Chapter 3

Nondimensionalisation

The first and arguably the most important step in the analysis of a system of differential equations.

It involves scaling each variable (dependent and independent) by a typical or reference value, leaving a nondimensional variable whose typical scale is $O(1)$.

Nondimensionalisation or problem normalisation has several important uses:

1. It identifies the dimensionless groups (ratios of dimensional parameters) which control the solution behaviour.

2. Terms in the equations are now dimensionless and so allows comparison of their sizes.

   This allows the identification of the important (i.e. dominant) terms in the equations and their interaction in different regimes, giving insight into the structure of solutions and the dominant physical mechanisms at work.

   In particular, negligible terms can be identified leading to simplification in many circumstances.

3. It allows estimates of the effects of additional features to the original model through the new dimensional group(s) associated with the additional term(s). This allows measurement of the effect of the physical feature(s) in the model.

4. Finally, it can reduce the number of parameters ocurring in the problem by forming the nondimensional parameters or dimensionless groups.

3.1 Scaling

If an equation has a variable u, say, then we nondimensionalise that variable by writing, for example,

$$u = [u]\bar{u}$$

where $[u]$ is the chosen scale (with the same dimensions as u) and $\bar{u}$ is the corresponding dimensionless variable. If a system of equations describes a real process then it is dimensionally homogeneous i.e. consistent. The process of nondimensionalisation will necessarily give a set of equations, each of whose terms is dimensionless, after division through by the dimensions of the equations. It is then possible to compare terms in a meaningful way.

The art of nondimensionalisation lies in the choice of scales. There is no right or wrong way to do it (other than to only partially nondimensionalise the equations) and in more complicated problems, the choice of scales can be the difficult part of the analysis. The basic principle is that the scales must ultimately be chosen self-consistently by balancing terms in the equations. Because the purpose is to obtain 'properly scaled' equations in which the largest dimensionless terms are numerically of $O(1)$, the simplest choices arise when the scales can be chosen so that all the dimensionless parameters are $O(1)$.

This provides our rationale. Given no other information, one assumes a priori that dimensionless variables and their derivatives are $O(1)$, until we are forced to assume otherwise. It is only when inconsistencies arise that the process of rescaling becomes necessary. The generic situation in which this happens is where singular perturbation theory is appropriate. In general, not all dimensionless parameters can be choosen to be $O(1)$, in which case they are first assumed to be $O(1)$ and then the limit in which they become large is taken.

3.2 Examples

3.2.1 Example 1

The number of atoms $N(t)$ at time t of a radiactive substance is governed by the differential equation

$$\frac{dN}{dt} = -\lambda N$$

with an initial condition $N = N_0$ at $t = 0$. Here λ is a decay constant with units of $[\text{time}]^{-1}$.

We nondimensionalise as follows

$$N = N_0 \bar{N}, \qquad t = \frac{\bar{t}}{\lambda},$$

where N_0 is taken as the reference value for N and $1/\lambda$ for the time scale. This gives the dimensionless problem

$$\frac{d\bar{N}}{d\bar{t}} = -\bar{N} \qquad \text{with } \bar{N} = 1 \text{ at } \bar{t} = 0,$$

for $\bar{N}(\bar{t})$.

3.2.2 Example 2

The motion of a linearly damped pendulum is governed by the equation

$$\ell \frac{d^2\theta}{dt^2} + k\frac{d\theta}{dt} + g\sin\theta = 0, \tag{3.1}$$

with the initial conditions

$$\text{at t=0} \qquad \theta = \theta_0 \quad \text{and} \quad \frac{d\theta}{dt} = \omega_0. \tag{3.2}$$

Here $\theta(t)$ represents the angle that the pendulum makes to the vertical at time t, the initial angle being θ_0 and initial angular speed ω_0. The dimensional parameters are the length ℓ of the pendulum, the coefficient of resistance k and acceleration due to gravity g.

We nondimensionalise as follows

$$\theta = \theta_0 \bar{\theta}, \qquad t = \frac{\theta_0}{\omega_0}\bar{t},$$

using the initial values to give characteristic scales for the dependent variable θ and independent variable t. Thus we obtain the dimensionless problem

$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + K\frac{d\bar{\theta}}{d\bar{t}} + G\sin\left(\theta_0\bar{\theta}\right) = 0, \tag{3.3}$$

subject to

$$\text{at } \bar{t} = 0 \qquad \bar{\theta} = 1 \quad \text{and} \quad \frac{d\bar{\theta}}{d\bar{t}} = 1, \tag{3.4}$$

where we have introduced the dimensionless parameters

$$K = \frac{k\theta_0}{\omega_0 \ell}, \qquad G = \frac{g\theta_0}{\omega_0^2 \ell},$$

in addition to θ_0.

The dimensional problem has 5 parameters, the four dimensional parameters ℓ, k, g, ω_0 and the dimensionless parameter θ_0. The dimensionless problem has three dimensionless parameters K, G, θ_0.

If observational values are available for the parameters $\ell, k, g, \omega_0, \theta_0$ then these may be used to infer the sizes of the dimensionless groups K, G, θ_0. The sizes of these dimensionless parameters determine whether they are to be taken as $O(1)$ or a suitable limit needs to be considered in which their values are small or large. The latter cases lead to either a regular or singular perturbation problem.

As an example, suppose we have the situation in which $\theta_0 = \pi/4$ radians, $\omega_0 = 1$ radian per second, $\ell = 1$ m, $k = 0.1$ m/s and $g = 10$ m/s^2. Then $K \approx 0.079, G \approx 7.9, \theta_0 \approx 0.79$. Thus we are interested in analyzing the dimensionless problem in the limit $K \to 0$ with $G = O(1), \theta_0 = O(1)$ which should be regular.

If in contrast $k = 100$ m/s then we would have $K \approx 79, G \approx 7.9, \theta_0 \approx 0.79$. Thus we are now interested in analyzing the dimensionless problem in the limit $K \to \infty$ with $G = O(1), \theta_0 = O(1)$ which is singular.

3.2.3 Example 3

Consider the following boundary-value problem (BVP) for one-dimensional heat flow in a bar,

$$\text{in } 0 < x < \ell, \;\; t > 0 \qquad \rho c \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + q, \tag{3.5}$$

$$\text{at } x = 0 \qquad u = u_0, \tag{3.6}$$

$$\text{at } x = \ell \qquad -k \frac{\partial u}{\partial x} = h(u - u_i), \tag{3.7}$$

$$\text{at } t = 0 \qquad u = u_i, \tag{3.8}$$

where u_i is the initial temperature (which is also the external surrounding temperature), u_0 is the temperature at the end of the bar which is raised above u_i, h denotes the heat transfer coefficient, ρ, c, k denote the density, specific heat and conductivity of the bar respectively. The length of the bar is ℓ and q represents a constant heat source term. All variables are assumed dimensional, this being the statement of the dimensional problem.

We nondimensionalise as follows

$$x = \ell \bar{x}, \qquad t = \frac{\ell^2}{\kappa} \bar{t}, \qquad u = u_i + (u_0 - u_i)\bar{u},$$

with $\kappa = k/\rho c$ and introduce the two dimensionless parameters

$$Q = \frac{q\ell^2}{k(u_0 - u_i)}, \qquad H = \frac{h\ell}{k},$$

to obtain

$$\text{in } 0<\bar{x}<1,\ \ \bar{t}>0 \qquad \frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + Q, \tag{3.9}$$

$$\text{at } \bar{x}=0 \qquad \bar{u}=1, \tag{3.10}$$

$$\text{at } \bar{x}=1 \qquad -\frac{\partial \bar{u}}{\partial \bar{x}} = H\bar{u}, \tag{3.11}$$

$$\text{at } \bar{t}=0 \qquad \bar{u}=0. \tag{3.12}$$

It is assumed that H, Q are $O(1)$ quantities. If they are not, then the appropriate limit needs to be taken which will lead to either a regular or singular perturbation problem.

3.2.4 Example 4

Consider the Navier-Stokes equations of an incompressible Newtonian viscous fluid

$$\nabla.\mathbf{v} = 0,$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{v}\right) = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{F},$$

$$\rho c\left(\frac{\partial T}{\partial t} + (\mathbf{v}.\nabla)T\right) = k\nabla^2 T + \Phi,$$

where $\mathbf{v}$ is the velocity, p is the pressure, T is the temperature and are functions of spatial coordinates $\mathbf{x}$ and time t. Also,

$$\Phi = \frac{1}{2}\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2$$

is the viscous dissipation term. The dimensional parameters of density ρ, specific heat c, conductivity k and viscosity μ are assumed constant.

The body force per unit mass $\mathbf{F}$ is assumed to be gravity so that $\mathbf{F} = \mathbf{g} = g\bar{\mathbf{g}}$ with $g = |\mathbf{g}|$.

These equations arise from the application of the physical laws of conversation of mass, momentum and energy.

In the momentum equation, the physical effects modelled are: inertia, viscous forces, gravity respectively.

In the heat equation, the physical effects modelled are: heat convection (or advection), heat conduction, viscous dissipation respectively.

The problem is completed by specification of suitable boundary conditions to a give a well-posed BVP.

Consequently typical scales or reference values for the variables will be given.

Let U be a typical velocity scale

L be a typical length

$T_1 - T_0$ be a typical temperature deviation (T_0 is an ambient or equilibrium temperature)

then we may nondimensionalise as follows

$$\mathbf{x} = L\bar{\mathbf{x}}, \quad \mathbf{v} = U\bar{\mathbf{v}}, \quad t = \frac{L}{U}\bar{t}, \quad p = P\bar{p}, \quad T = T_0 + (T_1 - T_0)\bar{T},$$

where P is to be determined, which gives

$$\bar{\nabla}.\bar{\mathbf{v}} = 0,$$

$$\left(\frac{\partial \bar{\mathbf{v}}}{\partial \bar{t}} + (\bar{\mathbf{v}}.\bar{\nabla})\bar{\mathbf{v}}\right) = -\frac{P}{\rho U^2}\bar{\nabla}\bar{p} + \frac{\mu}{\rho U L}\bar{\nabla}^2\bar{\mathbf{v}} + \frac{Lg}{U^2}\bar{\mathbf{g}},$$

$$\left(\frac{\partial \bar{T}}{\partial \bar{t}} + (\bar{\mathbf{v}}.\bar{\nabla})\bar{T}\right) = \frac{k}{\rho c U L}\bar{\nabla}^2\bar{T} + \frac{\mu U}{\rho c L(T_1 - T_0)}\bar{\Phi},$$

where

$$\bar{\Phi} = \frac{1}{2}\left(\frac{\partial \bar{v}_i}{\partial \bar{x}_j} + \frac{\partial \bar{v}_j}{\partial \bar{x}_i}\right)^2, \qquad \bar{\nabla} = \left(\frac{\partial}{\partial \bar{x}_1}, \frac{\partial}{\partial \bar{x}_2}, \frac{\partial}{\partial \bar{x}_3}\right).$$

Introduce the dimensionless parameters

Reynolds number $Re = \dfrac{\rho U L}{\mu}$ compares effects of inertia and viscous forces

Froude number $Fr = \dfrac{U^2}{Lg}$ compares inertia and gravity

Peclet number $Pe = \dfrac{\rho c U L}{k} = \dfrac{UL}{\kappa}$ compares convection to conduction

Brinkman number $Br = \dfrac{\mu U^2}{k(T_1 - T_0)}$ compares viscous dissipation with heat conduction

Prandtl number $Pr = \dfrac{\mu c}{k} = \dfrac{Pe}{Re}$ compares viscous terms to those of heat conduction

Hence we have

$$\bar{\nabla}.\bar{\mathbf{v}} = 0,$$

$$\left(\frac{\partial \bar{\mathbf{v}}}{\partial \bar{t}} + (\bar{\mathbf{v}}.\bar{\nabla})\bar{\mathbf{v}}\right) = -\bar{\nabla}\bar{p} + \frac{1}{Re}\bar{\nabla}^2\bar{\mathbf{v}} + \frac{1}{Fr}\bar{\mathbf{g}},$$

$$\left(\frac{\partial \bar{T}}{\partial \bar{t}} + (\bar{\mathbf{v}}.\bar{\nabla})\bar{T}\right) = \frac{1}{Pe}\bar{\nabla}^2\bar{T} + \frac{Br}{Pe}\bar{\Phi},$$

where we have chosen $P = \rho U^2$ i.e. the pressure scale is based on inertial forces.

The choice of pressure scale is not unique and could have been based on viscous forces so that

$$P = \frac{\mu U}{L} = \frac{\rho U^2}{Re}.$$

The appropriate scalings are dictated by the problem and the numerical values of the dimensionless parameters.

The identification of dimensionless parameters allows model similitude i.e. a model will represent (dynamically similar to) a practical situation if the values of the dimensionless parameters are the same for both.

The limits of large and small Reynolds number Re are of particular interest.

Simplification of the equations by neglecting a term multiplied by a small dimensionless parameter is the first step in a systematic procedure for obtaining an asymptotic expansion for the full solution in terms of the small parameter.

If we consider the above model with $1/Fr = Br = 0$ and no temperature variation then

$$\bar{\mathbf{v}} = \bar{\mathbf{v}}(\bar{\mathbf{x}}, \bar{t}, Re) \quad \text{and} \quad \bar{p} = \bar{p}(\bar{\mathbf{x}}, \bar{t}, Re).$$

If $Re \ll 1$ then we may expand in regular powers of Re to obtain the asymptotic expansions

$$\bar{p} = \bar{p}_0(\bar{\mathbf{x}}, \bar{t}) + Re\,\bar{p}_1(\bar{\mathbf{x}}, \bar{t}) + \ldots, \qquad \bar{\mathbf{v}} = \bar{\mathbf{v}}_0(\bar{\mathbf{x}}, \bar{t}) + Re\,\bar{\mathbf{v}}_1(\bar{\mathbf{x}}, \bar{t}, Re) + \ldots,$$

where $\bar{p}_0$ and $\bar{\mathbf{v}}_0$ are the solutions to the simplified model with $Re = 0$.

This is a regular perturbation procedure and may be used if $\bar{p}(\bar{\mathbf{x}}, \bar{t}, Re)$ and $\bar{\mathbf{v}}(\bar{\mathbf{x}}, \bar{t}, Re)$ are regular functions of Re near $Re = 0$.

In some circumstances the expansion is invalid and a singular perturbation procedure must be used. In these cases the solution of the problem with $Re = 0$ and the solution for $Re \neq 0$ but $Re \ll 1$ are very different.

An example is the high Reynolds number flow $Re \gg 1$. A regular expansion in powers of $1/Re$ gives the inviscid flow equations at leading order. In this case the highest derivtive term $\bar{\nabla}^2\bar{\mathbf{v}}$ is neglected and consequently not all boundary conditions can be satisfied (usuall the no slip conditon). Thus thin regions develop where this term must be brought back, which are termed boundary layers.

3.2.5 Example 5

The one-phase Stefan problem for the temperature $T(x,t)$ in the water phase for a melting ice problem can be written as

$$\text{in } 0 < x < s(t),\ \ t > 0 \qquad \rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}, \tag{3.13}$$

$$\text{at } x = 0 \qquad T = T_0, \tag{3.14}$$

$$\text{at } x = s(t) \qquad T = 0, \quad -k\frac{\partial T}{\partial x} = L\rho \frac{ds}{dt}, \tag{3.15}$$

$$\text{at } t = 0 \qquad s = 0, \tag{3.16}$$

where $s(t)$ denotes the moving ice/water interface, L is the latent heat per unit mass, ρ, c, k are the density, specific heat and conductivity of the water. The water (liquid phase) occupies the region $0 < x < s(t)$ and the ice (solid phase) $x > s(t)$. Initially there is no water present i.e. s(0)=0. T_0 (> 0) is the temperature of the water at the fixed boundary $x = 0$.

We nondimensionalise as follows

$$x = \ell \bar{x}, \qquad s = \ell \bar{s}, \qquad t = \beta \bar{t}, \qquad T = T_0 \bar{T},$$

where the scales ℓ and β have to be found. The system (3.13)–(3.16) becomes

$$\text{in } 0 < \bar{x} < \bar{s}(\bar{t}),\ \ \bar{t} > 0 \qquad \frac{\partial \bar{T}}{\partial \bar{t}} = \frac{k}{\rho c}\frac{\beta}{\ell^2}\frac{\partial^2 \bar{T}}{\partial \bar{x}^2}, \tag{3.17}$$

$$\text{at } \bar{x} = 0 \qquad \bar{T} = 1, \tag{3.18}$$

$$\text{at } \bar{x} = \bar{s}(\bar{t}) \qquad \bar{T} = 0, \quad -\frac{\partial \bar{T}}{\partial \bar{x}} = \frac{L\rho}{kT_0}\frac{\ell^2}{\beta}\frac{d\bar{s}}{d\bar{t}}, \tag{3.19}$$

$$\text{at } \bar{t} = 0 \qquad \bar{s} = 0, \tag{3.20}$$

The governing equation (3.17) suggests

$$\beta = \frac{\rho c}{k}\ell^2,$$

and the Stefan condition on the moving interface then suggests introducing the dimensionless parameter λ where

$$\lambda = \frac{L\rho}{kT_0}\frac{\ell^2}{\beta} = \frac{L}{cT_0},$$

which is commonly termed the Stefan number. The dimensionless problem is thus

$$\text{in } 0 < \bar{x} < \bar{s}(\bar{t}),\ \ \bar{t} > 0 \qquad \frac{\partial \bar{T}}{\partial \bar{t}} = \frac{\partial^2 \bar{T}}{\partial \bar{x}^2}, \tag{3.21}$$

$$\text{at } \bar{x} = 0 \qquad \bar{T} = 1, \tag{3.22}$$

$$\text{at } \bar{x} = \bar{s}(\bar{t}) \qquad \bar{T} = 0, \quad -\frac{\partial \bar{T}}{\partial \bar{x}} = \lambda\frac{d\bar{s}}{d\bar{t}}, \tag{3.23}$$

$$\text{at } \bar{t} = 0 \qquad \bar{s} = 0, \tag{3.24}$$

where the length scaling ℓ remains arbitrary; this is not unexpected given that the original problem had no inherent spatial length scale. The fact that ℓ is not fixed leads to the existence of a similarity solution of this problem termed the Neumann solution (one of the few explicit solutions that exist for moving boundary problems).

3.3 Dimensional analysis

The topic of dimensional analysis formalises the procedure of nondimensionalisation. An important theorem in which is the Buckingham Pi theorem:

If n quantities $Q_1, Q_2, \ldots, Q_n$ (dependent, independent variables and parameters) involving r separate fundamental dimensional components (usually $r = 3$, these being mass, length, time, i.e. M,L,T, but see the table below) are related by a unique dimensionally consistent function $f(Q_1, \ldots, Q_n) = 0$, then we can find $n - r$ dimensionless combinations of Q_i, say $\Pi_j(Q_1, \ldots, Q_n)$, $j = 1, \ldots, n - r$, such that the solution can be expressed as $F(\Pi_1, \ldots, \Pi_{n-r}) = 0$.

As a first illustration of this theorem, consider Example 2. Let $f(\theta, t, \ell, k, g, \theta_0, \omega_0) = 0$ be the solution of the IVP (3.1)–(3.2). There are two dimensionless quantities (θ, θ_0) and $n = 5$ dimensional quantities involving $r = 2$ fundamental dimensions (L,T). Thus the solution can be expressed in terms of θ, θ_0 and $n - r = 3$ dimensionless quantities namely $F(\theta, \theta_0, \bar{t}, K, G) = 0$ as shown by (3.3)–(3.4).

As a second illustration of this theorem, consider Example 3. Let $f(u - u_i, x, t, \rho c, k, q, h, \ell, u_0 - u_i) = 0$ be the solution of the BVP (3.5)–(3.8). There are $n = 9$ dimensional quantities involving $r = 4$ fundamental dimensions (M,L,T,Θ). Thus the solution can be expressed in terms of $n - r = 5$ dimensionless quantities namely $F(\bar{u}, \bar{x}, \bar{t}, Q, H) = 0$ as shown by (3.9)–(3.12).

As a third illustration, we consider Example 4. Let $f(\mathbf{v}, p, T - T_0, \mathbf{x}, t, \rho, \mu, c, k, T_1 - T_0, g, L, U) = 0$ be the solution of the dimensionless governing equations. There are $n = 17$ dimensional quantities involving $r = 4$ fundamental dimensions (M,L,T,Θ). Thus the solution can be expressed in terms of $n - r = 13$ dimensionless quantities namely $F(\bar{\mathbf{v}}, \bar{p}, \bar{T}, \bar{\mathbf{x}}, \bar{t}, Re, Fr, Pe, Br) = 0$ as shown by the dimensionless equations.

As a fourth illustration, we consider Example 5. Let $f(T, s, x, t, \rho, c, k, L, T_0) = 0$ be the solution of the moving boundary problem (3.13)–(3.16). There are $n = 9$ dimensional quantities involving $r = 4$ fundamental dimensions (M,L,T,Θ). Thus the solution can be expressed in terms of $n - r = 5$ dimensionless quantities namely $F(\bar{T}, \bar{s}, \bar{x}, \bar{t}, \lambda) = 0$ as shown by (3.21)–(3.24).

Note 1. The function f may be a solution of a PDE, a BVP or IVP.

Note 2. If a PDE involves fewer fundamental dimensions than dimensional quantities, it must admit a simplified solution in accordance with the Buckingham Pi Theorem.

The International System (SI) of fundamental units are:

Fundamental Dimension	Base Unit
Length (L)	metre, m
Mass (M)	kilogram, kg
Time (T)	second, s
Electric current (A)	ampere, A
Thermodynamic temperature (Θ)	kelvin, K
Amount of substance (X)	mole, mol
Luminous intensity (I)	candela, cd