Useful Properties:

1. Set $f(t) = 1$ then $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

2. If $f : \mathbb{R} \to \mathbb{R}$ is continuous in an interval containing $a \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$$

3. For $f(t) = e^{-st}$ with $s \in \mathbb{R}$ fixed,

$$\int_{-\infty}^{\infty} e^{-st}\,\delta(t)\,dt = e^{-0} = 1.$$

$$\therefore \mathcal{L}\{\delta(t)\}(s) = \int_{0}^{\infty} \delta(t)\,e^{-st}\,dt = \int_{-\infty}^{\infty} \delta(t)\,e^{-st}\,dt = 1$$

Hence $\boxed{\mathcal{L}\{\delta(t)\}(s) = 1}$ or $\mathcal{L}^{-1}\{1\}(t) = \delta(t)$.

4. Further,

$$(f * \delta)(t) = \int_{\tau=0}^{t} f(t-\tau)\,\delta(\tau)\,d\tau = \int_{\tau=-\infty}^{\infty} f(t-\tau)\,\delta(\tau)\,d\tau = f(t)$$

Thus $f(t) * \delta(t) = \delta(t) * f(t) = f(t)$.

convolution is commutative

5. For $T \geq 0$:

$$\mathcal{L}\{\delta(t-T)\}(s) = \int_0^\infty \delta(t-T)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t-T)\, e^{-st}\, dt = e^{-sT}.$$

Thus $\mathcal{L}\{\delta(t-T)\}(s) = e^{-sT}$ or $\mathcal{L}^{-1}\{e^{-sT}\}(t) = \delta(t-T)$

6. Finally,

$$\delta(t-T) * f(t) = \int_{\tau=0}^{t} \delta(t-T-\tau)\, f(\tau)\, d\tau$$

$$\stackrel{(*)}{=} \begin{cases} 0, & \text{if } t < T \\ f(t-T), & \text{if } t \geq T. \end{cases}$$

Thus $\delta(t-T) * f(t) = f(t-T)\, H(t-T)$.

(*) Note: $\delta(t-T-\tau) = 0$ if $\tau \neq t-T$.

Example Solve

$$\ddot{y} + 2\dot{y} + y = \delta(t-1), \quad y(0) = 2, \quad \dot{y}(0) = 3.$$

Answer: Take LT of the ODE:

$$s^2\hat{y} - sy(0) - \dot{y}(0) + 2(s\hat{y} - y(0)) + \hat{y} = e^{-s}$$

$$\therefore (s^2 + 2s + 1)\hat{y} = e^{-s} + 2s + 7$$

$$\therefore \hat{y} = \frac{e^{-s}}{(s+1)^2} + \frac{2s+7}{(s+1)^2} = \frac{e^{-s}}{(s+1)^2} + \frac{5}{(s+1)^2} + \frac{2}{(s+1)}$$

Inverting using standard transforms:

$$y(t) = \delta(t-1) * te^{-t} + 5te^{-t} + 2e^{-t}$$

$$= (t-1)e^{-(t-1)} H(t-1) + (5t+2)e^{-t}.$$

$\square$

Note: $e^{-s} \cdot \frac{1}{(s+1)^2} \equiv \hat{f}(s)\,\hat{g}(s)$ where $\hat{f}(s) = e^{-s}$

$\hat{g}(s) = \frac{1}{(s+1)^2} \equiv \hat{h}(s+1)$

Now, $\mathcal{L}^{-1}\{\hat{f}(s)\} = \delta(t-1)$

By damping formula: $\mathcal{L}^{-1}\{\hat{h}(s+1)\} = e^{-t} h(t) = te^{-t}$

since $\hat{h}(s) = \frac{1}{s^2}$ and $\mathcal{L}^{-1}\{\hat{h}(s)\} = t$

2.5. Final Value Theorem.

Theorem: Let $g : [0,\infty) \to \mathbb{R}$ satisfy

$$|g(t)| \leq M e^{-\alpha t}$$

for some $\alpha > 0, M > 0$ (i.e. the function g is exponentially decaying). Then

$$\int_{t=0}^{\infty} g(t)\,dt = \lim_{t\to\infty} (g * H)(t) = \mathcal{L}\{g(t)\}(0).$$

Proof:

$$\mathcal{L}\{g\}(0) = \int_{\tau=0}^{\infty} g(\tau)\,d\tau = \lim_{t\to\infty} \int_{\tau=0}^{t} g(\tau)\,d\tau$$

$$\boxed{\begin{aligned}\sigma &= t-\tau \\ d\sigma &= -d\tau\end{aligned}} \qquad = \lim_{t\to\infty} -\int_{\sigma=t}^{0} g(t-\sigma)\,d\sigma$$

$$= \lim_{t\to\infty} \int_{\sigma=0}^{t} g(t-\sigma) H(\sigma)\,d\sigma$$

$$= \lim_{t\to\infty} (g * H)(t).$$

Note: "∞" is "$+\infty$" here.

$\square$.

END OF LECTURE NOTES

Notes on Damping & Delay formulas

Damping formula: $a \in \mathbb{C}$

$$\mathcal{L}\{e^{-at} f(t)\}(s) = \hat{f}(s+a) = \mathcal{L}\{f(t)\}(s+a) \qquad (1)$$

$$\mathcal{L}^{-1}\{\hat{f}(s+a)\}(t) = e^{-at} f(t) = e^{-at}\mathcal{L}^{-1}\{\hat{f}(s)\}(t) \qquad (1)^*$$

Examples:

1. $\mathcal{L}\{e^{-at}t^n\}(s) = \mathcal{L}\{t^n\}(s+a) = \dfrac{n!}{(s+a)^{n+1}}$ $\quad n \in \mathbb{N}$, $a \in \mathbb{R}$

holds for $\mathrm{Re}(s+a) > 0$ ie. $\mathrm{Re}(s) > -a$ since

$\mathcal{L}\{t^n\}(s) = \dfrac{n!}{s^{n+1}}$ $\quad \mathrm{Re}(s) > 0$.

2. $\mathcal{L}\{e^{-\lambda t}\cos(at)\}(s) = \mathcal{L}\{\cos(at)\}(s+\lambda) = \dfrac{s+\lambda}{(s+\lambda)^2+a^2}$, $\lambda \in \mathbb{R}$, $a \in \mathbb{R}$

holds for $\mathrm{Re}(s+\lambda) > 0$ ie $\mathrm{Re}(s) > -\lambda$ using

$\mathcal{L}\{\cos(at)\}(s) = \dfrac{s}{s^2+a^2}$ for $\mathrm{Re}(s) > 0$.

3. $\mathcal{L}^{-1}\left\{\dfrac{s+a}{(s+a)^2+b^2}\right\}(t) = e^{-at}\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+b^2}\right\}(t) = e^{-at}\cos(bt).$

4. $\mathcal{L}^{-1}\left\{\dfrac{5}{(s+1)^2}\right\}(t) = e^{-t}\mathcal{L}^{-1}\left\{\dfrac{5}{s^2}\right\}(t) = e^{-t}5t = 5te^{-t}$

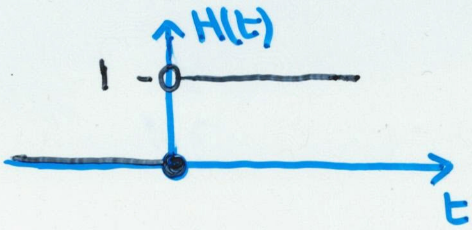
NB. $\mathcal{L}^{-1}\left\{\dfrac{5}{s^2}\right\} = 5\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\}$ holds.

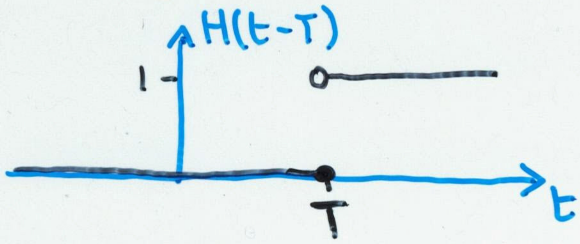
Delay formula: $T>0$

$$\mathcal{L}\{f(t-T)H(t-T)\}(s) = e^{-sT}\hat{f}(s) = e^{-sT}\mathcal{L}\{f(t)\}(s) \qquad (2)$$

$$\mathcal{L}^{-1}\{e^{-sT}\hat{f}(s)\}(t) = f(t-T)H(t-T) = \mathcal{L}^{-1}\{\hat{f}(s)\}(t-T)\,H(t-T) \qquad (2)^*$$

$$H(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 0 \end{cases} \qquad H(t-T) = \begin{cases} 0 & t \le T \\ 1 & t > T \end{cases}$$





Examples:

1. $\mathcal{L}\{(t-T)H(t-T)\}(s) = e^{-sT}\mathcal{L}\{t\}(s) = e^{-sT}\cdot\frac{1}{s^2}$

2. $\mathcal{L}\{H(t-T)\}(s) = e^{-sT}\mathcal{L}\{1\}(s) = e^{-sT}\cdot\frac{1}{s}$

3. $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t-1)\,H(t-1) = (t-1)H(t-1)$

   since $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) = t$.

4. $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+1)^2}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t-1)\,H(t-1) = (t-1)e^{-(t-1)}H(t-1)$

   using $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t) \underset{\uparrow}{=} e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) = te^{-t}$

   damping formula

<u>NB</u>. Alternatively: $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+1)^2}\right\}(t) = \delta(t-1) * te^{-t}$

as the example in §2.4.