MA20220 Revision Checklist

- Give the form of the most general system of first-order linear ordinary differential equations. Show how to rewrite this (if possible) in the standard form of a first-order system of linear ordinary differential equations. When is the system called (i) autonomous (ii) homogeneous (iii) inhomogeneous?
- Define the matrix exponential function $\exp(tA) : \mathbb{R} \to \mathbb{C}^{n\times n}$, where $A \in \mathbb{C}^{n\times n}$ is a given matrix. State two properties of $\exp(tA)$ (concerning convergence and derivative).
- Let $A \in \mathbb{C}^{n\times n}$ and $\mathbf{x}_0 \in \mathbb{C}^n$. What is the solution of $\dot{x}(t) = Ax(t)$ with $x(t_0) = \mathbf{x}_0$? What other property does this solution possess?
- Let $I \subset \mathbb{R}$ be an interval. Define that it means for the vector functions $y_1, \dots, y_m : I \to \mathbb{C}^n$ to be linearly independent and linearly dependent.
- Let $A \in \mathbb{C}^{n\times n}$ and consider the system $\dot{x}(t) = Ax(t)$, where $x : \mathbb{R} \to \mathbb{C}^n$. Show that there exist n linearly independent solutions of this system, and that any other solution can be written in terms of these solutions.
- Show that $x(t) = e^{\lambda t}v$, $v \in \mathbb{C}^n$ is a solution of $\dot{x}(t) = Ax(t)$ if and only if λ is an eigenvalue of A and v a corresponding eigenvector.
- How is the characteristic polynomial $A \in \mathbb{C}^{n\times n}$ sometimes denoted? How is the set of eigenvalues of A denoted?
- Define what it means for $v \in \mathbb{C}^n$ to be a generalised eigenvector of order $m \in \mathbb{N}$ with respect to $\lambda \in \operatorname{Spec} A$.
- Let v be a generalised eigenvector of order $m \geq 2$ with respect to $\lambda \in \operatorname{Spec} A$. Show that

$$v_{m-k} := (A - \lambda I)^k v$$

  is a generalised eigenvector of order $m - k$, for $k = 1, \dots, m - 1$.
- Define the geometric and algebraic multiplicities of $\lambda \in \operatorname{Spec} A$, where $A \in \mathbb{C}^{n\times n}$. Define a simple eigenvalue.
- Let $A \in \mathbb{C}^{n\times n}$ and $v \in \mathbb{C}^n$ be a generalised eigenvector of order m, with respect to $\lambda \in \operatorname{Spec} A$. Define v_i, for $1 \leq i \leq m-1$, and show that they are linearly independent. Further show that the functions

$$x_k(t) = \sum_{i=0}^{k-1} \frac{t^i}{i!} v_{k-i}, \quad 1 \leq k \leq m,$$

  form a set of m linearly independent solutions of $\dot{x}(t) = Ax(t)$.
- Define a fundamental system and a fundamental matrix. Show that for a fundamental matrix $\Phi(t)$, $\dot{\Phi}(t) = A\Phi(t)$. Why does $\Phi^{-1}(t)$ exist for all $t \in \mathbb{R}$?
- Prove that if $\Phi(t)$ and $\Psi(t)$ are two fundamental matrices for the system $\dot{x}(t) = Ax(t)$ then $\exists C \in \mathbb{C}^{n\times n}$ such that $\Phi(t) = \Psi(t)C$.
- Show that the matrix function $\Phi(t) = \exp(tA)$ is a fundamental matrix for the system $\dot{x}(t) = Ax(t)$. Show that $\Phi(t) = \Psi(t)\Psi_0^{-1}\Phi_0$ solves the initial value problem

$$\dot{\Phi}(t) = A\Phi(t), \quad \Phi(t_0) = \Phi_0,$$

  where $\Psi(t)$ is a fundamental matrix with $\Psi(t_0) = \Psi_0$. State the unique solution to this initial value problem.
- Derive a formula for $\exp(tA)$ in terms of an arbitrary fundamental matrix of $\dot{x}(t) = Ax(t)$.
- Derive a formula for $\exp(tA)$, where $A \in \mathbb{C}^{n\times n}$ is diagonalisable.

- Non-homogeneous (or inhomogeneous) systems: consider the system

$$\dot{x}(t) = Ax(t) + g(t), \quad A \in \mathbb{C}^{n\times n}, \quad x, g : \mathbb{R} \to \mathbb{C}^n. \tag{1}$$

  Derive a formula for the solution of this system, assuming that it takes the form $x(t) = \Phi(t)u(t)$, where $u : \mathbb{R} \to \mathbb{C}^n$. Derive the same result via the Laplace transform. Given the initial value problem (1) with $x(t_0) = \mathbf{x}_0 \in \mathbb{C}^n$, determine the solution.

- Define the Laplace transform of a function $f : [0, \infty) \to \mathbb{R}$. If $g : \mathbb{R} \to \mathbb{C}$ and $z \in \mathbb{C}$, what can be said about

$$\int g(t)dt \quad \text{and} \quad |e^z|?$$

- Define what it means for $f : [0, \infty) \to \mathbb{R}$ to be of exponential order.

- Suppose $f : [0, \infty) \to \mathbb{R}$ is piecewise continuous and of exponential order with constants α, M. Show that $\hat{f}(s)$ exists for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \alpha$.

- Let $f(t), g(t)$ be of exponential order. Show that for $\operatorname{Re} s$ sufficiently large, the properties of linearity, transform of derivative, transform of integral and the damping and delay formulae hold. Show inductively that

$$\mathcal{L}\left\{f^{(n)}(t)\right\}(s) = s^n \hat{f}(s) - \sum_{i=0}^{n-1} s^{n-1-i} f^{(i)}(0).$$

- Derive the Laplace transform of $\cos at$, $\sin at$, $t^n (n \in \mathbb{N})$ and $e^{-\lambda t}\cos at$.

- Define the convolution integral of f and g.

- Given $\hat{f}(s)$ and $\hat{g}(s)$, derive

$$\mathcal{L}^{-1}\{\hat{f}(s)\hat{g}(s)\}.$$

- Consider the initial value problem $\dot{\Phi}(t) = A\Phi(t)$, $\Phi(0) = I$. Show that

$$\Phi(t) = \exp(tA) = \mathcal{L}^{-1}\{(sI - A)^{-1}\}.$$

- Let $\epsilon > 0$. Define $\delta_\epsilon(t)$; for continuous $f : \mathbb{R} \to \mathbb{R}$, derive a formula for

$$\int_{-\infty}^{\infty} f(t)\delta_\epsilon(t)dt.$$

  State the mean value theorem for integrals. Hence show that

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}} f(t)\delta_\epsilon(t)dt = f(0).$$

- Define the Dirac Delta "function". Derive the properties

$$\int_{\mathbb{R}} \delta(t)dt = 1; \quad \int_{\mathbb{R}} f(t)\delta(t-a)dt = f(a); \quad \int_{\mathbb{R}} e^{-st}\delta(t)dt = 1 \Rightarrow \mathcal{L}^{-1}\{1\} = \delta(t),$$

  and $(f \star \delta)(t) = f(t)$. Find expressions for $\mathcal{L}\{\delta(t-T)\}(s)$ and $\delta(t-T) \star f(t)$.

- State and prove the final value theorem.

- Given the general form of the system (with output) considered in the control theory section of this course. Find $x(t)$ by the variation of parameters formula, and find $y(t)$. Show that with $\xi = 0$, $\hat{y}(s) = G(s)\hat{u}(s)$, where $G(s) = c^T(sI - A)^{-1}b$. Define the state of the system. What is G the Laplace transform of? What are G and g called?

- Define $\mathbb{R}[s]$, $\mathbb{R}(s)$, a proper rational function and a strictly proper rational function. Give equivalent conditions for $R \in \mathbb{R}(s)$ to be proper and improper. Define a pole and a zero of $R \in \mathbb{R}(s)$. Define coprime polynomials. Let $R \in \mathbb{R}(s)$; derive equivalent conditions for $z \in \mathbb{C}$ to be a zero and a pole of R.

- Let $G \in \mathbb{R}(s)$ be a transfer function given by

$$G(s) = c^T(sI - A)^{-1}b + d.$$

Show that if $p \in \mathbb{C}$ is a pole of G, then p is also an eigenvalue of A, and that the converse is false.

- Define the equilibrium solution of

$$\dot{x}(t) = Ax(t), \quad A \in \mathbb{R}^{N \times N}.$$

What is 0 called?

- Define what it means for $\dot{x}(t) = Ax(t)$ to be asymptotically stable. Give an equivalent condition (with reasons) for the system to be asymptotically stable.

- Define BIBO stability, and show that the system is BIBO stable if and only if every pole of the transfer function

$$G(s) = c^T(sI - A)^{-1}b + d$$

has negative real part.

- Define what it means for $P \in \mathbb{R}[s]$ to be stable. State necessary conditions for $P \in \mathbb{R}[s]$ to be stable.

- Define the Hurwitz matrix associated with

$$P(s) = \sum_{i=0}^{N} a_i s^i \in \mathbb{R}[s], \quad a_N \neq 0.$$

Define the Hurwitz determinants associated with P.

- State the Hurwitz stability criterion.

- Assume that $\dot{x}(t) = Ax(t) + bu(t)$, $y(t) = c^T x(t)$ is BIBO stable and that $G(0) > 0$. Show that $\exists k^* > 0$ such that $\forall k \in (0, k^*)$, the feedback system is BIBO stable and $y(t) \to r$ as $t \to \infty$.