NB. This theorem gives a recipe for finding m lin. indep. solutions i.e.

$$x_1(t) = e^{\lambda t} v_1 , \quad x_2(t) = e^{\lambda t}(v_2 + t v_1),$$

$$x_3(t) = e^{\lambda t}\left(v_3 + t v_2 + \frac{t^2}{2} v_1\right), \ \ldots,$$

$$x_m(t) = e^{\lambda t}\left(v_m + t v_{m-1} + \cdots + \frac{t^{m-1}}{(m-1)!} v_1\right),$$

if we have a generalised eigenvector of order $m \geq 2$ (wrt the eigenvalue $\lambda \in \mathrm{spec}(A)$).

NB. It also tells us how to generate the v_k $k = 1, \ldots, m$.

- Start by finding the generalised eigenvector of order m by solving
$$(A - \lambda I)^m v = 0$$
and set $v_m = v$
- Then set $v_{m-1} = (A - \lambda I) v_m$
- Then set $v_{m-2} = (A - \lambda I) v_{m-1}$
- Continue until we obtain $v_1 = (A - \lambda I) v_2$

Qn: When do we need generalised eigenvectors and how many?

Answer:

- When the geometric multiplicity for a given $\lambda \in \text{spec}(A)$ is LESS THAN the algebraic multiplicity.

- The number (of lin. indep. eigenvectors) to be found in total = algebraic multiplicity for the given $\lambda \in \text{spec}(A)$

Why?: See Primary Decomposition Theorem (later)

- Use generalised eigenvectors of order UPTO

$$d+1 \equiv \underbrace{n_\lambda}_{\text{algebraic mult.}} - \underbrace{m_\lambda}_{\text{geometric mult.}} + 1 \qquad \text{for a given } \lambda.$$

- Solve $(A-\lambda I)^{d+1} v = 0$.

NB

$$(A-\lambda I)v_1 = (A-\lambda I)^m v = 0$$
$$(A-\lambda I)^2 v_2 = (A-\lambda I)^m v = 0$$
$$\vdots$$
$$(A-\lambda I)^k v_k = (A-\lambda I)^m v = 0$$
$$\vdots$$
$$(A-\lambda I)^m v_m = (A-\lambda I)^m v = 0$$

NB

$$x_k(t) = e^{\lambda t} \sum_{i=0}^{k-1} \frac{t^i}{i!} v_{k-i} = e^{tA} v_k$$

since
$$e^{tA} = e^{\lambda t I} e^{t(A-\lambda I)} v_k$$
$$= e^{\lambda t} \cdot I \cdot \sum_{i=0}^{\infty} \frac{t^i}{i!} (A-\lambda I)^i v_k$$
$$= e^{\lambda t} \sum_{i=0}^{k-1} \frac{t^i}{i!} v_{k-i}$$

using $(A-\lambda I)^i v_k = 0$ for $i \geq k$

$(A-\lambda I)^i v_k = v_{k-i}$ for $i = 0, 1, \dots, k-1$.

Example Consider $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$$\Pi_A(\lambda) = (1-\lambda)^2(3-\lambda) \implies \lambda = 1, 3.$$

For $\lambda = 1$: Solve $(A - \lambda I)\hat{v} = 0 \iff \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$

$$\therefore \left.\begin{aligned} 0 &= 0 \\ a_1 + 2a_2 &= 0 \\ a_2 &= 0 \end{aligned}\right\} \Rightarrow \hat{v} = \begin{pmatrix} 0 \\ 0 \\ a_3 \end{pmatrix} = a_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ \forall a_3 \in \mathbb{C}$$

(Take $a_3 = 1$ wlog) so $\hat{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Only one lin. indep. eigenvector and thus $\lambda = 1$ has geometric multiplicity one
but algebraic " two

Hence seek a generalised eigenvector v_2 s.t.

$$(A - I)v_2 \neq 0 \quad \text{and} \quad (A - I)^2 v_2 = 0.$$

Now $(A - I)^2 v_2 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0.$

$$\therefore \left.\begin{aligned} 0 &= 0 \\ 2b_1 + 4b_2 &= 0 \\ b_1 + 2b_2 &= 0 \end{aligned}\right\} \Rightarrow v_2 = \begin{pmatrix} -2b_2 \\ b_2 \\ b_3 \end{pmatrix} = b_2\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + b_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Wlog take $b_2 = 1$, $b_3 = 0$ so $v_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ is a gen. eigenvector of order 2.

Then set $v_1 := (A-\lambda I)v_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

which (unsurprisingly) agrees with $\hat{v}$, since $(A-\lambda I)v_1 = (A-\lambda I)^2 v_2 = 0$.

For $\lambda = 3$: Solve $(A-3I)\hat{v}_3 = 0$

$$\therefore \begin{pmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = 0 \iff \begin{cases} -2d_1 = 0 \\ d_1 = 0 \\ d_2 - 2d_3 = 0 \end{cases}$$

$$\therefore \hat{v}_3 = \begin{pmatrix} 0 \\ 2d_3 \\ d_3 \end{pmatrix} = d_3 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad \text{taking } d_3 = 1 \text{ (wlog)}$$

Thus lin. indep. solutions to $\dot{x} = Ax$ are

$$x_1(t) = e^t v_1 = \begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix}, \quad x_2(t) = e^t(v_2 + t v_1) = \begin{pmatrix} -2e^t \\ e^t \\ te^t \end{pmatrix}$$

and $x_3(t) = e^{3t}\hat{v}_3 = \begin{pmatrix} 0 \\ 2e^{3t} \\ e^{3t} \end{pmatrix}$

with the general solution being (use (1.6))

$$x(t) = \sum_{i=1}^{3} c_i x_i(t)$$

for arbitrary constants $c_i \in \mathbb{C}$ $i = 1,2,3$.

Example Find three lin. indep. solutions of

$$\dot{x}(t) = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{A} x(t) .$$

Here $\Pi_A(\lambda) = \det(A - \lambda I) = (1-\lambda)^3 \Rightarrow \lambda = 1$.

For $\lambda = 1$: $A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Solve $(A - I)\hat{v} = 0 \iff \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$

$$\iff \left.\begin{matrix} a_2 = 0 \\ a_3 = 0 \\ 0 = 0 \end{matrix}\right\} \Rightarrow \hat{v} = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(Setting $a_1 = 1$)

Thus only one l.i. eigenvector

Hence $\lambda = 1$ has geometric multiplicty 1

algebraic " 3.

Hence we seek a generalised eigenvector v_3 of order 3:

$$(A - I)v_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} v_3 \neq 0, \quad (A-I)^2 v_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v_3 \neq 0,$$

and $(A - I)^3 v_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v_3 = 0$.

Thus $v_3 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is completely arbitrary, subject to

$$(A-I)^2 v_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_3 \\ 0 \\ 0 \end{pmatrix} \neq 0 \quad \text{i.e. } b_3 \neq 0, \text{ and}$$

$$(A-I) v_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_2 \\ b_3 \\ 0 \end{pmatrix} \neq 0 .$$

Take $b_3 = 1$, $b_2 = b_1 = 0$ (not a unique choice)

⌈Other choices include: $b_1 = 0$, $b_2 = b_3 = 1$ or $b_1 = b_2 = b_3 = 1$ etc
But NOT: $b_1 = 1$, $b_2 = b_3 = 0$ nor $b_2 = 1$, $b_1 = b_3 = 0$ etc⌋

Thus $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and then we set

$$v_2 = (A-I) v_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$v_1 = (A-I) v_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Thus 3 lin. indep. solutions are

$$x_1(t) = e^t v_1$$

$$x_2(t) = e^t (v_2 + t v_1)$$

$$x_3(t) = e^t \left(v_3 + t v_2 + \frac{t^2}{2} v_1\right) .$$

1.4 Fundamental Matrices

Assume that for $A \in \mathbb{C}^{n\times n}$ (constant matrix) we have found n lin. indep. solutions $x_1(t), \dots, x_n(t)$ of the system $\dot{x} = Ax$. Such n functions constitute a fundamental system.

Given a fundamental system:

(i) $\exists$ constants $c_i \in \mathbb{C}$ s.t. any other solution $x(t)$ can be written as

$$x(t) = c_1 x_1(t) + \cdots + c_n x_n(t).$$

(ii) Define a fundamental matrix

$$\Phi(t) = \left(x_1(t) \middle| \cdots \middle| x_n(t) \right) \in \mathbb{C}^{n\times n}$$

whose ith column is the vector function $x_i(t)$.

Then $\dot{\Phi} = \left(\dot{x}_1(t) \,|\, \cdots \,|\, \dot{x}_n(t) \right)$

$$= \left(A x_1(t) \,|\, \cdots \,|\, A x_n(t) \right)$$

$$\overset{(*)}{=} A \left(x_1(t) \,|\, \cdots \,|\, x_n(t) \right) = A \Phi$$

$$\therefore \quad \dot{\Phi} = A \Phi$$

Note: $\Phi^{-1}(t)$ exists $\forall t \in \mathbb{R}$, since the columns are lin. indep. ($\Rightarrow \det \Phi \neq 0$).

Note: Do a simple 2x2 example if necessary to convince yourself of (*) e.g. take

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \Phi = \begin{array}{c} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ x_1 & x_2 \end{array} \end{array}$$