Example: For $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ we computed solutions of $\dot{x} = Ax$: $x_1(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $x_2(t) = e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (see Example with (1.8) in §1.3).

Since $x_1(t)$ and $x_2(t)$ are lin. indep., they constitute a fundamental system. Hence

$$\Phi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$ is a fundamental matrix.

Lemma: If $\Phi(t)$ and $\Psi(t)$ are two fundamental matrices of $\dot{x} = Ax$, then $\exists$ a constant matrix $C \in \mathbb{C}^{n\times n}$ s.t.

$$\Phi(t) = \Psi(t)\, C \quad \forall t \in \mathbb{R}.$$

Proof: Write $\Phi(t) = (x_1(t) | \cdots | x_n(t))$

$$\Psi(t) = (y_1(t) | \cdots | y_n(t))$$

Since $y_1(t), \ldots, y_n(t)$ are a fundamental system, $\exists$ constants $c_{ij} \in \mathbb{C}$ s.t.

$$x_j(t) = c_{1j}\, y_1(t) + c_{2j}\, y_2(t) + \cdots + c_{nj}\, y_n(t) \quad (*)$$

for each $j = 1, \ldots, n$. Now consider the kth component in the column vector $x_j(t)$; this is the kjth entry of $\Phi(t)$ i.e. $\Phi_{kj}(t)$ say.

Take the kth component of $(*)$:

$$\Phi_{kj}(t) = c_{1j}\, \Psi_{k1}(t) + c_{2j}\, \Psi_{k2}(t) + \cdots + c_{nj}\, \Psi_{kn}(t)$$

$$= \Psi_{k1} c_{1j} + \Psi_{k2} c_{2j} + \cdots + \Psi_{kn} c_{nj} \quad (\dagger)$$

the last expression being the kjth entry of ΨC.
This holds $\forall\, k, j \in (1, \ldots, n)$ and thus $\Phi = \Psi C$.

Note: $(\dagger)$ is

$$\underbrace{\begin{pmatrix} \vdots & \vdots & & \vdots \\ \Psi_{k1} & \Psi_{k2} & \cdots & \Psi_{kn} \\ \vdots & \vdots & & \vdots \end{pmatrix}}_{k\text{th row in } \Psi} \underbrace{\begin{pmatrix} \cdots & c_{1j} & \cdots \\ \cdots & c_{2j} & \cdots \\ & \vdots & \\ \cdots & c_{nj} & \cdots \end{pmatrix}}_{j\text{th column of } C.}$$

How to find a fundamental matrix?

Theorem: The map $\Phi(t) = \exp(tA)$ is a fundamental matrix for the system $\dot{\Phi} = A\Phi$.

Proof: Consider $\frac{d}{dt}\exp(tA) = \frac{d}{dt}\sum_{k=0}^{\infty}\frac{1}{k!}(tA)^k$

$$= \frac{d}{dt}\left(I + tA + \frac{t^2}{2!}A^2 + \cdots\right) \qquad A^0 = I$$

$$= A + \frac{2}{2!}tA^2 + \frac{3}{3!}t^2A^3 + \cdots$$

$$= A\left(I + tA + \frac{t^2}{2!}A^2 + \cdots\right) = A\exp(tA)$$

$\therefore\ \dot{\Phi} = A\Phi$ is satisfied.

To show that Φ is a fundamental matrix:

Write $\Phi = (x_1(t)|\cdots|x_n(t))$ where $x_i(t)$ is the ith column of Φ.

We have to show that the x_i

a) satisfy $\dot{x}_i = Ax_i$, $i = 1, \ldots, n$

and b) are lin. indep.

To show a):

Let $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\dots$, $e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

Then $x_i(t) = \Phi(t)\, e_i$

$$\therefore \; \dot{x}_i(t) = \dot{\Phi}(t)\, e_i = A\,\Phi(t)\, e_i = A\, x_i(t)$$

For b): $x_i(0) = \underset{\substack{\shortparallel \\ I}}{\Phi(0)}\, e_i = I\, e_i = e_i$ for $i = 1, \dots, n$

follows from the series definition of $\exp(tA)$ with $t=0$.

Thus the set $\{x_i(0)\}_{i=1}^{n}$ are lin. indep., since the e_i $i = 1, \dots, n$ are a set of basis vectors (for $\mathbb{C}^n$) and hence it follows that $\{x_i(t)\}_{i=1}^{n}$ are lin. indep.

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1.5. IVPs Revisited

Let $A \in \mathbb{C}^{n\times n}$ and seek $\Phi : \mathbb{R} \to \mathbb{C}^{n\times n}$ s.t.

$$\dot{\Phi} = A\Phi \quad \text{and} \quad \Phi(t_0) = \Phi_0 \in \mathbb{C}^{n\times n} \qquad (1.9)$$

If $\Psi(t)$ is a fundamental matrix and $\Psi(t_0) = \Psi_0$, set

$$\Phi(t) := \Psi(t)\,\Psi_0^{-1}\,\Phi_0$$

Claim: $\Phi(t)$ solves the IVP (1.9).

Proof: a) $\dot{\Phi} = \dot{\Psi}\,\Psi_0^{-1}\,\Phi_0 = A\,\Psi(t)\,\Psi_0^{-1}\,\Phi_0 = A\,\Phi(t)$

b) $\Phi(t_0) = \Psi(t_0)\,\Psi_0^{-1}\,\Phi_0 = \Psi_0\,\Psi_0^{-1}\,\Phi_0 = \Phi_0$ $\square$

Corollary: The unique solution of the IVP (1.9) is given by

$$\Phi(t) = \exp\big((t-t_0)A\big)\,\Phi_0 .$$

Proof: Take $\Psi(t) = \exp(tA)$ which is a fundamental matrix (see previous section).

Hence $\Psi_0 = \Psi(t_0) = \exp(t_0 A)$ is invertible and

$$\Psi_0^{-1} = \left(\exp(t_0 A)\right)^{-1} \underset{\uparrow}{=} \exp(-t_0 A)$$

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$$\therefore \Phi(t) = \Psi(t)\,\Psi_0^{-1}\,\Phi_0 = \exp(tA)\exp(-t_0 A)\,\Phi_0$$

$$\underset{\uparrow}{=} \exp((t-t_0)A)\,\Phi_0$$

(use series definition of exp).

Example: Solve $\dot{\Phi} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \Phi$, $\Phi(0) = I$. (1.10)

Solution: By the example of §1.4

$$\Psi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \text{ is a fundamental matrix}$$

Now $\Psi_0 = \Psi(0) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$, $\Psi_0^{-1} = -\frac{1}{4}\begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}$

$$\therefore \Phi(t) = \Psi(t)\Psi_0^{-1}\Phi_0 = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$$

□