MATHEMATICS OF COMPUTATION
Volume 00, Number 0, Pages 000–000
S 0025-5718(XX)0000-0

A FILON-CLENSHAW-CURTIS-SMOLYAK RULE FOR MULTI-DIMENSIONAL OSCILLATORY INTEGRALS WITH APPLICATION TO A UQ PROBLEM FOR THE HELMHOLTZ EQUATION

ZHIZHANG WU, IVAN G. GRAHAM, DINGJIONG MA, AND ZHIWEN ZHANG

Abstract. In this paper, we combine the Smolyak technique for multi-dimensional interpolation with the Filon-Clenshaw-Curtis (FCC) rule for one-dimensional oscillatory integration, to obtain a new Filon-Clenshaw-Curtis-Smolyak (FCCS) rule for oscillatory integrals with linear phase over the $d-$dimensional cube $[-1,1]^d$. By combining stability and convergence estimates for the FCC rule with error estimates for the Smolyak interpolation operator, we obtain an error estimate for the FCCS rule, consisting of the product of a Smolyak-type error estimate multiplied by a term that decreases with $\mathcal{O}(k^{-\widetilde{d}})$, where k is the wavenumber and $\widetilde{d}$ is the number of oscillatory dimensions. If all dimensions are oscillatory, a higher negative power of k appears in the estimate. As an application, we consider the forward problem of uncertainty quantification (UQ) for a one-space-dimensional Helmholtz problem with wavenumber k and a random heterogeneous refractive index, depending in an affine way on d i.i.d. uniform random variables. After applying a classical hybrid numerical-asymptotic approximation, expectations of functionals of the solution of this problem can be formulated as a sum of oscillatory integrals over $[-1,1]^d$, which we compute using the FCCS rule. We give numerical results for the FCCS rule which illustrate its theoretical properties and show that the accuracy of the UQ algorithm improves when both k and the order of the FCCS rule increase. We also give results for both the quadrature and UQ problems when the underlying FCCS rule uses a dimension-adaptive Smolyak interpolation. These show increasing accuracy for the UQ problem as both the adaptive tolerance decreases and k increases, requiring very modest increase in work as the stochastic dimension increases, for a case when the affine expansion in random variables decays quickly.

1. Introduction

In this paper, we formulate and analyse a novel numerical method for computing the multi-dimensional oscillatory integral

$$\mathcal{I}^{k,d,\mathbf{a}}f := \int_{[-1,1]^d} f(\boldsymbol{y})\exp(\mathrm{i}k\mathbf{a}\cdot\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \tag{1.1}$$

where $k>0$ is a parameter, which may be large, and $\mathbf{a}=(a_1,...,a_d)^\top\in\mathbb{R}^d$ is a fixed vector. As an application of this, we solve an uncertainty quantification problem for the Helmholtz equation (modelling frequency-domain linear waves),

2020 *Mathematics Subject Classification.* Primary 35C20, 42B20, 65D30, 65D32, 65D40.

via a hybrid numerical asymptotic method, yielding increasing accuracy as the frequency increases.

Background. The computation of oscillatory integrals is a classical problem in applied mathematics (e.g., [40]), which has enjoyed considerable recent interest. By combining numerical and asymptotic techniques, research has focussed on providing methods which work well for moderate k, but remain accurate (or even improve in accuracy) as the parameter k (proportional to frequency) gets large. While there has been strong interest in this topic in the recent past (partly driven by applications in high-frequency scattering e.g., [4, 17, 16]), most methods proposed in this context are appropriate only for relatively low-dimensional oscillatory integration problems.

On the other hand there is a considerable literature on multi-dimensional integration for the non-oscillatory version of (1.1), where k is small - here we mention just [14, 32, 3, 9, 29, 43] as exemplars of the huge literature on this topic. However, since the accuracy of these rules depends on the derivatives of the integrand, their direct application to the whole integrand in (1.1) will incur an error which in general will blow up strongly with increasing k.

Quite a large portion of research on oscillatory integration in the low-dimensional case (mostly $d = 1$) is concerned with Filon-type methods. In [20, 21, 22, 24, 42] the analysis concentrates on accelerating the convergence as $k \to \infty$.

The basic 1D method central to the current paper is [11], which proves stability and algebraic convergence (with respect to the number of function evaluations) for the Filon-Clenshaw-Curtis rule, explicit in the regularity required of f. The convergence is superalgebraic if $f \in C^\infty$ and the error estimate features a negative power of k as $k \to \infty$. The range of application of this approach was considerably extended in recent work [25]. Extensions of Filon methods to hp approximation and the handling of nonlinear phase functions (again in 1D) are given in [28, 10, 25, 27, 26].

A complexity study of quadrature rules for oscillatory integration in one dimension has been carried out in [33], with a recent review in [31]. We comment on the relation of these results to ours in Remark 2.6 later in the paper.

To extend the approach of [11] to the higher dimensional case, the factor $f(\boldsymbol{y})$ in (1.1) should be approximated by some linear combination of simple basis functions, with coefficients which can be computed easily from f, and then this approximation should be integrated analytically against the oscillatory factor $\exp(\mathrm{i}k\mathbf{a}\cdot\boldsymbol{y})$. There are relatively few papers on the higher dimensional oscillatory case. Exceptions are [23, 19] (see also [8]) which include discussion of generalization of a Filon-type method to problem (1.1), making use of function values and derivatives at vertices of the boundary and proving increasing accuracy as $k \to \infty$, but without explicit error estimates showing how the error depends on the number of function evaluations, the regularity of f or the dimension d.

Overview of Algorithm. Our method for (1.1) essentially extends the 1D 'Filon-Clenshaw-Curtis' approach to the multidimensional case by applying Smolyak-type interpolation to the non-oscillatory part of the integrand in (1.1). Since we shall allow $\mathbf{a}$ to have entries of either sign and possibly small, we introduce the following notation in order to identify the oscillatory and non-oscillatory dimensions in (1.1).

Notation 1.1. For $a \in \mathbb{R}$ and $k > 0$ we define

$$\widetilde{a} = \begin{cases} a & \text{if} \quad k|a| \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

and we set $\widehat{a} = a - \widetilde{a}$. (We note that $\widetilde{a}$ and $\widehat{a}$ depend on k as well as a.)

Applying Notation 1.1 to each component of the vector $\mathbf{a}$ in (1.1) we obtain the decomposition $k\mathbf{a} = k\widehat{\mathbf{a}} + k\widetilde{\mathbf{a}}$, where each component of $k\widehat{\mathbf{a}}$ has modulus bounded above by 1. and thus $\widehat{f}(\boldsymbol{y}) := f(\boldsymbol{y})\exp(\mathrm{i}k\widehat{\mathbf{a}}\cdot\boldsymbol{y})$ is the non-oscillatory part of the integrand in (1.1). We then rewrite (1.1) as

$$\mathcal{I}^{k,d,\mathbf{a}} f = \int_{[-1,1]^d} \widehat{f}(\boldsymbol{y})\exp(\mathrm{i}k\widetilde{\mathbf{a}}\cdot\boldsymbol{y})\mathrm{d}\boldsymbol{y}. \tag{1.2}$$

Our quadrature rule for (1.2) (and hence (1.1)) is then defined by replacing the factor $\widehat{f}$ by its classical Smolyak polynomial interpolant $\mathcal{Q}^{r,d}\widehat{f}$ of maximum level r (formally defined in (3.3)). This sparse grid interpolant employs separable polynomial interpolation at points on sparse grids generated by a nested sequence of 1-d grids. Here we use, at level ℓ, the points:

$$\{0\} \text{ for } \ell = 1, \quad \text{and} \quad \left\{ t_{j,\ell} := \cos\left(\frac{j\pi}{n_\ell}\right) \right\}_{j=0}^{n_\ell} \quad \text{for } \ell \geq 2, \tag{1.3}$$

where $n_\ell = 2^{\ell-1}$, i.e. the mid-point rule is used at level $\ell = 1$ and $2^{\ell-1}+1$ Clenshaw-Curtis points are used at level $\ell \geq 2$. Using this, we approximate (1.2) by

$$\mathcal{I}^{k,d,\mathbf{a},r} f := \int_{[-1,1]^d} (\mathcal{Q}^{r,d}\widehat{f})(\boldsymbol{y})\exp(\mathrm{i}k\widetilde{\mathbf{a}}\cdot\boldsymbol{y})\mathrm{d}\boldsymbol{y}. \tag{1.4}$$

This is a d-dimensional version of the 1D Filon-Clenshaw-Curtis (FCC) rule from [11].

Since, in each dimension the interpolant on the Clenshaw-Curtis grid can be written as a linear combination of Chebyshev polynomials of the first kind of degree n (here denoted by T_n), the integral (1.4) can be computed exactly, provided the $k-$dependent 'weights'

$$W_n(ka_j) := \int_{-1}^{1} T_n(y)\exp(\mathrm{i}ka_j y)dy, \quad n = 0,\ldots,n_\ell, \quad \ell \geq 2, \quad j = 1,\ldots d \tag{1.5}$$

are known. A stable algorithm for computing these weights (for any k, a_j and n), and its analysis, are given in [11]. (The weight for the case $\ell = 1$ is trivial to compute.) Since the cost of computing the weights for an $M+1$ point rule in 1-d and with a fixed choice of k and a_j has complexity $\mathcal{O}(M\,\log M)$ (see [11, Remark 5.4]), and since the weights for each dimension can be computed independently, the cost of computing the weights (1.5) (on a serial computer) grows with $\mathcal{O}(d\,2^{r-1}\log(2^{r-1}))$ as dimension d or the maximum level r increases. Weight computation in each dimension could be done in parallel.

Main results of the paper. In order to prove an error estimate, we assume that

$$f \in \mathcal{W}^{p,d} := \left\{ f : [-1,1]^d \to \mathbb{R} : \frac{\partial^{|\mathbf{s}|} f}{\partial \boldsymbol{y}^{\mathbf{s}}} \in C([-1,1]^d), \text{ for all } |\mathbf{s}|_\infty \leq p \right\} \tag{1.6}$$

for some positive integer p, where $\|\cdot\|_\infty$ denotes the uniform norm over $[-1,1]$, $\mathbf{s} = (s_1,...,s_d) \in \mathbb{N}_0^d$ are the multi-indices of order $|\mathbf{s}| = s_1 + \cdots + s_d$,

$$\frac{\partial^{|\mathbf{s}|} f}{\partial \boldsymbol{y}^{\mathbf{s}}} = \frac{\partial^{|\mathbf{s}|} f}{\partial y_1^{s_1}\cdots\partial y_d^{s_d}},$$

and $|\mathbf{s}|_\infty = \max_{1\le i\le d} s_i$. Thus $\mathcal{W}^{p,d}$ is here defined as the space of $d-$variate functions on $[-1,1]^d$, whose mixed partial derivatives of up to order p in each dimension are continuous. We introduce the norm on $\mathcal{W}^{p,d}$:

$$\|f\|_{\mathcal{W}^{p,d}} := \max_{\mathbf{s}\in\mathbb{N}_0^d\,:\,|\mathbf{s}|_\infty\le p} \left\| \frac{\partial^{|\mathbf{s}|} f}{\partial \boldsymbol{y}^{\mathbf{s}}} \right\|_{\infty,[-1,1]^d}. \tag{1.7}$$

We note that $\mathcal{W}^{p,1}$ is just the usual space $C^p[-1,1]$ with norm given by

$$\|f\|_{\mathcal{W}^{p,1}} = \max\{\|f^{(j)}\|_{\infty,[-1,1]},\quad j=0,\dots,p\}.$$

By combining the properties of the Smolyak algorithm and the FCC rule, together with the regularity assumption (1.6), we prove in §4 the following error estimates.

Theorem 1.2. *For each $p\ge 1$ and $d\ge 1$ there is a constant $C_{d,p}>0$ such that, for all $\mathbf{a}\in\mathbb{R}^d$, $k>0$, and r sufficiently large, we have*

$$\begin{aligned} &|\mathcal{I}^{k,d,\mathbf{a}} f - \mathcal{I}^{k,d,\mathbf{a},r} f| \\ &\le C_{d,p} \left(\prod_{\substack{j\in\{1,\dots,d\}\\ k|a_j|\ge 1}} k|a_j| \right)^{-1} \left(\log^{d-1} N(r,d)\right)^p \left(\tfrac{1}{N(r,d)}\right)^{p-1} \|f\|_{\mathcal{W}^{p,d}}, \end{aligned} \tag{1.8}$$

where $N(r,d)$ is the number of point evaluations of f used in the quadrature rule (1.4).

While this result is explicit in k and $\mathbf{a}$, we can obtain the following better estimate, for sufficiently large k, although this is implicit in its dependence on $\mathbf{a}$ and requires more regularity on f.

Theorem 1.3. *Suppose $\mathbf{a}\in\mathbb{R}^d$ with $a_j\ne 0$ for each j. Then, for each $p\ge 1$ and $d\ge 1$, there is a constant $C_{d,p,\mathbf{a}}>0$ such that, for r sufficently large, we have*

$$|\mathcal{I}^{k,d,\mathbf{a}} f - \mathcal{I}^{k,d,\mathbf{a},r} f| \;\le\; C_{d,p,\mathbf{a}}\, k^{-(d+1)} \left(\log^{d-1} N(r,d)\right)^p \left(\frac{1}{N(r,d)}\right)^{p-1} \|f\|_{\mathcal{W}^{p+3,d}}, \tag{1.9}$$

The third and fourth terms on the right-hand sides of (1.8) and (1.9) are standard in the analysis of Smolyak-type approximation methods, having a power of $\log N$ which grows with dimension. While this is good for moderate (but not very high) dimension d, our estimates (1.8), (1.9) also contain additional factors containing potentially high negative powers of k, giving a substantial advantage over traditional tensor product rules for these integrals: If a $d-$dimensional tensor product of the standard one-dimensional (non-Filon) Clenshaw-Curtis rule were used to approximate (1.1) then (for fixed p and d), the standard error estimate would give $\mathcal{O}(k^p N^{-p/d})$ as $k, N\to\infty$ (where N is the total number of points used - see Example 3.1 for a precise statement). This is vastly inferior to (1.8), (1.9) when either d or k is large. The estimate (1.8) can be seen as a generalisation of the concept of *universality* discussed in [32, eqn. (7)] to the case of oscillatory integrals.

When computing oscillatory integrals, one should always bear in mind that the integral itself is usually decaying in modulus as k increases. In 1d, simple arguments (see, e.g., Remark 2.6) show that $\|\mathcal{I}^{k,1,a}\|_{\mathcal{W}^{p,1}} \sim k^{-1}$, provided $a\ne 0$ and this argument can be extended to show that $\|\mathcal{I}^{k,d,\mathbf{a}}\|_{\mathcal{W}^{p,d}} \sim k^{-\widetilde{d}}$, where $\widetilde{d}$ is the number

of oscillatory dimensions (i.e., the number of components a_j of $\mathbf{a}$ such that $k|a_j| \geq 1$). For this reason, it is natural to also consider the *normalized error*, which, for simplicity, we define here as

$$k^{\widetilde{d}} \left| \mathcal{I}^{k,d,\mathbf{a}} f - \mathcal{I}^{k,d,\mathbf{a},r} f \right| , \tag{1.10}$$

(although a slightly different definition of this concept is used in complexity theory – see Remark 2.6). Then Theorem 1.2 shows that the absolute error of the FCCS rule decays with at least the same order $\mathcal{O}(k^{-\widetilde{d}})$ as $\mathcal{I}^{k,d,a} f$ itself (as $k \to \infty$), whereas Theorem 1.3 essentially shows that, under certain mild conditions, the normalized error of the FCCS rule actually decays with at least order $\mathcal{O}(k^{-1})$ as $k \to \infty$.

It may be noticed that the estimate (1.8) contains the factor $N(r,d)^{1-p}$, for functions with p mixed derivatives, whereas in the classical non-oscillatory case (e.g., [32]), this is replaced by the better factor $N(r,d)^{-p}$. Since we consider here the oscillatory case, it is important to have negative powers of k in the estimate, and for this we pay with one less negative power of $N(r,d)$. Nevertheless we also point out that the following alternative to (1.8)

$$|\mathcal{I}^{k,d,\mathbf{a}} f - \mathcal{I}^{k,d,\mathbf{a},r} f| \leq C_{d,p} \left(\log^{d-1} N(r,d) \right)^{p+1} \left(\frac{1}{N(r,d)} \right)^{p} \|f\|_{\mathcal{W}^{p,d}}, \tag{1.11}$$

valid for all k, can also be proved, thus recovering the usual power of $N(r,d)$, but losing decay with respect to k. Theorems 1.2, and 1.3 will be proved in §4, and there we shall also give a sketch of the proof of (1.11).

In proving (1.8), (1.9) we make no special assumption concerning the decay of the derivatives of f with respect to increasing dimension. Numerical experiments for the non-oscillatory case (e.g., [15, 30]) have shown that if a suitable decay rate is present, then dimension-adaptive algorithms will give better results as d increases. Theory underpinning the idea of the dimension-adaptive algorithms is given in [5, 37]. In §6.1 we give computations using both a standard and a dimension-adaptive method (the latter adapted from the algorithm in [30]) for the oscillatory quadrature problem. These demonstrate the theoretical properties of the former in general and the efficiency of the latter in cases where the importance of the dimensions is decaying.

As an application of our FCCS rule, we consider the UQ problem for the one-space-dimensional Helmholtz boundary-value problem:

$$Lu(x) := u''(x) + k^2 n^2(x) u(x) = F(x), \quad 0 < x < 1 \tag{1.12}$$

$$B_L u(x) := u(0) = u_L, \tag{1.13}$$

$$B_R u(x) := u'(1) - ikn_\infty u(1) = 0, \tag{1.14}$$

where k is the wave number, u_L and $n_\infty > 0$ are constants, F is a smooth function and n is the (generally variable) refractive index, here assumed sufficiently smooth and uniformly bounded above and below by positive numbers. The boundary conditions model a sound-soft scattering boundary at $x = 0$ and a simple absorbing boundary condition at $x = 1$. In our UQ problem, n is assumed to be a random field of the form

$$n(x, \boldsymbol{y}) = n_0(x) + \sum_{j=1}^{d} n_j(x) y_j, \tag{1.15}$$

where $\boldsymbol{y} \in [-1,1]^d$ are uniform i.i.d. random variables and the quantity of interest (QoI) is a linear functional (with respect to the x variable) of the solution u. Although 1-d in space, this problem still has some considerable difficulties for large k, because the solution $u(x,\boldsymbol{y})$ suffers oscillations with respect not only to the spatial variable x but also to the random variable $\boldsymbol{y}$ as $k \to \infty$, and the latter phenomenon poses considerable difficulty for UQ algorithms at high wavenumber. The structure of these oscillations is explained in more detail in §5.2.

However (because we are in 1d in space), for each fixed choice of $\boldsymbol{y}$, the resulting deterministic Helmholtz problem can be solved (with accuracy up to any negative order in k) using an asymptotic method (with some numerical approximation), originally proposed by Aziz, Kellogg and Stephens [1], in which the work involved is independent of k. This provides us with an asymptotic ansatz for the solution of the random Helmholtz problem, and (it turns out that) the expected value of the QoI then can be expressed as a sum of oscillatory and non-oscillatory integrals of the form (1.1) with respect to the random variables, which we can compute with $k-$independent accuracy using our FCCS rule.

In §6.2 we give numerical results for the Helmholtz UQ problem, using the the numerical-asymptotic approximation for the Helmholtz problem and comparing the standard and dimension-adaptive methods for the multi-dimensional integrals. In the dimension-adaptive case, for an example where the expansion (1.15) decays exponentially, we observe increasing accuracy of the results as both the adaptive tolerance decreases and the wave number k increases, with very modest growth in complexity with increasing dimension.

This is significant, since it is known that for Helmholtz problems in any space dimension, derivatives with respect to the random parameters $\boldsymbol{y}$ of the solution blow up as k increases, thus enforcing strong constraints on UQ methods in general at high wavenumber. For example, in [13], conditions ensuring convergence of first order randomized Quasi-Monte Carlo methods for a Helmholtz problem in any space dimension were studied. There, to ensure a dimension-independent optimal error estimate, one requires $\sum_{j=1}^{\infty} \|n_j\|_{W^{1,\infty}}^{2/3} = \mathcal{O}(k^{-2/3})$, (i.e., the amplitude of the allowed randomness must decrease as k increases). Strong constraints on the allowed amplitude of the randomness also appear in the so-called multi-modes method described in [12]. We impose no such constraint in our computations. Quasi-Monte Carlo methods for random Helmholtz problems in 2d were studied in [35, Chapter 4], where it was observed (for moderate wavenumbers) that the number of quadrature points needed to ensure a bounded error as k increased apparently grew exponentially in k.

The blow-up (as k increases) of the derivatives with respect to $\boldsymbol{y}$ of the Helmholtz solution is directly related to the width of the region in $\mathbb{C}^d$ in which (the complex extension of) u is holomorphic. This region of holomorphy is analysed in detail in the recent paper [38] for trapping and non-trapping geometries in any dimension and with general random perturbations of a deterministic base problem. In particular it is shown there that the estimates in [13] (for a non-trapping case and the expansion (1.15)) are sharp.

The remainder of the paper is organized as follows. In §2, we recall the Filon-Clenshaw-Curtis (FCC) rule for the 1d oscillatory integral and give some basic theory for it. In §3, we combine the 1d quadrature with the Smolyak algorithm to

obtain our new FCCS rule for the multi-dimensional oscillatory integrals. In §4, we give the error analysis for the FCCS rule, proving Theorems 1.2 and 1.3. The application to the UQ problem for the Helmholtz equation is given in §5. In §6, we present numerical examples to demonstrate the performance of the FCCS rule and its application to the UQ problem, while §7 provides a short concluding section.

A longer version of the current paper can be found in [41].

2. The FCC quadrature rule for 1D problems

In this section, we briefly review from [11] the Filon-Clenshaw-Curtis (FCC) rule for approximating the one-dimensional integral

$$I^{\omega} g := \int_{-1}^{1} g(y) \exp(\mathrm{i}\omega y) dy, \tag{2.1}$$

for $\omega \in \mathbb{R}$, where $g \in C^p[-1,1]$ for some integer $p \geq 1$.

In the oscillatory case ($|\omega| \geq 1$), the FCC quadrature rule is obtained by replacing the factor g in (2.1) by its degree N polynomial interpolant at the Clenshaw-Curtis points $\cos(j\pi/N)$, $j = 0, ..., N$ for $N \geq 1$ (extrema of the Chebyshev polynomial of the first kind $T_N(y) := \cos\big(N \arccos(y)\big)$). The interpolant is expressed in terms of the basis $\{T_n : n = 0, \ldots, N\}$, and the products of these basis functions with the oscillatory function $\exp(i\omega y)$ are integrated exactly to obtain the quadrature weights.

Starting with the nested set of quadrature points (1.3), the FCC approximation to (2.1) is then

$$I^{\omega,\ell}_{\mathrm{FCC}} g := \int_{-1}^{1} (Q^{\ell} g)(y) \exp(\mathrm{i}\omega y) dy, \tag{2.2}$$

where $(Q^1 g) = g(0)$ and, for $\ell \geq 2$, $Q^{\ell} g$ is the polynomial of degree n_ℓ satisfying

$$(Q^{\ell} g)(t_{j,\ell}) = g(t_{j,\ell}), \quad j = 0, \ldots, n_\ell. \tag{2.3}$$

It is a classical result that, for $\ell \geq 2$, $Q^{\ell} g$ can be written as

$$(Q^{\ell} g)(y) = {\sum_{n=0}^{n_\ell}}'' a_{n,\ell}(g) T_n(y), \tag{2.4}$$

where the notation $\sum''$ means that the first and the last terms in the sum are to be halved, and the coefficients $a_{n,\ell}(g)$ are given by the discrete cosine transform:

$$a_{n,\ell}(g) \;=\; \frac{2}{n_\ell} {\sum_{j=0}^{n_\ell}}'' \cos\left(\frac{jn\pi}{n_\ell}\right) g(t_{j,\ell}), \quad n = 0, \ldots, n_\ell. \tag{2.5}$$

Substituting (2.4) into (2.2) for $\ell \geq 2$, we obtain the quadrature rule

$$I^{\omega,\ell}_{\mathrm{FCC}}\, g := \begin{cases} W_0(\omega) g(0), & \ell = 1, \\ {\displaystyle\sum_{n=0}^{n_\ell}}'' W_n(\omega) a_{n,\ell}(g), & \ell \geq 2, \end{cases} \tag{2.6}$$

where the weights

$$W_n(\omega) = \int_{-1}^{1} T_n(y) \exp(\mathrm{i}\omega y)\, dy, \quad n = 0, \ldots, n_\ell \tag{2.7}$$

have to be computed. An algorithm for computing $W_n(\omega)$ for $\ell \geq 2$ is given and shown to be stable for all n_ℓ and ω in [11].

In the case where $\omega = 0$, the weights are known analytically:

$$W_n(0) = \int_{-1}^{1} T_n(y)dy = \begin{cases} 0, & n \text{ is odd}, \\ \dfrac{2}{1-n^2}, & n \text{ is even}, \end{cases} \tag{2.8}$$

and these provide us with the classical standard Clenshaw-Curtis (CC) rule [6]:

$$\int_{-1}^{1} g(y)\mathrm{d}y \;\approx I^{\ell}_{\mathrm{CC}}\, g \;:=\; \int_{-1}^{1} Q^{\ell} g \;=\; \begin{cases} 2g(0), & \ell = 1, \\ \sum\limits_{n=0}^{n_\ell}{}'' W_n(0) a_{n,\ell}(g), & \ell \geq 2. \end{cases} \tag{2.9}$$

When $|\omega| < 1$ the integral (2.1) is considered to be non-oscillatory, and can be approximated directly with the CC rule. Hence our 1D quadrature method is:

Definition 2.1. For $\omega \in \mathbb{R}$ and $\ell \geq 1$, we define the approximation $I^{\omega,\ell} g$ to the integral (2.1) by

$$I^{\omega,\ell} g = \begin{cases} I^{\omega,\ell}_{FCC}\, g, & \text{when } \; |\omega| \geq 1, \\ I^{\ell}_{CC}\left(g(\cdot)\exp(\mathrm{i}\omega\cdot)\right), & \text{when } \; |\omega| < 1. \end{cases} \tag{2.10}$$

Remark 2.2. Given the values $\{g(t_{j,\ell}) : j = 1, \ldots 2^{\ell-1}\}$, the quadrature rule $I^{\omega,\ell} g$ can be computed with complexity $\mathcal{O}(n_\ell \log n_\ell)$, using FFT (see, e.g., [11, Remarks 2.1, 5.4]).

The following simple proposition uses Notation 1.1 to give a unified expression for (2.10) and will be useful later.

Proposition 2.3. Let $g \in C[-1,1]$ and $\ell \geq 1$. Then the rule defined in (2.10) can be written

$$I^{ka,\ell} g = \int_{-1}^{1} (Q^{\ell}\widehat{g})(y)\, \exp(\mathrm{i}k\widetilde{a}y)\mathrm{d}y, \quad \text{where} \quad \widehat{g}(y) = g(y)\exp(\mathrm{i}k\widehat{a}y). \tag{2.11}$$

The following error estimate is a minor variation on the result in [11, Corollary 2.3].

Theorem 2.4. *Define $\eta(1) = 0$, $\eta(2) = 3$. Then, for $p \in \mathbb{N}$, $p > 1$, and $s \in \{1, 2\}$, there exists a constant C_p such that, for all $\ell \geq 2$ and $g \in \mathcal{W}^{p+\eta(s),1}$, the quadrature rule (2.10) has the error estimate:*

$$\left| I^{\omega} g - I^{\omega,\ell} g \right| \;\leq\; C_p \, \min\left\{1, |\omega|^{-s}\right\} \left(\frac{1}{n_\ell}\right)^{p-1} \|g\|_{\mathcal{W}^{p+\eta(s),1}}, \tag{2.12}$$

for all $\omega \in \mathbb{R}$, where $\min\left\{1, |\omega|^{-s}\right\} := 1$, if $\omega = 0$. Moreover for $\ell = 1$ and any $p > 1$ we have

$$\left| I^{\omega} g - I^{\omega,\ell} g \right| \;\leq\; C_p \, \min\left\{1, |\omega|^{-1}\right\} \left(\frac{1}{n_\ell}\right)^{p-1} \|g\|_{\mathcal{W}^{p,1}}. \tag{2.13}$$

Proof. When $\ell \geq 2$, we have $n_\ell \geq 2$. A slight variation of [11, Theorem 2.2] then shows that there exists a constant $C > 0$ such that, for all $p > 1$, the estimate

$$\left|I^\omega g - I^{\omega,\ell} g\right| \;\leq\; C|\omega|^{-s} \left(\frac{1}{n_\ell}\right)^{p-1} \|g_c\|_{H^{p+\eta(s)}}, \tag{2.14}$$

holds, for $|\omega| \geq 1$, where $g_c(\theta) := g(\cos\theta)$ is the cosine transform of g and $\|\cdot\|_{H^p}$ is the usual univariate Sobolev norm of order p. The estimate in [11, Theorem 2.2] is only stated for positive ω, but the negative case can be trivially obtained from this by replacing y by $-y$ and ω by $-\omega$ in (2.1).

For $|\omega| < 1$, we proceed by estimating the error in the classical Clenshaw-Curtis rule applied to $\widehat{g}(y) := g(y)\exp(\mathrm{i}\omega y)$ by

$$\begin{aligned} \left|I^\omega g - I^{\omega,\ell} g\right| &= \left|\int_{-1}^{1} (I - Q^\ell)\widehat{g}\right| \leq \sqrt{2}\|(I - Q^\ell)\widehat{g}\|_{L^2[-1,1]} \\ &\leq \sqrt{2}\left(\int_0^\pi |((I - Q^\ell)\widehat{g})_c(\theta)|^2 \mathrm{d}\theta\right)^{1/2} \\ &\leq C\left(\frac{1}{n_\ell}\right)^p \|\widehat{g}_c\|_{H^p}. \end{aligned} \tag{2.15}$$

(In the last step, we used [11, eq (15)]).

The constants C appearing in (2.14) and (2.15) are independent of ℓ and ω as well as p, but to complete the proof we need to estimate the Sobolev norms on the right-hand sides of (2.14) and (2.15) in terms of the $\mathcal{W}^{p,1}$ norm of g. This is where the p-dependence appears. Since all the derivatives of $\cos\theta$ are bounded above by 1, the Faa di Bruno formula (e.g. [18])) readily yields

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^p (g_c(\theta))\right| \;\leq\; \sum_{\mathcal{P}} |g^{(|\mathcal{P}|)}(\cos\theta)|, \quad \theta \in [-\pi, \pi],$$

where the sum is over all partitions $\mathcal{P}$ of the set $\{1, \ldots, p\}$, with $|\mathcal{P}|$ denoting the number of subsets in $\mathcal{P}$. Since $|\mathcal{P}| \leq p$ and the number of all such partitions is B_p (the pth *Bell number*) it follows that $\|g_c\|_{H^p} \leq CB_p\|g\|_{\mathcal{W}^{p,1}}$. Also, since $\widehat{g}_c(\theta) = g_c(\theta)\exp(\mathrm{i}\omega\cos\theta)$, an application of the Leibniz rule shows that

$$\|\widehat{g}_c\|_{H^p} \leq K_p\|g_c\|_{H^p} \leq CB_pK_p\|g\|_{\mathcal{W}^{p,1}},$$

and combining this with (2.15), completes the proof of (2.12).

To obtain (2.13) for $|\omega| \geq 1$, we integrate by parts to obtain

$$\begin{aligned} I^\omega g - I^{\omega,1} g &= \int_{-1}^{1} (g(y) - g(0))\exp(\mathrm{i}\omega y)\mathrm{d}y \\ &= \frac{1}{\mathrm{i}\omega}\left([(g(y) - g(0))\exp(\mathrm{i}\omega y)]_{-1}^{1} - \int_{-1}^{1} g'(y)\exp(\mathrm{i}\omega y)\mathrm{d}y\right), \end{aligned}$$

from which the required estimate follows, since $n_1 = 1$. When $|\omega| < 1$, the proof is trivial. □

Remark 2.5 (Dependence on p). In the proof above, the constant C_p in (2.12) grows quickly with p, in fact with the order of the growth of the Bell number B_p. This is the price we pay for an estimate to be uniform in ℓ. (Uniformity of the estimate with respect to ℓ is required in the proof of Theorem 4.2, because in the Smolyak construction, low order approximations in some dimensions are combined with high

order in others, so we need estimates for all orders.) If (2.12) were only required to hold for ℓ sufficiently large (relative to p), then the constant C_p can be bounded independently of p (see, for example, [10, Remark 2.4]).

Remark 2.6 (Relation to results from complexity theory). The results in Theorem 2.4 imply that, for $p \geq 2$,

$$\begin{aligned} &\sup_{\|g\|_{\mathcal{W}^{p,1}} \leq 1} \left|I^{\omega} g - I^{\omega,\ell} g\right| \lesssim |\omega|^{-1} \left(\tfrac{1}{n_\ell}\right)^{p-1} \\ \text{and} \quad &\sup_{\|g\|_{\mathcal{W}^{p+3,1}} \leq 1} \left|I^{\omega} g - I^{\omega,\ell} g\right| \lesssim |\omega|^{-2} \left(\tfrac{1}{n_\ell}\right)^{p-1}, \end{aligned} \tag{2.16}$$

where $\lesssim$ indicates that a generic multiplicative constant independent of ω, p and ℓ is omitted. These are bounds for the "*worst-case error*" as studied in complexity theory - see [33], [31], where the authors develop this theory for the one-dimensional oscillatory integral $I^{\omega} g$ (albeit for integer ω and in Sobolev spaces, rather than the spaces of functions with continuous partial derivatives considered here). In that literature the *normalized error* is defined as the quotient of the absolute error in (2.16) and the norm of I^{ω}. Since trivial integration by parts yields

$$I^{\omega} g = \frac{1}{\mathrm{i}\omega} \left([\exp(\mathrm{i}\omega y) g(y)]_{-1}^{1} - \int_{-1}^{1} \exp(\mathrm{i}\omega y) g'(y) \mathrm{d}y \right),$$

we have $\|I^{\omega}\|_{\mathcal{W}^{p,1}} \lesssim |\omega|^{-1}$, for all $p \geq 1$. Moreover, this bound is sharp in terms of ω dependence. In particular:

$$\text{(a)} \quad I^{\omega} 1 = \frac{2}{\omega} \sin \omega \quad \text{and} \quad \text{(b)} \quad I^{\omega} y = \frac{2}{\mathrm{i}\omega} \left(\cos \omega - \frac{\sin \omega}{\omega} \right).$$

Using (a) for ω bounded away from integer multiples of π and (b) otherwise, we obtain $|\omega|^{-1} \lesssim \|I^{\omega}\|_{\mathcal{W}^{p,1}}$, for all $p \geq 0$. Thus (2.16) can be rewritten:

$$\begin{aligned} &\sup_{\|g\|_{\mathcal{W}^{p,1}} \leq 1} \left|I^{\omega} g - I^{\omega,\ell} g\right| \lesssim \left(\tfrac{1}{n_\ell}\right)^{p-1} \|I^{\omega}\|_{\mathcal{W}^{p,1}} \\ \text{and} \quad &\sup_{\|g\|_{\mathcal{W}^{p+3,1}} \leq 1} \left|I^{\omega} g - I^{\omega,\ell} g\right| \lesssim |\omega|^{-1} \left(\tfrac{1}{n_\ell}\right)^{p-1} \|I^{\omega}\|_{\mathcal{W}^{p+3,1}}, \end{aligned} \tag{2.17}$$

yielding bounds on the normalized error (as defined in [33], [31]). While both bounds in (2.16) show that as $|\omega| \to \infty$, the (absolute) error gets smaller and in some sense the problem becomes 'easier', the second bound in (2.17) shows that the same is true for the normalized error, provided we are working in spaces of sufficiently smooth functions.

Compared to these results, the detailed analysis in [33] has some major differences: While (2.16) gives error estimates for our specific FCC rule, [33] studies the infimum of this worst case error (absolute or normalized) over all quadrature rules using a given number of point evaluations of f (thus including the FCC rule as a special case). Moreover [33] gives both upper and lower error bounds (essential in complexity theory). Nevertheless [33] contains some qualitative statements which are comparable to ours, in particular (a) When considering absolute errors, the problem becomes easier for large $|\omega|$ ([33, Corollary 16]); (b) When considering normalized errors the problem also becomes easier for large $|\omega|$, but this is only true when g has enough Sobolev regularity [33, Corollary 18].

While the results in [33] only apply in one dimension, the purpose of this paper is to study the high-dimensional FCCS rule. Here we show that our method maintains the aforementioned advantages for computing $\mathcal{I}^{k,d,\mathbf{a}} f$: as $|k|$ grows, the

accuracy of our method improves when considering the absolute error criterion; see Theorem 1.2, and also improves when considering normalized error, under certain mild additional regularity conditions; see Theorem 1.3.

3. An FCCS rule for multi-dimensional integrals

The direct application of the tensor product version of a conventional 1D rule to the multi-dimensional problem (1.1) will give very poor results as d or k increases, firstly because of the high oscillation and secondly because of the curse of dimensionality. The difficulty is illustrated by the following simple example.

Example 3.1. Suppose the integral (1.1) is approximated by the tensor product of the 1d Clenshaw-Curtis rule (2.9), using $n+1$ points in each coordinate direction, so that the integrand is evaluated at $N := (n+1)^d$ points. Then, in the special case $f(\boldsymbol{y}) = 1$ and $\mathbf{a} = (1, 0, 0, \ldots, 0)^\top$, the error is

$$\begin{aligned}\int_{[-1,1]^{d-1}} \int_{-1}^{1} (I - P^n)\Big(\exp(\mathrm{i}k\cdot)\Big)(y_1)\,\mathrm{d}y_1\mathrm{d}y_2\ldots\mathrm{d}y_d \\ = \; 2^{d-1}\int_{-1}^{1}(I-P^n)\Big(\exp(\mathrm{i}k\cdot)\Big)(y_1)\,\mathrm{d}y_1,\end{aligned}$$

where P^n denotes the polynomial interpolant at $n+1$ Clenshaw-Curtis points in 1d (i.e. the operator Q^ℓ in (2.4), with n_ℓ replaced by n). The error estimate for this is $k^p\, n^{-p} = k^p\, \mathcal{O}(N^{-p/d})$ for any p.

We alleviate the problem of growth with respect to k by adopting the Filon approach described above. Then, to reduce the effect of dimension (encapsulated in the $N^{-p/d}$ term), we approximate (1.1) by replacing $\widehat{f}$ in (1.2) by its Smolyak interpolant $\mathcal{Q}^{r,d}\widehat{f}$ defined as follows.

Using the 1D interpolation operator Q^ℓ in (2.3), and the nested sequence of Clenshaw-Curtis grids in (1.3), we define the difference operator D^ℓ (for $\ell \geq 1$) by

$$D^\ell g := (Q^\ell - Q^{\ell-1})g, \quad \text{with} \quad Q^0 g := 0. \tag{3.1}$$

To define the Smolyak interpolation operator, it is convenient to define the index set $\Lambda(q,d)$, for integers q, d with $q \geq d$ by

$$\Lambda(q,d) \;=\; \{\boldsymbol{\ell} \in \mathbb{N}^d : \mathbf{1} \leq \boldsymbol{\ell},\, |\boldsymbol{\ell}| \leq q\},$$

where $\mathbf{1} = (1,\ldots,1)^\top$ and $|\boldsymbol{\ell}| = \ell_1 + \ell_2 + \ldots + \ell_d$. By [39, p.13], the cardinality of $\Lambda(q,d)$ is given by the binomial coefficient:

$$\#\Lambda(q,d) = \begin{pmatrix} q \\ d \end{pmatrix}. \tag{3.2}$$

Smolyak's formula for interpolating any function $f : \mathbb{R}^d \to \mathbb{R}$, with maximum level $r \in \mathbb{N} := \{1, 2, 3, \ldots\}$, is then given (e.g., in [39, eq (10)] or [14, p.214]) by

$$(\mathcal{Q}^{r,d} f)(\boldsymbol{y}) := \sum_{\boldsymbol{\ell}\in\Lambda(r+d-1,d)} (D^{\ell_1} \otimes \ldots \otimes D^{\ell_d}) f(\boldsymbol{y}). \tag{3.3}$$

See [3, Prop 6] for a discussion of the interpolatory properties of $\mathcal{Q}^{r,d}$. The notation $D^{\ell_1} \otimes \ldots \otimes D^{\ell_d}$ indicates that we apply D^{ℓ_j} with respect to variable y_j, for each $j = 1, \ldots, d$. Note that, since $\boldsymbol{\ell} \in \Lambda(r+d-1, d)$ in (3.3), we have $d \leq |\boldsymbol{\ell}| \leq r+d-1$ and also $1 \leq \ell_j \leq r$ for each $j = 1, \ldots, d$.

Then, to define the FCCS rule for (1.2) (and hence (1.1)), we replace $\widehat{f}$ in (1.2) by $(\mathcal{Q}^{r,d}\widehat{f})$, thus obtaining (1.4). An alternative formula for $\mathcal{Q}^{r,d}\widehat{f}$ is obtained using the *combination technique* ([39, Lemma 1], [14, Section 4.1]). This allows the formula (3.3) to be written in terms of Q^l instead of D^l; the result is:

$$(\mathcal{Q}^{r,d}\widehat{f})(\boldsymbol{y}) := \sum_{\substack{\boldsymbol{\ell}\geq\mathbf{1}\\ r\leq|\boldsymbol{\ell}|\leq r+d-1}} (-1)^{r+d-|\boldsymbol{\ell}|-1}\binom{d-1}{|\boldsymbol{\ell}|-r}(Q^{\ell_1}\otimes\ldots\otimes Q^{\ell_d})\widehat{f}(\boldsymbol{y}). \tag{3.4}$$

A useful observation from this is (when $d=1$),

$$\mathcal{Q}^{r,1}\widehat{f} = Q^r\widehat{f} \tag{3.5}$$

Then, inserting (3.4) into (1.2), we obtain the following approximation of (1.1):

Proposition 3.2.

$$\mathcal{I}^{k,d,\mathbf{a},r}f = \sum_{\substack{\boldsymbol{\ell}\geq\mathbf{1}\\ r\leq|\boldsymbol{\ell}|\leq r+d-1}} (-1)^{r+d-|\boldsymbol{\ell}|-1}\binom{d-1}{|\boldsymbol{\ell}|-r}(I^{\omega_1,\ell_1}\otimes\cdots\otimes I^{\omega_d,\ell_d})f, \tag{3.6}$$

where

$$\omega_j = ka_j \quad \text{for each} \quad j=1,\ldots,d. \tag{3.7}$$

Proof. By (1.4) and (3.4), it is sufficient to prove that

$$\int_{[-1,1]^d}(Q^{\ell_1}\otimes\ldots\otimes Q^{\ell_d})\widehat{f}(\boldsymbol{y})\exp(\mathrm{i}k\widetilde{\mathbf{a}}\cdot\boldsymbol{y})\mathrm{d}\boldsymbol{y} = (I^{\omega_1,\ell_1}\otimes\cdots\otimes I^{\omega_d,\ell_d})f, \tag{3.8}$$

with ω_j as given in (3.7). The proof of (3.8) is obtained by induction on the dimension d. For $d=1$, we have, directly from Proposition 2.3,

$$\int_{-1}^{1}(Q^{\ell_1}\widehat{f})(y)\exp(\mathrm{i}k\widetilde{a}_1 y)\mathrm{d}y = I^{ka_1,\ell_1}f = I^{\omega_1,\ell_1}f. \tag{3.9}$$

Now suppose (3.8) holds for dimension d and, considering dimension $d+1$, we introduce new notation as follows. For $\boldsymbol{y}\in[-1,1]^{d+1}$ and $\mathbf{a}\in\mathbb{R}^{d+1}$, we write $\boldsymbol{y}=(\boldsymbol{y}^*,y_{d+1})$ and $\mathbf{a}=(\mathbf{a}^*,a_{d+1})$, where $\boldsymbol{y}^*\in[-1,1]^d$ and $\mathbf{a}^*\in\mathbb{R}^d$. Moreover, for any $f\in C([-1,1]^{d+1})$ and any fixed $\boldsymbol{y}^*\in[-1,1]^d$, we write $f(\boldsymbol{y}^*,\cdot)$ to denote the univariate function $y_{d+1}\mapsto f(\boldsymbol{y}^*,y_{d+1})=f(\boldsymbol{y})$. Using $I^{\omega,\ell}[f(\boldsymbol{y}^*,\cdot)]$ to denote the application of the quadrature rule (2.10) to $f(\boldsymbol{y}^*,\cdot)$, we also define the $d-$variate functions

$$F^{\omega,\ell}(\boldsymbol{y}^*) = I^{\omega,\ell}[f(\boldsymbol{y}^*,\cdot)], \quad \text{and } \widehat{F}^{\omega,\ell}(\boldsymbol{y}^*) = F^{\omega,\ell}(\boldsymbol{y}^*)\exp(\mathrm{i}k\widehat{\mathbf{a}}^*\cdot\boldsymbol{y}^*), \tag{3.10}$$

for all $\boldsymbol{y}^*\in[-1,1]^d$. Then, we have

$$\begin{aligned}&\int_{[-1,1]^{d+1}}(Q^{\ell_1}\otimes\ldots\otimes Q^{\ell_d}\otimes Q^{\ell_{d+1}})\widehat{f}(\boldsymbol{y})\exp(\mathrm{i}k\widetilde{\mathbf{a}}\cdot\boldsymbol{y})\mathrm{d}\boldsymbol{y}\\ &= \int_{[-1,1]^d}(Q^{\ell_1}\otimes\ldots\otimes Q^{\ell_d})\left(\int_{-1}^{1}(Q^{\ell_{d+1}}\widehat{f}(\boldsymbol{y}^*,\cdot)\exp(\mathrm{i}k\widetilde{a}_{d+1}\cdot))(y_{d+1})\mathrm{d}y_{d+1}\right)\\ &\qquad\qquad \times\exp(\mathrm{i}k\widetilde{\mathbf{a}}^*\cdot\boldsymbol{y}^*)\mathrm{d}\boldsymbol{y}^*\end{aligned} \tag{3.11}$$

By Proposition 2.3, we have

$$\int_{-1}^{1}(Q^{\ell_{d+1}}\widehat{f}(\boldsymbol{y}^*,\cdot)\exp(\mathrm{i}k\widetilde{a}_{d+1}\cdot))(y_{d+1})\mathrm{d}y_{d+1}$$
$$= I^{\omega_{d+1},\ell_{d+1}}\left[f(\boldsymbol{y}^*,\cdot)\right]\exp(\mathrm{i}k\widehat{\mathbf{a}}^*\cdot\boldsymbol{y}^*) = \widehat{F}^{\omega_{d+1},\ell_{d+1}}(\boldsymbol{y}^*).$$

Inserting this into (3.11) and using the inductive hypothesis (i.e., that (3.8) holds), we obtain

$$\begin{aligned}\int_{[-1,1]^{d+1}}&(Q^{\ell_1}\otimes\ldots\otimes Q^{\ell_d}\otimes Q^{\ell_{d+1}})\widehat{f}(\boldsymbol{y})\exp(\mathrm{i}k\widetilde{\mathbf{a}}\cdot\boldsymbol{y})\mathrm{d}\boldsymbol{y}\\ &= \int_{[-1,1]^d}\left((Q^{\ell_1}\otimes\ldots\otimes Q^{\ell_d})\widehat{F}^{\omega_{d+1},\ell_{d+1}}\right)(\boldsymbol{y}^*)\exp(\mathrm{i}k\widetilde{\mathbf{a}}^*\cdot\boldsymbol{y}^*)\mathrm{d}\boldsymbol{y}^*\\ &= \left(I^{\omega_1,\ell_1}\otimes\ldots\otimes I^{\omega_d,\ell_d}\right)F^{\omega_{d+1},\ell_{d+1}}. \qquad (3.12)\end{aligned}$$

The fact that (3.8) holds for dimension $d+1$ then follows by the defintion of $F^{\omega_{d+1},\ell_{d+1}}$ in (3.10). □

4. Error analysis of the FCCS rule

In this section, we shall provide an error estimate of the FCCS rule (3.6) for approximating (1.1) (equivalently (1.2)). Before this we need several preliminary results.

Lemma 4.1. (i) Recall $I^\omega g$ defined in (2.1). Then, for any $\omega\in\mathbb{R}$ and $g\in\mathcal{W}^{1,1}$,

$$|I^\omega g| \;\le\; 4\min\{1,|\omega|^{-1}\}\,\|g\|_{\mathcal{W}^{1,1}},$$

where $\min\{1,|\omega|^{-1}\} := 1$ when $\omega = 0$.
(ii) Let $g\in\mathcal{W}^{p,1}$, with $p>1$, let $a\in\mathbb{R}$, and define $\widetilde{a},\widehat{a}$ as in Notation 1.1 and set $\widehat{g}(y) = g(y)\exp(\mathrm{i}k\widehat{a}y)$. Then, for all $k\ge 0$, and $\ell\ge 1$, we have

$$\left|\int_{-1}^{1}(D^\ell\widehat{g})(y)\exp(\mathrm{i}k\widetilde{a}y)\mathrm{d}y\right| \;\le\; C_p'\min\{1,|\omega|^{-1}\}\left(\frac{1}{n_{\ell-1}}\right)^{p-1}\|g\|_{\mathcal{W}^{p,1}}, \qquad (4.1)$$

where $\omega = ka$, $C_p' = 2\max\{C_p,2\}$, with C_p as in Proposition 2.4 and we have set $n_0 = 1$.

Moreover, for $k\ge 0$ and $\ell\ge 3$,

$$\left|\int_{-1}^{1}(D^\ell\widehat{g})(y)\exp(\mathrm{i}k\widetilde{a}y)\mathrm{d}y\right| \;\le\; C_p'\min\{1,|\omega|^{-2}\}\left(\frac{1}{n_{\ell-1}}\right)^{p-1}\|g\|_{\mathcal{W}^{p+3,1}}. \qquad (4.2)$$

Proof. (i) For $\omega\neq 0$. we use integration by parts to obtain

$$I^\omega g \;=\; \int_{-1}^{1}g(y)\exp(\mathrm{i}\omega y)\mathrm{d}y \;=\; \frac{1}{\mathrm{i}\omega}[g(y)\exp(\mathrm{i}\omega y)]_{-1}^{1} - \frac{1}{\mathrm{i}\omega}\int_{-1}^{1}g'(y)\exp(\mathrm{i}\omega y)\mathrm{d}y.$$

Thus

$$|I^\omega g| \;\le\; \frac{2}{|\omega|}||g||_\infty + \frac{2}{|\omega|}||g'||_\infty \;\le\; \frac{4}{|\omega|}\,||g||_{\mathcal{W}^{1,1}}.$$

On the other hand, a direct estimate yields $|I^\omega g|\le 2\|g\|_\infty$ for all ω, and part (i) follows.

(ii) For $\ell \geq 2$, by the definition of D^ℓ, Proposition 2.3 and Theorem 2.4 (with $s = 1$),

$$\begin{aligned}\left|\int_{-1}^{1}(D^\ell \widehat{g})(y)\exp(\mathrm{i}k\widetilde{a}y)\mathrm{d}y\right| &= \left|\int_{-1}^{1}(Q^\ell - Q^{\ell-1})\widehat{g}(y)\exp(\mathrm{i}k\widetilde{a}y)\mathrm{d}y\right| \\ &= |I^{\omega,\ell}g - I^{\omega,\ell-1}g| \leq |I^{\omega}g - I^{\omega,\ell}g| + |I^{\omega}g - I^{\omega,\ell-1}g| \qquad (4.3)\\ &\leq C_p \min\{1,|\omega|^{-1}\}\left[\left(\frac{1}{n_\ell}\right)^{p-1} + \left(\frac{1}{n_{\ell-1}}\right)^{p-1}\right]\|g\|_{\mathcal{W}^{p,1}} \\ &= C_p \min\{1,|\omega|^{-1}\}\left[\left(\frac{1}{n_{\ell-1}}\right)^{p-1}(2^{1-p}+1)\right]\|g\|_{\mathcal{W}^{p,1}} \\ &\leq 2C_p \min\{1,|\omega|^{-1}\}\left(\frac{1}{n_{\ell-1}}\right)^{p-1}\|g\|_{\mathcal{W}^{p,1}}. \qquad (4.4)\end{aligned}$$

For $\ell = 1$, we use Proposition 2.3 to obtain,

$$\left|\int_{-1}^{1}(D^1\widehat{g})(y)\exp(\mathrm{i}k\widetilde{a}y)\mathrm{d}y\right| = |I^{\omega,1}g| = \begin{cases} |g(0)|\left|\int_{-1}^{1}\exp(\mathrm{i}\omega y)dy\right| & \text{for} \quad |\omega| \geq 1, \\ 2|g(0)|, & \text{for} \quad |\omega| < 1, \end{cases}$$

which gives $\left|\int_{-1}^{1}(D^1\widehat{g})(y)\exp(\mathrm{i}k\widetilde{a}y)\mathrm{d}y\right| \leq 2\min\{1,|\omega|^{-1}\}||g||_\infty$. The last two estimates yield (4.1).

To obtain (4.2) we proceed as in the case $\ell \geq 2$, but when estimating (4.3) we can use Theorem 2.4 with $s = 2$ instead of $s = 1$. □

Theorem 4.2. *Let C'_p be as in Lemma 4.1 (ii) and, for $\mathbf{a} \in \mathbb{R}^d$, set $\omega_j = ka_j$. Then for fixed $p > 1$, $r \geq 1$, and $f \in \mathcal{W}^{p,d}$,*

$$|\mathcal{I}^{k,d,\mathbf{a}}f - \mathcal{I}^{k,d,\mathbf{a},r}f| \leq \prod_{j=1}^{d}\left(C'_p \min\{1,|\omega_j|^{-1}\}\right)\binom{r+d-1}{d-1}2^{-(r-d)(p-1)}\|f\|_{\mathcal{W}^{p,d}}, \qquad (4.5)$$

where $\min\{1,|\omega_j|^{-1}\} := 1$ *when* $\omega_j = 0$.

Proof. This is proved by induction on d. The argument follows that of [39] and [14], although these references analysed tensor versions of standard quadrature rules and not Filon rules for oscillatory integrals, as considered here.

Note that for $d = 1$, we have, by (1.1) and (2.1), $\mathcal{I}^{k,1,\mathbf{a}}f = I^{\omega_1}f$ with $\omega_1 = ka_1$. By (1.4) with $d = 1$, then (3.5) and Proposition 2.3, we have $\mathcal{I}^{k,1,\mathbf{a},r}f = I^{\omega_1,r}f$. Then applying Theorem 2.4 with $s = 1$, we have

$$|\mathcal{I}^{k,1,\mathbf{a}}f - \mathcal{I}^{k,1,\mathbf{a},r}f| = |I^{\omega_1}f - I^{\omega_1,r}f| \leq C_p \min\{1,|\omega_1|^{-1}\}\left(\frac{1}{n_r}\right)^{p-1}\|f\|_{\mathcal{W}^{p,1}}. \qquad (4.6)$$

Since $C_p \leq C'_p$, this yields (4.5) for $d = 1$.

Now suppose (4.5) holds for dimension d and consider dimension $d+1$. We adopt the notation used in the proof of Proposition 3.2, and, for $\boldsymbol{\ell} \in \mathbb{N}^{d+1}$, we write $\boldsymbol{\ell} = (\boldsymbol{\ell}^*, \ell_{d+1})$ with $\boldsymbol{\ell}^* \in \mathbb{N}^d$. Also, recalling (2.1), we introduce, for $f \in C([-1,1]^{d+1})$,

$$F^\omega(\boldsymbol{y}^*) := I^\omega[f(\boldsymbol{y}^*,\cdot)]. \qquad (4.7)$$

We then estimate the error by the sum of two terms:

$$\begin{aligned} |\mathcal{I}^{k,d+1,\mathbf{a}} f - \mathcal{I}^{k,d+1,\mathbf{a},r} f| \;&\leq\; |\mathcal{I}^{k,d+1,\mathbf{a}} f - \mathcal{I}^{k,d,\mathbf{a}^*,r} F^{\omega_{d+1}}| \\ &\quad + |\mathcal{I}^{k,d,\mathbf{a}^*,r} F^{\omega_{d+1}} - \mathcal{I}^{k,d+1,\mathbf{a},r} f| \\ &=:\; |T_1| + |T_2|. \end{aligned} \tag{4.8}$$

Considering $|T_1|$ first, we use the equality $\mathcal{I}^{k,d+1,\mathbf{a}} f = \mathcal{I}^{k,d,\mathbf{a}^*} F^{\omega_{d+1}}$ and the inductive hypothesis to obtain

$$\begin{aligned} |T_1| \;&= \left| \left(\mathcal{I}^{k,d,\mathbf{a}^*} - \mathcal{I}^{k,d,\mathbf{a}^*,r} \right) F^{\omega_{d+1}} \right| \\ &\leq \begin{pmatrix} r+d-1 \\ d-1 \end{pmatrix} \prod_{j=1}^{d} \left(C_p' \min\{1, |\omega_j|^{-1}\} \right) 2^{-(r-d)(p-1)} \|F^{\omega_{d+1}}\|_{\mathcal{W}^{p,d}}. \end{aligned} \tag{4.9}$$

Then, using Lemma 4.1 (i) and a little manipulation one can see that

$$\|F^{\omega_{d+1}}\|_{W^{p,d}} \;\leq\; 4 \min\{1, |\omega_{d+1}|^{-1}\} \|f\|_{W^{p,d+1}} \;\leq\; C_p' \min\{1, |\omega_{d+1}|^{-1}\} \|f\|_{W^{p,d+1}}. \tag{4.10}$$

Hence, using this in (4.9) we obtain

$$|T_1| \;\leq \begin{pmatrix} r+d-1 \\ d-1 \end{pmatrix} \prod_{j=1}^{d+1} \left(C_p' \min\{1, |\omega_j|^{-1}\} \right) 2^{-(r-d)(p-1)} \|f\|_{\mathcal{W}^{p,d+1}}. \tag{4.11}$$

To estimate T_2, we first apply [39, eq. (11)], to write

$$\begin{aligned} \mathcal{Q}^{r,d+1} \;&=\; \sum_{\boldsymbol{\ell} \in \Lambda(r+d,d+1)} (D^{\ell_1} \otimes \ldots \otimes D^{\ell_{d+1}}) \\ &=\; \sum_{\boldsymbol{\ell}^* \in \Lambda(r+d-1,d)} (D^{\ell_1^*} \otimes \ldots \otimes D^{\ell_d^*}) \otimes Q^{r+d-|\boldsymbol{\ell}^*|}. \end{aligned}$$

Then, with notation as in (3.10) and proceeding as in (3.12), we have

$$\begin{aligned} \mathcal{I}^{k,d+1,\mathbf{a},r} f &= \int_{[-1,1]^{d+1}} (\mathcal{Q}^{r,d+1} \widehat{f})(\boldsymbol{y}) \exp(\mathrm{i} k \widetilde{\mathbf{a}} \cdot \boldsymbol{y}) \mathrm{d}\boldsymbol{y} \\ &= \sum_{\boldsymbol{\ell}^* \in \Lambda(r+d-1,d)} \int_{[-1,1]^d} (D^{\ell_1^*} \otimes \ldots \otimes D^{\ell_d^*}) \widehat{F}^{\omega_{d+1}, r+d-|\boldsymbol{\ell}^*|}(\boldsymbol{y}^*) \\ &\qquad\qquad \times \exp(\mathrm{i} k \widetilde{\mathbf{a}}^* \cdot \boldsymbol{y}^*) \mathrm{d}\boldsymbol{y}^*. \end{aligned} \tag{4.12}$$

Also, by (1.4) and subsequently (3.3), we have

$$\begin{aligned} \mathcal{I}^{k,d,\mathbf{a}^*,r} F^{\omega_{d+1}} &= \int_{[-1,1]^d} (\mathcal{Q}^{r,d} \widehat{F}^{\omega_{d+1}})(\boldsymbol{y}^*) \exp(\mathrm{i} k \widetilde{\mathbf{a}}^* \cdot \boldsymbol{y}^*) \mathrm{d}\boldsymbol{y}^* \\ &= \sum_{\boldsymbol{\ell}^* \in \Lambda(r+d-1,d)} \int_{[-1,1]^d} (D^{\ell_1^*} \otimes \ldots \otimes D^{\ell_d^*}) \widehat{F}^{\omega_{d+1}}(\boldsymbol{y}^*) \\ &\qquad\qquad \times \exp(\mathrm{i} k \widetilde{\mathbf{a}}^* \cdot \boldsymbol{y}^*) \mathrm{d}\boldsymbol{y}^*. \end{aligned} \tag{4.13}$$

Thus, combining (4.12) and (4.13), we obtain

$$\begin{aligned} T_2 &= \mathcal{I}^{k,d,\mathbf{a}^*,r} F^{\omega_{d+1}} - \mathcal{I}^{k,d+1,\mathbf{a},r} f \\ &= \sum_{\boldsymbol{\ell}^* \in \Lambda(r+d-1,d)} \int_{[-1,1]^d} (D^{\ell_1^*} \otimes \ldots \otimes D^{\ell_d^*}) \left(\widehat{F}^{\omega_{d+1}} - \widehat{F}^{\omega_{d+1}, r+d-|\boldsymbol{\ell}^*|} \right)(\boldsymbol{y}^*) \\ &\qquad \times \exp(\mathrm{i}k\widetilde{\mathbf{a}}^* \cdot \boldsymbol{y}^*) \mathrm{d}\boldsymbol{y}^*. \end{aligned} \tag{4.14}$$

Now, to estimate $|T_2|$, we consider any function $G \in \mathcal{W}^{p,d}$, any $\mathbf{a} \in \mathbb{R}^d$ and $k > 0$, and define $\widehat{G}(\boldsymbol{y}) = G(\boldsymbol{y}) \exp(\mathrm{i}k\widehat{\mathbf{a}}.\boldsymbol{y})$, with $\widehat{\mathbf{a}}$ as defined in Notation 1.1 as in (1.2). By induction on dimension d, using Lemma 4.1 (ii) at each step, we obtain the estimate

$$\begin{aligned} &\left| \int_{[-1,1]^d} (D^{\ell_1^*} \otimes \ldots \otimes D^{\ell_d^*}) \, \widehat{G}(\boldsymbol{y}^*) \exp(\mathrm{i}k\widetilde{\mathbf{a}}^* \cdot \boldsymbol{y}^*) \mathrm{d}\boldsymbol{y}^* \right| \\ &\qquad \leq \prod_{j=1}^d \left(C_p' \min\{1, |\omega_j|^{-1}\} \left(\frac{1}{n_{\ell_j^*-1}} \right)^{p-1} \right) \|G\|_{\mathcal{W}^{p,d}} \\ &\qquad \leq \left[\prod_{j=1}^d \left(C_p' \min\{1, |\omega_j|^{-1}\} \right) \right] 2^{(2d-|\boldsymbol{\ell}^*|)(p-1)} \|G\|_{\mathcal{W}^{p,d}}, \end{aligned} \tag{4.15}$$

where, in the last step, we used the fact that

$$\prod_{j=1}^d \frac{1}{n_{\ell_j^*-1}} \leq \prod_{j=1}^d 2^{2-\ell_j^*} = 2^{2d-|\boldsymbol{\ell}^*|}.$$

Thus, to complete the estimate of (4.14) we need to estimate $\|F^{\omega_{d+1}} - F^{\omega_{d+1}, r+d-|\boldsymbol{\ell}^*|}\|_{\mathcal{W}^{p,d}}$. Recalling that F^{ω} is given by (4.7) and $F^{\omega,\ell}$ is given in (3.10), we apply Theorem 2.4 with $s = 1$ to obtain, for any $j = 1, \ldots, d$, and any $q = 0, \ldots, p$,

$$\begin{aligned} \left| \partial_j^q (F^{\omega_{d+1}} - F^{\omega_{d+1}, r+d-|\boldsymbol{\ell}^*|})(\boldsymbol{y}^*) \right| &= \left| \left(I^{\omega_{d+1}} - I^{\omega_{d+1}, r+d-|\boldsymbol{\ell}^*|} \right) \left[(\partial_j^q f)(\boldsymbol{y}^*, \cdot) \right] \right| \\ &\leq C_p \min\{1, |\omega_{d+1}|^{-1}\} \, 2^{-(r+d-|\boldsymbol{\ell}^*|-1)(p-1)} \, \|(\partial_j^q f)(\boldsymbol{y}^*, \cdot)\|_{\mathcal{W}^{p,1}}, \end{aligned}$$

(where we used the fact that $r+d-|\boldsymbol{\ell}^*| \geq 1$ in (4.14), because $\boldsymbol{\ell}^* \in \Lambda(r+d-1, d)$). Hence

$$\|F^{\omega_{d+1}} - F^{\omega_{d+1}, r+d-|\boldsymbol{\ell}^*|}\|_{\mathcal{W}^{p,d}} \leq C_p \min\{1, |\omega_{d+1}|^{-1}\} \, 2^{-(r+d-|\boldsymbol{\ell}^*|-1)(p-1)} \, \|f\|_{\mathcal{W}^{p,d+1}}. \tag{4.16}$$

Combining (4.14), (4.15) and (4.16), using $C_p \leq C_p'$ and then the cardinality formula (3.2), we obtain

$$\begin{aligned} |T_2| &\leq \left[\prod_{j=1}^{d+1} \left(C_p' \min\{1, |\omega_j|^{-1}\} \right) \right] \\ &\qquad \times \left(\sum_{\boldsymbol{\ell}^* \in \Lambda(r+d-1,d)} \left(2^{2d-|\boldsymbol{\ell}^*|} 2^{-(r+d-|\boldsymbol{\ell}^*|-1)} \right)^{p-1} \right) \|f\|_{\mathcal{W}^{p,d+1}} \\ &= \binom{r+d-1}{d} \left[\prod_{j=1}^{d+1} \left(C_p' \min\{1, |\omega_j|^{-1}\} \right) \right] 2^{-(r-d-1)(p-1)} \|f\|_{\mathcal{W}^{p,d+1}}. \end{aligned} \tag{4.17}$$

Then, combining (4.11), (4.17) and using the elementary identity $\binom{q}{d} + \binom{q}{d-1} = \binom{q+1}{d}$, with $q = r+d-1$, we have shown that the estimate (4.5) holds for dimension $d+1$. □

Remark 4.3. (i) For those values of $\boldsymbol{\ell}^*$ satisfying $r+d-|\boldsymbol{\ell}^*| \geq 2$, an application of Theorem 2.4 shows that (4.16) holds with $|\omega_{d+1}|^{-1}$ replaced by $|\omega_{d+1}|^{-2}$ and $\mathcal{W}^{p,d+1}$ replaced by $\mathcal{W}^{p+3,d+1}$.

(ii) If any $\ell_j^* \geq 3$ then (for that particular j), an application of Theorem 4.1 (ii), shows that $|\omega_j|^{-1}$ can be replaced by $|\omega_j|^{-2}$ and $\mathcal{W}^{p,d}$ replaced by $\mathcal{W}^{p+3,d}$ in (4.15).

While Theorem 4.2 gives an estimate for the error which is explicit in $\mathbf{a}$ and k, the following theorem gives a simpler (and higher order in k^{-1}) at the cost of a stronger regularity requirement.

Theorem 4.4. *Let $\mathbf{a} \in \mathbb{R}^d$ with $\mathbf{a}_j \neq 0$ for each j. Let $p \geq 1$ and $d \geq 1$. Then there exists a constant $C_{p,d,\mathbf{a}}$ such that, for all $r \geq d+1$ and $f \in \mathcal{W}^{p+3,d}$,*

(4.18)
$$|\mathcal{I}^{k,d,\mathbf{a}} f - \mathcal{I}^{k,d,\mathbf{a},r} f| \;\leq\; C_{p,d,\mathbf{a}}\; k^{-(d+1)} \begin{pmatrix} r+d-1 \\ d-1 \end{pmatrix} 2^{-(r-d)(p-1)} \|f\|_{\mathcal{W}^{p+3,d}}.$$

Proof. We follow the proof of Theorem 4.2, indicating differences only briefly. This proof is less technical since we do not require explicitness with respect to $\mathbf{a}$. Throughout the proof $C_{p,d,\mathbf{a}}$ denotes a generic constant which may depend on $p, d, \mathbf{a}$, and whose value may vary from line to line.

Step 1 For the case $d = 1$, we have $r \geq 2$. Then we recall (4.6), but this time we use Theorem 2.4 with $s = 2$ to obtain

$$|\mathcal{I}^{k,1,\mathbf{a}} f - \mathcal{I}^{k,1,\mathbf{a},r} f| \;=\; |I^{\omega_1} f - I^{\omega_1,r} f| \;\leq\; C_p (k|a|)^{-2} \left(\frac{1}{n_r}\right)^{p-1} \|f\|_{\mathcal{W}^{p+3,1}},$$

which yields (4.18) for $d = 1$.

Step 2 Now assuming (4.18) holds for d we consider the corresponding result for dimension $d+1$. In this case we are assuming

$$r \geq d+2. \tag{4.19}$$

Again we introduce the splitting (4.8).

Step 2a Analogously to (4.9) and (4.10) we obtain, via the inductive hypothesis,

$$\begin{aligned} |T_1| \;&\leq\; C_{p,d,\mathbf{a}}\, k^{-(d+1)} \begin{pmatrix} r+d-1 \\ d-1 \end{pmatrix} 2^{-(r-d)(p-1)} \|F^{\omega_{d+1}}\|_{\mathcal{W}^{p+3,d}} \\ &\leq\; C_{p,d,\mathbf{a}}\, k^{-(d+2)} \begin{pmatrix} r+d-1 \\ d-1 \end{pmatrix} 2^{-(r-d)(p-1)} \|f\|_{\mathcal{W}^{p+3,d+1}}, \end{aligned} \tag{4.20}$$

where the additional power of k comes from the estimate of $\|F^{\omega_{d+1}}\|_{\mathcal{W}^{p+3,d+1}}$ – analgous to (4.10).

Step 2b The estimate of $|T_2|$ (starting from (4.14)) is slightly more complicated. Note first that (4.14) is a sum of terms, each corresponding to a different choice of $\boldsymbol{\ell}^*$. In all cases $r + d - |\boldsymbol{\ell}^*| \geq 1$. The key estimates for each of the summands in (4.14) are given in (4.15) and (4.16). In the following discussion we will discuss

only asymptotic decay as $k \to \infty$ of the terms in (4.15) and (4.16), the dependence on other variables is the same as in the proof of Theorem 4.2.

If, in fact, $r + d - |\boldsymbol{\ell}^*| \geq 2$ then, via Remark 4.3 (i), we obtain

$$\|F^{\omega_{d+1}} - F^{\omega_{d+1}, r+d-|\boldsymbol{\ell}^*|}\|_{\mathcal{W}^{p,d}} \;\leq\; C_{p,d,\mathbf{a}}\, k^{-2}\, 2^{-(r+d-|\boldsymbol{\ell}^*|-1)(p-1)} \|f\|_{\mathcal{W}^{p+3,d+1}},$$

i.e., one additional negative power of k compared with (4.16). In this case, the corresponding summand in (4.14) can be estimated as a product of d terms of $\mathcal{O}(k^{-1})$ (analogous to (4.15)) and one of $\mathcal{O}(k^{-2})$, yielding $\mathcal{O}(k^{-(d+2)})$ overall for that summand.

On the other hand, if $r + d - |\boldsymbol{\ell}^*| = 1$, then, by (4.19), $|\boldsymbol{\ell}^*| = r + d - 1 \geq 2d + 1$. Since also $\boldsymbol{\ell}^* \geq \mathbf{1}$, it follows that at least one ℓ_j^* must be ≥ 3. For this j, Remark 4.3 (ii) can be applied. Thus (4.15) has an estimate of $\mathcal{O}(k^{-d-1})$ (or better) and combining this with the standard $\mathcal{O}(k^{-1})$ estimate for (4.16), we again obtain $\mathcal{O}(k^{-d-2})$ overall for that summand as well.

Thus overall (4.14) has an $\mathcal{O}(k^{-(d+2)})$ estimate in the case of dimension $d + 1$. This completes the induction argument. □

Remark 4.5. The result in Theorem 4.4 is important if one is interested in computing (1.1) to high normalized accuracy (in the sense of (1.10)) for large k. This is because in general the oscillatory integral (1.1) decays with order $\mathcal{O}(k^{-d})$ as k increases, and Theorem 4.4 shows that the normalized error in computing this with the FCCS rule still decays with increasing k.

We are now ready to complete the proofs of the main theorems - Theorem 1.2 and 1.3, stated in the Introduction.

Proof of Theorems 1.2, 1.3. The proofs just require estimating the term $\binom{r+d-1}{d-1} 2^{-(r-d)(p-1)}$, which appears in the estimates in Theorems 4.2 and 4.4.

The reference [32, Lemma 4] gives an asymptotic formula (due to Müller-Gronbach) for the number of nodes $N(r, d)$ used in the quadrature rule (1.4). Using this, and a little manipulation, we obtain

$$N(r,d) \;\approx\; \frac{1}{(d-1)!\, 2^d} \left(1 + \frac{d-1}{r}\right)^{d-1} \left[2^r\, r^{d-1}\right],$$

where $\approx$ means that the ratio of the left-hand side and the right-hand side tends to 1 as $r \to \infty$. Hence from this it follows that, for sufficiently large r,

$$\frac{1}{(d-1)!\, 2^{d-1}} \left[2^r\, r^{d-1}\right] \geq N(r,d) \;\geq\; \frac{1}{(d-1)!\, 2^{d+1}} \left[2^r\, r^{d-1}\right]. \tag{4.21}$$

In particular, for sufficiently large r, we have the inequalities

$$\text{(i)} \quad 2^{-(r-d)} \;\leq\; \frac{2}{(d-1)!} \frac{r^{d-1}}{N(r,d)} \quad \text{and} \quad \text{(ii)} \quad 2^r \;\leq\; N(r,d). \tag{4.22}$$

Moreover, as is easily shown,

$$\lim_{r\to\infty} \frac{1}{r^{d-1}} \binom{r+d-1}{d-1} = \frac{1}{(d-1)!},$$

and so, for sufficiently large r,

$$\binom{r+d-1}{d-1} \leq \frac{2}{(d-1)!} r^{d-1}. \tag{4.23}$$

Putting together (4.22) (i) and (4.23) and then using (4.22)(ii), we obtain

$$\begin{aligned}\binom{r+d-1}{d-1}2^{-(r-d)(p-1)} &\leq \left(\frac{2}{(d-1)!}\right)^p (r^{d-1})^p \left(\frac{1}{N(r,d)}\right)^{p-1} \\ &\leq \left(\frac{2}{(d-1)!\log^{d-1}2}\right)^p (\log^{d-1} N(r,d))^p \frac{1}{N(r,d)^{p-1}}.\end{aligned}$$

Combining this with Theorems 4.2, 4.4, we complete the proof of Theorems 1.2, 1.3.

Remark 4.6 (Higher rate of decay as $k \to \infty$). We remark that the better decay rate as $k \to \infty$ obtained in Theorem 1.3 could be obtained for smaller r if, instead of the mid-point rule when $\ell = 1$ (see (1.3)), we employ the two-point Clenshaw-Curtis rule. This fact is illustrated in Example 1 of §6. While this observation is useful for problems of moderate dimension d, it is less interesting for higher d because the number of nodes used in the Smolyak interpolant grows quickly with d if a rule with more than one point is employed at level 1 (see the statement after equation(4) in [3], or [32, equation (25)]).

Remark 4.7 (Relation to classical estimates for sparse grid integration). As mentioned in §1, the result in Theorem 1.2 can be improved with respect to its dependence on $N(r,d)$ (see (1.11)). The proof of (1.11) can be obtained by following the proof above of Theorem 1.2, but using the following alternative estimate to those given in Theorem 2.4:

$$|\mathcal{I}^{\omega} g - \mathcal{I}^{\omega,\ell} g| \;\leq\; C_p \left(\frac{1}{n_\ell}\right)^p \|g\|_{\mathcal{W}^{p,1}}, \quad \text{for} \quad \ell \geq 1 \quad \text{and} \quad \omega \in \mathbb{R}. \tag{4.24}$$

(This obtains a better power of $1/n_\ell$ at the expense of no decay with respect to $|\omega|$.) Then repeating the proof of Theorem 1.2 using this estimate instead of Theorem 2.4 yields a proof of (1.11).

5. Application to a UQ problem for the Helmholtz equation

In uncertainty quantification for problems governed by PDEs, one typically wants to compute the output statistics (e.g. expectation or higher order moments) of some quantity of interest (QoI - typically a functional of the PDE solution), given the statistical properties of the random input data (e.g., coefficients) of the PDE. The required output moments are usually written as a multi-dimensional integral, with dimension determined by the number of random parameters in the model. For problems governed by the Helmholtz equation, the solution is usually oscillatory with respect to both the physical variable(s) and the random parameters. – see, e.g., [35, Chapter 5], [13, §4], [38].

In this paper we restrict to problem (1.12) – (1.14) and we first deal with the oscillation with respect to x by applying a 'hybrid numerical-asymptotic' method in physical space. For each random parameter $\boldsymbol{y}$, this yields an expression for the solution $u(x,\boldsymbol{y})$ (increasingly accurate as k increases) which identifies the principal oscillations with respect to x, and the only parts to be computed numerically are smooth functions of x. For this Helmholtz problem, it turns out that the approximation also yields the principal oscillations with respect to $\boldsymbol{y}$ and integration with respect to $\boldsymbol{y}$ requires integration of multidimensional oscillatory integrals of the type discussed earlier in this paper. (See e.g., [4], for a general discussion of hybrid

numerical-asymptotic methods in the context of Helmholtz problems.) Later we will study the case when n is a random field, but we start here with the deterministic case in order to explain the hybrid numerical-asymptotic method.

5.1. **The deterministic problem and its hybrid numerical-asymptotic solution.** In this subsection we shall explain briefly how to obtain an asymptotic approximation for the solution u of (1.12) – (1.14) which is increasingly accurate as k increases. Its implementation will involve the numerical solution of problems which are well-behaved with respect to k. We refer to the extended preprint of this paper [41] (see also [1]) for more details.

To introduce the asymptotic approximation, we define $N(x) = \int_0^x n(x')\mathrm{d}x'$ and the sequence

$$F_2 = F/n^2 \quad \text{and} \quad F_{2j+2} = -F_{2j}''/n^2, \quad \text{for} \quad j = 1, 2, \ldots. \tag{5.1}$$

Then, we consider the 'approximate ray-expansion':

$$\widetilde{u}^m \;:=\; \sum_{j=0}^{2m} k^{-j} r_j \;+\; \sum_{j=1}^{m} k^{-2j} F_{2j}, \quad \text{for} \quad m \geq 0, \tag{5.2}$$

where $r_j := \mu_j \xi + \nu_j \xi^{-1}$ with $\xi(x) = \exp(\mathrm{i}kN(x))$ and μ_j, ν_j satisfying

$$2n\mu_j' + n'\mu_j \;=\; \mathrm{i}\mu_{j-1}'', \quad j = 0, 1, \ldots, m \tag{5.3}$$

$$2n\nu_j' + n'\nu_j \;=\; -\mathrm{i}\nu_{j-1}'', \quad j = 0, 1, \ldots, m \tag{5.4}$$

$$\text{with} \quad \mu_{-1} = \nu_{-1} = 0. \tag{5.5}$$

We provide a theorem describing the approximation property of (5.2). A proof is given in [41].

Theorem 5.1. *Assume that $m \geq 1$, $F \in C^{2m}[0,1]$ and $n \in C^{2m+2}[0,1]$. Then for k sufficiently large, and for $j = 0, \ldots, 2m$, there exist unique μ_j, ν_j satisfying (5.3) – (5.5), together with the boundary conditions:*

$$B_L r_0 = u_L, \;\; B_R r_0 = 0; \tag{5.6}$$

and,

$$\left.\begin{array}{ll} B_L r_{2j-1} = 0, & B_R r_{2j-1} = 0 \\ B_L r_{2j} = -F_{2j}(0), & B_R r_{2j} = -B_R F_{2j} \end{array}\right\}, \quad \textit{for} \quad j = 1, \ldots, m. \tag{5.7}$$

Moreover, with the approximation to u then defined by (5.2) there exists a constant C independent of k such that, for sufficiently large k,

$$\|u - \widetilde{u}^m\|_{H^1(0,1)} \;\leq\; Ck^{-2m}. \tag{5.8}$$

Remark 5.2. By taking $m = 1$ in Theorem 5.1, and under the assumptions $n \in C^4[0,1]$ and $F \in C^2[0,1]$, we obtain the approximation (valid for k sufficiently large):

$$\widetilde{u}^1 = \widetilde{\mu}\xi + \widetilde{\nu}\xi^{-1} + \widetilde{F}, \tag{5.9}$$

where

$$\xi = \exp(\mathrm{i}kN), \quad \widetilde{\mu} = \mu_0 + k^{-1}\mu_1 + k^{-2}\mu_2, \quad \widetilde{\nu} = \nu_0 + k^{-1}\nu_1 + k^{-2}\nu_2,$$
$$\text{and} \quad \widetilde{F} = k^{-2}F_2 = k^{-2}F/n^2. \tag{5.10}$$

With this approximation we have the error estimate $||u - \widetilde{u}^1||_{H^1(0,1)} = \mathcal{O}(k^{-2})$. We use this approximation in the numerical experiment in the following section. In

practice, we need numerical approximations of $\widetilde{\mu}, \widetilde{\nu}$ and $\widetilde{F}$ in (5.10). We provide a way of computing $\widetilde{u}^1$ with high accuracy in Appendix A.

5.2. **The random problem.** We now introduce the random model by assuming that $n(x)$ depends in an affine way on d i.i.d. random variables $\boldsymbol{y} = (y_1, ..., y_d)$ with $y_i \in U[-1, 1]$. That is, we assume

$$n(x, \boldsymbol{y}) = n_0(x) + \sum_{j=1}^{d} n_j(x) y_j, \tag{5.11}$$

where $n_j(x) \in C^4[-1, 1]$ for $j = 0, 1, \ldots, d$. We also assume that the expansion functions satisfy, for some constant $C > 0$,

$$\min_{x\in[0,1]} n_0(x) - \sum_{j=1}^{d} ||n_j||_{\infty,[0,1]} \ \geq \ C \tag{5.12}$$

This condition ensures that

$$C \ \leq \ n(x, \boldsymbol{y}) \ \leq \ \sum_{j=0}^{d} \|n_j\|_{\infty,[0,1]}, \quad \text{for all} \quad x \in [0, 1], \quad \boldsymbol{y} \in U[-1, 1]^d. \tag{5.13}$$

With the parametrization of n given in (5.11), we have

$$N(x, \boldsymbol{y}) = N_0(x) + \sum_{j=1}^{d} N_j(x) y_j, \quad \text{with} \quad N_j(x) = \int_0^x n_j(x')\mathrm{d}x', \quad j = 0, \ldots, d, \tag{5.14}$$

and hence

$$\xi(x, \boldsymbol{y}) = \exp(\mathrm{i}kN_0(x)) \exp(\mathrm{i}k\mathbf{a}(x) \cdot \boldsymbol{y}), \tag{5.15}$$

where $\mathbf{a} \in C[0, 1]^d$ is the real vector-valued function with components given by

$$a_j(x) = N_j(x), \quad j = 1, \ldots, d. \tag{5.16}$$

In UQ applications one is often interested in computing the expectation or higher moments of a Quantity of Interest, typically a functional of the solution u. Since our aim here is to provide an application of the use of the FCCS rule combined with the hybrid numerical-asymptotic method we restrict attention to the computation of the the expectation of $u(x)$ at any given point $x \in [0, 1]$ which we can approximate by $\mathbb{E}[\widetilde{u}^1(x)]$, with $\widetilde{u}^1$ given in (5.9), (5.10) above. To express this quantity neatly, for any smooth enough function ξ defined on $[0, 1] \times [-1, 1]^d$, we define the integrals

$$(\mathcal{I}^{\pm\mathbf{a}}\xi)(x) = 2^{-d} \int_{[-1,1]^d} \xi(x, \boldsymbol{y}) \exp(\pm \mathrm{i}k\mathbf{a}(x)\boldsymbol{y}) d\boldsymbol{y}$$

$$\text{and} \quad (\mathcal{I}\xi)(x) = 2^{-d} \int_{[-1,1]^d} \xi(x, \boldsymbol{y}) d\boldsymbol{y}.$$

Using this notation we have

$$\mathbb{E}[\widetilde{u}^1(x)] = \quad \exp(\mathrm{i}kN_0(x)) \, (\mathcal{I}^{+\mathbf{a}}\widetilde{\mu})(x) \tag{5.17}$$

$$+ \exp(-\mathrm{i}kN_0(x)) \, (\mathcal{I}^{-\mathbf{a}}\widetilde{\nu})(x) \tag{5.18}$$

$$+ \ (\mathcal{I}\widetilde{F})(x) \tag{5.19}$$

The integral (5.19) is not oscillatory and can be computed using the standard Clenshaw-Curtis-Smolyak rule, but (5.17) and (5.18) need to be computed using our FCCS rule. The values of $\widetilde{\mu}(x,\boldsymbol{y})$ and $\widetilde{\nu}(x,\boldsymbol{y})$ at the multidimensional quadrature nodes are computed using the procedure described in Appendix A.

At this point we remark that, although the functions μ_j, ν_j are non-oscillatory with respect to spatial variable x, the coefficients α_j^ℓ appearing in (A.1) do depend on k (via application of the boundary conditions (5.6), (5.7)), and so $\widetilde{\mu},\widetilde{\nu}$ could potentially have derivatives (with respect to the random parameters $\boldsymbol{y}$) which depend on k. We investigate this point in §6.2 and show that, under reasonable assumptions, the application of the FCCS rule to (5.17), (5.18), where $\widetilde{\mu},\widetilde{\nu}$ are approximated by the Smolyak interpolation can be justified and works well in practice.

Remark 5.3 (Possible use of asymptotic expansions for integrals). When all components of $k\mathbf{a}$ are large enough, one could consider using the known asymptotic expansions of oscillatory integrals with linear phase (found, e.g., in [8, §2.4]) to approximate (5.17), (5.18), as an alternative to the FCCS rule. However in our UQ application, the vector $\mathbf{a}(x)$ can easily have components which are small or even vanish, and the location of these "non-oscillatory" directions can vary with x in a complicated way. Because of this, even when k is large, some of the products $ka_j(x)$ may still be quite small (or even vanish) and the corresponding integrals (5.17), (5.18) will not be (highly) oscillatory in the corresponding dimensions. Moreover the location of such components also depends on the choice of expansion functions n_j. Thus in order to have a method which works robustly in k and $\mathbf{a}$ we prefer to use the FCCS rule, and we have designed this so that it works robustly independently of the size of $ka_j(x)$ (see (1.2)). Moreover, to obtain high accuracy in the asymptotic evaluation of oscillatory integrals, it is necessary to compute partial derivatives of the smooth function f (in (1.1)), a task which would be complicated in the context of our UQ application.

5.3. **More general Quantities of Interest.** In the computations in the following section we compute the expected value of the linear functional $u(1)$. This has a certain physical meaning, since the boundary condition (1.14) at the point $x = 1$ corresponds to an approximation of the Sommerfeld radiation condition at this point. Thus the value $u(1)$ constitutes an approximation to the value of the wavefield in the 'far field', a quantity of some physical interest.

The approximate computation of $\mathbb{E}(u)$ can easily be extended to general linear functionals on $H^1(0,1)$. For example, to approximate the expected value of $Gu := (u,g)_{L^2(0,1)}$, with $g \in L^2(0,1)$, we compute

$$\begin{aligned}\mathbb{E}\left[\int_0^1 \widetilde{u}^1(x,\cdot)\overline{g}(x)\mathrm{d}x\right] &= \int_0^1 \mathbb{E}\left[\widetilde{u}^1(x,\cdot)\right]\overline{g}(x)\mathrm{d}x \\ &= \int_0^1 \exp(\mathrm{i}kN_0(x))(\mathcal{I}^{+\mathbf{a}}\widetilde{\mu})(x)\overline{g}(x)\mathrm{d}x + \int_0^1 \exp(-\mathrm{i}kN_0(x))(\mathcal{I}^{-\mathbf{a}}\widetilde{\nu})(x)\overline{g}(x)\mathrm{d}x \\ &\qquad + \int_0^1 (\mathcal{I}\widetilde{f})(x)\overline{g}(x)\mathrm{d}x,\end{aligned} \tag{5.20}$$

so that additional 1d oscillatory integrals now appear in the first and second terms of (5.20). Similar but slightly more complicated terms appear in the computation

of the general linear functional on $H^1(0,1)$: $Gu := (u', g')_{L^2(0,1)} + (u, g)_{L^2(0,1)}$, for any $g \in H^1(0,1)$.

Some nonlinear functionals can also be handled by the above technique. For example, to compute the variance of $u(x)$, we need also $\mathbb{E}[u(x)^2]$. We approximate this by the expectation of

$$(\widetilde{u}^1(x))^2 = \widetilde{\mu}^2\xi^2 + \widetilde{\nu}^2\xi^{-2} + 2(\widetilde{\mu}\widetilde{F})\xi + 2(\widetilde{\nu}\widetilde{F})\xi^{-1} + \left(\widetilde{F}^2 + 2\widetilde{\mu}\widetilde{\nu}\right),$$

and the integral with respect to $\boldsymbol{y}$ of all these terms can be done by the FCCS rule. (The last two terms are not oscillatory.) Similar arguments can be used to approximate the expected value of $|u(x)|^2 = u(x)\overline{u(x)}$.

6. Numerical Experiments

Our code for the following numerical examples was based on the Sparse Grids Matlab Kit, available at `https://sites.google.com/view/sparse-grids-kit`. See also [2, 36]. Here we give a selection of numerical results illustrating the theory of the paper. More results are available in [41].

6.1. **Multi-dimensional quadrature.** In this subsection we illustrate the convergence properties of the Filon-Clenshaw-Curtis-Smolyak rule. We shall compute the following error indicators, defined as:

$$e^{k,d,\mathbf{a},r}(f) = \left|\mathcal{I}^{k,d,\mathbf{a}}f - \mathcal{I}^{k,d,\mathbf{a},r}f\right| \quad \text{and} \quad E^{k,d,\mathbf{a},r}(f) = \frac{\left|\mathcal{I}^{k,d,\mathbf{a}}f - \mathcal{I}^{k,d,\mathbf{a},r}f\right|}{|\mathcal{I}^{k,d,\mathbf{a}}f|},$$

with $\mathcal{I}^{k,d,\mathbf{a}}f$ and $\mathcal{I}^{k,d,\mathbf{a},r}f$ defined in (1.2) and (3.6) respectively. Here $e^{k,d,\mathbf{a},r}(f)$ is the absolute error and $E^{k,d,\mathbf{a},r}(f)$ is a proxy for the normalized error (as defined in (1.10)). We use the proxy $E^{k,d,\mathbf{a},r}(f)$ because in some experiments the number of oscillatory dimensions $\widetilde{d}$ changes within the experiment and the quantity $k^{-\widetilde{d}}$ is well represented by $|\mathcal{I}^{k,d,\mathbf{a}}f|$ according to the discussion above (1.10).

In the first two examples we compute (1.1) for the case $d = 3$ with

$$f(\boldsymbol{y}) = \cos(my_1y_2y_3) \tag{6.1}$$

and various choices of $\mathbf{a}$. The exact value of $\mathcal{I}^{k,d,\mathbf{a}}f$ is taken to be $\mathcal{I}^{k,d,\mathbf{a},10}f$, and this value is used to compute the error indicators. We repeated with exact value computed with $r = 12$ and observed no changes in the results.

Example 1 - asymptotic decay as $k \to \infty$, with fixed r. We fix $\mathbf{a} = (1,1,1)^\top$ and study the behaviour of the error indicators as $k \to \infty$. In general, the decay rate of the exact and approximate integrals as $k \to \infty$ can be quite delicate. To see this, consider the following model 1D integral with integration by parts:

$$\int_{-1}^{1} \exp(\mathrm{i}ky)g(y)\mathrm{d}y = \frac{1}{\mathrm{i}k}\left[g(1)\exp(\mathrm{i}k) - g(-1)\exp(-\mathrm{i}k)\right] - \frac{1}{\mathrm{i}k}\int_{-1}^{1}\exp(\mathrm{i}ky)g'(y)\mathrm{d}y. \tag{6.2}$$

A second integration by parts shows that the second term on the right-hand side of (6.2) is $\mathcal{O}(k^{-2})$, while the first term takes the form $C(k)k^{-1}$, so is dominant in general. However in general the factor $C(k)$ can vary considerably with respect to k, leading to possibly irregular behaviour as $k \to \infty$. However by taking $k = 2\ell\pi + \pi/4$, $\ell = 2, 4, 8, \ldots, 128$, $C(k)$ turns out to be the $k-$independent constant $C(k) = ((1-\mathrm{i})g(1) + (1+\mathrm{i})g(-1))/\sqrt{2}$, thus ensuring (excluding special cases of g) a regular $O(k^{-1})$ decay for the dominant term in (6.2). We use this sequence of

wavenumbers in the experiments below. With f as given in (6.1) with $m = 2$, Table 1 illustrates that $|\mathcal{I}^{k,d,\mathbf{a}}f|$ decays with $\mathcal{O}(k^{-d})$. Also, $E^{k,d,\mathbf{a},r}(f)$ remains bounded with respect to k when $r = 3$ and decays with order about $\mathcal{O}(k^{-1})$ for $r = 4$, as predicted by Theorem 1.3.

			$r=3$				$r=4$			
k	$\lvert\mathcal{I}^{k,d,\mathbf{a}}(f)\rvert$	α	$e^{k,d,\mathbf{a},r}(f)$	α	$E^{k,d,\mathbf{a},r}(f)$	α	$e^{k,d,\mathbf{a},r}(f)$	α	$E^{k,d,\mathbf{a},r}(f)$	α
13.35	1.06(-03)		2.25(-03)		2.12(+00)		2.35(-04)		2.21(-01)	
25.92	1.04(-04)	3.50	2.66(-04)	3.22	2.56(+00)	-0.29	1.88(-05)	3.80	1.81(-01)	0.30
51.05	1.12(-05)	3.29	3.24(-05)	3.11	2.90(+00)	-0.18	1.28(-06)	3.97	1.14(-01)	0.68
101.32	1.28(-06)	3.16	4.00(-06)	3.05	3.12(+00)	-0.11	8.22(-08)	4.00	6.42(-02)	0.84
201.85	1.52(-07)	3.09	4.96(-07)	3.03	3.26(+00)	-0.06	5.20(-09)	4.00	3.41(-02)	0.92
402.91	1.86(-08)	3.05	6.18(-08)	3.01	3.33(+00)	-0.03	3.27(-10)	4.00	1.76(-02)	0.96
805.03	2.29(-09)	3.02	7.71(-09)	3.01	3.36(+00)	-0.02	2.05(-11)	4.00	8.94(-03)	0.98

TABLE 1. Error indicators for $\mathbf{a} = (1, 1, 1)^\top$, f given by (6.1) with $m = 2$, increasing k, for $r = 3$ and $r = 4$. The decay rate is conjectured to be $Ck^{-\alpha}$, with values of α computed from successive pairs of computations

We recall that the rule analysed in this paper uses the mid-point rule at level 1, and Clenshaw-Curtis grids thereafter (see (1.3)). By Remark 4.6, a better asymptotic decay of $E^{k,d,\mathbf{a},r}(f)$ with respect to k – even for small r – can be obtained if we use instead the two-point Clenshaw-Curtis rule at level 1. Results using this rule for the same case as in Table 1 are given in Table 2. Here $E^{k,d,\mathbf{a},r}(f)$ is observed to decay with $\mathcal{O}(k^{-2})$ when $r = 4$ although the behaviour for $r = 3$ has not stabilized yet. Overall we observe several orders of magnitude improvement in accuracy for large k. However we recall that the number of quadrature points grows more rapidly with dimension when the level 1 rule has more than one point and so this method may not be appropriate for higher dimensions. But if d is not too big, this variant should be useful if accurate results for very high k are required.

	$r=3$				$r=4$			
k	$e^{k,d,\mathbf{a},r}(f)$	α	$E^{k,d,\mathbf{a},r}(f)$	α	$e^{k,d,\mathbf{a},r}(f)$	α	$E^{k,d,\mathbf{a},r}(f)$	α
13.35	6.65(-05)		6.27(-02)		2.05(-05)		1.93(-02)	
25.92	2.57(-06)	4.90	2.47(-02)	1.40	8.37(-07)	4.82	8.06(-03)	1.32
51.05	5.36(-08)	5.71	4.79(-03)	2.24	2.86(-08)	4.98	2.56(-03)	1.69
101.32	1.03(-09)	5.77	8.05(-04)	2.60	9.25(-10)	5.01	7.23(-04)	1.84
201.85	2.19(-10)	2.25	1.43(-03)	-0.83	2.93(-11)	5.01	1.92(-04)	1.92
402.91	1.88(-11)	3.55	1.01(-03)	0.50	9.19(-13)	5.01	4.94(-05)	1.96
805.03	1.34(-12)	3.82	5.83(-04)	0.79	2.85(-14)	5.02	1.24(-05)	2.00

TABLE 2. Error indicators for f given in (6.1) with $m = 2$, $\mathbf{a} = (1, 1, 1)^\top$, for $r = 3$ and $r = 4$ using Clenshaw-Curtis 2 point rule on level 1. The decay rate is conjectured to be $Ck^{-\alpha}$, with values of α computed from successive pairs of computations

Example 2 - Robustness to variation in the elements of a. Recall that integral $\mathcal{I}^{k,d,\mathbf{a}}(f)$ is only oscillatory in the jth dimension when $k|a_j| \geq 1$, and if this is not true then the standard quadrature rule (and not its Filon variant) is

applied in that dimension. The standard rule is Clenshaw-Curtis when $\ell \geq 2$ and the mid-point rule when $\ell = 1$. Here we show that the algorithm proposed works stably when $k|a_j|$ passes through the value 1 or when $a_j = 0$. In this example we consider examples with $\mathbf{a} = (0.01, 1, 1)^\top$ and $\mathbf{a} = (0, 1, 1)^\top$. In the first case $\mathcal{I}^{k,d,\mathbf{a}}(f)$ is not oscillatory in the y_1 direction unless $k \geq 100$ and in the second case it is never oscillatory in the y_1 direction. Results are in Table 3. The method behaves robustly with respect to the value of a_1.

	$\mathbf{a} = (0.01, 1, 1)^\top$			$\mathbf{a} = (0, 1, 1)^\top$		
	$k = 25.92$	$k = 101.32$	$k = 201.85$	$k = 25.92$	$k = 101.32$	$k = 201.85$
$\lvert\mathcal{I}^{k,d,\mathbf{a}}(f)\rvert$:	2.30(-3)	1.68(-4)	3.96(-5)	2.30(-3)	1.70(-4)	4.38(-5)
$E^{k,d,\mathbf{a},r}(f)$:						
$r = 4$	1.96(-1)	1.34(-1)	5.42(-2)	1.80(-1)	1.64(-1)	1.63(-1)
$r = 5$	2.41(-2)	7.00(-3)	3.54(-3)	2.47(-2)	7.97(-3)	4.87(-3)
$r = 6$	1.37(-4)	2.70(-4)	4.57(-6)	2.11(-4)	3.88(-4)	2.21(-4)
$r = 7$	1.30(-5)	2.13(-5)	1.92(-5)	1.56(-5)	1.53(-5)	1.09(-5)
$r = 8$	2.05(-6)	4.46(-7)	1.59(-7)	2.12(-6)	8.60(-7)	2.48(-7)

TABLE 3. Values of $E^{k,d,\mathbf{a},r}(f)$ as r increases, with $d = 3, m = 2$ and two different choices of $\mathbf{a}$.

In the next example we illustrate the advantage of a dimension-adaptive implementation of our FCCS rule when some dimensions of the problem are less important than others.

Example 3 – Dimension-adaptive methods. It is well-known (e.g., [15, 30]) that if the dimensions can be ordered so that higher dimensions become less and less important than lower dimensions, then dimension-adaptive tensor product methods will be more efficient than standard procedures. This observation is relevant to the UQ problem considered in the next subsection. To illustrate this here, we consider the integral

$$\mathcal{I}(x) := \int_{[-1,1]^d} n^{-1/2}(x, \boldsymbol{y}) \exp(\mathrm{i}k\mathbf{a}(x).\boldsymbol{y})\mathrm{d}y, \tag{6.3}$$

where n given in (5.11), with

$$n_0(x) = 1 \quad \text{and} \quad n_j(x) = \exp(-j)\sin(j\pi x), \quad \text{for} \quad x \in [0, 1], \tag{6.4}$$

and so, by (5.16),

$$\begin{aligned} a_j(x) \;=\; N_j(x) \;&=\; \int_0^x n_j(x')\mathrm{d}x' \;=\; \exp(-j)\int_0^x \sin(j\pi x')\mathrm{d}x' \\ &= \frac{1}{j\pi}\exp(-j)\,(1 - \cos(j\pi x)). \end{aligned}$$

Since the functions μ_0, ν_0 (which constitute the principal parts of $\widetilde{\mu}, \widetilde{\nu}$ in (5.10)) are $\boldsymbol{y}$-dependent multiples of $n^{-1/2}$, the computation of (6.3) is a good test for the UQ computation considered in the following subsection. In this example we choose $k = 101.53$ and $x = 1/2$ and we compare the performance of the 'standard' FCCS rule (i.e., the rule analysed above) with an adaptive version where the approximation $\mathcal{Q}^{r,d}\widehat{f}$ in (1.4) is replaced by a dimension-adaptive procedure.

Our dimension-adaptive algorithm is implemented using the Sparse Grids Matlab kit [30, 36] and uses an adaptive procedure motivated by discussions in [15, 30].

The complete adaptive procedure is given in Algorithm 1. In Algorithm 1, $\mathbf{G}, \mathbf{L}$ and $\mathbf{R_L}$ are all multi-index sets. A multi-index set $\mathbf{L}$ is said to be downward closed (see, e.g., [7]) if $\forall \boldsymbol{\ell} \in \mathbf{L}, \boldsymbol{\ell} - \mathbf{e}_j \in \mathbf{L}$ for $1 \le j \le d$ such that $\ell_j > 1$. Moreover, $\mathcal{I}_{\mathbf{L}} f$ defined in Line 2 of Algorithm 1 is the FCCS rule based on the generalized sparse grids over the multi-index set $\mathbf{L}$; see also [15, Section 3.1], [14, Section 5.2] and [30, Equations (3) and (4)]. Then $\mathbf{G}$ contains the indices, over which the newest integral is computed, $\mathbf{L}$ contains the indices, which have been explored and whose neighbours (defined in 5 of of Algorithm 1) are to be explored, and $\mathbf{R_L}$ contains the indices, which have been explored and are candidates to be chosen to be added into $\mathbf{L}$. Also note that $\mathbf{G} = \mathbf{L} \cup \mathbf{R_L}$. For more detailed discussions of the dimension-adaptive algorithm, we refer readers to [15, 30].

Algorithm 1 Adaptive FCCS rule

Inputs: maximum number of sparse grids $N_{\max}$, tolerance τ.

1: Let $\mathbf{L} = \{\mathbf{1}\}, \mathbf{G} = \{\mathbf{1}\}, \mathbf{R_L} = \emptyset, \boldsymbol{\ell} = \mathbf{1}$.

2: Let $\mathcal{I}_{\text{old}} = \mathcal{I}_{\mathbf{L}} f := \sum_{\mathbf{j} \in \mathbf{L}} \int_{[-1,1]^d} (D^{j_1} \otimes \cdots \otimes D^{j_d}) \widehat{f}(\boldsymbol{y}) \exp(\mathrm{i} k \widetilde{a} \cdot \boldsymbol{y}) \mathrm{d}\boldsymbol{y} = \sum_{\mathbf{j} \in \mathbf{L}} c_{\mathbf{j}} (I^{\omega_1, j_1} \otimes \cdots \otimes I^{\omega_d, j_d}) f$, where $c_{\mathbf{j}} = \sum_{\mathbf{k} \in \{0,1\}^d, (\mathbf{k}+\mathbf{j}) \in \mathbf{L}} (-1)^{|\mathbf{k}|}$.

3: Let $\mathscr{P}_{\text{old}} = \text{pts}(\mathcal{I}_{\text{old}})$, where $\text{pts}(\mathcal{I}_{\text{old}})$ is the set of points used in computing $\mathcal{I}_{\text{old}}$, and $N = |\mathscr{P}_{\text{old}}|$, where $|\mathscr{P}_{\text{old}}|$ is the cardinality of $\mathscr{P}_{\text{old}}$, and $P_{\max} = +\infty$.

4: **while** $N < N_{\max}$ **and** $P_{\max} \ge \tau$ **do**

5: $\quad \boldsymbol{\mathcal{N}_g} = \text{neigh}(\boldsymbol{\ell}, \mathbf{L}) := \{\mathbf{j} \in \mathbb{N}^d \backslash \mathbf{L} : |\mathbf{j} - \boldsymbol{\ell}| = 1\}$.

6: $\quad$ **for** $\mathbf{j} \in \boldsymbol{\mathcal{N}_g}$ such that $\mathbf{L} \cup \{\mathbf{j}\}$ is downward closed **do**

7: $\quad\quad \mathbf{G} = \mathbf{G} \cup \{\mathbf{j}\}$.

8: $\quad\quad \mathcal{I}_{\text{new}} = \mathcal{I}_{\mathbf{G}} f$.

9: $\quad\quad \mathscr{P}_{\text{new}} = \text{pts}(\mathcal{I}_{\text{new}})$.

10: $\quad\quad N = N + |\mathscr{P}_{\text{new}} \backslash \mathscr{P}_{\text{old}}|$.

11: $\quad\quad$ Compute the *local profit* for $\mathbf{j}$: $P_{\mathbf{j}} = |\mathcal{I}_{\text{new}} - \mathcal{I}_{\text{old}}| / |\mathcal{I}_{\text{new}}|$.

12: $\quad\quad R_{\mathbf{L}} = R_{\mathbf{L}} \cup \{\mathbf{j}\}$.

13: $\quad\quad \mathcal{I}_{\text{old}} = \mathcal{I}_{\text{new}}, \mathscr{P}_{\text{old}} = \mathscr{P}_{\text{new}}$.

14: $\quad$ **end for**

15: $\quad$ Compute the *global profit* $P_{\max} = \max\{P_{\mathbf{j}} : \mathbf{j} \in \boldsymbol{\mathcal{N}_g}$ such that $\mathbf{L} \cup \{\mathbf{j}\}$ is downward closed$\}$.

16: $\quad$ Choose the $\boldsymbol{\ell}$ from $R_{\mathbf{L}}$ with the highest profit $P_{\boldsymbol{\ell}}$.

17: $\quad \mathbf{L} = \mathbf{L} \cup \{\boldsymbol{\ell}\}, R_{\mathbf{L}} = R_{\mathbf{L}} \backslash \{\boldsymbol{\ell}\}$.

18: **end while**

19: $\mathcal{I}^{\tau} = \mathcal{I}_{\text{new}}$.

Outputs: the integral $\mathcal{I}^{\tau}$.

To compute the error indicators, a reference value for $\mathcal{I}(x)$ is computed by 'brute force', using the tensor product Gauss-Legendre rule with 25 Gauss points in each of the d dimensions.

Results are given in Tables 4 – 6. The tables show the substantial advantage of the adaptive method in terms of the number of function evaluations over the standard method when the dimensions have decreasing importance, a situation often encountered in UQ applications.

	adaptive	standard		
	$\tau = 10^{-4}$	$r = 4$	$r = 5$	$r = 6$
$E^{k,d,\mathbf{a},r}(f)$	1.15 (-7)	8.37 (-6)	1.34 (-7)	7.21 (-10)
number of function evaluations	53	137	401	1105

TABLE 4. Comparison of the dimension-adaptive and standard FCCS rule for $\mathcal{I}(1/2)$, $d = 4$

	adaptive	standard		
	$\tau = 10^{-6}$	$r = 4$	$r = 5$	$r = 6$
$E^{k,d,\mathbf{a},r}(f)$	9.33 (-8)	8.46 (-6)	1.41 (-7)	8.64 (-10)
number of function evaluations	129	389	1457	4865

TABLE 5. Comparison of the dimension-adaptive and standard FCCS rule for $\mathcal{I}(1/2)$, $d = 6$

	adaptive	standard		
	$\tau = 10^{-6}$	$r = 4$	$r = 5$	$r = 6$
$E^{k,d,\mathbf{a},r}(f)$	1.17 (-7)	8.46 (-6)	1.41 (-7)	7.85 (-10)
number of function evaluations	151	849	3937	15713

TABLE 6. Comparison of the dimension-adaptive and standard FCCS rules for $\mathcal{I}(1/2)$, $d = 8$

6.2. **UQ problem for the Helmholtz equation.** In this subsection we consider the Helmholtz problem (1.12) – (1.14) with random $n = n(x, \boldsymbol{y})$ given by (5.11), but with F a function of x only, so that the solution $u = u(x, \boldsymbol{y})$ depends on x and $\boldsymbol{y}$ (and also on the frequency k). Then formulae (5.17) – (5.18) show that $\mathbb{E}[u(x)]$ can be written as a sum of oscillatory integrals with kernels given in equation (5.10). These integrals are (formally) in a form suitable for approximation by our FCCS rule, but in order to predict more precisely how well this will work, some further analysis is needed to investigate the regularity of $\widetilde{\mu}$ and $\widetilde{\nu}$ with respect to $\boldsymbol{y}$. Since the principal components of $\widetilde{\mu}$, $\widetilde{\nu}$ (i.e. those components which are $\mathcal{O}(1)$ as $k \to \infty$) are μ_0, ν_0 respectively, we restrict the discussion here to the analysis of the regularity of μ_0, ν_0 with respect to $\boldsymbol{y}$. In the following discussion we make the simplifying assumptions that (in the random problem), the Dirichlet data $u_L = u(0)$ is a constant independent of $\boldsymbol{y}$ and that $n_\infty = n(1, \boldsymbol{y})$ is a positive constant for all $\boldsymbol{y} \in [-1, 1]^d$. Then it can be shown, after some algebra, that

$$\mu_0(x, \boldsymbol{y}) = \alpha_0^1(\boldsymbol{y}) n(x, \boldsymbol{y})^{-1/2}, \quad \text{and} \quad \nu_0(x, \boldsymbol{y}) = \alpha_0^2(\boldsymbol{y}) n(x, \boldsymbol{y})^{-1/2}, \tag{6.5}$$

where the functions α_0^j are given by

$$\alpha_0^2(\boldsymbol{y}) = \frac{u_L \sqrt{n(0, \boldsymbol{y})} n'(1, \boldsymbol{y})}{2\mathrm{i}} \frac{\exp(\mathrm{i}kN(1, \boldsymbol{y}))}{n'(1, \boldsymbol{y}) \sin(kN(1, \boldsymbol{y})) - 2kn_\infty^2 \exp(-\mathrm{i}kN(1, \boldsymbol{y}))} \tag{6.6}$$

and

$$\alpha_0^1(\boldsymbol{y}) = u_L\sqrt{n(0,\boldsymbol{y})} - \alpha_0^2(\boldsymbol{y}). \tag{6.7}$$

From this we see that:

(i) If $n'(1,\boldsymbol{y}) = 0$ (i.e., $n(x,\boldsymbol{y})$ is a constant function of x near $x = 1$ for all $\boldsymbol{y}$), then $\alpha_0^2(\boldsymbol{y}) = 0$, $\alpha_0^1(\boldsymbol{y}) = u_L\sqrt{n(0,\boldsymbol{y})}$, and there is no $k-$dependent oscillation with respect to $\boldsymbol{y}$ in $\alpha_0^1(\boldsymbol{y})$. The component r_0 in the expansion (5.2) corresponds to a wave moving from left to right across the domain;
(ii) If $n'(1,\boldsymbol{y}) \neq 0$ then, while α_0^1 and α_0^2 both potentially have $k-$dependent oscillations with respect to $\boldsymbol{y}$, the *amplitude* of their oscillatory components decays with $\mathcal{O}(1/k)$ as k increases.

These facts allow us to apply the FCCS rule directly to the integrals (5.17) and (5.18) without any further splitting of their kernels. We do this in the following example for a case where $n'(1,\boldsymbol{y})$ is not the zero function, and observe good results.

In the following two examples we consider computing $\mathbb{E}[u(1)]$ for the problem (1.12) – (1.14) with $u_L = 1, n_\infty = 1, F(x) = x$, with n given by (5.11), and (6.4). So in this case $n(1,\boldsymbol{y}) = n_\infty = 1$ for all $\boldsymbol{y}$ and formulae (6.6), (6.7) hold.

Using the asymptotic approximation described in §5.1, this can be approximated by $\mathbb{E}[\widetilde{u}^1(1)]$, where $\widetilde{u}^1$ is defined in Remark 5.2. The formulae in (5.17), (5.18) and (5.19) show that $\mathbb{E}[\widetilde{u}^1(1)]$ can be written as a sum of three multidimensional integrals, the first two of which are oscillatory.

The functions $\widetilde{\mu}$ and $\widetilde{\nu}$ appearing in (5.17) and (5.18) are obtained using the formulae (5.9), (5.10), (A.1), requiring the solution of a system of ODEs that are non-oscillatory with respect to x. To solve these we use the method described in Appendix A to do this with parameters chosen as $M = 1, L = 1024$ and $M_G = 10$. These parameters are chosen to give very accurate values of $\widetilde{\mu}$ and $\widetilde{\nu}$ and it is not the purpose of this paper to investigate the most efficient choice of these parameters, since this question is not related to our main task here, namely to find methods which are efficient in terms of k and d dependence.

We then study the performance of both the standard and the adaptive methods for approximating the integrals appearing in (5.17), (5.18) and (5.19).

Example 4 - The standard FCCS method. For $r \geq 1$, we denote by $\mathbb{E}^{k,r}[\widetilde{u}^1(x)]$ the approximation of $\mathbb{E}[\widetilde{u}^1(x)]$ obtained by applying the FCCS rule to (5.17), (5.18) and (5.19) with maximum level r. Since (5.19) is not oscillatory, the FCCS rule just corresponds to a standard sparse grid quadrature on the hierarchy of grids (1.3).

r	$k=8$	$k=16$	$k=32$	$k=64$
5	5.86 (-3)	1.18 (-4)	8.83 (-4)	8.09 (-4)
6	5.83 (-3)	2.48 (-5)	5.06 (-5)	3.19 (-4)
7	5.83 (-3)	2.78 (-5)	6.89 (-6)	1.13 (-4)
8	5.83 (-3)	2.79 (-5)	5.83 (-6)	1.85 (-6)
9	5.83 (-3)	2.79 (-5)	5.79 (-6)	8.52 (-7)
10	5.83 (-3)	2.79 (-5)	5.79 (-6)	3.65 (-7)
11	5.83 (-3)	2.79 (-5)	5.79 (-6)	3.50 (-7)
12	5.83 (-3)	2.79 (-5)	5.79 (-6)	3.50 (-7)
α		7.71	2.27	4.05

TABLE 7. $|\mathbb{E}^{k,r}[\widetilde{u}^1(1)]-\mathbb{E}[u(1)]|$ for $d=4$ as r and k vary. The last row shows the decay rate $\mathcal{O}(k^{-\alpha})$ with α computed from successive pairs of errors of $r=12$.

We first consider $d=4$. A reference value is computed by applying the continuous piecewise linear finite element method to the full $k-$dependent boundary-value problem (1.12) – (1.14) with spatial mesh size $h=(2^{16}+1)^{-1}$. This is done for sample points $\boldsymbol{y}$ chosen on the grid formed as the tensor product of the 1d Gauss-Legendre rule with 50 Gauss points in each of the d dimensions. This is an expensive method, but it provides a very accurate $\mathbb{E}[u(1)]$, and is done only once to allow us to study errors. The absolute error $|\mathbb{E}^{k,r}[\widetilde{u}^1(1)]-\mathbb{E}[u(1)]|$ is shown in Table 7. To verify the reliability of these errors we have repeated the experiment again with $h=(2^{17}+1)^{-1}$ and found that the absolute differences between the reference solutions computed by $h=(2^{16}+1)^{-1}$ and $h=(2^{17}+1)^{-1}$ are smaller than the errors shown in Table 7.

Recall that the method we are studying has an error with respect to k (due to the asymptotic approximation) and with respect to r (from the sparse grid approximation). Hence we see convergence as both k and r increase by reading diagonally across the table, e.g., starting from $r=7$ and $k=8$ we see the sequence: $5.83(-3),\ 2.79(-5),\ 5.79(-6),\ 3.65(-7)$, and similarly for other diagonals.

Since, for $r=12$ there is not much error in the oscillatory integrals, we see decay of the error as k increases. For small r, on the other hand, the rows of Table 7 do not exhibit steady decay with respect to k due to the error in the oscillatory integrals.

The computation of the exact reference value used in Table 7 is costly and not feasible for higher dimensions or wavenumbers. Instead, in Table 8 we study the error proxy

$$|\mathbb{E}^{k,r}[\widetilde{u}^1(1)]-\mathbb{E}^{k,r+4}[\widetilde{u}^1(1)]| \tag{6.8}$$

for $d=4$ and higher values of k. Table 7 tells us that we should use this proxy cautiously, since (for example) computing this quantity for the column corresponding to $k=8$ will give values uniformly of order 10^{-6} where the true error is much larger. However reading Table 8 diagonally we still see quite convincing convergence of this proxy to zero as k,r both increase simultaneously, although the convergence is not always monotonic.

r	$k=32$	$k=64$	$k=128$
4	2.17 (-3)	5.35 (-4)	4.04 (-5)
5	8.77 (-4)	8.09 (-4)	5.43 (-5)
6	4.48 (-5)	3.19 (-4)	1.02 (-4)
7	2.22 (-6)	1.13 (-4)	5.19 (-5)
8	1.30 (-7)	1.50 (-6)	5.84 (-5)

TABLE 8. Values of the 'Error proxy' (6.8) for $d=4$ for various r and k

In Table 9 we study the 'error proxy':

$$|\mathbb{E}^{k,r}[\widetilde{u}^1(1)] - \mathbb{E}^{k,r+2}[\widetilde{u}^1(1)]| \tag{6.9}$$

for the case $d=6$, and observe a similar diagonal behaviour.

r	$k=32$	$k=64$	$k=128$
4	2.21 (-3)	2.05 (-4)	1.25 (-4)
5	8.89 (-4)	9.17 (-4)	4.03 (-5)
6	4.20 (-5)	3.31 (-4)	1.54 (-4)
7	2.12 (-6)	1.08 (-4)	5.13 (-5)
8	1.20 (-7)	1.71 (-6)	5.63 (-5)

TABLE 9. Values of the 'Error proxy' (6.9) for $d=6$ for various r and k

Example 5 - The dimension adaptive algorithm. Finally we consider the dimension adaptive method for the UQ problem. In this case, for any given k, $\mathbb{E}[u(1)]$ is computed by approximating the three integrals (5.17), (5.18) and (5.19), using the dimension adaptive method. To do this we introduce a tolerance parameter τ. Since the formulae (6.6), (6.7) show that the amplitude of ν_0 appearing in the integral (5.18) decays with $\mathcal{O}(1/k)$, and (recall Example 3 above), the adaptive method aims to control the normalized error (and not the absolute error) in the approximate integral, we use a smaller tolerance τ for integrals (5.17) and (5.19) and larger tolerance $k\tau$ for integral (5.18), and the resulting approximation of $\mathbb{E}[\widetilde{u}^1(x)]$ is denoted by $\mathbb{E}^{k,\tau}[\widetilde{u}^1(x)]$. For $d=6,8,10$, we display values of the error proxy:

$$|\mathbb{E}^{k,\tau}[\widetilde{u}^1(1)] - \mathbb{E}^{k,\tau/4}[\widetilde{u}^1(1)]|. \tag{6.10}$$

We also let $N_{\widetilde{\mu}}, N_{\widetilde{\nu}}, N_{\widetilde{f}}$ denote, respectively, the number of grid points in the adaptive sparse grids used for computing integrals (5.17), (5.18) and (5.19) and set $N_{tot} = N_{\widetilde{\mu}} + N_{\widetilde{\nu}} + N_{\widetilde{f}}$.

τ	$k = 32$	$k = 64$	$k = 128$	$k = 256$
0.01	3.84 (-5)	1.13 (-4)	2.77 (-5)	2.78 (-7)
0.005	6.37 (-5)	1.86 (-4)	2.57 (-5)	7.76 (-6)
0.0025	7.01 (-5)	5.98 (-5)	3.12 (-5)	3.19 (-7)
0.00125	2.64 (-4)	1.81 (-6)	6.63 (-5)	7.19 (-6)
0.01	(21, 27, 13, 61)	(21, 75, 13, 109)	(13, 15, 13, 41)	(13, 13, 13, 39)
0.005	(49, 43, 13, 105)	(21, 75, 13, 109)	(13, 21, 13, 47)	(21, 13, 13, 47)
0.0025	(53, 53, 13, 119)	(81, 75, 13, 169)	(13, 149, 13, 175)	(21, 15, 13, 49)
0.00125	(53, 77, 13, 143)	(141, 81, 13, 235)	(21, 149, 13, 183)	(39, 15, 13, 67)
0.000625	(53, 77, 13, 143)	(149, 141, 13, 303)	(31, 157, 13, 201)	(55, 15, 13, 83)
0.0003125	(91, 77, 13, 181)	(219, 141, 13, 373)	(167, 277, 13, 457)	(55, 45, 13, 113)

TABLE 10. Results of the dimension adaptive method for $d = 6$ for various τ and k. Top panel: Values of the 'Error proxy' (6.10). Bottom panel: $(N_{\widetilde{\mu}}, N_{\widetilde{\nu}}, N_{\widetilde{f}}, N_{tot})$

τ	$k = 32$	$k = 64$	$k = 128$	$k = 256$
0.01	3.84 (-5)	1.13 (-4)	2.77 (-5)	2.78 (-7)
0.005	6.37 (-5)	1.86 (-4)	2.57 (-5)	7.76 (-6)
0.0025	7.01 (-5)	5.98 (-5)	3.12 (-5)	3.19 (-7)
0.00125	2.64 (-4)	1.81 (-6)	6.63 (-5)	7.19 (-6)
0.01	(25, 31, 17, 73)	(25, 79, 17, 121)	(17, 19, 17, 53)	(17, 17, 17, 51)
0.005	(53, 47, 17, 117)	(25, 79, 17, 121)	(17, 25, 17, 59)	(25, 17, 17, 59)
0.0025	(57, 57, 17, 131)	(85, 79, 17, 181)	(17, 153, 17, 187)	(25, 19, 17, 61)
0.00125	(57, 81, 17, 155)	(145, 85, 17, 247)	(25, 153, 17, 195)	(43, 19, 17, 79)
0.000625	(57, 81, 17, 155)	(153, 145, 17, 315)	(35, 161, 17, 213)	(59, 19, 17, 95)
0.0003125	(95, 81, 17, 193)	(223, 145, 17, 385)	(171, 281, 17, 469)	(59, 49, 17, 125)

TABLE 11. Results of the dimension adaptive method for $d = 8$ for various τ and k. Top panel: Values of the 'Error proxy' (6.10). Bottom panel: $(N_{\widetilde{\mu}}, N_{\widetilde{\nu}}, N_{\widetilde{f}}, N_{tot})$

τ	$k = 32$	$k = 64$	$k = 128$	$k = 256$
0.01	3.84 (-5)	1.13 (-4)	2.77 (-5)	2.78 (-7)
0.005	6.37 (-5)	1.86 (-4)	2.57 (-5)	7.76 (-6)
0.0025	7.01 (-5)	5.98 (-5)	3.12 (-5)	3.19 (-7)
0.00125	2.64 (-4)	1.81 (-6)	6.63 (-5)	7.19 (-6)
0.01	(29, 35, 21, 85)	(29, 83, 21, 133)	(21, 23, 21, 65)	(21, 21, 21, 63)
0.005	(57, 51, 21, 129)	(29, 83, 21, 133)	(21, 29, 21, 71)	(29, 21, 21, 71)
0.0025	(61, 61, 21, 143)	(89, 83, 21, 193)	(21, 157, 21, 199)	(29, 23, 21, 73)
0.00125	(61, 85, 21, 167)	(149, 89, 21, 259)	(29, 157, 21, 207)	(47, 23, 21, 91)
0.000625	(61, 85, 21, 167)	(157, 149, 21, 327)	(39, 165, 21, 225)	(63, 23, 21, 107)
0.0003125	(99, 85, 21, 205)	(227, 149, 21, 397)	(175, 285, 21, 481)	(63, 53, 21, 137)

TABLE 12. Results of the dimension adaptive method for $d = 10$ for various τ and k. Top panel: Values of the 'Error proxy' (6.10). Bottom panel: $(N_{\widetilde{\mu}}, N_{\widetilde{\nu}}, N_{\widetilde{f}}, N_{tot})$

In Tables 10 – 12 we again observe (mostly) diagonal convergence as $\tau \to 0$ and $k \to \infty$. For fixed k and τ we see only very modest growth in the amount of work

as the dimension increases. In fact for $\tau = 0.00125$ and any fixed k a linear least squares fit on the data here suggests the number of function evaluations grows at most like $\mathcal{O}(d^{0.1})$ as d increases. For fixed τ and d, we see some initial growth of the work as k increases, but this seems to reduce substantially as k gets higher.

7. Conclusion

We have proposed and analysed a Filon-Clenshaw-Curtis-Smolyak sparse grid-based rule for $d-$dimensional oscillatory integrals $\int_{[-1,1]^d} f(\boldsymbol{y}) \exp(\mathrm{i}k\mathbf{a}.\boldsymbol{y})\mathrm{d}\boldsymbol{y}$, which is accurate for all $k > 0$, where $\mathbf{a} \in \mathbb{R}^d$ may have components with large, small or vanishing modulus. We give novel upper bounds for the absolute error. These take the form of a Smolyak-type error estimate multiplied by the factor $k^{-\widetilde{d}}$, where $\widetilde{d}$ denotes the number of oscillatory dimensions. The Smolyak estimate is explicit in N (the number of function evaluations), the maximum level r of the sparse grids and the regularity of f in spaces of functions with continuous mixed derivatives, The estimates imply that the normalized error (defined here to be the absolute error multiplied by $k^{\widetilde{d}}$) converges quickly as N grows, (assuming enough regularity of f) and is bounded with respect to k. On the other hand if all components of $\mathbf{a}$ are non-vanishing, we also show that the normalized error decays with order $\mathcal{O}(k^{-1})$, as $k \to \infty$.

We apply the rule to the forward problem of UQ for a one-space-dimensional Helmholtz problem with a random refractive index n, depending in an affine way on d i.i.d. uniform random variables. An asymptotic approximation shows, that for large k, expectations of linear functionals of the solution can be formulated as a sum of multidimensional oscillatory integrals which are computed using the FCCS rule. Numerical experiments show that this yields a UQ algorithm with increasing accuracy as k increases.

There are a number of interesting directions for possible future work. It seems feasible to extend to the case of additively separable non-linear phase, where $\mathbf{a} \cdot \boldsymbol{y}$ is replaced by $g(\boldsymbol{y}) = g_1(y_1) + \ldots + g_d(y_d)$, and each g_i may have stationary points or singularities. Then one could consider incorporating methods for 1D oscillatory integrals with nonlinear phase (such as [10, 34, 8, 25]) in the dimensions with the stationarity. More general nonlinear phases with stationary points would present a bigger challenge. It would be interesting, also, to investigate whether the asymptotic method for the 1D Helmholtz problem could be replaced by a practical 'hybrid' method, based on a solution ansatz like (5.9), but for which solutions are accurate for all ranges of k. The extension of the asymptotic method to higher physical space dimension represents a particular challenge, since the current method makes use of the fact that in 1D waves can only move from left-to-right or right-to-left along the domain. On the other hand the 1D in space model is still interesting in the UQ context; in fact this is the first time that the oscillatory behaviour of Helmholtz solutions with respect to random perturbations of the refractive index has been explicitly displayed and exploited.

Acknowledgement

The research of I.G. Graham was supported by UK EPSRC grant EP/S003975/1. The research of Z. Zhang is supported by the National Natural Science Foundation of China (Project 12171406), Hong Kong RGC grants (Projects 17300318, 17307921

and 17304324), the Outstanding Young Researcher Award of HKU (2020–21), and Seed Funding for Strategic Interdisciplinary Research Scheme 2021/22 (HKU). The computations were performed using research computing facilities offered by Information Technology Services, the University of Hong Kong.

We wish to thank Prof Víctor Domínguez (Public University of Navarra) for sharing with us his codes for implementing the 1D Filon-Clenshaw-Curtis rules.

We also thank the anonymous referees for their insightful comments.

Appendix

Appendix A. Computation of $\widetilde{u}^1$

The computation of $\widetilde{F}$ is easy, and thus we need to compute $\widetilde{\mu}$ and $\widetilde{\nu}$ in order to obtain $\widetilde{u}^1$, which is then reduced to the computation of $\{\mu_j, \nu_j\}_{j=0}^2$. We know from §5.1 that μ_j and ν_j satisfy the ODE system (5.3) – (5.5) together with the boundary conditions (5.6) and (5.7) for $j = 0, 1, 2$. The ODE system (5.3) – (5.5) admits the following solutions:

$$\mu_j(x) = \alpha_j^1 \mu_j^G(x) + \mu_j^P(x), \quad \nu_j(x) = \alpha_j^2 \nu_j^G(x) + \nu_j^P(x), \tag{A.1}$$

where

$$\mu_j^G(x) = \nu_j^G(x) = n(x)^{-\frac{1}{2}} \tag{A.2}$$

are the general solutions and

(A.3)

$$\mu_j^P(x) = \frac{i}{2} n(x)^{-\frac{1}{2}} \int_0^x \mu_{j-1}''(s) n(s)^{-\frac{1}{2}} ds, \quad \nu_j^P(x) = -\frac{i}{2} n(x)^{-\frac{1}{2}} \int_0^x \nu_{j-1}''(s) n(s)^{-\frac{1}{2}} ds$$

are the particular solutions, with the coefficients α_j^1 and α_j^2 determined by the boundary conditions (5.6) and (5.7). We let

$$\mathcal{I}_j^\mu(x) = \int_0^x \mu_{j-1}''(s) n(s)^{-\frac{1}{2}} ds, \quad \mathcal{I}_j^\nu(x) = \int_0^x \nu_{j-1}''(s) n(s)^{-\frac{1}{2}} ds. \tag{A.4}$$

In general, we do not have the analytic expressions for μ_j and ν_j, since we do not always have an explicit expression for the integrals in (A.4). We sometimes need to resort to numerical integration.

Partition of $[0, 1]$. Let M and L be positive integers with $H = \frac{1}{M}$ and $h = \frac{1}{LM}$, where L is assumed to be even for convenience. We partition $[0, 1]$ into $0 = x_0 < x_1 < \ldots < x_{M-1} < x_M = 1$, where $x_m = mH$ for $m = 0, 1, \ldots, M$. For each interval $[x_m, x_{m+1}]$, we further partition it into $x_m = x_m^0 < x_m^1 < \ldots < x_m^{L-1} < x_m^L = x_{m+1}$, where $x_m^\ell = x_m + \ell h$ for $\ell = 0, 1, \ldots, L$. We aim to obtain the values of $\widetilde{\mu}$ and $\widetilde{\nu}$ at x_m for $m = 0, 1, \ldots, M$.

Computation of μ_0 and ν_0. Note that $\mu_0^P = \nu_0^P = 0$ since $\mu_{-1} = \nu_{-1} = 0$. Then $\mu_0(x)$ and $\nu_0(x)$ can be analytically obtained with the coefficients α_0^1 and α_0^2 determined by the boundary condition (5.6).

Computation of μ_1 and ν_1. Note that $\mu_0''(x) n(x)^{-\frac{1}{2}}$ and $\nu_0''(x) n(x)^{-\frac{1}{2}}$, i.e., the integrands of $\mathcal{I}_1^\mu$ and $\mathcal{I}_1^\nu$, are analytically available. We obtain the values of $\mathcal{I}_1^\mu(x_m^\ell)$ and $\mathcal{I}_1^\nu(x_m^\ell)$ for each m and ℓ by successively applying an M_G-point Gauss quadrature rule to the integration on $[x_m^\ell, x_m^{\ell+1}]$. Then we obtain the values of $\mu_1(x_m^\ell)$ and $\nu_1(x_m^\ell)$ for each $m = 0, 1, \ldots, M$ and $\ell = 0, 1, \ldots, L$, with the coefficients α_1^1 and α_1^2 determined by the boundary condition (5.7) for $j = 1$.

Computation of μ_2 and ν_2. Note that

$$\begin{aligned}\left(\mu_1^P(x)\right)'' = \frac{i}{2}\Bigg[&\left(\frac{3}{4}n(x)^{-\frac{5}{2}}n'(x)^2 - \frac{1}{2}n(x)^{-\frac{3}{2}}n''(x)\right)\mathcal{I}_1^{\mu}(x)\\ &- \frac{3}{2}n(x)^{-2}n'(x)\mu_0''(x) + n(x)^{-1}\mu_0'''(x)\Bigg].\end{aligned} \tag{A.5}$$

Hence $\mu_1''(x)n(x)^{-\frac{1}{2}}$ is available at x_m^ℓ for each $m = 0, 1, \ldots, M$ and $\ell = 0, 1, \ldots, L$, and so is $\nu_1''(x)n(x)^{-\frac{1}{2}}$. We obtain the values of $\mathcal{I}_2^{\mu}(x_m)$ and $\mathcal{I}_2^{\nu}(x_m)$ for each m by successively applying the composite Simpson's rule to the integration on $[x_m, x_{m+1}]$ using the values of $\mu_1''(x)n(x)^{-\frac{1}{2}}$ and $\nu_1''(x)n(x)^{-\frac{1}{2}}$ at x_m^ℓ for $\ell = 0, 1, \ldots, L$. The we obtain the values of $\mu_2(x_m)$ and $\nu_2(x_m)$ for each $m = 0, 1, \ldots, M$, with the coefficients α_2^1 and α_2^2 determined by the boundary condition (5.7) for $j = 2$.

References

1. A. K. Aziz, R. B. Kellogg, and A. B. Stephens, *A two point boundary value problem with a rapidly oscillating solution*, Numer. Math. **53** (1988), no. 1, 107–121.
2. J. Bäck, F. Nobile, L. Tamellini, and R. Tempone, *Stochastic spectral Galerkin and collocation methods for PDEs with random coefficients: a numerical comparison*, Spectral and High Order Methods for Partial Differential Equations (J.S. Hesthaven and E.M. Ronquist, eds.), Lecture Notes in Computational Science and Engineering, vol. 76, Springer, 2011, Selected papers from the ICOSAHOM '09 conference, June 22-26, Trondheim, Norway, pp. 43–62.
3. V. Barthelmann, E. Novak, and K. Ritter, *High dimensional polynomial interpolation on sparse grids*, Adv. Comput. Math. **12** (2000), no. 4, 273–288.
4. S. N. Chandler-Wilde, I. G. Graham, S. Langdon, and E. A. Spence, *Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering*, Acta Numer. **21** (2012), 89–305.
5. A. Chkifa, A. Cohen, and C. Schwab, *High-dimensional adaptive sparse polynomial interpolation and applications to parametric PDEs*, Found. Comput. Math. **14** (2014), no. 4, 601–633.
6. C. W. Clenshaw and A. R. Curtis, *A method for numerical integration on an automatic computer*, Numer. Math. **2** (1960), no. 1, 197–205.
7. B. A. Davey and H. A. Priestley, *Introduction to lattices and order*, Cambridge University Press, 2012.
8. A. Deaño, D. Huybrechs, and A. Iserles, *Computing highly oscillatory integrals*, SIAM, 2018.
9. J. Dick, F. Y. Kuo, and I. H. Sloan, *High-dimensional integration: the quasi-Monte Carlo way*, Acta Numer. **22** (2013), 133–288.
10. V. Domínguez, I. G. Graham, and T. Kim, *Filon-Clenshaw-Curtis rules for highly-oscillatory integrals with algebraic singularities and stationary points*, SIAM J. Numer. Anal. **51** (2013), no. 3, 1542–1566.
11. V. Domínguez, I. G. Graham, and V. P. Smyshlyaev, *Stability and error estimates for Filon-Clenshaw-Curtis rules for highly oscillatory integrals*, IMA J. Numer. Anal. **31** (2011), no. 4, 1253–1280.
12. X. Feng, J. Lin, and C. Lorton, *An efficient numerical method for acoustic wave scattering in random media*, SIAM/ASA J. Uncertainty Quantification **3** (2015), no. 1, 790–822.
13. M. Ganesh, F. Y. Kuo, and I. H. Sloan, *Quasi-Monte Carlo finite element analysis for wave propagation in heterogeneous random media*, SIAM/ASA J Uncertainty Quantification **9** (2021), 106–134.
14. T. Gerstner and M. Griebel, *Numerical integration using sparse grids*, Numer. Algorithms **18** (1998), no. 3, 209–232.
15. ———, *Dimension-adaptive tensor-product quadrature*, Computing **71** (2003), 65–87.
16. A. Gibbs, D. P. Hewett, D. Huybrechs, and E. Parolin, *Fast hybrid numerical-asymptotic boundary element methods for high frequency screen and aperture problems based on least-squares collocation*, SN Partial Differ. Equat. Appl. **1** (2020), no. 4, 1–26.
17. S. P. Groth, D. P. Hewett, and S. Langdon, *A hybrid numerical–asymptotic boundary element method for high frequency scattering by penetrable convex polygons*, Wave Motion **78** (2018), 32–53.

18. M. Hardy, *Combinatorics of partial derivatives*, Electron. J. Combin. **13** (2006), 1–13.
19. D. Huybrechs and S. Vandewalle, *The construction of cubature rules for multivariate highly oscillatory integrals*, Math. Comp. **76** (2007), no. 260, 1955–1980.
20. A. Iserles, *On the numerical quadrature of highly-oscillating integrals I: Fourier transforms*, IMA J. Numer. Anal. **24** (2004), no. 3, 365–391.
21. ———, *On the numerical quadrature of highly-oscillating integrals II: Irregular oscillators*, IMA J. Numer. Anal. **25** (2005), no. 1, 25–44.
22. A. Iserles and S. Nørsett, *On quadrature methods for highly oscillatory integrals and their implementation*, BIT Numer. Math. **44** (2004), no. 4, 755–772.
23. ———, *Quadrature methods for multivariate highly oscillatory integrals using derivatives*, Math. Comp. **75** (2006), no. 255, 1233–1258.
24. A Iserles, S. Nørsett, and S. Olver, *Highly oscillatory quadrature: The story so far*, Numerical mathematics and advanced applications, Springer, 2006, pp. 97–118.
25. G. Maierhofer, A. Iserles, and N. Peake, *Recursive moment computation in Filon methods and application to high-frequency wave scattering in two dimensions*, IMA J. Numer. Anal. **43** (2022), 3169–3211.
26. H. Majidian, *Efficient computation of oscillatory integrals by exponential transformations*, BIT **61** (2021), 1337–1365.
27. ———, *Modified Filon-Clenshaw-Curtis rules for oscillatory integrals with a nonlinear oscillator*, Electron. Trans. Numer. Anal. **54** (2021), 276–295.
28. J. M. Melenk, *On the convergence of Filon quadrature*, J. Comput. Appl. Math. **234** (2010), no. 6, 1692–1701.
29. F. Nobile, L. Tamellini, and R. Tempone, *Convergence of quasi-optimal sparse-grid approximation of Hilbert-space-valued functions: application to random elliptic PDEs*, Numer. Math. **134** (2016), no. 2, 343–388.
30. F. Nobile, L. Tamellini, F. Tesei, and R. Tempone, *An adaptive sparse grid algorithm for elliptic PDEs with lognormal diffusion coefficient*, Sparse Grids and Applications-Stuttgart 2014, Springer, 2016, pp. 191–220.
31. E. Novak, *Optimal algorithms for numerical integration: Recent results and open problems*, Monte Carlo and Quasi-Monte Carlo Methods 2022 (A. Hinrichs, P. Kritzer, and F. Pillichshammer, eds.), Springer Verlag, 2024, pp. 105–132.
32. E. Novak and K. Ritter, *Simple cubature formulas with high polynomial exactness*, Constr. Approx. **15** (1999), no. 4, 499–522.
33. E. Novak, M. Ullrich, and H. Woźniakowski, *Complexity of oscillatory integration for univariate Sobolev spaces*, Journal of Complexity **31** (2015), 15–41.
34. S. Olver, *Moment-free numerical approximation of highly oscillatory integrals with stationary points*, European J. Appl. Math. **18** (2007), no. 4, 435–447.
35. O. Pembery, *The Helmholtz equation in heterogeneous and random media: Analysis and numerics*, Ph.D. thesis, University of Bath, 2020.
36. C. Piazzola and L. Tamellini, *Algorithm 1040: The Sparse Grids Matlab Kit-a Matlab implementation of sparse grids for high-dimensional function approximation and uncertainty quantification*, ACM Trans. Math. Software **50** (2024), no. 7, 1–22.
37. C. Schillings and C. Schwab, *Sparse, adaptive Smolyak quadratures for Bayesian inverse problems*, Inverse Probl. **29** (2013), no. 6, 065011.
38. E. A. Spence and J. Wunsch, *Wavenumber-explicit parametric holomorphy of Helmholtz solutions in the context of uncertainty quantification*, SIAM/ASA J. Uncertain. Quantif. **11** (2023), no. 2, 567–590.
39. G. W. Wasilkowski and H. Wozniakowski, *Explicit cost bounds of algorithms for multivariate tensor product problems*, J. Complexity **11** (1995), no. 1, 1–56.
40. Roderick Wong, *Asymptotic approximations of integrals*, Classics in Applied Mathematics 24, SIAM, 2001.
41. Z. Wu, I.G.Graham, D. Ma, and Z. Zhang, *A Filon-Clenshaw-Curtis-Smolyak rule for multi-dimensional oscillatory integrals with application to a UQ problem for the Helmholtz equation*, arXiv:2208.10078 (2024).
42. S. Xiang, *Efficient Filon-type methods for* $\int_a^b f(x)e^{i\omega g(x)}dx$, Numer. Math. **105** (2007), no. 4, 633–658.
43. J. Zech and C. Schwab, *Convergence rates of high dimensional Smolyak quadrature*, ESAIM: Math. Model. Numer. Anal. **54** (2020), no. 4, 1259–1307.

Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong SAR, China.
Email address: wuzz@hku.hk

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK.
Email address: I.G.Graham@bath.ac.uk

Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong SAR, China.
Email address: martin35@connect.hku.hk

Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong SAR, China.
Email address: zhangzw@hku.hk