MA20220 Ordinary Differential Equations (ODEs) and Control

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ODEs	Control
$\approx$ 14 Lectures	$\simeq$ 7 Lectures

- Web Page: people.bath.ac.uk/masjde
  Reading List, Course Notes, Problem Sheets (+Hints), Past Exam Papers

- Schedule:

  Lectures - Mon + Wed ; Tutorial (Week 2+)

  11 Problem Sheets : Issued @ Wed

  Due in following Thurs 4pm to Tutors

Course Structure

3 Parts

ODEs

I. Systems of Linear ODEs

$$\dot{x}(t) = A\,x(t) + g(t) \qquad (*)$$

A $n \times n$ matrix

$x(t), g(t)$ n-dimensional vectors

II. Laplace Transforms (LT)

LT transforms ODE into algebraic equation whose solutions are then transformed/inverted with inverse LT to yield solutions of original ODE.

Control

III. Control Theory

Controlling the dynamic behaviour of $(*)$ by suitable choice of $g(t)$.

0. Revision

In science and engineering, mathematical models often yield equations containing an unknown function together with some of its derivatives. Such an equation is termed a differential equation (DE).

Notation: Derivatives of a function x of scalar argument t i.e. $x(t)$

$$\frac{dx}{dt} \equiv \dot{x} \equiv x' \equiv x^{(1)} \quad ; \quad \frac{d}{dt}x(t) \equiv \dot{x}(t) \text{ etc.}$$

$$\frac{d^2x}{dt^2} \equiv \ddot{x} \equiv x'' = x^{(2)} \quad ; \quad \frac{d^nx}{dt^n} \equiv x^{(n)} \text{ etc.}$$

x dependent variable, t independent variable.

Examples:

1. Free fall of a body of mass m (constant):

$$m\frac{d^2h}{dt^2} = -mg \qquad (0.1)$$

$h(t)$ height, mg gravitational force

Solution of (0.1): (0.1) $\Rightarrow \ddot{h} = -g$

Integrate twice: $\dot{h} = -gt + c_1$

$$h = -\tfrac{1}{2}gt^2 + c_1 t + c_2 \qquad (0.2)$$

c_1, c_2 constants of integration

$c_2 = h(0)$ initial height

$c_1 = \dot{h}(0)$ initial velocity

2. Radioactive decay:

Rate of decay is proportional to the amount $x(t)$ of radioactive substance at time t:

$$\dot{x}(t) = -K\, x(t), \quad K > 0 \text{ constant} \qquad (0.3)$$

Solution: $$x(t) = C e^{-Kt} \qquad (0.4)$$

$C = x(0)$ initial amount

Remark: Solutions of DEs are not unique. $\forall\, C_1, C_2, C$ have a solution.
These are often found from initial conditions (IC)

NB. A DE is termed an ordinary differential equation (ODE) if the dependent variable (i.e. h, x above) is a function of only one independent variable (ie t in above)

Often deal with initial value problems (IVP):

ODE + IC

Example: Solve the IVP

$$\dot{y}(t) + a y(t) = 0 \,, \quad y(0) = 2 \qquad \text{(a constant)}$$

Solution: $$\frac{\dot{y}}{y} = -a$$

$$\therefore \quad \frac{d}{dt}\left(\ln y(t)\right) = -a.$$

$$\therefore \quad \ln y(t) = -at + c$$

$$\therefore \quad y(t) = e^{-at+c} = e^{-at}\underbrace{e^{c}}_{=C} = C e^{-at}$$

But $y(0) = 2 \Rightarrow C e^{-a.0} = 2 \Rightarrow C = 2$

$\therefore \quad y(t) = 2e^{-at}$ is the solution to the IVP.

Notation:

- The order of an ODE is given by the highest derivative occuring in it e.g (0.1) is a second order ODE, (0.3) is a first order ODE.

- An ODE is linear if the dependent variable or its derivatives occur to the first power only in any ~~each~~ term e.g (0.1) and (0.3) are linear ODEs

[$m\frac{d^2h}{dt^2} + \left(\frac{dh}{dt}\right)^2 = -mg$, $\dot{y} + ay^2 = 0$ are nonlinear ODEs]

1. Systems of linear ODEs

1.1 Writing ODEs as First-Order Systems

Any linear ODE (of any order) can be written in terms of a first order system.

Example: $\ddot{x} + 2\dot{x} + 3x = t \qquad (1.1)$

Let $X(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$ 2-d vector function

Then $\dot{X} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -2\dot{x} - 3x + t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ t \end{pmatrix}}_{g(t)}$

$\therefore \; \dot{X} = AX + g \qquad (1.2)$

(More examples Problem Sheet 1)

The system (1.2) and the equation (1.1) are equivalent i.e. x solves (1.1) iff X solves (1.2).

The most general system of first order is of the form

$$B(t)\,\dot{X}(t) + C(t)\,X(t) = f(t)$$

where $B(t), C(t) \in \mathbb{C}^{n\times n}$ and $X(t), f(t) \in \mathbb{C}^n$.

If $B^{-1}(t)$ exists $\forall t$, then we rewrite as

$$\dot{X}(t) = -B^{-1}(t)\,C(t)\,X(t) + B^{-1}(t)\,f(t)$$

i.e. $$\dot{X}(t) = A(t)\,X(t) + g(t) \qquad (1.3)$$

with $A(t) = -B^{-1}(t)\,C(t)$, $g(t) = B^{-1}(t)\,f(t)$.

Definitions:

1) Equation (1.3) is called the standard form of a first order system of ODEs.

2) If both $A(t)$ and $g(t)$ do not depend on t, so that all entries of both A and g are constants, the system is called autonomous.
If $g \equiv 0$, the system (1.3) is called homogeneous, otherwise ($g \neq 0$) inhomogeneous.
(non-homogeneous)