to $\mathbf{v}$ (it is easy to check that $\mathbf{v}\cdot\mathbf{w} = 0$) and also satisfies $|\mathbf{v}\times\mathbf{w}| = 1$. We can then express $\mathbf{T}^P$ using dyadic products of $\mathbf{v}$ and $\mathbf{w}$ as follows[2]:

$$\mathbf{T}^P = -\frac{1-\beta}{\mathrm{We}}\mathbf{I} + \lambda\mathbf{v}\mathbf{v}^T + \mu(\mathbf{v}\mathbf{w}^T + \mathbf{w}\mathbf{v}^T) + \nu\mathbf{w}\mathbf{w}^T \tag{24}$$

In component form the Cartesian and natural stress bases are related as follows:

$$T_{11} = -\frac{(1-\beta)}{\mathrm{We}} + \lambda u^2 - \frac{2\mu uv}{u^2+v^2} + \frac{\nu v^2}{(u^2+v^2)^2} \tag{25}$$

$$T_{12} = \lambda uv + \frac{\mu(u^2-v^2)}{u^2+v^2} - \frac{\nu uv}{(u^2+v^2)^2} \tag{26}$$

$$T_{22} = -\frac{(1-\beta)}{\mathrm{We}} + \lambda v^2 + \frac{2\mu uv}{u^2+v^2} + \frac{\nu u^2}{(u^2+v^2)^2} \tag{27}$$

Under this change of variables, equations (19), (20) and (21) become:

$$\lambda + \mathrm{We}\mathbf{v}\cdot\nabla\lambda + (a+1)\mathrm{We}\mu\nabla\cdot\mathbf{w} - \frac{(a-1)}{|\mathbf{v}|^2}\mathrm{We}\lambda\mathbf{v}\cdot\nabla(|\mathbf{v}|^2) = \frac{(1-\beta)}{|\mathbf{v}|^2}\left(\frac{1}{\mathrm{We}} - \frac{(a-1)}{|\mathbf{v}|^2}\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\right) \tag{28}$$

$$\mu + \mathrm{We}\mathbf{v}\cdot\nabla\mu + \left(\frac{(a-1)}{2}\lambda\mathrm{We}|\mathbf{v}|^4 + \frac{(a+1)}{2}\nu\mathrm{We}\right)\nabla\cdot\mathbf{w} = (1-\beta)(a-1)|\mathbf{v}|^2\nabla\cdot\mathbf{w} \tag{29}$$

$$\nu + \mathrm{We}\mathbf{v}\cdot\nabla\nu + \mathrm{We}(a-1)\left(\mu|\mathbf{v}|^4\nabla\cdot\mathbf{w} - \frac{\nu}{|\mathbf{v}|^2}\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\right) = (1-\beta)\left(\frac{|\mathbf{v}|^2}{\mathrm{We}} - (a-1)\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\right) \tag{30}$$

Or alternatively, stated in order to make the effects of the slip parameter a clearer:

$$\lambda + \mathrm{We}\mathbf{v}\cdot\nabla\lambda + 2\mathrm{We}\mu\nabla\cdot\mathbf{w} + (a-1)\left(\frac{1}{|\mathbf{v}|^4}\left((1-\beta) - \mathrm{We}\lambda|\mathbf{v}|^2\right)\mathbf{v}\cdot\nabla(|\mathbf{v}|^2) + \mathrm{We}\mu\nabla\cdot\mathbf{w}\right) = \frac{1-\beta}{\mathrm{We}}\frac{1}{|\mathbf{v}|^2} \tag{31}$$

$$\mu + \mathrm{We}\mathbf{v}\cdot\nabla\mu + \mathrm{We}\nu\nabla\cdot\mathbf{w} + (a-1)\left(\frac{\lambda\mathrm{We}}{2}|\mathbf{v}|^4 - (1-\beta)|\mathbf{v}|^2 + \frac{\mathrm{We}\nu}{2}\right)\nabla\cdot\mathbf{w} = 0 \tag{32}$$

$$\nu + \mathrm{We}\mathbf{v}\cdot\nabla\nu + (a-1)\left(\mathrm{We}\mu|\mathbf{v}|^4\nabla\cdot\mathbf{w} + \left((1-\beta) - \frac{\nu\mathrm{We}}{|\mathbf{v}|^2}\right)(\mathbf{v}\cdot\nabla(|\mathbf{v}|^2))\right) = \frac{1-\beta}{\mathrm{We}}|\mathbf{v}|^2 \tag{33}$$

Here it is very easy to see that for the case $a = 1$ the above equations agree with published results for the UCM model[2]. Where $\nabla\cdot\mathbf{w}$ is given by:

$$\nabla\cdot\mathbf{w} = \frac{1}{|\mathbf{v}|^4}\left((v^2 - u^2)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + 4uv\frac{\partial u}{\partial x}\right) \tag{34}$$

And the momentum equations 17 and 18 become:

$$\mathrm{Re}\mathbf{v}\cdot\nabla u = -\frac{\partial p}{\partial x} + \mathbf{v}\cdot\nabla(\lambda u) + \left(\mu u\mathbf{w} - (\mu\mathbf{v} + \nu\mathbf{w})\frac{v}{|\mathbf{v}|^2}\right) \tag{35}$$

$$\mathrm{Re}\mathbf{v}\cdot\nabla v = -\frac{\partial p}{\partial y} + \mathbf{v}\cdot\nabla(\lambda v) + \left(\mu v\mathbf{w} + (\mu\mathbf{v} + \nu\mathbf{w})\frac{u}{|\mathbf{v}|^2}\right) \tag{36}$$

We can solve (25) - (27) simultaneously to get expressions for λ, μ and ν:

$$\lambda = \frac{1}{\mathrm{We}(u^2+v^2)^2}((1-\beta)(u^2+v^2) + \mathrm{We}(u^2T_{11} + 2uvT_{12} + v^2T_{22})) \tag{37}$$

$$\mu = \frac{1}{(u^2+v^2)}((u^2 - v^2)T_{12} + uv(T_{22} - T_{11})) \tag{38}$$

$$\nu = \frac{1}{\mathrm{We}}((1-\beta)(u^2+v^2) + \mathrm{We}(u^2T_{22} - 2uvT_{12} + v^2T_{11})) \tag{39}$$

We note these for use later.

3 Viscometric Behaviour

We begin the analysis by considering a simple shear flow with constant shear rate $\dot{\gamma}$, i.e. the velocity is given by $\mathbf{v} = (\dot{\gamma}y, 0)$. We thus have:

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \qquad \nabla\mathbf{v} = \begin{pmatrix} 0 & \dot{\gamma} \\ 0 & 0 \end{pmatrix} \tag{40}$$

and so the constitutive equations become:

$$T_{11} - \dot{\gamma}\mathrm{We}(a+1)T_{12} = 0 \tag{41}$$

$$T_{12} - \frac{\dot{\gamma}\mathrm{We}}{2}((a+1)T_{22} + (a-1)T_{11}) = \dot{\gamma}(1-\beta) \tag{42}$$

$$T_{22} - \dot{\gamma}\mathrm{We}(a-1)T_{12} = 0 \tag{43}$$

The cases $a = \pm 1$ will require a little special care. First the case $a = -1$ which corresponds to the lower convected Maxwell model. Here the above equations become:

$$T_{11} = 0 \tag{44}$$

$$T_{12} = \dot{\gamma}(1-\beta) \tag{45}$$

$$T_{22} = -2(1-\beta)\dot{\gamma}^2\mathrm{We} \tag{46}$$

For $\dot{\gamma} = O(1)$ we have that $T_{11} = 0$, $T_{12} = O(1)$ and $T_{22} = O(\mathrm{We})$. If, on the other hand, we consider the case $a = 1$ corresponding to the upper convected Maxwell model, we find the following:

$$T_{11} = 2(1-\beta)\dot{\gamma}^2\mathrm{We} \tag{47}$$

$$T_{12} = \dot{\gamma}(1-\beta) \tag{48}$$

$$T_{22} = 0 \tag{49}$$

So we have that $T_{11} = O(\mathrm{We})$, $T_{12} = O(1)$ and $T_{22} = 0$. Having dealt with these two cases, we can consider a more general case $-1 < a < 1$. This is the case we will be most interested as analysis has already been done for the upper and lower convected Maxwell models. For this case we have:

$$T_{11} = \frac{\dot{\gamma}^2\mathrm{We}(a+1)(1-\beta)}{1-\dot{\gamma}^2\mathrm{We}^2(a^2-1)} \tag{50}$$

$$T_{22} = \frac{\dot{\gamma}^2\mathrm{We}(a-1)(1-\beta)}{1-\dot{\gamma}^2\mathrm{We}^2(a^2-1)} \tag{51}$$

$$T_{12} = \frac{\dot{\gamma}(1-\beta)}{1-\dot{\gamma}^2\mathrm{We}^2(a^2-1)} \tag{52}$$

So we can deduce that $T_{11} = O(\mathrm{We}^{-1})$, $T_{12} = O(\mathrm{We}^{-2})$ and $T_{22} = O(\mathrm{We}^{-1})$. We can actually do slightly better than this by rewriting our expressions for T_{11}, T_{12} and T_{22} in the following manner:

$$T_{11} = \frac{-(1-\beta)}{(a-1)\mathrm{We}}\left[1 - \frac{1}{\dot{\gamma}^2(a^2-1)\mathrm{We}^2}\right]^{-1} \tag{53}$$

$$T_{22} = \frac{-(1-\beta)}{(a+1)\mathrm{We}}\left[1 - \frac{1}{\dot{\gamma}^2(a^2-1)\mathrm{We}^2}\right]^{-1} \tag{54}$$

$$T_{12} = \frac{-(1-\beta)}{\dot{\gamma}\mathrm{We}^2(a^2-1)}\left[1 - \frac{1}{\dot{\gamma}^2(a^2-1)\mathrm{We}^2}\right]^{-1} \tag{55}$$

Doing this is only valid for $a \neq \pm 1$ else we will have a zero in the denominator of each expression. We can then expand the square brackets to obtain the following expressions:

$$T_{11} = \frac{-(1-\beta)}{(a-1)\mathrm{We}}\left[1 + \frac{1}{\dot{\gamma}^2(a^2-1)\mathrm{We}^2} + O\left(\frac{1}{\mathrm{We}^4}\right)\right] \tag{56}$$

$$T_{22} = \frac{-(1-\beta)}{(a+1)\mathrm{We}}\left[1 + \frac{1}{\dot{\gamma}^2(a^2-1)\mathrm{We}^2} + O\left(\frac{1}{\mathrm{We}^4}\right)\right] \tag{57}$$

$$T_{12} = \frac{-(1-\beta)}{\dot{\gamma}\mathrm{We}^2(a^2-1)}\left[1 + \frac{1}{\dot{\gamma}^2(a^2-1)\mathrm{We}^2} + O\left(\frac{1}{\mathrm{We}^4}\right)\right] \tag{58}$$

So as $\mathrm{We} \to \infty$ we have the asymptotic relations:

$$T_{11} \sim -\frac{(1-\beta)}{\mathrm{We}(a-1)} \tag{59}$$

$$T_{22} \sim -\frac{(1-\beta)}{\mathrm{We}(a+1)} \tag{60}$$

$$T_{12} \sim -\frac{(1-\beta)}{\dot{\gamma}\mathrm{We}^2(a^2-1)} \tag{61}$$

As We $\to \infty$. These are in agreement to what we derived above. We can also substitute (56) and (57) into (42) to obtain:

$$T_{12} - \frac{\dot{\gamma}\mathrm{We}}{2}\left\{-\frac{(1-\beta)}{\mathrm{We}}\left[1+\frac{1}{\dot{\gamma}^2(a^2-1)\mathrm{We}^2}+O\left(\frac{1}{\mathrm{We}^4}\right)\right]-\frac{(1-\beta)}{\mathrm{We}}\left[1+\frac{1}{\dot{\gamma}^2(a^2-1)\mathrm{We}^2}+O\left(\frac{1}{\mathrm{We}^4}\right)\right]\right\} = \dot{\gamma}(1-\beta)$$

$$T_{12} + \dot{\gamma}(1-\beta)\left\{1+\frac{1}{\dot{\gamma}^2(a^2-1)\mathrm{We}^2}+O\left(\frac{1}{\mathrm{We}^4}\right)\right\} = \dot{\gamma}(1-\beta) \tag{62}$$

The right hand side will cancel with the first term in the braces, and so we have:

$$T_{12} \sim -\frac{(1-\beta)}{\dot{\gamma}\mathrm{We}^2(a^2-1)} \tag{63}$$

Which is consistent with 61. We can see from the above working that as $a \to 1$ and We $\to \infty$ there is a clash of scales. To get around this problem and to allow us to transition between cases more smoothly we shall introduce the new parameter $b = (1-a)\mathrm{We}^2 = O(1)$. Then we have that $a = 1 - \frac{b}{\mathrm{We}^2}$. We can now substitute this into the expressions above for the components of $\mathbf{T}$ to get:

$$T_{11} = \frac{\dot{\gamma}^2(1-\beta)(2\mathrm{We}-b)}{1+\dot{\gamma}^2(2\mathrm{We}-b)b} \tag{64}$$

$$T_{12} = \frac{\dot{\gamma}(1-\beta)}{1+\dot{\gamma}^2(2\mathrm{We}-b)b} \tag{65}$$

$$T_{22} = \frac{-\dot{\gamma}^2(1-\beta)b}{1+\dot{\gamma}^2(2\mathrm{We}-b)b} \tag{66}$$

Which means we have that

$$T_{11} \approx \begin{cases} \frac{(1-\beta)}{b} & \text{if } 2\mathrm{We} \gg b \gg 1 \\ 2\mathrm{We}\dot{\gamma}^2(1-\beta) & \text{if } b \ll 1 \end{cases} \tag{67}$$

$$T_{12} \approx \begin{cases} \frac{1-\beta}{\dot{\gamma}(2\mathrm{We}-b)b} & \text{if } 2\mathrm{We} \gg b \gg 1 \\ \dot{\gamma}^2(1-\beta) & \text{if } b \ll 1 \end{cases} \tag{68}$$

$$T_{22} \approx \begin{cases} -\frac{(1-\beta)}{(2\mathrm{We}-b)} & \text{if } 2\mathrm{We} \gg b \gg 1 \\ 0 & \text{if } b \ll 1 \end{cases} \tag{69}$$

The upper bound for b comes from noting the following:

$$2\mathrm{We} - b = \mathrm{We}(2-(1-a)) = \mathrm{We}(1+a) \geq 0$$

In the limit as We gets large. As $b \to 0$ and so $a \to 1$ we find that $T_{11} = O(\mathrm{We})$, $T_{12} = O(1)$ and $T_{22} = 0$ which is in agreement with published results on the upper convected Maxwell model[8]. We are going to concentrate on the case where $We \to \infty$ and $a \neq \pm 1$ as analysis has already been done for the cases $a = \pm 1$ which correspond to the Upper and Lower Convected Maxwell models. See appendix for viscometric stress plots.

4 High Weissenberg Limit

We will look at a steady two-dimensional flow near a flat boundary, so the domain is the half plane $y > 0$. We are interested in solutions for small y and large We. The viscometric behaviour derived in the previous section suggests the following scaling of variables.

$$x = \bar{x}(= O(1)) \qquad y = \delta\bar{y} \qquad \psi = \delta^n\bar{\psi}$$
$$T_{11} = \delta^{n_1}\bar{T}_{11} \qquad T_{12} = \delta^{n_2}\bar{T}_{12} \qquad T_{22} = \delta^{n_3}\bar{T}_{22} \qquad p = \delta^{n_4}\bar{p}$$

Here n, n_1, n_2 and n_3 are yet to be determined and $\delta = \delta(\mathrm{We})$ (the 'gauge') is assumed small. We will determine it by balancing terms in the constitutive equations. Now for the case $a \neq \pm 1$ we have from the viscometric behaviour that:

$$\delta^{n_1} = \mathrm{We}^{-1} \qquad \delta^{n_2} = \mathrm{We}^{-2} \qquad \delta^{n_3} = \mathrm{We}^{-1}$$

We can also scale the velocity components using the scaled stream function:

$$u = \frac{\partial \psi}{\partial y} = \frac{\delta^n}{\delta}\frac{\partial \bar{\psi}}{\partial \bar{y}} = \delta^{n-1}\bar{u} \tag{70}$$

$$v = -\frac{\partial \psi}{\partial x} = -\delta^n \frac{\partial \bar{\psi}}{\partial \bar{x}} = \delta^n \bar{v} \tag{71}$$

Substituting this into the constitutive equations we obtain:

$$\frac{1}{\mathrm{We}}\bar{T}_{11} + \mathrm{We}\left(\frac{\delta^{n-1}}{\mathrm{We}}\left(\bar{u}\frac{\partial \bar{T}_{11}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}_{11}}{\partial \bar{y}} - 2a\frac{\partial \bar{u}}{\partial \bar{x}}\bar{T}_{11}\right) - (a-1)\frac{\delta^n}{\mathrm{We}^2}\frac{\partial \bar{v}}{\partial \bar{x}}\bar{T}_{12} - (a+1)\frac{\delta^{n-2}}{\mathrm{We}^2}\frac{\partial \bar{u}}{\partial \bar{y}}\bar{T}_{12}\right) = 2(1-\beta)\delta^{n-1}\frac{\partial \bar{u}}{\partial \bar{x}} \tag{72}$$

$$\frac{1}{\mathrm{We}}\bar{T}_{22} + \mathrm{We}\left(\frac{\delta^{n-1}}{\mathrm{We}}\left(\bar{u}\frac{\partial \bar{T}_{22}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}_{22}}{\partial \bar{y}} - 2a\frac{\partial \bar{v}}{\partial \bar{y}}\bar{T}_{22}\right) - (a-1)\frac{\delta^{n-2}}{\mathrm{We}^2}\frac{\partial \bar{u}}{\partial \bar{y}}\bar{T}_{12} - (a+1)\frac{\delta^n}{\mathrm{We}^2}\frac{\partial \bar{v}}{\partial \bar{x}}\bar{T}_{12}\right) = 2(1-\beta)\delta^{n-1}\frac{\partial \bar{v}}{\partial \bar{y}} \tag{73}$$

$$\begin{aligned}\frac{1}{\mathrm{We}^2}\bar{T}_{12} + \mathrm{We}\left(\frac{\delta^{n-1}}{\mathrm{We}^2}\left(\bar{u}\frac{\partial \bar{T}_{12}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}_{12}}{\partial \bar{y}}\right) - \frac{\delta^{n-2}}{\mathrm{We}}\frac{\partial \bar{u}}{\partial \bar{y}}\left(\frac{a-1}{2}\bar{T}_{11} + \frac{a+1}{2}\bar{T}_{22}\right) - \frac{\delta^n}{\mathrm{We}}\frac{\partial \bar{v}}{\partial \bar{x}}\left(\frac{a+1}{2}\bar{T}_{11} + \frac{a-1}{2}\bar{T}_{22}\right)\right) \\ = (1-\beta)\left(\delta^{n-2}\frac{\partial \bar{u}}{\partial \bar{y}} + \delta^n\frac{\partial \bar{v}}{\partial \bar{x}}\right)\end{aligned} \tag{74}$$

Since $\delta \ll 1$, the fullest balance is when

$$\frac{\delta^{n-1}}{\mathrm{We}} = \frac{\delta^{n-2}}{\mathrm{We}^2} \Rightarrow \delta = \frac{1}{\mathrm{We}} \tag{75}$$

and additionally $\frac{1}{\mathrm{We}} = \delta^{n-1}$ which allows us to conclude that $n = 2$. Then keeping only the leading order terms we obtain the following constitutive boundary layer equations.

$$\bar{T}_{11} + \bar{u}\frac{\partial \bar{T}_{11}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}_{11}}{\partial \bar{y}} - 2a\frac{\partial \bar{u}}{\partial \bar{x}}\bar{T}_{11} - (a+1)\frac{\partial \bar{u}}{\partial \bar{y}}\bar{T}_{12} = 2(1-\beta)\frac{\partial \bar{u}}{\partial \bar{x}} \tag{76}$$

$$\bar{T}_{22} + \bar{u}\frac{\partial \bar{T}_{22}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}_{22}}{\partial \bar{y}} - 2a\frac{\partial \bar{v}}{\partial \bar{y}}\bar{T}_{22} - (a-1)\frac{\partial \bar{u}}{\partial \bar{y}}\bar{T}_{12} = 2(1-\beta)\frac{\partial \bar{v}}{\partial \bar{y}} \tag{77}$$

$$-\frac{a-1}{2}\bar{T}_{11} - \frac{a+1}{2}\bar{T}_{22} = (1-\beta) \tag{78}$$

At first glance this system seems a little unexpected, it is however possible to recover a viscometric flow. Let:

$$\bar{\psi} = \tfrac{1}{2}\dot{\gamma}\bar{y}^2 \qquad \bar{u} = \dot{\gamma}\bar{y} \qquad \bar{v} = 0$$

Then we can substitute these into our boundary layer equations to obtain:

$$\begin{aligned}\bar{T}_{11} - (a+1)\dot{\gamma}\bar{T}_{12} &= 0 \\ \bar{T}_{22} - (a-1)\dot{\gamma}\bar{T}_{12} &= 0 \\ -(a-1)\bar{T}_{11} - (a+1)\bar{T}_{22} &= 2(1-\beta)\end{aligned} \tag{79}$$

From these we can get $\bar{T}_{12} = -\frac{(1-\beta)}{\dot{\gamma}(a^2-1)}$ which is the highest order term in (61) and so we recover viscometric behaviour. We can also scale the momentum equations in a similar manner, with just the scaling for p remaining to be found explicitly.

$$\frac{\mathrm{Re}}{\mathrm{We}^2}\bar{\mathbf{v}} \cdot \bar{\nabla}\bar{u} = -\delta^{n_4}\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{1}{\mathrm{We}}\left(\frac{\partial \bar{T}_{11}}{\partial \bar{x}} + \frac{\partial \bar{T}_{12}}{\partial \bar{y}}\right) \tag{80}$$

$$\frac{\mathrm{Re}}{\mathrm{We}^3}\bar{\mathbf{v}} \cdot \bar{\nabla}\bar{v} = -\mathrm{We}\delta^{n_4}\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{1}{\mathrm{We}^2}\frac{\partial \bar{T}_{12}}{\partial \bar{x}} + \frac{\partial \bar{T}_{22}}{\partial \bar{y}} \tag{81}$$

The fullest balance here is when $\delta^{n_4} = \frac{1}{\mathrm{We}}$ so we conclude $n_4 = 1$. Thus keeping the leading order terms , we obtain the boundary layer momentum equations.

$$-\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial \bar{T}_{11}}{\partial \bar{x}} + \frac{\partial \bar{T}_{12}}{\partial \bar{y}} = 0 \tag{82}$$

$$-\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\partial \bar{T}_{22}}{\partial \bar{y}} = 0 \tag{83}$$

By multiplying (76) and (77) appropriately then subtracting one from the other we are able to eliminate $\bar{T}_{12}$. We can then use (78) to obtain a boundary layer equation just in terms of $\bar{T}_{11}$. Applying the continuity equation, this reduces to:

$$\bar{T}_{11} + \bar{\mathbf{v}} \cdot \bar{\nabla} \bar{T}_{11} = \frac{(1-\beta)}{(1-a)} \tag{84}$$

We can replace $\bar{\mathbf{v}} \cdot \bar{\nabla}$ with $\frac{\partial}{\partial s}$ where s is measured along streamlines. Then we have a simple ODE for T_{11} in s:

$$\bar{T}_{11} + \frac{\partial \bar{T}_{11}}{\partial s} = \frac{(1-\beta)}{(1-a)} \tag{85}$$

We can immediately recover the particular solution:

$$\bar{T}_{11} = \frac{(1-\beta)}{(1-a)}$$

The general solution may be found using the integrating factor and we obtain:

$$\bar{T}_{11} = \frac{(1-\beta)}{(1-a)} + c(\psi)e^{-s} \tag{86}$$

Where $c(\psi)$ is constant along streamlines.

We now seek to find the natural stress formulations of the boundary layer equations, we first however need to find scalings for λ, μ and ν. We do this by attempting to balance the relation that links cartesian and natural stress formulations.

$$\begin{aligned} \frac{\bar{T}_{11}}{\mathrm{We}} &= -\frac{(1-\beta)}{\mathrm{We}} + \lambda \frac{\bar{u}^2}{\mathrm{We}^2} - 2\frac{\mu}{\mathrm{We}}\frac{\bar{v}}{\bar{u}} + \nu \frac{\bar{v}^2}{\bar{u}^4} \\ \bar{T}_{11} &= -(1-\beta) + \frac{\lambda}{\mathrm{We}}\bar{u}^2 - 2\mu\frac{\bar{v}}{\bar{u}} + \nu \mathrm{We}\frac{\bar{v}^2}{\bar{u}^4} \end{aligned} \tag{87}$$

$$\begin{aligned} \frac{\bar{T}_{12}}{\mathrm{We}^2} &= \lambda \frac{\bar{u}\bar{v}}{\mathrm{We}^3} + \mu - \nu \frac{\bar{v}}{\bar{u}^3}\mathrm{We} \\ \bar{T}_{12} &= \lambda \frac{\bar{u}\bar{v}}{\mathrm{We}} + \mathrm{We}^2\mu - \nu \mathrm{We}^3 \frac{\bar{v}}{\bar{u}^3} \end{aligned} \tag{88}$$

$$\begin{aligned} \frac{\bar{T}_{22}}{\mathrm{We}} &= -\frac{(1-\beta)}{\mathrm{We}} + \frac{\lambda}{\mathrm{We}^4}\bar{v}^2 + 2\bar{\mu}\mathrm{We}\frac{\bar{v}}{\bar{u}} + \nu \frac{\mathrm{We}^2}{\bar{u}^2} \\ \bar{T}_{22} &= -(1-\beta) + \frac{\lambda}{\mathrm{We}^3}\bar{v}^2 + 2\mu\frac{\bar{v}}{\bar{u}} + \nu \frac{\mathrm{We}^3}{\bar{u}^2} \end{aligned} \tag{89}$$

The fullest balance seems to come when we scale in the following manner:

$$\lambda = \mathrm{We}\bar{\lambda} \qquad \mu = \frac{1}{\mathrm{We}^2}\bar{\mu} \qquad \nu = \frac{1}{\mathrm{We}^3}\bar{\nu}$$

Then at leading order we have:

$$\bar{T}_{11} = -(1-\beta) + \bar{\lambda}\bar{u}^2 \tag{90}$$

$$\bar{T}_{22} = -(1-\beta) + \frac{\bar{\nu}}{\bar{u}^2} \tag{91}$$

$$\bar{T}_{12} = \bar{\lambda}\bar{u}\bar{v} + \bar{\mu} - \bar{\nu}\frac{\bar{v}}{\bar{u}^3} \tag{92}$$

We can substitute these expressions into the boundary layer equations, and we can also substitute our scalings into (31) - (33) to obtain natural stress boundary layer equations.

$$\bar{\lambda} + \bar{\mathbf{v}} \cdot \bar{\nabla}\bar{\lambda} - (a+1)\frac{\bar{\mu}}{\bar{u}^2}\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{(a-1)}{\bar{u}^2}\bar{\mathbf{v}} \cdot \bar{\nabla}(\bar{u}^2)\left(\frac{(1-\beta)}{\bar{u}^2} - \bar{\lambda}\right) = \frac{(1-\beta)}{\bar{u}^2} \tag{93}$$

$$\frac{(1-a)}{2}\frac{\lambda}{\bar{u}^2} - \frac{(a+1)}{2}\bar{\nu}\bar{u}^2 = (1-\beta)(1-a)\bar{u}^4 \tag{94}$$

$$\bar{\nu} + \bar{\mathbf{v}} \cdot \bar{\nabla}\bar{\nu} - (a-1)\bar{\mu}\bar{u}^2\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{(a-1)}{\bar{u}^2}\bar{\mathbf{v}} \cdot \bar{\nabla}(\bar{u}^2)(\bar{u}^2(1-\beta) - \bar{\nu}) = (1-\beta)\bar{u}^2 \tag{95}$$

We seek a similarity solution for the constitutive equations. We start with:

$$\xi = \bar{y}\bar{x}^{-q_0} \qquad \bar{\psi} = \bar{x}^{q_1} f(\xi) \qquad \bar{T}_{ij} = \bar{x}^{q_{ij}} t_{ij}(\xi)$$

This then also gives us:

$$\bar{u} = \frac{\partial \bar{\psi}}{\partial \bar{y}} = \bar{x}^{q_1}\frac{df}{d\xi}\frac{\partial \xi}{\partial \bar{y}} = \bar{x}^{(q_1 - q_0)} f'$$

$$\bar{v} = -\frac{\partial \bar{\psi}}{\partial \bar{x}} = -q_1\bar{x}^{q_1 - 1} f - \bar{x}^{q_1}\frac{df}{d\xi}\frac{\partial \xi}{\partial \bar{x}} = \bar{x}^{q_1 - 1}(q_0\xi f' - q_1 f)$$

From (78) we can immediately recover that $q_{11} = q_{22} = 0$. Then substituting into (76) and (77) we also learn that $q_1 = q_0 + 1$ and $q_{12} = q_0 - 1$, and the similarity solution is given by:

$$\xi = \frac{\bar{y}}{\bar{x}^{q_0}}, \qquad \bar{\psi} = \bar{x}^{q_0+1} f(\xi)$$

$$\bar{T}_{11} = t_{11}(\xi), \qquad \bar{T}_{12} = \bar{x}^{q_0 - 1} t_{12}(\xi), \qquad \bar{T}_{22} = t_{22}(\xi) \tag{96}$$

Which gives us the following system of ODEs:

$$t_{11}(\xi) - (1+q_0)f(\xi)t'_{11}(\xi) - 2a(f'(\xi) - q_0\xi f''(\xi))t_{11}(\xi) - (a+1)f''(\xi)t_{12}(\xi) = 2(1-\beta)(q_0\xi f''(\xi) - f'(\xi)) \tag{97}$$

$$t_{22}(\xi) - (1+q_0)f(\xi)t'_{22}(\xi) - 2a(q_0\xi f''(\xi) - f'(\xi))t_{22}(\xi) - (a-1)f''(\xi)t_{12}(\xi) = 2(1-\beta)(q_0\xi f''(\xi) - f'(\xi)) \tag{98}$$

$$\frac{1-a}{2}t_{11}(\xi) - \frac{1+a}{2}t_{22}(\xi) = (1-\beta) \tag{99}$$

Multiplying and adding (97) and (98) appropriately then substituting in an expression for t_{22} obtained from (99) we can obtain the following single ode to be solved for t_{11} and f:

$$2(1-\beta)(a-1)(f' - q_0\xi f'') + (a-1)(t_{11} - (1+q_0)f t'_{11}) + (1-\beta) = 0 \tag{100}$$

5 Low Weissenberg Limit

We will look again at a steady two-dimensional flow near a flat boundary, so the domain is the half plane $y > 0$. We are interested in solutions for small y and small We. We once again assume the variables will scale in the following manner:

$$x = \bar{x}(= O(1)) \qquad y = \delta\bar{y} \qquad \psi = \delta^n\bar{\psi}$$
$$T_{11} = \delta^{n_1}\bar{T}_{11} \qquad T_{12} = \delta^{n_2}\bar{T}_{12} \qquad T_{22} = \delta^{n_3}\bar{T}_{22} \qquad p = \delta^{n_4}\bar{p}$$

Here n, n_1, n_2 and n_3 are yet to be determined and $\delta = \delta(\text{We})$ (the 'gauge') is assumed small. We will determine it by balancing terms in the constitutive equations. Now for the case $a \neq \pm 1$ we have from the viscometric behaviour that:

$$\delta^{n_1} = \mathrm{We}^1 \qquad \delta^{n_2} = \mathrm{We}^0 = 1 \qquad \delta^{n_3} = \mathrm{We}^1$$

We can also scale the velocity components using the scaled stream function:

$$u = \frac{\partial \psi}{\partial y} = \frac{\delta^n}{\delta}\frac{\partial \bar{\psi}}{\partial \bar{y}} = \delta^{n-1}\bar{u} \tag{101}$$

$$v = -\frac{\partial \psi}{\partial v} = -\delta^n \frac{\partial \bar{\psi}}{\partial \bar{x}} = \delta^n \bar{v} \tag{102}$$

Substituting this into the constitutive equations we once again obtain:

$$\mathrm{We}\bar{T}_{11} + \mathrm{We}\left(\delta^{n-1}\mathrm{We}\left(\bar{u}\frac{\partial \bar{T}_{11}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}_{11}}{\partial \bar{y}} - 2a\frac{\partial \bar{u}}{\partial \bar{x}}\bar{T}_{11}\right) - (a-1)\delta^n \frac{\partial \bar{v}}{\partial \bar{x}}\bar{T}_{12} - (a+1)\delta^{n-2}\frac{\partial \bar{u}}{\partial \bar{y}}\bar{T}_{12}\right) = 2(1-\beta)\delta^{n-1}\frac{\partial \bar{u}}{\partial \bar{x}} \tag{103}$$

$$\mathrm{We}\bar{T}_{22} + \mathrm{We}\left(\delta^{n-1}\mathrm{We}\left(\bar{u}\frac{\partial \bar{T}_{22}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}_{22}}{\partial \bar{y}} - 2a\frac{\partial \bar{v}}{\partial \bar{y}}\bar{T}_{22}\right) - (a-1)\delta^{n-2} \frac{\partial \bar{u}}{\partial \bar{y}}\bar{T}_{12} - (a+1)\delta^{n}\frac{\partial \bar{v}}{\partial \bar{x}}\bar{T}_{12}\right) = 2(1-\beta)\delta^{n-1}\frac{\partial \bar{v}}{\partial \bar{y}} \tag{104}$$

$$\begin{aligned}\bar{T}_{12} + \mathrm{We}\left(\delta^{n-1}\left(\bar{u}\frac{\partial \bar{T}_{12}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}_{12}}{\partial \bar{y}}\right) - \delta^{n-2}\mathrm{We}\frac{\partial \bar{u}}{\partial \bar{y}}\left(\frac{a-1}{2}\bar{T}_{11} + \frac{a+1}{2}\bar{T}_{22}\right) - \delta^n\mathrm{We}\frac{\partial \bar{v}}{\partial \bar{x}}\left(\frac{a+1}{2}\bar{T}_{11} + \frac{a-1}{2}\bar{T}_{22}\right)\right) \\ = (1-\beta)\left(\delta^{n-2}\frac{\partial \bar{u}}{\partial \bar{y}} + \delta^n \frac{\partial \bar{v}}{\partial \bar{x}}\right)\end{aligned} \tag{105}$$

Here both We and δ are small, so the fullest balance is when

$$\delta^{n-1} = \mathrm{We} \quad \& \quad \delta^{n-2} = 1 \quad \Rightarrow \quad \delta = \mathrm{We} \quad \& \quad n = 2 \tag{106}$$

Thus keeping just the highest order terms we obtain the low Weissenberg number boundary layer constitutive equations:

$$\bar{T}_{11} - (a+1)\frac{\partial \bar{u}}{\partial \bar{y}}\bar{T}_{12} = 2(1-\beta)\frac{\partial \bar{u}}{\partial \bar{x}} \tag{107}$$

$$\bar{T}_{22} - (a-1)\frac{\partial \bar{u}}{\partial \bar{y}}\bar{T}_{12} = 2(1-\beta)\frac{\partial \bar{v}}{\partial \bar{y}} \tag{108}$$

$$\bar{T}_{12} = (1-\beta)\frac{\partial \bar{u}}{\partial \bar{y}} \tag{109}$$

From this it is easy to obtain the following system of non-linear pdes:

$$\begin{aligned}\bar{T}_{11} - (1-\beta)(a+1)\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 &= 2(1-\beta)\frac{\partial \bar{u}}{\partial \bar{x}} \\ \bar{T}_{22} - (1-\beta)(a-1)\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 &= 2(1-\beta)\frac{\partial \bar{v}}{\partial \bar{y}}\end{aligned} \tag{110}$$

6 Extended Stretching Solution

From the constituive relation for $\mathbf{T}^P$ ($\mathbf{T}^P + \mathrm{We}\,\overset{\square}{\mathbf{T}^{\mathbf{P}}} = 2(1-\beta)\mathbf{D}$) we can recover that for large We we must have $\overset{\square}{\mathbf{T}^{\mathbf{P}}} = 0$. This can be written in component form as the system:

$$u\frac{\partial T_{11}}{\partial x} + v\frac{\partial T_{11}}{\partial y} - 2a\frac{\partial u}{\partial x}T_{11} - (a-1)\frac{\partial v}{\partial x}T_{12} - (a+1)\frac{\partial u}{\partial y}T_{12} = 0 \tag{111}$$

$$u\frac{\partial T_{22}}{\partial x} + v\frac{\partial T_{22}}{\partial y} - 2a\frac{\partial v}{\partial y}T_{22} - (a-1)\frac{\partial u}{\partial y}T_{12} - (a+1)\frac{\partial v}{\partial x}T_{12} = 0 \tag{112}$$

$$u\frac{\partial T_{12}}{\partial x}+v\frac{\partial T_{12}}{\partial y}-\left(\frac{a+1}{2}\frac{\partial u}{\partial y}+\frac{a-1}{2}\frac{\partial v}{\partial x}\right)T_{22}-\left(\frac{a-1}{2}\frac{\partial u}{\partial y}+\frac{a+1}{2}\frac{\partial v}{\partial x}\right)T_{11}=0 \tag{113}$$

Doing as before we can obtain a natural stress formulation for this new system. The natural stress formulations read:

$$\mathbf{v}\cdot\nabla\lambda+2\mu\nabla\cdot\mathbf{w}+(a-1)\left(\mu\nabla\cdot\mathbf{w}-\frac{\lambda}{|\mathbf{v}|^2}\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\right)=0 \tag{114}$$

$$2\mathbf{v}\cdot\nabla\mu+2\nu\nabla\cdot\mathbf{w}+(a-1)(\lambda|\mathbf{v}|^4+\nu)\nabla\cdot\mathbf{w}=0 \tag{115}$$

$$\mathbf{v}\cdot\nabla\nu+(a-1)\left(|\mathbf{v}|^4\mu\nabla\cdot\mathbf{w}+\frac{\nu}{|\mathbf{v}|^2}\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\right)=0 \tag{116}$$

Motivated by Renardy's work for the case $a=1$[10], we will try the case $\mu=0$. Equations (114) and (116) then become respectively:

$$\mathbf{v}\cdot\nabla\lambda-(a-1)\frac{\lambda}{|\mathbf{v}|^2}\mathbf{v}\cdot\nabla|\mathbf{v}|^2=0 \tag{117}$$

$$\mathbf{v}\cdot\nabla\nu+(a-1)\frac{\nu}{|\mathbf{v}|^2}\mathbf{v}\cdot\nabla|\mathbf{v}|^2=0 \tag{118}$$

We can solve this by making use of the following identity:

$$\frac{1}{\lambda}\mathbf{v}\cdot\nabla\lambda=(\mathbf{v}\cdot\nabla)\log\lambda$$

This gives us the following particular solution.

$$\lambda=\lambda_0(\psi)|\mathbf{v}|^{2(a-1)}\qquad\mu=0\qquad\nu=\nu_0(\psi)|\mathbf{v}|^{2(1-a)} \tag{119}$$

Where $\lambda_0(\psi)$ and $\nu_0(\psi)$ are arbitrary functions of streamlines of the flow, i.e. λ_0 and ν_0 are constant along a streamline, though may take different values on distinct streamlines.

7 To Do

- Natural stress boundary layer equations
- Hybrid boundary layer equations from limits $a\to1^-$ and $a\to-1^+$

A High Weissenberg Boundary Layer Equations for Cases $a=\pm1$

As we obtained different viscometric behaviour for the cases $a=\pm1$ we would expect different scalings and thus different boundary layer equations. We will look, as before, at a steady two-dimensional flow near a flat boundary, so the domain is the half plane $y>0$. We are interested in solutions for small y and large We.

The viscometric behaviour derived in the previous section for the first case $a=1$ (corresponding to the Upper Convected model) suggests the following scaling of variables.

$$x=\bar{x}(=O(1))\qquad y=\delta\bar{y}\qquad\psi=\delta^n\bar{\psi}$$
$$T_{11}=\delta^{n_1}\bar{T}_{11}\qquad T_{12}=\delta^{n_2}\bar{T}_{12}\qquad T_{22}=\delta^{n_3}\bar{T}_{22}\qquad p=\delta^{n_4}\bar{p}$$

Here n, n_1, n_2 and n_3 are yet to be determined and $\delta=\delta(\mathrm{We})$ (the 'gauge') is assumed small. We will determine it by balancing terms in the constitutive equations. We have from the viscometric behaviour that:

$$\delta^{n_1}=\mathrm{We}^1\qquad\delta^{n_2}=\mathrm{We}^0=1$$

We cannot conclude anything about the scaling for T_{22} as the viscometric behaviour gives $T_{22}=0$ so we cannot say anything about δ^{n_3} at the moment. We can also scale the velocity components using the scaled stream function:

$$u=\frac{\partial\psi}{\partial y}=\frac{\delta^n}{\delta}\frac{\partial\bar{\psi}}{\partial\bar{y}}=\delta^{n-1}\bar{u} \tag{120}$$

$$v=-\frac{\partial\psi}{\partial x}=-\delta^n\frac{\partial\bar{\psi}}{\partial\bar{x}}=\delta^n\bar{v} \tag{121}$$

We can now substitute the scaled variables into the constitutive equations to get:

$$\mathrm{We}\bar{T}_{11}+\mathrm{We}\left(\delta^{n-1}\mathrm{We}\left(\bar{u}\frac{\partial\bar{T}_{11}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{T}_{11}}{\partial\bar{y}}-2\frac{\partial\bar{u}}{\partial\bar{x}}\bar{T}_{11}\right)-2\delta^{n-2}\frac{\partial\bar{u}}{\partial\bar{y}}\bar{T}_{12}\right)=2(1-\beta)\delta^{n-1}\frac{\partial\bar{u}}{\partial\bar{x}} \tag{122}$$

$$\delta^{n_3}\bar{T}_{22}+\mathrm{We}\left(\delta^{n_3+n-1}\left(\bar{u}\frac{\partial\bar{T}_{22}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{T}_{22}}{\partial\bar{y}}-2\frac{\partial\bar{v}}{\partial\bar{y}}\bar{T}_{22}\right)-2\delta^{n}\frac{\partial\bar{v}}{\partial\bar{x}}\bar{T}_{12}\right)=2(1-\beta)\frac{1}{\mathrm{We}}\frac{\partial\bar{v}}{\partial\bar{y}} \tag{123}$$

$$\bar{T}_{12}+\mathrm{We}\left(\frac{1}{\mathrm{We}}\left(\bar{u}\frac{\partial\bar{T}_{12}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{T}_{12}}{\partial\bar{y}}\right)-\delta^{n-2}\frac{\partial\bar{u}}{\partial\bar{y}}\delta^{n_3}\bar{T}_{22}-\delta^{n}\frac{\partial\bar{v}}{\partial\bar{x}}\mathrm{We}\bar{T}_{11}\right)=(1-\beta)\left(\delta^{n-2}\frac{\partial\bar{u}}{\partial\bar{y}}+\delta^{n}\frac{\partial\bar{v}}{\partial\bar{x}}\right) \tag{124}$$

The fullest balance comes when the following holds:

$$\delta=\tfrac{1}{\mathrm{We}}\qquad n=2\qquad n_3=1$$

Then keeping just the highest order terms, we obtain the UCM high Weissenberg number boundary layer equations:

$$\bar{T}_{11}+\bar{u}\frac{\partial\bar{T}_{11}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{T}_{11}}{\partial\bar{y}}-2\frac{\partial\bar{u}}{\partial\bar{x}}\bar{T}_{11}-2\frac{\partial\bar{u}}{\partial\bar{y}}\bar{T}_{12}=0 \tag{125}$$

$$\bar{T}_{22}+\bar{u}\frac{\partial\bar{T}_{22}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{T}_{22}}{\partial\bar{y}}-2\frac{\partial\bar{v}}{\partial\bar{y}}-2\frac{\partial\bar{v}}{\partial\bar{x}}\bar{T}_{12}=2(1-\beta)\frac{\partial\bar{v}}{\partial\bar{y}} \tag{126}$$

$$T_{12}+\bar{u}\frac{\partial\bar{T}_{12}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{T}_{12}}{\partial\bar{y}}-\frac{\partial\bar{u}}{\partial\bar{y}}\bar{T}_{22}-\frac{\partial\bar{v}}{\partial\bar{x}}\bar{T}_{11}=(1-\beta)\frac{\partial\bar{u}}{\partial\bar{y}} \tag{127}$$

The viscometric behaviour derived for the first case $a=-1$ (corresponding to the Lower Convected model) suggests the following scaling of variables.

$$x=\bar{x}(=O(1))\qquad y=\delta\bar{y}\qquad \psi=\delta^n\bar{\psi}$$
$$T_{11}=\delta^{n_1}\bar{T}_{11}\qquad T_{12}=\delta^{n_2}\bar{T}_{12}\qquad T_{22}=\delta^{n_3}\bar{T}_{22}\qquad p=\delta^{n_4}\bar{p}$$

Here n, n_1, n_2 and n_3 are yet to be determined and $\delta=\delta(\mathrm{We})$ (the 'gauge') is assumed small. We will determine it by balancing terms in the constitutive equations. We have from the viscometric behaviour that:

$$\delta^{n_2}=\mathrm{We}^0=1\qquad \delta^{n_3}=\mathrm{We}^1$$

We cannot conclude anything about the scaling for T_{11} as the viscometric behaviour gives $T_{11}=0$ so we cannot say anything about δ^{n_1} at the moment. We can also scale the velocity components using the scaled stream function:

$$u=\frac{\partial\psi}{\partial y}=\frac{\delta^n}{\delta}\frac{\partial\bar{\psi}}{\partial\bar{y}}=\delta^{n-1}\bar{u} \tag{128}$$

$$v=-\frac{\partial\psi}{\partial x}=-\delta^n\frac{\partial\bar{\psi}}{\partial\bar{x}}=\delta^n\bar{v} \tag{129}$$

We can now substitute the scaled variables into the constitutive equations to get:

$$\delta^{n_1}\bar{T}_{11}+\mathrm{We}\left(\delta^{n-1}\delta^{n_1}\left(\bar{u}\frac{\partial\bar{T}_{11}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{T}_{11}}{\partial\bar{y}}+2\frac{\partial\bar{u}}{\partial\bar{x}}\bar{T}_{11}\right)+2\delta^{n}\frac{\partial\bar{v}}{\partial\bar{x}}\bar{T}_{12}\right)=2(1-\beta)\delta^{n-1}\frac{\partial\bar{u}}{\partial\bar{x}} \tag{130}$$

$$\mathrm{We}\bar{T}_{22}+\mathrm{We}\left(\delta^{n-1}\mathrm{We}\left(\bar{u}\frac{\partial\bar{T}_{22}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{T}_{22}}{\partial\bar{y}}+2\frac{\partial\bar{v}}{\partial\bar{y}}\bar{T}_{22}\right)+2\delta^{n-2}\frac{\partial\bar{u}}{\partial\bar{y}}\bar{T}_{12}\right)=2(1-\beta)\delta^{n-1}\frac{\partial\bar{v}}{\partial\bar{y}} \tag{131}$$

$$\bar{T}_{12}+\mathrm{We}\left(\delta^{n-1}\left(\bar{u}\frac{\partial\bar{T}_{12}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{T}_{12}}{\partial\bar{y}}\right)+\delta^{n_1}\delta^{n-2}\frac{\partial\bar{u}}{\partial\bar{y}}\bar{T}_{11}+\mathrm{We}\delta^{n}\frac{\partial\bar{v}}{\partial\bar{x}}\bar{T}_{22}\right)=(1-\beta)\left(\delta^{n-2}\frac{\partial\bar{u}}{\partial\bar{y}}+\delta^{n}\frac{\partial\bar{v}}{\partial\bar{x}}\right) \tag{132}$$

The fullest balance here is when:

$$\delta=\tfrac{1}{\mathrm{We}}\qquad n=2\qquad n_1=1$$

As before we keep just the leading order terms to obtain the high Weissenberg boundary layer equations for the LCM model.

$$\bar{T}_{11} + \bar{u}\frac{\partial \bar{T}_{11}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}_{11}}{\partial \bar{y}} + 2\frac{\partial \bar{u}}{\partial \bar{x}}\bar{T}_{11} + 2\frac{\partial \bar{v}}{\partial \bar{x}}\bar{T}_{12} = 2(1-\beta)\frac{\partial \bar{v}}{\partial \bar{y}} \quad (133)$$

$$\bar{T}_{22} + \bar{u}\frac{\partial \bar{T}_{22}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}_{22}}{\partial \bar{y}} + 2\frac{\partial \bar{v}}{\partial \bar{y}}\bar{T}_{22} + 2\frac{\partial \bar{u}}{\partial \bar{y}}\bar{T}_{12} = 0 \quad (134)$$

$$\bar{T}_{12} + \bar{u}\frac{\partial \bar{T}_{12}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}_{12}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{y}}\bar{T}_{11} + \frac{\partial \bar{v}}{\partial \bar{x}}\bar{T}_{22} = (1-\beta)\frac{\partial \bar{u}}{\partial \bar{y}} \quad (135)$$

A Viscometric Stress Plots

The following plots show the relation between shear rate $\dot{\gamma}$ and the stress tensor components T_{11}, T_{12} and T_{22}.

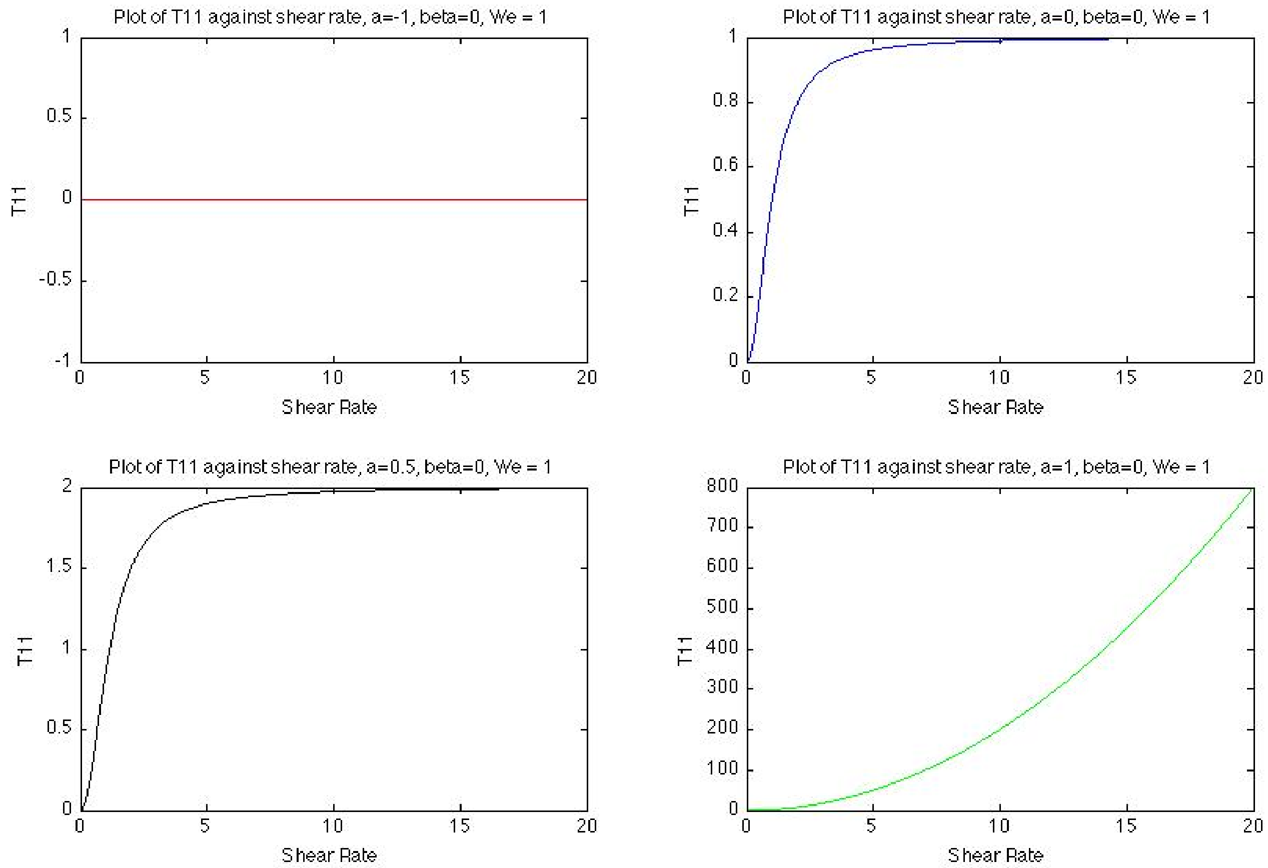

Figure 1: Plots of T_{11} against $\dot{\gamma}$ for different values of a

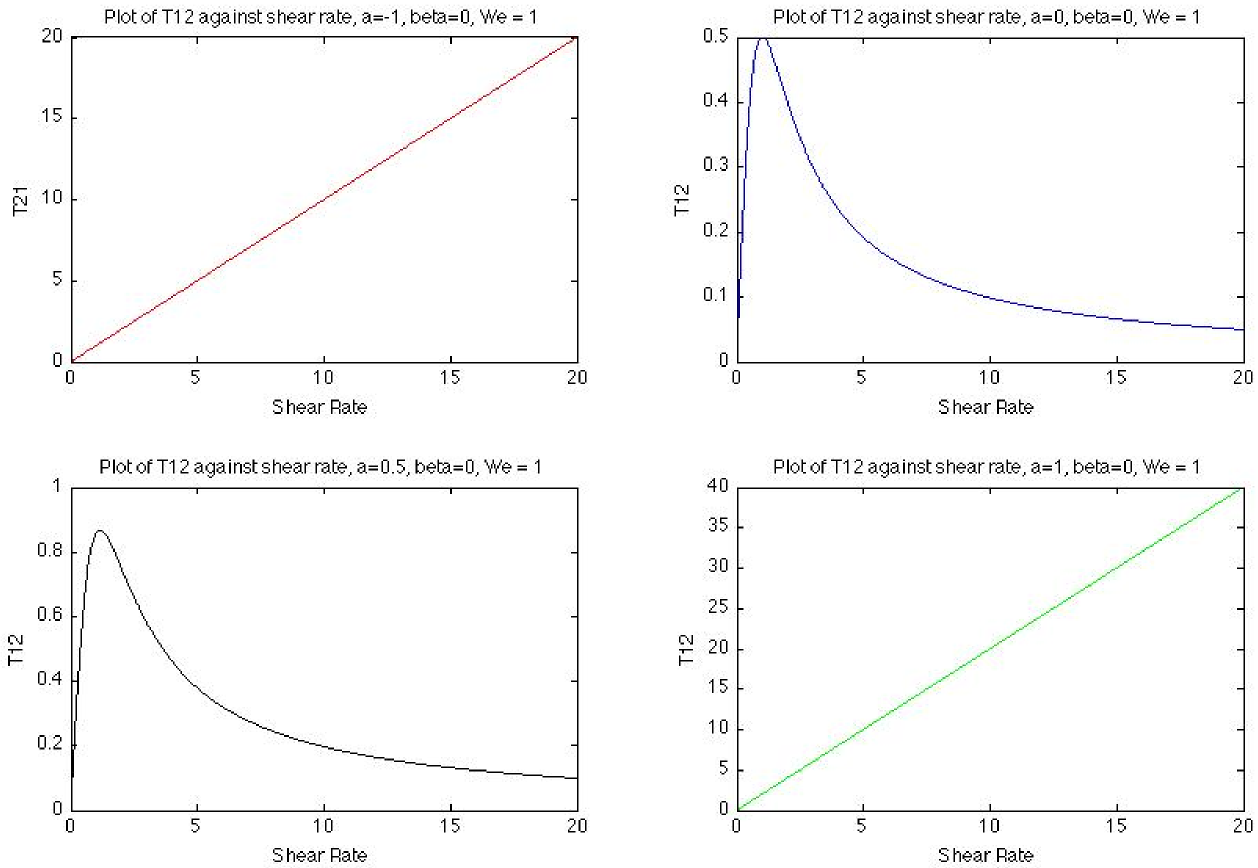

Figure 2: Plots of T_{12} against $\dot{\gamma}$ for different values of a

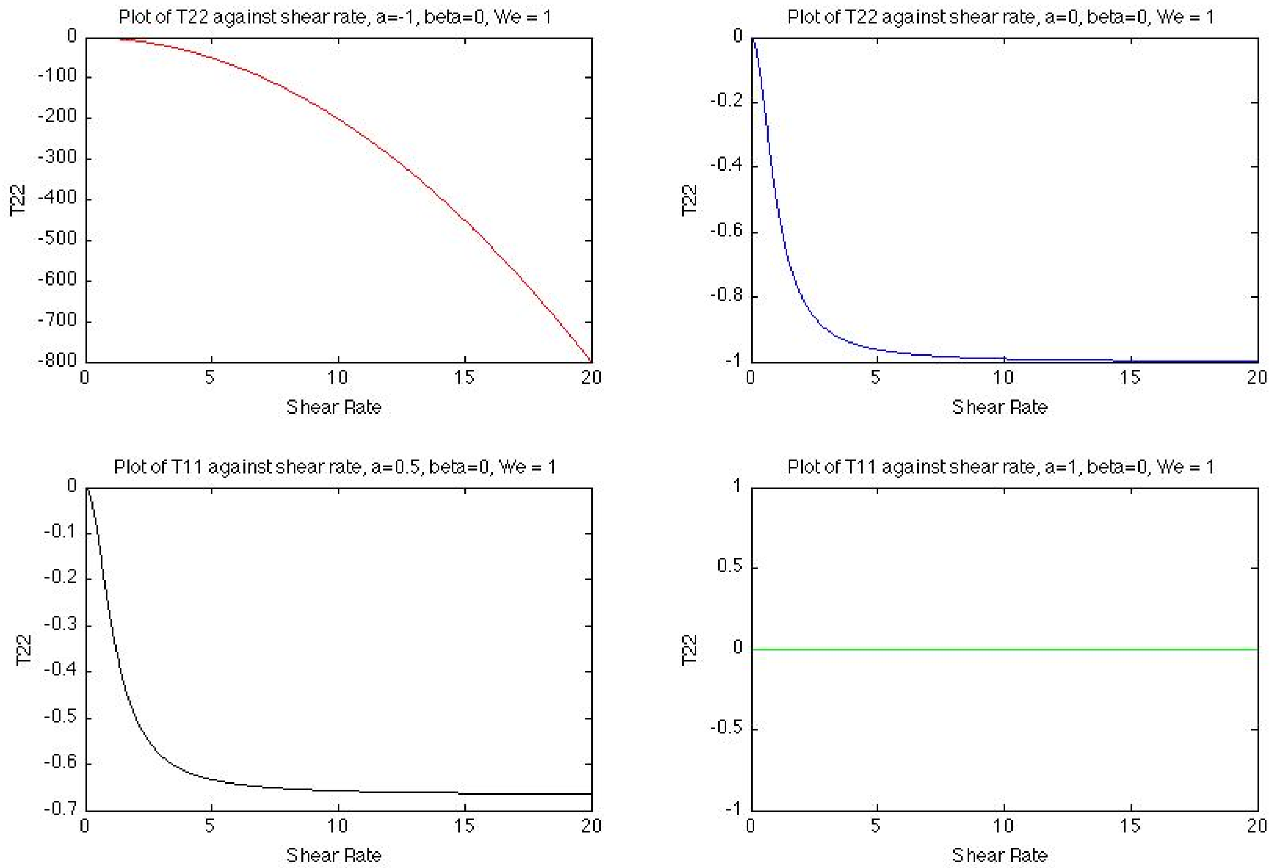

Figure 3: Plots of T_{22} against $\dot{\gamma}$ for different values of a

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