Absolute stability and input-to-state stability of Lur'e systems

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Summary

We analyse stability properties of Lur'e systems, that is, feedback interconnections consisting of a linear state-space system in the forward path and a static nonlinearity in the feedback path. Classical absolute stability theory infers the stability of the Lur'e system by combining assumptions on frequency-domain properties of the linear system and sector data properties of the nonlinearity. The influential Aizerman's conjecture hypothesizes that if a Lur'e system is stable for all linear output feedback matrices in a given sector, then it is stable for all nonlinear output feedback maps in the same sector. While in its original form Aizerman's conjecture is known to be false, we show that a complexified version of Aizerman's conjecture that uses ball data instead of sector data is true. Then we show that, under slightly more restrictive assumptions on the nonlinearity, input-to-state stability holds. This adds to the growing body of work linking the areas of absolute stability and input-to-state stability. In contrast to most previous work, we place emphasis on the discrete-time case (although the continuous-time case is analysed as well), allow nonzero feedthrough and deal with multivariable systems. A key role in proving input-to-state stability for Lur'e systems is played by novel estimates involving comparison functions, which allow us to construct ISS-Lyapunov functions. Finally, we consider discrete-time input-output Lur'e systems, that is, feedback interconnections consisting of a linear higher-order matrix difference equation relating input and output in the forward path and a static nonlinearity in the feedback path. Absolute stability theory is rarely developed in an input-output setting and when it is, the results are usually reminiscent of the small-gain theorem. We extend results from behavioural theory pertaining to relating linear state-space and input-output systems and obtain a number of new results. We thus obtain extensions of absolute stability and input-to-state stability results to an unfamiliar, input-output, setting.

Contents

0.1 Notation

Let $\mathbb{N}_0$ be the set of integers greater than or equal to 0.

Let X be a vector space over a field $\mathbb{F}$ (= $\mathbb{R}$ or $\mathbb{C}$). Denote the space of all functions from $\mathbb{N}_0$ to X by $X^{\mathbb{N}_0}$. We can also view $X^{\mathbb{N}_0}$ as the space of all sequences with entries in X. We will interchange these two views depending on what is more convenient. Note that, by defining addition in $X^{\mathbb{N}_0}$ and scalar multiplication by elements in $\mathbb{F}$ pointwise, $X^{\mathbb{N}_0}$ becomes a vector space over $\mathbb{F}$. Similarly, for a linear map $M\colon X \to Y$, we can define its action on $X^{\mathbb{N}_0}$ pointwise to obtain a linear map $M\colon X^{\mathbb{N}_0} \to Y^{\mathbb{N}_0}$.

The **left-shift** $\mathcal{L}\colon X^{\mathbb{N}_0} \to X^{\mathbb{N}_0}$ is defined by $(\mathcal{L}x)(t) = x(t+1)$ for all $t \in \mathbb{N}_0$.

For $r > 0, z_0 \in \mathbb{C}$, define $\mathbb{E}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| > r\}$ and denote $\mathbb{E} := \mathbb{E}(0, 1)$. Similarly define $\mathbb{D}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ and set $\mathbb{D} := \mathbb{D}(0, 1)$. We define the **punctured disc** $\mathbb{D}'(z_0, r)$ **at** z_0 **of radius** r by $\mathbb{D}'(z_0, r) := \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$.

Denote by $\mathbb{C}_-$ the **open left half-plane** of the complex plane; that is, $\mathbb{C}_- := \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$. Similarly $\mathbb{C}_+$ will denote the **open right half-plane** of the complex plane; that is, $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.

For a normed vector space X, a point $x_0 \in X$ and $r > 0$, define $\mathbb{B}(x_0, r) := \{x \in X : \|x - x_0\| < r\}$. For $X = \mathbb{C}^{m\times p}$ and $K \in X$, we will sometimes write $\mathbb{B}_{\mathbb{C}}(K, r)$.

Denote by $\mathbb{F}[z]$ the ring of polynomials in the complex variable z with coefficients in $\mathbb{F}$. Denote the field of fractions of the ring $\mathbb{F}[z]$ by $\mathbb{F}(z)$, we will call elements in $\mathbb{F}(z)$ **rational functions**. Note that for any rational function $f \in \mathbb{F}(z)$, there exists a unique representation $f = \frac{p}{q}$, where $p, q \in \mathbb{F}[z]$ have no common factors and p is monic (that is, the leading coefficient is equal to 1). If $\deg p \le \deg q$, then f is said to be **proper**. If $\deg p < \deg q$, then f is said to be **strictly proper**. A rational function matrix $G \in \mathbb{F}(z)^{p\times m}$ is said to be proper (resp. strictly proper) if every element of G is proper (resp. strictly proper).

In what follows we will use the standard inner product on $\mathbb{F}^m$ given by $\langle x, y\rangle := \sum_{j=1}^{m} x_j \overline{y_j}$. Note that contrary to some conventions, it is linear in the first variable. We will also use the norm induced by this inner product (this is usually known as the 2-norm). For $M \in \mathbb{F}^{p\times m}$ define $\|M\| := \sup\{\|Mx\| : x \in \mathbb{F}^m, \|x\| = 1\}$. This is the 2-norm induced operator norm.

A square matrix $\Pi \in \mathbb{F}^{m\times m}$ is said to be a **projection** if $\Pi^2 = \Pi$. It is well-known that then the vector space $\mathbb{F}^m$ can be decomposed in the direct sum $\mathbb{F}^m = \operatorname{im}\Pi \oplus \ker\Pi$. We call Π an **orthogonal projection** if $\operatorname{im}\Pi$ is orthogonal to $\ker\Pi$, that is, for $\xi \in \operatorname{im}\Pi$ and $\mu \in \ker\Pi$, we have $\langle \xi, \mu\rangle = 0$.

If $P \in \mathbb{F}^{n\times n}$ is self-adjoint, then we will say that the function $V \colon \mathbb{F}^n \to \mathbb{R}$ defined by $V(\xi) := \langle P\xi, \xi\rangle$ is a **quadratic form**.

We will use the shorthand I_n for $n \times n$ identity matrices; the subscript will be omitted, when it can easily be inferred from context.

Let $T : X \to X$ be a bounded linear operator on a Banach space X. We define the **spectrum** of T, from here on $\sigma(T)$, to be the set of points λ in the complex plane for which $\lambda I - T$ is not bijective. If $X = \mathbb{F}^n$, then the spectrum of T coincides with the set of its eigenvalues.

Let A and B be two sets. The **Minkowski sum** $A+B$ of A and B is defined as $A + B := \{a + b \colon a \in A, b \in B\}$.

Consider an open set $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C} \cup \{\infty\}$. We say that f is **holomorphic in** Ω if it is complex differentiable at every point of Ω. We say that f is **meromorphic in** Ω if it is holomorphic in $\Omega \setminus P$, where - for some index set I - we can write $P = \cup_{\alpha\in I}\{z_\alpha\} \subseteq \Omega$ with z_α's isolated. We call z_α's the **poles** of f. These definitions can be extended in the obvious way to matrix maps $F : \Omega \to (\mathbb{C} \cup \{\infty\})^{p\times m}$.

Consider a function $V \colon \mathbb{F}^n \to \mathbb{R}$. We define its **gradient** $\nabla V \colon \mathbb{F}^n \to \mathbb{F}^n$ as $(\nabla V(\xi))_i = \frac{\partial V}{\partial \xi_i}(\xi)$ for $1 \le i \le n$.

For $0 < \omega \le \infty$, we denote by $C([0, \omega), \mathbb{F}^n)$ the set of all continuous maps $x \colon [0, \omega) \to \mathbb{F}^n$ and we set $C(\mathbb{F}^n) := \cup_{\omega\in(0,\infty]}C([0, \omega), \mathbb{F}^n)$. Similarly, we denote by $C^1([0, \omega), \mathbb{F}^n)$ the set of all continuously differentiable maps $x \colon [0, \omega) \to \mathbb{F}^n$ and we set $C^1(\mathbb{F}^n) := \cup_{\omega\in(0,\infty]}C^1([0, \omega), \mathbb{F}^n)$.

We will say that a map $f : \mathbb{F}^p \to \mathbb{F}^m$ is **locally Lipschitz** if for each $\xi \in \mathbb{F}^p$, there exist $r, L > 0$ such that $\|f(\xi_1) - f(\xi_2)\| \le L \|\xi_1 - \xi_2\|$ for all $\xi_1, \xi_2 \in \mathbb{B}_{\mathbb{F}}(\xi, r)$.

For $u \in (\mathbb{F}^m)^{\mathbb{N}_0}$, we define $\|u\|_\infty := \sup_{t\in\mathbb{N}_0} \|u(t)\|$ and we denote by $l^\infty(\mathbb{F}^m)$ the set of all $u \in (\mathbb{F}^m)^{\mathbb{N}_0}$ such that $\|u\|_\infty < \infty$. It is well-known that $l^\infty(\mathbb{F}^m)$ with the norm $\|\cdot\|_\infty$ is a Banach space.

For $0 < \omega \le \infty$, we denote by $AC([0, \omega), \mathbb{F}^m)$ the set of all **absolutely continuous** maps $u \colon [0, \omega) \to \mathbb{F}^m$ and set $AC(\mathbb{F}^m) := \cup_{\omega\in(0,\infty]}AC([0, \omega), \mathbb{F}^m)$ We will use the phrase "almost everywhere" (abbreviated as "a.e.") in the measure-theoretic sense.

For $u \colon [0, \omega) \to \mathbb{F}^m$, where $0 < \omega \le \infty$, we set $\|u\|_\infty := \operatorname{ess\,sup}_{t\in[0,\omega)} \|u(t)\|$. We denote by $L^\infty([0, \omega), \mathbb{F}^m)$ the set of all maps $u \colon [0, \omega) \to \mathbb{F}^m$ such that $\|u\|_\infty < \infty$ and we set $L^\infty(\mathbb{F}^m) := \cup_{\omega\in(0,\infty]}L^\infty([0, \omega), \mathbb{F}^m)$. We also define $L^\infty_{\mathrm{loc}}(\mathbb{F}^m) := \{u \in L^\infty(\mathbb{F}^m) : \operatorname{ess\,sup}_{t\in[0,\omega)\cap K} \|u(t)\| < \infty \;\forall \,\text{compact}\, K \subset [0, \infty)\}$.

Symbol Use

Lowercase Latin alphabet letters will be used for vectors in $\mathbb{R}^m$ and elements in $(\mathbb{F}^m)^{\mathbb{N}_0}, [0,\infty), \mathbb{N}_0, \mathbb{C}$. The letter t will almost exclusively be used to denote the time variable: in discrete-time $t \in \mathbb{N}_0$ and in continuous-time $t \in [0,\infty)$. In some instances e_i's will be used to denote the standard basis in $\mathbb{F}^m$. As is common, f, g, h will all be used to represent functions.

Uppercase Latin alphabet letters will be used for elements in sets $\mathbb{F}[z]^{p\times m}$, $\mathbb{F}(z)^{p\times m}$ and $\mathbb{F}^{p\times m}$.

Lowercase Greek letters α, β, σ and γ will be used for $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions (for definitions see Chapter 5). Other Greek letters will be used for elements in $\mathbb{F}^m$, except for (i) θ, which will denote an element in the interval $[0, 2\pi)$ or a particular map in Part III, (ii) ε, which will denote an element in $(0,\infty)$, and (iii) ω, which will denote an element in $(0,\infty]$.

Chapter 1

Introduction

In this thesis we link two well-established areas of nonlinear control theory: absolute stability and input-to-state stability (from now on, ISS). Classical absolute stability theory, the origins of which date back to the late 1940s and the seminal paper by Lur'e and Postnikov [44], is concerned with Lur'e systems: that is, feedback interconnections consisting of a linear state-space system in the forward path and a nonlinearity in the feedback path. In continuous-time these can be represented as

$$\dot{x}(t) = Ax(t) + Bf(y(t)), \qquad y(t) = Cx(t) + Df(y(t)), \tag{1.1}$$

where A, B, C, D are real matrices of appropriate dimensions and f is a nonlinearity. Absolute stability of Lur'e systems has been studied extensively and forms an integral part of nonlinear control theory, see e.g. Vidyasagar [56], Khalil [36] or Haddad and Chellaboina [23] for textbook treatments in the continuous-time setting (the latter includes a chapter on the discrete-time case as well) and Haddad and Bernstein [22], Gonzaga [18], Alamo [3] or Ahmad [1] for some publications in the discrete-time setting. Also, an overview of the area is provided in Liberzon's survey article [40], which collects just shy of 500 references from work on absolute stability theory. We should note that in literature the nonlinearity f is often time-varying; the assumptions in stability results then have to hold uniformly in the time variable.

An important landmark, the celebrated Aizerman's conjecture from [2], has been a major influence on the development of absolute stability theory and, arguably, its starting point. In the single-input single-output setting it hypothesizes that if, for a given linear system (A, B, C, D), the Lur'e interconnection is globally asymptotically stable for *all linear* output feedbacks F that satisfy the sector condition $a \leq F \leq b$, then, in fact, the Lur'e interconnection is globally asymptotically stable for *all nonlinear* output feedback maps f that satisfy the same sector condition $a\xi^2 \leq f(\xi)\xi \leq b\xi^2$ for all $\xi \in \mathbb{R}$, see Figure 1.1. While a counterexample to the classical Aizerman's

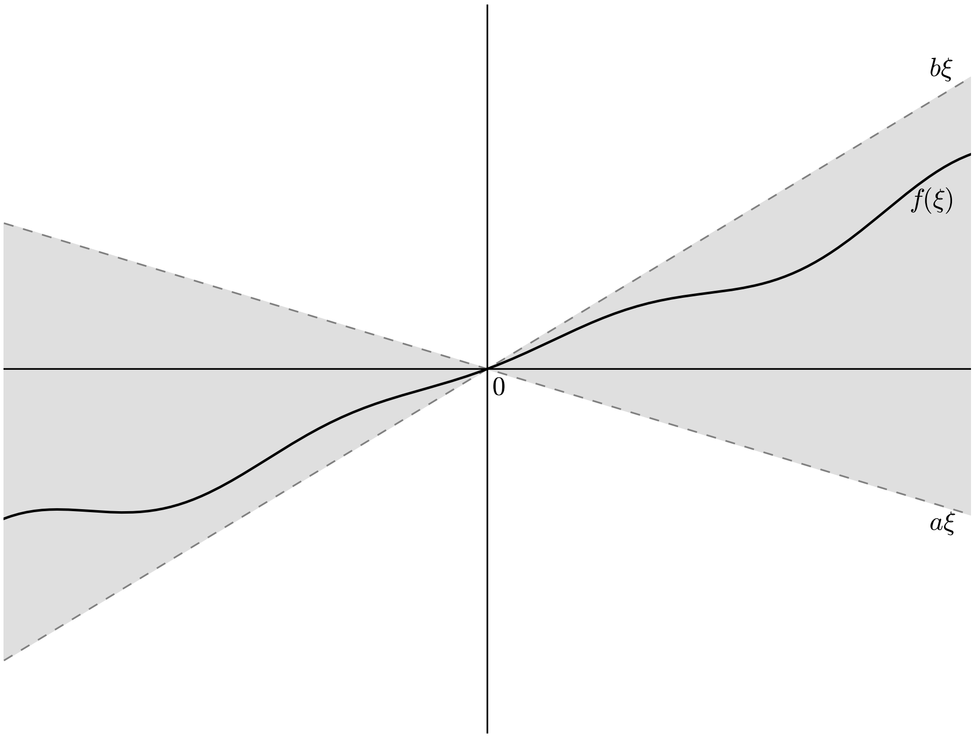

Figure 1.1: Sector condition

conjecture was constructed by Pliss [46] and Fitts [15], it was shown in Hinrichsen and Pritchard [25] that a stability radius version of Aizerman's conjecture holds true over the complex field. This modified conjecture plays a prominent role in the present thesis and a version of it is the first result we present that is of some novelty, see Propositions 4.2.1 and 8.2.1 for the discrete-time and the continuous-time cases, respectively. A slightly simplified version of our results states that if, for a given multivariable linear system (A, B, C, D), the Lur'e interconnection is globally asymptotically stable for *all complex linear* output feedback maps F that satisfy the norm condition $\|F\xi\| < r\,\|\xi\|$ for some $r > 0$ and for all $\xi \in \mathbb{C}^p \setminus \{0\}$, then, in fact, the Lur'e interconnection is globally asymptotically stable for *all nonlinear* output feedback maps f that satisfy the same norm condition $\|f(\xi)\| < r\,\|\xi\|$ for all $\xi \in \mathbb{R}^p$. This result displays some differences in our overall approach from previous work: (i) we adopt an Aizerman viewpoint in our results, (ii) we allow nonzero feedthrough and consider multivariable systems, and (iii) we analyse both the discrete-time and the continuous-time cases. As a corollary we obtain a result strongly reminiscent of the classical circle criterion for multivariable systems, see Propositions 4.3.2 and 8.4.1 for the discrete-time and the continuous-time cases respectively. Note that, in contrast to existing work, we develop our results over the complex field as well as the real field and thus allow complex-valued nonlinearities.

While our absolute stability results are of some novelty and offer interesting perspectives on a classical hypothesis, the main results of this thesis pertain to input-to-state stability. ISS theory has been developed over the last 25 years and it provides a natural stability framework for nonlinear systems with inputs, merging, in a sense, Lyapunov and input-output approaches to stability and encapsulating the robustness of a globally asymptotically stable equilibrium with respect to bounded disturbances. See Jiang and Wang [31] and Sontag [52] for an overview of ISS in discrete-time and continuous-time, respectively. Recent developments demonstrate that, under slightly stronger assumptions than those in results from absolute stability theory, we in fact obtain ISS, see Arcak and Teel [7], Jayawardhana, Logemann and Ryan [29, 30], Bruin, Doris, van de Wouw, Heemels and Nijmeijer [12] and Yang, Zhang and Huang [63]. In line with the above references, we study Lur'e systems with forcing, however we also admit nonzero feedthrough and consider both the discrete-time case

$$\begin{aligned} x(t+1) &= Ax(t) + B(f(y(t)) + d(t)) \\ y(t) &= Cx(t) + D(f(y(t)) + d(t)) \end{aligned} \tag{1.2}$$

and the continuous-time case

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(f(y(t)) + d(t)) \\ y(t) &= Cx(t) + D(f(y(t)) + d(t)). \end{aligned} \tag{1.3}$$

Here the forcing d could represent a target trajectory or a disturbance. The main results in this thesis are Aizerman-like ISS criteria for Lur'e systems with forcing (1.2) and (1.3). These state that if, for a given multivariable linear system (A, B, C, D), the Lur'e interconnection is ISS for *all complex linear* output feedback maps F that satisfy the norm condition $\|F\xi\| < r\,\|\xi\|$ for some $r > 0$ and for all $\xi \in \mathbb{C}^p \backslash \{0\}$, then, in fact, the Lur'e interconnection is ISS for *all nonlinear* output feedback maps f that satisfy a similar norm condition $\|f(\xi)\| < r\,\|\xi\| - \alpha(\|\xi\|)$ for an appropriate comparison function α and for all $\xi \in \mathbb{R}^p$, see Theorems 5.3.1 and 9.2.1. We stress that the class of comparison functions allowed above is quite wide and that it admits functions that increase slowly. In the continuous-time setting we obtain a number of results from Jayawardhana, Logemann and Ryan [29, 30] as consequences, see Corollaries 9.3.2 and 9.3.3.

A central tool for proving ISS in previous work that derives ISS criteria from assumptions similar to ones made in absolute stability results is a Lyapunov characterization of input-to-state stability (see e.g. Sontag [52] for the continuous-time result or Jiang and Wang [31, 32] for the discrete-time result). In Arcak and Teel [7] and Jayawardhana, Logemann and Ryan [29, 30] an ISS-Lyapunov function is constructed by combining a quadratic form obtained from the positive real lemma with a quadratic form obtained from solving a Lyapunov equation associated to the underlying linear system (A, B, C, D). This is where our approach differs - while we do construct an ISS-Lyapunov function to prove ISS, we use the bounded real lemma for a quadratic form instead. This allows us to analyse different classes of systems and requires new methods of estimation to be developed, which results in a treatment of $\mathcal{K}_\infty$ functions (see §5.1) and novel results in Lemma 5.1.11 and Proposition 5.1.15.

Lastly, we will also consider linear discrete-time input-output systems defined by higher-order difference equations

$$\sum_{j=0}^{k} P_j y(t+j) = \sum_{j=0}^{k} Q_j u(t+j), \tag{1.4}$$

where P_j's and Q_j's are matrices of appropriate dimensions. By closing the feedback loop via $u(t) := f(y(t)) + d(t)$, where f is some nonlinearity and d is a forcing, which could represent e.g. a target trajectory or a disturbance, we obtain a class of input-output Lur'e systems

$$\sum_{j=0}^{k} P_j y(t+j) = \sum_{j=0}^{k} Q_j (f(y(t+j)) + d(t+j)). \tag{1.5}$$

Some absolute stability aspects of the continuous-time single-input single-output (from here on, SISO) counterpart of (1.5) have been studied in Brockett and Willems [10], where a Popov criterion type result is obtained. Other approaches include Haddad and Chellaboina [23], Desoer and Vidyasagar [14], who study classical input-output systems (that is, causal maps from the input space to the output space) in the context of absolute stability. Their results revolve around the small-gain theorem and norm approximations, and they typically establish classical input-output stability in the l^p sense. We obtain a different class of results by applying behavioural theory to the study of (1.4). The main role is played by an extension of known results from behavioural theory, which relate tuples (u, y) that satisfy (1.5) to triples (u, x, y) that satisfy

$$x(t+1) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t),$$

for an appropriate linear state-space system (A, B, C, D), see Theorem 11.4.5 and its corollaries. It allows us to apply, in effect, Lyapunov techniques from Part I of the thesis to input-output Lur'e systems (1.5) and obtain both absolute stability results, see Propositions 12.1.7, 12.1.10 and Corollary

12.1.8, as well as input-to-output stability results, see Theorem 12.1.13 and Corollaries 12.1.14, 12.1.15.

This thesis is structured as follows: in Part I we analyse discrete-time Lur'e systems (1.2). We state some preliminary results on quadratic forms and stabilizing linear output feedback matrices in §2 and §3 before applying them to absolute stability problems in §4 and to ISS problems in §5. In Part II we analyse continuous-time Lur'e systems (1.3). The results on quadratic forms and linear output feedback matrices are collected in §7 and then applied to studying absolute stability in §8, while §9 is concerned with ISS questions. Finally, in Part III we study discrete-time input-output Lur'e systems (1.5). We state and prove preliminary results on realization theory in §11. This is then applied to absolute stability and input-to-output stability problems in §12. We collect some proofs and state some unused, yet interesting, results in the appendix.

Part I

Stability of discrete-time Lur'e systems

Since the general ideas of Part I of this thesis were already explained in the Introduction, we shall not repeat ourselves, but will instead explain how this part is organized. Chapter 2 starts with a short section, §2.1, which defines linear state-space systems. Then we exhibit two ways of constructing quadratic functions for stability analysis: one from bounded real lemma in §2.2 and one from output injection in §2.3. Then in §2.4 we state and prove a result on ω-limit sets. Chapter 3 is devoted to analysing stabilizing linear output feedback matrices. After the definition and some simple consequences in §3.1, we relate a ball condition to the bounded real property, the positive real property and the complex stability radius (see Lemma 3.2.7, Proposition 3.2.12 and Corollary 3.2.17, respectively). We then use an equivalence of a ball condition and the bounded real property to construct a quadratic form, which will be used in both absolute stability and ISS analysis, see Lemma 3.2.8. In Chapter 4 we analyse the absolute stability of Lur'e systems without forcing and obtain a result that proves the complexified version of Aizerman's conjecture, see Proposition 4.2.1. This is in turn then used to prove a result similar to the classical circle criterion, see Proposition 4.3.2. In Chapter 5 we finally turn to ISS of Lur'e systems and obtain - in Theorem 5.3.1 - sufficient conditions for ISS from assumptions similar to the ones made in the complexified version of Aizerman's conjecture. It is then used to provide sufficient conditions for ISS from assumptions similar to the ones made in the circle criterion, see Proposition 5.4.1. We state which results are original, provide references to similar existing work and discuss possible future avenues of exploration in §6.

We should remark that since existing similar work [7, 29, 30] is conducted in a continuous-time setting, we defer detailed comparisons with our results to Part II, where we analyse continuous-time systems.

Chapter 2

Preliminaries

In this chapter we collect preliminaries on linear state-space systems. After defining linear state-space systems and their behaviours in §2.1, we will look at the quadratic forms offered to us by the use of the bounded real lemma and the positive real lemma in §2.2 and by the use of an output injection in §2.3. Finally, we will note some ω-limit set properties in §2.4.

2.1 Linear state-space systems

Definition 2.1.1. We call a matrix quadruple of dimensions $(A, B, C, D) \in \mathbb{F}^{n \times n} \times \mathbb{F}^{n \times m} \times \mathbb{F}^{p \times n} \times \mathbb{F}^{p \times m}$ a **linear state-space system** or sometimes, a linear system. The set of all linear systems of this format are denoted by $\Sigma(m, n, p; \mathbb{F})$.

The **transfer function** of (A, B, C, D), usually denoted by G, is defined as $C(zI - A)^{-1}B + D \in \mathbb{F}(z)^{p \times m}$.

We say that a linear state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is controllable if $\operatorname{rank} \begin{pmatrix} B & AB & \dots & A^{n-1} \end{pmatrix} = n$ and we say that it is observable if $\operatorname{rank} \begin{pmatrix} C^* & A^*C^* & \dots & A^{*n-1}C^* \end{pmatrix}^* = n$. We will sometimes say that a controllable and observable linear system is minimal. If there exists $K \in \mathbb{F}^{m \times n}$ such that $\sigma(A + BK) \subseteq \mathbb{D}$, then we say that (A, B, C, D) is stabilizable. We say that (A, B, C, D) is detectable if (A^*, C^*, B^*, D^*) is stabilizable.

Somewhat unusually for stability analysis, we will make use of the notion of a behaviour; we introduce it for linear systems first. Recall the definition of the left-shift operator $\mathcal{L}$ from §0.1.

Definition 2.1.2. For $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ we define the **behaviour**

of the linear state-space system (A, B, C, D) as

$$\begin{aligned}\mathcal{B}(A,B,C,D) := \Big\{(u,x,y) \in (\mathbb{F}^m)^{\mathbb{N}_0} \times (\mathbb{F}^n)^{\mathbb{N}_0} \times (\mathbb{F}^p)^{\mathbb{N}_0} :\\ \mathcal{L}x = Ax + Bu\\ y = Cx + Du\Big\}.\end{aligned}$$

We call an element $(u, x, y) \in \mathcal{B}(A, B, C, D)$ a **trajectory**.

Note that the above definition of a trajectory is simply saying that $(u, x, y) \in \mathcal{B}(A, B, C, D)$ are such that

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \qquad \forall\, t \in \mathbb{N}_0. \end{aligned}$$

The concept of a behaviour allows us to easily apply quadratic forms obtained from linear techniques (e.g. Lemmas 2.2.1, 3.2.8, 3.2.13) to both absolute stability analysis (see the proof of Proposition 4.2.1) and ISS analysis (see the proof of Theorem 5.3.1).

2.2 Bounded real lemma

Existing ISS results for Lur'e systems from [7, 29, 30] are proved using an ISS-Lyapunov function obtained by the use of the positive real lemma. This result, which is sometimes called the Kalman-Yakubovich-Popov lemma, allows one to guarantee the existence of a quadratic form that is useful in stability analysis by assuming a frequency-domain condition for a controllable and observable state-space system. Moreover, Lyapunov functions obtained by similar methods are known to be useful for absolute stability analysis of Lur'e systems, see e.g. [22, 23, 36]. Instead of the positive real lemma we will use the bounded real lemma to construct a quadratic form, which will be used in both absolute stability and ISS analysis. This approach allows us to obtain stability results for new classes of systems. Moreover, we prove the bounded real lemma for stabilizable and detectable linear state-space systems, which allows us to relax the common assumption that the underlying linear state-space system of a Lur'e system must be controllable and observable.

Before we move on to the main part of this section, we state a lemma that will be useful for the analysis of quadratic forms.

Lemma 2.2.1. *Consider a positive semi-definite matrix* $P = P^* \in \mathbb{F}^{n\times n}$ *and define a quadratic form* $V : \mathbb{F}^n \to [0, \infty)$ *by* $V(\xi) := \langle P\xi, \xi\rangle$.

Then $V^{-1}(0) = \ker P$. *Moreover, there exists a projection* $\Pi\colon \mathbb{F}^n \to \mathbb{F}^n$ *such that* $\ker \Pi = \ker P$ *and* $V(\xi) \geq c\,\|\Pi\xi\|^2$ *for all* $\xi \in \mathbb{F}^n$.

Proof. By the complex and real spectral theorems (see e.g. [8]), there exists $M \in \mathbb{F}^{n\times n}$ such that M^*PM is a diagonal matrix and such that $M^{-1} = M^*$. We can then rewrite $\langle P\xi, \xi\rangle = \langle M^*PM\mu, \mu\rangle$, where $\mu = M^*\xi$. Since M^*PM is diagonal, if $\xi \in V^{-1}(0)$, then $\mu \in \ker M^*PM$, which in turn gives us $\xi \in \ker P$, so that $V^{-1}(0) \subseteq \ker P$. On the other hand, $\ker P \subseteq V^{-1}(0)$ follows trivially from the definition of V and hence we have $V^{-1}(0) = \ker P$.

Let Π be the orthogonal projection onto $(\ker P)^\perp$ along $\ker P = V^{-1}(0)$ (see e.g. §6 from [8] for the construction of Π). Then $\ker \Pi = V^{-1}(0)$. Moreover since $P = P\Pi$ and $P = P^*$, it follows that $V(\xi) = V(\Pi\xi)$ for all $\xi \in \mathbb{F}^n$. Finally, since $\ker \Pi = V^{-1}(0)$, the seminorm $\xi \mapsto \sqrt{V(\xi)}$ becomes a norm once restricted to $(V^{-1}(0))^\perp = (\ker P)^\perp = \operatorname{im} \Pi$. Hence there exists a positive c such that $V(\xi) = V(\Pi\xi) \geq c\,\|\Pi\xi\|^2$ for all $\xi \in \mathbb{F}^n$, completing the proof. □

Definition 2.2.2. The **Hardy space** $H^\infty(\mathbb{E}, \mathbb{C}^{p\times m})$ is the space of all bounded analytic functions $F\colon \mathbb{E} \to \mathbb{C}^{p\times m}$ with the norm given by

$$\|F\|_{H^\infty} = \sup_{z\in\mathbb{E}} \|F(z)\|.$$

We will usually use the shorthand H^∞ for $H^\infty(\mathbb{E}, \mathbb{C}^{p\times m})$.

Consider a linear state-space system (A, B, C, D) and denote its transfer function by G. The bounded real lemma is a name given to a set of results that provide characterization of the properties $\|G\|_{H^\infty} < 1$ and $\|G\|_{H^\infty} \leq 1$ in terms of existence of solutions to certain matrix equations. More precisely, it states that under the above assumptions, one can construct matrices P, W, L (with P self-adjoint) that satisfy the bounded real equations

$$\begin{aligned} A^*PA - P + C^*C &= -L^*L, \\ A^*PB + C^*D &= -L^*W, \\ B^*PB + D^*D &= I - W^*W. \end{aligned}$$

There seems to be no definitive version of the bounded real lemma in the literature. Haddad and Chellaboina [23], Anderson and Vongpanitlerd [5] and Anderson [4] assume strict inequality $\|G\|_{H^\infty} < 1$ and that the underlying linear system is controllable and observable and infer that $P > 0$. Haddad and Bernstein [22] assume nonstrict inequality $\|G\|_{H^\infty} \leq 1$ and that the underlying linear system is controllable and observable and infer that $P > 0$. Finally, Wimmer [62]) relaxes the minimality of (A, B, C, D) to stabilizability and detectability at the price of only being able to guarantee that $P \geq 0$ and making an additional assumption on the transfer function of

(A, B, C, D). We will use the latter two versions of the bounded real lemma in this thesis.

Lemma 2.2.3. *Consider a stabilizable and detectable system* $(A, B, C, D) \in \Sigma(m, n, p, \mathbb{F})$ *and denote its transfer function by* G. *Assume that* $\|G\|_{H^\infty} \leq 1$ *and* $\|G(z_0)\| < 1$ *for some* $z_0 \in \mathbb{C}$ *with* $|z_0| = 1$.

Then there exist matrices L, W *and a positive semi-definite* $P = P^* \in \mathbb{F}^{n\times n}$ *such that*

$$\begin{aligned} A^*PA - P + C^*C &= -L^*L, \\ A^*PB + C^*D &= -L^*W, \\ B^*PB + D^*D &= I - W^*W. \end{aligned}$$

Proof. By stabilizability, detectability and $\|G\|_{H^\infty} \leq 1$ we have $\sigma(A) \subseteq \mathbb{D}$. Hence we can apply Theorem 5.3 from Wimmer [62] (technically Wimmer only proves it for the case, when the underlying field is complex, but an inspection reveals that it can be extended to the real field as well) to see that there exists a positive semi-definite $P = P^* \in \mathbb{F}^{n\times n}$ such that $I - D^*D - B^*PB$ is positive definite and such that P is a solution of the Riccati equation

$$\begin{aligned} A^*PA - P + (A^*PB + C^*D)[I - D^*D - B^*PB]^{-1}(B^*PA + D^*C) \\ + C^*C = 0. \end{aligned}$$

Since $I - D^*D - B^*PB$ is positive definite, there exists $W = W^* > 0$ such that $W^*W = I - D^*D - B^*PB$ (see e.g. Theorem 7.27 from [8]). The proof is then complete if we set $L := -(W^*)^{-1}(B^*PA + D^*C)$. $\square$

We provide a full proof of the continuous-time counterpart of Lemma 2.2.3, see Lemma 7.3.1 and its proof in Appendix §C. We remark that the assumption "$\|G(z_0)\| < 1$ for some $z_0 \in \mathbb{C}$ with $|z_0| = 1$" is similar to the assumption made in the continuous-time case, namely "$\|D\| = \lim_{z\to\infty} \|G(z)\| < \|G\|_{H^\infty}$", see Lemma 7.3.1.

We will also use the more common version of the bounded real lemma, which assumes that the linear system is minimal, but allows us to omit the assumption that there exists $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that $\|G(z_0)\| < 1$.

Lemma 2.2.4. *Consider a controllable and observable system* $(A, B, C, D) \in \Sigma(m, n, p, \mathbb{F})$ *and assume that its transfer function* G *satisfies* $\|G\|_{H^\infty} \leq 1$.

Then there exist matrices L, W *and a positive semi-definite* $P = P^* \in \mathbb{F}^{n\times n}$

such that

$$\begin{aligned} A^*PA - P + C^*C &= -L^*L, \\ A^*PB + C^*D &= -L^*W, \\ B^*PB + D^*D &= I - W^*W. \end{aligned}$$

Proof. In the real case this is Lemma 3.1 from [22] and an inspection of their proof reveals that it can be easily extended to the complex case. □

The bounded real lemma allows us to construct a quadratic form that will be useful in stability analysis.

Lemma 2.2.5. *Consider a stabilizable and detectable system* $(A, B, C, D) \in \Sigma(m, n, p, \mathbb{F})$ *and denote its transfer function by* G*. Assume that* $G \in H^\infty$ *and that there exists* $z_0 \in \mathbb{C}$ *with* $|z_0| = 1$ *such that* $\|G(z_0)\| < \|G\|_{H^\infty}$*.*

Then there exists a positive semi-definite $P = P^* \in \mathbb{F}^{n\times n}$ *such that the quadratic form* $V : \mathbb{F}^n \to [0, \infty)$ *defined by* $V(\xi) := \langle P\xi, \xi\rangle$ *satisfies*

$$V(x(t+1)) - V(x(t)) \le \|u(t)\|^2 - \|G\|_{H^\infty}^{-2} \|y(t)\|^2$$

for all $t \in \mathbb{N}_0$ *and for all trajectories* $(u, x, y) \in \mathcal{B}(A, B, C, D)$*.*

Moreover there exists a projection $\Pi\colon \mathbb{F}^n \to \mathbb{F}^n$ *and a positive* c *such that* $\ker \Pi \subseteq \ker C$ *and* $V(\xi) \ge c \|\Pi\xi\|^2$ *for all* $\xi \in \mathbb{F}^n$*.*

Proof. Set $\rho := \|G\|_{H^\infty}^{-1}$ and consider the (stabilizable and detectable) linear state-space system $(A, \rho B, C, \rho D)$. Its transfer function ρG satisfies the assumptions of Lemma 2.2.3, so there exist matrices L, W and a positive semi-definite $P = P^* \in \mathbb{F}^{n\times n}$ such that

$$A^*PA - P + C^*C = -L^*L, \tag{2.2.1a}$$
$$\rho A^*PB + \rho C^*D = -L^*W, \tag{2.2.1b}$$
$$\rho^2 B^*PB + \rho^2 D^*D = I - W^*W. \tag{2.2.1c}$$

Now define $V(\xi) := \langle P\xi, \xi\rangle$, pick an arbitrary $(u, x, y) \in \mathcal{B}(A, B, C, D)$ and use the difference equations for state and output to obtain

$$\begin{aligned} V(x(t+1)) - V(x(t)) &= \langle P(Ax(t) + Bu(t)), Ax(t) + Bu(t)\rangle \\ &\quad - \langle Px(t), x(t)\rangle \\ &= \langle (A^*PA - P)x(t), x(t)\rangle + \langle A^*PBu(t), x(t)\rangle \\ &\quad + \langle x(t), A^*PBu(t)\rangle + \langle B^*PBu(t), u(t)\rangle . \end{aligned}$$

Therefore, by using equations (2.2.1a) - (2.2.1c), we can obtain

$$\begin{aligned}
&V(x(t+1)) - V(x(t)) = \\
&\qquad = - \|Cx(t)\|^2 - \|Lx(t)\|^2 - \langle Du(t), Cx(t)\rangle - \frac{1}{\rho}\langle Wu(t), Lx(t)\rangle \\
&\qquad\quad - \langle Cx(t), Du(t)\rangle - \frac{1}{\rho}\langle Lx(t), Wu(t)\rangle \\
&\qquad\quad - \|Du(t)\|^2 + \frac{1}{\rho^2}\|u(t)\|^2 - \frac{1}{\rho^2}\|Wu(t)\|^2 \\
&\qquad = - \left\|Lx(t) + \frac{1}{\rho}Wu(t)\right\|^2 + \frac{1}{\rho^2}\|u(t)\|^2 - \|y(t)\|^2 \\
&\qquad \le \frac{1}{\rho^2}\|u(t)\|^2 - \|y(t)\|^2
\end{aligned}$$

for all $t \in \mathbb{N}_0$. Thus $\rho^2 V$ satisfies the required estimate.

Now, by Lemma 2.2.1, we know that $V^{-1}(0) = \ker P$ and that there exists a projection $\Pi\colon \mathbb{F}^n \to \mathbb{F}^n$ and a positive c such that $\ker \Pi = \ker P = V^{-1}(0)$ and $V(\xi) \ge c\|\Pi\xi\|^2$ for all $\xi \in \mathbb{F}^n$. Pick $\xi \in V^{-1}(0)$ and use equation (2.2.1a) to see that

$$0 = \langle PA\xi, A\xi\rangle = -\|C\xi\|^2 - \|L\xi\|^2 .$$

Hence $\xi \in \ker C$ and consequently $\ker \Pi = V^{-1}(0) \subseteq \ker C$ completing the proof. □

An almost identical line of reasoning leads us to the following lemma.

Lemma 2.2.6. *Consider a controllable and detectable system* $(A, B, C, D) \in \Sigma(m, n, p, \mathbb{F})$ *and assume that its transfer function* $G \in H^\infty$.

Then there exists a positive semi-definite $P = P^* \in \mathbb{F}^{n\times n}$ *such that the quadratic form* $V : \mathbb{F}^n \to [0, \infty)$ *defined by* $V(\xi) := \langle P\xi, \xi\rangle$ *satisfies*

$$V(x(t+1)) - V(x(t)) \le \|u(t)\|^2 - \|G\|_{H^\infty}^{-2}\|y(t)\|^2$$

for all $t \in \mathbb{N}_0$ *and for all trajectories* $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Moreover there exists a projection $\Pi\colon \mathbb{F}^n \to \mathbb{F}^n$ *and a positive* c *such that* $\ker \Pi \subseteq \ker C$ *and* $V(\xi) \ge c\|\Pi\xi\|^2$ *for all* $\xi \in \mathbb{F}^n$.

Our focus in this section is the bounded real lemma, however, as mentioned at the start of this section, it is the positive real lemma, which seems to be used more often in stability analysis of Lur'e systems. In §2.3 we will compare the respective quadratic forms, so we now state the relevant positive real lemma results.

Definition 2.2.7. Consider a linear system $(A, B, C, D) \in \Sigma(m, n, m; \mathbb{F})$ with an equal number of inputs and outputs, and denote its transfer function by G. We say that G is (discrete-time) **positive real**, if $(G(z))^* + G(z) \geq 0$ for all $z \in \mathbb{E}$ which are not poles of $G(z)$.

We say that G is **strictly positive real** if there exists $\rho > 1$ such that $G(z\rho)$ is positive real.

We say that G is **strongly positive real** if it is strictly positive real and $D^* + D > 0$, where $D := \lim_{s\to\infty} G(s)$.

Some consequences of these definitions are collected in the appendix, see §A.1. The following version of the positive real lemma is proved in [28].

Lemma 2.2.8 (Positive real lemma)**.** *Consider a controllable and observable $(A, B, C, D) \in \Sigma(m, n, m, \mathbb{R})$ and denote its transfer function G.*

Then G is positive real if and only if there exist matrices L, W and a positive definite $P = P^* \in \mathbb{R}^{n\times n}$ such that:

$$
\begin{aligned}
A^*PA - P &= -LL^*, \\
A^*PB - C^* &= -LW, \\
D + D^* - B^*PB &= W^*W.
\end{aligned}
$$

Much like the bounded real lemma, the positive real lemma can be used to construct a quadratic form useful in stability analysis. We relegate the proof of the following result to the appendix, see §A.1.

Lemma 2.2.9. *Consider a controllable and observable linear state-space system $(A, B, C, D) \in \Sigma(m, n, m, \mathbb{F})$ and assume that its transfer function G is positive real.*

Then there exists a positive definite $P = P^* \in \mathbb{F}^{n\times n}$ such that the quadratic form $V : \mathbb{F}^n \to [0, \infty)$ defined by $V(\xi) := \langle P\xi, \xi\rangle$ satisfies

$$
V(x(t+1)) \leq V(x(t)) + \frac{1}{2}\left[\|u(t) + y(t)\|^2 - \|u(t) - y(t)\|^2\right]
$$

for all $t \in \mathbb{N}_0$ and for all trajectories $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Note that the quadratic forms V obtained from Lemmas 2.2.9 and 2.2.5 offer very different estimates on $V(x(t + 1)) - V(x(t))$. However the following result establishes a link between positive real and bounded real (that is, $\|G\|_{H^\infty} \leq 1$) transfer functions.

Lemma 2.2.10. *Let $G \in \mathbb{F}(z)^{m\times m}$; the following are equivalent:*

(a) G is positive real,

(b) $I + G$ is invertible and $\left\|(I - G)(I + G)^{-1}\right\|_{H^\infty} \leq 1$.

Proof. First we note that if G is positive real, then $2 \cdot \mathrm{Re}\,\langle (I + G(z))\xi, \xi\rangle = \langle (2I + G(z) + G(z)^*)\xi, \xi\rangle > 0$ for all $\xi \in \mathbb{F}^m \setminus \{0\}$ and for all $z \in \mathbb{E}$, which implies that $I + G(z)$ is invertible for all $z \in \mathbb{E}$ and hence $I + G$ is invertible.

Thus it now suffices to show that if $I + G$ is invertible, then G is positive real if, and only if, $\left\|(I - G)(I + G)^{-1}\right\|_{H^\infty} \leq 1$. To this end, set $F := (I - G)(I + G)^{-1}$ and note that $G = (I - F)(I + F)^{-1}$. Thus G is positive real if, and only if,

$$(I - F(z))(I + F(z))^{-1} + \left[(I - F(z))(I + F(z))^{-1}\right]^* \geq 0 \qquad \forall\, z \in \mathbb{E}. \tag{2.2.2}$$

Premultiply this by $(I + F(z))^*$ and postmultiply by $I + F(z)$ (note that these two operations do not change the positive definiteness of the matrix on the left hand side of inequality (2.2.2)) to see that G is positive real if, and only if,

$$(I + F(z))^*(I - F(z)) + (I + F(z))^*(I - F(z)) \geq 0 \qquad \forall\, z \in \mathbb{E}.$$

This simplifies to

$$\begin{aligned} & & I \geq (F(z))^* F(z) & \qquad \forall\, z \in \mathbb{E} \\ &\iff & \|\xi\|^2 \geq \|F(z)\xi\|^2 & \qquad \forall\, \xi \in \mathbb{F}^m,\ \forall\, z \in \mathbb{E} \\ &\iff & 1 \geq \|F\|_{H^\infty}. & \end{aligned}$$ □

Results similar to this lemma are known: see Anderson [4], Guiver [20] or Hinrichsen and Pritchard [25]. The map $G \mapsto (I - G)(I + G)^{-1}$ is sometimes called the Cayley transform.

2.3 Output injection

In this section we obtain a quadratic form via a technique sometimes called "an output injection". In the continuous-time setting, this is a well-known technique, see the references provided in §7.4.

Lemma 2.3.1. *Consider $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and assume that it is detectable. Then there exists a positive definite $P = P^* \in \mathbb{F}^{n \times n}$ and $\delta > 0$ such that the quadratic form $V\colon \mathbb{F}^n \to [0, \infty)$ defined by $V(\xi) := \langle P\xi, \xi\rangle$ satisfies*

$$V(x(t + 1)) - V(x(t)) \leq -\delta \|x(t)\|^2 + \|y(t)\|^2 + \|u(t)\|^2$$

for all $t \in \mathbb{N}_0$ and for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Proof. By detectability of (C, A), there exists $H \in \mathbb{F}^{n\times m}$ such that $\sigma(A + HC) \subset \mathbb{D}$. It is a well-known fact from linear algebra that $\sigma(M) = \overline{\sigma(M^*)}$, where overline denotes the complex conjugation (see e.g. Proposition 4.4.4v) from [9]).

Hence there exists a solution $Q = Q^* > 0$ of the discrete-time Lyapunov equation

$$(A + HC)^* Q(A + HC) - Q = -I. \tag{2.3.1}$$

Now consider the quadratic form $V_Q(\xi) := \langle Q\xi, \xi\rangle$. We can use equation (2.3.1) to see that:

$$\begin{aligned}
&V_Q(x(t+1)) - V_Q(x(t)) \\
&\qquad = \langle (A^*QA - Q)x(t), x(t)\rangle + \langle QAx(t), Bu(t)\rangle \\
&\qquad\quad + \langle QBu(t), Ax(t)\rangle + \langle QBu(t), Bu(t)\rangle \\
&\qquad = -\|x(t)\|^2 - \langle QAx(t), HCx(t)\rangle \\
&\qquad\quad - \langle QHCx(t), HCx(t)\rangle - \langle QHCx(t), Ax(t)\rangle \\
&\qquad\quad + \langle QAx(t), Bu(t)\rangle + \langle QBu(t), Ax(t)\rangle + \langle QBu(t), Bu(t)\rangle
\end{aligned}$$

for all $t \in \mathbb{N}_0$ and $(u, x, y) \in \mathcal{B}(A, B, C, D)$. Our aim now is to estimate the right hand side of this expression in such a way that we do not accrue more than a "$-1/2\,\|x(t)\|^2$" term in total. To this end, we substitute $Cx(t) = y(t) - Du(t)$ and apply the Cauchy-Schwarz inequality, the property of the operator norm $\|M\xi\| \leq \|M\|\,\|\xi\|$ and the simple inequality $ab = \frac{a}{c}bc \leq \frac{1}{c^2}a^2 + c^2b^2$ to see that there exist positive δ, c_1, c_2 such that

$$V_Q(x(t+1)) - V_Q(x(t)) \leq -\delta\,\|x(t)\|^2 + c_1\,\|y(t)\|^2 + c_2\,\|u(t)\|^2$$

for all $t \in \mathbb{N}_0$ and $(u, x, y) \in \mathcal{B}(A, B, C, D)$. The function $\frac{1}{\max\{c_1,c_2\}}V_Q$ has all the sought properties. $\square$

2.4 ω-limit set

Typically, ω-limit sets are defined for initial value problems, however we use them in a slightly unusual context, so it will be useful to define an ω-limit set for an element $v \in (\mathbb{F}^m)^{\mathbb{N}_0}$.

Definition 2.4.1. For $v \in (\mathbb{F}^m)^{\mathbb{N}_0}$ we define the ω-**limit set of** v, denoted Ω_v, by

$$\Omega_v := \{\xi \in \mathbb{F}^m : \exists (t_j)_{j\in\mathbb{N}_0} \subset \mathbb{N}_0 \text{ s.t. } t_j \to \infty \text{ and } v(t_j) \to \xi \text{ as } j \to \infty\}.$$

For a point $\xi \in \mathbb{F}^m$ and a nonempty set $\mathbb{S} \subset \mathbb{F}^m$ we define the **distance between ξ and $\mathbb{S}$** as

$$\operatorname{dist}(\xi, \mathbb{S}) := \inf\{\|\xi - \mu\| : \mu \in \mathbb{S}\}.$$

Lemma 2.4.2. *Consider a bounded $v \in (\mathbb{F}^m)^{\mathbb{N}_0}$; then Ω_v is nonempty and*

$$\lim_{t\to\infty} \operatorname{dist}(v(t), \Omega_v) = 0.$$

Proof. Since v is bounded, by the Bolzano-Weierstrass theorem, there exists a convergent subsequence of $(v(t))_{t\in\mathbb{N}_0}$. Its limit is clearly in Ω_v, so that Ω_v is indeed nonempty.

Now suppose on the contrary, that $\lim_{t\to\infty} \operatorname{dist}(v(t), \Omega_v) = 0$ does not hold. Then there exists $\varepsilon > 0$ and a subsequence $(t_j)_{j\in\mathbb{N}_0}$ of $\mathbb{N}_0$ such that $t_j \to \infty$ and $\operatorname{dist}(v(t_j), \Omega_v) > \varepsilon$ for all $j \in \mathbb{N}_0$. However, by the Bolzano-Weierstrass theorem, $(v(t_j))_{j\in\mathbb{N}_0}$ has a convergent subsequence, so there exists a subsequence $(t_{j_i})_{i\in\mathbb{N}_0}$ of $(t_j)_{j\in\mathbb{N}_0}$ such that $t_{j_i} \to \infty$ and $\lim_{i\to\infty} v(t_{j_i}) = \xi$ for some $\xi \in \mathbb{F}^m$. Hence, by definition, $\xi \in \Omega_v$ which in turn contradicts $\operatorname{dist}(v(t_{j_i}), \Omega_v) > \varepsilon$ for all $i \in \mathbb{N}_0$ and thus completes the proof. $\square$

Chapter 3

Stabilization by linear output feedback

This short chapter is concerned with the study of linear output feedback matrices of state-space systems (A, B, C, D). In §3.1 we define what these are and note a simple consequence, sometimes called the loop shift technique, see Lemma 3.1.4. In §3.2 we introduce the set $\mathbb{S}_{\mathbb{C}}(A, B, C, D)$, which contains all linear output feedback matrices that stabilize (A, B, C, D). As we will see in Proposition 3.2.12, balls of stabilizing output feedback matrices being contained in $\mathbb{S}_{\mathbb{C}}(A, B, C, D)$ (from here on, we will reference this as the ball condition) provide us with an alternative characterization of the bounded real property $\|G\|_{H^\infty} \leq 1$. In combination with the bounded real lemma, the ball condition provides us with a quadratic form, which will turn out to be useful for stability analysis in later chapters, see Lemma 3.2.8. Then, in Proposition 3.2.12, we observe that the ball condition also admits an equivalent positive-real characterization. Finally, we note a connection with the complex stability radius in Corollary 3.2.16.

3.1 Linear output feedback

Consider a linear state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and denote its transfer function by G. Pick $K \in \mathbb{F}^{m \times p}$ and consider a trajectory $(u, x, y) \in \mathcal{B}(A, B, C, D)$. If we set $v := u - Ky$, then

$$\mathcal{L}x = Ax + Bu = Ax + Bv + BKy \tag{3.1.1}$$

$$y = Cx + Du = Cx + Dv + DKy. \tag{3.1.2}$$

If we assume that $I - DK$ is invertible, then we can rewrite (3.1.2) as

$$y = (I_p - DK)^{-1}Cx + (I_p - DK)^{-1}Dv$$

and in turn substitute this in (3.1.1) to obtain

$$\mathcal{L}x = \left[A + BK(I_p - DK)^{-1}C\right] x + \left[B + BK(I_p - DK)^{-1}D\right] v.$$

Hence if we define

$$\begin{aligned} A_K &:= A + BK(I_p - DK)^{-1}C, \quad B_K := B + BK(I_p - DK)^{-1}D, \\ C_K &:= (I_p - DK)^{-1}C, \quad D_K := (I_p - DK)^{-1}D, \end{aligned} \tag{3.1.3}$$

then

$$\mathcal{L}x = A_K x + B_K v \tag{3.1.4}$$

$$y = C_K x + D_K v. \tag{3.1.5}$$

This straightforward observation is at the basis of the loop shifting technique, see e.g. Green and Limebeer [19], and motivates the following definition.

Definition 3.1.1. Consider $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$. We define the set $\mathbb{A}_{\mathbb{C}}(A, B, C, D)$ of **admissible output feedback matrices** by

$$\mathbb{A}_{\mathbb{C}}(A, B, C, D) := \{K \in \mathbb{F}^{m\times p} \colon \det(I_p - DK) \neq 0\}.$$

Note that $\mathbb{A}_{\mathbb{C}}(A, B, C, D)$ only depends on the matrix D, so we will usually write $\mathbb{A}_{\mathbb{C}}(D)$ instead.

We note the following straightforward consequence of Sylvester's determinant theorem.

Lemma 3.1.2. *Let $D \in \mathbb{F}^{p\times m}$, $K \in \mathbb{F}^{m\times p}$; then $\det(I_p - DK) \neq 0$ if, and only if, $\det(I_m - KD) \neq 0$. In particular, $K \in \mathbb{A}_{\mathbb{C}}(D)$ if, and only if, $D \in \mathbb{A}_{\mathbb{C}}(K)$.*

This lemma allows us to make an observation on transfer functions, which will be useful below.

Lemma 3.1.3. *Consider a state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and denote its transfer function by G.*

Then for all $K \in \mathbb{A}_{\mathbb{C}}(D)$, we have

$$(I_p - GK)^{-1}G = G(I_m - KG)^{-1}.$$

Proof. Since $K \in \mathbb{A}_{\mathbb{C}}(D)$ and $D = G(\infty)$, $\det(I_p - G(\infty)K) \neq 0$. Hence as $I_p - GK$ is a rational matrix function, it is invertible in $\mathbb{F}(z)^{p\times p}$ and hence $(I_p - GK)^{-1}$ exists. Also, by Lemma 3.1.2, $D \in \mathbb{A}_{\mathbb{C}}(K)$ and thus, identically as above $(I_m - KG)^{-1}$, exists.

Clearly $G(I_m - KG) = (I_p - GK)G$; multiply this on the left by $(I_p - GK)^{-1}$ and on the right by $(I_m - KG)^{-1}$ to obtain

$$(I_p - GK)^{-1}G = G(I_m - KG)^{-1}. \qquad \square$$

For a given state-space system (A, B, C, D) with transfer function $G \in \mathbb{F}(z)^{p\times m}$ and with $K \in \mathbb{A}_{\mathbb{C}}(D)$, let us define

$$G^K := (I_p - GK)^{-1}G = G(I_m - KG)^{-1}.$$

Note that for $K \in \mathbb{A}_{\mathbb{C}}(D)$ we can also define this for a constant matrix:

$$D^K := (I_p - DK)^{-1}D = D(I_m - KD)^{-1}, \tag{3.1.6}$$

where Lemma 3.1.3 implies the equality of the two different representations.

Remark: note that $(I_p - GK)^{-1}G$ might be well-defined even if we do not require $K \in \mathbb{A}_{\mathbb{C}}(D)$, however if we do not require this, then $(I_p - GK)^{-1}G$ does not correspond to the transfer function of a linear state-space system. Hence we will *always* require that $K \in \mathbb{A}_{\mathbb{C}}(D)$, when we write down G^K.

From now on we will always define A_K, B_K, C_K, D_K by equation (3.1.3).

Lemma 3.1.4 (Loop shift lemma)**.** *Consider $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$, denote its transfer function by G and assume $K \in \mathbb{A}_{\mathbb{C}}(D)$.*

(a) A trajectory $(u, x, y) \in \mathcal{B}(A, B, C, D)$ if, and only if, $(u - Ky, x, y) \in \mathcal{B}(A_K, B_K, C_K, D_K)$.

(b) The transfer function of (A_K, B_K, C_K, D_K) is G^K.

(c) The linear system (A, B, C, D) is stabilizable and detectable if, and only if, (A_K, B_K, C_K, D_K) is.

Proof. (a): necessity follows from (3.1.4) and (3.1.5). Sufficiency can be obtained by reversing the implications that lead from equations (3.1.1) and (3.1.2) to equations (3.1.4) and (3.1.5). This is straightforward, so we omit the details.

(b): by the definition of the transfer function, we only need to check that

$$C_K(zI_n - A_K)^{-1}B_K + D_K = G(z)(I_m - KG(z))^{-1}. \tag{3.1.7}$$

Note that, by Lemma 3.1.2, $D \in \mathbb{A}_{\mathbb{C}}(K)$. Equation (3.1.7) can then be obtained via a long calculation which uses $\det(I_p - GK) \neq 0$, $(I_p - DK)^{-1}D$

$= D(I_m - KD)^{-1}$, Lemma 3.1.3 and equation (3.1.6) numerous times:

$$\begin{aligned}
&C_K(zI_n - A_K)^{-1}B_K + D_K = \\
&=C_K\left[zI_n - A - BK^DC\right]^{-1}B(I_m - KD)^{-1} + D_K \\
&=C_K\left[zI_n - A - B(I_m - KD)^{-1}KC\right]^{-1}B(I_m - KD)^{-1} + D_K \\
&=C_K\left[I_n - \underbrace{(zI_n - A)^{-1}B(I_m - KD)^{-1}}KC\right]^{-1} \\
&\qquad\qquad\qquad \times \underbrace{(zI_n - A)^{-1}B(I_m - KD)^{-1}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + D_K \\
&=C_K(zI_n - A)^{-1}B(I_m - KD)^{-1} \\
&\qquad\qquad\qquad \times \left[I_m - KC(zI_n - A)^{-1}B(I_m - KD)^{-1}\right]^{-1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + D_K \\
&=(I_p - DK)^{-1}[G(z) - D]\left[(I_m - KD) - K(G(z) - D)\right]^{-1} + D_K \\
&=(I_p - DK)^{-1}\left[G(z) - D + D(I_m - KG(z))\right](I_m - KG(z))^{-1} \\
&=G(z)(I_m - KG(z))^{-1}.
\end{aligned}$$

(c): by the Hautus Lemma for stabilizability and detectability (see e.g. Theorem 4.5.6 from [37]), it is sufficient to show that rank $\begin{pmatrix}\lambda I_n - A_K & B_K\end{pmatrix} = n$ and rank $\begin{pmatrix}\lambda I_n - A_K \\ C_K\end{pmatrix} = n$ for all $\lambda \in \mathbb{E}$. Now pick $\lambda \in \mathbb{E}$ and suppose there exists $v \in \mathbb{F}^n$ such that $v^*(\lambda I_n - A_K) = 0_n$ and $v^*B_K = 0_m$. Since $B_K = B(I_m - KD)^{-1}$, we obtain $0_m = v^*B_K(I_m - KD) = v^*B$. Consequently, $\lambda v^* = v^*A_K = v^*A + v^*BK(I_p - DK)^{-1}C = v^*A$, so that $v^*(\lambda I_n - A) = 0_n$ and $v^*B = 0_m$. Thus by the stabilizability of (A, B, C, D), we must have $v = 0_n$, so that rank $\begin{pmatrix}\lambda I_n - A_K & B_K\end{pmatrix} = n$, whence (A_K, B_K, C_K, D_K) is stabilizable. The converse can be proven in a similar manner or by applying a loop shift of $-K$. Detectability can be shown in an almost identical way, so we omit the details. □

Before embarking on the study of stabilizing output feedback matrices, we prove a result on "consecutive loop shifts".

Lemma 3.1.5. *Consider $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and denote its transfer function by G. Let $K, M \in \mathbb{F}^{m\times p}$ and assume that $K \in \mathbb{A}_{\mathbb{C}}(D)$.*

Then $M \in \mathbb{A}_{\mathbb{C}}(D_K)$ if, and only if, $K + M \in \mathbb{A}_{\mathbb{C}}(D)$. Furthermore, if one of these holds, then $\left(G^K\right)^M = G^{K+M}$.

Proof. Since $K \in \mathbb{A}_{\mathbb{C}}(D)$, the matrix $I - DK$ is invertible, so we have

$$(I - DK)^{-1}(I - DK - DM) = I - (I - DK)^{-1}DM = I - D^KM.$$

Hence $I - DK - DM$ is invertible if, and only if, $I - D^K M$ is. Since $D_K = D^K$, we see that $M \in \mathbb{A}_{\mathbb{C}}(D_K)$ if, and only if, $M + K \in \mathbb{A}_{\mathbb{C}}(D)$.

If either $M \in \mathbb{A}_{\mathbb{C}}(D_K)$ or $K + M \in \mathbb{A}_{\mathbb{C}}(D)$ holds, then, by the first part of this lemma, both statements hold. Hence G^K, G^{K+M} are well-defined. Moreover, a simple calculation shows us that

$$\begin{aligned} G^{K+M} &= (I - G(K+M))^{-1}G \\ &= (I - GK - GM))^{-1}(I - GK)(I - GK)^{-1}G \\ &= (I - G^K M)^{-1}G^K \\ &= (G^K)^M. \end{aligned}$$

□

3.2 Stabilization by linear output feedback

In this section we will define a concept that plays a central role in this thesis: the set of stabilizing output feedback matrices. For a stabilizable and detectable state-space system (A, B, C, D), it consists of all matrices K such that the state-space system (A_K, B_K, C_K, D_K), where the matrices are given by equation (3.1.3), is asymptotically stable, that is, $\sigma(A_K) \subseteq \mathbb{D}$. We will explore the consequences of assuming a "ball condition", namely, that a matrix ball (in the norm sense) is contained in the set of stabilizing output feedback matrices. As we will see in Lemma 3.2.7, this ball condition admits an equivalent characterization in terms of an inequality involving the Hardy norm of the transfer function of (A, B, C, D). Via the bounded real lemma, we will, in Lemma 3.2.8, construct a quadratic form, which will lead to absolute stability and ISS results in Chapters 4 and 5. Finally, we will also note some connections between the ball condition, a positive real condition and an inequality involving the stability radius as developed by Hinrichsen and Pritchard in e.g. [25].

Definition 3.2.1. Consider a linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and denote its transfer function by G. We define the (discrete-time) set of **stabilizing linear output feedback matrices of** (A, B, C, D) as

$$\mathbb{S}_{\mathbb{C}}(A, B, C, D) := \{K \in \mathbb{A}_{\mathbb{C}}(D) : G^K \in H^{\infty}_{p \times m}(\mathbb{E}; \mathbb{C}^{p \times m})\}.$$

Since $D = \lim_{|z| \to \infty} G(z)$, we can see that the transfer function G describes $\mathbb{S}_{\mathbb{C}}(A, B, C, D)$ completely. Hence we will write $\mathbb{S}_{\mathbb{C}}(G)$ for $\mathbb{S}_{\mathbb{C}}(A, B, C, D)$.

We note the following well-known fact, see e.g. Theorem 2 from [42].

Lemma 3.2.2. *Consider a linear state-space system (A, B, C, D) and denote its transfer function by G.*

Then $\sigma(A) \subseteq \mathbb{D}$ if, and only if, (A, B, C, D) is stabilizable and detectable, and $G \in H^\infty$.

As a simple consequence, we obtain the following result.

Lemma 3.2.3. *Consider a stabilizable and detectable system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and assume that its transfer function G is in H^∞.*

Then there exists a positive c and $a \in (0, 1)$ such that

$$\|x(t)\| \leq ca^t \|x(0)\| + c \max_{0 \leq s \leq t} \|u(s)\|$$

for all $t \in \mathbb{N}_0$ and for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Proof. By Lemma 3.2.2, we have $\sigma(A) \subseteq \mathbb{D}$. The required estimate then follows from the observation that

$$x(t) = A^t x(0) + \sum_{s=0}^{t-1} A^{t-1-s} B u(s)$$

for all $t \in \mathbb{N}_0$ and for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$. □

Before we go on to prove the main result of this section, we need the following useful lemma on matrices.

Lemma 3.2.4. *Let $D \in \mathbb{F}^{p \times m} \setminus \{0\}$.*

(a) If $M \in \mathbb{F}^{m \times p}$ and $\det(I_p - DM) = 0$, then $\frac{1}{\|D\|} \leq \|M\|$.

(b) There exists $M \in \mathbb{F}^{m \times p}$ such that $\|D\| = \frac{1}{\|M\|}$ and $\det(I_p - DM) = 0$.

Proof. (a): since $\det(I_p - DM) = 0$, there exists $\xi \in \mathbb{F}^p \setminus \{0\}$ such that $(I_p - DM)\xi = 0$. Hence $DM\xi = \xi$, so that $\|DM\| \geq 1$. Combine this with the well-known operator norm inequality $\|DM\| \leq \|D\| \|M\|$, to obtain $\frac{1}{\|M\|} \leq \|D\|$.

(b): let $\xi \in \mathbb{F}^m \setminus \{0\}$ be such that $\|\xi\| = 1$ and $\|D\xi\| = \|D\|$ and define $M := \|D\|^{-2} \xi\xi^* D^*$. Then we can use the definition of the operator norm, the homogeneity of norms and $\|D\xi\| = \|D\|$ to obtain

$$\begin{aligned}
\|M\| &= \sup_{\|\mu\|=1} \|M\mu\| = \sup_{\|\mu\|=1} \frac{1}{\|D\|^2} \|\xi\xi^* D^* \mu\| \\
&= \sup_{\|\mu\|=1} \frac{|\xi^* D^* \mu|}{\|D\|^2} \|\xi\| = \frac{1}{\|D\|^2} \left| \xi^* D^* \frac{D\xi}{\|D\|} \right| \\
&= \frac{1}{\|D\|}.
\end{aligned}$$

It is easy to see that $(I_p - DM)D\xi = D\xi \left(1 - \frac{\eta^*\eta}{\|\eta\|^2}\right) = 0$, where we have set $\eta := D\xi \neq 0$. Therefore $\det(I_p - DM) = 0$, showing that M is the sought matrix. □

A somewhat obvious consequence of Lemma 3.1.5 is the following result.

Lemma 3.2.5. *Consider $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$, denote its transfer function by G and let $K \in \mathbb{A}_{\mathbb{C}}(D)$.*

Then $\mathbb{S}_{\mathbb{C}}(G) + \{-K\} = \mathbb{S}_{\mathbb{C}}\left(G^K\right)$, where "+" is the Minkowski sum (see §0.1 for the obvious definition).

Proof.

$$\begin{aligned}
\mathbb{S}_{\mathbb{C}}(G) + \{-K\} =& \{M \in \mathbb{A}_{\mathbb{C}}(D) : G^M \in H^\infty_{p\times m}\} + \{-K\} \\
=& \{M - K \in \mathbb{C}^{m\times p} : M \in \mathbb{A}_{\mathbb{C}}(D),\ G^M \in H^\infty_{p\times m}\} \\
=& \{M \in \mathbb{C}^{m\times p} : M + K \in \mathbb{A}_{\mathbb{C}}(D),\ G^{M+K} \in H^\infty_{p\times m}\} \\
& \text{[by Lemma 3.1.5]} \\
=& \{M \in \mathbb{C}^{m\times p} : M \in \mathbb{A}_{\mathbb{C}}(D^K),\ (G^K)^M \in H^\infty_{p\times m}\} \\
=& \mathbb{S}_{\mathbb{C}}\left(G^K\right).
\end{aligned}$$

□

Recall that, for $K \in \mathbb{F}^{m\times p}$, we use the following notation to denote the ball of complex matrices of radius r around K:

$$\mathbb{B}_{\mathbb{C}}(K, r) := \{M \in \mathbb{C}^{m\times p} : \|M - K\| < r\}.$$

The observation $\mathbb{B}_{\mathbb{C}}(K, r) = \mathbb{B}_{\mathbb{C}}(0, r) + \{K\}$, where $0 = 0_{m\times p}$, gives us a simple corollary of Lemma 3.2.5.

Corollary 3.2.6. *Consider $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$, denote its transfer function by G and let $K \in \mathbb{F}^{m\times p}$. Then*

$$\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G) \iff \mathbb{B}_{\mathbb{C}}(0, r) \subseteq \mathbb{S}_{\mathbb{C}}\left(G^K\right)$$

We are now ready to prove the main result of this section, which provides an equivalent characterization of the ball condition $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.

Lemma 3.2.7. *Consider $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$, denote its transfer function by G and let $K \in \mathbb{F}^{m\times p}$. Then*

$$\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G) \iff \left\|G^K\right\|_{H^\infty} \leq \frac{1}{r}.$$

Proof. By Corollary 3.2.6, it suffices to show that $\mathbb{B}_{\mathbb{C}}(0,r) \subseteq \mathbb{S}_{\mathbb{C}}\left(G^K\right)$ if, and only if, $\left\|G^K\right\|_{H^\infty} \leq \frac{1}{r}$. Thus, without loss of generality, it suffices to show that $\mathbb{B}_{\mathbb{C}}(0,r) \subseteq \mathbb{S}_{\mathbb{C}}(G) \iff \|G\|_{H^\infty} \leq \frac{1}{r}$.

Let us first show that $\mathbb{B}_{\mathbb{C}}(0,r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ implies $\|G\|_{H^\infty} \leq \frac{1}{r}$. To this end, suppose on the contrary, that $\|G\|_{H^\infty} > \frac{1}{r}$. Then there exists $|z_0| > 1$ with $\|G(z_0)\| > \frac{1}{r}$. Hence, by Lemma 3.2.4 (b), there exists $M \in \mathbb{C}^{m\times p}$ with $\|M\| = \frac{1}{\|G(z_0)\|} < r$ and $\det(I - G(z_0)M) = 0$. However as $M \in \mathbb{B}_{\mathbb{C}}(0,r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$, we have $G^M \in H^\infty$ and we can use the triangle inequality and matrix algebra to obtain

$$\begin{aligned}\left\|(I-GM)^{-1}\right\|_{H^\infty} &\leq \left\|(I-GM)^{-1} - I\right\|_{H^\infty} + 1\\ &= \left\|(I-GM)^{-1}GM\right\|_{H^\infty} + 1\\ &\leq \left\|G^M\right\|_{H^\infty}\|M\| + 1 < \infty.\end{aligned}$$

This contradicts $\det(I - G(z_0)M) = 0$ and thus completes the first half of the proof.

Let us now show that $\|G\|_{H^\infty} \leq \frac{1}{r}$ implies $\mathbb{B}_{\mathbb{C}}(0,r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$. Pick $M \in \mathbb{C}^{m\times p}\setminus\mathbb{S}_{\mathbb{C}}(G)$ so that either $\det(I-GM) = 0$ or the rational function matrix $(I-GM)^{-1}$ is defined, but the Hardy norm of G^M is infinity. If the former holds, then pick any $z_0 \in \mathbb{E}$ and note that, by Lemma 3.2.4 (a),

$$\|M\| \geq \frac{1}{\|G(z_0)\|} \geq \frac{1}{\|G\|_{H^\infty}} \geq r,$$

so that in turn $M \notin \mathbb{B}_{\mathbb{C}}(0,r)$. If however the latter holds, then there exists $z_0 \in \mathbb{E} \cup \{\infty\}$ such that $\lim_{z\to z_0}\left\|(I - G(z)M)^{-1}G(z)\right\| = \infty$. Since $\|G(z)\| \leq \frac{1}{r}$ for all $z \in \mathbb{E}$, this implies $\det(I - G(z_0)M) = 0$. As before, by Lemma 3.2.4 (a),

$$\|M\| \geq \frac{1}{\|G(z_0)\|} \geq \frac{1}{\|G\|_{H^\infty}} \geq r,$$

so that again $M \notin \mathbb{B}_{\mathbb{C}}(0,r)$. Hence in both cases we have $(\mathbb{C}^{m\times p} \setminus \mathbb{S}_{\mathbb{C}}(G)) \cap \mathbb{B}_{\mathbb{C}}(0,r) = \emptyset$ and so $\mathbb{B}_{\mathbb{C}}(0,r) \subset \mathbb{S}_{\mathbb{C}}(G)$ which concludes the second, and final, part of the proof. $\square$

Lemma 3.2.7 implies that the largest matrix ball centered on $K \in \mathbb{F}^{m\times p}$ and contained in $\mathbb{S}_{\mathbb{C}}(G)$ has radius $\left\|G^K\right\|_{H^\infty}^{-1}$. This observation shows that $\mathbb{S}_{\mathbb{C}}(G)$ is closely related to the stability radius as defined in the work of Hinrichsen and Pritchard, see [25, 27]. We elaborate on this connection in Corollary 3.2.17.

We now use Lemma 3.2.7 to obtain a quadratic form from the ball condition $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.

Lemma 3.2.8. *Consider a stabilizable and detectable linear state-space system* $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ *and denote its transfer function by* G. *Assume that, for some* $K \in \mathbb{F}^{m\times p}$ *and* $r > 0$, *we have* $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ *and that there exists* $z_0 \in \mathbb{C}$ *with* $|z_0| = 1$ *such that* $\left\|G^K(z_0)\right\| < \frac{1}{r}$.

Then there exists a positive semi-definite $P = P^* \in \mathbb{F}^{n\times n}$ *such that the quadratic form* $V : \mathbb{F}^n \to [0, \infty)$ *defined by* $V(\xi) := \langle P\xi, \xi\rangle$ *satisfies*

$$V(x(t+1)) - V(x(t)) \leq \|u(t) - Ky(t)\|^2 - r^2 \|y(t)\|^2$$

for all $t \in \mathbb{N}_0$ *and for all trajectories* $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Moreover, there exists a projection $\Pi\colon \mathbb{F}^n \to \mathbb{F}^n$ *and a positive* c *such that* $\ker \Pi \subseteq \ker C$ *and* $V(\xi) \geq c \|\Pi\xi\|^2$ *for all* $\xi \in \mathbb{F}^n$.

Proof. By Lemma 3.1.4, we know that (A_K, B_K, C_K, D_K) is stabilizable and detectable. Note that, by Lemma 3.2.7, we have $\left\|G^K\right\|_{H^\infty} \leq \frac{1}{r}$, so - in combination with our assumption that there exists $|z_0| = 1$ such that $\left\|G^K(z_0)\right\| < \frac{1}{r}$ - we can apply Lemma 2.2.5 to see that there exists a positive semi-definite $P = P^* \in \mathbb{F}^{n\times n}$ such that the quadratic form $V(\xi) := \langle P\xi, \xi\rangle$ satisfies

$$\begin{aligned} V(x(t+1)) - V(x(t)) &\leq \|u(t)\|^2 - \left\|G^K\right\|_{H^\infty}^{-2} \|y(t)\|^2 \\ &\leq \|u(t)\|^2 - r^2 \|y(t)\|^2 \end{aligned}$$

for all $t \in \mathbb{N}_0$ and for all trajectories $(u, x, y) \in \mathcal{B}(A_K, B_K, C_K, D_K)$. Moreover, there exists a projection $\Pi\colon \mathbb{F}^n \to \mathbb{F}^n$ and a positive c such that $\ker \Pi \subseteq \ker C$ and $V(\xi) \geq c \|\Pi\xi\|^2$ for all $\xi \in \mathbb{F}^n$.

Finally, we use Lemma 3.1.4 to see that if $(u, x, y) \in \mathcal{B}(A, B, C, D)$, then $(u - Ky, x, y) \in \mathcal{B}(A_K, B_K, C_K, D_K)$, which in turn completes the proof. ☐

Similarly, we obtain a version of the above for minimal systems.

Lemma 3.2.9. *Consider a controllable and observable linear state-space system* $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$, *denote its transfer function by* G, *and - for some* $K \in \mathbb{F}^{m\times p}$ *and* $r > 0$ *- assume that* $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.

Then there exists a positive definite $P = P^* \in \mathbb{F}^{n\times n}$ *such that the quadratic form* $V : \mathbb{F}^n \to [0, \infty)$ *defined by* $V(\xi) := \langle P\xi, \xi\rangle$ *satisfies*

$$V(x(t+1)) - V(x(t)) \leq \|u(t) - Ky(t)\|^2 - r^2 \|y(t)\|^2$$

for all $t \in \mathbb{N}_0$ *and for all trajectories* $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Moreover, there exists a projection $\Pi\colon \mathbb{F}^n \to \mathbb{F}^n$ *and a positive* c *such that* $\ker \Pi \subseteq \ker C$ *and* $V(\xi) \geq c \|\Pi\xi\|^2$ *for all* $\xi \in \mathbb{F}^n$.

In Chapter 4 we will prove a proposition that is similar to a well-known result in absolute stability theory, the circle criterion. Our main tool in the proof of said proposition will be Lemma 3.2.8 and thus, essentially, the bounded real lemma. To the best of the author's knowledge this is a nonstandard route as the circle criterion is typically proved using the positive real lemma, see Haddad and Bernstein [22] or Haddad and Chellaboina [23]. A drawback of this approach is that in order to apply the positive real lemma one has to assume that the underlying linear system is minimal. The use of the bounded real lemma will allow us to relax this assumption in our version of the circle criterion, Proposition 4.3.2.

We know from Lemma 2.2.10 that a bounded real property is related to a positive real property. This will enable us to relate the ball condition $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ to a positive real condition. First, we need the following lemma.

Lemma 3.2.10. *Consider $(A,B,C,D) \in \Sigma(m,n,m,\mathbb{F})$, denote its transfer function by G and let $\lambda \in \mathbb{C} \setminus \{0\}$ be such that $\lambda I \in \mathbb{A}_{\mathbb{C}}(D)$.*

Then $\|G\|_{H^\infty} \leq \frac{1}{|\lambda|}$ if, and only if, $I + 2\lambda G^{\lambda I}$ is positive real.

Proof. Since $\lambda I \in \mathbb{A}_{\mathbb{C}}(D)$, we have $\det(I - \lambda D) \neq 0$ and hence $I - \lambda G$ is invertible. Set $F := (I + \lambda G)(I - \lambda G)^{-1}$ and note that $-\lambda G = (I - F)(I + F)^{-1}$. By Lemma 2.2.10, $\left\|(I - F)(I + F)^{-1}\right\|_{H^\infty} \leq 1$ if, and only if, F is positive real. The observation that $F = I + 2\lambda G^{\lambda I}$ completes the proof. □

It is interesting to note that the same technique gives us a slightly stronger result.

Corollary 3.2.11. *Consider $(A,B,C,D) \in \Sigma(m,n,m,\mathbb{F})$, denote its transfer function by G and let $K \in \mathbb{F}^{m\times m}$ be such that $K \in \mathbb{A}_{\mathbb{C}}(D)$.*

Then $\|KG\|_{H^\infty} \leq 1$ if, and only if, $I + 2KG^K$ is positive real.

Lemmas 3.2.7 and 3.2.10 allow us to relate the ball condition $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ to a bounded real and to a positive real property.

Proposition 3.2.12. *Consider $(A,B,C,D) \in \Sigma(m,n,m,\mathbb{F})$, denote its transfer function by G and let $r > 0$, $K \in \mathbb{F}^{m\times m}$. Then the following are equivalent:*

(a) $\mathbb{B}_{\mathbb{C}}(K,r) \subset \mathbb{S}_{\mathbb{C}}(G)$,

(b) $\left\|G^K\right\|_{H^\infty} \leq \frac{1}{r}$,

(c) there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = r$, $\lambda I + K \in \mathbb{A}_{\mathbb{C}}(D)$ and $I + 2\lambda G^{\lambda I + K}$ is positive real.

Proof. The equivalence of (a) and (b) is the statement of Lemma 3.2.7.

The equivalence of (a) and (c) follows from Lemmas 3.2.10 and 3.1.5 as long as we can show the existence of $\lambda \in \mathbb{C}$ such that $|\lambda| = r$ and $\lambda I + K \in \mathbb{A}_{\mathbb{C}}(D)$. We proceed to do this.

By Lemma 3.1.5, $\lambda I + K \in \mathbb{A}_{\mathbb{C}}(D)$ if, and only if, $\lambda I \in \mathbb{A}_{\mathbb{C}}(D_K)$ which in turn is equivalent to $\det(I - \lambda D_K) \neq 0$. However $\det(I - \lambda D_K)$ is a nonzero polynomial and a nonzero polynomial can only have a finite number of zeros. Hence there exists $\lambda \in \mathbb{C}$ with $|\lambda| = r$ and $\lambda I \in \mathbb{A}_{\mathbb{C}}(D_K)$. □

We demonstrated in Lemma 3.2.8, the proof of which combined Lemma 3.2.7 and the bounded real lemma, that, by assuming the ball condition $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$, we can obtain a quadratic form V that provides us with an estimate on $V(x(t+1)) - V(x(t))$ for trajectories $(u, x, y) \in \mathcal{B}(A, B, C, D)$. We now compare it with the quadratic form obtained from the same ball condition, however for the proof we use instead Proposition 3.2.12 and the positive real lemma. Note that this lemma will not be used in this thesis, so we relegate its proof to the appendix, see §A.1.

Lemma 3.2.13. *Consider a controllable and observable linear state-space system $(A, B, C, D) \in \Sigma(m, n, m; \mathbb{R})$, denote its transfer function by G and - for some $K \in \mathbb{F}^{m \times m}$ and $r > 0$ - assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.*

Then there exists a positive-definite $P = P^* \in \mathbb{F}^{n \times n}$ such that the quadratic form $V : \mathbb{F}^n \to [0, \infty)$ defined by $V(\xi) := \langle P\xi, \xi \rangle$ satisfies

$$V(x(t+1)) - V(x(t)) \leq \|u(t) - Ky(t)\|^2 - r^2 \|y(t)\|^2$$

for all $t \in \mathbb{N}_0$ and for all trajectories $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

If we compare the above lemma with Lemma 3.2.8, then we can see that the quadratic forms obtained from the ball condition $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ and either the bounded real lemma or the positive real lemma both offer us precisely the same estimates on $V(x(t+1)) - V(x(t))$. The bounded real lemma allows us to relax the controllability and observability conditions to stabilizability and detectability. Hence we will favour it over the positive real lemma in this thesis.

It is interesting to note a connection between $\mathbb{S}_{\mathbb{C}}(G)$ and the well-known concept (especially in the continuous-time setting) of the structured stability radius.

Definition 3.2.14. Consider $(A, B, C, 0) \in \Sigma(m, n, p; \mathbb{F})$ with $\sigma(A) \subseteq \mathbb{D}$. The (discrete-time) **complex structured stability radius of A with respect to weights B and C** is defined as

$$r_{\mathbb{C}}(A; B, C) := \inf\{\|M\| : M \in \mathbb{C}^{m \times p} \text{ and } \sigma(A + BMC) \not\subseteq \mathbb{D}\}.$$

It is a measure of how robust to perturbations is the simple family of perturbed systems

$$x(t+1) = (A + BMC)x(t).$$

Proposition 2.1 from [27] gives us the following result.

Proposition 3.2.15. *Consider $(A, B, C, 0) \in \Sigma(m, n, p, \mathbb{C})$ and denote by G its transfer function. Then*

$$r_{\mathbb{C}}(A; B, C) = \|G\|_{H^\infty}^{-1}.$$

This leads us to the following equivalences.

Corollary 3.2.16. *Consider $(A, B, C, 0) \in \Sigma(m, n, p, \mathbb{F})$, denote its transfer function by G and let $K \in \mathbb{F}^{m\times p}$. Then the following are equivalent:*

(a) $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$,

(b) $\left\|G^K\right\|_{H^\infty} \leq \frac{1}{r}$,

(c) $r_{\mathbb{C}}(A_K; B_K, C_K) \geq r$.

Corollary 3.2.17. *Consider $(A, B, C, 0) \in \Sigma(m, n, p; \mathbb{F})$. Then we have $r_{\mathbb{C}}(A; B, C) = \sup\{r \geq 0 : \mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)\}$.*

Before moving on to stability analysis of Lur'e systems we note two interesting properties of $\mathbb{S}_{\mathbb{C}}(G)$.

Lemma 3.2.18. *$\mathbb{S}_{\mathbb{C}}(G)$ is an open set.*

Proof. If $M \in \mathbb{S}_{\mathbb{C}}(G)$, then, by Corollary 3.2.7, $\mathbb{B}_{\mathbb{C}}\left(M, \left\|G^M\right\|^{-1}\right) \subseteq \mathbb{S}_{\mathbb{C}}(G)$. □

If we set

$$\mathbb{M} := \{K \in \mathbb{C}^{m\times m} : K + K^* \text{ is negative definite}\},$$

then we can obtain an alternative characterization of positive real functions. Since the following lemma is not used in this document, we relegate its proof to the appendix.

Lemma 3.2.19. *Consider $G \in \mathbb{F}(z)^{m\times m}$.*

$\mathbb{M} \subseteq \mathbb{S}_{\mathbb{C}}(G)$ if, and only if, G is positive-real.

Chapter 4

Absolute stability of Lur'e systems

In this chapter we will be demonstrating how the tools we have developed in the previous two chapters can be used to obtain absolute stability results for Lur'e systems

$$\begin{aligned} x(t+1) &= Ax(t) + Bf(y(t)), \\ y(t) &= Cx(t) + Df(y(t)) \qquad \forall\, t \in \mathbb{N}_0. \end{aligned} \tag{4.1}$$

We already mentioned in the Introduction that our approach is inspired by the complexified Aizerman's conjecture, which was proved (in a continuous-time setting for $D = 0$) in Hinrichsen and Pritchard [25]. It states that if the Lur'e interconnection is globally asymptotically stable for *all complex linear* output feedback matrices F that satisfy the norm condition $\|F(\xi)\| < r\,\|\xi\|$ for some $r > 0$ and for all $\xi \in \mathbb{C}^p\backslash\{0\}$, then, in fact, the Lur'e interconnection is globally asymptotically stable for *all nonlinear* output feedback maps f that satisfy the same norm condition $\|f(\xi)\| < r\,\|\xi\|$. By combining quadratic forms obtained from Lemmas 2.3.1 and 3.2.8 we will prove an extension of this result in a discrete-time setting, see Proposition 4.2.1, which we will sometimes refer to as the Aizerman version of the circle criterion.

We will also consider a well-known result in absolute stability theory, the circle criterion. It states that if the underlying linear system (A, B, C, D) is controllable and observable and if its transfer function G is such that - for some matrices K_1, K_2 - the rational function matrix $(I - K_2G)(I - K_1G)^{-1}$ is strongly positive real, then the Lur'e system (4.1) is globally asymptotically stable for all nonlinearities f that satisfy $\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \leq 0$, see e.g. Theorem 5.1 from Haddad and Bernstein [22]. Usually the circle criterion is proved using a Lyapunov function obtained from the positive real lemma, however we prove it using Proposition 4.2.9 and hence, in effect, the bounded real lemma. We also remark that one can obtain a version of the circle criterion by using Proposition 4.2.1 and thus relax the assumptions on the underlying linear system to stabilizability and detectability, however due to an extra condition this introduces, we do not do it in this thesis.

We note that typically absolute stability theory is concerned with Lur'e systems, where the nonlinearities f are time-dependent. However our ultimate aim is the study of input-to-state stability, where time-invariant nonlinearities are used. Moreover, most results in this chapter extend, in trivial manner, to time-dependent nonlinearities f as long as the assumptions on them are satisfied uniformly in the time variable. The only exception is Proposition 4.2.1 (b) and its corollaries.

This chapter is organized as follows. In §4.1 we define precisely what we mean by a Lur'e system and collect some notions of stability that we will be interested in. §4.2 is then devoted to proving the Aizerman version of the circle criterion and discussing some of its consequences as well as some of the assumptions made in its statement. We note a consequence, a result strongly reminiscent of the circle criterion, in §4.3.

4.1 Lur'e systems

Definition 4.1.1. Consider $(A, B, C, D) \in \Sigma(m, n, p, \mathbb{F})$ and let $f : \mathbb{F}^p \to \mathbb{F}^m$ be some map. We say that (A, B, C, D, f) is a **Lur'e system**.

We define its **behaviour** $\mathcal{B}(A, B, C, D, f)$ as

$$\begin{aligned} &\mathcal{B}(A, B, C, D, f) \\ &\qquad := \Big\{ (x, y) \in (\mathbb{F}^n)^{\mathbb{N}_0} \times (\mathbb{F}^p)^{\mathbb{N}_0} : (f \circ y, x, y) \in \mathcal{B}(A, B, C, D) \Big\} . \end{aligned}$$

A pair $(x, y) \in \mathcal{B}(A, B, C, D, f)$ is called a **trajectory**.

Our definition of the behaviour of (A, B, C, D, f) merely states that if a trajectory $(x, y) \in \mathcal{B}(A, B, C, D, f)$, then

$$\begin{aligned} x(t+1) &= Ax(t) + Bf(y(t)) \\ y(t) &= Cx(t) + Df(y(t)) \qquad \forall\, t \in \mathbb{N}_0. \end{aligned} \tag{4.1.1}$$

Note that, for a general f and a given $\xi \in \mathbb{F}^n$, we cannot guarantee that there exist trajectories $(x, y) \in \mathcal{B}(A, B, C, D, f)$ such that $x(0) = \xi$. Neither can we guarantee that if a trajectory exists, that it is unique. Existence depends on the surjectivity of the inverse of $I - Df$; while uniqueness depends on the injectivity of the inverse of $I - Df$.

We will be considering the following three notions of stability.

Definition 4.1.2. Consider a Lur'e system (A, B, C, D, f).

1. If there exists a positive c such that
$$\|x(t)\| \le c\,\|x(0)\|$$
$$\|y(t)\| \le c\,\|x(0)\| \qquad \forall\, t \in \mathbb{N}_0$$
and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$, then we say that the Lur'e system (A, B, C, D, f) is **globally stable**.

2. If (A, B, C, D, f) is globally stable and if
$$\lim_{t\to\infty} x(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} y(t) = 0$$
for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$, then we say that the Lur'e system (A, B, C, D, f) is **globally asymptotically stable**.

3. If (A, B, C, D, f) is globally stable and if there exist $c > 0$ and $a \in (0, 1)$ such that
$$\|x(t)\| \le ca^t\,\|x(0)\|$$
$$\|y(t)\| \le ca^t\,\|x(0)\| \qquad \forall\, t \in \mathbb{N}_0$$
and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$, then we say that the Lur'e system (A, B, C, D, f) is **globally exponentially stable**.

4.2 Aizerman version of the circle criterion

We now apply the quadratic form obtained from a ball condition in Lemma 3.2.8 to stability analysis of Lur'e systems. Recall the shorthand $D^K := (I - DK)^{-1}D$.

Proposition 4.2.1 (Aizerman version of the circle criterion)**.** *Consider a Lur'e system (A, B, C, D, f), assume that the underlying linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable and denote its transfer function by G. For $K \in \mathbb{F}^{m\times p}$ and $r > 0$, assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ and that there exists $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that $\left\|G^K(z_0)\right\| < \frac{1}{r}$.*

(a) If $\left\|D^K\right\| < \frac{1}{r}$ and
$$\|f(\xi) - K\xi\| \le r\,\|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p, \tag{4.2.1}$$
then the Lur'e system (A, B, C, D, f) is globally stable.

(b) If f is continuous, $\left\|D^K\right\| < \frac{1}{r}$ and
$$\|f(\xi) - K\xi\| < r\,\|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p \setminus \{0\}, \tag{4.2.2}$$
then the Lur'e system (A, B, C, D, f) is globally asymptotically stable.

(c) If there exists $\delta > 0$ such that

$$\|f(\xi) - K\xi\| \leq (r - \delta)\|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p, \tag{4.2.3}$$

then the Lur'e system (A, B, C, D, f) is globally exponentially stable.

Note that if $F \in \mathbb{B}_{\mathbb{C}}(K, r)$, then clearly $\|F\xi - K\xi\| < r\|\xi\|$ for all $\xi \in \mathbb{F}^p$. Hence Proposition 4.2.1 (b) can be interpreted as saying that if the Lur'e system (A, B, C, D, F) is globally asymptotically stable for *all complex linear* output feedback matrices F such that $\|F\xi - K\xi\| < r\|\xi\|$, then the Lur'e system (A, B, C, D, f) is globally asymptotically stable for *all nonlinear* output feedback maps f such that $\|f(\xi) - K\xi\| < r\|\xi\|$. Therefore, Proposition 4.2.1 (b) can be seen as saying that the Aizerman conjecture is true over the complex field. For a similar result in a continuous-time setting, see Theorem 5.6.22 from Hinrichsen and Pritchard [25]. Indeed, their work has inspired our results in this section with the only real novelty being Proposition 4.2.1 (a) and the extension to systems with feedthrough. We defer a detailed comparison to Part II, where we consider continuous-time Lur'e systems.

Proof of Proposition 4.2.1. By Lemma 3.2.8, there exists a positive semi-definite matrix $P = P^* \in \mathbb{F}^{n\times n}$ such that the function $V : \mathbb{F}^n \to [0, \infty)$ given by $V(\xi) := \langle P\xi, \xi\rangle$ satisfies

$$V(x(t+1)) - V(x(t)) \leq \|u(t) - Ky(t)\|^2 - r^2\|y(t)\|^2 \qquad \forall\, t \in \mathbb{N}_0$$

and for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$. Moreover, it guarantees the existence of a projection $\Pi\colon \mathbb{F}^n \to \mathbb{F}^n$ and $c_1 > 0$ such that $\ker\Pi \subseteq \ker C$ and $V(\xi) \geq c_1\|\Pi\xi\|^2$ for all $\xi \in \mathbb{F}^n$. On the other hand, there exists $c_2 > 0$ such that $V(\xi) \leq c_2\|\xi\|^2$ for all $\xi \in \mathbb{F}^n$. By definition, $(x, y) \in \mathcal{B}(A, B, C, D, f)$ if, and only if, $(f \circ y, x, y) \in \mathcal{B}(A, B, C, D)$, so that

$$V(x(t+1)) - V(x(t)) \leq \|f(y(t)) - Ky(t)\|^2 - r^2\|y(t)\| \qquad \forall\, t \in \mathbb{N}_0 \tag{4.2.4}$$

and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$.

Proof of (a): (4.2.4) and (4.2.1) allow us to estimate

$$V(x(t+1)) - V(x(t)) \leq \|f(y(t)) - Ky(t)\|^2 - r^2\|y(t)\| \leq 0 \qquad \forall\, t \in \mathbb{N}_0 \tag{4.2.5}$$

and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$. Hence $t \mapsto V(x(t))$ is a nonincreasing function, so that $V(x(t)) \leq V(x(0)) \leq c_2\|x(0)\|^2$ for all $t \in \mathbb{N}_0$. Since $\ker\Pi \subseteq \ker C$, it follows that $C\Pi = C$, so - upon setting $c_3 := \sqrt{\frac{c_2}{c_1}}$ - we have

$$\begin{aligned}\|Cx(t)\| = \|C\Pi x(t)\| &\leq \|C\| \, \|\Pi x(t)\| \\ &\leq \|C\| \frac{1}{\sqrt{c_1}} \sqrt{V(x(t))} \leq c_3 \|C\| \, \|x(0)\| \qquad \forall\, t \in \mathbb{N}_0\end{aligned} \tag{4.2.6}$$

and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$.

Define (A_K, B_K, C_K, D_K) by equation (3.1.3). Then, by Lemma 3.1.4, we know that $(x, y) \in \mathcal{B}(A, B, C, D, f)$ if, and only if, $(f \circ y - Ky, x, y) \in \mathcal{B}(A_K, B_K, C_K, D_K)$, so that $y(t) = C_K x(t) + D_K[f(y(t)) - Ky(t)]$ for all $t \in \mathbb{N}_0$ and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$. Taking norms and using $C_K = (I - DK)^{-1}C$ and $\left\|D^K\right\| < \frac{1}{r}$ alongside assumption (4.2.1) shows that there exists a positive c_4 such that

$$\|y(t)\| \leq c_4 \|Cx(t)\| \leq c_3 c_4 \|C\| \, \|x(0)\| \qquad \forall\, t \in \mathbb{N}_0 \tag{4.2.7}$$

and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$.

Thus the proof of (a) is complete if we can show that $\|x(t)\| \leq c \|x(0)\|$ for some positive c. By Lemma 3.1.4, we know that (A_K, B_K, C_K, D_K) is a stabilizable and detectable system and moreover, as $K \in \mathbb{S}_{\mathbb{C}}(G)$, we have $G^K \in H^\infty$. Therefore, we can apply Lemma 3.2.3, to see that that there exists a positive c_5 such that $\|x(t)\| \leq c_5 \|x(0)\| + c_5 \max_{0 \leq s \leq t} \|u(s)\|$ for all $t \in \mathbb{N}_0$ and for all $(u, x, y) \in \mathcal{B}(A_K, B_K, C_K, D_K)$. Since $(x, y) \in \mathcal{B}(A, B, C, D, f)$ if, and only if, $(f \circ y - Ky, x, y) \in \mathcal{B}(A_K, B_K, C_K, D_K)$, we can use (4.2.7) and (4.2.1) to obtain

$$\begin{aligned}\|x(t)\| &\leq c_5 \|x(0)\| + c_5 \max_{0 \leq s \leq t} \|f(y(s)) - Ky(s)\| \\ &\leq c_5 (1 + r c_3 c_4 \|C\|) \, \|x(0)\|,\end{aligned}$$

which completes the proof of (a).

Proof of (b): by (a) we already know that there exists a positive c such that $\|x(t)\| \leq c \|x(0)\|$ and $\|y(t)\| \leq c \|x(0)\|$, so that (A, B, C, D, f) is globally stable. Hence we only need to show global attractivity of 0. To this end, let $(x, y) \in \mathcal{B}(A, B, C, D, f)$. We note that it is sufficient to show that

$$\lim_{t \to \infty} y(t) = 0. \tag{4.2.8}$$

Indeed, if (4.2.8) holds, then $f(y(t)) - Ky(t) \to 0$ as $t \to \infty$, which combined with the asymptotic stability of A_K and the fact that $(f \circ y - Ky, x, y) \in \mathcal{B}(A_K, B_K, C_K, D_K)$ implies $\lim_{t \to \infty} x(t) = 0$.

To establish (4.2.8), recall that y is bounded. Thus by Lemma 2.4.2, the omega limit set Ω_y of y is nonempty and $\operatorname{dist}(y(t), \Omega_y) \to 0$ as $t \to \infty$.

Consequently, it suffices to show that $\Omega_y = \{0\}$. Consider $\xi \in \Omega_y$, so that there exists a sequence $(t_k)_{k\in\mathbb{N}_0}$ in $\mathbb{N}_0$ such that $t_k \to \infty$ and $y(t_k) \to \xi$ as $k \to \infty$. Since $t \mapsto V(x(t))$ is a non-negative non-increasing function, $\lim_{t\to\infty} V(x(t))$ exists. In particular $V(x(t_k+1)) - V(x(t_k)) \to 0$ as $k \to \infty$. We now use (4.2.4) for $t = t_k$, continuity of f and let $k \to \infty$ to obtain

$$0 \leq \|f(\xi) - K\xi\|^2 - r^2 \|\xi\|^2 .$$

Together with (4.2.2) this implies $\xi = 0$, completing the proof of statement (b).

Proof of (c): this can be shown using exponential weighting arguments, see e.g. Theorem 15 from Jayawardhana, Logemann and Ryan [30], however we will exhibit an alternative proof that uses Lyapunov arguments.

First, recall equation (4.2.4) and use assumption (4.2.3) to obtain

$$V(x(t+1)) - V(x(t)) \leq (r-\delta)^2 \|y(t)\| - r^2 \|y(t)\| \leq -\delta^2 \|y(t)\|^2$$

for all $t \in \mathbb{N}_0$ and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$.

Since (A, B, C, D) is detectable, Lemma 2.3.1 guarantees the existence of a positive definite matrix $Q = Q^* \in \mathbb{F}^{n\times n}$ and a positive $\delta_1 > 0$ such that the function $U \colon \mathbb{F}^n \to [0, \infty)$ defined by $U(\xi) := \langle Q\xi, \xi\rangle$ satisfies

$$U(x(t+1)) - U(x(t)) \leq -\delta_1 \|x(t)\|^2 + \|y(t)\|^2 + \|u(t)\|^2$$

for all $t \in \mathbb{N}_0$ and for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$. Therefore by the definition of a trajectory of a Lur'e system, the simple estimate $\|f(\xi)\| \leq (\|K\|+r) \|\xi\|$ and inequality $2a^2 + 2b^2 \geq (a + b)^2$, we have

$$\begin{aligned} U(x(t+1)) - U(x(t)) &\leq -\delta_1 \|x(t)\|^2 + \|y(t)\|^2 + \|f(y(t))\|^2 \\ &\leq -\delta_1 \|x(t)\|^2 + (1 + 2\|K\|^2 + 2r^2) \|y(t)\|^2 \end{aligned}$$

for all $t \in \mathbb{N}_0$ and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$. Thus if we set $b := \frac{\delta^2}{1+2\|K\|^2+2r^2}$, $\delta_2 := \delta_1 b$ and define $W := bU + V$, then

$$W(x(t+1)) - W(x(t)) \leq -\delta_2 \|x(t)\|^2 \tag{4.2.9}$$

for all $t \in \mathbb{N}_0$ and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$. Obviously, W is positive definite and so $\xi \mapsto \sqrt{W(\xi)}$ defines a norm on $\mathbb{F}^n$. Hence there exist positive constants c_6 and c_7 such that $c_6 \|\xi\|^2 \leq W(\xi) \leq c_7 \|\xi\|^2$ for all $\xi \in \mathbb{F}^n$. If we combine this with estimate (4.2.9), then we obtain

$$W(x(t+1)) \leq \left(1 - \frac{\delta_2}{c_7}\right) W(x(t))$$

for all $t \in \mathbb{N}_0$ and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$. Upon setting $c_8 := \sqrt{c_7/c_6}$ and $a := \sqrt{1 - \delta_2/c_7} < 1$, we obtain that, for every $(x, y) \in \mathcal{B}(A, B, C, D, f)$,

$$\|x(t)\| \leq c_8 a^t \|x(0)\| \qquad \forall\, t \in \mathbb{N}_0,$$

completing the proof of statement (c). $\square$

By picking $K = 0$ in the Aizerman version of the circle criterion and by using Lemma 3.2.7 to see that $\mathbb{B}_{\mathbb{C}}(0, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ is equivalent to $\|G\|_{H^\infty} \leq \frac{1}{r}$, we obtain the following corollary, which resembles the small-gain theorem.

Corollary 4.2.2. *Consider a Lur'e system (A, B, C, D, f), assume that the underlying linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable and let its transfer function G be such that there exists $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that $\|G(z_0)\| < \|G\|_{H^\infty}$.*

(a) If

$$\|G\|_{H^\infty} \|f(\xi)\| \leq \|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p,$$

and if $\|D\| \sup_{\xi \in \mathbb{F}^p} \frac{\|f(\xi)\|}{\|\xi\|} < 1$, then the Lur'e system (A, B, C, D, f) is globally stable.

(b) If f is continuous,

$$\|G\|_{H^\infty} \|f(\xi)\| < \|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p \setminus \{0\},$$

and if $\|D\| \sup_{\xi \in \mathbb{F}^p} \frac{\|f(\xi)\|}{\|\xi\|} < 1$, then the Lur'e system (A, B, C, D, f) is globally asymptotically stable.

(c) If

$$\|G\|_{H^\infty} \sup_{\xi \in \mathbb{F}^p} \frac{\|f(\xi)\|}{\|\xi\|} < 1,$$

then the Lur'e system (A, B, C, D, f) is globally exponentially stable.

Statement (c) is a time-invariant version of the small gain theorem (see Theorem 3.1 from Haddad and Bernstein [22]), but extended to cases, where the underlying linear system is only stabilizable and detectable (as opposed to controllable and observable), although we do require an extra assumption on the unit circle.

Note that the statements of Proposition 4.2.1 and Corollary 4.2.2 are simpler for Lur'e systems that have no feedthrough in the underlying linear system as the inequalities involving D^K and D, respectively, are automatically satisfied.

It follows from Lemma 3.2.7 that if $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$, then $\left\|G^K\right\|_{H^\infty} \leq \frac{1}{r}$ and hence in particular $\left\|D^K\right\| \leq \frac{1}{r}$. However, to satisfy assumptions of Proposition 4.2.1 (a) and (b), we need to have $\left\|D^K\right\| < \frac{1}{r}$ and there must exist $|z_0| = 1$ such that $\left\|G^K(z_0)\right\| < \frac{1}{r}$.

The following example shows that there exists a Lur'e system (A, B, C, D, f) such that $\left\|D^K\right\| = \frac{1}{r}$ and $\left\|G^K(z_0)\right\| = \frac{1}{r}$ for all $|z_0| = 1$ and the conclusions of the Aizerman version of the circle criterion (a) and (b) do not hold.

Example 4.2.3. Consider the controllable and observable linear system

$$\left(\begin{pmatrix} 0 & -1 \\ -\frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\right) \in \Sigma(2,2,2;\mathbb{R})$$

and note that $\|D\| = 2$ and the transfer function G is given by $\begin{pmatrix} \frac{z}{z^2-\frac{1}{2}} & 0 \\ 0 & 2 \end{pmatrix}$. Therefore, we have $\|G\|_{H^\infty} = 2$, so that $\mathbb{B}_{\mathbb{C}}\left(0, \frac{1}{2}\right) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ and clearly $\|G(z_0)\| = 2$ for all $|z_0| = 1$.

Let us now define $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(\xi) := \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \xi.$$

It is easy to check that f satisfies $\|f(\xi)\| \leq \frac{1}{2}\|\xi\|$ for all $\xi \in \mathbb{R}^2$. However, if we write down the output equation for a trajectory $(x, y) \in \mathcal{B}(A, B, C, D, f)$, then we obtain

$$\begin{aligned} \begin{pmatrix} y(t)_1 \\ y(t)_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t)_1 \\ x(t)_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} y(t)_1 \\ y(t)_2 \end{pmatrix} \\ &= \begin{pmatrix} x(t)_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y(t)_2 \end{pmatrix}. \end{aligned}$$

Hence $y(t)_2$ is arbitrary, so we cannot hope to extend the Aizerman version of the circle criterion to incorporate the case when both $\left\|D^K\right\| = \frac{1}{r}$ and $\left\|G^K(z_0)\right\| = \frac{1}{r}$ for all $|z_0| = 1$.

We note the following equivalent characterizations of the statement $\left\|D^K\right\| < \frac{1}{r}$.

Lemma 4.2.4. *Consider $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and denote its transfer function by G. Assume that for some $K \in \mathbb{F}^{m \times p}$ and $r > 0$ we have $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.*

Then $\left\|D^K\right\| < \frac{1}{r}$ if, and only if, the map $z \mapsto \left\|G^K(z)\right\|$ with domain $\mathbb{E}$ is not the constant function $\frac{1}{r}$.

Proof. Sufficiency is trivial, so we only need to prove necessity.

By Proposition 3.2.12, $\left\|G^K\right\|_{H^\infty} \leq \frac{1}{r}$, so that $\left\|D^K\right\| \leq \frac{1}{r}$. Thus we are done if we can rule out the possibility $\left\|D^K\right\| = \frac{1}{r}$. To this end, suppose on the contrary that $\frac{1}{r} = \left\|D^K\right\| = \lim_{z\to\infty}\left\|G^K(z)\right\|$. Pick $\xi \in \mathbb{C}^m$ such that $\|\xi\| = 1$ and $\left\|D^K\xi\right\| = \left\|D^K\right\|$. Set $\mu := \frac{1}{\|D^K\xi\|}D^K\xi$, and define a map $g\colon \mathbb{E} \to \mathbb{C}$ by $g(z) := \left\langle G^K(z)\xi, \mu\right\rangle$. Since $\left\|G^K\right\|_{H^\infty} < \infty$, the rational function matrix G^K does not have poles in $\mathbb{E}$, so g is analytic in $\mathbb{E}$. Hence if we define a map h on $\mathbb{D}$ by $h(z) := g\left(z^{-1}\right)$ for $z \neq 0$ and $h(0) := \lim_{z\to\infty} g(z)$, then h is analytic on $\mathbb{D}$. Now let us use the Cauchy-Schwarz inequality to infer that for all $z \in \mathbb{D}$ we have

$$\begin{aligned}
|h(z)| &= |g(z^{-1})| \\
&= |\left\langle G^K(z^{-1})\xi, \mu\right\rangle| \\
&\leq \left\|G^K(z^{-1})\right\| \|\xi\| \|\mu\| \\
&\leq \left\|G^K\right\|_{H^\infty} \leq \frac{1}{r}.
\end{aligned} \tag{4.2.10}$$

However, by the choice of μ and ξ, $|h(0)| = \lim_{z\to\infty}\left\|G^K(z)\right\| = \left\|D^K\right\| = \frac{1}{r}$. If we combine this with (4.2.10), then we see that $|h|$ attains its maximum in the interior of $\mathbb{D}$. Thus, by the maximum modulus principle (see e.g. §16.2 from Priestley [48]), $|h|$ is constant in $\mathbb{D}$. Since $|h(0)| = \frac{1}{r}$, this means that $|h(z)| = \frac{1}{r}$ in $\mathbb{D}$. Substitute this in (4.2.10) to see that $\left\|G^K(z)\right\| = \frac{1}{r}$ for all $z \in \mathbb{E}$, which contradicts the assumptions of this lemma and thus completes the proof. □

The above characterization allows us to obtain a simple restatement of the Aizerman version of the circle criterion for SISO Lur'e systems.

Corollary 4.2.5. *Consider a Lur'e system (A, B, C, D, f) and assume that the underlying linear system $(A, B, C, D) \in \Sigma(1, n, 1; \mathbb{F})$ is stabilizable and detectable, and that its transfer function g is not a constant (that is, $g \in \mathbb{F}(z) \setminus \mathbb{F}$). For $k \in \mathbb{F}$ and $r > 0$, assume that $\mathbb{B}_{\mathbb{C}}(k, r) \subseteq \mathbb{S}_{\mathbb{C}}(g)$ and that there exists $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that $\left|g^k(z_0)\right| < \left\|g^k\right\|_{H^\infty}$.*

(a) If

$$\|f(\xi) - k\xi\| \leq r\|\xi\| \qquad \forall\, \xi \in \mathbb{F},$$

then the Lur'e system (A, B, C, D, f) is globally stable.

(b) If f is continuous and

$$\|f(\xi) - k\xi\| < r\|\xi\| \qquad \forall\, \xi \in \mathbb{F} \setminus \{0\},$$

then the Lur'e system (A, B, C, D, f) is globally asymptotically stable.

(c) If there exists $\delta > 0$ such that

$$\|f(\xi) - k\xi\| \leq (r - \delta)\,\|\xi\| \qquad \forall\, \xi \in \mathbb{F},$$

then the Lur'e system (A, B, C, D, f) is globally exponentially stable.

We will now look at some examples that demonstrate that the Aizerman version of the circle criterion is not a conservative result.

We will consider linear systems of the form $(a, b, c, 0) \in \Sigma(1, 1, 1, \mathbb{C})$ so that the requirement $\|D^K\| < \frac{1}{r}$ is satisfied automatically. Also, we will assume $K = 0$, so that $r = \|g\|_{H^\infty}^{-1}$. Since $g(z) = \frac{bc}{z-a}$, we have $g \in H^\infty$ if, and only if, $a \in \mathbb{D}$ and if so, then $\|g\|_{H^\infty} = \frac{|bc|}{1-|a|}$ giving us $r = \frac{1-|a|}{|bc|}$. Moreover, if $a \neq 0$, then clearly there will exist $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that $\|g(z_0)\| < \|g\|_{H^\infty}$. For our examples we will take $b = c = 1$ and $a = \frac{1}{2}$, so that $r = \frac{1}{2}$ and the evolution of the state is given by

$$x(t+1) = \frac{1}{2}x(t) + f(x(t)) \qquad \forall\, t \in \mathbb{N}_0.$$

Example 4.2.6. Let $\varepsilon > 0$ and define $f_1 \colon \mathbb{R} \to \mathbb{R}$ by

$$f_1(\xi) := \left(\frac{1}{2} + \varepsilon\right)\xi.$$

It is obvious that $|x(t+1)| = (1+\varepsilon)|x(t)| = \ldots = (1+\varepsilon)^{t+1}|x(0)|$ for all $(x, y) \in \mathcal{B}(\frac{1}{2}, 1, 1, 0, f_1)$, so that the Lur'e system $(\frac{1}{2}, 1, 1, 0, f_1)$ is not globally stable.

Example 4.2.7. Define $f_2 \colon \mathbb{R} \to \mathbb{R}$ by

$$f_2(\xi) := \frac{1}{2}\xi.$$

Since $\|f(\xi)\| = \frac{1}{2}\|\xi\|$ for all $\xi \in \mathbb{R}$, Aizerman version of the circle criterion (a) shows us that the Lur'e system $(\frac{1}{2}, 1, 1, 0, f_2)$ is globally stable. However $x(t) = x(0)$ for all $t \in \mathbb{N}_0$ and for all $(x, y) \in \mathcal{B}(\frac{1}{2}, 1, 1, 0, f_2)$, so that the Lur'e system $(\frac{1}{2}, 1, 1, 0, f_2)$ is not globally asymptotically stable.

Example 4.2.8. Define $f_3 \colon \mathbb{R} \to \mathbb{R}$ by

$$f_3(\xi) = \begin{cases} 0, & \text{if } |\xi| \geq 1 \\ \xi(\frac{1}{2} - |\xi|), & \text{if } |\xi| < 1. \end{cases}$$

Since $\|f_3(\xi)\| < \frac{1}{2}\|\xi\|$ for all $\xi \in \mathbb{R}$, Aizerman version of the circle criterion (b) implies that the Lur'e system $(\frac{1}{2}, 1, 1, 0, f_3)$ is globally asymptotically stable. We now show that it is not globally exponentially stable. To this end, suppose on the contrary that the Lur'e system $(\frac{1}{2}, 1, 1, 0, f_3)$ is globally exponentially stable. Then there exist constants $c > 0$ and $a \in (0,1)$ such that $|x(t)| \leq ca^t|x(0)|$ for all $t \in \mathbb{N}_0$ and for all $(x,y) \in \mathcal{B}(\frac{1}{2}, 1, 1, 0, f_3)$. Pick $\varepsilon > 0$ such that $(1+\varepsilon)a < 1$; by global asymptotic stability of $(\frac{1}{2}, 1, 1, 0, f_3)$, there exists $t \in \mathbb{N}_0$ such that $|x(t)| < (1-(1+\varepsilon)a)$. Hence $|x(t+1)| = |x(t)|(1-|x(t)|) > |x(t)|(1+\varepsilon)a$. Since we already noted that $|x(t+1)| < |x(t)| < 1-(1+\varepsilon)a$, we can repeat this procedure to see that for all $k \in \mathbb{N}$ we have

$$|x(t+k)| > |x(t)|(1+\varepsilon)^k a^k.$$

Combine this with global exponential stability to see that for all $k \in \mathbb{N}$ and for all $(x,y) \in \mathcal{B}(\frac{1}{2}, 1, 1, 0, f_3)$ we must have $ca^{t+k}|x(0)| \geq |x(t+k)| > |x(t)|(1+\varepsilon)^k a^k$. Hence

$$ca^t|x(0)| \geq |x(t)|(1+\varepsilon)^k$$

for all $k \in \mathbb{N}$ and for all $(x,y) \in \mathcal{B}(\frac{1}{2}, 1, 1, 0, f_3)$, which is a contradiction: pick any trajectory with $x(0) \neq 0$ (and therefore $x(t) \neq 0$ for all $t \in \mathbb{N}_0$). Thus $(\frac{1}{2}, 1, 1, 0, f_3)$ is not globally exponentially stable.

Finally we note an adaptation of the Aizerman version of the circle criterion to the case when the underlying linear system is controllable and observable. Its proof is identical in everything except it uses Lemma 3.2.9 instead of Lemma 3.2.8. Note that in this version of the result we do not need to assume that there exists $|z_0| = 1$ such that $\left\|G^K(z_0)\right\| < \frac{1}{r}$.

Proposition 4.2.9. *Consider a Lur'e system (A,B,C,D,f), assume that the underlying linear system $(A,B,C,D) \in \Sigma(m,n,p;\mathbb{F})$ is controllable and observable and denote its transfer function by G. Let $K \in \mathbb{F}^{m\times p}$, $r > 0$ and assume that $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.*

(a) If $\left\|D^K\right\| < \frac{1}{r}$ and

$$\|f(\xi) - K\xi\| \leq r\,\|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p,$$

then the Lur'e system (A,B,C,D,f) is globally stable.

(b) If f is continuous, $\left\|D^K\right\| < \frac{1}{r}$ and

$$\|f(\xi) - K\xi\| < r\,\|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p \setminus \{0\},$$

then the Lur'e system (A,B,C,D,f) is globally asymptotically stable.

(c) If there exists $\delta > 0$ such that

$$\|f(\xi) - K\xi\| \leq (r-\delta)\,\|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p,$$

then the Lur'e system (A,B,C,D,f) is globally exponentially stable.

4.3 "Standard" version of the circle criterion

In this section we will state and prove another corollary of Proposition 4.2.1, the statement of which resembles the classical circle criterion, see Theorem 5.1 from [22]. First, we need a lemma.

Lemma 4.3.1. *Let $L \in \mathbb{F}^{m\times p}$ be left-invertible and set $L^\sharp := (L^*L)^{-1}L^*$. Then*

(a) $LL^\sharp$ is the orthogonal projection onto $\operatorname{im} L$,

(b) $\|LL^\sharp\| = 1$, and

(c) for any $\delta > 0$ there exists $\delta_1 > 0$ such that

$$\|\xi\|^2 - \delta \left\|L^\sharp \xi\right\|^2 \le (1-\delta_1)^2 \|\xi\|^2 \qquad \forall\, \xi \in \operatorname{im} L.$$

Proof. (a) is well-known as $LL^\sharp$ is the Moore-Penrose pseudoinverse of L, see Proposition 6.1.6. from [9], and (b) follows trivially from (a).

To obtain (c), we note that $\operatorname{im} L = (\ker L^*)^\perp = (\ker L^\sharp)^\perp$. Clearly, there exists $c > 0$ such that $\|L^\sharp\xi\| \ge c\|\xi\|$ for all $\xi \in (\ker L^\sharp)^\perp$ and without loss of generality we can assume that $1 - \delta c^2 > 0$. Thus

$$\|\xi\|^2 - \delta \left\|L^\sharp \xi\right\|^2 \le \sqrt{(1-\delta c^2)}^2 \|\xi\|^2 \qquad \forall\, \xi \in \operatorname{im} L. \qquad \square$$

With this is hand, we can now prove the following.

Proposition 4.3.2 ("Standard" Version of the Circle Criterion)**.** *Consider a Lur'e system (A, B, C, D, f) and assume that the underlying linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is controllable and observable and denote its transfer function by G. Let $K_1, K_2 \in \mathbb{F}^{m\times p}$ and assume that $K_1 \in \mathbb{A}_{\mathbb{C}}(D)$, $(I - K_2G)(I - K_1G)^{-1}$ is positive real and that - for $K := \frac{1}{2}(K_1 + K_2)$ and $L := \frac{1}{2}(K_1 - K_2)$ - we have $\|LD^K\| < 1$.*

(a) If $\ker(K_1 - K_2) = \{0\}$ and if

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \le 0 \qquad \forall\, \xi \in \mathbb{F}^p, \tag{4.3.1}$$

then the Lur'e system (A, B, C, D, f) is globally stable.

(b) If

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle < 0 \qquad \forall\, \xi \in \mathbb{F}^p \setminus \{0\}, \tag{4.3.2}$$

then the Lur'e system (A, B, C, D, f) is globally asymptotically stable.

(c) If for some positive δ we have

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \le -\delta \|\xi\|^2 \qquad \forall\, \xi \in \mathbb{F}^p, \tag{4.3.3}$$

then the Lur'e system (A, B, C, D, f) is globally exponentially stable.

Proof. By rewriting K_1 and K_2 in terms of K and L, we obtain

$$\begin{aligned}\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle &= \operatorname{Re}\langle f(\xi) - (K+L)\xi, f(\xi) - (K-L)\xi\rangle \\ &= \|f(\xi) - K\xi\|^2 + \operatorname{Re}\langle f(\xi) - K\xi, L\xi\rangle \\ &\quad - \operatorname{Re}\langle L\xi, f(\xi) - K\xi\rangle - \|L\xi\|^2 \\ &= \|f(\xi) - K\xi\|^2 - \|L\xi\|^2 \qquad \forall\, \xi \in \mathbb{F}^p.\end{aligned} \tag{4.3.4}$$

Note that for (b) and (c) this implies that $\ker L = \{0\}$, while in (a) we explicitly assume $\ker L = \{0\}$. Therefore we always have $\ker L = \{0\}$. Thus L^*L is invertible and $L^\sharp := (L^*L)^{-1}L^* \in \mathbb{F}^{p\times m}$ is a left-inverse of L.

We can then check that

$$\begin{aligned}(I - K_2G)(I - K_1G)^{-1} &= (I - K_1G + 2LG)(I - K_1G)^{-1} \\ &= I + 2LG^{K_1},\end{aligned}$$

so, by Lemma 2.2.10, $\left\|-LG^{K_1}(I + LG^{K_1})^{-1}\right\|_{H^\infty} \le 1$. On the other hand, $-LG^{K_1}(I + LG^{K_1})^{-1} = L(-L^\sharp L)G^{K_1}(I - L(-L^\sharp L)G^{K_1})^{-1} = (LG^{K_1})^{-LL^\sharp}$. Hence by Proposition 3.2.12, we see that $\mathbb{B}_{\mathbb{C}}(-LL^\sharp, 1) \subseteq \mathbb{S}_{\mathbb{C}}\left(LG^{K_1}\right)$. This suggests looking at a state-space system that has transfer function LG^{K_1}.

Consider $(x, y) \in \mathcal{B}(A, B, C, D, f)$; then, by Lemma 3.1.4, we have

$$\begin{aligned}x(t+1) &= A_{K_1}x(t) + B_{K_1}(f(y(t)) - K_1y(t)) \\ y(t) &= C_{K_1}x(t) + D_{K_1}(f(y(t)) - K_1y(t))\end{aligned}$$

for all $t \in \mathbb{N}_0$. Left-multiplication by L of the output equation and the use of $I = L^\sharp L$ then gives us

$$\begin{aligned}x(t+1) =& A_{K_1}x(t) + B_{K_1}\Big(f(L^\sharp Ly(t)) - K_1L^\sharp Ly(t)\Big) \\ Ly(t) =& LC_{K_1}x(t) + LD_{K_1}\Big(f(L^\sharp Ly(t)) - K_1L^\sharp Ly(t)\Big) \qquad \forall\, t \in \mathbb{N}_0.\end{aligned}$$

Define $g(\xi) := f(L^\sharp\xi) - K_1L^\sharp\xi$, so that $(x, y) \in \mathcal{B}(A, B, C, D, f)$ if, and only if, $(x, Ly) \in \mathcal{B}(A_{K_1}, B_{K_1}, LC_{K_1}, LD_{K_1}, g)$. Since L is left-invertible, it thus suffices to show that - under the assumptions (4.3.1), (4.3.2) and (4.3.3) respectively - the Lur'e system $(A_{K_1}, B_{K_1}, LC_{K_1}, LD_{K_1}, g)$ is globally stable, globally asymptotically stable and globally exponentially stable respectively.

We will now do just that by applying the Aizerman version of the circle criterion to the Lur'e system $(A_{K_1}, B_{K_1}, LC_{K_1}, LD_{K_1}, g)$.

We have already checked that $\mathbb{B}_{\mathbb{C}}(-LL^\sharp, 1) \subseteq \mathbb{S}_{\mathbb{C}}\left(LG^{K_1}\right)$ and Lemma 3.1.4 together with an application of the Hautus tests for stabilizability and detectability shows that $(A_{K_1}, B_{K_1}, LC_{K_1}, LD_{K_1})$ (which is clearly a realization of LG^{K_1}) is stabilizable and detectable. Observe that

$$\begin{aligned}\left(LG^{K_1}\right)^{-LL^\sharp} &= LG(I-K_1G)^{-1}\left(I + LG(I-K_1G)^{-1}\right)^{-1}\\ &= LG^K,\end{aligned}$$

so that we obtain $\left\|\left(LD^K\right)\right\|^{-LL^\sharp} < 1$.

Thus to apply Proposition 4.2.9, we only need to check that g satisfies appropriate inequalities. By definition of g and equation (4.3.4), we have

$$\begin{aligned}\left\|g(\xi)+LL^\sharp\xi\right\|^2 &= \left\|f(L^\sharp\xi) - K_1L^\sharp\xi + LL^\sharp\xi\right\|^2\\ &= \left\|f(L^\sharp\xi) - KL^\sharp\xi\right\|^2\\ &= \operatorname{Re}\left\langle f(L^\sharp\xi) - K_1L^\sharp\xi, f(L^\sharp\xi) - K_2L^\sharp\xi\right\rangle + \left\|LL^\sharp\xi\right\|^2\end{aligned} \tag{4.3.5}$$

for all $\xi \in \mathbb{F}^m$.

Thus the use of equation (4.3.1) and Lemma 4.3.1 (b) allows us to establish the required inequality, so that we can apply Proposition 4.2.9 (a) to the Lur'e system $(A_{K_1}, B_{K_1}, LC_{K_1}, LD_{K_1}, g)$ to complete the proof of (a).

Now to prove (b) we consider two separate cases: if $\xi \notin \ker L^\sharp$, then the use of equations (4.3.5) and (4.3.2) combined with Lemma 4.3.1 (b) gives us $\left\|g(\xi)+LL^\sharp\xi\right\|^2 < \left\|LL^\sharp\xi\right\|^2 \le \|\xi\|^2$, while if $\xi \in \ker L^\sharp$, then $g(\xi)+LL^\sharp\xi = 0$. Thus $\left\|g(\xi)+LL^\sharp\xi\right\|^2 < \|\xi\|^2$ for all $\xi \in \mathbb{F}^m \setminus \{0\}$. Hence an application of Proposition 4.2.9 (b) to the Lur'e system $(A_{K_1}, B_{K_1}, LC_{K_1}, LD_{K_1}, g)$ completes the proof of (b).

Finally, to prove (c), we use Lemma 4.3.1 (a) to infer that we can decompose $\xi \in \mathbb{F}^m$ as $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \operatorname{im} L$ and $\xi_2 \in (\operatorname{im} L)^\perp = \ker L^\sharp$ (the last equality follows from Theorem 2.4.3 from [9]). Then the use of equations (4.3.5), (4.3.3) and an application of Lemma 4.3.1 (c) shows us that there exists $\delta_1 > 0$ such that

$$\begin{aligned}\left\|g(\xi) + LL^{\sharp}\xi\right\|^2 &= \left\|g(\xi_1) + LL^{\sharp}\xi_1\right\|^2 \\ &\le \left\|LL^{\sharp}\xi_1\right\|^2 - \delta\left\|L^{\sharp}\xi_1\right\| \\ &\le (1-\delta_1)^2 \left\|\xi_1\right\|^2 \\ &\le (1-\delta_1)^2 \left\|\xi\right\|^2 \qquad\qquad \forall\, \xi \in \mathbb{F}^m.\end{aligned}$$

Hence an application of Proposition 4.2.9 (c) completes the proof of (c). □

We remark that one can obtain a counterpart of Proposition 4.3.2 for stabilizable and detectable systems by using Proposition 4.2.1 instead of 4.2.9. However then one has to introduce an extra condition in Proposition 4.3.2, namely that - for $H := (I - K_2G)(I - K_1G)^{-1}$ - there exists $|z_0| = 1$ such that $H(z_0) + H(z_0)^* > 0$. The sufficiency of this assumption follows from the strict version of Lemma 2.2.10 (applied to the constant matrix $H(z_0)$). Since this result is not used in the present document and it introduces no other new ideas, we omit it.

We can see that Proposition 4.3.2 is similar to the well-known circle criterion. In fact, Theorem 5.1 from Haddad and Bernstein [22] is equivalent to a time-variant version of Proposition 4.3.2 (c). This equivalence is not obvious as they make stronger assumptions on $(I - K_2G)(I - K_1G)^{-1}$, but weaker assumptions on the nonlinearity f. However, one can check that these two assumptions imply that a related Lur'e system satisfies assumptions made in Proposition 4.3.2 and vice versa, which can then be used to show that these two results are equivalent. This is a lengthy calculation, so we will not perform it here.

Chapter 5

Input-to-state stability of Lur'e systems

In this chapter we finally reach the crux of Part I. We will see that under assumptions similar to the ones we made in absolute stability results, that is, Propositions 4.2.1 and 4.3.2, we in fact obtain input-to-state stability (from now on, ISS) of Lur'e systems with forcing

$$\begin{aligned} x(t+1) &= Ax(t) + B(f(y(t)) + d(t)) \\ y(t) &= Cx(t) + D(f(y(t)) + d(t)) \qquad \forall\, t \in \mathbb{N}_0. \end{aligned}$$

Introduced by Sontag in [50] (1989), ISS is a recent notion of stability, which provides a natural framework for stability analysis of nonlinear systems with inputs, merging, in a sense, Lyapunov and input-output approaches to stability. ISS has attracted a great deal of interest and its properties have been analysed in numerous papers, for an overview in discrete-time setting see Jiang and Wang [31], for an overview in continuous-time see Sontag [52].

There has been some interest in proving ISS for continuous-time Lur'e systems from assumptions that are similar to ones made in absolute stability theory, see e.g. Arcak and Teel [7], Jayawardhana, Logemann and Ryan [29] and [30], Bruin et al. [12]. All of the above make use of the ISS-Lyapunov function characterization of ISS and construct an appropriate ISS-Lyapunov function. [7, 30, 29] use the positive real lemma and output injection to obtain two quadratic forms, which are then used as building blocks for an ISS-Lyapunov function. [12] analyses a standard Lur'e-Postnikov Lyapunov function to obtain a result resembling the Popov criterion. Our approach is inspired by the former of these two, however there are major differences: (i) we use the bounded real lemma instead of the positive real lemma and hence our results apply to a different class of systems (we will obtain some of the results from [29, 30] as corollaries in Part II, where we deal with continuous-time systems), (ii) we allow nonzero feedthrough and multivariable systems and (iii) we consider the discrete-time case. Due to the use of the bounded real lemma to construct a quadratic form, we need new methods of estimation in establishing that our construction is indeed an ISS-Lyapunov function. Therefore we include a treatment of $\mathcal{K}_\infty$ functions and prove some novel results for them, see Lemma 5.1.11 and Proposition 5.1.15.

Finally, we will note that, under the assumptions made in the small-gain theorem, we can obtain a stronger version of ISS, which we call exponential input-to-state stability. Moreover, this result can be obtained without the use of ISS-Lyapunov functions.

This chapter is organized as follows: we introduce comparison function classes $\mathcal{K}, \mathcal{K}_\infty$ and $\mathcal{KL}$ in §5.1 and then prove some results on estimates involving comparison functions. In §5.2 we introduce Lur'e systems with forcing and define ISS for them. We also note an ISS-Lyapunov characterization of ISS. Then in §5.3 and §5.4 we state and prove results that guarantee ISS under assumptions similar to the ones made in absolute stability results from Chapter 4, see Theorem 5.3.1 and Proposition 5.4.1. Finally, in §5.5 we consider exponential ISS and show that it is guaranteed under the assumptions made in the small-gain theorem.

5.1 Function classes $\mathcal{K}, \mathcal{K}_\infty$ and $\mathcal{KL}$

In this section we introduce three standard classes of comparison functions: $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$. These function classes are convenient for defining stability concepts for nonautonomous differential equations (see e.g. §4.5, §4.8 and §4.9 from [36]) and are central in defining ISS. We will also prove some properties of $\mathcal{K}_\infty$ functions, which will then be used in ISS analysis.

Contrary to the rest of this document, in §5.1 we will sometimes use the symbols x and y to donate elements in $[0, \infty)$.

We denote by $\mathcal{K} \subset C([0,\infty))$ the set of continuous functions which are strictly increasing and are zero at zero:

$$\mathcal{K} := \{\alpha \in C([0,\infty)) : \alpha \text{ is strictly increasing and } \alpha(0) = 0\}.$$

We denote by $\mathcal{K}_\infty$ the set of $\mathcal{K}$ functions which are not bounded:

$$\mathcal{K}_\infty := \left\{\alpha \in \mathcal{K} : \lim_{s\to\infty} \alpha(s) = \infty\right\}.$$

Finally, we denote by (discrete-time)-$\mathcal{KL}$ the set of functions in two variables, $\beta\colon [0,\infty) \times \mathbb{N}_0 \to [0,\infty)$ with the following properties: if $\beta \in \mathcal{KL}$, then, for each fixed t, $\beta(\cdot, t) \in \mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is non-increasing and $\lim_{t\to\infty} \beta(s,t) = 0$.

Remark: the set of discrete-$\mathcal{KL}$ functions is just the obvious adaptation of the usual $\mathcal{KL}$ functions to discrete-time systems.

The following properties of $\mathcal{K}$ and $\mathcal{K}_\infty$ functions are straightforward, so we omit the proofs.

Lemma 5.1.1. *$\mathcal{K}$ is closed under addition, multiplication, composition and taking the minimum (or maximum) of a finite number of $\mathcal{K}$ functions.*

Lemma 5.1.2. *$\mathcal{K}_\infty$ is closed under addition, multiplication, composition and taking the minimum (or the maximum) of a finite number of $\mathcal{K}_\infty$ functions; moreover, it is also closed under inversion.*

From here on we will use the lattice notation for minima and maxima: $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. It is convenient to use these shorthands when dealing with $\mathcal{K}$ functions.

Lemma 5.1.3. *Let $\alpha \in \mathcal{K}$ and suppose that $a,\, b \geq 0$.*

Then $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$ and $\alpha(a \vee b) = \alpha(a) \vee \alpha(b)$.

Consider functions $U, V : \mathbb{F}^n \to [0, \infty)$. We will say that U and V are **$\mathcal{K}_\infty$-equivalent**, denoted $U \sim V$, if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(U(\xi)) \leq V(\xi) \leq \alpha_2(U(\xi)) \qquad \forall\, \xi \in \mathbb{F}^n.$$

Lemma 5.1.4. *$\mathcal{K}_\infty$-equivalence is an equivalence relation.*

Proof. Reflexivity and transitivity are trivial. Now suppose $U \sim V$, so that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(U(\xi)) \leq V(\xi) \leq \alpha_2(U(\xi))$ for all $\xi \in \mathbb{F}^n$. By Lemma 5.1.2, $\alpha_1^{-1}, \alpha_2^{-1} \in \mathcal{K}_\infty$. Since $\mathcal{K}_\infty$ functions are increasing, we have $U(\xi) \leq \alpha_1^{-1}(V(\xi))$ and $\alpha_2^{-1}(V(\xi)) \leq U(\xi)$, which completes the proof. □

We now record some results that will be useful in arguments involving $\mathcal{K}_\infty$ functions.

Lemma 5.1.5. *If $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, then $\alpha_1 + \alpha_2$ and $\alpha_1 \vee \alpha_2$ are $\mathcal{K}_\infty$-equivalent.*

Proof. This follows from the observation that $a \vee b \leq a + b \leq 2(a \vee b)$ for all $a, b \geq 0$. □

The following result is Lemma B.1 from Jiang and Wang [31], however we present an alternative proof as the proof presented in the article omitted some details and was difficult to penetrate.

Lemma 5.1.6. *If $\alpha \in \mathcal{K}_\infty$, then there exists $\alpha_1 \in \mathcal{K}_\infty$ such that $\alpha_1 \leq \alpha$ on $[0, \infty)$ and such that $id - \alpha_1 \in \mathcal{K}_\infty$.*

Proof. Let us set

$$\alpha_1(x) := \min_{s \in [0,x]} \left\{ \alpha(s) + \frac{1}{2}(x - s) \right\}$$

and let us prove that it has all the required properties. Note that continuous functions over compact intervals attain their minima, so α_1 is well-defined.

<u>$0 \leq \alpha_1 \leq \alpha$ on $[0, \infty)$:</u>

that $\alpha_1 \leq \alpha$ on $[0, \infty)$ follows from the evaluation of $\alpha(s) + \frac{1}{2}(x - s)$ at $s = x$. $0 \leq \alpha_1$ follows from the estimate $\alpha(s) + \frac{1}{2}(x - s) \geq 0 + 0$, when $s \in [0, x]$.

<u>α_1 is strictly increasing:</u>

let us fix $x \geq 0$, $\delta > 0$ and set $m \in [0, x + \delta]$ to be such that $\alpha_1(x + \delta) = \alpha(m) + \frac{1}{2}(x + \delta - m)$. If $m = x + \delta$, then $\alpha_1(x + \delta) = \alpha(x + \delta) > \alpha(x) \geq \alpha_1(x)$. If $m \in [x, x + \delta)$, then $\alpha_1(x + \delta) = \alpha(m) + \frac{1}{2}(x + \delta - m) > \alpha(m) \geq \alpha(x) \geq \alpha_1(x)$. Finally, if $m \in [0, x)$, then $\alpha_1(x + \delta) = \alpha(m) + \frac{1}{2}(x - m) + \frac{1}{2}\delta \geq \alpha_1(x) + \frac{1}{2}\delta > \alpha_1(x)$.

<u>α_1 is continuous:</u>

fix $x \geq 0$ and $\delta > 0$; straightforward estimations then give us

$$\begin{aligned}
\alpha_1(x + \delta) &= \min_{s \in [0, x+\delta]} \left\{ \alpha(s) + \frac{1}{2}(x + \delta - s) \right\} \\
&= \min_{s \in [0,x]} \left\{ \alpha(s) + \frac{1}{2}(x + \delta - s) \right\} \wedge \min_{s \in [x, x+\delta]} \left\{ \alpha(s) + \frac{1}{2}(x + \delta - s) \right\} \\
&= \left(\alpha_1(x) + \frac{1}{2}\delta \right) \wedge \min_{s \in [x, x+\delta]} \left\{ \alpha(s) + \frac{1}{2}(x + \delta - s) \right\} \\
&\leq \alpha_1(x) + \frac{1}{2}\delta.
\end{aligned}$$

Since x and δ were arbitrary, this shows right-continuity of α_1.

Similarly, for $x > 0$ and $x - \delta \geq 0$ (also $\delta > 0$), we can see that

$$\begin{aligned}
\alpha_1(x) &= \min_{s \in [0, x-\delta]} \left\{ \alpha(s) + \frac{1}{2}(x - \delta - s) + \frac{1}{2}\delta \right\} \wedge \min_{s \in [x-\delta, x]} \left\{ \alpha(s) + \frac{1}{2}(x - s) \right\} \\
&= \left(\alpha_1(x - \delta) + \frac{1}{2}\delta \right) \wedge \min_{s \in [x-\delta, x]} \left\{ \alpha(s) + \frac{1}{2}(x - s) \right\} \\
&\leq \alpha_1(x - \delta) + \frac{1}{2}\delta.
\end{aligned}$$

This shows that α_1 is left-continuous and hence it must be continuous.

$\alpha_1 \in \mathcal{K}_\infty$:

by combining the above properties, we have $\alpha_1 \in \mathcal{K}$, so we only need to show that $\lim_{x\to\infty} \alpha_1(x) = \infty$. This follows from

$$\begin{aligned}
\alpha_1(2x) &= \min_{s\in[0,2x]} \left\{\alpha(s) + x - \frac{1}{2}s\right\} \\
&= \min_{s\in[0,x]} \left\{\alpha(s) + x - \frac{1}{2}s\right\} \wedge \min_{s\in[x,2x]} \left\{\alpha(s) + x - \frac{1}{2}s\right\} \\
[\text{as } \alpha \in \mathcal{K}] &\geq \min_{s\in[0,x]} \left\{x - \frac{1}{2}s\right\} \wedge \min_{s\in[x,2x]} \left\{\alpha(x) + x - \frac{1}{2}s\right\} \\
&= \frac{1}{2}x \wedge \alpha(x).
\end{aligned}$$

Since $\alpha \in \mathcal{K}_\infty$, this shows that $\lim_{x\to\infty} \alpha_1(x) = \infty$.

$id - \alpha_1 \in \mathcal{K}_\infty$:

Clearly $id - \alpha_1$ is a continuous function and $(id - \alpha_1)(0) = 0$. The estimate $\alpha_1(x) = \min_{s\in[0,x]} \left\{\alpha(s) + \frac{1}{2}(x - s)\right\} \leq 0 + \frac{1}{2}x$ shows that $(id - \alpha_1)(x) \geq \frac{1}{2}x$, so that $\lim_{x\to\infty}(id - \alpha_1)(x) = \infty$. Hence we only need to show that $id - \alpha_1$ is strictly increasing.

This follows from the observation that, for $y > x$, we have

$$\begin{aligned}
&(id - \alpha_1)(y) - (id - \alpha_1)(x) \\
&\qquad = \frac{1}{2}y - \min_{s\in[0,y]} \left\{\alpha(s) - \frac{1}{2}s\right\} - \frac{1}{2}x + \min_{s\in[0,x]} \left\{\alpha(s) - \frac{1}{2}s\right\} \\
&\qquad \geq \frac{1}{2}(y - x) + 0,
\end{aligned}$$

where we have used the observation that both minima are ≤ 0 and the smaller one is being subtracted from the larger one. This completes the proof. $\square$

We now state and prove a few more simple results on estimates involving $\mathcal{K}_\infty$ functions.

Lemma 5.1.7. *Let $\alpha, \gamma \in \mathcal{K}_\infty$ and assume that $id - \alpha \in \mathcal{K}_\infty$. Then there exist $\tilde{\alpha}, \tilde{\gamma} \in \mathcal{K}_\infty$ such that $\tilde{\alpha} < id$ on $(0, \infty)$ and*

$$(id - \alpha)(x) + \gamma(y) \leq \tilde{\alpha}(x) \vee \tilde{\gamma}(y)$$

for all $x, y \geq 0$.

Proof. Set $\tilde{\alpha} := \frac{1}{2}id + \frac{1}{2}(id - \alpha)$ and $\tilde{\gamma} := (id - \alpha) \circ \alpha^{-1} \circ (2\gamma) + \gamma$. Then, by Lemma 5.1.2, $\tilde{\alpha}, \tilde{\gamma} \in \mathcal{K}_\infty$. Moreover it is easy to see that $\tilde{\alpha} = id - \frac{1}{2}\alpha < id$ on $(0, \infty)$.

Now, if $\gamma(y) \leq \frac{1}{2}\alpha(x)$, then

$$\begin{aligned}(id - \alpha)(x) + \gamma(y) &\leq (id - \alpha)(x) + \frac{1}{2}\alpha(x) \\ &= \tilde{\alpha}(x) \\ &\leq \tilde{\alpha}(x) \vee \tilde{\gamma}(y).\end{aligned}$$

If however $\gamma(y) > \frac{1}{2}\alpha(x)$, then $x < \alpha^{-1}(2\gamma(y))$, so that

$$\begin{aligned}(id - \alpha)(x) + \gamma(y) &< \tilde{\gamma}(y) \\ &\leq \tilde{\alpha}(x) \vee \tilde{\gamma}(y).\end{aligned}$$

This completes the proof. □

Lemma 5.1.8. *If $\alpha \in \mathcal{K}_\infty$ and $\varepsilon > 0$, then there exists $k > 0$ such that*

$$\alpha(x + y) \leq \alpha\left((1 + \varepsilon)x\right) + \alpha(ky)$$

for all $x, y \geq 0$. In particular $k = 1 + \frac{1}{\varepsilon}$ works.

Remark: This result cannot be extended to $\varepsilon = 0$. Consider e.g. $s \mapsto s^2$.

Proof. It suffices to find $k > 0$ such that $x + y \leq (1 + \varepsilon)x \vee ky$ for all $x, y \geq 0$. Note that

$$(x + y) + \varepsilon(x + y) = (1 + \varepsilon)x + \varepsilon\left(1 + \frac{1}{\varepsilon}\right)y.$$

Hence either $x + y \leq (1 + \varepsilon)x$ or $\varepsilon(x + y) \leq \varepsilon\left(1 + \frac{1}{\varepsilon}\right)y$. Thus $k = 1 + \frac{1}{\varepsilon}$ has the required properties. □

Lemma 5.1.9. *If $\alpha \in \mathcal{K}_\infty$, then there exists $\gamma \in \mathcal{K}_\infty$ such that*

$$xy \leq x\alpha(x) + \gamma(y) \qquad \forall\, x, y \geq 0.$$

Proof. If $y \leq \alpha(x)$, then $xy \leq x\alpha(x)$; and if $y > \alpha(x)$, then $x < \alpha^{-1}(y)$, so that $xy < y\alpha^{-1}(y)$. Hence $\gamma(y) := y\alpha^{-1}(y)$ satisfies all the requirements. □

For what is to come, it will be useful to single out the following subset of $\mathcal{K}_\infty$.

Definition 5.1.10. If $\alpha \in \mathcal{K}_\infty$ is such that, for all $\varepsilon > 0$,

$$\lim_{x\to\infty} \{\alpha((1+\varepsilon)x) - \alpha(x)\} = \infty, \tag{GC}$$

then we say that α satisfies the growth condition (GC). We denote the set of all $\mathcal{K}_\infty$ functions that satisfy the growth condition (GC) by $\mathcal{K}_\infty^{\mathrm{GC}}$.

Note that if $a > 0$, then the function $\alpha(x) := x^a$ is in $\mathcal{K}_\infty^{\mathrm{GC}}$, while the function $\gamma(x) := \log(1+x)$ is not.

$\mathcal{K}_\infty^{\mathrm{GC}}$ functions admit the following characterization.

Lemma 5.1.11. *$\alpha \in \mathcal{K}_\infty^{\mathrm{GC}}$ if, and only if, for each $\varepsilon > 0$, there exists $\eta \in \mathcal{K}_\infty$ such that*

$$\alpha(x-y) \le \alpha\left((1+\varepsilon)x\right) - \eta(y) \qquad \forall\, x \ge y \ge 0.$$

Remark: we emphasize that η depends on ε.

Proof. We show necessity first. Let $\varepsilon > 0$; then, by our assumption, there exists $\eta \in \mathcal{K}_\infty$ such that

$$\eta(y) \le \alpha\left(\left(1 + \frac{\varepsilon}{2}\right)x\right) - \alpha(x-y),$$

for all $x \ge y \ge 0$. Now fix $y \ge 0$; then for x large enough we have $\frac{\varepsilon}{2}x \ge (1+\varepsilon)y$, so that

$$\begin{aligned}\lim_{x\to\infty}\{\alpha((1+\varepsilon)x) - \alpha(x)\} &= \lim_{x\to\infty}\{\alpha((1+\varepsilon)(x-y)) - \alpha(x-y)\}\\ &\ge \lim_{x\to\infty}\{\alpha\left(\left(1+\frac{\varepsilon}{2}\right)x\right) - \alpha(x-y)\}\\ &\ge \eta(y).\end{aligned}$$

Since y was arbitrary, we must have $\lim_{x\to\infty}\{\alpha((1+\varepsilon)x) - \alpha(x)\} = \infty$, whence $\alpha \in \mathcal{K}_\infty^{\mathrm{GC}}$.

To prove sufficiency, assume that (GC) holds. Set $\Delta := \{(x,y) \in \mathbb{R}\times\mathbb{R} : x \ge y \ge 0\}$ and consider the continuous function $g : \Delta \to [0,\infty)$ given by

$$g(x,y) := \alpha\left((1+\varepsilon)x\right) - \alpha(x-y).$$

Also, define $\eta : [0,\infty) \to [0,\infty)$ by

$$\eta(y) := \inf_{x\in[y,\infty)} g(x,y) = \inf_{x\in[0,\infty)} g(y+x,y).$$

By (GC), for each fixed $y \ge 0$, we have $\lim_{x\to\infty} g(x,y) = \infty$. Therefore, by continuity of g, the set $G(y) := \{x \ge 0 : g(y+x,y) = \eta(y)\}$ is non-empty and compact for all $y \ge 0$. We now define $l(y) := \min G(y)$, so that

$$\eta(y) = g(y + l(y), y) \qquad \forall\, y \ge 0. \tag{5.1.1}$$

It is easy to see that η satisfies the required inequality, so it suffices to show that $\eta \in \mathcal{K}_\infty$.

To this end, we note that $g \geq 0$ and $g(0,0) = 0$, so $\eta(0) = 0$. To show that η is strictly increasing, fix $y \geq 0$, $\delta > 0$ and set $a := l(y+\delta)$. Then

$$\begin{aligned}\eta(y+\delta) &= \min_{x\in[0,a]} g(y+\delta+x, y+\delta) \\ &= \min_{x\in[0,a]} \left[\alpha\left((1+\varepsilon)(y+\delta+x)\right) - \alpha(x)\right]. \end{aligned} \tag{5.1.2}$$

Now, for every $x \geq 0$, we have $\alpha\left((1+\varepsilon)(y+\delta+x)\right) - \alpha(\delta+x) = g(y+\delta+x, y) \geq \eta(y)$, and hence $\alpha\left((1+\varepsilon)(y+\delta+x)\right) - \alpha(x) \geq \eta(y) + \alpha(\delta+x) - \alpha(x)$. If we use this in equation (5.1.2), then we obtain

$$\eta(y+\delta) \geq \eta(y) + \min_{x\in[0,a]} \left[\alpha(\delta+x) - \alpha(x)\right] > \eta(y),$$

where the second, strict, inequality follows from the fact that α is continuous and strictly increasing. Since $y \geq 0$ and $\delta > 0$ were arbitrary, we have shown that η is strictly increasing.

We proceed to prove that $\eta(y) \to \infty$ as $y \to \infty$. This follows from

$$\begin{aligned}\lim_{y\to\infty} \eta(y) &= \lim_{y\to\infty} \inf_{x\in[0,\infty)} \left[\alpha\left((1+\varepsilon)(y+x)\right) - \alpha(x)\right] \\ &\geq \lim_{y\to\infty} \inf_{x\in[0,\infty)} \left[\alpha\left((1+\varepsilon)(y+x)\right) - \alpha(y+x)\right] \\ &= \liminf_{x\to\infty} \left[\alpha((1+\varepsilon)x) - \alpha(x)\right] \\ &= \lim_{x\to\infty} \left[\alpha((1+\varepsilon)x) - \alpha(x)\right] \\ &= \infty.\end{aligned}$$

Thus it only remains to prove that η is continuous. It is well-known that continuity and sequential continuity are equivalent, so we pick $y \geq 0$ and let $(y_i)_{i\in\mathbb{N}_0}$ be a sequence in $[0,\infty)$ such that $y_i \to y$ as $i \to \infty$. Hence it is sufficient to show that

$$\limsup_{i\to\infty} \eta(y_i) \leq \eta(y) \leq \liminf_{i\to\infty} \eta(y_i). \tag{5.1.3}$$

Set $x_i := y_i + l(y) \geq y_i$; then the continuity of g and equation (5.1.1) guarantee that $\lim_{i\to\infty} g(x_i, y_i) = g(y + l(y), y) = \eta(y)$. Now $\eta(y_i) \leq g(x_i, y_i)$ for all $i \in \mathbb{N}$ and thus,

$$\limsup_{i\to\infty} \eta(y_i) \leq \limsup_{i\to\infty} g(x_i, y_i) = \lim_{i\to\infty} g(x_i, y_i) = \eta(y), \tag{5.1.4}$$

giving us the first inequality.

To prove the second one, for $j \in \mathbb{N}$, we define $\eta_j := \inf_{i \geq j} \eta(y_i)$. Thus, for every $j \in \mathbb{N}$, there exists an integer $i_j \geq j$ such that $\eta(y_{i_j}) - \eta_j \leq \frac{1}{j}$. Upon setting $z_j := y_{i_j}$, we have $\lim_{j\to\infty} z_j = y$, so that

$$\lim_{j\to\infty} \eta(z_j) = \lim_{j\to\infty} \eta_j = \liminf_{i\to\infty} \eta(y_i). \tag{5.1.5}$$

Hence, by equation (5.1.1), it follows that, for all $j \in \mathbb{N}$,

$$\eta(z_j) = g(l(z_j) + z_j, z_j) \geq \alpha\left((1+\varepsilon)(l(z_j) + z_j)\right) - \alpha(l(z_j) + z_j).$$

Boundedness of $(\eta(z_j))_{j\in\mathbb{N}_0}$ together with the growth condition (GC) implies that $(l(z_j))_{j\in\mathbb{N}_0}$ is bounded. Thus, there exists a convergent subsequence $(l(z_{j_k}))_{k\in\mathbb{N}_0}$ and we denote its limit by l. By equation (5.1.5), we have that

$$\lim_{k\to\infty} z_{j_k} = y \qquad \text{and} \qquad \lim_{k\to\infty} \eta\left(z_{j_k}\right) = \liminf_{i\to\infty} \eta(y_i),$$

and thus, by equation (5.1.1) and by the continuity of g, we have

$$\liminf_{i\to\infty} \eta(y_i) = \lim_{k\to\infty} g\left(z_{j_k} + l(z_{j_k}), z_{j_k}\right) = g(y + l, y) \geq \eta(y).$$

Together with equation (5.1.4) this shows that (5.1.3) holds, which in turn completes the proof. ☐

The following lemma shows how we will use this result.

Lemma 5.1.12. *Let $\alpha \in \mathcal{K}_\infty$ and define $\tilde{\alpha} \in \mathcal{K}_\infty$ by $\tilde{\alpha}(s) := \sqrt{s}\alpha(\sqrt{s})$. Then for every $\varepsilon > 0$ there exists $\eta \in \mathcal{K}_\infty$ such that*

$$\tilde{\alpha}(s_1 - s_2) \leq \tilde{\alpha}((1+\varepsilon)s_1) - \eta(s_2) \qquad \forall\, s_1 \geq s_2 \geq 0.$$

Proof. It is easy to check that $\tilde{\alpha} \in \mathcal{K}_\infty^{\mathrm{GC}}$. The proof is then complete after an application of Lemma 5.1.11. ☐

Let us now state and prove some other results that will be useful in the stability analysis of Lur'e systems.

Lemma 5.1.13. *If $\alpha \in \mathcal{K}$ and $\beta \in \mathcal{KL}$, then $\alpha \circ \beta \in \mathcal{KL}$.*

Proof. We need to check that for fixed $t \in \mathbb{N}_0$ the function $(\alpha \circ \beta)(\cdot, t)$ is in $\mathcal{K}$ and that for fixed $s \in [0, \infty)$ we have $\lim_{t\to\infty}(\alpha \circ \beta)(s,t) = 0$.

By definition, for a fixed $t \in \mathbb{N}_0$, the function $\beta(\cdot, t)$ is in $\mathcal{K}$ and, by Lemma 5.1.1, a composition of a $\mathcal{K}$ function with a $\mathcal{K}$ function is in $\mathcal{K}$, which gives us the first statement.

The second statement follows from the continuity of α:

$$\lim_{t\to\infty} (\alpha \circ \beta)(s,t) = \alpha\left(\lim_{t\to\infty} \beta(s,t)\right) = \alpha(0) = 0.$$ ☐

The following result will be useful for an ISS version of the circle criterion, see Proposition 5.4.1.

Lemma 5.1.14. *Let $\alpha \in \mathcal{K}_\infty$, $L \in \mathbb{F}^{m\times p}$ be left-invertible and set $L^\sharp := (L^*L)^{-1}L^*$. Then there exists $\alpha_1 \in \mathcal{K}_\infty$ with $\alpha_1 < id$ on $(0,\infty)$ such that*

$$\left\|LL^\sharp\xi\right\|^2 - \left\|L^\sharp\xi\right\| \alpha\left(\left\|L^\sharp\xi\right\|\right) \leq (\|\xi\| - \alpha_1(\|\xi\|))^2 \qquad \forall \xi \in \mathbb{F}^m.$$

Proof. By Lemma 4.3.1, we know that $LL^\sharp$ is the orthogonal projection onto $\operatorname{im} L$. Let us decompose $\mathbb{F}^m = \operatorname{im}(L) \oplus \operatorname{im}(L)^\perp$ and, for an arbitrary $\xi \in \mathbb{F}^m$, write $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \operatorname{im}(L)$ and $\xi_2 \in \operatorname{im}(L)^\perp$. Hence $\left\|LL^\sharp\xi\right\| = \|\xi_1\|$. Moreover, as $\operatorname{im} L = (\ker L^*)^\perp = (\ker L^\sharp)^\perp$ (use e.g. Theorem 2.4.3 from [9] for the first equality), there exists $c > 0$ such that $\left\|L^\sharp\xi\right\| = \left\|L^\sharp\xi_1\right\| \geq c\|\xi_1\|$ for all $\xi \in \mathbb{F}^m$. Thus, upon noting that $\|\xi\|^2 = \|\xi_1\|^2 + \|\xi_2\|^2$, we have

$$\left\|LL^\sharp\xi\right\|^2 - \left\|L^\sharp\xi\right\| \alpha\left(\left\|L^\sharp\xi\right\|\right) \leq \|\xi\|^2 - \|\xi_2\|^2 - c\|\xi_1\|\,\alpha\left(c\|\xi_1\|\right) \qquad (5.1.6)$$

for all $\xi \in \mathbb{F}^m$.

On the other hand, if $\alpha_1 \in \mathcal{K}_\infty$, then

$$\begin{aligned}(\|\xi\| - \alpha_1(\|\xi\|))^2 &= \|\xi\|^2 - 2\|\xi\|\,\alpha_1(\|\xi\|) + \alpha_1(\|\xi\|)^2 \\ &\geq \|\xi\|^2 - 2\|\xi\|\,\alpha_1(\|\xi\|)\end{aligned}$$

for all $\xi \in \mathbb{F}^m$. Hence, in view of estimate (5.1.6), it suffices to find $\alpha_1 \in \mathcal{K}_\infty$ such that $2\|\xi\|\,\alpha_1(\|\xi\|) \leq \|\xi_2\|^2 + c\|\xi_1\|\,\alpha(c\|\xi_1\|)$ for all $\xi \in \mathbb{F}^m$. Or, equivalently,

$$2\sqrt{s_1^2 + s_2^2} \cdot \alpha_1\left(\sqrt{s_1^2+s_2^2}\right) \leq cs_1\alpha(cs_1) + s_2^2 \qquad (5.1.7)$$

for all $s_1, s_2 \geq 0$. We define $\alpha_1 \in \mathcal{K}_\infty$ by $\alpha_1(s) := \frac{1}{4}\cdot\big(c\alpha(cs/2) \wedge s/2\big)$ for $s \geq 0$. Clearly, $\alpha_1 < id$ on $(0,\infty)$. Moreover, since $(s_1+s_2)^2 \geq s_1^2 + s_2^2$ and $s_1 + s_2 \leq 2s_1 \vee 2s_2$ for all $s_1, s_2 \geq 0$, we have

$$\begin{aligned}2\sqrt{s_1^2+s_2^2}\cdot\alpha_1\left(\sqrt{s_1^2+s_2^2}\right) &\leq 2(s_1+s_2)\alpha_1(s_1+s_2) \\ &\leq (4s_1 \vee 4s_2)\cdot\alpha(2s_1 \vee 2s_2) \\ &\leq 4s_1\alpha_1(2s_1) + 4s_2\alpha_1(2s_2) \\ &= cs_1\alpha(cs_1) \wedge s_1^2 + cs_2\alpha(cs_2) \wedge s_2^2 \\ &\leq cs_1\alpha(cs_1) + s_2^2\end{aligned}$$

for all $s_1, s_2 \geq 0$, which completes the proof. □

The following result will only be used in Part II.

Proposition 5.1.15. *Let $\alpha \in \mathcal{K}_\infty$ and suppose there exists $c_1 > 0$ such that $\alpha(s) \leq c_1 s$ for all $s \in [0, \infty)$. Then there exist, $f \in C([0,\infty),[0,c_1])$ and $\eta \in \mathcal{K}_\infty$ such that*

$$\eta(s) \leq s^2 f(s^2) \leq s\alpha(s) \qquad \forall\, s \in [0, \infty)$$

and $\int_0^\infty f(s)\,\mathrm{d}s = \infty$.

Proof. First define $\tilde{f} : (0,\infty) \to (0, c_1]$ by $\tilde{f}(s) := \frac{\alpha(\sqrt{s})}{\sqrt{s}}$. As a composition of continuous functions, $\tilde{f}$ is itself continuous. Thus we can define a continuous function $f \in C([0,\infty),[0,c_1])$ by

$$f(s) := \begin{cases} s \wedge \tilde{f}(s), & \text{if } s > 0 \\ 0, & \text{if } s = 0. \end{cases}$$

Then $s^2 f(s^2) = s^3 \wedge s\alpha(s)$ for all $s \geq 0$. Hence, if we define $\eta \in \mathcal{K}_\infty$ by $\eta(s) := s^3 \wedge s\alpha(s)$, then $\eta(s) \leq s^2 f(s^2) \leq s\alpha(s)$ for all $s \geq 0$.

The proof is completed by observing that

$$\int_0^\infty f(s)\,\mathrm{d}s \geq \int_0^\infty \tilde{f}(s)\,\mathrm{d}s > \int_{\alpha^{-1}(1)}^\infty \tilde{f}(s)\,\mathrm{d}s > \int_{\alpha^{-1}(1)}^\infty \frac{\alpha(\alpha^{-1}(1))}{\sqrt{s}}\,\mathrm{d}s = \infty.$$

□

5.2 Input-to-state stability

In this section we define ISS for Lur'e systems

$$\begin{aligned} x(t+1) &= Ax(t) + B(f(y(t)) + d(t)) \\ y(t) &= Cx(t) + D(f(y(t)) + d(t)). \end{aligned} \tag{5.2.1}$$

It is well-known that ISS is equivalent to the existence of a continuously differentiable ISS-Lyapunov function, see Jiang and Wang [31]. However, in the main result of this chapter, Theorem 5.3.1, we construct an ISS-Lyapunov function for (5.2.1), which could have a discontinuous derivative. Since we are working in discrete-time, the assumption that the ISS-Lyapunov function should be continuously differentiable seems surplus to requirements. Indeed, in Proposition 5.2.4 we show that the existence of a continuous ISS-Lyapunov function with a possibly discontinuous derivative still guarantees ISS of (5.2.1).

Definition 5.2.1. Consider a Lur'e system (A, B, C, D, f), where the underlying linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$. We define the **behaviour**

with disturbances of (A, B, C, D, f) as

$$\begin{aligned}\mathcal{B}_{\mathrm{d}}(A, B, C, D) \\ := \Big\{(d, x, y) \in (\mathbb{F}^m)^{\mathbb{N}_0} \times (\mathbb{F}^n)^{\mathbb{N}_0} \times (\mathbb{F}^p)^{\mathbb{N}_0} \\ : (f \circ y + d, x, y) \in \mathcal{B}(A, B, C, D)\Big\}.\end{aligned}$$

The behaviour with disturbances of (A, B, C, D, f) thus consists of triples $(d, x, y) \in (\mathbb{F}^m)^{\mathbb{N}_0} \times (\mathbb{F}^n)^{\mathbb{N}_0} \times (\mathbb{F}^p)^{\mathbb{N}_0}$ that satisfy (5.2.1).

Definition 5.2.2. Consider a Lur'e system (A, B, C, D, f), where the underlying linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$. It is said to be (globally) **input-to-state stable** if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all trajectories $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$ with $d \in l^\infty(\mathbb{F}^m)$ we have

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|d\|_\infty) \qquad \forall\, t \in \mathbb{N}_0. \tag{5.2.2}$$

Note that, via Lemma 5.1.5, equation (5.2.2) is equivalent to

$$\|x(t)\| \leq \beta(\|x(0)\|, t) \vee \gamma(\|d\|_\infty) \qquad \forall\, t \in \mathbb{N}_0. \tag{5.2.3}$$

We now introduce ISS-Lyapunov functions. In contrast to existing discrete-time work (see e.g. [31, 32]), we do not require ISS-Lyapunov functions to be continuously differentiable.

Definition 5.2.3. Consider a Lur'e system (A, B, C, D, f), where the underlying linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$. A function $V : \mathbb{F}^n \to [0, \infty)$ is called an **ISS-Lyapunov function for** (A, B, C, D, f) if V and $\|.\|$ are $\mathcal{K}_\infty$-equivalent and if there exist $\alpha, \sigma \in \mathcal{K}_\infty$ such that for all trajectories $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$ we have

$$V(x(t+1)) - V(x(t)) \leq -\alpha(\|x(t)\|) + \sigma(\|d(t)\|) \qquad \forall\, t \in \mathbb{N}_0. \tag{5.2.4}$$

The existence of an ISS-Lyapunov function implies ISS.

Proposition 5.2.4. *If there exists an ISS-Lyapunov function for a Lur'e system (A, B, C, D, f), then the Lur'e system (A, B, C, D, f) is ISS.*

Proof. Let V be an ISS-Lyapunov function for (A, B, C, D, f). By the $\mathcal{K}_\infty$-equivalence of V and $\|.\|$ and the alternative characterization of ISS property given in equation (5.2.3), it suffices to show that there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$V(x(t+1)) \leq \beta(V(x(0)), t) \vee \gamma(\|d\|_\infty) \qquad \forall\, t \in \mathbb{N}_0$$

and for all $(d, x, y) \in \mathcal{B}_\mathrm{d}(A, B, C, D, f)$ with $d \in l^\infty(\mathbb{F}^m)$.

Since V is an ISS-Lyapunov function, there exist $\alpha, \sigma \in \mathcal{K}_\infty$ such that

$$V(x(t+1)) - V(x(t)) \leq -\alpha(V(x(t))) + \sigma(\|d(t)\|) \qquad \forall\, t \in \mathbb{N}_0$$

and for all $(d, x, y) \in \mathcal{B}_\mathrm{d}(A, B, C, D, f)$. Furthermore, by Lemma 5.1.6, we can assume - without loss of generality - that $id - \alpha \in \mathcal{K}_\infty$. Lemma 5.1.7 shows us that there exist $\rho, \gamma \in \mathcal{K}_\infty$ with $\rho < id$ on $(0, \infty)$ such that $(id - \alpha)(s_1) + \sigma(s_2) \leq \rho(s_1) \vee \gamma(s_2)$ for all $s_1, s_2 \geq 0$. Hence we can estimate

$$\begin{aligned} V(x(t+1)) &\leq (id - \alpha)\,(V(x(t)) + \sigma(\|d\|_\infty) \\ &\leq \rho(V(x(t))) \vee \gamma(\|d\|_\infty) \\ &\leq \rho\Big(\rho(V(x(t-1))) \vee \gamma(\|d\|_\infty)\Big) \vee \gamma(\|d\|_\infty) \\ &\leq \rho^2(V(x(t-1))) \vee \gamma(\|d\|_\infty) \\ &\quad\;\vdots \\ &\leq \rho^{t+1}(V(x(0))) \vee \gamma(\|d\|_\infty) \end{aligned} \tag{5.2.5}$$

for all $t \in \mathbb{N}_0$ and for all $(d, x, y) \in \mathcal{B}_\mathrm{d}(A, B, C, D, f)$ with $d \in l^\infty(\mathbb{F}^m)$.

Thus we are done if we can show that $\beta(s, t) := \rho^{t+1}(s)$ defines a $\mathcal{KL}$ function. But this is straightforward: by Lemma 5.1.1, $\beta(\cdot, t)$ is a $\mathcal{K}$ function for each fixed $t \in \mathbb{N}_0$. Now fix $s > 0$; then $0 < \rho^t(s) < \rho^{t-1}(s) < \ldots < \rho(s) < s$, so the sequence $(\rho^t(s))_{t\in\mathbb{N}_0}$ is decreasing and bounded from below. Therefore it has a limit, which must be a fixed point of ρ. Since $\rho < id$ on $(0, \infty)$, this fixed point is 0. Thus $\beta(s, \cdot)$ is decreasing and $\lim_{t\to\infty} \beta(s, t) = 0$, which completes the proof. $\square$

5.3 Ball condition assumptions

In this and the following section we will obtain the main results in this part of the thesis. We will see that under assumptions similar to the ones we made in absolute stability results, that is, Propositions 4.2.1 and 4.3.2, we in fact obtain ISS of Lur'e systems with forcing

$$\begin{aligned} x(t+1) &= Ax(t) + B(f(y(t)) + d(t)) \\ y(t) &= Cx(t) + D(f(y(t)) + d(t)) \qquad \forall\, t \in \mathbb{N}_0. \end{aligned} \tag{5.3.1}$$

Similarly as in Chapter 4, we will first obtain a result that resembles Aizerman's conjecture, see Theorem 5.3.1, which will then be used to obtain a result that resembles the circle criterion, see Proposition 5.4.1. As in the absolute stability case, we will conduct stability analysis by using quadratic forms obtained from: (i) a "ball condition" and the bounded real lemma, and (ii) output injection. We will use $\mathcal{K}_\infty$ results from §5.1 to prove that a suitable combination of the two quadratic forms is an ISS-Lyapunov function for the Lur'e system (5.3.1).

Theorem 5.3.1. *Consider a Lur'e system (A, B, C, D, f), assume that the underlying state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable and denote its transfer function by G. Furthermore, let $r > 0$, $K \in \mathbb{F}^{m \times p}$ and assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ and that there exists $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that $\left\|G^K(z_0)\right\| < \frac{1}{r}$.*

If there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\|f(\xi) - K\xi\| \leq r\|\xi\| - \alpha(\|\xi\|) \qquad \forall\, \xi \in \mathbb{F}^p, \tag{5.3.2}$$

then the Lur'e system (A, B, C, D, f) is ISS.

Proof. In view of Proposition 5.2.4, it suffices to find an ISS-Lyapunov function for the Lur'e system (A, B, C, D, f). We will do this in two steps - firstly we will obtain a quadratic form from the condition $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ and then we will obtain and modify a quadratic form obtained via output injection. The sum of these two will be the eventual ISS-Lyapunov function for (A, B, C, D, f).

Since $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ and since there exists $|z_0| = 1$ such that $\left\|G^K(z_0)\right\| < \frac{1}{r}$, by Lemma 3.2.8, there exists a positive semi-definite $P = P^* \in \mathbb{F}^{n \times n}$ such that the function $V \colon \mathbb{F}^n \to [0, \infty)$ defined by $V(\xi) := \langle P\xi, \xi\rangle$ satisfies

$$V(x(t+1)) - V(x(t)) \leq \|u(t) - Ky(t)\|^2 - r^2\|y(t)\|^2 \qquad \forall\, t \in \mathbb{N}_0$$

and for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$. By the definition of the behaviour with disturbances of (A, B, C, D, f), we thus have

$$V(x(t+1)) - V(x(t)) \leq \|f(y(t)) - Ky(t) + d(t)\|^2 - r^2\|y(t)\|^2 \tag{5.3.3}$$

for all $t \in \mathbb{N}_0$ and for all $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$.

Using equation (5.3.2) we can estimate

$$\begin{aligned} \|f(\xi) - K\xi\|^2 - r^2\|\xi\|^2 &\leq -2\alpha(\|\xi\|)r\|\xi\| + \alpha(\|\xi\|)^2 \\ &\leq -2\alpha(\|\xi\|)r\|\xi\| + \alpha(\|\xi\|)r\|\xi\| \\ &= -r\|\xi\|\,\alpha(\|\xi\|) \qquad\qquad \forall\, \xi \in \mathbb{F}^p. \end{aligned}$$

By Lemma 5.1.9, there exists $\tilde{\gamma} \in \mathcal{K}_\infty$ such that $s_1 s_2 \leq \frac{1}{4}s_1\alpha(s_1) + \tilde{\gamma}(s_2)$ for all $s_1, s_2 \geq 0$. Therefore, if we define $\gamma \in \mathcal{K}_\infty$ by $\gamma(s) := 2r\tilde{\gamma}(s) + s^2$ and use (5.3.2) alongside the Cauchy-Schwarz inequality, then we can estimate

$$\begin{aligned}\|f(\xi) - K\xi + \mu\|^2 - r^2\|\xi\|^2 &\leq \|f(\xi) - K\xi\|^2 + 2\|f(\xi) - K\xi\|\,\|\mu\| \\ &\quad + \|\mu\|^2 - r^2\|\xi\|^2 \\ &\leq -r\|\xi\|\,\alpha(\|\xi\|) + 2r\|\xi\|\,\|\mu\| + \|\mu\|^2 \\ &\leq -\frac{r}{2}\|\xi\|\,\alpha(\|\xi\|) + \gamma(\|\mu\|)\end{aligned}$$

for all $\xi \in \mathbb{F}^p$, $\mu \in \mathbb{F}^m$. If we use this estimate in (5.3.3), then we obtain

$$V(x(t+1)) - V(x(t)) \leq -\frac{r}{2}\|y(t)\|\,\alpha(\|y(t)\|) + \gamma(\|d(t)\|) \qquad \forall\, t \in \mathbb{N}_0$$

and for all $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$. Upon defining $\alpha_1 \in \mathcal{K}_\infty$ by $\alpha_1(s) := (r/2)\sqrt{s}\alpha(\sqrt{s})$ for all $s \geq 0$, it follows that

$$V(x(t+1)) - V(x(t)) \leq -\alpha_1\left(\|y(t)\|^2\right) + \gamma(\|d(t)\|) \qquad \forall\, t \in \mathbb{N}_0 \quad (5.3.4)$$

and for all $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$.

Now we obtain a quadratic form from output injection and then modify it. Since (A, B, C, D) is detectable, we can use Lemma 2.3.1 to see that there exists a positive definite $Q = Q^* \in \mathbb{F}^{n\times n}$ and $\delta > 0$ such that the function $U_1 : \mathbb{F}^n \to [0, \infty)$ defined by $U_1(\xi) = \langle Q\xi, \xi\rangle$ satisfies $U_1(x(t+1)) - U_1(x(t)) \leq -\delta\|x(t)\|^2 + \|y(t)\|^2 + \|u(t)\|^2$ for all $t \in \mathbb{N}_0$ and for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$. Thus we can use the definition of a behaviour with disturbances of (A, B, C, D, f) to see that

$$U_1(x(t+1)) - U_1(x(t)) \leq -\delta\|x(t)\|^2 + \|y(t)\|^2 + \|f(y(t)) + d(t)\|^2 \quad (5.3.5)$$

for all $t \in \mathbb{N}_0$ and for all $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$. We now use assumption (5.3.2) as well as the simple estimate $\|\xi_1 + \xi_2\|^2 \leq 2\|\xi_1\|^2 + 2\|\xi_2\|^2$ to arrive at $\|\xi\|^2 + \|f(\xi) + \mu\|^2 \leq (4\|K\|^2 + 4r^2 + 1)\|\xi\|^2 + 2\|\mu\|^2$ for all $\xi \in \mathbb{F}^p, \mu \in \mathbb{F}^m$. If we set $c_1 := 4\|K\|^2 + 4r^2 + 1$ and $\delta_1 := \frac{\delta}{\max\{c_1, 2\}}$, then equation (5.3.5) implies that the function $U_2 := \frac{1}{\max\{c_1,2\}}U_1$ satisfies

$$U_2(x(t+1)) - U_2(x(t)) \leq -\delta_1\|x(t)\|^2 + \|y(t)\|^2 + \|d(t)\|^2 \qquad (5.3.6)$$

for all $t \in \mathbb{N}_0$ and for all $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$.

We will now construct a function W, using U_2 so that $V + W$ is an ISS-Lyapunov function. Thus we want to find W such that there exist $\eta_1, \gamma_1 \in \mathcal{K}_\infty$ that satisfy

$$W(x(t+1)) - W(x(t)) \leq -\eta_1(\|x(t)\|) + \alpha_1(\|y(t)\|) + \gamma_1(\|d(t)\|),$$

for all $t \in \mathbb{N}_0$ and for all $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$.

Let us now define W. Pick $c_2 \geq \delta_1$ such that $U_2(\xi) \leq c_2 \|\xi\|^2$ for all $\xi \in \mathbb{F}^n$ and choose $b > 1$ such that $a := b\left(1 - \frac{\delta_1}{2c_2}\right) < 1$. Note that $a \geq b/2 > 0$. Now, by Lemma 5.1.12, there exists $\eta \in \mathcal{K}_\infty$ such that

$$\alpha_1(s_1 - s_2) \leq \alpha_1(bs_1) - \eta(s_2) \tag{5.3.7}$$

for all $s_1 \geq s_2 \geq 0$. Furthermore, Lemma 5.1.8 guarantees the existence of $k > 1$ such that

$$\alpha(as_1 + s_2) \leq \alpha_1(s_1) + \alpha_1(ks_2) \tag{5.3.8}$$

for all $s_1, s_2 \geq 0$. We now set $c := \frac{1}{2bk}$ and define $W\colon \mathbb{F}^n \to [0,\infty)$ by $W(\xi) := \alpha_1\left(cU_2(\xi)\right)$. The use of estimates (5.3.6) and (5.3.7) gives us

$$\begin{aligned}
W(x(t+1)) =& \alpha_1\left(cU_2(x(t+1))\right)\\
\leq& \alpha_1\left(c\left[U_2(x(t)) - \delta_1\|x(t)\|^2 + \|y(t)\|^2 + \|d(t)\|^2\right]\right)\\
\leq& \alpha_1\left(bc\left[U_2(x(t)) - \delta_1\|x(t)\|^2/2 + \|y(t)\|^2 + \|d(t)\|^2\right]\right)\\
&- \eta\left(c\delta_1\|x(t)\|^2/2\right)
\end{aligned}$$

for all $t \in \mathbb{N}_0$ and for all $(d,x,y) \in \mathcal{B}_{\mathrm{d}}(A,B,C,D,f)$. By our choice of a, we see that $b\left(U_2(\xi) - \frac{\delta_1}{2}\|\xi\|^2\right) \leq aU_2(\xi)$ for all $\xi \in \mathbb{F}^n$. For $\delta_2 := c\delta_1/2$, this implies that

$$W(x(t+1)) \leq \alpha_1\left(acU_2(x(t)) + bc\|y(t)\|^2 + bc\|d(t)\|^2\right) - \eta\left(\delta_2\|x(t)\|^2\right) \tag{5.3.9}$$

for all $t \in \mathbb{N}_0$ and for all $(d,x,y) \in \mathcal{B}_{\mathrm{d}}(A,B,C,D,f)$. The use of estimate (5.3.8) as well as $\alpha_1(s_1+s_2) \leq \alpha_1(2s_1 \vee 2s_2) \leq \alpha_1(2s_1) + \alpha_1(2s_2)$ for all $s_1, s_2 \geq 0$ gives us

$$\begin{aligned}
&\alpha_1\left(acU_2(x(t)) + bc\|y(t)\|^2 + bc\|d(t)\|^2\right)\\
&\qquad\leq \alpha_1(cU_2(x(t))) + \alpha_1\left(bck\|y(t)\|^2 + bck\|d(t)\|^2\right)\\
&\qquad\qquad\leq W(x(t)) + \alpha_1(\|y(t)\|^2) + \alpha_1(\|d(t)\|^2)
\end{aligned}$$

for all $t \in \mathbb{N}_0$ and for all $(d,x,y) \in \mathcal{B}_{\mathrm{d}}(A,B,C,D,f)$. In combination with (5.3.9), this yields

$$W(x(t+1)) - W(x(t)) \leq -\eta(\delta_2\|x(t)\|^2) + \alpha_1(\|y(t)\|^2) + \alpha_1(\|d(t)\|^2)$$

for all $t \in \mathbb{N}_0$ and for all $(d,x,y) \in \mathcal{B}_{\mathrm{d}}(A,B,C,D,f)$. This and the estimate (5.3.4) shows that $V+W$ is an ISS-Lyapunov function for the Lur'e system (A,B,C,D,f), provided we prove that $V+W$ is $\mathcal{K}_\infty$-equivalent to $\|\cdot\|$. This is straightforward. Since V is a positive semi-definite quadratic form and U_2 is a positive definite quadratic form, there exist positive c_3, c_4 and c_5 such that $0 \le V(\xi) \le c_3 \|\xi\|^2$ and $c_4 \|\xi\|^2 \le U_2(\xi) \le c_5 \|\xi\|^2$ for all $\xi \in \mathbb{F}^n$. Thus

$$\alpha_1(cc_4 \|\xi\|^2) \le V(\xi) + W(\xi) \le c_3 \|\xi\|^2 + \alpha_1(cc_5 \|\xi\|^2)$$

for all $\xi \in \mathbb{F}^n$, which completes the proof. □

If $F \in \mathbb{B}_{\mathbb{C}}(K, r)$, then we can define $\alpha \in \mathcal{K}_\infty$ by $\alpha(s) := (r - \|F - K\|)s$ to see that Theorem 5.3.1 implies the Lur'e system (A, B, C, D, F) is ISS. Hence Theorem 5.3.1 can be interpreted as stating that if the Lur'e system (A, B, C, D, F) is ISS for *all complex linear* feedback matrices F such that $\|F\xi - K\xi\| < r\|\xi\|$, then the Lur'e system (A, B, C, D, f) is ISS for *all nonlinear* output feedback maps f such that $\|f(\xi) - K\xi\| < r\|\xi\| - \alpha(\|\xi\|)$ for some $\alpha \in \mathcal{K}_\infty$. Thus Theorem 5.3.1 is an Aizerman-like result, but for input-to-state stability.

By picking $K = 0$ in Theorem 5.3.1 and by using Lemma 3.2.7 to see that $\mathbb{B}_{\mathbb{C}}(0, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ is equivalent to $\|G\|_{H^\infty} \le \frac{1}{r}$, we obtain the following corollary, which can be seen as a small-gain version of Theorem 5.3.1.

Corollary 5.3.2. *Consider a Lur'e system (A, B, C, D, f), assume that the underlying state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable and let its transfer function by G be such that there exists $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that $\|G(z_0)\| < \|G\|_{H^\infty}$.*

If there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\|G\|_{H^\infty} \|f(\xi)\| \le \|\xi\| - \alpha(\|\xi\|) \qquad \forall\, \xi \in \mathbb{F}^p,$$

then the Lur'e system (A, B, C, D, f) is ISS.

Note that the assumptions in Theorem 5.3.1 on α cannot be relaxed as illustrated by the following example.

Example 5.3.3. Define $\alpha \in \mathcal{K} \setminus \mathcal{K}_\infty$ by $\alpha(s) := \frac{1}{2} - e^{-s}$ and $f\colon \mathbb{R} \to \mathbb{R}$ by $f(\xi) := \frac{1}{2}\xi - \operatorname{sgn}(\xi)\alpha(|\xi|)$. Note that the linear state-space system $(\frac{1}{2}, 1, 1, 0) \in \Sigma(1, 1, 1; \mathbb{R})$ has transfer function $g(z) = \frac{1}{z - \frac{1}{2}}$, so that $\|g\|_{H^\infty} = 2$ and therefore $\mathbb{B}_{\mathbb{C}}(0, \frac{1}{2}) \subseteq \mathbb{S}_{\mathbb{C}}(g)$. Also, one can check that $\|g(i)\| < 2$. However, there does not exist $\alpha_1 \in \mathcal{K}_\infty$ such that $|f(\xi)| \le \frac{1}{2}|\xi| - \alpha_1(|\xi|)$ and hence we cannot apply Theorem 5.3.1. Moreover, the Lur'e system $(\frac{1}{2}, 1, 1, 0, f)$ is described by

$$x(t+1) = \frac{1}{2}x(t) + f(x(t)) + d(t),$$

so that, if we pick $d(t) = 1 + \varepsilon$ for some positive ε, then $x(t+1) \ge x(t) + \varepsilon$. Hence $(\frac{1}{2}, 1, 1, 0, f)$ is not ISS and demonstrates that the assumption (5.3.2) cannot be relaxed to allow $\alpha \in \mathcal{K}$.

We note that the assumptions on the underlying linear state-space system in Theorem 5.3.1 are identical to the ones in the Aizerman version of the circle criterion (Proposition 4.2.1). On the other hand, the assumption on the nonlinearity, (5.3.2), is stronger than assumption (4.2.2) in (b) from Proposition 4.2.1, which guarantees global asymptotic stability of (A, B, C, D, f), yet weaker than assumption (4.2.3) in (c) from Proposition 4.2.1, which guarantees global exponential stability of (A, B, C, D, f). In particular, the latter observation means that the conditions on (A, B, C, D, f) that guarantee global exponential stability are sufficient for input-to-state stability. Moreover, we shall see in §5.5 that these assumptions guarantee an even stronger version of stability that we will call exponential input-to-state stability.

Sector-bounded nonlinearities f have been considered in previous work on ISS for continuous-time SISO Lur'e systems, see Theorem 17 from Jayawardhana, Logemann and Ryan [29]. The following corollary of Theorem 5.3.1 is an extension of the discrete-time counterpart of Theorem 17 from [29] (to be precise, it is an extension of Theorem 17 under hypothesis (H1), where we borrow a label from [29]).

Corollary 5.3.4. *Consider a real SISO Lur'e system (A, b, c, d, f), assume that the underlying linear system is stabilizable and detectable and denote its transfer function by g. Let $k_1 < k_2$, assume that $k_1 \neq d^{-1}$ and that $\frac{1-k_2 g}{1-k_1 g}$ is positive real. Moreover, assume that, for $k := \frac{1}{2}(k_1 + k_2)$, there exists $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that $\left|g^k(z_0)\right| < \left\|g^k\right\|_{H^\infty}$.*

If there exists $\alpha \in \mathcal{K}_\infty$ such that

$$k_1\xi^2 + \xi\alpha(|\xi|) \leq f(\xi)\xi \leq k_2\xi^2 - \xi\alpha(|\xi|) \qquad \forall\, \xi \in \mathbb{R}, \tag{5.3.10}$$

then the Lur'e system (A, b, c, d, f) is ISS.

Proof. Let us set $r := \frac{1}{2}(k_2 - k_1) > 0$. Note that (5.3.10) implies $\xi\alpha(|\xi|) \leq r\xi^2$ or, equivalently, $\alpha(|\xi|) \leq r|\xi|$ for all $\xi \in \mathbb{R}$. Moreover, the assumption (5.3.10) can be rewritten as

$$-r\xi^2 + \xi\alpha(|\xi|) \leq (f(\xi) - k\xi)\xi \leq r\xi^2 - \xi\alpha(|\xi|) \qquad \forall \xi \in \mathbb{R}.$$

Therefore, $|f(\xi) - k\xi||\xi| \leq (r|\xi| - \alpha(|\xi|))\,|\xi|$, so that

$$|f(\xi) - k\xi| \leq r|\xi| - \alpha(|\xi|) \qquad \forall\, \xi \in \mathbb{R}.$$

On the other hand,

$$\frac{1 - k_2 g}{1 - k_1 g} = 1 - 2rg^{-k_1} = 1 - 2rg^{k-r}$$

and since $k_1 \neq d^{-1}$, we have $k_1 = k - r \in \mathbb{A}_{\mathbb{C}}(d)$, so that, by Proposition 3.2.12, $\mathbb{B}_{\mathbb{C}}(k, r) \subseteq \mathbb{S}_{\mathbb{C}}(g)$. Hence we can apply Theorem 5.3.1 to see that (A, b, c, d, f) is indeed ISS. $\square$

The sector condition (5.3.10) admits the intuitively appealing visualization depicted in Figure 5.1. In the figure we have picked $\alpha(s) := \min\{s, \sqrt{s}\}$, which clearly increases sublinearly. There are numerous characterizations of positive realness, some of which can be especially simple and sometimes - for a given linear state-space system - can be determined experimentally, see e.g. Lemma 6.1 and the discussion in §7.1 from Kailath [34].

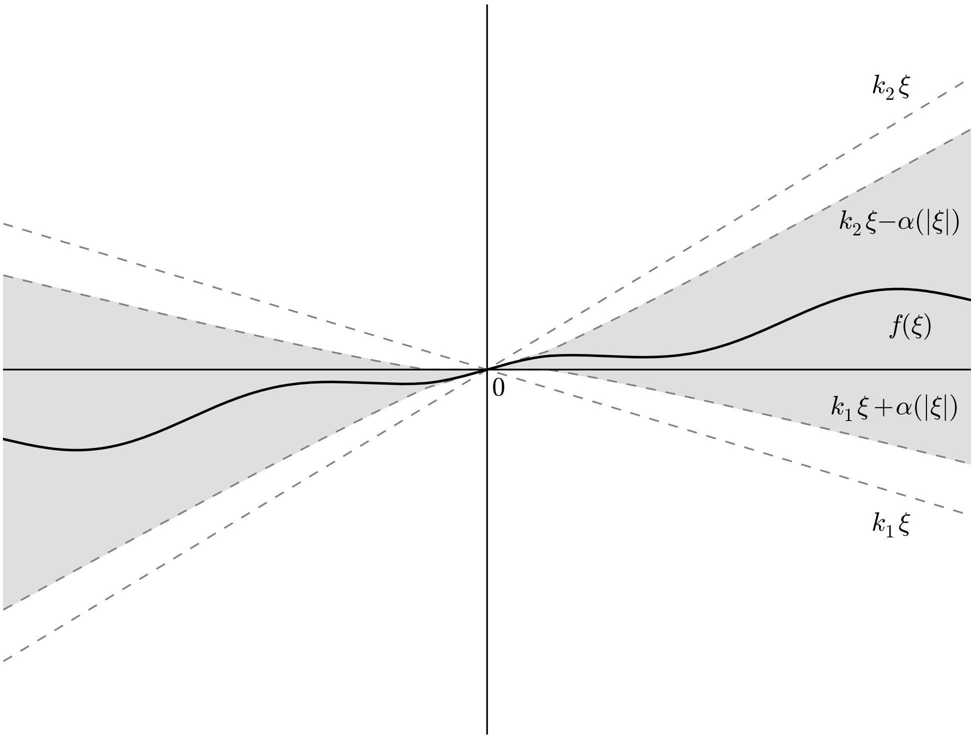

Figure 5.1: A sector bounded nonlinearity

We note that, by assuming that the underlying linear system is controllable and observable, we can drop an assumption on the transfer function in Theorem 5.3.1. We omit the proof of the following result as it is identical to the proof of Theorem 5.3.1, except that it uses Lemma 3.2.9 instead of Lemma 3.2.8.

Proposition 5.3.5. *Consider a Lur'e system (A, B, C, D, f), assume that the underlying state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable and let its transfer function G - for some $r > 0$ and $K \in \mathbb{F}^{m \times p}$ - satisfy $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.*

If there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\|f(\xi) - K\xi\| \leq r\,\|\xi\| - \alpha(\|\xi\|) \qquad \forall\, \xi \in \mathbb{F}^p,$$

then the Lur'e system (A, B, C, D, f) *is ISS.*

5.4 Positive real assumptions

A corollary of Proposition 5.3.5 is a result that makes assumptions similar to the ones in the circle criterion, Proposition 4.3.2.

Proposition 5.4.1. *Consider a Lur'e system* (A, B, C, D, f) *and assume that the underlying linear system* $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ *is controllable and observable and denote its transfer function by* G. *Let* $K_1, K_2 \in \mathbb{F}^{m\times p}$ *and assume that* $K_1 \in \mathbb{A}_{\mathbb{C}}(D)$ *and that* $(I - K_2G)(I - K_1G)^{-1}$ *is positive real.*

If there exists $\alpha \in \mathcal{K}_\infty$ *such that*

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \leq -\|\xi\|\,\alpha(\|\xi\|) \qquad \forall\, \xi \in \mathbb{F}^p, \tag{5.4.1}$$

then the Lur'e system (A, B, C, D, f) *is ISS.*

Proof. We note that we will have to repeat some computations from the proof of Proposition 4.3.2.

By rewriting K_1 and K_2 in terms of K and L, we obtain

$$\begin{aligned}
&\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \\
&\qquad = \operatorname{Re}\langle f(\xi) - (K+L)\xi, f(\xi) - (K-L)\xi\rangle \\
&\qquad = \|f(\xi) - K\xi\|^2 + \operatorname{Re}\langle f(\xi) - K\xi, L\xi\rangle \\
&\qquad\quad - \operatorname{Re}\langle L\xi, f(\xi) - K\xi\rangle - \|L\xi\|^2 \\
&\qquad = \|f(\xi) - K\xi\|^2 - \|L\xi\|^2 \qquad\qquad \forall\, \xi \in \mathbb{F}^p.
\end{aligned} \tag{5.4.2}$$

Note that in conjunction with equation (5.4.1) this implies $\ker L = \{0\}$. Thus L^*L is invertible and $L^\sharp := (L^*L)^{-1}L^* \in \mathbb{F}^{p\times m}$ is a left inverse of L.

We can check that

$$\begin{aligned}
(I - K_2G)(I - K_1G)^{-1} &= (I - K_1G + 2LG)(I - K_1G)^{-1} \\
&= I + 2LG^{K_1},
\end{aligned}$$

so that, by Lemma 2.2.10, we have $\left\|-LG^{K_1}(I + LG^{K_1})^{-1}\right\|_{H^\infty} \leq 1$. On the other hand, $-LG^{K_1}(I + LG^{K_1})^{-1} = L(-L^\sharp L)G^{K_1}(I - L(-L^\sharp L)G^{K_1})^{-1} = (LG^{K_1})^{-LL^\sharp}$. Hence a use of Proposition 3.2.12 implies that $\mathbb{B}_{\mathbb{C}}(-LL^\sharp, 1) \subseteq \mathbb{S}_{\mathbb{C}}\left(LG^{K_1}\right)$. This suggests considering a state-space system that has transfer function LG^{K_1}.

Consider $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$, so that, by Lemma 3.1.4, we have

$$\begin{aligned} x(t+1) &= A_{K_1}x(t) + B_{K_1}(f(y(t)) - K_1 y(t) + d(t)) \\ y(t) &= C_{K_1}x(t) + D_{K_1}(f(y(t)) - K_1 y(t) + d(t)) \end{aligned}$$

for all $t \in \mathbb{N}_0$. Left-multiplication by L of the output equation and the use of $I = L^\sharp L$ gives us

$$\begin{aligned} x(t+1) &= A_{K_1}x(t) + B_{K_1}(f(L^\sharp Ly(t)) - K_1 L^\sharp Ly(t) + d(t)) \\ Ly(t) &= LC_{K_1}x(t) + LD_{K_1}(f(L^\sharp Ly(t)) - K_1 L^\sharp Ly(t) + d(t) \end{aligned}$$

for all $t \in \mathbb{N}_0$. Define $g : \mathbb{F}^m \to \mathbb{F}^m$ by $g(\xi) := f(L^\sharp \xi) - K_1 L^\sharp \xi$. Then we can see that $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$ if, and only if, $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A_{K_1}, B_{K_1}, LC_{K_1}, LD_{K_1}, g)$. Since L is left-invertible, it thus suffices to show that the Lur'e system $(A_{K_1}, B_{K_1}, LC_{K_1}, LD_{K_1}, g)$ is ISS.

We have already checked that $\mathbb{B}_{\mathbb{C}}(-LL^\sharp, 1) \subseteq \mathbb{S}_{\mathbb{C}}\left(LG^{K_1}\right)$ and Lemma 3.1.4 together with an application of the Hautus tests for stabilizability and detectability shows that $(A_{K_1}, B_{K_1}, LC_{K_1}, LD_{K_1})$ (which is clearly a realization of LG^{K_1}) is stabilizable and detectable.

Thus we are only left with checking that the relevant inequality holds for g. The use of the definition of g together with equation (5.4.2) shows us that

$$\begin{aligned} \left\| g(\xi) + LL^\sharp \xi \right\|^2 &= \left\| f(L^\sharp \xi) - K_1 L^\sharp \xi + LL^\sharp \xi \right\|^2 \\ &= \left\| f(L^\sharp \xi) - KL^\sharp \xi \right\|^2 \\ &= \operatorname{Re}\left\langle f(L^\sharp \xi) - K_1 L^\sharp \xi, f(L^\sharp \xi) - K_2 L^\sharp \xi \right\rangle + \left\| LL^\sharp \xi \right\|^2 \\ &\leq -\alpha\left(\left\| L^\sharp \xi \right\|\right)\left\| L^\sharp \xi \right\| + \left\| LL^\sharp \xi \right\|^2 \end{aligned}$$

for all $\xi \in \mathbb{F}^m$. By Lemma 5.1.14, we know that there exists $\alpha_1 \in \mathcal{K}_\infty$ with $\alpha_1 < id$ on $(0, \infty)$ such that

$$\left\| LL^\sharp \xi \right\|^2 - \alpha\left(\left\| L^\sharp \xi \right\|\right)\left\| L^\sharp \xi \right\| \leq (\|\xi\| - \alpha_1(\|\xi\|))^2$$

for all $\xi \in \mathbb{F}^m$. Thus

$$\left\| g(\xi) + LL^\sharp \xi \right\| \leq \|\xi\| - \alpha_1(\|\xi\|)$$

for all $\xi \in \mathbb{F}^m$ and hence an application of Proposition 5.3.5 implies that the Lur'e system $(A_{K_1}, B_{K_1}, LC_{K_1}, LD_{K_1}, g)$ is ISS, which in turn implies that the Lur'e system (A, B, C, D, f) is ISS. This completes the proof. $\square$

As before, one can obtain 5.4.1, when the underlying linear system is stabilizable and detectable, but then an additional assumption has to be made, namely that - for $H := (I - K_2 G)(I - K_1 G)^{-1}$ - there exists $|z_0| = 1$ such that $H(z_0) + H(z_0)^* > 0$.

We note that the assumptions on the linear state-space system in Proposition 5.4.1 are identical to the ones in the circle criterion, Proposition 4.3.2. On the other hand, the assumption on the nonlinearity, (5.4.1), is stronger than in Proposition 4.3.2 (b), which guarantees global asymptotic stability of (A, B, C, D, f), yet weaker than in Proposition 4.3.2 (c), which guarantees global exponential stability of (A, B, C, D, f). In particular, we see that assumptions that guarantee global exponential stability in Proposition 4.3.2 also guarantee input-to-state stability and, as we shall see in §5.5, even exponential input-to-state stability.

If we pick $K_1 = 0$, then we obtain a simple corollary of Proposition 5.4.1. It is interesting to note that it is a discrete-time version of Theorem 3.5 from [29] (more precisely, it is a version of Theorem 3.5 under hypothesis (H3), where we borrow the label from [30]). We make a more precise comparison in continuous-time, see Corollary 9.3.3.

Corollary 5.4.2. *Consider a Lur'e system (A, B, C, D, f) and assume that the underlying linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is controllable and observable and denote its transfer function by G. Let $K \in \mathbb{F}^{m \times p}$ and assume that $I - K_2 G$ is positive real.*

If there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\operatorname{Re}\langle f(\xi), f(\xi) - K_2 \xi \rangle \leq - \|\xi\| \, \alpha(\|\xi\|) \qquad \forall \, \xi \in \mathbb{F}^p,$$

then the Lur'e system (A, B, C, D, f) is ISS.

Finally, we note that we can use Proposition 5.4.1 to obtain an alternative proof of Corollary 5.3.4.

Corollary 5.4.3. *Consider a SISO Lur'e system (A, b, c, d, f) and assume that the underlying linear system $(A, b, c, d) \in \Sigma(1, n, 1; \mathbb{R})$ is controllable and observable and denote its transfer function by g. Let $k_1 < k_2$, assume that $k_1 \neq d^{-1}$ and that $\frac{1-k_2 g}{1-k_1 g}$ is positive real.*

If there exists $\alpha \in \mathcal{K}_\infty$ such that

$$k_1 \xi^2 + \xi \alpha(|\xi|) \leq f(\xi)\xi \leq k_2 \xi^2 - \xi \alpha(|\xi|) \qquad \forall \, \xi \in \mathbb{R},$$

then the Lur'e system (A, b, c, d, f) is ISS.

Proof. Let us set $k := \frac{1}{2}(k_1 + k_2)$ and $r := \frac{1}{2}(k_2 - k_1) > 0$. Then, as in the proof of Corollary 5.3.4, we have $\alpha(|\xi|) \leq r|\xi|$ and

$$|f(\xi) - k\xi| \leq r|\xi| - \alpha(|\xi|) \qquad \forall \, \xi \in \mathbb{R}.$$

If we use this and equation (5.4.2), then we obtain

$$\begin{aligned}\operatorname{Re}\langle f(\xi)-k_1\xi, f(\xi)-k_2\xi\rangle &= |f(\xi)-k\xi|^2 - r^2|\xi|^2 \\ &\le -2r|\xi|\alpha(|\xi|)+\alpha(|\xi|)^2 \\ &\le -r|\xi|\alpha(|\xi|) \qquad \forall\, \xi\in\mathbb{R}.\end{aligned}$$

Hence we can apply Proposition 5.4.1 to infer that (A, b, c, d, f) is ISS. □

5.5 Exponential ISS

In this section we note that exponential weighting arguments allow us to prove that if we strengthen the assumptions of Theorem 5.3.1 by picking $\alpha(s) = \delta s$, for some positive δ, then we obtain a stronger version of stability, which we will call exponential input-to-state stability. It is also interesting to note that in contrast to most ISS-related results, this can be proved without ISS-Lyapunov function techniques. Therefore, these results might generalize to the infinite-dimensional setting.

Definition 5.5.1. Consider a Lur'e system (A, B, C, D, f), where the underlying linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$. It is said to be (globally) **exponentially input-to-state stable** if there exist $c_1, c_2 > 0$ and $a \in (0, 1)$ such that for all $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$ with $d \in l^\infty(\mathbb{F}^m)$ we have

$$\|x(t)\| \le c_1 a^t \|x(0)\| + c_2 \|d\|_\infty \qquad \forall\, t\in\mathbb{N}_0.$$

We define a family of operators $(\pi_T)_{T\in\mathbb{N}_0}$ on $(\mathbb{F}^m)^{\mathbb{N}_0}$ as

$$(\pi_T u)(t) := \begin{cases} u(t) & \text{if } t\le T \\ 0 & \text{otherwise.}\end{cases}$$

For $u \in (\mathbb{F}^m)^{\mathbb{N}_0}$, we define $\|u\|_2 := \sqrt{\sum_{j=0}^{\infty}\|u(j)\|^2}$ and we denote by $l^2(\mathbb{F}^m)$ the set of all $u \in (\mathbb{F}^m)^{\mathbb{N}_0}$ such that $\|u\|_2 < \infty$. It is well-known that $l^2(\mathbb{F}^m)$ with the norm $\|\cdot\|_2$ is a Banach space.

The following result will be important for us.

Lemma 5.5.2. *Consider a stabilizable and detectable linear state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and assume that its transfer function, G, is in H^∞. Then there exists $c > 0$ such that*

$$\|\pi_t y\|_2 \le c\|x(0)\| + \|G\|_{H^\infty}\|\pi_t u\|_2$$

for all $t \in \mathbb{N}_0$ and for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Proof. Let us define an input-output map $\mathcal{G}\colon (\mathbb{F}^m)^{\mathbb{N}_0} \to (\mathbb{F}^p)^{\mathbb{N}_0}$ by

$$
\begin{aligned}
(\mathcal{G}u)(0) &= Du(0) \\
(\mathcal{G}u)(t) &= \sum_{j=0}^{t-1} CA^{t-1-j}Bu(j) + Du(t), \qquad \text{for } t \in \mathbb{N}.
\end{aligned}
$$

It is well-known that the restriction of $\mathcal{G}$ to $l^2(\mathbb{F}^m)$ is a bounded map with respect to the l^2 norms. Moreover, if we denote by $\|\cdot\|_{2,2}$ the l^2-norm induced operator norm, then $\|\mathcal{G}\|_{2,2} = \|G\|_{H^\infty}$, see e.g. Theorem 2.3.28 from Hinrichsen and Pritchard [25].

Now note that if $(u, x, y) \in \mathcal{B}(A, B, C, D)$, then $y(t) = CA^t x(0) + (\mathcal{G}u)(t)$. Therefore, if we define $w \in (\mathbb{F}^p)^{\mathbb{N}_0}$ by $w(t) := CA^t x(0)$, then $\pi_t y = \pi_t w + \mathcal{G}(\pi_t u)$ for all $t \geq 0$. Since (A, B, C, D) is stabilizable and detectable, and since $G \in H^\infty$, by Lemma 3.2.2, we have $\sigma(A) \subseteq \mathbb{D}$, so that we can define $c := \sqrt{\sum_{j=0}^{\infty} \|CA^t\|^2} < \infty$. This gives us the required estimate:

$$
\begin{aligned}
\|\pi_t y\|_2 &\leq \|\pi_t w\|_2 + \|\mathcal{G}\|_{2,2} \|\pi_t u\|_2 \\
&\leq c\,\|x(0)\| + \|G\|_{H^\infty} \|\pi_t u\|_2 .
\end{aligned}
$$
□

We use this result and the exponential weighting technique to obtain the following result. Note that the assumptions are the same as in Theorem 5.3.1, except that we pick $\alpha(s) := \delta s$ and do not require there to exist $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that $\|G^K(z_0)\| < \frac{1}{r}$.

Proposition 5.5.3. *Consider a Lur'e system (A, B, C, D, f), assume that the underlying state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable and denote its transfer function by G. Furthermore, let $r > 0$, $K \in \mathbb{F}^{m\times p}$ and assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.*

If, for some $\delta > 0$,

$$
\|f(\xi) - K\xi\| \leq (r - \delta)\|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p, \tag{5.5.1}
$$

then the Lur'e system (A, B, C, D, f) is exponentially ISS.

Proof. By Lemma 3.2.7, we have $\|G^K\|_{H^\infty} \leq \frac{1}{r}$, so, by the continuity of entries of G, there exists $s > 1$ small enough, so that $\sup_{z\in s^{-1}\mathbb{E}} \|G^K(z)\| < \frac{1}{r-\delta}$. If we set $G^K_s(z) := G^K(\frac{z}{s})$, then, equivalently, $\|G^K_s\|_{H^\infty} < \frac{1}{r-\delta}$. Note that, if we define (A_K, B_K, C_K, D_K) by (3.1.3), then G^K_s is the transfer function of (sA_K, sB_K, C_K, D_K), which - via the use of the Hautus test - is easily seen to be stabilizable and detectable. Therefore, by Lemma 5.5.2, there exists a positive c_1 such that

$$
\|\pi_t y\|_2 \leq c_1 \|x(0)\| + \|G^K_s\|_{H^\infty} \|\pi_t u\|_2 \tag{5.5.2}
$$

for all $t \in \mathbb{N}_0$ and for all $(u, x, y) \in \mathcal{B}(sA_K, sB_K, C_K, D_K)$.

Now let us pick an arbitrary $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$. Then $(f \circ y + d, x, y) \in \mathcal{B}(A, B, C, D)$ and hence, by Lemma 3.1.4,

$$\begin{aligned} x(t+1) &= A_K x(t) + B_K(f(y(t)) - Ky(t) + d(t)) \\ y(t) &= C_K x(t) + D_K(f(y(t)) - Ky(t) + d(t)). \end{aligned}$$

If we multiply the first equation by s^{t+1}, the second one by s^t and define $f_s \in (\mathbb{F}^m)^{\mathbb{N}_0}$, $d_s \in (\mathbb{F}^m)^{\mathbb{N}_0}$, $x_s \in (\mathbb{F}^n)^{\mathbb{N}_0}$ and $y_s \in (\mathbb{F}^p)^{\mathbb{N}_0}$ by $f_s(t) := s^t f(y(t))$, $d_s(t) := s^t d(t)$, $x_s(t) := s^t x(t)$ and $y_s(t) := s^t y(t)$, then

$$\begin{aligned} x_s(t+1) &= (sA_K)x_s(t) + (sB_K)(f_s(t) - Ky_s(t) + d_s(t)) \\ y_s(t) &= C_K x_s(t) + D_K(f_s(t) - Ky_s(t) + d_s(t)), \end{aligned}$$

so that $(f_s - Ky_s + d_s, x_s, y_s) \in \mathcal{B}(sA_K, sB_K, C_K, D_K)$. Hence, by (5.5.2) and the triangle inequality, we have

$$\begin{aligned} \|\pi_t y_s\|_2 &\le c_1 \|x_s(0)\| + \left\|G_s^K\right\|_{H^\infty} \|\pi_t(f_s - Ky_s + d_s)\|_2 \\ &\le c_1 \|x_s(0)\| + \left\|G_s^K\right\|_{H^\infty} \|\pi_t(f_s - Ky_s)\|_2 + \left\|G_s^K\right\|_{H^\infty} \|\pi_t d_s\|_2 \end{aligned}$$

for all $t \in \mathbb{N}_0$. By assumption (5.5.1), we have $\|\pi_t(f_s - Ky_s)\|_2 \le (r - \delta)\|\pi_t y_s\|_2$, whence if we set $c_2 := \frac{c_1}{1 - \|G_s^K\|_{H^\infty}(r-\delta)}$ and $c_3 := \frac{\|G_s^K\|_{H^\infty}}{1 - \|G_s^K\|_{H^\infty}(r-\delta)}$, then

$$\|\pi_t y_s\|_2 \le c_2 \|x_s(0)\| + c_3 \|\pi_t d_s\|_2 \tag{5.5.3}$$

for all $t \in \mathbb{N}_0$.

Note that, since (sA_K, sB_K, C_K, D_K) is stabilizable, detectable and since its transfer function G_s^K is in H^∞, we have $\sigma(sA_K) \subseteq \mathbb{D}$ and hence we can set $c_4 := \sup_{t \in \mathbb{N}_0} \left\|(sA_K)^t\right\| < \infty$ and $c_5 := \sqrt{\sum_{j=0}^\infty \|(sA_K)^j\|^2} < \infty$. Now, since $(f_s - Ky_s + d_s, x_s, y_s) \in \mathcal{B}(sA_K, sB_K, C_K, D_K)$, we have $x_s(t) = (sA_K)^t x_s(0) + \sum_{j=0}^{t-1}(sA_K)^{t-1-j}(sB_K)\left(f_s(j) - Ky_s(j) + d_s(j)\right)$ for all $t \in \mathbb{N}_0$. Therefore, we can use the definitions of f_s, y_s, d_s, the triangle inequality, assumption (5.5.1) and Hölder's inequality to obtain

$$\begin{aligned} \|x_s(t)\| &= \left\|(sA_K)^t x_s(0) + \sum_{j=0}^{t-1}(sA_K)^{t-1-j}(sB_K)s^j(f(y(j)) - Ky(j) + d(j))\right\| \\ &\le c_4 \|x_s(0)\| + \sum_{j=0}^{t-1} \left\|(sA_K)^{t-1-j}(sB_K)\right\| \left(\left\|s^j r y(j)\right\| + \left\|s^j d(j))\right\|\right) \\ &\le c_4 \|x_s(0)\| + \|sB_K\| \sum_{j=0}^{t-1} \left\|(sA_K)^{t-1-j}\right\| (r\|y_s(j)\| + \|d_s(j)\|) \\ &\le c_4 \|x_s(0)\| + \|sB_K\| c_5(r\|\pi_{t-1}y_s\|_2 + \|\pi_{t-1}d_s\|_2). \end{aligned}$$

Since $x_s(0) = x(0)$ and $\sigma(sA_K) \subseteq \mathbb{D}$, the use of (5.5.2) shows us that, if we set $c_6 := c_4 + \|sB_K\| c_5 r c_2$ and $c_7 := \|sB_K\| c_5(1 + rc_3)$, then

$$\|x_s(t)\| \leq c_6 \|x_s(0)\| + c_7 \|\pi_{t-1} d_s\|_2 . \tag{5.5.4}$$

We now set $c_8 := \sqrt{\frac{s^{-2}}{1-s^{-2}}}$ and estimate the l^2 norm of d_s as

$$\begin{aligned} s^{-t} \|\pi_{t-1} d_s\|_2 &= \sqrt{\sum_{j=0}^{t-1} s^{2j-2t} \|d(j)\|^2} \\ &\leq \|d\|_\infty \sqrt{\sum_{i=1}^{\infty} s^{-2i}} \qquad = c_8 \|d\|_\infty . \end{aligned}$$

Finally, by using the definition of x_s and (5.5.4), we obtain

$$\|x(t)\| \leq c_6 s^{-t} \|x(0)\| + c_7 c_8 \|d\|_\infty$$

for all $t \in \mathbb{N}_0$. Since the constants c_6, c_7, c_8 and s do not depend on the particular trajectory $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$, we conclude that the Lur'e system (A, B, C, D, f) is exponentially ISS. $\square$

It is interesting to note that if we pick $K = 0$ in Proposition 5.5.3, then it shows that the assumptions made in the small-gain theorem actually guarantee exponential ISS.

Corollary 5.5.4. *Consider a Lur'e system (A, B, C, D, f), assume that the underlying state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable, and denote its transfer function by G. If*

$$\|G\|_{H^\infty} \cdot \sup_{\xi \in \mathbb{F}^p \setminus 0} \frac{\|f(\xi)\|}{\|\xi\|} < 1,$$

then the Lur'e system (A, B, C, D, f) is exponentially ISS.

We can also obtain an exponential ISS adaptation of the "standard" version of circle criterion.

Corollary 5.5.5. *Consider a Lur'e system (A, B, C, D, f) and assume that the underlying linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable and denote its transfer function by G. Let $K_1, K_2 \in \mathbb{F}^{m \times p}$ and assume that $K_1 \in \mathbb{A}_{\mathbb{C}}(D)$ and that $(I - K_2 G)(I - K_1 G)^{-1}$ is positive real.*

If there exists $\delta > 0$ such that

$$\operatorname{Re} \langle f(\xi) - K_1 \xi, f(\xi) - K_2 \xi \rangle \leq -\delta \|\xi\|^2 \qquad \forall\, \xi \in \mathbb{F}^p, \tag{5.5.5}$$

then the Lur'e system (A, B, C, D, f) is exponentially ISS.

We omit the proof, as it is similar to the proof of Proposition 5.4.1, but we note that the key step in the proof is showing that, for any $\delta \geq 0$, there exists $\delta_1 \in (0,1)$ such that

$$\left\|LL^\sharp \xi\right\|^2 - \delta \left\|L^\sharp \xi\right\|^2 \leq (1-\delta_1)\left\|\xi\right\|^2 \qquad \forall \xi \in \mathbb{F}^m, \tag{5.5.6}$$

where $L := \frac{1}{2}(K_1 - K_2)$ is left invertible with left-inverse $L^\sharp = (L^*L)^{-1}L^*$ and $LL^\sharp$ is the orthogonal projection onto $\operatorname{im} L$. As in the proof of Lemma 5.1.14, an arbitrary $\xi \in \mathbb{F}^m$ can be decomposed as $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \operatorname{im} LL^\sharp = (\ker LL^\sharp)^\perp$ and $\xi_2 \in \ker LL^\sharp = \ker L^\sharp$. Hence there exists $c > 0$ such that $\left\|L^\sharp \xi\right\|^2 \geq c\left\|\xi_1\right\|^2$ for all $\xi \in \mathbb{F}^m$. Without loss of generality, $\delta c < 1$, whence

$$\begin{aligned}\left\|LL^\sharp \xi\right\|^2 - \delta \left\|L^\sharp \xi\right\|^2 &\leq \left\|\xi_1\right\|^2 - \delta c\left\|\xi_1\right\|^2 \\ &\leq (1-\delta c)\left\|\xi\right\|^2 \qquad \forall \xi \in \mathbb{F}^m,\end{aligned}$$

which shows that (5.5.6) holds with $\delta_1 := \delta c$.

Chapter 6

Notes, references and future work

6.1 Notes and references

The main result in Part I is Theorem 5.3.1, which shows that Lur'e systems with forcing are ISS under assumptions similar to ones made in results from absolute stability analysis and to the best of author's knowledge this result is new. Proving ISS from assumptions typical of absolute stability results is not a novel idea: this has been pursued in a continuous-time setting in Arcak and Teel [7], Jayawardhana, Logemann and Ryan [29, 30] and Bruin et al. [12], however we stress that Theorem 5.3.1 is more than just a discrete-time counterpart of known results. As indicated previously, we use the bounded real lemma instead of the positive real lemma for the construction of quadratic forms, which allows us to obtain stability results for different classes of systems. We also analyse multivariable systems with feedthrough. Part II of this thesis deals with continuous-time systems and we defer detailed comparisons with [7, 29, 30, 12] until then.

$\mathcal{K}_\infty$ results from §5.1 played a prominent role in proving Theorem 5.3.1 as they were used to establish crucial estimates in proving that a certain function was in fact an ISS-Lyapunov function for the Lur'e system at hand. While some of these results are standard, Lemmas 5.1.11 and 5.1.12 seem to be original results, and they seem to pave the way for a novel technique of constructing ISS-Lyapunov functions. We should remark that there are sources, which provide a comprehensive overview of comparison function results and techniques made accessible by them, see e.g. Kellett [35].

Proposition 5.5.3, which proves exponential ISS under the assumptions of the small-gain theorem seems to be new, however, its proof introduces no new techniques and a similar result was proved in the continuous-time setting in Jayawardhana et al. [30].

The main result in §4 is Proposition 4.2.1, which proves that a certain version of Aizerman's conjecture holds true. This viewpoint is inspired by Theorem 5.6.22 from Hinrichsen and Pritchard [25], which is the continuous-time version of Proposition 4.2.1 (b) for systems with no feedthrough. However, Proposition 4.2.1 (a) - (c) shows a transition of modes of stability as we change the assumptions on the nonlinearity while Examples 4.2.6, 4.2.7 and 4.2.8 show that this transition is, in a sense, conservative.

6.2 Future work

As mentioned in §6.1, we have proved new results on comparison functions, which, in essence, has provided us with a new way of constructing ISS-Lyapunov functions. It seems plausible that this construction could be applied to other absolute stability results, e.g. the Popov criterion. Bruin et al. [12] have already obtained a Popov-like criterion that guarantees ISS in a continuous-time setting. However, they seem to be using a classical Lur'e-Postnikov Lyapunov function, which is then shown to be an ISS-Lyapunov function for the system at hand under suitable assumptions. It would be interesting to see whether any of these assumptions could be relaxed by using the technique we used in the proof of Theorem 5.3.1, namely taking a quadratic form V and then composing it with another function h to construct part of an ISS-Lyapunov function.

In §5.5 we saw that, under the assumptions made in the small-gain theorem, we obtain a stronger version of ISS, namely exponential ISS. Moreover, we were able to prove the key result, Proposition 5.5.3, without the use of ISS-Lyapunov functions. Therefore, it seems plausible that a version of this result would hold in the infinite-dimensional setting.

In the statement of the bounded real lemma, we introduced the assumption that there exists $|z_0| = 1$ such that $\|G(z_0)\| < 1$. We have not found an example that would demonstrate the necessity of this assumption for the conclusions of Lemma 2.2.3 to hold. It would be interesting to determine whether it is needed for the bounded real lemma.

Finally, an obvious next step to take is to see whether we can obtain continuous-time counterparts of results in Part I. That is the content of Part II.

Part II

Stability of continuous-time Lur'e systems

In this part of the thesis we seek continuous-time counterparts to results from Part I. We will be concerned with continuous-time Lur'e systems with forcing: that is, systems with a linear state-space system (A, B, C, D) in the forward path and a static nonlinearity f alongside a forcing d in the feedback path

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(f(y(t)) + d(t)), \\ y(t) &= Cx(t) + D(f(y(t)) + d(t)), \end{aligned} \tag{7.1}$$

where the forcing d could represent e.g. a disturbance or a target trajectory.

Similarly as in Part I, we will first consider standard Lur'e systems, that is,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bf(y(t)), \\ y(t) &= Cx(t) + Df(y(t)), \end{aligned} \tag{7.2}$$

and obtain absolute stability results for them. Lur'e systems (7.2) have been studied extensively in the literature, see Haddad and Chellaboina [23], Vidyasagar [56] or Khalil [36] for textbook treatments. An overview of the area is presented in the survey article Liberzon [40], which collects almost 500 references on absolute stability theory. We should note that in the literature the nonlinearity f is often assumed to be time-variant, whereupon the same stability results as in the time-invariant case are obtained as long as assumptions on f are satisfied uniformly with respect to the time variable.

It is well-known that for Lur'e systems without feedthrough (that is, $D = 0$) a complexified version of Aizerman's conjecture holds. More precisely, Hinrichsen and Pritchard in [25] prove that if, for a given multivariable linear system $(A, B, C, 0)$, the Lur'e interconnection (7.2) is globally asymptotically stable for *all complex linear* output feedback maps F that satisfy the norm condition $\|F\xi\| < r\,\|\xi\|$ for some $r > 0$ and for all $\xi \in \mathbb{C}^p \setminus \{0\}$, then, in fact, the Lur'e interconnection is globally asymptotically stable for *all nonlinear* output feedback maps f that satisfy the same norm condition $\|f(\xi)\| < r\,\|\xi\|$ for all $\xi \in \mathbb{F}^p \setminus \{0\}$ (here $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, either of which is allowed to be the underlying field of the system (7.2)). We offer a slight extension of their results to systems with nonzero feedthrough, which demonstrates a transition of modes of stability as we change the assumptions on the nonlinearity f, see Proposition 8.2.1. It is then used to prove a version of the well-known circle criterion, see Proposition 8.4.1. As in the discrete-time setting, we use the bounded real lemma instead of (the more commonly used) positive real lemma.

After this, we will turn our attention to input-to-state stability of Lur'e systems with forcing (7.1). Recent developments demonstrate that, under slightly stronger assumptions than those made in results from absolute stability theory, we in fact obtain ISS. For example, a result sometimes called the positivity theorem (see Haddad and Chellaboina [23]) guarantees global asymptotic stability of the Lur'e system (7.2) as long as the transfer function of the underlying linear state-space system (A, B, C, D) is strictly positive real and the nonlinearity f satisfies $\langle f(\xi), \xi \rangle \leq 0$ for all $\xi \in \mathbb{R}^p$. Now, Arcak and Teel [7] obtain ISS of the Lur'e system (7.1) by strengthening the assumption on f to (i) $\langle f(\xi), \xi \rangle \leq -\|\xi\| \alpha(\|\xi\|)$ for some $\alpha \in \mathcal{K}_\infty$ and for all $\xi \in \mathbb{R}^p$, and (ii) $\langle f(\xi), \xi \rangle \geq \|f(\xi)\|$ for all $\|\xi\| \geq r$, where r is some positive constant.

Jayawardhana, Logemann and Ryan [29] prove a number of results, including one, which resembles the circle criterion (see e.g. [23, 36]). More precisely, if we set G to be the transfer function of (A, B, C, D), then they assume that, for some real numbers $a < b$ and $\delta > 0$, the rational function matrix $(I + bG)(I + aG)^{-1} + \delta I$ is positive real and that $G(I + aG)^{-1} \in H^\infty$. Upon adding an assumption that the nonlinearity satisfies $\langle a\xi - f(\xi), b\xi - f(\xi) \rangle \leq 0$, they obtain ISS of the Lur'e system (7.1).

Other notable approaches are Bruin et al. [12], who look to ensure ISS from assumptions typically seen in Popov's criterion and Yang et al. [63], who consider Lur'e descriptor systems and formulate their results in terms of linear matrix inequalities.

We will follow in the footsteps of the above work in obtaining ISS results from assumptions similar to those typically seen in absolute stability theory, however we will be taking an Aizerman's conjecture viewpoint. The main result in this part of the thesis, Theorem 9.2.1, states roughly that if, for a given linear state-space system (A, B, C, D), the Lur'e interconnection (7.1) is globally asymptotically stable for *all complex linear* output feedback maps F that satisfy the norm condition $\|F\xi\| < r \|\xi\|$ for some positive r and for all $\xi \in \mathbb{C}^p \setminus \{0\}$, then the Lur'e interconnection is input-to-state stable for *all nonlinear* output feedback maps f that, for some $\alpha \in \mathcal{K}_\infty$, satisfy the norm condition $\|f(\xi)\| \leq r \|\xi\| - \alpha(\|\xi\|)$ for all $\xi \in \mathbb{F}^p$. As a corollary we will obtain a result, which resembles the circle criterion and allows us to obtain a number of results from [29] as corollaries, see Proposition 9.3.1 and the subsequent Corollaries 9.3.2 and 9.3.3.

We should remark that much of the presentation in this part of the thesis mirrors its discrete-time counterparts. Therefore, we will at times omit proofs that do not require development of new techniques. Also, commentary will sometimes be brief, if the reasoning given in Part I is unchanged.

This part of the thesis is organized as follows. In Chapter 7 we collect all the preliminaries and it is close in content and presentation to Chapter 2 from Part I: we introduce linear systems, describe how to obtain quadratic forms from the bounded real lemma and output injection, and note a fact on ω-limit sets. However, in contrast to Chapter 2, we derive some results on differentiating functions arising from quadratic forms in §7.2. Also, we condense the continuous-time equivalent of Chapter 3 to §7.6 as the results and the proofs are similar. After this, we devote Chapter 8 to absolute stability results and the presentation is parallel to its discrete-time counterpart, Chapter 4: we introduce Lur'e systems in §8.1 and then proceed to obtain stability criteria from ball condition assumptions in §8.2 and positive-real assumptions in §8.4. Chapter 9 is the main chapter in this part of the thesis as it deals with input-to-state stability of continuous-time Lur'e systems with forcing. Again, the presentation mirrors its discrete-time counterpart and we obtain ISS results from ball assumptions in §9.2 and from positive-real assumptions in §9.3.

Chapter 7

Preliminaries

In this chapter we collect the preliminaries required for the rest of Part II. In §7.1 we define linear state-space systems and their behaviours. After devoting §7.2 to results on differentiating functions that arise from quadratic forms we describe methods of constructing quadratic forms. In §7.3 we use the bounded real lemma and the positive real lemma, while §7.4 is devoted to the technique of "output injection". In §7.5 we introduce a slightly nonstandard notion of an ω-limit set for a function and prove one result on it. Then in §7.6 we state the continuous-time counterparts to results on linear output feedback from Chapter 3.

7.1 Linear state-space systems

Definition 7.1.1. We call a matrix quadruple $(A, B, C, D) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m} \times \mathbb{F}^{p\times n} \times \mathbb{F}^{p\times m}$ an **m-input, p-output linear state-space system**. The set of all linear systems of this format is denoted by $\Sigma(m, n, p; \mathbb{F})$.

We note that definitions of controllability and observability are unchanged in the continuous-time setting, we do need to amend the definitions of stabilizability and detectability however. If there exists $K \in \mathbb{F}^{m\times n}$ such that $\sigma(A + BK) \subseteq \mathbb{C}_-$, then we say that (A, B, C, D) is stabilizable. We say that (A, B, C, D) is detectable if (A^*, C^*, B^*, D^*) is stabilizable.

As in Part I, behaviours provide a convenient language for what we have in mind.

Definition 7.1.2. Consider $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$. We define the

behaviour of the linear state-space system (A, B, C, D) as

$$\mathcal{B}(A,B,C,D) := \left\{ (u,x,y) \in L^\infty_{\mathrm{loc}}(\mathbb{F}^m) \times AC(\mathbb{F}^n) \times L^\infty_{\mathrm{loc}}(\mathbb{F}^p) : \begin{array}{ll} \dot{x}(t) = Ax(t) + Bu(t) & \text{a.e.} \\ y(t) = Cx(t) + Du(t) & \end{array} \right\}.$$

This notation does not use different symbols for discrete-time behaviours and continuous-time behaviours, but hopefully that will not cause confusion as we will exclusively use continuous-time behaviours in Part II.

As a straightforward consequence of this definition, we note the following lemma.

Lemma 7.1.3. *Consider* $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ *and let* $(u, x, y) \in \mathcal{B}(A, B, C, D)$*. Then for any* $t \geq t_1 \geq 0$ *we have*

$$x(t) = e^{A(t-t_1)}x(t_1) + \int_{t_1}^{t} e^{A(t-s)}Bu(s)\,\mathrm{d}s.$$

7.2 Functions arising from quadratic forms

We will now make some observations that will be useful for analysis of functions arising from quadratic forms.

Consider $x \in AC(\mathbb{F}^n)$ and a function $V\colon \mathbb{F}^n \to [0,\infty)$. For asymptotic stability we are usually concerned with finding V such that it is a Lyapunov function, or equivalently, such that $\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \leq 0$. If the underlying field is real ($\mathbb{F} = \mathbb{R}$), then this derivative admits the following useful characterization: $\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \langle \nabla V(x(t)), \dot{x}(t)\rangle$, where ∇V is the gradient of V. We will sometimes write $\nabla_{\mathbb{R}}$ for ∇.

We now develop a similar expression for the case when the underlying field is complex. We do this in the obvious way, by identifying $\mathbb{C}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$, which we identify in turn with $\mathbb{R}^{2n}$. This is straightforward yet somewhat nonstandard so we spell out the (arguably) trivial details.

We define a bijective map ϕ in the following way:

$$\begin{aligned} \phi\colon \qquad \mathbb{C}^n &\longrightarrow \mathbb{R}^n \times \mathbb{R}^n \\ \xi &\longmapsto (\operatorname{Re}\xi, \operatorname{Im}\xi), \end{aligned}$$

where the real and imaginary parts of ξ are taken entrywise. In a slight abuse of notation we will sometimes use ϕ for the map between $\mathbb{C}^m$ and $\mathbb{R}^m \times \mathbb{R}^m$ as well as between $\mathbb{C}^{m+n}$ and $\mathbb{R}^{m+n} \times \mathbb{R}^{m+n}$.

Lemma 7.2.1. *Consider the map* $\phi : \mathbb{C}^n \to \mathbb{R}^n \times \mathbb{R}^n$ *defined by* $\phi(\xi) := (\operatorname{Re}\xi, \operatorname{Im}\xi)$. *Then*

$$\langle \phi(\xi_1), \phi(\xi_2) \rangle_{\mathbb{R}^{2n}} = \operatorname{Re} \langle \xi_1, \xi_2 \rangle_{\mathbb{C}^n} \tag{7.2.1}$$

for all $\xi_1, \xi_2 \in \mathbb{C}^n$.

Proof. This is a simple check: let $a_1, a_2, b_1, b_2 \in \mathbb{R}^n$ be such that $\phi(\xi_1) = (a_1, b_1)$ and $\phi(\xi_2) = (a_2, b_2)$. Then we can calculate

$$\begin{aligned}
\operatorname{Re} \langle \xi_1, \xi_2 \rangle_{\mathbb{C}^n} =& \frac{1}{2} \left[\langle a_1 + ib_1, a_2 + ib_2 \rangle_{\mathbb{C}^n} + \langle a_2 + ib_2, a_1 + ib_1 \rangle_{\mathbb{C}^n} \right] \\
=& \langle a_1, a_2 \rangle_{\mathbb{R}^n} + \langle b_1, b_2 \rangle_{\mathbb{R}^n} + \\
&+ \frac{1}{2} \left[-i \langle a_1, b_2 \rangle_{\mathbb{R}^n} + i \langle b_1, a_2 \rangle_{\mathbb{R}^n} - i \langle a_2, b_1 \rangle_{\mathbb{R}^n} + i \langle b_2, a_1 \rangle_{\mathbb{R}^n} \right] \\
=& \langle a_1, a_2 \rangle_{\mathbb{R}^n} + \langle b_1, b_2 \rangle_{\mathbb{R}^n} \\
=& \langle \phi(\xi_1), \phi(\xi_2) \rangle_{\mathbb{R}^{2n}} ,
\end{aligned}$$

where we have used $\langle a, b \rangle_{\mathbb{R}^n} = \langle b, a \rangle_{\mathbb{R}^n}$. □

Note that Lemma 7.2.1 shows that, for the standard 2-norms on $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, ϕ is an isometry: $\|\phi(\xi)\|_{\mathbb{R}^{2n}} = \|\xi\|_{\mathbb{C}^n}$ for all $\xi \in \mathbb{C}^n$.

Definition 7.2.2. We say that a function $V : \mathbb{C}^n \to \mathbb{R}$ is **continuously differentiable** if it is continuously differentiable in the $\mathbb{R}^{2n}$ sense, that is, if $V \circ \phi^{-1} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable.

Definition 7.2.3. Suppose that a function $V : \mathbb{C}^n \to \mathbb{R}$ is continuously differentiable. Let ϕ be the bijection between $\mathbb{C}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$ we defined above. We then define the **complex gradient** $\nabla_{\mathbb{C}} V : \mathbb{C}^n \to \mathbb{C}^n$ of V by $\nabla_{\mathbb{C}} V = \phi^{-1} \circ \left[\nabla_{\mathbb{R}} (V \circ \phi^{-1}) \right] \circ \phi$.

The utility of the above definitions can be seen in the following lemma.

Lemma 7.2.4. *Consider a continuously differentiable* $V : \mathbb{F}^n \to \mathbb{R}$ *and* $x \in AC(\mathbb{F}^n)$. *Then* $V \circ x \in AC(\mathbb{R})$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = \operatorname{Re} \langle (\nabla_{\mathbb{F}} V)(x(t)), \dot{x}(t) \rangle_{\mathbb{F}^n} \qquad \text{a.e.}$$

Proof. A composition of a continuously differentiable function with an absolutely continuous function is an absolutely continuous function, see e.g. Theorem 3.68 from Leoni [39]. This proves the first statement.

If $\mathbb{F} = \mathbb{R}$, then the second statement follows directly from an application of the chain rule.

If $\mathbb{F} = \mathbb{C}$, then we use the previously defined map ϕ to obtain functions whose domain and image spaces are real:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left((V \circ \phi^{-1}) \circ (\phi \circ x)\right)(t) \\
\text{[by the chain rule]} &= \left\langle \left(\nabla_{\mathbb{R}}(V \circ \phi^{-1})\right)(\phi(x(t))), \frac{\mathrm{d}}{\mathrm{d}t}(\phi \circ x)(t) \right\rangle_{\mathbb{R}^n \times \mathbb{R}^n} \\
&= \langle \phi\left(\nabla_{\mathbb{C}} V\right)(x(t)), \phi(\dot{x}(t))\rangle_{\mathbb{R}^n \times \mathbb{R}^n} \\
\text{[by Lemma 7.2.1]} &= \operatorname{Re}\langle \nabla_{\mathbb{C}} V(x(t)), \dot{x}(t)\rangle_{\mathbb{C}^n}. \qquad \square
\end{aligned}$$

We will usually be concerned with quadratic forms V given by $V(\xi) := \langle P\xi, \xi\rangle$, so it is be convenient to describe $\frac{\mathrm{d}}{\mathrm{d}t}V(x(t))$ for such V.

Lemma 7.2.5. *Consider a matrix $P = P^* \in \mathbb{F}^{n\times n}$ and a quadratic form $V : \mathbb{F}^n \to \mathbb{R}$ defined by $V(\xi) := \langle P\xi, \xi\rangle$. Then V is continuously differentiable and $\nabla_{\mathbb{F}} V = 2P$.*

Proof. If $\mathbb{F} = \mathbb{R}$, then straightforward differentiation and the use of $P^T = P^* = P$ gives us $\nabla_{\mathbb{R}} V(\xi) = 2P\xi$.

If $\mathbb{F} = \mathbb{C}$, then the calculation of $\nabla_{\mathbb{C}} V$ is slightly more complicated. For $a, b \in \mathbb{R}^n$, we set $\nabla_a V(a + ib) := \left(\frac{\partial V(a+ib)}{\partial a_1}, \frac{\partial V(a+ib)}{\partial a_2}, \ldots, \frac{\partial V(a+ib)}{\partial a_n}\right)$ and $\nabla_b V(a+ib) := \left(\frac{\partial V(a+ib)}{\partial b_1}, \frac{\partial V(a+ib)}{\partial b_2}, \ldots, \frac{\partial V(a+ib)}{\partial b_n}\right)$ to see that - for $\xi = a+ib$ - the complex gradient can be written as $\nabla_{\mathbb{C}} V(\xi) = (\nabla_a + i\nabla_b)V(a + ib)$.

Now fix $l \in \{1, 2, \ldots, n\}$ and use $P^* = P$ to calculate

$$\begin{aligned}
\frac{\partial V}{\partial a_l}(a + ib) =& \frac{\partial}{\partial a_l}\sum_{j,k} P_{jk}(a_k + ib_k)\overline{(a_j + ib_j)} \\
=& \sum_j P_{jl}\overline{(a_j + ib_j)} + \sum_k P_{lk}(a_k + ib_k) \\
=& \overline{[P(a + ib)]_l} + [P(a + ib)]_l
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial b_l}V(a + ib) =& \frac{\partial}{\partial b_l}\sum_{j,k} P_{jk}(a_k + ib_k)\overline{(a_j + ib_j)} \\
=& i\sum_j P_{jl}\overline{(a_j + ib_j)} - i\sum_k P_{lk}(a_k + ib_k) \\
=& i\overline{[P(a + ib)]_l} - i[P(a + ib)]_l.
\end{aligned}$$

Hence $\nabla_a V(a+ib) = \overline{P(a + ib)} + P(a+ib)$ and $i\nabla_b V(a+ib) = -\overline{P(a + ib)} + P(a + ib)$, which in turn means that $\nabla_{\mathbb{C}} V(\xi) = 2P\xi$ for all $\xi \in \mathbb{C}^n$.

Hence we have shown that $\nabla_{\mathbb{F}} V = 2P$ and thus V is continuously differentiable, which completes the proof. $\square$

We combine Lemmas 7.2.4 and 7.2.5 in the following result.

Corollary 7.2.6. *Consider a matrix $P = P^* \in \mathbb{F}^{n\times n}$ and a quadratic form $V\colon \mathbb{F}^n \to \mathbb{R}$ defined by $V(\xi) := \langle P\xi, \xi\rangle$. Then V is continuously differentiable. Moreover, if $x \in AC(\mathbb{F}^n)$, then $V \circ x \in AC(\mathbb{R})$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = \mathrm{Re}\,\langle 2Px(t), \dot{x}(t)\rangle_{\mathbb{F}^n} \qquad \text{a.e.}$$

7.3 Bounded real lemma

The ISS results for Lur'e systems from Arcak and Teel [7], Jayawardhana, Logemann and Ryan [29, 30] are proved using an ISS-Lyapunov function obtained from the positive real lemma, which guarantees the existence of a quadratic form useful for stability analysis by assuming a frequency-domain condition for a controllable and observable linear state-space system. The utility of such quadratic forms has also been demonstrated in absolute stability analysis, see Haddad and Bernstein [21], Haddad and Chellaboina [23].

In Chapters 8 and 9 we will perform stability analysis in both the absolute stability and ISS setting using a quadratic form obtained from the bounded real lemma. This approach will enable us to obtain stability results for new classes of systems.

As in the discrete-time setting, we will prove the bounded real lemma for stabilizable and detectable linear state-space systems, which relaxes a common assumption made in the bounded real lemma, namely that the underlying linear system is controllable and observable: see Lemma 7.3.1.

We define the (continuous-time) Hardy space $H^\infty(\mathbb{C}_+; \mathbb{C}^{p\times m})$ as the set of all bounded analytic functions $G\colon \mathbb{C}_+ \to \mathbb{C}^{p\times m}$ with the norm given by

$$\|G\|_{H^\infty} = \sup_{s\in\mathbb{C}_+} \|G(s)\|\,.$$

Lemma 7.3.1 (Bounded Real Lemma)**.** *Consider a stabilizable and detectable linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and assume that its transfer function G satisfies $\|G\|_{H^\infty} \leq 1$ and $\|D\| < 1$.*

Then there exist matrices L, W and a positive semi-definite $P = P^* \in \mathbb{F}^{n\times n}$

such that

$$\begin{aligned} A^*P + PA &= -C^*C - L^*L \\ PB &= -C^*D - L^*W \\ D^*D &= I - W^*W. \end{aligned}$$

We should note that there are various versions of the bounded real lemma, but most of them assume the strict inequality $\|G\|_{H^\infty} < 1$ (e.g. Theorem 3.7.1 from Green and Limebeer [19] or Theorem 5.3.25 from Hinrichsen and Pritchard [25]). To the best of the author's knowledge Lemma 7.3.1 in its present form is only claimed in Remark 5.3.27 from [25], where the proof is postponed to the unpublished second volume and in its present form it is only proved in the case when $D = 0$ in Theorem 3.3 from Hinrichsen and Pritchard [24]. Seeing as this result is central to our arguments, we prove this version of the bounded real lemma in the appendix, see §C.

We now use it to obtain a quadratic form that we will use in stability analysis.

Lemma 7.3.2. *Consider a stabilizable and detectable linear state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and assume that, for some positive r, its transfer function G satisfies $\|D\| < \|G\|_{H^\infty} \leq \frac{1}{r}$.*

Then there exists a positive semi-definite $P = P^* \in \mathbb{F}^{n \times n}$ such that the quadratic form $V : \mathbb{F}^n \to [0, \infty)$ defined by $V(\xi) := \langle P\xi, \xi \rangle$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq \|u(t)\|^2 - r^2 \|y(t)\|^2 \qquad \text{a.e.}$$

for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Moreover, there exists a projection $\Pi : \mathbb{F}^n \to \mathbb{F}^n$ and a positive c such that $\ker \Pi \subseteq \ker C$ and $V(\xi) \geq c \|\Pi\xi\|^2$ for all $\xi \in \mathbb{F}^n$.

Proof. Set $\rho := \|G\|_{H^\infty}^{-1}$ and consider the (stabilizable and detectable) linear state-space system $(A, \rho B, C, \rho D)$, whose transfer function ρG satisfies $\|\rho G\|_{H^\infty} = 1$. Apply Lemma 7.3.1, to see that there exist matrices L, W and a positive semi-definite $P = P^* \in \mathbb{F}^{n \times n}$ such that

$$A^*P + PA = -C^*C - L^*L, \tag{7.3.1a}$$

$$\rho PB = -\rho C^*D - L^*W, \tag{7.3.1b}$$

$$\rho^2 D^*D = I - W^*W. \tag{7.3.1c}$$

Now, consider the positive semi-definite quadratic form $U(\xi) := \langle P\xi, \xi \rangle$. We pick an arbitrary trajectory $(u, x, y) \in \mathcal{B}(A, B, C, D)$. By Corollary 7.2.6, U is continuously differentiable, $U \circ x$ is absolutely continuous and

$\frac{\mathrm{d}}{\mathrm{d}t}U(x(t)) = \mathrm{Re}\,\langle 2Px(t), \dot{x}(t)\rangle$ almost everywhere. The use of bounded real equations (7.3.1a) - (7.3.1c) and the technique of completing the square gives us

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}U(x(t)) &= \mathrm{Re}\,\langle 2Px(t), Ax(t) + Bu(t)\rangle \\
&= \langle (A^*P + PA)x(t), x(t)\rangle + \langle x(t), PBu(t)\rangle + \langle PBu(t), x(t)\rangle \\
&= -\left\|Cx(t)\right\|^2 - \left\|Lx(t)\right\|^2 - \langle Cx(t), Du(t)\rangle - \frac{1}{\rho}\langle Lx(t), Wu(t)\rangle \\
&\quad - \langle Du(t), Cx(t)\rangle - \frac{1}{\rho}\langle Wu(t), Lx(t)\rangle \\
&= -\left\|Lx(t) + \frac{1}{\rho}Wu(t)\right\|^2 + \frac{1}{\rho^2}\left\|Wu(t)\right\|^2 \\
&\quad - \left\|Cx(t) + Du(t)\right\|^2 + \left\|Du(t)\right\|^2 \\
&= -\left\|Lx(t) + \frac{1}{\rho}Wu(t)\right\|^2 + \frac{1}{\rho^2}\left\|u(t)\right\|^2 - \left\|y(t)\right\|^2 \\
&\leq \frac{1}{\rho^2}\left\|u(t)\right\|^2 - \left\|y(t)\right\|^2 \qquad \text{a.e.}
\end{aligned}$$

Thus $V := \rho^2 U$ has all the required properties.

Now, by Lemma 2.2.1, we know that $V^{-1}(0) = \ker P$ and that there exists a projection $\Pi\colon \mathbb{F}^n \to \mathbb{F}^n$ and a positive c such that $\ker\Pi = \ker P = V^{-1}(0)$ and $V(\xi) \geq c\left\|\Pi\xi\right\|^2$ for all $\xi \in \mathbb{F}^n$. Pick $\xi \in V^{-1}(0)$ and use equation (7.3.1a) to see that

$$0 = \langle P\xi, A\xi\rangle + \langle A\xi, P\xi\rangle = -\left\|C\xi\right\|^2 - \left\|L\xi\right\|^2.$$

Hence $\xi \in \ker C$ and consequently $\ker\Pi = V^{-1}(0) \subseteq \ker C$ completing the proof. □

As mentioned at the start of this section, the positive real lemma seems to be used more frequently in absolute stability analysis. In §7.6 we will make comparisons between the two quadratic forms, so we state the relevant positive real lemma results here.

Definition 7.3.3. A rational function matrix $G \in \mathbb{F}(s)^{m\times m}$ is said to be (continuous-time) **positive real** if $(G(s))^* + G(s) \geq 0$ for all $s \in \mathbb{C}_+$ which are not poles of $G(s)$.

We say that G is **strictly positive real** if there exists a positive ε such that $G(s-\varepsilon)$ is positive real.

We say that G is **strongly positive real** if it is strictly positive real and $D^* + D > 0$, where $D := \lim_{s\to\infty} G(s)$.

Lemma 7.3.4 (Positive Real Lemma)**.** *Consider a controllable and observable* $(A, B, C, D) \in \Sigma(m, n, m; \mathbb{R})$ *and denote by* G *its transfer function.*

Then G *is (continuous-time) positive real if and only if there exist matrices* L, W *and a positive definite matrix* $P^* = P > 0$ *such that:*

$$\begin{aligned} A^*P + PA &= -L^*L, \\ PB - C^* &= -L^*W, \\ D + D^* &= W^*W. \end{aligned}$$

Proof. This is Theorem 5.13 from [23]. □

It is interesting to note that even though the complex field equivalent of Lemma 7.3.4 is claimed in the unpublished [26] (Corollary 9.4.15), we were not able to locate a published reference.

We can use the positive real lemma to construct a quadratic form useful in stability analysis. We relegate its proof to the appendix, see §A.2.

Lemma 7.3.5. *Consider a controllable and observable linear state-space system* $(A, B, C, D) \in \Sigma(m, n, m; \mathbb{F})$ *and denote by* G *its transfer function.*

If G *is positive real, then there exists a positive definite* $P = P^* \in \mathbb{F}^{n\times n}$ *such that the quadratic form* $V : \mathbb{F}^n \to [0, \infty)$ *defined by* $V(\xi) := \langle P\xi, \xi\rangle$ *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq \frac{1}{2}\left[\|u(t) + y(t)\|^2 - \|u(t) - y(t)\|^2\right] \qquad \text{a.e.}$$

for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Finally, we note a well-known relation between positive real and bounded real functions due to the Moebius transform (sometimes called the Cayley transform).

Lemma 7.3.6. *Consider* $G \in \mathbb{F}(s)^{m\times m}$*; the following are equivalent*

(a) G *is positive real,*

(b) $I + G$ *is invertible and* $\left\|(I - G)(I + G)^{-1}\right\|_{H^\infty} \leq 1$.

Proof. in slightly different language this is Theorem 4 from [4], but it can also be derived directly in the same way as we did in discrete-time in Lemma 2.2.10. □

7.4 Output injection

We now obtain a quadratic form using a technique sometimes called "an output injection". For real SISO systems without feedthrough, this construction is employed in Angeli [6], Arcak and Teel [7] and Jayawardhana, Logemann and Ryan [30].

Lemma 7.4.1. *Consider a detectable $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$. Then there exists a positive definite $Q = Q^* \in \mathbb{F}^{n\times n}$ such that, for some $\delta > 0$, the quadratic form $V : \mathbb{F}^n \to [0, \infty)$ defined by $V(\xi) := \langle Q\xi, \xi\rangle$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq -\delta \left\| x(t) \right\|^2 + \left\| y(t) \right\|^2 + \left\| u(t) \right\|^2 \qquad \text{a.e.}$$

for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Proof. By detectability of (C, A), there exists $H \in \mathbb{F}^{n\times m}$ such that $\sigma(A + HC) \subset \mathbb{C}_-$. It is a well-known fact from linear algebra that $\sigma(M) = \overline{\sigma(M^*)}$, where overline denotes the complex conjugation, see Proposition 4.4.4v) from [9].

Hence there exists a solution $Q = Q^* > 0$ of the continuous-time Lyapunov equation

$$(A + HC)^* Q + Q(A + HC) = -I. \tag{7.4.1}$$

Now consider the positive definite quadratic form $V_Q(\xi) := \langle Q\xi, \xi\rangle$ and pick an arbitrary trajectory $(u, x, y) \in \mathcal{B}(A, B, C, D)$. By Corollary 7.2.6, V_Q is continuously differentiable, $V_Q \circ x$ is absolutely continuous and $\frac{\mathrm{d}}{\mathrm{d}t} V_Q(x(t)) = \operatorname{Re} \langle 2Qx(t), \dot{x}(t)\rangle$ almost everywhere. Now we can use equation (7.4.1) to see that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} V_Q(x(t)) =& \frac{1}{2} [\langle 2Qx(t), Ax(t)\rangle + \langle 2Qx(t), Bu(t)\rangle \\
&+ \langle Ax(t), 2Qx(t)\rangle + \langle Bu(t), 2Qx(t)\rangle] \\
=& \langle Qx(t), (A + HC)x(t)\rangle - \langle Qx(t), HCx(t)\rangle + \langle Qx(t), Bu(t)\rangle \\
&\langle (A + HC)x(t), Qx(t)\rangle - \langle HCx(t), Qx(t)\rangle + \langle Bu(t), Qx(t)\rangle \\
=& - \left\| x(t) \right\|^2 - \langle Qx(t), H(y(t) - Du(t))\rangle + \langle Qx(t), Bu(t)\rangle \\
&- \langle H(y(t) - Du(t)), Qx(t)\rangle + \langle Bu(t), Qx(t)\rangle \qquad \text{a.e.}
\end{aligned}$$

Now an application of the Cauchy-Schwarz inequality, subsequent use of the property of the operator norm that $\|M\xi\| \leq \|M\| \, \|\xi\|$ and finally the use of the simple inequality $ab = \frac{a}{c} bc \leq \frac{1}{c^2} a^2 + c^2 b^2$ shows us that there exist positive δ, c_1, c_2 - independent of (u, x, y) - such that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_Q(x(t)) \leq -\delta \left\| x(t) \right\|^2 + c_1 \left\| y(t) \right\|^2 + c_2 \left\| u(t) \right\|^2 \qquad \text{a.e.}$$

We can then see that $\frac{1}{\max\{c_1, c_2\}} V_Q$ has all the required properties. $\square$

7.5 ω-limit sets

ω-limit sets are usually defined for initial value problems, however for our intended application in the proof of Proposition 8.2.1 it is more convenient to define and use the ω-limit set of a function $x : [0, \infty) \to \mathbb{F}^n$.

Definition 7.5.1. Let $x\colon [0, \infty) \to \mathbb{F}^n$ be some map. We define the **ω-limit set of x** as

$$\Omega_x := \left\{ \xi \in \mathbb{F}^n : \exists\, (t_k)_{k\in\mathbb{N}_0} \subseteq [0, \infty) \text{ s.t. } \lim_{k\to\infty} t_k = \infty \text{ and } \lim_{k\to\infty} x(t_k) = \xi \right\}.$$

Recall that for a nonempty $\mathbb{S} \subseteq \mathbb{F}^n$ and $\xi \in \mathbb{F}^n$ we defined the **distance between $\mathbb{S}$ and ξ** as

$$\operatorname{dist}(\xi, \mathbb{S}) := \inf\{\|\xi - \mu\| : \mu \in \mathbb{S}\}.$$

Lemma 7.5.2. *Let $x\colon [0, \infty) \to \mathbb{F}^n$ be a bounded map, so that there exists $c > 0$ such that $\|x(t)\| \le c$ for all $t \in [0, \infty)$. Then Ω_x is nonempty and*

$$\lim_{t\to\infty} \operatorname{dist}\left(x(t), \Omega_x\right) = 0.$$

Proof. Since x is bounded, by the Bolzano-Weierstrass theorem for finite-dimensional vector spaces, there is a convergent subsequence of $(x(t))_{t\in\mathbb{N}_0}$. Its limit is clearly in Ω_x, so that Ω_x is indeed nonempty.

Now suppose on the contrary, that $\lim_{t\to\infty} \operatorname{dist}(x(t), \Omega_x) = 0$ does not hold. Then there exists $\varepsilon > 0$ and a subset $\{t_k\}_{k\in\mathbb{N}_0}$ of $[0, \infty)$ with $\lim_{k\to\infty} t_k = \infty$ such that $\operatorname{dist}(x(t_k), \Omega_x) > \varepsilon$ for all $k \in \mathbb{N}_0$. However, by the Bolzano-Weierstrass theorem, $x(t_k)$ has a convergent subsequence, or, in other words, there exists a subset $(t_{k_j})_{j\in\mathbb{N}_0}$ of $(t_k)_{k\in\mathbb{N}_0}$ such that $\lim_{j\to\infty} t_{k_j} = \infty$ and such that $\lim_{j\to\infty} x(t_{k_j}) = \xi$ for some $\xi \in \mathbb{F}^n$. Hence, by definition, $\xi \in \Omega_x$ which in turn contradicts our initial assumption. $\square$

7.6 Stabilization by output feedback

This section replicates the results of Chapter 3 for the continuous-time setting.

Important! We will not write down explicit proofs for any of the results in this section and any commentary will be brief. The reason is that these results follow in the same way as the corresponding results in Chapter 3 and the motivation for them is similar.

We will first introduce the technique of loop shifting, see Lemma 7.6.5. Then we will define the set of stabilizing output feedback matrices $\mathbb{S}_{\mathbb{C}}\,(A, B, C, D)$

and see how the ball condition $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(A,B,C,D)$ can be characterized using a bounded real condition, see Lemma 7.6.12. This will be used to obtain a quadratic form from the ball condition in Corollary 7.6.13. Finally, we will note a connection with the complex stability radius in Corollary 7.6.17.

Definition 7.6.1. Consider a linear system $(A,B,C,D) \in \Sigma(m,n,p;\mathbb{F})$. We define the set $\mathbb{A}_{\mathbb{C}}(A,B,C,D)$ of **admissible output feedback matrices for** (A,B,C,D) as

$$\mathbb{A}_{\mathbb{C}}(A,B,C,D) := \{K \in \mathbb{C}^{m\times p} : \det(I_p - DK) \neq 0\}.$$

Note that $\mathbb{A}_{\mathbb{C}}(A,B,C,D)$ only depends on the matrix D, so we will usually write $\mathbb{A}_{\mathbb{C}}(D)$ instead.

We re-state some consequences of the definition of $\mathbb{A}_{\mathbb{C}}$.

Lemma 7.6.2. *Let $D \in \mathbb{F}^{p\times m}$ and $N \in \mathbb{F}^{m\times p}$.*

Then $\det(I_p - DN) \neq 0 \iff \det(I_m - ND) \neq 0$.

Corollary 7.6.3. $N \in \mathbb{A}_{\mathbb{C}}(D) \iff D \in \mathbb{A}_{\mathbb{C}}(N)$.

Lemma 7.6.4. *Consider a state-space system $(A,B,C,D) \in \Sigma(m,n,p;\mathbb{F})$ and denote its transfer function by G.*

Then for all $K \in \mathbb{A}_{\mathbb{C}}(D)$, we have

$$(I_p - GK)^{-1}G = G(I_m - KG)^{-1}.$$

In view of this lemma, for a given state-space system (A,B,C,D) with transfer function $G \in \mathbb{F}(z)^{p\times m}$ and for $K \in \mathbb{A}_{\mathbb{C}}(D)$, we (again) define

$$G^K := (I_p - GK)^{-1}G = G(I_m - KG)^{-1}.$$

Note that for $K \in \mathbb{A}_{\mathbb{C}}(D)$ we can also define this operation for a constant matrix:

$$D^K := (I_p - DK)^{-1}D = D(I_m - KD)^{-1}. \tag{7.6.1}$$

Remark: note that $(I_p - GK)^{-1}G$ might be well-defined even if we do not require $K \in \mathbb{A}_{\mathbb{C}}(D)$, however in this case $(I_p - GK)^{-1}G$ is not the transfer function of any state-space system, so we shall *always* require that $K \in \mathbb{A}_{\mathbb{C}}(D)$ when we write down G^K.

Recall that for $K \in \mathbb{A}_{\mathbb{C}}(D)$ we can define

$$\begin{aligned} A_K &:= A + BK(I_p - DK)^{-1}C, \quad B_K := B + BK(I_p - DK)^{-1}D, \\ C_K &:= (I_p - DK)^{-1}C, \quad D_K := (I_p - DK)^{-1}D. \end{aligned} \tag{7.6.2}$$

One can then check that for $(u, x, y) \in \mathcal{B}(A, B, C, D)$ we have $Ax + Bu = A_K x + B_K[u - Ky]$ and $Cx + Du = C_K x + D_K[u - Ky]$. This observation gives rise to the following lemma.

Lemma 7.6.5. *Consider a linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and denote its transfer function by G. Let $K \in \mathbb{A}_{\mathbb{C}}(D)$ and let (A_K, B_K, C_K, D_K) be defined by equation (7.6.2).*

(a) A trajectory $(u, x, y) \in \mathcal{B}(A, B, C, D)$ if, and only if, $(u - Ky, x, y) \in \mathcal{B}(A_K, B_K, C_K, D_K)$.

(b) The transfer function of (A_K, B_K, C_K, D_K) is G^K.

(c) The linear system (A, B, C, D) is controllable and observable if, and only if, (A_K, B_K, C_K, D_K) is.

(d) The linear system (A, B, C, D) is stabilizable and detectable if, and only if, (A_K, B_K, C_K, D_K) is.

Lemma 7.6.6. *Consider a linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and denote its transfer function by G. Let $K, M \in \mathbb{F}^{m \times p}$ and $K \in \mathbb{A}_{\mathbb{C}}(D)$.*

Then $M \in \mathbb{A}_{\mathbb{C}}(D_K) \iff K + M \in \mathbb{A}_{\mathbb{C}}(D)$. Furthermore, if one of these holds, then $\left(G^K\right)^M = G^{K+M}$.

Definition 7.6.7. Consider a linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and denote its transfer function by G. We define the (continuous-time) set of **stabilizing linear output feedback matrices of** (A, B, C, D) as

$$\mathbb{S}_{\mathbb{C}}\left(A, B, C, D\right) := \{K \in \mathbb{A}_{\mathbb{C}}(D) : G^K \in H^\infty(\mathbb{C}_+; \mathbb{C}^{p \times m})\}.$$

Since $D = \lim_{|z| \to \infty} G(z)$, the transfer function G describes $\mathbb{S}_{\mathbb{C}}\left(A, B, C, D\right)$ completely. Hence we will write $\mathbb{S}_{\mathbb{C}}\left(G\right)$ for $\mathbb{S}_{\mathbb{C}}\left(A, B, C, D\right)$.

We note the following well-known fact.

Lemma 7.6.8. *Consider a linear state-space system (A, B, C, D) and denote its transfer function by G.*

Then $\sigma\left(A\right) \subseteq \mathbb{C}_-$ if, and only if, (A, B, C, D) is stabilizable and detectable, and $G \in H^\infty$.

Recall the following useful lemma on matrices.

Lemma 7.6.9. *Let $D \in \mathbb{F}^{p \times m}$ with $D \neq 0_{p \times m}$. Then*

(a) If $M \in \mathbb{F}^{m \times p}$ and $\det(I_p - DM) = 0$, then

$$\frac{1}{\|D\|} \leq \|M\|;$$

(b) There exists $M \in \mathbb{F}^{m \times p}$ *such that* $\|D\| = \frac{1}{\|M\|}$ *and* $\det(I_p - DM) = 0$.

Lemma 7.6.10. *Consider a linear system* $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$*, denote its transfer function by* G *and assume that* $K \in \mathbb{A}_{\mathbb{C}}(D)$.

Then $\mathbb{S}_{\mathbb{C}}(G) - K = \mathbb{S}_{\mathbb{C}}\left(G^K\right)$.

Remark: Of course, $\mathbb{S}_{\mathbb{C}}(G) - K := \{M - K : M \in \mathbb{S}_{\mathbb{C}}(G)\}$.

Corollary 7.6.11. *Consider a linear system* $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ *and let* $K \in \mathbb{F}^{m \times p}$. *Then*

$$\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G) \iff \mathbb{B}_{\mathbb{C}}(0, r) \subseteq \mathbb{S}_{\mathbb{C}}\left(G^K\right).$$

As in Part I, an important role in construction of quadratic forms for stability analysis is played by the following lemma. It implies that the largest matrix ball centered on $K \in \mathbb{F}^{m \times p}$ and contained in $\mathbb{S}_{\mathbb{C}}(G)$ has radius $\left\|G^K\right\|_{H^\infty}^{-1}$. This demonstrates that $\mathbb{S}_{\mathbb{C}}(G)$ is closely related to the stability radius as defined in the work of Hinrichsen and Pritchard, see [25]. We will elaborate on this connection in Corollary 7.6.17.

Lemma 7.6.12. *Consider a linear system* $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ *and denote its transfer function by* G.

Then $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G) \iff \left\|G^K\right\|_{H^\infty} \leq \frac{1}{r}$.

The following result is a simple corollary, which assumes the ball condition $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ and constructs a quadratic form using the bounded real lemma.

Corollary 7.6.13. *Consider a stabilizable and detectable linear state-space system* $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$*, denote its transfer function by* G *and let* $r > 0$*,* $K \in \mathbb{F}^{m \times p}$*. Furthermore, assume that* $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ *and* $\left\|D^K\right\| < \left\|G^K\right\|_{H^\infty}$.

Then there exists a positive semi-definite $P = P^* \in \mathbb{F}^{n \times n}$ *such that the quadratic form* $V : \mathbb{F}^n \to [0, \infty)$ *defined by* $V(\xi) := \langle P\xi, \xi\rangle$ *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq \|u(t) - Ky(t)\|^2 - r^2 \|y(t)\|^2 \qquad \text{a.e.}$$

for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Moreover, there exists a projection $\Pi : \mathbb{F}^n \to \mathbb{F}^n$ *and a positive* c *such that* $\ker \Pi \subseteq \ker C$ *and* $V(\xi) \geq c \|\Pi\xi\|^2$ *for all* $\xi \in \mathbb{F}^n$.

In Chapter 8 we will prove a proposition that is similar to a well-known result in absolute stability theory, the circle criterion. Our main tool in proving this result will be Corollary 7.6.13 and thus, essentially, the bounded real lemma. To the best of the author's knowledge this is a nonstandard route as the circle criterion is typically proved using the positive real lemma, see e.g. Haddad and Bernstein [21], Anderson and Vongpanitlerd [5] or Khalil [36]. As a consequence of the use of the positive real lemma in the standard proofs of the circle criterion, it is generally assumed that the underlying linear system is real (that is, $\mathbb{F} = \mathbb{R}$) and minimal. The use of the bounded real lemma will allow us to relax these assumptions in our version of the circle criterion, see Proposition 8.4.1.

We know from Lemma 7.3.6 that the bounded real property is related to the positive real property, which leads to the following proposition.

Proposition 7.6.14. *Consider $(A, B, C, D) \in \Sigma(m, n, m; \mathbb{R})$, denote its transfer function by G and let $r > 0$, $K \in \mathbb{R}^{m\times m}$. The following are equivalent:*

(a) $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$,

(b) $\left\|G^K\right\|_{H^\infty} \leq \frac{1}{r}$,

(c) there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = r$, $\lambda I + K \in \mathbb{A}_{\mathbb{C}}(D)$ and $I + 2\lambda G^{\lambda I+K}$ is positive real.

This proposition allows us to compare the quadratic form obtained from the ball condition $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ and the bounded real lemma to the one obtained from the same ball condition $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$, but the positive real lemma.

Lemma 7.6.15. *Consider a controllable and observable linear state-space system $(A, B, C, D) \in \Sigma(m, n, m; \mathbb{R})$, denote its transfer function by G and let $r > 0$, $K \in \mathbb{R}^{m\times m}$. Furthermore assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.*

Then there exists a positive definite $P = P^* \in \mathbb{R}^{n\times n}$ such that the quadratic form $V : \mathbb{R}^n \to [0, \infty)$ defined by $V(\xi) := \langle P\xi, \xi\rangle$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq \|u(t) - Ky(t)\|^2 - r^2 \|y(t)\|^2 \qquad \text{a.e.}$$

for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Thus one can use either the bounded real lemma or the positive real lemma to obtain quadratic forms from the ball condition $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$: both approaches provide identical estimates on $\frac{\mathrm{d}}{\mathrm{d}t} V(x(t))$. However, the positive real lemma seems more restrictive in that we have to assume both the minimality of the state-space system and an equal number of inputs and outputs, hence we elect to use the bounded real lemma in this thesis.

Finally, we note a connection between $\mathbb{S}_{\mathbb{C}}(G)$ and the well-known concept of the structured stability radius.

Definition 7.6.16. Consider a linear state-space system $(A, B, C, 0) \in \Sigma(m, n, p; \mathbb{F})$ with $\sigma(A) \subseteq \mathbb{C}_-$. We define the (continuous-time) **complex structured stability radius of** A **with respect to weights** B **and** C as

$$r_{\mathbb{C}}(A; B, C) := \inf\{\|K\| : K \in \mathbb{C}^{m \times p} \text{ and } \sigma(A_K) \nsubseteq \mathbb{C}_-\}.$$

Corollary 7.6.17. *Consider a stabilizable and detectable* $(A, B, C, 0) \in \Sigma(m, n, p; \mathbb{F})$. *Then* $\mathbb{B}_{\mathbb{C}}(0, r) \subseteq \mathbb{S}_{\mathbb{C}}(G) \iff r_{\mathbb{C}}(A; B, C) \geq r$. *Moreover,* $r_{\mathbb{C}}(A; B, C) = \sup\{r \geq 0 : \mathbb{B}_{\mathbb{C}}(0, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)\}$.

Chapter 8

Absolute stability of Lur'e systems

We will now put the tools developed in Chapter 7 to use in absolute stability analysis of Lur'e systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bf(y(t)) \\ y(t) &= Cx(t) + Df(y(t)). \end{aligned} \tag{8.1}$$

Similarly as in Part I, our approach is inspired by the complexified Aizerman's conjecture, which was proved for systems with no feedthrough, that is, $D = 0$, in Hinrichsen and Pritchard [25]. It states that if the Lur'e interconnection (8.1) is globally asymptotically stable for *all complex linear* output feedback matrices F that satisfy the norm condition $\|F(\xi)\| < r\,\|\xi\|$ for some $r > 0$ and for all $\xi \in \mathbb{C}^p \setminus \{0\}$, then the Lur'e interconnection is globally asymptotically stable for *all nonlinear* output feedback maps f that satisfy the same norm condition $\|f(\xi)\| < r\,\|\xi\|$. We will use the quadratic forms obtained from Lemma 7.4.1 and Corollary 7.6.13 to prove an extension of this result in Proposition 8.2.1, which we will sometimes refer to as the Aizerman version of the circle criterion.

We will also consider the well-known circle criterion, which is a stability criterion for Lur'e interconnections, where the underlying linear system (A, B, C, D) is controllable and observable and - for some matrices K_1, K_2 - its transfer function G is such that $(I - K_2G)(I - K_1G)^{-1}$ is strongly positive real. The circle criterion then states that if the nonlinearity f satisfies $\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \leq 0$, then the Lur'e system (8.1) is globally asymptotically stable, see e.g. Theorem 5.1 from Haddad and Bernstein [21]. The circle criterion is usually proved by using a quadratic form obtained from the positive real lemma, however we prove it by using the Aizerman version of the circle criterion and hence, in effect, the bounded real lemma, which allows us to relax the assumptions on the underlying nonlinear system to stabilizability and detectability.

Finally, we note that most results in this chapter extend easily to time-dependent nonlinearities as long as the relevant assumptions are satisfied uniformly in the time variable. The only exception is Proposition 8.2.1 (b) and its corollaries.

This chapter is organized as follows. In §8.1 we define Lur'e systems and discuss a related initial value problem. Then we define the notions of stability we will be interested in. After this, in §8.2, we prove the Aizerman version of the circle criterion and discuss some of its consequences as well as some of the assumptions made in its statement. We note a straightforward consequence of it in §8.3 and a more substantial consequence, which resembles the classical circle criterion in §8.4.

8.1 Lur'e systems

If we assume that $I - Df$ is bijective, then the Lur'e system (8.1) gives rise to the initial value problem

$$\dot{x}(t) = Ax(t) + Bf \circ (I - Df)^{-1}(Cx(t)), \qquad x(0) = \xi \in \mathbb{F}^n. \tag{8.1.1}$$

Standard ordinary differential equations theory (see e.g. §4.6 from Logemann and Ryan [43]) guarantees the existence of a unique maximal solution of (8.1.1) as long as $f \circ (I - Df)^{-1}$ is locally Lipschitz. That is, there exists a continuously differentiable x, defined on some maximal interval $[0, \omega) \subseteq [0, \infty)$, such that if x_1 is any other solution of (8.1.1), defined on $[0, \omega_1) \subseteq [0, \infty)$, then $[0, \omega_1) \subseteq [0, \omega)$ and $x_1 = x$ on $[0, \omega_1)$. The unique solution x with the above property is called the maximal solution. Furthermore, it is well-known that, if $x\colon [0, \omega) \to \mathbb{F}^n$ is a maximal solution of (8.1.1) and if $\omega < \infty$, then

$$\lim_{t \to \omega} \|x(t)\| = \infty. \tag{8.1.2}$$

Definition 8.1.1. Consider $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and let $f\colon \mathbb{F}^p \to \mathbb{F}^m$ be locally Lipschitz. If $I - Df$ is bijective and $(I - Df)^{-1}$ is locally Lipschitz then we call the quintuple (A, B, C, D, f) a **Lur'e system**.

We define the **behaviour of Lur'e system** (A, B, C, D, f) as

$$\mathcal{B}(A,B,C,D,f) := \Big\{ (x,y) \in C^1(\mathbb{F}^n) \times C(\mathbb{F}^p) : \\ x \text{ is a maximal solution of (8.1.1) and} \\ y(t) = Cx(t) + Df(y(t)) \Big\}.$$

In the case of zero feedthrough (that is, $D = 0$), the assumptions made in the definition of a Lur'e system simplify considerably.

A straightforward consequence of this definition is the following result.

Lemma 8.1.2. *Consider a Lur'e system (A, B, C, D, f).*

If $(x, y) \in \mathcal{B}(A, B, C, D, f)$, then $(f \circ y, x, y) \in \mathcal{B}(A, B, C, D)$.

We note that, when x is continuously differentiable, the estimate in Corollary 7.2.6 holds for all t on the interval of definition of x. Therefore, so do the estimates obtained in Corollaries 7.4.1 and 7.6.13.

We will be exploring the following stability properties.

Definition 8.1.3. Consider a Lur'e system (A, B, C, D, f).

1. If there exists a positive c such that

$$\begin{aligned} \|x(t)\| &\leq c\,\|x(0)\| \\ \|y(t)\| &\leq c\,\|x(0)\| \qquad \forall\, t \in [0, \omega) \end{aligned}$$

   and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$, where $[0, \omega)$ is the maximal interval of solution of the initial value problem (8.1.1), then we say that the Lur'e system (A, B, C, D, f) is (continuous-time) **globally stable**. Note that, this then implies $\omega = \infty$.

2. If (A, B, C, D, f) is globally stable and if

$$\lim_{t\to\infty} x(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} y(t) = 0$$

   for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$, then we say that the Lur'e system (A, B, C, D, f) is (continuous-time) **globally asymptotically stable**.

3. If (A, B, C, D, f) is globally stable and if there exist positive a and c such that

$$\begin{aligned} \|x(t)\| &\leq ce^{-at}\,\|x(0)\| \\ \|y(t)\| &\leq ce^{-at}\,\|x(0)\| \qquad \forall\, t \in [0, \infty) \end{aligned}$$

and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$, the we say that the Lur'e system (A, B, C, D, f) is (continuous-time) **globally exponentially stable**.

8.2 Aizerman version of the circle criterion

We will now apply the quadratic form obtained from the ball condition $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ to the stability analysis of Lur'e systems. Recall the shorthand $D^K := (I - DK)^{-1}D$.

Proposition 8.2.1 (Aizerman version of the circle criterion)**.** *Consider a Lur'e system (A, B, C, D, f); assume that the underlying linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable and denote its transfer function by G. Let $K \in \mathbb{F}^{m \times p}$, $r > 0$ and assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.*

(a) If $\|D^K\| < \|G^K\|_{H^\infty}$ and

$$\|f(\xi) - K\xi\| \leq r\,\|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p, \tag{8.2.1}$$

then the Lur'e system (A, B, C, D, f) is globally stable.

(b) If $\|D^K\| < \|G^K\|_{H^\infty}$ and

$$\|f(\xi) - K\xi\| < r\,\|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p \setminus \{0\}, \tag{8.2.2}$$

then the Lur'e system (A, B, C, D, f) is globally asymptotically stable.

(c) If there exists $\delta > 0$ such that

$$\|f(\xi) - K\xi\| \leq (r - \delta)\,\|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p, \tag{8.2.3}$$

then the Lur'e system (A, B, C, D, f) is globally exponentially stable.

Note that if $F \in \mathbb{B}_{\mathbb{C}}(K, r)$, then clearly $\|F\xi - K\xi\| < r\,\|\xi\|$ for all $\xi \in \mathbb{F}^p$. Hence Proposition 8.2.1 (b) can be interpreted as saying that if the Lur'e system (A, B, C, D, F) is globally asymptotically stable for *all complex linear* output feedback matrices F such that $\|F\xi - K\xi\| < r\,\|\xi\|$, then the Lur'e system (A, B, C, D, f) is globally asymptotically stable for *all nonlinear* output feedback maps f such that $\|f(\xi) - K\xi\| < r\,\|\xi\|$. Hence Proposition 8.2.1 (b) can be seen as saying that Aizerman's conjecture is true over the complex field. Note that this is not a new observation as an identical statement (although for systems with no feedthrough) is Theorem 5.6.22 from Hinrichsen and Pritchard [25]. Indeed, their work has inspired and guided our results in this section and (arguably) the only real novelty in this section is Proposition 8.2.1 (a) and the extension to systems with feedthrough. The latter rests on proving a version of the bounded real lemma (Lemma 7.3.1), which seems to be unavailable in the literature.

Proof of Proposition 8.2.1. By Corollary 7.6.13, there exists a positive semi-definite matrix $P = P^* \in \mathbb{F}^{n\times n}$ such that the quadratic form $V : \mathbb{F}^n \to [0,\infty)$ given by $V(\xi) := \langle P\xi, \xi\rangle$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \le \|u(t) - Ky(t)\|^2 - r^2 \|y(t)\|^2$$

for all $(u,x,y) \in \mathcal{B}(A,B,C,D)$. Moreover, there exists a projection $\Pi\colon \mathbb{F}^n \to \mathbb{F}^n$ and a positive c_1 such that $\ker \Pi \subseteq \ker C$ and $V(\xi) \ge c_1 \|\Pi\xi\|^2$ for all $\xi \in \mathbb{F}^n$. We note that since $\ker \Pi \subseteq \ker C$, it follows that $C\Pi = C$ and hence

$$\|C\xi\|^2 = \|C\Pi\xi\|^2 \le \|C\|^2 \|\Pi\xi\|^2 \le \frac{\|C\|^2}{c_1}V(\xi) \qquad \forall \xi \in \mathbb{F}^n. \tag{8.2.4}$$

Also, by Lemma 8.1.2,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \le \|f(y(t)) - Ky(t)\|^2 - r^2 \|y(t)\| \tag{8.2.5}$$

for all $(x,y) \in \mathcal{B}(A,B,C,D,f)$.

Let us first prove that trajectories in $\mathcal{B}(A,B,C,D,f)$ are global. To this end, we pick $(x,y) \in \mathcal{B}(A,B,C,D,f)$ and set $[0,\omega)$ to be the domain of x. In all cases (a) - (c), inequality (8.2.5) implies that $\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \le 0$, so that $V(x(t)) \le V(x(0))$ for all $t \in [0,\omega)$. By the definition of V, there exists $c_2 > 0$ such that $V(\xi) \le c_2 \|\xi\|^2$ for all $\xi \in \mathbb{F}^n$, so that, by (8.2.4),

$$\|Cx(t)\|^2 \le \frac{\|C\|^2}{c_1}V(x(t)) \le \frac{\|C\|^2}{c_1}V(x(0)) \le c_3 \|x(0)\|^2 \qquad \forall\, t \in [0,\omega), \tag{8.2.6}$$

where we have set $c_3 := \frac{\|C\|^2 c_2}{c_1}$. Since $y(t) = (I - Df)^{-1}(Cx(t))$ and since $(I - Df)^{-1}$ is Lipschitz continuous, y is bounded. Moreover, as $\dot{x}(t) = Ax(t) + Bf(y(t))$ and as f is Lipschitz continuous, by Gronwall's lemma, $x(t)$ is bounded for all finite t. Therefore, equation (8.1.2) implies that $\omega = \infty$.

Let us now prove (a). Equation (8.2.5) and assumption (8.2.1) give us

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \le \|f(y(t)) - Ky(t)\|^2 - r^2 \|y(t)\| \le 0 \qquad \forall\, t \in [0,\infty) \tag{8.2.7}$$

and for all $(x,y) \in \mathcal{B}(A,B,C,D,f)$. Let us define $(A_K, B_K, C_K, D_K) \in \Sigma(m,n,p;\mathbb{F})$ by (7.6.2). By Lemmas 7.6.5 and 8.1.2, we know that if $(x,y) \in \mathcal{B}(A,B,C,D,f)$, then $(f \circ y - Ky, x, y) \in \mathcal{B}(A_K, B_K, C_K, D_K)$, so that $y(t) = C_K x(t) + D_K[f(y(t)) - Ky(t)]$ for all $t \in [0,\infty)$ and for all $(x,y) \in \mathcal{B}(A,B,C,D,f)$. Taking norms and using $C_K = (I - DK)^{-1}C$ and $\|D^K\| <$

$\frac{1}{r}$ alongside assumption (8.2.1) and estimate (8.2.6) shows that there exists a positive c_4 such that

$$\|y(t)\| \leq c_4 \|Cx(t)\| \leq \sqrt{c_3}c_4 \|C\| \|x(0)\| \qquad \forall\, t \in [0,\infty) \tag{8.2.8}$$

and for all $(x,y) \in \mathcal{B}(A,B,C,D,f)$.

Thus we are done if we can show that $\|x(t)\| \leq c\|x(0)\|$ for some positive c. To this end, recall that if $(x,y) \in \mathcal{B}(A,B,C,D,f)$, then $(f\circ y - Ky, x, y) \in \mathcal{B}(A_K,B_K,C_K,D_K)$ and use Lemma 7.1.3 to see that

$$x(t) = e^{A_K t}x(0) + \int_0^t e^{A_K(t-s)}B_K[f(y(s)) - Ky(s)]\,\mathrm{d}s \qquad \forall\, t \in [0,\infty).$$

By Lemma 7.6.5, we know that (A_K,B_K,C_K,D_K) is stabilizable and detectable; since $G^K \in H^\infty$, Lemma 7.6.8 then implies that $\sigma(A_K) \subset \mathbb{C}_-$ and hence there exists a positive c_5 such that $\left\|e^{A_K t}\right\| + \int_0^\infty \left\|e^{A_K s}\right\| \mathrm{d}s \leq c_5$ for all $t \in [0,\infty)$. Therefore, the use of (8.2.8) gives us

$$\begin{aligned}\|x(t)\| &\leq c_5\|x(0)\| + c_5\|B_K\| \sup_{0\leq s\leq t}\|f(y(s)) - Ky(s)\| \\ &\leq (c_5 + c_5\|B_K\| r c_3 c_4 \|C\|)\|x(0)\| \qquad \forall\, t \in [0,\infty),\end{aligned}$$

which completes the proof of (a).

To prove (b) we pick an arbitrary $(x,y) \in \mathcal{B}(A,B,C,D,f)$. By (a), we already know (A,B,C,D,f) is globally stable, so that there exists $c_6 > 0$ such that $\|x(t)\| \leq c_6\|x(0)\|$ and $\|y(t)\| \leq c_6\|x(0)\|$. Moreover, by Lemma 8.1.2, $(f\circ y, x, y) \in \mathcal{B}(A,B,C,D)$. Since x is continuously differentiable, for arbitrary $t \geq 0$ and $\tau > 0$, the fundamental theorem of calculus gives us

$$\begin{aligned}\|x(t+\tau) - x(t)\| &= \left\|\int_t^{t+\tau} Ax(s) + Bf(y(s))\,\mathrm{d}s\right\| \\ &\leq \tau c_6\Big(\|A\|\|x(0)\| + \|B\|(\|K\| + r)\|y(0)\|\Big),\end{aligned}$$

so that x is uniformly continuous. Now as $(I - Df)^{-1}$ is locally Lipschitz, there exists $L > 0$ such that $\left\|(I-Df)^{-1}(\xi_1) - (I-Df)^{-1}(\xi_2)\right\| \leq L\|\xi_1 - \xi_2\|$ for all $\xi_1, \xi_2 \in \mathbb{B}_{\mathbb{F}}(0, c_6\|x(0)\|)$. Since $y(t) = (I-Df)^{-1}(Cx(t))$ and $x(t) \in \mathbb{B}_{\mathbb{F}}(0, c_6\|x(0)\|)$ for all $t \geq 0$, the uniform continuity of x also implies the uniform continuity of y.

We will first show that $\lim_{t\to\infty} y(t) = 0$. By Lemma 7.5.2, we know that $\lim_{t\to\infty} \operatorname{dist}(y(t), \Omega_y) = 0$, so it suffices to show that $\Omega_y = \{0\}$. For this we employ an argument borrowed from [30]. Suppose on the contrary, that there exists a nonzero $\xi_0 \in \Omega_y$ and pick $\varepsilon > 0$ such that $0 \notin \overline{\mathbb{B}_{\mathbb{F}}(\xi_0,\varepsilon)}$. Hence, by continuity of f, there exists $c_7 > 0$ such that $\|f(\xi) - K\xi\|^2 - r^2\|\xi\|^2 \leq -c_7$ for all $\xi \in \mathbb{B}_{\mathbb{F}}(\xi_0,\varepsilon)$. Furthermore let $(t_k)_{k\in\mathbb{N}_0} \subseteq [0,\infty)$ be such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} y(t_k) = \xi_0$; set $m \in \mathbb{N}_0$ to be such that $\|y(t_k) - \xi_0\| \leq \varepsilon/2$ for all $k \geq m$. By the Bolzano-Weierstrass theorem, we can assume, without loss of generality, that $(x(t_k))_{k\in\mathbb{N}_0}$ converges and we denote this limit by χ. Now note that by equation (8.2.5) and assumption (8.2.2), the function $t \mapsto V(x(t))$ is decreasing and bounded from below by 0, so it converges to a limit. By uniform continuity of x and y, there exists $\tau > 0$ such that we have $\|x(t+\tau_1) - x(t)\| \leq \varepsilon/2$ and $\|y(t+\tau_1) - y(t)\| \leq \varepsilon/2$ for all $t \geq 0$ and for all $0 \leq \tau_1 \leq \tau$. Thus, by (8.2.5), we have

$$(V \circ x)'(t_k + \tau_1) \leq -c_7 \qquad \forall\, k \geq m, \quad \forall\, \tau_1 \in [0, \tau] \tag{8.2.9}$$

and hence integration gives us

$$V(x(t_k + \tau_1)) \leq -c_7\tau_1 + V(x(t_k))$$

for all $k \geq m$. This however contradicts the convergence of $t \mapsto V(x(t))$ and in turn shows that $\lim_{t\to\infty} y(t) = 0$.

Finally, note that by Lemma 7.1.3 for any $t > t_1 \geq 0$ we have

$$x(t) = e^{A_K(t-t_1)}x(t_1) + \int_{t_1}^{t} e^{A_K(t-s)}B_K[f(y(s)) - Ky(s)]\,\mathrm{d}s.$$

As in (a), $\sigma(A_K) \subset \mathbb{C}_-$ and by continuity of f we have $\lim_{t\to\infty} f(y(t)) - Ky(t) = 0$. Hence $\lim_{t\to\infty} x(t) = 0$, which completes the proof of (b).

Finally, (c) can be proved using exponential weighting arguments similar to the ones used in the proof of Theorem 15 from [30], however we will exhibit an alternative proof that uses the quadratic form given by Lemma 7.4.1.

First, recall equation (8.2.5) and use assumption (8.2.3) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \leq \|f(y(t)) - Ky(t)\|^2 - r^2\,\|y(t)\| \leq -\delta\,\|y(t)\|^2 \qquad \forall\, t \in [0, \infty)$$

and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$.

Now, by Lemmas 8.1.2 and 7.4.1, there exists a positive definite $Q \in \mathbb{F}^{n\times n}$ such that, for some $\delta_1 > 0$, the function $U\colon \mathbb{F}^n \to [0, \infty)$ defined by $U(\xi) := \langle Q\xi, \xi\rangle$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}U(x(t)) \leq -\delta_1\,\|x(t)\|^2 + \|y(t)\|^2 + \|f(y(t))\|^2 \qquad \forall\, t \in [0, \infty)$$

and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$. We can use the simple estimate $\|f(\xi)\| \leq (\|K\| + r)\,\|\xi\|$ to see that if we set $b := \frac{\delta}{1+(\|K\|+r)^2}$ and $\delta_2 := \delta_1 b$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(bU + V\right)(x(t)) \leq -\delta_2\,\|x(t)\|^2 \qquad \forall\, t \in [0, \infty)$$

and for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$. Define $h(t) := (bU + V)(x(t))$ and note that, by definitions of U and V, there exist positive c_8 and c_9 such that $c_8 \|x(t)\|^2 \leq h(t) \leq c_9 \|x(t)\|^2$. Therefore,

$$\dot{x}(t) \leq -\delta_2 \|x(t)\|^2 \leq -\frac{\delta_2}{c_9} h(t) \qquad \forall\, t \in [0, \infty)$$

and subsequently, by Gronwall's inequality, $h(t) \leq h(0)e^{-at}$ for all $t \geq 0$, where $a := \delta_2/c_9$. Hence

$$\|x(t)\|^2 \leq \frac{c_9}{c_8} e^{-at} \|x(0)\|^2 \qquad \forall t \in [0, \infty)$$

and an application of estimate (8.2.8) completes the proof. □

Note that for systems with no feedthrough, the Aizerman version of the circle criterion takes a slightly simpler form. Also, we will not pursue this here, but time-variant versions of Proposition 8.2.1 (a) and (c) hold as long as the nonlinearity f satisfies the relevant assumptions uniformly in the time variable.

By picking $K = 0$ and by using Lemma 7.6.12 to see that $\mathbb{B}_{\mathbb{C}}(0, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ is equivalent to $\|G\|_{H^\infty} \leq \frac{1}{r}$, we obtain the following corollary, which resembles the small-gain theorem.

Corollary 8.2.2. *Consider a Lur'e system (A, B, C, D, f), assume that the underlying linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable and denote its transfer function by G.*

(a) If $\|D\| < \|G\|_{H^\infty}$ and

$$\|G\|_{H^\infty} \|f(\xi)\| \leq \|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p,$$

then the Lur'e system (A, B, C, D, f) is globally stable.

(b) If $\|D\| < \|G\|_{H^\infty}$ and

$$\|G\|_{H^\infty} \|f(\xi)\| < \|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p \setminus \{0\},$$

then the Lur'e system (A, B, C, D, f) is globally asymptotically stable.

(c) If

$$\|G\|_{H^\infty} \sup_{\xi \in \mathbb{F}^p} \frac{\|f(\xi)\|}{\|\xi\|} < 1,$$

then the Lur'e system (A, B, C, D, f) is globally exponentially stable.

Statement (c) is a time-invariant extension of the small gain theorem to the case when the underlying state-space system is only stabilizable and detectable. For a standard version of the small-gain theorem, see Theorem 3.1 from Haddad and Bernstein [21].

The following examples demonstrate that Proposition 8.2.1 is not conservative.

Example 8.2.3. Consider the stabilizable and detectable one-dimensional SISO system $(-1, 1, 1, 0) \in \Sigma(1, 1, 1; \mathbb{R})$. We can calculate $G(s) = \frac{1}{s+1}$, so that $\|G\|_{H^\infty} = 1$ and thus $\mathbb{B}_{\mathbb{C}}(0, 1) \subseteq \mathbb{S}_{\mathbb{C}}(G)$. Thus the Aizerman version of the circle criterion (a) shows us that if the nonlinearity f satisfies $|f(\xi)| \leq |\xi|$ for all $\xi \in \mathbb{R}$, then the Lur'e system $(-1, 1, 1, 0, f)$ is globally stable. Indeed, we can pick $f(\xi) := \xi$ to obtain

$$\dot{x}(t) = 0,$$

which defines a Lur'e system that is globally stable, but not globally asymptotically stable.

Example 8.2.4. Again, consider the stabilizable and detectable SISO system $(-1, 1, 1, 0) \in \Sigma(1, 1, 1; \mathbb{R})$; we know that $\mathbb{B}_{\mathbb{C}}(0, 1) \subseteq \mathbb{S}_{\mathbb{C}}(G)$. Now set $f(\xi) := \xi - \min\left\{\xi^2, \frac{\xi}{2}\right\}$ and note that, upon considering ξ close to 0, we can see there does not exist a $\delta > 0$ such that $|f(\xi)| < (1 - \delta)|\xi|$. Thus we can apply the Aizerman version of circle criterion (b) to infer asymptotic stability of $(-1, 1, 1, 0, f)$, but not the Aizerman version of the circle criterion (c). Indeed, one can check that if $x(0) = 1/2$, then the unique solution of the initial value problem

$$\dot{x}(t) = -x(t) + f(x(t)) \qquad \forall\, t \in [0, \infty),\ x(0) = 1/2$$

is given by $x(t) = \frac{1}{2+t}$, which clearly does not decay exponentially.

Note that the statements (a) - (c) of Proposition 8.2.1 indicate a gradual change in modes of stability: for a given underlying linear system, stronger assumptions on the nonlinearity f result in a stronger mode of stability.

8.3 A note on matrix stability

We now note a simple consequence of the Aizerman version of the circle criterion applied to matrix stability theory. Recall the definition of the structured complex stability radius

$$r_{\mathbb{C}}(A; B, C) := \inf\{\|K\| : K \in \mathbb{C}^{m \times p} \text{ and } \sigma(A_K) \not\subseteq \mathbb{C}_-\}.$$

It is well-known that there exists a "destabilizing" output feedback matrix $F \in \mathbb{C}^{m\times p}$ of minimal norm, that is, $A+BFC$ is not globally asymptotically stable and $\|F\| = r_{\mathbb{C}}(A; B, C)$. The following corollary of Aizerman version of the circle criterion shows that the application of a "destabilizing" output feedback matrix of minimal norm results in a marginally stable closed-loop system. Somewhat surprisingly, this result does not seem to be available in the literature.

Corollary 8.3.1. *Consider $(A, B, C, 0) \in \Sigma(m, n, p; \mathbb{F})$ with $\sigma(A) \subseteq \mathbb{C}_-$ and let $F \in \mathbb{C}^{m\times p}$ be such that $\sigma(A+BFC) \not\subseteq \mathbb{C}_-$ and $\|F\| = r_{\mathbb{C}}(A; B, C)$. Then $\sigma(A+BFC) \subseteq \overline{\mathbb{C}_-}$ and all $\lambda \in \sigma(A+BFC)$ with $\operatorname{Re}\lambda = 0$ are semisimple.*

Proof. This follows from the Aizerman version of the circle criterion (a) applied to the Lur'e system $(A, B, C, 0, F)$. □

8.4 "Standard" version of the circle criterion

Using the Aizerman version of the circle criterion we now obtain a result that is reminiscent of the circle criterion (compare with Theorem 5.1 from Haddad and Bernstein [21]).

Proposition 8.4.1. *Consider a Lur'e system (A, B, C, D, f) and assume that the underlying linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable and denote its transfer function by G. Let $K_1, K_2 \in \mathbb{F}^{m\times p}$ and assume that $K_1 \in \mathbb{A}_{\mathbb{C}}(D)$, $(I - K_2G)(I - K_1G)^{-1}$ is positive real and that - for $K := \frac{1}{2}(K_1 + K_2)$ and $L := \frac{1}{2}(K_1 - K_2)$ - we have $\left\|LD^K\right\| < 1$.*

(a) If $\ker(K_1 - K_2) = \{0\}$ and if

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \le 0 \qquad \forall\, \xi \in \mathbb{F}^p, \tag{8.4.1}$$

then the Lur'e system (A, B, C, D, f) is globally stable.

(b) If

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle < 0 \qquad \forall\, \xi \in \mathbb{F}^p, \tag{8.4.2}$$

then the Lur'e system (A, B, C, D, f) is globally asymptotically stable.

(c) If for some positive δ we have

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \le -\delta\,\|\xi\|^2 \qquad \forall\, \xi \in \mathbb{F}^p, \tag{8.4.3}$$

then the Lur'e system (A, B, C, D, f) is globally exponentially stable.

Remark: note that the requirement $\left\|LD^K\right\| < 1$ is satisfied trivially if $D = 0$.

We omit the proof of Proposition 8.4.1 as it follows, *mutatis mutandis*, in the same way as Proposition 4.3.2.

Also, Proposition 8.4.1 (c), when restricted to controllable and observable underlying linear systems, is equivalent to Theorem 5.1 from Haddad and Bernstein [21], however much like in the discrete-time case and for the same reasons, this is not straightforward to demonstrate, so we omit a detailed comparison.

Chapter 9

Input-to-state stability of Lur'e systems

In this chapter we finally arrive at the essence of Part II. We will see that under assumptions similar to the ones we made in absolute stability results, namely Propositions 8.2.1 and 8.4.1, we in fact obtain input-to-state stability (from here onwards, ISS) of Lur'e systems with forcing

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(f(y(t)) + d(t)) \\ y(t) &= Cx(t) + D(f(y(t)) + d(t)). \end{aligned} \tag{9.1}$$

As mentioned previously, ISS is a recent notion of stability, which provides a natural framework for stability analysis of nonlinear systems with inputs. Ever since its inception in Sontag [50] (1989), ISS has been a busy area of research as evidenced by the growing body of work collected in the overview article Sontag [52], which collects 128 references to ISS-related papers.

Recall the introduction of Part II, which describes a number of approaches to proving ISS for Lur'e systems under assumptions similar to those made in absolute stability results. All of them make use of the ISS-Lyapunov function characterization of ISS and construct an appropriate ISS-Lyapunov function. [7, 29, 30] use the positive real lemma and output injection to obtain two quadratic forms, which are then combined into an ISS-Lyapunov function. [12] analyses a standard Lur'e-Postnikov Lyapunov function. We will follow in the footsteps of the above work in obtaining ISS results, however we will use the bounded real lemma instead of the positive real lemma for one of our quadratic forms. As a consequence, our results will apply to a different class of systems and we will have to develop new estimates involving comparison functions to establish that our construction yields an ISS-Lyapunov function, see Theorem 9.2.1. It will turn out that, by making precisely the same assumptions on the underlying state-space system (A, B, C, D) as in

Proposition 8.2.1, but assuming that, for some $\alpha \in \mathcal{K}_\infty$, the norm condition $\|f(\xi)\| \leq r\,\|\xi\| - \alpha(\|\xi\|)$ is satisfied for all $\xi \in \mathbb{F}^p$, we in fact obtain ISS of (9.1). As a corollary we will obtain a result, which resembles the circle criterion and allows us to obtain a number of results from Jayawardhana et al. [29] as corollaries, see Proposition 9.3.1 and the subsequent Corollaries 9.3.2 and 9.3.3.

Finally, we will introduce a stronger version of ISS that we call exponential input-to-state stability. It will turn out that under the assumptions made in the small-gain theorem we obtain exponential ISS.

This chapter is organized as follows: we introduce Lur'e systems with forcing and define ISS in §9.1. We also note an ISS-Lyapunov characterization of ISS. Then in §9.2 and 9.3 we state and prove results that guarantee ISS under assumptions similar to the ones made in absolute stability results in Chapter 8. Finally, in §5.5 we introduce exponential ISS and show that it is guaranteed under the assumptions made in the small-gain theorem.

9.1 Input-to-state stability

If we assume that $I - Df$ is invertible, then - for a prescribed $d \in L^\infty_{\mathrm{loc}}(\mathbb{F}^m)$ - the Lur'e system with forcing (9.1) gives rise to the initial value problem

$$\begin{aligned}\dot{x}(t) = Ax(t) + Bf \circ (I - Df)^{-1}(Cx(t) + Dd(t)) + Bd(t)&\\ x(0) = \xi \in \mathbb{F}^n.&\end{aligned} \qquad (9.1.1)$$

Standard ordinary differential equations theory (see e.g. Appendix C from Sontag [51]) tells us that, as long as f and $(I - Df)^{-1}$ are locally Lipschitz, (9.1.1) admits a unique maximal solution $x \in AC(\mathbb{F}^n)$. That is, it admits $x \in AC(\mathbb{F}^n)$, defined on some maximal interval $[0, \omega) \subseteq [0, \infty)$, such that it solves (9.1.1) almost everywhere and such that if x_1 is any other solution of (9.1.1), defined on $[0, \omega_1) \subseteq [0, \infty)$, then $[0, \omega_1) \subseteq [0, \omega)$ and $x_1 = x$ on $[0, \omega_1)$. The unique solution x with the above property is called the maximal solution. Furthermore, it is well-known that, if $x\colon [0, \omega) \to \mathbb{F}^n$ is a maximal solution of (9.1.1) and if $\omega < 0$, then

$$\lim_{t \to \omega} \|x(t)\| = \infty.$$

For a more detailed treatment, see the Appendix, §D.

Definition 9.1.1. Consider a Lur'e system (A, B, C, D, f) with the underlying linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and $f\colon \mathbb{F}^p \to \mathbb{F}^m$. We define the **behaviour with disturbances of** (A, B, C, D, f) as

$$\mathcal{B}_{\mathrm{d}}(A,B,C,D,f) := \Big\{ (d,x,y) \in L^\infty_{\mathrm{loc}}(\mathbb{F}^m) \times AC(\mathbb{F}^n) \times L^\infty_{\mathrm{loc}}(\mathbb{F}^n) : \\ x \text{ is a maximal solution of (9.1.1) and} \\ y(t) = Cx(t) + D(f(y(t)) + d(t)) \Big\}.$$

The following lemma is a simple consequence of the definition of trajectories.

Lemma 9.1.2. *Consider a Lur'e system* (A, B, C, D, f). *If* $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$, *then* $(f \circ y + d, x, y) \in \mathcal{B}(A, B, C, D)$.

For $t \geq 0$ we define a family of projection operators $\pi_t \colon L^\infty_{\mathrm{loc}}([0,\infty), \mathbb{F}^m) \to L^\infty_{\mathrm{loc}}([0,\infty), \mathbb{F}^m)$ as

$$(\pi_t d)(s) := \begin{cases} d(s), & \text{when } s \leq t, \\ 0, & \text{when } s > t. \end{cases}$$

We now define ISS for Lur'e systems. We denote by (continuous-time)-$\mathcal{KL}$ the set of functions in two variables, $\beta \colon [0,\infty) \times [0,\infty) \to [0,\infty)$ with the following properties: if $\beta \in \mathcal{KL}$, then, for each fixed t, $\beta(\cdot, t) \in \mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is non-increasing and $\lim_{t\to\infty} \beta(s,t) = 0$.

Definition 9.1.3. Let (A, B, C, D, f) be a Lur'e system. We say that it is (globally) **input-to-state stable** if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$ we have

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|\pi_t d\|_\infty) \qquad \forall\, t \in [0,\infty). \tag{9.1.2}$$

Here we have assumed that trajectories $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$ are global. Even if we did not do that, one can see that the estimate (9.1.2) would force any maximal solution of the initial value problem (9.1.1) to be global.

Remark: equation (9.1.2) is equivalent to

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\sup_{0\leq s\leq t} \|d(s)\|).$$

Recall that we call two functions $U, V \colon \mathbb{F}^n \to [0,\infty)$ $\mathcal{K}_\infty$-equivalent if there exist $\alpha, \gamma \in \mathcal{K}_\infty$ such that $\alpha(U(\xi)) \leq V(\xi) \leq \gamma(U(\xi))$ for all $\xi \in \mathbb{F}^n$.

As in discrete-time, there exists a useful dissipation characterization of ISS.

Definition 9.1.4. Let (A, B, C, D, f) be a Lur'e system with $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$. A positive-definite continuously differentiable function $V : \mathbb{F}^n \to [0, \infty)$ is said to be an **ISS-Lyapunov function for** (A, B, C, D, f) if V and $\|\cdot\|_{\mathbb{F}^n}$ are $\mathcal{K}_\infty$-equivalent and if there exist $\alpha, \gamma \in \mathcal{K}_\infty$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq -\alpha(\|x(t)\|) + \gamma(\|d(t)\|) \qquad \text{a.e.}$$

for all $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$.

We omit the proof of the following theorem, which says that the existence of an ISS-Lyapunov function implies ISS. In slightly different frameworks this has been proved in Sontag [50] and Logemann and Ryan [43].

Theorem 9.1.5. *Consider a Lur'e system (A, B, C, D, f). If there exists an ISS-Lyapunov function for it, then (A, B, C, D, f) is ISS.*

It is interesting to note that the converse result is true as well, see Lin, Sontag and Wang [41].

9.2 Ball condition assumptions

We are now ready to prove our main result of this part, an ISS criterion under assumptions similar to the ones made in the Aizerman version of the circle criterion. We will achieve this by combining quadratic forms obtained from (i) a "ball condition" and the bounded real lemma, and (ii) output injection.

Theorem 9.2.1. *Consider a Lur'e system (A, B, C, D, f) and assume that the underlying state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable, and assume that its transfer function G satisfies $\|D^K\| < \|G^K\|_{H^\infty}$. Furthermore, let $r > 0$, $K \in \mathbb{F}^{m \times p}$ and assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.*

If there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\|f(\xi) - K\xi\| \leq r\|\xi\| - \alpha(\|\xi\|) \qquad \forall\, \xi \in \mathbb{F}^p, \tag{9.2.1}$$

then the Lur'e system (A, B, C, D, f) is input-to-state stable.

Remark: the assumption $\|D^K\| < \|G^K\|_{H^\infty}$ is satisfied, for example, if $D = 0$.

Proof. By Theorem 9.1.5, it is sufficient to exhibit an ISS-Lyapunov function for (A, B, C, D, f). We do this by constructing two functions V and W and then showing that $V + W$ is an ISS-Lyapunov function.

Since $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$, Corollary 7.6.13 provides us with the existence of a positive semi-definite $P = P^* \in \mathbb{F}^{n\times n}$ such that the quadratic form $V : \mathbb{F}^n \to [0,\infty)$ defined by $V(\xi) = \langle P\xi, \xi\rangle$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq \|u(t) - Ky(t)\|^2 - r^2 \|y(t)\|^2 \qquad \text{a.e.}$$

for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Hence, by Lemma 9.1.2, for arbitrary $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq \|f(y(t)) + d(t) - Ky(t)\|^2 - r^2 \|y(t)\|^2 \qquad \text{a.e.} \tag{9.2.2}$$

Using (9.2.1) we can estimate

$$\begin{aligned} \|f(\xi) - K\xi\|^2 - r^2 \|\xi\|^2 &\leq -2\alpha(\|\xi\|)r \|\xi\| + \alpha(\|\xi\|)^2 \\ &\leq -2\alpha(\|\xi\|)r \|\xi\| + \alpha(\|\xi\|)r \|\xi\| \\ &= -r \|\xi\| \alpha(\|\xi\|) \qquad \forall\, \xi \in \mathbb{F}^p. \end{aligned}$$

Thus, by the Cauchy-Schwarz inequality as well as Lemma 5.1.9, there exists $\gamma \in \mathcal{K}_\infty$ such that

$$\begin{aligned} \|f(\xi) - K\xi + \mu\|^2 &- r^2 \|\xi\|^2 \\ &\leq \|f(\xi) - K\xi\|^2 + 2 \|f(\xi) - K\xi\| \|\mu\| \\ &\quad + \|\mu\|^2 - r^2 \|\xi\|^2 \\ &\leq - r \|\xi\| \alpha(\|\xi\|) + 2r \|\xi\| \|\mu\| + \|\mu\|^2 \\ &\leq - \frac{r}{2} \|\xi\| \alpha(\|\xi\|) + \gamma(\|\mu\|) \qquad \forall\, \xi \in \mathbb{F}^p,\ \mu \in \mathbb{F}^m. \end{aligned}$$

If we use this estimate in (9.2.2), then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq -\frac{r}{2} \|y(t)\| \alpha(\|y(t)\|) + \gamma(\|d(t)\|) \qquad \text{a.e.} \tag{9.2.3}$$

for all $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$.

On the other hand, (A, B, C, D) is detectable, so Lemma 7.4.1 guarantees the existence of a positive definite $Q = Q^* \in \mathbb{F}^{n\times n}$ and $\delta_1 > 0$ such that the quadratic form $U_1 : \mathbb{F}^n \to [0,\infty)$ defined by $U_1(\xi) = \langle Q\xi, \xi\rangle$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t} U_1(x(t)) \leq -\delta_1 \|x(t)\|^2 + \|y(t)\|^2 + \|u(t)\|^2$ almost everywhere for all trajectories $(u, x, y) \in \mathcal{B}(A, B, C, D)$. Equivalently, by Lemma 9.1.2, for $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} U_1(x(t)) \leq -\delta_1 \|x(t)\|^2 + \|y(t)\|^2 + \|f(y(t)) + d(t)\|^2 \qquad \text{a.e.} \tag{9.2.4}$$

Set $c_1 := 4\|K\|^2 + 4r^2 + 1$ and use (9.2.1) as well as $\|\xi_1 + \xi_2\|^2 \leq 2\|\xi_1\|^2 + 2\|\xi_2\|^2$ repeatedly to arrive at the estimate $\|\xi\|^2 + \|f(\xi) + \mu\|^2 \leq c_1 \|\xi\|^2 +$

$2\|\mu\|^2$ for all $\xi \in \mathbb{F}^p, \mu \in \mathbb{F}^m$. Using equation (9.2.4) we can thus see that - for $\delta := \frac{\delta_1}{\max\{c_1,2\}}$ - the function $U := \frac{1}{\max\{c_1,2\}}U_1$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}U(x(t)) \leq -\delta\|x(t)\|^2 + \|y(t)\|^2 + \|d(t)\|^2 \qquad \text{a.e.} \tag{9.2.5}$$

for all $(d,x,y) \in \mathcal{B}_{\mathrm{d}}(A,B,C,D,f)$.

We will now complete the proof by constructing a function $h : [0,\infty) \to [0,\infty)$ such that $V + h \circ U$ is an ISS-Lyapunov function for (A,B,C,D,f). We will specify the function h later, but we will require it to have two properties, which will be used in the estimations below. When we finally define an appropriate h, we will then check that it has said properties. We require h to satisfy:

$$h \in C^1([0,\infty)) \tag{9.2.6}$$

$$\exists\, c_2 > 0 \quad \text{such that} \quad 0 \leq h'(s) \leq c_2 \qquad \forall\, s \in [0,\infty). \tag{9.2.7}$$

Now let us analyse the properties of $h \circ U$. Since h and U are both continuously differentiable we can use the chain rule to see that so is $h \circ U$ and moreover $[\nabla_{\mathbb{F}}(h\circ U)](\xi) = h'(U(\xi))\nabla_{\mathbb{F}}U(\xi)$ for all $\xi \in \mathbb{F}^n$. We now use Lemma 7.2.4 as well as equations (9.2.7), (9.2.3) and (9.2.5) to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\big(V+h\circ U\big)(x(t))\\ &\leq -\frac{r}{2}\|y(t)\|\,\alpha(\|y(t)\|) + \gamma(\|d(t)\|)\\ &\qquad + h'(U(x(t)))[-\delta\|x(t)\|^2 + \|y(t)\|^2 + \|d(t)\|^2] \qquad \text{a.e.}\end{aligned} \tag{9.2.8}$$

for all $(d,x,y) \in \mathcal{B}_{\mathrm{d}}(A,B,C,D,f)$.

Recall that it suffices to find h such that $V + h \circ U$ is an ISS-Lyapunov function for (A,B,C,D,f). In view of equation (9.2.8), it is sufficient to find h satisfying (9.2.6) and (9.2.7) such that for some $\eta, \gamma_1 \in \mathcal{K}_\infty$ we have

$$\begin{aligned}h'(U(x(t)))[-\delta\|x(t)\|^2 + \|y(t)\|^2 + \|d(t)\|^2] - \frac{r}{2}\|y(t)\|\,\alpha(\|y(t)\|) + \gamma(\|d(t)\|)\\ \leq -\eta(\|x(t)\|) + \gamma_1(\|d(t)\|) \qquad \text{a.e.}\end{aligned} \tag{9.2.9}$$

for all $(d,x,y) \in \mathcal{B}_{\mathrm{d}}(A,B,C,D,f)$. We now make a change in notation, which goes against the conventions established this far in the present document. However, we believe it is the easiest way to do the proof at hand. From here until the end of this proof x will be an element of $\mathbb{F}^n$, $y \in \mathbb{F}^p$ and $d \in \mathbb{F}^m$. Now, by (9.2.9), the proof of this theorem is complete if we can find h satisfying (9.2.6) and (9.2.7), such that $V + h \circ U$ is $\mathcal{K}_\infty$-equivalent to $\|\cdot\|_{\mathbb{F}^n}$ and such that for some $\eta, \gamma_1 \in \mathcal{K}_\infty$ we have

$$\begin{aligned}h'(U(x))[-\delta\|x\|^2 + \|y\|^2 + \|d\|^2] - \frac{r}{2}\|y\|\,\alpha(\|y\|) + \gamma(\|d\|)\\ \leq -\eta(\|x\|) + \gamma_1(\|d\|)\end{aligned}$$

for all $x \in \mathbb{F}^n$, $d \in \mathbb{F}^m$ and for $y \in \mathbb{F}^p$ with $y = Cx + D(f(y)+d)$ (equivalently, $y = (I - Df)^{-1}(Cx + Dd)$).

Set $\gamma_1(s) := \gamma(s) + c_2 s^2 + \gamma_2(s)$ for $\gamma_2 \in \mathcal{K}_\infty$ to be determined later, to see that h has to satisfy

$$\eta(\|x\|) + \|y\|^2 h'(U(x)) \leq \delta \|x\|^2 h'(U(x)) + \frac{r}{2} \|y\| \alpha(\|y\|) + \gamma_2(\|d\|) \tag{9.2.10}$$

for all $x \in \mathbb{F}^n$, $d \in \mathbb{F}^m$ and for $y = Cx + D(f(y) + d)$. The key idea now is the observation that it is clearly sufficient to satisfy both

$$\eta(\|x\|) \leq \frac{\delta}{2} \|x\|^2 h'(U(x)) \qquad \text{and} \tag{9.2.11}$$

$$\|y\|^2 h'(U(x)) \leq \max\left\{\frac{\delta}{2} \|x\|^2 h'(U(x)),\ \frac{r}{2} \|y\| \alpha(\|y\|)\right\} + \gamma_2(\|d\|) \tag{9.2.12}$$

for all $x \in \mathbb{F}^n$, $d \in \mathbb{F}^m$ and for $y = Cx + D(f(y) + d)$.

We will first look to satisfy inequality (9.2.12). If $\|y\|^2 \leq \frac{\delta}{2} \|x\|^2$, then $\|y\|^2 h'(U(x)) \leq \frac{\delta}{2} \|x\|^2 h'(U(x))$ and thus inequality (9.2.12) holds. Hence it is sufficient to find h such that for $\|y\|^2 > \frac{\delta}{2} \|x\|^2$ we have

$$\|y\|^2 h'(U(x)) \leq \frac{r}{2} \|y\| \alpha(\|y\|) + \gamma_2(\|d\|) \tag{9.2.13}$$

for all $x \in \mathbb{F}^n$, $d \in \mathbb{F}^m$ and for $y := Cx + D(f(y) + d)$. Since $\|y\|^2 > \frac{\delta}{2} \|x\|^2$, if we set $c_3 := \frac{r}{2}\sqrt{\frac{\delta}{2}}$ and $c_4 := \sqrt{\frac{\delta}{2}}$, then we obtain $\frac{r}{2} \|y\| \alpha(\|y\|) > c_3 \|x\| \alpha(c_4 \|x\|)$. Moreover, we can use $\left\|D^K\right\| < \frac{1}{r}$ (since (A, B, C, D) is stabilizable and detectable, this follows from Lemmas 7.6.5 and 7.6.12) to see that there exist positive constants c_5, c_6 such that $\|y\|^2 \leq c_5 \|x\|^2 + c_6 \|d\|^2$. Set $\gamma_2(s) := c_2 c_6 s^2$ to see that we have (9.2.13) as long as

$$c_5 \|x\| h'(U(x)) \leq c_3 \alpha(c_4 \|x\|) \tag{9.2.14}$$

for all $x \in \mathbb{F}^n$. We will now define h' that satisfies (9.2.14) in two steps. By the definition of U, we know that there exist positive constants c_7, c_8 such that $c_7 \|x\|^2 \leq U(x) \leq c_8 \|x\|^2$ for all $x \in \mathbb{F}^n$. Now set $\tilde{h}'\colon (0, \infty) \to \left(0, \frac{r c_3 c_4}{c_5}\sqrt{\frac{c_7}{c_8}}\right]$ to be

$$\tilde{h}'(s) := \frac{c_3 \sqrt{c_7} \alpha\left(\frac{c_4 \sqrt{s}}{\sqrt{c_8}}\right)}{c_5 \sqrt{s}}$$

and define a (clearly continuous) function $h'\colon [0, \infty) \to \left[0, \frac{r c_3 c_4}{c_5}\sqrt{\frac{c_7}{c_8}}\right]$ as

$$h'(s) := \begin{cases} \min\{s, \tilde{h}'(s)\} & \text{for } s > 0, \\ 0 & \text{for } s = 0. \end{cases} \tag{9.2.15}$$

If we then define h in the obvious way as $h(v) := \int_0^v h'(s)\,\mathrm{d}s$, then it satisfies both (9.2.6) and (9.2.7). We can also check that this h satisfies (9.2.14). The case $x = 0$ follows trivially, so assume $x \in \mathbb{F}^n \setminus \{0\}$. Then

$$\begin{aligned} h'(U(x)) \leq \tilde{h}'(U(x)) &= \frac{c_3\sqrt{c_7}\alpha\left(\frac{c_4\sqrt{U(x)}}{\sqrt{c_8}}\right)}{c_5\sqrt{U(x)}} \\ &\leq \frac{c_3\sqrt{c_7}\alpha\left(\frac{c_4\sqrt{c_8}\|x\|}{\sqrt{c_8}}\right)}{c_5\sqrt{c_7}\,\|x\|} \\ &= \frac{c_3\alpha(c_4\,\|x\|)}{c_5\,\|x\|} \end{aligned}$$

and inequality (9.2.14) follows.

Moreover, we can see that

$$\begin{aligned} \frac{\delta}{2}\|x\|^2 h'(U(x)) &\geq \frac{\delta}{2}\|x\|^2 \cdot \min\left\{ c_7\,\|x\|^2, \frac{c_3\sqrt{c_7}\alpha\left(\frac{c_4\sqrt{U(x)}}{\sqrt{c_8}}\right)}{c_5\sqrt{U(x)}} \right\} \\ &\geq \min\left\{ \frac{\delta c_7}{2}\|x\|^4, \frac{\delta c_3\sqrt{c_7}\,\|x\|\,\alpha\left(c_4\frac{\sqrt{c_7}}{\sqrt{c_8}}\,\|x\|\right)}{2c_5\sqrt{c_8}} \right\}, \end{aligned} \tag{9.2.16}$$

which is clearly a $\mathcal{K}_\infty$ function and thus (9.2.11) is satisfied if we define $\eta \in \mathcal{K}_\infty$ in the obvious way as the map on the right hand side of (9.2.16).

Hence the proof is complete, if we can show that $V + h \circ U$ is $\mathcal{K}_\infty$-equivalent to $\|\cdot\|_{\mathbb{F}^n}$. From the construction of V, we know that there exists a positive constant c_9 such that $0 \leq V(x) \leq c_9\,\|x\|^2$ for all $x \in \mathbb{F}^n$. It is then easy to obtain $V(x) + h(U(x)) \leq c_9\,\|x\|^2 + \int_0^{c_8\|x\|^2} c_2\,\mathrm{d}s = c_9\,\|x\|^2 + c_2c_8\,\|x\|^2$ for all $x \in \mathbb{F}^n$. For a lower bound, define $\sigma\colon [0,\infty) \to [0,\infty)$ by $\sigma(v) := \int_0^{c_7v^2} h'(s)\,\mathrm{d}s$ and note that $h(U(x)) \geq \sigma(\|x\|)$. Thus we only need to show that $\sigma \in \mathcal{K}_\infty$. Notice that, by the first fundamental theorem of calculus, σ is a continuous function and since $h' > 0$ on $(0,\infty)$, σ is strictly increasing, so that $\sigma \in \mathcal{K}$. Now $\sigma \in \mathcal{K}_\infty$ follows from

$$\begin{aligned} \int_0^\infty h'(s)\,\mathrm{d}s &= \int_0^\infty \min\left\{ s, \frac{c_3\sqrt{c_7}\alpha\left(\frac{c_4\sqrt{s}}{\sqrt{c_8}}\right)}{c_5\sqrt{s}} \right\} \mathrm{d}s \\ &> \int_1^\infty \min\left\{ 1, \frac{c_3\sqrt{c_7}\alpha\left(\frac{c_4}{\sqrt{c_8}}\right)}{c_5\sqrt{s}} \right\} \mathrm{d}s = \infty. \end{aligned}$$

$\square$

As the following example shows we cannot relax the assumption $\alpha \in \mathcal{K}_\infty$ in Theorem 9.2.1 to $\alpha \in \mathcal{K}$.

Example 9.2.2. Consider the stabilizable and detectable linear system from Example 8.2.3 - $(-1, 1, 1, 0) \in \Sigma(1, 1, 1; \mathbb{R})$ - and recall that its transfer function G was such that $\mathbb{B}_{\mathbb{C}}(0, 1) \subseteq \mathbb{S}_{\mathbb{C}}(G)$. Now consider the deadzone nonlinearity

$$f(\xi) := \begin{cases} \xi + 1, & \text{if } \xi < -1 \\ 0, & \text{if } -1 \leq \xi \leq 1 \\ \xi - 1, & \text{if } \xi > 1. \end{cases}$$

Note that for every $\alpha \in \mathcal{K}$ with $\alpha(s) \leq \min\{1, s\}$, (for example, $\alpha(s) := 1 - e^{-s}$) we have

$$|f(\xi)| \leq |\xi| - \alpha(|\xi|),$$

while there clearly does not exist $\alpha \in \mathcal{K}_\infty$ such that this inequality holds. If we then pick forcing d such that $d(t) = 2$ for all $t \geq 0$ and $x(0) \geq -1$, then $\dot{x}(t) \geq 1$ and thus the Lur'e system $(-1, 1, 1, 0, f)$ is not ISS.

If $F \in \mathbb{B}_{\mathbb{C}}(K, r)$, then we can define $\alpha \in \mathcal{K}_\infty$ by $\alpha(s) := (r - \|F - K\|)s$ to see that Theorem 9.2.1 implies the Lur'e system (A, B, C, D, F) is ISS. Hence Theorem 9.2.1 states that if the Lur'e system (A, B, C, D, F) is ISS for *all complex linear* feedback matrices F such that $\|F\xi - K\xi\| < r\,\|\xi\|$, then the Lur'e system (A, B, C, D, f) is ISS for *all nonlinear* output feedback maps f such that $\|f(\xi) - K\xi\| < r\,\|\xi\| - \alpha(\|\xi\|)$ for some $\alpha \in \mathcal{K}_\infty$.

By picking $K = 0$ in Theorem 9.2.1 and by using Lemma 7.6.12 to see that $\mathbb{B}_{\mathbb{C}}(0, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ is equivalent to $\|G\|_{H^\infty} \leq \frac{1}{r}$, we obtain a small-gain version of Theorem 9.2.1.

Corollary 9.2.3. *Consider a Lur'e system (A, B, C, D, f), assume that the underlying state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable and denote its transfer function by G.*

If there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\|G\|_{H^\infty} \|f(\xi)\| \leq \|\xi\| - \alpha(\|\xi\|) \qquad \forall\, \xi \in \mathbb{F}^p,$$

then the Lur'e system (A, B, C, D, f) is ISS.

Example 9.2.4. Consider forced oscillations

$$\ddot{z}(t) + 4\dot{z}(t) + 5z(t) + f(z(t), \dot{z}(t)) + d(t) = 0,$$

which can be modelled by a Lur'e system, where the underlying linear state-space system is

$$\left(\begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \in \Sigma(2, 1, 2; \mathbb{R}).$$

One can check that the transfer function is given by $G(s) = \frac{1}{s^2+4s+5}\begin{pmatrix}1\\ s\end{pmatrix}$ and hence $\|G\|_{H^\infty} = 1/3.598$ (see Example 5.3.12 from [25]). Therefore, by Theorem 9.2.1, the forced oscillations are ISS as long as there exists $\alpha \in \mathcal{K}_\infty$ such that the nonlinearity f satisfies

$$\|f(\xi,\eta)\| \leq 3.598 \times \|(\xi,\eta)\| - \alpha(\|(\xi,\eta)\|) \qquad \forall\,(\xi,\eta) \in \mathbb{R}^2.$$

As a special case consider the forced Liénard system, where $f(\xi,\eta) = (g(\xi)-4)\eta$, so that

$$\ddot{z}(t) + g(z(t))\dot{z}(t) + 5z(t) + d(t) = 0. \tag{9.2.17}$$

Thus in particular if $|g(\xi) - 4| < 3.5$ for all $\xi \in \mathbb{R}$, then the Liénard system (9.2.17) is ISS.

Recall Proposition 8.2.1 (b), which guarantees the asymptotic stability of certain unforced Lur'e systems. Theorem 9.2.1 demonstrates that under only slightly stronger assumptions (yet weaker than Proposition 9.2.1 (c), which guarantee global exponential stability) we obtain the much stronger input-to-state stability of the associated class of forced Lur'e systems.

The construction of the ISS-Lyapunov function $W = h \circ U + V$ is inspired by a similar technique used in [7, 30, 29], however the context and hence the technical details are different. [7, 30, 29] use the positive real lemma to obtain V, while we use the bounded real lemma. Hence while the function U is (essentially) identical to the one used in [7, 30, 29], the estimates that we require $h \circ U$ to satisfy are different. The key novelty is the construction of an appropriate h that satisfies inequality (9.2.10).

In previous work on ISS for Lur'e systems one can find SISO results for systems with a sector-bounded nonlinearity. Therefore, it is interesting to note that the following corollary is an extension of Theorem 17 from Jayawardhana, Logemann and Ryan [30] (we should note that they admit set-valued nonlinearities, but the rest of the statement is unchanged). More precisely, if we use the shorthand established in [30], then our corollary is an extension of Theorem 17 as it reads under hypothesis (H1)).

Corollary 9.2.5. *Consider a SISO Lur'e system (A, b, c, d, f) and assume that the underlying linear system $(A, b, c, d) \in \Sigma(1, n, 1; \mathbb{R})$ is stabilizable and detectable and denote its transfer function by g. Let $k_1 < k_2$, assume that $k_1 \neq d^{-1}$ and $\frac{1-k_2 g}{1-k_1 g}$ is positive real.*

If there exists $\alpha \in \mathcal{K}_\infty$ such that

$$k_1\xi^2 + \xi\alpha(|\xi|) \leq f(\xi)\xi \leq k_2\xi^2 - \xi\alpha(|\xi|) \qquad \forall\,\xi \in \mathbb{R},$$

then the Lur'e system (A, b, c, d, f) is ISS.

We omit the proof of this result as it is identical, *mutatis mutandis*, to the proof of Corollary 5.3.4. We direct the reader to Figure 5.1 for a visual representation of a sector condition, where we have picked $\alpha(s) := \min\{s, \sqrt{s}\}$. Another appealing feature of Corollary 9.2.5 is that there are a number of characterizations of the positive real property, which in the SISO case can be especially simple, see §7.1 from Kailath [34] or §5.8 from Haddad and Chellaboina [23].

9.3 Positive real assumptions

As in the section on absolute stability, we can use Theorem 9.2.1 to obtain a result reminiscent of the circle criterion. It will allow us to obtain as corollaries a number of results from Jayawardhana et al. [29, 30].

Proposition 9.3.1. *Consider a Lur'e system (A, B, C, D, f) and assume that the underlying linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable and denote its transfer function by G. Let $K_1, K_2 \in \mathbb{F}^{m\times p}$ and assume that $(I - K_2G)(I - K_1G)^{-1}$ is positive real and that - for $K := \frac{1}{2}(K_1 + K_2)$ and $L := \frac{1}{2}(K_1 - K_2)$ - we have $\|LD^K\| < 1$.*

If there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \leq -\|\xi\|\,\alpha(\|\xi\|) \qquad \forall\, \xi \in \mathbb{F}^p, \tag{9.3.1}$$

then the Lur'e system (A, B, C, D, f) is ISS.

Remark: note that the requirement $\|LD^K\| < 1$ is satisfied trivially if $D = 0$.

We omit the proof of Proposition 9.3.1 as it is identical, *mutatis mutandis*, to the proof of Proposition 5.4.1.

The following result is a restatement of Theorem 17 from [30]. The only change is that we work with differential equations instead of differential inclusions that allow set valued nonlinearities f.

Corollary 9.3.2 (Theorem 17 from [30], hypotheses (H1))**.** *Consider a Lur'e system $(A, b, c, 0, f)$ and assume that the underlying linear system $(A, b, c, 0) \in \Sigma(1, n, 1; \mathbb{R})$ is controllable and observable and denote its transfer function by G. Let $b > 0$ and assume that $I + bG$ is positive real.*

If there exists $\alpha \in \mathcal{K}_\infty$ and a positive $\delta < b$ such that

$$|\xi|\alpha(|\xi|) \leq -f(\xi)\xi \leq (b - \delta)\xi^2 \qquad \forall\, \xi \in \mathbb{R},$$

then the Lur'e system $(A, b, c, 0, f)$ is ISS.

Proof. Pick $K_1 = 0$, $K_2 = -b$, so that, by Proposition 9.3.1, it suffices to show that there exists $\alpha_1 \in \mathcal{K}_\infty$ such that

$$(f(\xi) + b\xi)f(\xi) \leq -|\xi|\alpha_1(|\xi|) \qquad \forall\, \xi \in \mathbb{R}.$$

This follows from the two inequalities in the assumptions. We rewrite them as $|\xi|\alpha(|\xi|) \leq -f(\xi)\xi$ and $\delta\xi^2 \leq \xi(f(\xi) + b\xi)$ and multiply to obtain (note that numbers on both sides of the inequalities are non-negative, so multiplication does not change the sign of the inequality)

$$\delta|\xi|\alpha(|\xi|) \leq -f(\xi)(f(\xi) + b\xi) \qquad \forall\, \xi \in \mathbb{R}.$$ □

Notice that Corollary 9.2.5 is an extension of Corollary 9.3.2.

A related result (although in the multi-input multi-output setting) appears in Jayawardhana et al. [29] as Theorem 3.5 under hypothesis (H3). We should remark that they also look at differential inclusions instead of differential equations.

Corollary 9.3.3 (Theorem 3.5 from [29], hypothesis (H3))**.** *Consider a Lur'e system $(A, B, C, 0, f)$ and assume that the underlying linear system $(A, B, C, 0) \in \Sigma(m, n, m; \mathbb{R})$ is controllable and observable and denote its transfer function by G. Let b and δ be positive and assume that the rational function matrix $\frac{\delta}{b}I + G$ is positive real.*

If there exists $\phi \in \mathcal{K}_\infty$ such that

$$\max\left\{\|\xi\|\,\phi(\|\xi\|),\ \frac{1}{b}\,\|f(\xi)\|^2\right\} \leq -\langle \xi, f(\xi)\rangle \qquad \forall\, \xi \in \mathbb{R}^m,$$

then the Lur'e system $(A, B, C, 0, f)$ is ISS.

Proof. Note that $I + \frac{b}{\delta}G$ is positive real if, and only if, $\frac{\delta}{b}I + G$ is positive real. Hence, if we pick $K_1 = 0$ and $K_2 = -\frac{b}{\delta}I$, by Proposition 9.3.1, it suffices to show that there exists $\alpha \in \mathcal{K}_\infty$ such that $\left\langle f(\xi), f(\xi) + \frac{b}{\delta}\xi\right\rangle \leq -\|\xi\|\,\alpha(\|\xi\|)$ for all $\xi \in \mathbb{R}^p$. We pick $\alpha := \frac{b(1-\delta)}{\delta}\phi$ and estimate

$$\begin{aligned}\langle f(\xi), b\xi\rangle &\leq -\max\left\{\frac{\delta}{1-\delta}\,\|\xi\|\,\alpha(\xi),\, \|f(\xi)\|^2\right\}\\ &\leq -\delta\,\|\xi\|\,\alpha(\xi) - \delta\,\|f(\xi)\|^2 \qquad \forall\, \xi \in \mathbb{R}^p.\end{aligned}$$

Thus $\left\langle f(\xi), f(\xi) + \frac{b}{\delta}\xi\right\rangle \leq -\|\xi\|\,\alpha(\|\xi\|)$ for all $\xi \in \mathbb{R}^p$, which completes the proof. □

9.4 Exponential ISS

In this section we note that exponential weighting arguments allow us to prove that if in the assumptions of Theorem 9.2.1 we pick $\alpha(s) = \delta s$ for some positive δ, then we in fact obtain a stronger version of stability that we will call exponential input-to-state stability. It is also interesting to note that in contrast to most ISS-related results, this can be proved without Lyapunov function techniques. Therefore, these results might generalize to the infinite-dimensional setting.

Definition 9.4.1. Consider a Lur'e system (A, B, C, D, f). We say that it is (globally) **exponentially input-to-state stable** if there exist $c_1, c_2, a > 0$ such that for all $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$ we have

$$\|x(t)\| \leq c_1 e^{-at} \|x(0)\| + c_2 \|\pi_t d\|_\infty \qquad \forall\, t \in [0, \infty).$$

We use this result and the exponential weighting technique to obtain the following result. Note that the assumptions are the same as in Theorem 9.2.1, except that we pick $\alpha(s) := \delta s$.

We omit the proof of the following proposition as it is identical, *mutatis mutandis*, to the proof of its discrete-time counterpart, Proposition 5.5.3.

Proposition 9.4.2. *Consider a Lur'e system (A, B, C, D, f), assume that the underlying state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable and denote its transfer function by G. Furthermore, let $r > 0$, $K \in \mathbb{F}^{m\times p}$ and assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.*

If, for some $\delta > 0$,

$$\|f(\xi) - K\xi\| \leq (r - \delta)\|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p,$$

then the Lur'e system (A, B, C, D, f) is exponentially ISS.

As in the discrete-time setting, if we pick $K = 0$ in Proposition 9.4.2, then it shows that the assumptions made in the small-gain theorem actually guarantee exponential ISS.

Corollary 9.4.3. *Consider a Lur'e system (A, B, C, D, f), assume that the underlying state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is stabilizable and detectable and denote its transfer function by G. If*

$$\|G\|_{H^\infty} \cdot \sup_{\xi \in \mathbb{F}^p \setminus 0} \frac{\|f(\xi)\|}{\|\xi\|} < 1,$$

then the Lur'e system (A, B, C, D, f) is exponentially ISS.

Chapter 10

Notes, references and future work

10.1 Notes and references

The main result in Part II was Theorem 9.2.1, which showed that Lur'e systems with forcing are ISS under assumptions similar to ones made in results from absolute stability analysis and to the best of author's knowledge this result is new. Proving ISS from assumptions typical of absolute stability results is not a novel idea: (i) Arcak and Teel [7] consider Lur'e systems similar to ones for which the positivity theorem holds (see Theorem 4.1 from Haddad and Bernstein [21]), (ii) Jayawardhana, Logemann and Ryan [29, 30] obtain a number of ISS results (including the one in (i)) under various assumptions for Lur'e systems with set-valued nonlinearities, (iii) Bruin et al. [12] obtain an ISS version of the Popov criterion. However, unlike [7, 29, 30], we used the bounded real lemma instead of the positive real lemma for the construction of quadratic forms, which resulted in Theorem 9.2.1 - an Aizerman-like ISS result that seems to be quite general, since we obtained in Corollary 9.3.3 a result from [30] and we extended a result from [29] in Corollary 9.2.5. Also, it enabled us to consider Lur'e systems with nonzero feedthrough.

Theorem 9.2.1 also allowed us to prove an ISS version of the well-known circle criterion in Proposition 9.3.1, which seems to be a novelty.

Proposition 9.4.2, which proves exponential ISS under the assumptions of the small-gain theorem seems to be new, however, its proof introduces no new techniques and a similar result was proved in Jayawardhana et al. [30].

Chapter 8 on absolute stability results consisted of slight refinements of known absolute stability results, see Corollary 8.2.2 and Proposition 8.4.1.

However, these were proved in a new way by using another slight extension of a known result from Hinrichsen and Pritchard [25], the Aizerman version of the circle criterion. Apart from the intuitively appealing Aizerman perspective, it displayed in its statements (a) - (c) a transition of modes of stability as we change the assumptions on the nonlinearity. Examples 8.2.3 and 8.2.4 then showed that this transition is, in a sense, conservative. Moreover, by restricting output feedback maps in Lur'e systems to matrices, we used Proposition 8.2.1 (a) to prove the seemingly novel Corollary 8.3.1.

10.2 Future work

There are three interesting avenues for exploration that we have not taken due to time constraints.

It is well-known that a stability concept, called integral input-to-state stability (from here on, iISS) is equivalent to the existence of an iISS-Lyapunov function, which for a Lur'e system (A, B, C, D, f) is a function $V\colon \mathbb{F}^n \to [0, \infty)$ that is $\mathcal{K}_\infty$-equivalent to $\|\cdot\|$ and that satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq -\rho(\|x(t)\|) + \gamma(\|d(t)\|)$$

for all t and for all trajectories $(d, x, y) \in \mathcal{B}(A, B, C, D, f)$, where $\gamma \in \mathcal{K}_\infty$, but ρ is only a continuous positive-definite function. Clearly, an ISS-Lyapunov function is an iISS-Lyapunov function, so, under the assumptions made in Theorem 9.2.1, we obtain iISS. However, it would be interesting to see whether the assumption on the nonlinearity, namely

$$\|f(\xi) - K\xi\| \leq r\,\|\xi\| - \alpha(\|\xi\|) \qquad \forall\, \xi \in \mathbb{F}^p,$$

could be relaxed from $\alpha \in \mathcal{K}_\infty$ to $\alpha \in \mathcal{K}$. We thank an anonymous conference paper referee for this suggestion.

The proof of Theorem 9.2.1 introduced a new way of providing estimates to establish that a function is an ISS-Lyapunov function, in effect allowing us to construct novel classes of ISS-Lyapunov functions. It would be interesting to see if this construction could be applied to other absolute stability results, e.g. the Popov criterion. Bruin et al. [12] have already obtained a Popov-like criterion that guarantees ISS in a continuous-time setting. However, they seem to be using a classical Lur'e-Postnikov Lyapunov function, which is then shown to be an ISS-Lyapunov function for the system at hand under suitable assumptions. It would be interesting to see whether any of these assumptions could be relaxed by using our techniques.

Finally, the proof of the main exponential ISS result, Proposition 9.4.2, did not make use of ISS-Lyapunov functions, therefore it seems likely that it could be generalized to the infinite-dimensional setting.

Part III

Stability of discrete-time input-output Lur'e systems

In Part III of this thesis we will consider linear discrete-time input-output systems defined by higher-order difference equations

$$\sum_{j=0}^{k} P_j y(t+j) = \sum_{j=0}^{k} Q_j u(t+j), \tag{11.1}$$

where P_j's and Q_j's are matrices of appropriate dimensions. Input-output systems of this form are obtained by modelling digital filters and computer controlled systems where the inputs and outputs are periodically sampled, see Desoer and Vidyasagar [14], Gabel and Roberts [16].

By closing the feedback loop via $u(t) := f(y(t))$, where f is some nonlinearity, we obtain a class of input-output Lur'e systems

$$\sum_{j=0}^{k} P_j y(t+j) = \sum_{j=0}^{k} Q_j f(y(t+j)), \tag{11.2}$$

which arise naturally from input-output systems (11.1) and can also model linear multistep methods from numerical analysis, see e.g. Coughlan, Hill and Logemann [11]. Input-output stability properties of related input-output systems are studied in Desoer and Vidyasagar [14] and Zames [64]. However, their results revolve around the small-gain theorem and norm approximations, and they typically establish input-to-output stability in the l^p sense $(1 \leq p \leq \infty)$.

Similarly as in Parts I and II, we will first analyse the absolute stability of input-output Lur'e systems (11.2) and obtain original results guaranteeing global asymptotic stability by combining frequency-domain assumptions with assumptions on the nonlinearity f.

After that we will consider forcing d and study input-output Lur'e systems with forcing

$$\sum_{j=0}^{k} P_j y(t+j) = \sum_{j=0}^{k} Q_j (f(y(t+j)) + d(t+j)), \tag{11.3}$$

where the forcing could represent a target trajectory or a disturbance. We will obtain a class of results, which guarantee input-to-output stability (this is a notion of stability that extends input-to-state stability to an input-output setting) by, again, combining frequency domain assumptions with assumptions on the nonlinearity f.

A major role in this part of the thesis is played by a result that we will call the realization theorem. Consider the input-output system (11.1), define $P(z) := \sum_{j=0}^{k} P_j z^j$ and $Q(z) := \sum_{j=0}^{k} Q_j z^j$, assume that P is invertible and $P^{-1}Q$ is proper. It is well-known from behavioural theory that there exists a state-space system (A, B, C, D) such that, for each tuple (u, y) that satisfies (11.1), there exists x such that $(u, x, y) \in \mathcal{B}(A, B, C, D)$, see Willems [58]. This has clear implications for stability analysis: if y satisfies (11.2), then there exists x such that $(f \circ y, x, y) \in \mathcal{B}(A, B, C, D)$ or, equivalently, $(x, y) \in \mathcal{B}(A, B, C, D, f)$. Therefore, one would expect to be able to obtain stability criteria for input-output Lur'e systems (11.2) and (11.3) by using absolute stability and ISS results from Part I as long as we can establish an appropriate connection between the initial values of y and the initial values of x. This connection will follow from the realization theorem, which will establish a linear relationship between the initial values of u and y that satisfy (11.1) and the initial value of x such that $(u, x, y) \in \mathcal{B}(A, B, C, D)$, see Theorem 11.4.8.

Finally, we will briefly consider image input-output systems given by

$$\begin{aligned} u(t) &= \sum_{j=0}^{k} S_j v(t+j) \\ y(t) &= \sum_{j=0}^{k} T_j v(t+j), \end{aligned} \tag{11.4}$$

where S_j's and T_j's are matrices of appropriate dimensions. They are closely related to input-output systems of the form (11.1). We will also study the associated Lur'e system

$$\begin{aligned} f(y(t)) &= \sum_{j=0}^{k} S_j v(t+j) \\ y(t) &= \sum_{j=0}^{k} T_j v(t+j), \end{aligned} \tag{11.5}$$

where f is some nonlinearity. In the continuous-time setting systems of this form have been studied in Brockett and Willems [10], where results resembling the Popov criterion are obtained. Using the relationship between input-output systems (11.4) and (11.1), we will be able to obtain stability results similar to the Aizerman version of the circle criterion and the standard version of the circle criterion.

This part of the thesis is organized as follows. We will initially analyse linear input-output systems (11.1) in Chapter 11, where we will collect preliminaries from realization theory and prove the realization theorem. This will be applied to input-output Lur'e systems (11.2), (11.3) and image input-output Lur'e systems (11.5) in Chapter 12.

Chapter 11

Linear input-output systems

In this chapter we will study linear input-output systems defined by higher-order difference equations

$$\sum_{j=0}^{k} P_j y(t+j) = \sum_{j=0}^{k} Q_j u(t+j), \tag{11.1}$$

where P_j's and Q_j's are matrices of appropriate dimensions. Our main goal will be to prove the realization theorem as outlined in the introduction of Part III. We will initially introduce linear input-output systems in §11.1, where we will also provide a characterization of tuples (u, y) that satisfy (11.1). After that, in §11.2, we will prove some results pertaining to Z-transforms of tuples that satisfy (11.1) and of triples $(u, x, y) \in \mathcal{B}(A, B, C, D)$. After collecting some facts from realization theory in §11.3, we will prove the realization theorem in §11.4. Finally, we will briefly consider image input-output systems given by

$$\begin{aligned} u(t) &= \sum_{j=0}^{k} S_j v(t+j) \\ y(t) &= \sum_{j=0}^{k} T_j v(t+j), \end{aligned} \tag{11.2}$$

where S_j's and T_j's are matrices of appropriate dimensions. In §11.5 we will prove that there is a close relationship between tuples (u, y) that satisfy (11.1) and triples (u, v, y) that satisfy (11.2).

11.1 Linear input-output systems

We will be considering input-output systems of the form

$$\sum_{j=0}^{k} P_j y(t+j) = \sum_{j=0}^{l} Q_j u(t+j) \qquad \forall\, t \in \mathbb{N}_0, \tag{11.1.1}$$

where $P_j \in \mathbb{F}^{p\times p}$ for $0 \leq j \leq k$ and $Q_j \in \mathbb{F}^{p\times m}$ for $0 \leq j \leq l$. If we set $P(z) := \sum_{j=0}^{k} P_j z^j$ and $Q(z) := \sum_{j=0}^{l} Q_j z^j$, then (11.1.1) can be described more concisely as $P(\mathcal{L})y = Q(\mathcal{L})u$, where $\mathcal{L}$ is the left-shift operator (see §0.1). If we consider the initial value problem

$$\begin{aligned} P(\mathcal{L})y =& Q(\mathcal{L})u, \\ & u \in (\mathbb{F}^m)^{\mathbb{N}_0}, \quad y(0), y(1), \dots, y(k-1) \in \mathbb{F}^p, \end{aligned} \tag{11.1.2}$$

then we instantly see the need to impose some conditions on the polynomial matrices P, Q to obtain the existence of solutions. For example, consider $P = \begin{pmatrix} z & 1 \\ 0 & 0 \end{pmatrix}$ and $Q = I_2$, so that the initial value problem is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} y(t+1) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y(t) = u(t) \qquad \forall\, t \in \mathbb{N}_0, \quad y(0) = \xi \in \mathbb{F}^2.$$

If we pick $u(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for all $t \in \mathbb{N}_0$, then it clearly has no solutions, no matter what $\xi \in \mathbb{F}^2$ we pick. Moreover, let us pick $u(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for all $t \geq 0$ and $\xi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then $y_1 \in (\mathbb{F}^2)^{\mathbb{N}_0}$ defined as $y_1(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $y_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $t \geq 1$ solves the initial value problem (11.1.2). However, so does $y_2 \in (\mathbb{F}^2)^{\mathbb{N}_0}$ defined as $y_2(2t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $y_2(2t+1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for all $t \geq 0$. Hence we can see that solutions are not necessarily unique, when they exist. The assumption $\det P \neq 0$ is sufficient for the existence of at least one solution of the initial value problem (11.1.1) for some initial conditions, see Theorem 8.3 from [17]. Hence we will restrict our attention to input-output systems that satisfy this condition.

Another desirable property for an input-output system is causality - we do not want $y(t)$ to depend on $u(t+1)$. In other words, the present should not depend on the future. Consider $P = I_2$ and $Q(z) = zI_2$; then the input-output system (11.1.1) reads

$$y(t) = u(t+1) \qquad \forall\, t \in \mathbb{N}_0$$

and hence the system is not causal. Again, this is not obvious, but a condition that ensures causality is that $P^{-1}Q \in \mathbb{F}(z)^{p\times m}$ should be proper, see e.g. Proposition 8.6 from [60]. The above considerations motivate the following definition.

Definition 11.1.1. Let $P \in \mathbb{F}[z]^{p\times p}$ and $Q \in \mathbb{F}[z]^{p\times m}$ be such that $\det P \neq 0$ and $P^{-1}Q \in \mathbb{F}(z)^{p\times m}$ is proper. We call the tuple (P, Q) a (discrete-time) **input-output system** and we write $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$.

We define the **behaviour** $\mathcal{B}(P, Q)$ of an input-output system (P, Q) as the set of all tuples $(u, y) \in (\mathbb{F}^m)^{\mathbb{N}_0} \times (\mathbb{F}^p)^{\mathbb{N}_0}$ that satisfy

$$P(\mathcal{L})y = Q(\mathcal{L})u.$$

We call a tuple $(u, y) \in \mathcal{B}(P, Q)$, a trajectory.

For a tuple $(u, y) \in \mathcal{B}(P, Q)$, we usually refer to u as the input and y as the output. Aspects of input-output systems of this form have been studied in §6.7 from [51] and in a slightly different form in [13]. There is more work in the continuous-time setting, see §3 from [47] and [58].

It is useful to note a characterization of trajectories in $\mathcal{B}(P, Q)$. Theorems 11.1.2, 11.1.3 and 11.1.4 can be proved in the same way as the corresponding continuous-time results from §3 in [47]. We omit the proofs as we only use them in one place: the proof of Lemma 11.2.7.

We will make use of the following shorthand. Let $P \in \mathbb{F}[z]^{p\times p}$ be given by $P(z) = \sum_{j=0}^{k} P_j z^j$ for some $P_j \in \mathbb{F}^{p\times p}$. For $m \in \mathbb{N}_0$, we define a polynomial matrix $P^{(m)} \in \mathbb{F}[z]^{p\times p}$ by $P^{(m)}(z) := \sum_{j=0}^{k} P_j j^m z^j$.

Theorem 11.1.2. *Consider a polynomial matrix $P \in \mathbb{C}[z]^{p\times p}$ and assume that $\det P \neq 0$ and that $\det P(z) = c\prod_{i=1}^{N}(z-\lambda_i)^{n_i}$, where $\lambda_i \neq \lambda_j$ for $i \neq j$. Then $y \in \ker P(\mathcal{L})$ if, and only if,*

$$y(t) = \sum_{i=1}^{N}\sum_{j=0}^{n_i-1} b_{ij} t^j \lambda_i^t, \tag{11.1.3}$$

where the vectors $b_{ij} \in \mathbb{C}^p$ satisfy the relations

$$\sum_{j=k}^{n_i-1} \binom{j}{k} P^{(j-k)}(\lambda_i) b_{ij} = 0 \tag{11.1.4}$$

for all $1 \leq i \leq N$, $0 \leq k \leq n_i - 1$.

The dimension of $\ker P(\mathcal{L})$ is $\deg\det P$.

By noting that zeros of real polynomials come in complex conjugate pairs, we could obtain a counterpart of Theorem 11.1.2 for the case when $\mathbb{F} = \mathbb{R}$. However, for our purposes the complex version will suffice.

Theorem 11.1.3. *Consider an input-output system* $(P,Q) \in \mathrm{IO}(m,p;\mathbb{C})$, *where* $\det P(z) = c\prod_{i=1}^{N}(z-\lambda_i)^{n_i}$ *and let the partial fractions expansion of* $P^{-1}Q$ *be given by*

$$(P^{-1}Q)(z) = A_0 + \sum_{i=1}^{N}\sum_{j=1}^{n_i} \frac{A_{ij}}{(z-\lambda_i)^j},$$

where $A_{ij} \in \mathbb{C}^{p\times m}$. *Let* $u \in (\mathbb{C}^m)^{\mathbb{N}_0}$ *be given and define* $y \in (\mathbb{C}^p)^{\mathbb{N}_0}$

$$\begin{aligned} y(0) &:= A_0 u(0) \\ y(t) &:= A_0 u(t) + \sum_{i=1}^{N}\sum_{j=1}^{n_i} A_{ij} \sum_{k=0}^{t-1} \binom{t-k-1}{j-1} \lambda_i^{t-k-j} u(k) \qquad \text{for } t \geq 1. \end{aligned} \tag{11.1.5}$$

Then $(u,y) \in \mathcal{B}(P,Q)$.

These two results allow us to characterize $\mathcal{B}(P,Q)$.

Theorem 11.1.4. *Consider an input-output system* $(P,Q) \in \mathrm{IO}(m,p;\mathbb{F})$ *and set* $\mathcal{B}_{\mathrm{i/o}}(P,Q) := \big\{(u, y_{\mathrm{i/o}}) : u \in (\mathbb{F}^m)^{\mathbb{N}_0}$ and $y_{\mathrm{i/o}}$ is given by (11.1.5)$\big\}$ *and* $\mathcal{B}_{\mathrm{hom}}(P,Q) := \{(0, y_{\mathrm{hom}}) : y_{\mathrm{hom}}$ is of the form (11.1.3)$\}$. *Then*

$$\mathcal{B}(P,Q) = \mathcal{B}_{\mathrm{i/o}}(P,Q) + \mathcal{B}_{\mathrm{hom}}(P,Q).$$

Note that while in Theorem 11.1.4 we allow $\mathbb{F} = \mathbb{R}$, in general $y_{\mathrm{i/o}}$ and y_{hom} will be in $(\mathbb{C}^p)^{\mathbb{N}_0}$.

Theorem 11.1.4 essentially says that given a $u \in (\mathbb{F}^m)^{\mathbb{N}_0}$, every solution of $P(\mathcal{L})y = Q(\mathcal{L})u$ can be written as

$$\begin{aligned} y(0) &= \sum_{i=1}^{N} b_{i0} + A_0 u(0), \qquad \text{and} \\ y(t) &= \sum_{i=1}^{N}\sum_{j=0}^{n_i-1} b_{ij} t^j \lambda_i^t + A_0 u(t) + \sum_{i=1}^{N}\sum_{j=1}^{n_i} A_{ij} \sum_{k=0}^{t-1} \binom{t-k-1}{j-1} \lambda_i^{t-k-j} u(k), \end{aligned} \tag{11.1.6}$$

for $t \geq 1$, where b_{ij}'s satisfy equation (11.1.4).

We define a family of operators $(\pi_T)_{T\in\mathbb{N}_0}$ on $(\mathbb{F}^m)^{\mathbb{N}_0}$ as

$$(\pi_T u)(t) = \begin{cases} u(t) & \text{if } t \leq T \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 11.1.5. *Consider* $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$ *and let* $(u, y) \in \mathcal{B}(P, Q)$. *Then for all* $T \in \mathbb{N}_0$ *there exists* $y_T \in (\mathbb{C}^p)^{\mathbb{N}_0}$ *such that*

$$P(\mathcal{L})y_T = Q(\mathcal{L})(\pi_T u)$$

and $y_T(t) = y(t)$ *for all* $t \leq T$.

Proof. By Theorem 11.1.4, there exist matrices $A_{ij} \in \mathbb{C}^{p\times m}$ and vectors $b_{ij} \in \mathbb{C}^p$ such that y is given by (11.1.6). Moreover, b_{ij}'s satisfy (11.1.4). Define

$$\begin{aligned} y_T(0) &= \sum_{i=1}^{N} b_{i0} + A_0(\pi_T u)(0), \qquad \text{and} \\ y_T(t) &= \sum_{i=1}^{N} \sum_{j=0}^{n_i-1} b_{ij} t^j \lambda_i^t \\ &\qquad + A_0(\pi_T u)(t) + \sum_{i=1}^{N} \sum_{j=1}^{n_i} A_{ij} \sum_{k=0}^{t-1} \binom{t-k-1}{j-1} \lambda_i^{t-k-j} (\pi_T u)(k), \end{aligned}$$

for $t \geq 1$. Then, by Theorem 11.1.4,

$$P(\mathcal{L})y_T = Q(\mathcal{L})(\pi_T u).$$

Since $(\pi_T u)(t) = u(t)$ for $t \leq T$, we have $y_T(t) = y(t)$ for all $t \leq T$. □

Note that in Corollary 11.1.5 we cannot say that $(\pi_T u, y_T) \in \mathcal{B}(P, Q)$ as y_T could be complex even if $\mathbb{F} = \mathbb{R}$. It would be desirable to obtain real y_T if $\mathbb{F} = \mathbb{R}$, however that would involve developing real counterparts to Theorems 11.1.3 and 11.1.4, which is quite involved. Fortunately the present, complex, version of Corollary 11.1.5 will turn out to be enough for our purposes, see the proof of Corollary 11.4.3.

11.2 The Z-transform

We will define the Z-transform, state some standard results without proof (they can all be completed by using standard power series theory) and finally describe the Z-transforms of trajectories in behaviours of both linear state-space systems and linear input-output systems.

Definition 11.2.1. For $y \in (\mathbb{F}^p)^{\mathbb{N}_0}$ we define the **Z-transform of** y as

$$\hat{y}(z) := \sum_{i\in\mathbb{N}_0} y(i) z^{-i} \tag{11.2.1}$$

for all $z \in \mathbb{C}$ for which this series converges.

For $y \in (\mathbb{F}^p)^{\mathbb{N}_0}$, set $r_y := \limsup_{i\to\infty} \|y(i)\|^{\frac{1}{i}}$. It is well-known that there exists a radius of convergence for power series and thus we obtain the following lemma.

Lemma 11.2.2. *Consider $y \in (\mathbb{F}^p)^{\mathbb{N}_0}$. If $r_y < \infty$, then the series* (11.2.1) *converges absolutely for all $|z| > r_y$ and diverges for all $|z| < r_y$.*

In light of Lemma 11.2.2, we will say that $y \in (\mathbb{F}^p)^{\mathbb{N}_0}$ is **Z-transformable** if $r_y < \infty$. Note that this is equivalent to saying that the series $\hat{y}(z)$ converges absolutely on $\mathbb{E}(0, r_y)$. It follows from the theory of absolutely convergent series that if y_1, y_2 are Z-transformable, then so is $y_1 + y_2$. Also, clearly, for a matrix $M \in \mathbb{F}^{m\times p}$, if y is Z-transformable, then so is My.

The following lemma shows that the Z-transform is injective.

Lemma 11.2.3. *Let $y, w \in (\mathbb{F}^p)^{\mathbb{N}_0}$ be Z-transformable. If $\hat{y}(z) = \hat{w}(z)$ on the intersection of the two regions of convergence, then $y = w$.*

The following characterization of the Z-transform of a left-shift of a given sequence $y \in (\mathbb{F}^p)^{\mathbb{N}_0}$ is well-known.

Lemma 11.2.4. *Let $y \in (\mathbb{F}^p)^{\mathbb{N}_0}$ be Z-transformable. Then for any $j \in \mathbb{N}$, $\mathcal{L}^j y$ is Z-transformable and*

$$(\widehat{L^j y})(z) = z^j \hat{y}(z) - \sum_{k=0}^{j-1} y(k) z^{j-k}. \tag{11.2.2}$$

Another powerful property of the Z-transform is that a convolution in the time-domain corresponds to multiplication in the Z-transform domain (this is sometimes called the frequency domain). We will use this powerful observation in the following way.

Lemma 11.2.5. *Let $G \in \mathbb{F}(z)^{p\times m}$ be proper and assume that its Laurent series expansion around ∞ is given by $G(z) = \sum_{i\in\mathbb{N}_0} G_i z^{-i}$, where $G_i \in \mathbb{F}^{p\times m}$; let $u \in (\mathbb{F}^m)^{\mathbb{N}_0}$ be Z-transformable.*

Then $y \in (\mathbb{F}^p)^{\mathbb{N}_0}$ is Z-transformable and given by

$$y(t) := \sum_{k=0}^{t} G_{t-k} u(k) \qquad \forall t \in \mathbb{N}_0$$

if, and only if, $\hat{y}(z) = G(z)\hat{u}(z)$.

With the preliminaries out of the way, we will now apply Z-transforms to study trajectories in linear state-space and input-output systems.

Lemma 11.2.6. *Consider $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$, denote its transfer function by G and let $u \in (\mathbb{F}^m)^{\mathbb{N}_0}$ be Z-transformable.*

Then $(u, x, y) \in \mathcal{B}(A, B, C, D)$ *if, and only if,* x *and* y *are Z-transformable and*

$$\hat{x}(z) = (zI - A)^{-1}B\hat{u}(z) + (zI - A)^{-1}z\xi$$
$$\hat{y}(z) = G(z)\hat{u}(z) + C(zI - A)^{-1}z\xi$$

for some $\xi \in \mathbb{F}^n$.

Proof. We note that - for an arbitrary $u \in (\mathbb{F}^m)^{\mathbb{N}_0}$ - $(u, x, y) \in \mathcal{B}(A, B, C, D)$ if, and only if, for some $\xi \in \mathbb{F}^n$ we have

$$\begin{aligned} x(t) &:= A^t\xi + \sum_{k=0}^{t-1} A^{t-1-k}Bu(k) \\ y(t) &:= CA^t\xi + Du(t) + \sum_{k=0}^{t-1} CA^{t-1-k}Bu(k) \qquad \forall\, t \in \mathbb{N}_0. \end{aligned} \tag{11.2.3}$$

Now let us prove sufficiency. As u is Z-transformable, equation (11.2.3) and straightforward estimates imply that if $(u, y, x) \in \mathcal{B}(A, B, C, D)$, then r_x and r_y are finite, so that x and y are Z-transformable. Since $\mathcal{L}x = Ax + Bu$ and $y = Cx + Du$, an application of Lemma 11.2.4 then gives us

$$\hat{x}(z) = (zI - A)^{-1}B\hat{u}(z) + (zI - A)^{-1}zx(0)$$
$$\hat{y}(z) = G(z)\hat{u}(z) + C(zI - A)^{-1}zx(0),$$

which completes the proof of sufficiency.

To prove necessity, we recall a standard result on Neumann series: $(zI - A)^{-1} = \sum_{i\in\mathbb{N}_0} A^i z^{-(i+1)}$ for all $|z| > \|A\|$. Thus

$$\hat{x}(z) = \left(\sum_{i\in\mathbb{N}_0} A^iBz^{-(i+1)}\right)\hat{u}(z) + \sum_{i\in\mathbb{N}_0} A^iz^{-i}\xi$$
$$\hat{y}(z) = \left(\sum_{i\in\mathbb{N}_0} (CA^iz^{-(i+1)}B + D)\right)\hat{u}(z) + C\sum_{i\in\mathbb{N}_0} A^iz^{-i}\xi,$$

so, by Lemma 11.2.5, we see that x and y satisfy equation (11.2.3). Hence $(u, x, y) \in \mathcal{B}(A, B, C, D)$ as required. □

Note that the sufficiency part of Lemma 11.2.6 is a generalization of equation (14) from §2.3.3 in [34].

We now similarly study the Z-transforms of trajectories of input-output systems. Consider an input-output system $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$, where

$P(z) = \sum_{j=0}^{\deg P} P_j z^j$ and $Q(z) = \sum_{j=0}^{\deg Q} Q_j z^j$. For $(u, y) \in \mathcal{B}(P, Q)$, we define a polynomial vector $\theta_{u,y} \in \mathbb{F}[z]^p$ by

$$\theta_{u,y}(z) := \sum_{i=1}^{\deg P} z^i \left[\sum_{j=i}^{\deg P} P_j y(j-i) \right] - \sum_{i=1}^{\deg Q} z^i \left[\sum_{j=i}^{\deg Q} Q_j u(j-i) \right]. \quad (11.2.4)$$

It will play an important role in describing Z-transforms of trajectories in $\mathcal{B}(P, Q)$.

Lemma 11.2.7. *Consider* $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$*, where* $P(z) = \sum_{j=0}^{\deg P} P_j z^j$ *and* $Q(z) = \sum_{j=0}^{\deg Q} Q_j z^j$.

If $(u, y) \in \mathcal{B}(P, Q)$ *is such that* u *is Z-transformable, then so is* y*. Moreover, for* $\theta_{u,y}$ *defined by* (11.2.4)*, we have*

$$\hat{y}(z) = (P^{-1}Q)(z)\hat{u}(z) + P^{-1}(z)\theta_{u,y}(z).$$

Proof. As $(u, y) \in \mathcal{B}(P, Q)$, by (11.1.6), we know that - for some matrices $A_{ij} \in \mathbb{C}^{p \times m}$ and vectors $b_{ij} \in \mathbb{C}^p$ - y is given by

$$\begin{aligned} y(0) &= \sum_{i=1}^{N} b_{i0} + A_0 u(0) \qquad \text{and} \\ y(t) &= \sum_{i=1}^{N} \sum_{j=0}^{n_i - 1} b_{ij} t^j \lambda_i^t + A_0 u(t) + \sum_{i=1}^{N} \sum_{j=1}^{n_i} A_{ij} \sum_{k=0}^{t-1} \binom{t-k-1}{j-1} \lambda_i^{t-k-j} u(k) \end{aligned}$$

for all $t \geq 1$. We omit the details, but via a straightforward estimation, one can check that r_y is finite and hence y is Z-transformable. Thus, by Lemma 11.2.4, so is $P(\mathcal{L})y$ and the region of convergence of its Z-transform is $\mathbb{E}(0, r_y)$, so that

$$\begin{aligned} (\widehat{P(\mathcal{L})y})(z) &= \sum_{j=0}^{\deg P} P_j (\widehat{\mathcal{L}^j y}) \\ &= \sum_{j=0}^{\deg P} P_j z^j \hat{y}(z) - \sum_{j=0}^{\deg P} P_j \sum_{k=0}^{j-1} y(k) z^{j-k} \\ &= P(z)\hat{y}(z) - \sum_{j=0}^{\deg P} P_j \sum_{k=0}^{j-1} y(k) z^{j-k} \\ &= P(z)\hat{y}(z) - \sum_{i=1}^{\deg P} z^i \sum_{j=i}^{\deg P} P_j y(j-i) \qquad \forall\, z \in \mathbb{E}(0, r_y). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
(\widehat{Q(\mathcal{L})u}(z) &= \sum_{j=0}^{\deg Q} Q_j(\widehat{\mathcal{L}^j u}) \\
&= \sum_{j=0}^{\deg Q} Q_j z^j \hat{u}(z) - \sum_{j=0}^{\deg Q} Q_j \sum_{k=0}^{j-1} u(k) z^{j-k} \\
&= Q(z)\hat{u}(z) - \sum_{j=0}^{\deg Q} Q_j \sum_{k=0}^{j-1} u(k) z^{j-k} \\
&= Q(z)\hat{u}(z) - \sum_{i=1}^{\deg Q} z^i \sum_{j=i}^{\deg Q} Q_j u(j-i) \qquad \forall z \in \mathbb{E}(0, r_u).
\end{aligned}$$

Let $\theta_{u,y}$ be defined by equation (11.2.4). Since $P(\mathcal{L})y = Q(\mathcal{L})u$, we thus have

$$P(z)\hat{y}(z) = Q(z)\hat{u}(z) + \theta_{u,y}(z) \qquad \forall\, z \in \mathbb{E}(0, r),$$

where $r = \max\{r_y, r_u\}$.

Finally, since P has a finite number of zeros, P^{-1} has only a finite number of poles and hence, for some $r_1 > 0$, $P^{-1}(z)$ is defined on $\mathbb{E}(0, r_1)$. Without loss of generality, $r \geq r_1$ and thus

$$\hat{y}(z) = (P^{-1}Q)(z)\hat{u}(z) + P^{-1}(z)\theta_{u,y}(z) \qquad \forall\, z \in \mathbb{E}(0, r). \qquad \square$$

11.3 Realization theory

We now collect standard terminology and facts from realization theory. These will be used in §11.4 to prove the realization theorem (see Theorem 11.4.8). Moreover, some results will be used in §11.5 to describe behaviours of image input-output systems.

Definition 11.3.1. Consider a rational function matrix $G \in \mathbb{F}(z)^{p \times m}$. A state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is said to be a **realization of** G if $G(z) = C(zI - A)^{-1}B + D$. The **order** of the realization (A, B, C, D) is defined as n.

This definition is standard, however we overload the word "realization" as we are primarily interested in input-output systems rather than rational function matrices.

Definition 11.3.2. Consider an input-output system $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$. A state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is said to be a **realization**

of (P, Q) if $(P^{-1}Q)(z) = C(zI - A)^{-1}B + D$. The **order** of the realization (A, B, C, D) is again n.

Note that a realization of an input-output system (P, Q) is also a realization of the rational function matrix $P^{-1}Q$. The following result is well known.

Theorem 11.3.3. *A rational function matrix G is realizable if, and only if, it is proper.*

Proof. This is Theorem 21.9 from [13] combined with the well-known fact that the transfer function of a linear state-space system is proper. □

Remark: [13] actually only deals with the case $\mathbb{F} = \mathbb{R}$, but the extension to $\mathbb{C}$ follows *mutatis mutandis*. This remark applies also to the other results that will be cited from [13].

Corollary 11.3.4. *Input-output systems are realizable.*

Definition 11.3.5. Let a rational function matrix $G \in \mathbb{F}(z)^{p\times m}$ be proper; its **McMillan degree** is defined as the minimal order among all realizations (A, B, C, D) of G.

Similarly, consider an input-output system $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$. We define its **McMillan degree** as the McMillan degree of $P^{-1}Q$.

Definition 11.3.6. Consider a proper $G \in \mathbb{F}(z)^{p\times m}$ and let (A, B, C, D) be a realization of G. If the order of the realization (A, B, C, D) is equal to the McMillan degree of G, then we call the state-space system (A, B, C, D) a **minimal realization of** G.

Given an input-output system (P, Q), we say that (A, B, C, D) is a **minimal realization of** (P, Q) if it is a minimal realization of $P^{-1}Q$.

Recall that a realization is minimal if, and only if, it is controllable and observable (see e.g. Theorem 21.13 from [13]).

While irrelevant for the arguments that follow, it is interesting to note that any two minimal realizations of a given proper rational function matrix (or of a given input-output system) are unique up to similarity transforms (see e.g. Theorem 21.16 from [13]).

We now gather some facts about polynomial matrices.

Definition 11.3.7. We call a polynomial matrix $V \in \mathbb{F}[z]^{p\times p}$ **unimodular** if $\det V = a$, for some $a \in \mathbb{F} \setminus \{0\}$.

Lemma 11.3.8. *The unimodular elements of $\mathbb{F}[z]^{p\times p}$ are precisely the polynomial matrices that are invertible in $\mathbb{F}[z]^{p\times p}$.*

Proof. This is Proposition 4.16 from [38]. □

Notation: we will write $\mathrm{U}\mathbb{F}[z]^{p\times p}$ for the unimodular elements of $\mathbb{F}[z]^{p\times p}$. Here "U" stands for "units", which is standard notation in algebra, see e.g. p. 83 from [38].

Definition 11.3.9. Suppose that polynomial matrices P, P_1 and R are such that $P = RP_1$. We say that R is a **left divisor** of P and we say that P is a **right multiple** of R.

Definition 11.3.10. Consider $P \in \mathbb{F}[z]^{p\times p}$ and $Q \in \mathbb{F}[z]^{p\times m}$. A polynomial matrix $R \in \mathbb{F}[z]^{p\times p}$ is said to be the **greatest common left divisor** of P and Q if

(a) R is a left divisor of both P and Q,

(b) R is a right multiple of every common left divisor of P and Q.

Lemma 11.3.11. *Consider $P \in \mathbb{F}[z]^{p\times p}$ and $Q \in \mathbb{F}[z]^{p\times m}$. Suppose there exists a greatest common left divisor $R \in \mathbb{F}[z]^{p\times p}$ of P and Q.*

Then the set of all greatest common left divisors of P and Q is $\{RV : V \in U\mathbb{F}[z]^{p\times p}\}$.

Proof. This is Lemma 2 from §4.1, [55]. □

Theorem 11.3.12. *Consider $P \in \mathbb{F}[z]^{p\times p}$ and $Q \in \mathbb{F}[z]^{p\times m}$.*

There exists a greatest common left divisor R of P and Q. Moreover there exist polynomial matrices X and Y - of appropriate dimensions - such that

$$PX + QY = R. \tag{11.3.1}$$

Proof. This is Theorem 7 from §4.1, [55]. □

Equation (11.3.1) is sometimes called the **left Bezout identity**.

Definition 11.3.13. Consider $P \in \mathbb{F}[z]^{p\times p}$ and $Q \in \mathbb{F}[z]^{p\times m}$. They are said to be **left coprime** if every greatest common left divisor of P and Q is unimodular.

By Theorem 11.3.12, equivalently, P and Q are left coprime if there exist polynomial matrices X and Y of appropriate dimensions such that $PX + QY = I$.

It is interesting to note that controllability of a linear state-space system (A, B, C, D) is equivalent to $zI - A$ and B being left coprime.

Lemma 11.3.14. *If a state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ is controllable, then there exist polynomial matrices $X \in \mathbb{F}[z]^{n\times n}$ and $Y \in \mathbb{F}[z]^{m\times n}$ such that*

$$(zI - A)X + BY = I.$$

Proof. This follows immediately if one combines Lemma 3.3.7 from [51] (this result is sometimes known as the Hautus Lemma or the Hautus controllability criterion) and Theorem 22.10 from [13]. □

Similarly we can define a greatest common right divisor.

Definition 11.3.15. Suppose that polynomial matrices S, S_1 and R are such that $S = S_1 R$. We say that R is a **right divisor** of S and we say that S is a **left multiple** of R.

Definition 11.3.16. Consider $S \in \mathbb{F}[z]^{m\times m}$ and $T \in \mathbb{F}[z]^{p\times m}$. A polynomial matrix $R \in \mathbb{F}[z]^{m\times m}$ is said to be the **greatest common right divisor** of S and T if

(a) R is a right divisor of both S and T,

(b) R is a left multiple of every common right divisor of S and T.

Lemma 11.3.17. *Consider $S \in \mathbb{F}[z]^{m\times m}$ and $T \in \mathbb{F}[z]^{p\times m}$. Suppose there exists a greatest common right divisor $R \in \mathbb{F}[z]^{m\times m}$ of S and T.*

Then the set of all greatest common right divisors of S and T is $\{RV : V \in U\mathbb{F}[z]^{m\times m}\}$.

Proof. This is Lemma 2 from §4.1, [55]. □

Theorem 11.3.18. *Consider $S \in \mathbb{F}[z]^{m\times m}$ and $Q \in \mathbb{F}[z]^{p\times m}$.*

There exists a greatest common right divisor R of S and T. Moreover there exist polynomial matrices X and Y - of appropriate dimensions - such that

$$XS + YT = R. \tag{11.3.2}$$

Proof. This is Theorem 7 from §4.1, [55]. □

Equation (11.3.2) is sometimes called the **right Bezout identity**.

Definition 11.3.19. Consider $S \in \mathbb{F}[z]^{m\times m}$ and $T \in \mathbb{F}[z]^{p\times m}$. They are said to be **right coprime** if every greatest common right divisor of S and T is unimodular.

By Theorem 11.3.18, equivalently, S and T are right coprime if there exist polynomial matrices X and Y - of appropriate dimensions - such that $XS + YT = I$.

It is reassuring to know that rational function matrices always admit left coprime and right coprime factorisations.

Lemma 11.3.20. *Consider a rational function matrix $G \in \mathbb{F}(z)^{p\times m}$. There exist*

- *left coprime* $P \in \mathbb{F}[z]^{p\times p}$ *and* $Q \in \mathbb{F}[z]^{p\times m}$ *such that* $P^{-1}Q = G$*,*
- *and right coprime* $S \in \mathbb{F}[z]^{m\times m}$ *and* $T \in \mathbb{F}[z]^{p\times m}$ *such that* $TS^{-1} = G$*.*

Proof. We only prove (a) as (b) can be proved in an identical manner. We first construct a factorisation $P^{-1}Q = G$ and then use it to construct a factorisation $P_1^{-1}Q_1 = G$ with P_1 and Q_1 left coprime.

We can write an arbitrary entry G_{ij} of G as $G_{ij} = n_{ij}/d_{ij}$, where $n_{ij}, d_{ij} \in \mathbb{F}[z]$ are coprime polynomials. Denote the least common multiple of d_{ij}'s by d. Then clearly $G_{ij} = (1/d) \cdot n_{ij} \cdot (d/d_{ij})$ and d/d_{ij} is a polynomial. Hence we can define a polynomial matrix $Q \in \mathbb{F}[z]^{p\times m}$ given by $Q_{ij} := n_{ij} \cdot (d/d_{ij})$ and $P := d \cdot I_p$ to see that $P^{-1}Q = G$. We now construct a left coprime factorization of G.

Let R be a greatest common left divisor of P and Q. Then by Lemma 11.3.11 there exist polynomial matrices X and Y - of appropriate dimensions - such that the left Bezout identity holds: $PX + QY = R$. Since R is a left divisor of both P and Q, there exist P_1 and Q_1 such that $P = RP_1$ and $Q = RQ_1$. Moreover as P is invertible, so is R and thus $P_1X + Q_1Y = I$, so that P_1 and Q_1 are left coprime. Finally $P_1^{-1}Q_1 = (RP_1)^{-1}RQ_1 = P^{-1}Q = G$, which completes the proof. □

The main reason for considering input-output systems (P, Q) with P and Q left coprime is the following lemma.

Lemma 11.3.21. *Consider an input-output system* $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$ *and assume that* P *and* Q *are left coprime. Then* $\deg\det P$ *is equal to the McMillan degree of* (P, Q)*.*

Proof. Suppose $P^{-1}Q$ is strictly proper. Then by Theorem 22.18 from [13] the McMillan degree of (P, Q) is equal to $\deg\det P$.

We now make the straightforward extension to the case when $P^{-1}Q$ is only proper. Set $D := \lim_{|z|\to\infty}(P^{-1}Q)(z)$, so that $P^{-1}Q - D = P^{-1}(Q - PD)$ is strictly proper. Note that (A, B, C, D) is a realization of (P, Q) if, and only if, $(A, B, C, 0)$ is a realization of $(P, Q - PD)$. Thus the McMillan degree of (P, Q) is the same as the McMillan degree of $(P, Q - PD)$, so the proof is complete if we can show that P and $Q - PD$ are left coprime.

This is straightforward. Since P and Q are left coprime, there exist polynomial matrices X and Y such that $PX + QY = I$, which is equivalent to $P(X + DY) + (Q - PD)Y = I$. An application of Theorem 11.3.12 completes the proof as $X + DY$ is clearly a polynomial matrix. □

In the light of this lemma we can see that the degree of $\det P$ will play a crucial role. Hence we need to develop a bit of theory to keep track of it.

Consider a rational function $r \in \mathbb{F}(z)$. Then there exist polynomials $p, q \in \mathbb{F}[z]$ such that $r = \frac{p}{q}$. The integer $\deg r := \deg p - \deg q$ is said to be the **degree of** r. It is straightforward to check that the degree of a rational function r does not depend on the choice of p and q. Now consider a rational function matrix M. The highest degree of the rational functions in the a-th row, denoted by $r_a(M)$, is called the **a-th row degree** of M.

Lemma 11.3.22. *If $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$, then $r_a(P) \geq r_a(Q)$ for all $1 \leq a \leq p$.*

Proof. Consider an arbitrary $a \in \{1, \dots, p\}$. Since $P^{-1}Q$ is proper, we have

$$r_a(P) \geq r_a(P(P^{-1}Q)) = r_a(Q). \qquad \square$$

Consider $P \in \mathbb{F}[z]^{p \times p}$, then upon considering the Leibniz formula for determinants one can see that $\deg \det P \leq \sum_{a=1}^{p} r_a(P)$. We say that the square matrix P is **row reduced** if this inequality is an equality.

It is reassuring to know that polynomial matrices can always be put into a row reduced form.

Lemma 11.3.23. *Let $P \in \mathbb{F}[z]^{p \times p}$ and assume that $\det P \neq 0$. Then there exists $V \in \mathrm{U}\mathbb{F}[z]^{p \times p}$ such that VP is row reduced and $\deg VP \leq \deg P$.*

Proof. With the obvious changes to generalize it to the case when we deal with the complex field and when we deal with row reduction instead of column reduction, this is Theorem 1 from [45]. $\square$

We note that if $\det P = 0$ then the conclusions of the previous lemma do not hold as $\begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix}$ clearly cannot be put into row reduced form.

11.4 Behaviours of input-output systems

We are now fully equipped to continue with our objective of proving the realization theorem. Initially, in §11.4.1 we will do it for a class of input-output systems for which this problem admits a considerably simpler solution. Then, in §11.4.2, we will solve the general case.

11.4.1 Behaviours of simple input-output systems

In this subsection we will prove the realization theorem for trajectories with Z-transformable inputs u, see Proposition 11.4.2. After that we will use

Corollary 11.1.5 to extend it to trajectories with non-Z-transformable inputs u.

Definition 11.4.1. Consider an input-output system $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$, set $k := \deg P$ and let $P(z) = \sum_{j=0}^{k} P_j z^j$. If P and Q are left coprime and $\det P_k \neq 0$, then we say that (P, Q) is a **simple input-output system**. We write $\mathrm{IO}_\mathrm{s}(m, p; \mathbb{F})$ for the set of simple input-output systems.

The significance of assumption $\det P_k \neq 0$ is evident in the proof of the main result of this subsection, Proposition 11.4.2. It allows us to find the order of a minimal realization of (P, Q).

Notation: for $v \in (\mathbb{F}^m)^{\mathbb{N}_0}$ and $k \in \mathbb{N}_0 \setminus \{0\}$, we define $v^k \in (\mathbb{F}^m)^k$ to be the first k vectors in v, that is, $v^k := (v(0), v(1), \ldots, v(k-1))$.

Proposition 11.4.2. *Consider $(P, Q) \in \mathrm{IO}_\mathrm{s}(m, p; \mathbb{F})$ and set $k := \deg P$.*

(a) The McMillan degree of (P, Q) is $pk = \deg\det P$.

(b) Let (A, B, C, D) be a minimal realization of (P, Q). Then for each $(u, y) \in \mathcal{B}(P, Q)$ with a Z-transformable u there exists a unique $x \in (\mathbb{F}^{pk})^{\mathbb{N}_0}$ such that $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

(c) Moreover, there exists a linear map $\phi\colon (\mathbb{F}^m)^k \times (\mathbb{F}^p)^k \to \mathbb{F}^{pk}$ such that, for all $(u, y) \in \mathcal{B}(P, Q)$ with a Z-transformable u and for the corresponding x such that $(u, x, y) \in \mathcal{B}(A, B, C, D)$, we have $\phi(u^k, y^k) = x(0)$.

(d) Finally, if $\mathbb{F} = \mathbb{C}$, but P and Q are real polynomial matrices, then the image of the restriction of ϕ to $(\mathbb{R}^m)^k \times (\mathbb{R}^p)^k$ is contained in $\mathbb{R}^{pk}$.

Remark: statement (c) is simply saying that $x(0)$ depends linearly on the first k values of u and y.

Remark: while statement (d) might seem out of place, it is needed for the later extension to non-Z-transformable trajectories. We will analyse non-Z-transformable trajectories $(u, y) \in \mathcal{B}(P, Q)$ by using Corollary 11.1.5 to obtain projections $(\pi_T u, y_T) \in \mathcal{B}(P, Q)$. However, Corollary 11.1.5 is only available over the complex field. Statement (d) will allow us to work around this inconvenience, for details see the proof of Corollary 11.4.3.

Proof of Proposition 11.4.2. Let us write $P(z) = \sum_{j=0}^{k} P_j z^j$ and $Q(z) = \sum_{j=0}^{k} Q_j z^j$ for appropriate matrices P_j and Q_j.

Since $\det P_k \neq 0$, it follows that $\deg\det P = pk$. Hence, by Lemma 11.3.21, the McMillan degree of (P, Q) is pk, which proves (a).

Consider an arbitrary trajectory $(u, y) \in \mathcal{B}(P, Q)$ with a Z-transformable u. Then by Lemma 11.2.7, y is Z-transformable and on some $\mathbb{E}(0, r_1)$ we have

$$\hat{y}(z) = (P^{-1}Q)(z)\hat{u}(z) + P^{-1}(z)\theta_{u,y}(z),$$

where $\theta_{u,y} \in \mathbb{F}[z]^p$ is given by

$$\theta_{u,y}(z) := \sum_{i=1}^{k} z^i \left[\sum_{j=i}^{k} P_j y(j-i)\right] - \sum_{i=1}^{k} z^i \left[\sum_{j=i}^{k} Q_j u(j-i)\right].$$

Here we have used the fact that $\deg Q \leq \deg P = k$ (this follows from the properness of $P^{-1}Q$) and - for $\deg Q < j \leq k$ - defined $Q_j := 0$, so the sums are easier to write down.

By Lemma 11.2.6 and the fact that (A, B, C, D) is a realization of (P, Q), a trajectory $(u, x, w) \in \mathcal{B}(A, B, C, D)$ if, and only if, x and w are Z transformable and

$$\begin{aligned}\hat{x}(z) &= (zI - A)^{-1}B\hat{u}(z) + (zI - A)^{-1}zx(0)\\ \hat{w}(z) &= (P^{-1}Q)(z)\hat{u}(z) + C(zI - A)^{-1}zx(0).\end{aligned}$$

By the properness of $P^{-1}Q$, by the Neumann series expansion of $(zI - A)^{-1}$ and regardless of $x(0)$, if we define $\hat{x}$ and $\hat{w}$ as above, then they are Z-transformable. Moreover, by the injectivity of the Z-transform, $(u, x, y) \in \mathcal{B}(A, B, C, D)$ as long as there exists $x(0) \in \mathbb{F}^{pk}$ such that

$$P^{-1}(z)\theta_{u,y}(z) = C(zI - A)^{-1}zx(0)$$

on $\mathbb{E}(0, r_2)$ for some $r_2 > 0$. By multiplying from the left by $P(z)$ we see that, equivalently, we are looking for $x(0) \in \mathbb{F}^{pk}$ such that

$$\sum_{i=1}^{k} z^i \left[\sum_{j=i}^{k} (P_j y(j-i) - Q_j u(j-i))\right] = P(z)C(zI - A)^{-1}zx(0) \quad (11.4.1)$$

on $\mathbb{E}(0, r_2)$.

Let us define a pk-dimensional vector subspace of $\mathbb{F}[z]^p$, $\mathbb{P} := \{q \in \mathbb{F}[z]^p : \deg q \leq k,\ q(0) = 0\}$, and a linear map

$$\begin{aligned}\psi_1\colon \quad & (\mathbb{F}^m)^k \times (\mathbb{F}^p)^k \to \mathbb{P}\\ & (\xi_0, \ldots, \xi_{k-1}, \mu_0, \ldots, \mu_{k-1}) \mapsto \sum_{i=1}^{k} z^i \left[\sum_{j=i}^{k} (P_j \xi_{j-i} - Q_j \mu_{j-i})\right].\end{aligned}$$

Now consider - for $1 \leq i \leq pk$ - the polynomial vectors $q_i(z) := P(z)C(zI - A)^{-1}ze_i \in \mathbb{F}[z]^p$, where $e_1, e_2, \ldots, e_{pk}$ are the standard basis vectors of $\mathbb{F}^{pk}$. We claim that if q_i's are a basis of $\mathbb{P}$, then the proof of this result is complete.

Indeed, if q_i's are a basis of $\mathbb{P}$, then the following linear map

$$\begin{aligned}\psi_2\colon \mathbb{F}^{pk} &\to \mathbb{P}\\ \xi &\mapsto P(z)C(zI - A)^{-1}\xi\end{aligned}$$

is a vector space isomorphism. Therefore ψ_2 is invertible and, since (11.4.1) reads $\psi_1(u^k, y^k) = \psi_2(x(0))$, the required $x(0) \in \mathbb{F}^{pk}$ is given by $x(0) = (\psi_2^{-1} \circ \psi_1)(u^k, y^k)$, which proves (b). As ψ_1 and ψ_2^{-1} are both linear maps, so is their composition $\phi := \psi_2^{-1} \circ \psi_1$; this proves (c). Finally, if P and Q are real polynomial matrices, then $\psi_1\left((\mathbb{R}^m)^k \times (\mathbb{R}^p)^k\right) \subseteq \mathbb{P} \cap \mathbb{R}[z]^p$ and $\psi_2(\mathbb{R}^{pk}) = \mathbb{P} \cap \mathbb{R}[z]^p$, whence $\phi\left((\mathbb{R}^m)^k \times (\mathbb{R}^p)^k\right) = (\psi_2^{-1} \circ \psi_1)\left((\mathbb{R}^m)^k \times (\mathbb{R}^p)^k\right) \subseteq \mathbb{R}^{pk}$, completing the proof of (d).

We will now check that q_i's form a basis of $\mathbb{P}$ by first establishing that q_i's are in $\mathbb{P}$ and then that q_i's are linearly independent.

By controllability of (A, B, C, D) and Lemma 11.3.14, we know that there exist polynomial matrices $X \in \mathbb{F}[z]^{pk \times pk}$ and $Y \in \mathbb{F}[z]^{m \times pk}$ such that $(zI - A)X(z) + BY(z) = I$. Multiply this from the left by $P(z)C(zI - A)^{-1}$ and use $C(zI - A)^{-1}B = (P^{-1}Q)(z) - D$ to obtain

$$\begin{aligned} P(z)C(zI - A)^{-1} &= P(z)CX(z) + P(z)\left[(P^{-1}Q)(z) - D\right]Y(z) \\ &= P(z)CX(z) + Q(z)Y(z) - P(z)DY(z). \end{aligned}$$

Thus $P(z)C(zI - A)^{-1} \in \mathbb{F}[z]^{p \times pk}$. Since $q_i(z) = P(z)C(zI - A)^{-1}ze_i$, we thus have $q_i \in \mathbb{F}[z]^p$ and $q_i(0) = 0$. Moreover note that

$$\lim_{|z| \to \infty} z^{-k} P(z)C(zI - A)^{-1} = P_k C \lim_{|z| \to \infty} (zI - A)^{-1} = 0.$$

Hence $\deg P(z)C(zI - A)^{-1} \leq k - 1$, so that indeed $q_i \in \mathbb{P}$.

Thus we are done if we show that $q_1, q_2, \ldots q_{pk}$ are linearly independent. To this end suppose that there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_{pk} \in \mathbb{F}$ such that $\sum_{i=1}^{pk} \lambda_i z P(z)C(zI - A)^{-1}e_i = 0$ for all $z \in \mathbb{C}$. If we let $\xi := \sum_{i=1}^{pk} \lambda_i e_i$, then, by the invertibility of $zP(z)$, this is equivalent to $C\left(I - \frac{A}{z}\right)^{-1}\xi = 0$ for all $z \in \mathbb{E}(0, r)$ for some $r > 0$. The Neumann series for $|z| > \|A\|$, give us $\left(I - \frac{A}{z}\right)^{-1} = \sum_{j=0}^{\infty} \frac{A^j}{z^j}$, so that $\sum_{j=0}^{\infty} \frac{CA^j\xi}{z^j} = 0$ and hence $0 = C\xi = CA\xi = CA^2\xi = \ldots$, which gives us

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{pk-1} \end{pmatrix} \xi = 0.$$

However (A, B, C, D) is observable, so we must have $\xi = 0$ and thus λ_i's are all 0. This in turn implies that q_i's are linearly independent and completes the proof. $\square$

Of course, arbitrary trajectories in $\mathcal{B}(P, Q)$ might consist of elements that are not Z-transformable. To prove a counterpart of Proposition 11.4.2 for arbitrary trajectories, we will consider possibly non-Z-transformable inputs u and then set them equal to 0 after a finite time $T \in \mathbb{N}_0$.

Corollary 11.4.3. *Let* $(P, Q) \in \mathrm{IO_s}(m, p; \mathbb{F})$ *and set* $k := \deg P$.

1. *The McMillan degree of* (P, Q) *is* $pk = \deg\det P$.
2. *Let* (A, B, C, D) *be a minimal realization of* (P, Q). *Then for each* $(u, y) \in \mathcal{B}(P, Q)$ *there exists a unique* $x \in (\mathbb{F}^{pk})^{\mathbb{N}_0}$ *such that* $(u, x, y) \in \mathcal{B}(A, B, C, D)$.
3. *Moreover, there exists a linear map* $\phi\colon (\mathbb{F}^m)^k \times (\mathbb{F}^p)^k \to \mathbb{F}^{pk}$ *such that, for all* $(u, y) \in \mathcal{B}(P, Q)$ *and for the corresponding* x *such that* $(u, x, y) \in \mathcal{B}(A, B, C, D)$, *we have* $\phi(u^k, y^k) = x(0)$.

Proof. (a) follows from Lemma 11.3.21, so we proceed to prove (b) and (c).

First, let us consider the case $\mathbb{F} = \mathbb{C}$ and set $\phi\colon (\mathbb{C}^m)^k \times (\mathbb{C}^p)^k \to \mathbb{C}^{pk}$ to be the map, the existence of which is guaranteed by Proposition 11.4.2. Let $T \in \mathbb{N}_0$, pick $(u, y) \in \mathcal{B}(P, Q)$ and set $u_T := \pi_T u$. By Corollary 11.1.5, we know that there exists $y_T \in (\mathbb{C}^p)^{\mathbb{N}_0}$ such that the trajectory $(u_T, y_T) \in \mathcal{B}(P, Q)$ and

$$y_T(t) = y(t) \qquad \forall\, t \le T. \tag{11.4.2}$$

Since u_T is clearly Z-transformable, the use of Proposition 11.4.2 shows us that there exists $x_T \in (\mathbb{C}^{pk})^{\mathbb{N}_0}$ such that $(u_T, x_T, y_T) \in \mathcal{B}(A, B, C, D)$. Moreover, by our choice of ϕ, we have $x_T(0) = \phi(u_T^k, y_T^k)$.

We now define $x \in (\mathbb{C}^{pk})^{\mathbb{N}_0}$ by

$$x(t) := \begin{cases} x_{k-1}(t), & \text{for } t \le k-1 \\ x_t(t), & \text{for } t \ge k. \end{cases}$$

Since $u^k = (u(0), u(1), \dots, u(k-1)) = u_{k-1}^k$ and $y^k = (y(0), y(1), \dots, y(k-1)) = y_{k-1}^k$, we have $x(0) = x_{k-1}(0) = \phi(u_{k-1}^k, y_{k-1}^k) = \phi(u^k, y^k)$. Therefore, if we prove that $(u, x, y) \in \mathcal{B}(A, B, C, D)$, then we have (b) and (c). Clearly, it suffices to check that

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \qquad \forall\, t \in \mathbb{N}_0. \end{aligned}$$

If $t \le k-2$, then this follows from the definition of x, equation (11.4.2), $(u_{k-1}, y_{k-1}) \in \mathcal{B}(P, Q)$ and the observation that, for $t \le k-2$, we have $u_{k-1}(t) = u(t)$:

$$\begin{aligned} x(t+1) = x_{k-1}(t+1) &= Ax_{k-1}(t) + Bu_{k-1}(t) = Ax(t) + Bu(t) \\ y(t) = y_{k-1}(t) \qquad &= Cx_{k-1}(t) + Du_{k-1}(t) = Cx(t) + Du(t). \end{aligned} \tag{11.4.3}$$

If $t \geq k-1$, then equation (11.4.3) still holds, provided that we show $x_{t+1}(t) = x_t(t)$. Since u, u_t and u_{t+1} agree up to time $k-1$, as do y, y_t and y_{t+1}, we have

$$x_t(0) = \phi(u_t^k, y_t^k) = \phi(u^k, y^k) = \phi(u_{t+1}^k, y_{t+1}^k) = x_{t+1}(0).$$

Moreover, $x_t(t)$ and $x_{t+1}(t)$ are given by (11.2.3), so, since u_t and u_{t+1} agree up to time t, we indeed have $x_t(t) = x_{t+1}(t)$, which completes the proof of (b).

The only difference of the proof of the case $\mathbb{F} = \mathbb{R}$ is that we need to check that $x \in (\mathbb{R}^{pk})^{\mathbb{N}_0}$. Since $x(0) = \phi(u^k, y^k)$ and since $u^k \in (\mathbb{R}^m)^k$, $y^k \in (\mathbb{R}^p)^k$, Proposition 11.4.2 (d) implies that $x(0) \in \mathbb{R}^{pk}$. Thus, by characterization (11.2.3), $x(t) \in \mathbb{R}^{pk}$ for all $t \in \mathbb{N}_0$, which completes the proof. □

11.4.2 Behaviours of input-output systems

In this subsection we will demonstrate that the conclusions of Corollary 11.4.3 are true even if we remove the assumption $\det P_k \neq 0$, see Theorem 11.4.8.

We will build towards the realization theorem, namely Theorem 11.4.8, in a series of steps. We will first introduce a realization called the observer-form realization and observe in Theorem 11.4.5 that, under some extra assumptions, it almost proves the realization theorem. In Corollaries 11.4.6 and 11.4.7 we will relax these extra assumptions and the realization theorem will follow.

Lemma 11.4.4 (The observer-form realization)**.** *Consider an input-output system $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$, assume that P is row reduced and that $P^{-1}Q$ is strictly proper. Set $n := \sum_{a=1}^{p} r_a(P)$ and define a block diagonal polynomial matrix $\Psi \in \mathbb{F}[z]^{p\times n}$ by*

$$\Psi(z) :=$$
$$\begin{pmatrix} z^{r_1(P)-1} & \dots & z & 1 & 0 & & & \dots & & & 0 \\ & 0 & & & z^{r_2(P)-1} & \dots & z & 1 & \dots & & 0 \\ & & & & & & & & \ddots & & \\ & 0 & & & & 0 & & & \dots & z^{r_p(P)-1} & \dots & z & 1 \end{pmatrix}, \tag{11.4.4}$$

where the matrix block is empty if the corresponding row degree is 0.

Then there exists a realization $(A, B, C, 0) \in \Sigma(m, n, p; \mathbb{F})$ of (P, Q) such that

$$\begin{pmatrix} \Psi(z) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} zI - A & B \\ -C & 0 \end{pmatrix} = \begin{pmatrix} P(z) & Q(z) \\ -I & 0 \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}. \tag{11.4.5}$$

Proof. The relevant linear state-space system is constructed on pp. 413 - 417 from [34]. Equation (38) on p. 417 from [34] is precisely our claim. $\square$

Theorem 11.4.5. *Consider $(P,Q) \in \mathrm{IO}(m,p;\mathbb{F})$ and set $k := \deg P$. Assume that P is row reduced, that $P^{-1}Q$ is strictly proper and set $n := \deg\det P$.*

(a) There exists a realization $(A,B,C,0) \in \Sigma(m,n,p;\mathbb{F})$ of (P,Q).

(b) For each $(u,y) \in \mathcal{B}(P,Q)$ with a Z-transformable u, there exists a unique $x \in (\mathbb{F}^n)^{\mathbb{N}_0}$ such that $(u,x,y) \in \mathcal{B}(A,B,C,0)$.

(c) Moreover, there exists a linear map $\phi\colon (\mathbb{F}^m)^k \times (\mathbb{F}^p)^k \to \mathbb{F}^{pk}$ such that, for all $(u,y) \in \mathcal{B}(P,Q)$ and for the corresponding x such that $(u,x,y) \in \mathcal{B}(A,B,C,D)$, we have $\phi(u^k,y^k) = x(0)$.

(d) Finally, if $\mathbb{F} = \mathbb{C}$, but P and Q are real polynomial matrices, then the image of the restriction of ϕ to $(\mathbb{R}^m)^k \times (\mathbb{R}^p)^k$ is contained in $\mathbb{R}^{pk}$.

Remark: note that the conclusions of (a) are not surprising at all, the important piece of information there is that the order of the realization $(A,B,C,0)$ is $\deg\det P$.

Proof. By row reducedness of P, we have $n = \sum_{a=1}^{p} r_a(P)$, so we can define $\Psi \in \mathbb{F}[z]^{p\times n}$ as in (11.4.4) to see that, by Lemma 11.4.4, there exists a realization $(A,B,C,0) \in \Sigma(m,n,p;\mathbb{F})$ of (P,Q) such that (11.4.5) holds. This completes the proof of (a). We note for further use that the top left entry of (11.4.5) gives us

$$z\Psi(z) = P(z)C(zI-A)^{-1}z. \tag{11.4.6}$$

Let us write $P(z) = \sum_{j=0}^{k} P_j z^j$ and $Q(z) = \sum_{j=0}^{k} Q_j z^j$, for appropriate matrices P_j and Q_j. Now consider a trajectory $(u,y) \in \mathcal{B}(P,Q)$ with a Z-transformable u. Then, by Lemma 11.2.7, y is Z-transformable and on some $\mathbb{E}(0,r_1)$ we have

$$\hat{y}(z) = (P^{-1}Q)(z)\hat{u}(z) + P^{-1}(z)\theta_{u,y}(z),$$

where $\theta_{u,y} \in \mathbb{F}[z]^p$ is given by

$$\theta_{u,y}(z) := \sum_{i=1}^{k} z^i \left[\sum_{j=i}^{k} P_j y(j-i)\right] - \sum_{i=1}^{k} z^i \left[\sum_{j=i}^{k} Q_j u(j-i)\right].$$

Here we have used the fact that $\deg Q \le \deg P = k$ (this follows from Lemma 11.3.22) and - for $\deg Q < j \le k$ - defined $Q_j := 0$, so the sums are easier to write down.

By Lemma 11.2.6, $(u, x, w) \in \mathcal{B}(A, B, C, 0)$ if, and only if, x and w are Z-transformable and

$$\begin{aligned}\hat{x}(z) &= (zI - A)^{-1}B\hat{u}(z) + (zI - A)^{-1}zx(0)\\ \hat{w}(z) &= (P^{-1}Q)(z)\hat{u}(z) + C(zI - A)^{-1}zx(0).\end{aligned}$$

By the strict properness of $P^{-1}Q$ and by the Neumann series expansion of $(zI - A)^{-1}$, if we define $\hat{x}$ and $\hat{w}$ as above, then they are Z-transformable. Moreover, by the injectivity of the Z-transform, $(u, x, y) \in \mathcal{B}(A, B, C, 0)$ as long as there exists $x(0) \in \mathbb{F}^n$ such that $P^{-1}(z)\theta_{u,y}(z) = C(zI - A)^{-1}zx(0)$. Hence, by (11.4.6), equivalently we have to find $x(0) \in \mathbb{F}^n$ such that

$$\theta_{u,y}(z) = z\Psi(z)x(0).$$

It is convenient to define $\gamma_{i,j}(u, y) := P_j y(j - i) - Q_j u(j - i)$, so that this reads

$$\theta_{u,y}(z) = \sum_{i=1}^{k} z^i \sum_{j=i}^{k} \gamma_{i,j}(u, y) =$$
$$\begin{pmatrix} z^{r_1(P)} & \dots & z^2 & z & & 0 & & & \dots & & & 0 \\ & 0 & & & z^{r_2(P)} & \dots & z^2 & z & \dots & & & 0 \\ & & & & & & & & \ddots & & & \\ & 0 & & & & 0 & & & \dots & z^{r_p(P)} & \dots & z^2 & z \end{pmatrix} x(0). \tag{11.4.7}$$

Now note crucially that if $r_a(\theta_{u,y}) \leq r_a(P)$ for all $a \in \{1, \dots, p\}$, then there is a unique $x(0)$ that satisfies equation (11.4.7) and it is given by

$$x(0) = \begin{array}{c} \left| \begin{array}{c} \sum_{j=r_1(P)}^{k} [\gamma_{r_1(P),j}(u, y)]_1 \\ \sum_{j=r_1(P)-1}^{k} [\gamma_{r_1(P)-1,j}(u, y)]_1 \\ \vdots \\ \sum_{j=1}^{k} [\gamma_{1,j}(u, y)]_1 \\ \sum_{j=r_2(P)}^{k} [\gamma_{r_2(P),j}(u, y)]_2 \\ \sum_{j=r_2(P)-1}^{k} [\gamma_{r_2(P)-1,j}(u, y)]_2 \\ \vdots \\ \sum_{j=1}^{k} [\gamma_{1,j}(u, y)]_2 \end{array} \right| \\ \vdots \\ \left| \begin{array}{c} \sum_{j=r_p(P)}^{k} [\gamma_{r_p(P),j}(u, y)]_p \\ \sum_{j=r_p(P)-1}^{k} [\gamma_{r_p(P)-1,j}(u, y)]_p \\ \vdots \\ \sum_{j=1}^{k} [\gamma_{1,j}(u, y)]_p \end{array} \right| \end{array} \tag{11.4.8}$$

Here subscripts for vectors in $\mathbb{F}^p$ denote the respective elements of the vector. Since $\gamma_{i,j}$'s depend linearly on the first $k-1$ values of u and y, equation (11.4.8) defines the required linear map ϕ. Thus (b) and (c) follow provided that $r_a(\theta_{u,y}) \leq r_a(P)$.

To prove this, set e_a to be the a-th basis vector of the canonical basis for $\mathbb{F}^p$ (or, depending on the context, for $\mathbb{F}^m$) and note that $e_a^* M$ is the a-th row of a matrix M; this shorthand will be useful in what is to come. We can now use Lemma 11.3.22 to estimate

$$\begin{aligned}
r_a(\theta_{u,y}) =& \max\left\{ i \in \{1,\dots,k\} : e_a^* \left[\sum_{j=i}^{k} P_j y(j-i) - Q_j u(j-i) \right] \neq 0 \right\} \\
\leq& \max\Big\{ i \in \{1,\dots,k\} : e_a^* P_j \neq 0_{1\times p} \text{ or} \\
& \qquad\qquad e_a^* Q_j \neq 0_{1\times m} \text{ for some } i \leq j \leq k \Big\} \\
=& \max\left\{ i \in \{1,\dots,k\} : e_a^* P_j \neq 0_{1\times p} \text{ for some } i \leq j \leq k \right\} \\
=& \max\left\{ i \in \{1,\dots,k\} : e_a^* P_i \neq 0_{1\times p} \right\} \\
=& r_a(P).
\end{aligned}$$

(d) follows from equation (11.4.8). □

The conclusions of Theorem 11.4.5 extend to trajectories, which might not be Z-transformable. *Mutatis mutandis*, the proof follows along the same lines as the proof of Corollary 11.4.3, so we omit it.

Corollary 11.4.6. *Consider $(P,Q) \in \mathrm{IO}(m,p;\mathbb{F})$ and set $k := \deg P$. Assume that P is row reduced, that $P^{-1}Q$ is strictly proper and set $n := \deg\det P$.*

(a) There exists a realization $(A,B,C,0) \in \Sigma(m,n,p;\mathbb{F})$ of (P,Q).

(b) For each $(u,y) \in \mathcal{B}(P,Q)$, there exists a unique $x \in (\mathbb{F}^n)^{\mathbb{N}_0}$ such that $(u,x,y) \in \mathcal{B}(A,B,C,0)$.

(c) Moreover, there exists a linear map $\phi\colon (\mathbb{F}^m)^k \times (\mathbb{F}^p)^k \to \mathbb{F}^{pk}$ such that, for all $(u,y) \in \mathcal{B}(P,Q)$ and for the corresponding x such that $(u,x,y) \in \mathcal{B}(A,B,C,D)$, we have $\phi(u^k,y^k) = x(0)$.

Now we will obtain a version of the above result for input-output systems, where P is not necessarily row reduced and where $P^{-1}Q$ is not necessarily strictly proper.

Corollary 11.4.7. *Consider $(P,Q) \in \mathrm{IO}(m,p;\mathbb{F})$ and set $k := \deg P$, $n := \deg\det P$.*

(a) There exists a realization $(A,B,C,D) \in \Sigma(m,n,p;\mathbb{F})$ of (P,Q).

(b) *For each* $(u, y) \in \mathcal{B}(P, Q)$*, there exists a unique* $x \in (\mathbb{F}^n)^{\mathbb{N}_0}$ *such that* $(u, x, y) \in \mathcal{B}(A, B, C, D)$*.*

(c) *Moreover, there exists a linear map* $\phi\colon (\mathbb{F}^m)^k \times (\mathbb{F}^p)^k \to \mathbb{F}^n$ *such that, for each* $(u, y) \in \mathcal{B}(P, Q)$ *and for the corresponding* x *such that* $(u, x, y) \in \mathcal{B}(A, B, C, D)$*, we have* $\phi(u^k, y^k) = x(0)$*.*

Proof. Let us set $D := \lim_{|z|\to\infty}(P^{-1}Q)(z)$. It is straightforward to verify that

$$(u, y) \in \mathcal{B}(P, Q) \iff (u, y - Du) \in \mathcal{B}(P, Q - PD). \tag{11.4.9}$$

By Lemma 11.3.23 there exists a unimodular V such that VP is row reduced and such that $\deg VP \leq \deg P = k$. Moreover, $(VP)^{-1}(V(Q - PD)) = P^{-1}(Q-PD)$ is a strictly proper rational function matrix and $\deg\det VP = \deg\det P = n$. Hence we can apply Corollary 11.4.6 to the input-output system $(VP, V(Q - PD)) \in \mathrm{IO}(m, p; \mathbb{F})$ to see that there exists a realization $(A, B, C, 0) \in \Sigma(m, n, p; \mathbb{F})$ of $(VP, V(Q - PD))$ such that for each $(u, y) \in \mathcal{B}(VP, V(Q - PD))$ there exists a unique $x \in (\mathbb{F}^n)^{\mathbb{N}_0}$ such that $(u, x, y) \in \mathcal{B}(A, B, C, 0)$. Moreover, if we set $l := \deg VP \leq k$, then there exists a linear map $\psi\colon (\mathbb{F}^m)^l \times (\mathbb{F}^p)^l \to \mathbb{F}^n$ such that, for all $(u, y) \in \mathcal{B}(VP, V(Q - PD))$ and for the corresponding x such that $(u, x, y) \in \mathcal{B}(A, B, C, 0)$, we have $\psi(u^l, y^l) = x(0)$.

It is easy to see that (A, B, C, D) is a realization of (P, Q), which proves (a). Now pick an arbitrary trajectory $(u, y) \in \mathcal{B}(P, Q)$. Then $(u, y - Du) \in \mathcal{B}(P, Q-PD) = \mathcal{B}(VP, V(Q-PD))$, where the equality of behaviours follows from the unimodularity of V. Hence there exists a unique $x \in (\mathbb{F}^n)^{\mathbb{N}_0}$ such that $(u, x, y - Du) \in \mathcal{B}(A, B, C, 0)$ and thus $(u, x, y) \in \mathcal{B}(A, B, C, D)$, which gives us (b). Moreover, by the definition of ψ, we have $x(0) = \psi(u^l, y^l - Du^l)$. Since $k \geq l$, we can define a linear map $\phi\colon (\mathbb{F}^m)^k \times (\mathbb{F}^p)^k \to \mathbb{F}^n$ by $\phi(u^k, y^k) := \psi(u^l, y^l - Du^l)$, which completes the proof of (c). □

We can guarantee that (A, B, C, D) is a minimal realization, if we assume that P and Q are left coprime.

Theorem 11.4.8 (Realization theorem)**.** *Consider an input-output system* $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$*, set* $k := \deg P$*,* $n := \deg\det P$ *and assume that* P *and* Q *are left coprime.*

(a) *The McMillan degree of* (P, Q) *is* n*.*

(b) *Let* (A, B, C, D) *be a minimal realization of* (P, Q)*. Then for each trajectory* $(u, y) \in \mathcal{B}(P, Q)$ *there exists a unique* $x \in (\mathbb{F}^n)^{\mathbb{N}_0}$ *such that* $(u, x, y) \in \mathcal{B}(A, B, C, D)$*,*

(c) *Moreover, there exists a linear map* $\phi\colon (\mathbb{F}^m)^k \times (\mathbb{F}^p)^k \to \mathbb{F}^n$ *such that, for each* $(u, y) \in \mathcal{B}(P, Q)$ *and for the corresponding* x *such that* $(u, x, y) \in \mathcal{B}(A, B, C, D)$*, we have* $\phi(u^k, y^k) = x(0)$.

Proof. By Corollary 11.4.7, there exists a realization (A, B, C, D) of (P, Q) of degree n such that for each $(u, y) \in \mathcal{B}(P, Q)$ there exists a unique $x \in (\mathbb{F}^n)^{\mathbb{N}_0}$ such that $(u, x, y) \in \mathcal{B}(A, B, C, D)$. Moreover, there exists a linear map $\phi\colon (\mathbb{F}^m)^k \times (\mathbb{F}^p)^k \to \mathbb{F}^n$ such that, for each $(u, y) \in \mathcal{B}(P, Q)$ and for the corresponding x such that $(u, x, y) \in \mathcal{B}(A, B, C, D)$, we have $\phi(u^k, y^k) = x(0)$. By Lemma 11.3.21, (A, B, C, D) is a minimal realization, so we only need to extend the above conclusions to any minimal realization of (P, Q). Since minimal realizations are unique up to a similarity transform (see e.g. Theorem 21.16 from [13]), this follows in a straightforward manner and we do not spell out the details. $\square$

Theorem 11.4.8 contains elements of novelty. Statement (a) is well-known (see e.g. §6.4 from Kailath [34]), but - to author's best knowledge - the only available proof of (b) is in the continuous-time setting (see Theorem 5.1 from [57], which has a short proof that we found difficult to penetrate) and (c) seems to be a new observation.

We finish this subsection with a characterization of trajectories in $\mathcal{B}(P, Q)$, where P and Q are not necessarily left coprime.

Proposition 11.4.9. *Consider* $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$ *and let* R *be a greatest common left divisor of* P *and* Q*, so that the left Bezout identity* (11.3.1) *holds:* $PX + QY = R$*. Set* P_1 *and* Q_1 *to be the polynomial matrices that satisfy* $P = RP_1$ *and* $Q = RQ_1$ *respectively.*

Then P_1 *and* Q_1 *are left coprime and*

$$\mathcal{B}(P, Q) = \mathcal{B}(P_1, Q_1) \oplus [(-Y(\mathcal{L}), X(\mathcal{L})) \ker R(\mathcal{L})] .$$

Proof. Since P is invertible, so is R; we here use the fact that $P, P_1, R \in \mathbb{F}[z]^{p\times p}$. Hence we can left-multiply the Bezout identity by R^{-1} to obtain

$$P_1X + Q_1Y = I, \tag{11.4.10}$$

which implies that P_1 and Q_1 are left coprime.

Let us now show that

$$\mathcal{B}(P, Q) = \mathcal{B}(P_1, Q_1) \oplus [(-Y(\mathcal{L}), X(\mathcal{L})) \ker R(\mathcal{L})] .$$

"$\subseteq$": Let $(u, y) \in \mathcal{B}(P, Q)$ and set $v := P_1(\mathcal{L})y - Q_1(\mathcal{L})u$. We can see that $v \in \ker R(\mathcal{L})$ as

$$\begin{aligned} R(\mathcal{L})v &= R(\mathcal{L})P_1(\mathcal{L})y - R(\mathcal{L})Q_1(\mathcal{L})u \\ &= P(\mathcal{L})y - Q(\mathcal{L})u = 0. \end{aligned}$$

Hence we can decompose (u, y) as

$$(u, y) = (u + Y(\mathcal{L})v, y - X(\mathcal{L})v) + (-Y(\mathcal{L}), X(\mathcal{L}))v,$$

so it remains to check that $(u+Y(\mathcal{L})v, y-X(\mathcal{L})v) \in \mathcal{B}(P_1, Q_1)$. But this is a straightforward calculation:

$$\begin{aligned} P_1(\mathcal{L})[y - X(\mathcal{L})v] - Q_1(\mathcal{L})[u + Y(\mathcal{L})v] = \\ &= P_1(\mathcal{L})y - Q_1(\mathcal{L})u - [P_1(\mathcal{L})X(\mathcal{L}) + Q_1(\mathcal{L})Y(\mathcal{L})]v \\ &= P_1(\mathcal{L})y - Q_1(\mathcal{L})u - v \\ &= 0. \end{aligned}$$

It remains to show that the sum is direct. If there exist two such decompositions, then - upon considering their difference - there also exists $(u_1, y_1) \in \mathcal{B}(P_1, Q_1)$ such that $(u_1, y_1) \in (-Y(\mathcal{L}), X(\mathcal{L})) \ker R(\mathcal{L})$. Hence there exists $v \in \ker R(\mathcal{L})$ such that $(u_1, y_1) = (-Y(\mathcal{L}), X(\mathcal{L}))v$. But we can use the Bezout identity to see that this v must satisfy

$$v = P_1(\mathcal{L})X(\mathcal{L})v + Q_1(\mathcal{L})Y(\mathcal{L})v = P_1(\mathcal{L})y_1 - Q_1(\mathcal{L})u_1 = 0,$$

so that $(u_1, y_1) = (-Y(\mathcal{L}), X(\mathcal{L}))v = (0, 0)$. Therefore we conclude that the decomposition is unique and the sum - direct.

"$\supseteq$": Let $(u, y) \in \mathcal{B}(P_1, Q_1)$ and $v \in \ker R(\mathcal{L})$. Then

$$\begin{aligned} P(\mathcal{L})(y + X(\mathcal{L})v) &= R(\mathcal{L})P_1(\mathcal{L})y + P(\mathcal{L})X(\mathcal{L})v \\ &= R(\mathcal{L})Q_1(\mathcal{L})u + R(\mathcal{L})v - Q(\mathcal{L})Y(\mathcal{L})v \\ &= Q(\mathcal{L})(u - Y(\mathcal{L})v), \end{aligned}$$

which shows that $(u - Y(\mathcal{L})v, y + X(\mathcal{L})v) \in \mathcal{B}(P, Q)$ and completes the proof. $\square$

Example 11.4.10. Consider $P(z) = z^2 - 2z$ and $Q(z) = z - 2$ so that $(P, Q) \in \mathrm{IO}(1, 1; \mathbb{R})$. A greatest common (left) divisor of P and Q is $R(z) = z - 2$ and

$$\ker R(\mathcal{L}) = \left\{ y \in \mathbb{F}^{\mathbb{N}_0} : y(t) = 2^t y(0) \right\}.$$

We can check that the (left) Bezout identity reads $(z^2-2)\cdot 0 + (z-2)\cdot 1 = z-2$, so - for $P_1(z) = z$ and $Q_1(z) = 1$ - we have $P = RP_1$ and $Q = RQ_1$. Thus, by Proposition 11.4.9, $(u, y) \in \mathcal{B}(P, Q)$ if, and only if, there exists $\xi \in \mathbb{F}$ such that

$$y(t+1) = u(t) + 2^{t+1}\xi \qquad \forall\, t \in \mathbb{N}_0.$$

11.5 Behaviours of image input-output systems

In this section we turn our attention to a class of input-output systems that we will call image input-output systems. Under the name "image representations", the behaviours of their continuous-time counterparts have been explored in §6.6 from Polderman and Willems [47]. As we will see in Lemmas 11.5.2 and 11.5.3, the trajectories of behaviours of image input-output systems are closely related to trajectories in behaviours of input-output systems.

Definition 11.5.1. Let $S \in \mathbb{F}[z]^{m\times m}$ and $T \in \mathbb{F}[z]^{p\times m}$ be such that $\det S \neq 0$ and $TS^{-1} \in \mathbb{F}(z)^{p\times m}$ is proper. We call the tuple (S,T) a (discrete-time) **image input-output system** and we write $(S,T) \in \mathrm{IO}_{\mathrm{im}}(m,p;\mathbb{F})$.

We define the **behaviour** $\mathcal{B}(S,T)$ of an image input-output system (S,T) as the set of all triples $(u,v,y) \in (\mathbb{F}^m)^{\mathbb{N}_0} \times (\mathbb{F}^m)^{\mathbb{N}_0} \times (\mathbb{F}^p)^{\mathbb{N}_0}$ such that

$$\begin{aligned} u &= S(\mathcal{L})v \\ y &= T(\mathcal{L})v. \end{aligned}$$

Lemma 11.5.2. *Let $(P,Q) \in \mathrm{IO}(m,p;\mathbb{F})$ and $(S,T) \in \mathrm{IO}_{\mathrm{im}}(m,p;\mathbb{F})$ be such that $P^{-1}Q = TS^{-1}$. If $(u,v,y) \in \mathcal{B}(S,T)$, then $(u,y) \in \mathcal{B}(P,Q)$.*

Proof. Since $(u,v,y) \in \mathcal{B}(S,T)$, we have $u = S(\mathcal{L})v$ and $y = T(\mathcal{L})v$. Hence, by using $QS = PT$, we can check that $P(\mathcal{L})y = (PT)(\mathcal{L})v = (QS)(\mathcal{L})v = Q(\mathcal{L})u$, which in turn means that $(u,y) \in \mathcal{B}(P,Q)$ as required. □

We can obtain a partial converse of this result by assuming more regularity on (P,Q) and (S,T).

Lemma 11.5.3. *Let $(P,Q) \in \mathrm{IO}(m,p;\mathbb{F})$ and $(S,T) \in \mathrm{IO}_{\mathrm{im}}(m,p;\mathbb{F})$ be such that $P^{-1}Q = TS^{-1}$. Assume further that P and Q are left coprime while S and T are right coprime. If $(u,y) \in \mathcal{B}(P,Q)$, then there exists a unique $v \in (\mathbb{F}^m)^{\mathbb{N}_0}$ such that $(u,v,y) \in \mathcal{B}(S,T)$. It is given - for some polynomial matrices X,Y - by $v = X(\mathcal{L})u + Y(\mathcal{L})y$.*

Remark: the proof of this lemma uses ideas from the proof of Lemma 6.4.2 in [34].

Proof. Since P and Q are left coprime and S and T are right coprime, by Theorems 11.3.12 and 11.3.18, there exist polynomial matrices X_1, Y_1, X_2, Y_2 of appropriate dimensions such that the Bezout identities hold: $PX_1 + QY_1 = I_p$ and $X_2S + Y_2T = I_m$. Now pick an arbitrary $(u,y) \in \mathcal{B}(P,Q)$ and set $v := X_2(\mathcal{L})u + Y_2(\mathcal{L})y$. We will show that $(u,v,y) \in \mathcal{B}(S,T)$ and hence X_2, Y_2 are the sought polynomial matrices.

Note that $T(\mathcal{L})v = (TX_2)(\mathcal{L})u + (TY_2)(\mathcal{L})y$ and $S(\mathcal{L})v = (SX_2)(\mathcal{L})u + SY_2(\mathcal{L})y$. We will now obtain alternative expressions for TX_2, TY_2, SX_2 and SY_2. Using the Bezout identities, it is straightforward to check that

$$\begin{pmatrix} X_2 & Y_2 \\ -Q & P \end{pmatrix} \begin{pmatrix} S & -Y_1 \\ T & X_1 \end{pmatrix} = \begin{pmatrix} I_m & Y_2X_1 - X_2Y_1 \\ 0 & I_p \end{pmatrix}. \tag{11.5.1}$$

We now note that

$$\begin{pmatrix} I_m & Y_2X_1 - X_2Y_1 \\ 0 & I_p \end{pmatrix} \begin{pmatrix} I_m & X_2Y_1 - Y_2X_1 \\ 0 & I_p \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_p \end{pmatrix}.$$

Hence - upon setting $X_3 := SX_2Y_1 - SY_2X_1 - Y_1$ and $Y_3 := TX_2Y_1 - TY_2X_1 + X_1$ - we have

$$\begin{aligned} \begin{pmatrix} X_2 & Y_2 \\ -Q & P \end{pmatrix} \begin{pmatrix} S & X_3 \\ T & Y_3 \end{pmatrix} &= \begin{pmatrix} X_2 & Y_2 \\ -Q & P \end{pmatrix} \begin{pmatrix} S & -Y_1 \\ T & X_1 \end{pmatrix} \begin{pmatrix} I_m & X_2Y_1 - Y_2X_1 \\ 0 & I_p \end{pmatrix} \\ &= \begin{pmatrix} I_m & 0 \\ 0 & I_p \end{pmatrix}. \end{aligned}$$

This in turn means that we also have

$$\begin{pmatrix} S & X \\ T & Y \end{pmatrix} \begin{pmatrix} X_2 & Y_2 \\ -Q & P \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_p \end{pmatrix}, \tag{11.5.2}$$

which allows us to verify that

$$\begin{aligned} T(\mathcal{L})v =&(TX_2)(\mathcal{L})u + (TY_2)(\mathcal{L})y \\ =&(YQ)(\mathcal{L})u + y - (YP)(\mathcal{L})y \\ =&y + Y(\mathcal{L})\left[Q(\mathcal{L})u - P(\mathcal{L})y\right] \\ =&y \end{aligned}$$

and

$$\begin{aligned} S(\mathcal{L})v =&(SX_2)(\mathcal{L})u + (SY_2)(\mathcal{L})y \\ =&u + (XQ)(\mathcal{L})u - (XP)(\mathcal{L})y \\ =&u + X(\mathcal{L})\left[Q(\mathcal{L})u - P(\mathcal{L})y\right] \\ =&u, \end{aligned}$$

so that $(u, v, y) \in \mathcal{B}(S, T)$ as claimed.

We are left with checking the uniqueness of v. Suppose that $(u, v_1, y) \in \mathcal{B}(S, T)$ and $(u, v_2, y) \in \mathcal{B}(S, T)$. Then $S(\mathcal{L})(v_1 - v_2) = u - u = 0$ and $T(\mathcal{L})(v_1 - v_2) = y - y = 0$. Hence, by right coprimeness of S and T, we have

$$\begin{aligned} 0 &= X_2(\mathcal{L})(S(\mathcal{L})(v_1 - v_2)) + Y_2(\mathcal{L})(T(\mathcal{L})(v_1 - v_2)) \\ &= (X_2S + Y_2T)(\mathcal{L})(v_1 - v_2) \\ &= v_1 - v_2. \end{aligned}$$

□

Remark: equation (11.5.1) can be used to obtain other polynomial matrix identities in the spirit of equation (11.5.2). For example, in a similar way we can see that

$$\begin{pmatrix} I_m & -Y_2X_1 \\ 0 & I_p \end{pmatrix} \begin{pmatrix} X_2 & Y_2 \\ -Q & P \end{pmatrix} \begin{pmatrix} S & X_1 \\ T & Y_1 \end{pmatrix} \begin{pmatrix} I_m & Y_1X_2 \\ 0 & I_p \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_p \end{pmatrix},$$

which in turn (after a lengthy calculation) implies that Y_1 and X_2 commute: $Y_1X_2 = X_2Y_1$. These ideas come from the proof of Lemma 6.4.2 from [34].

Chapter 12

Stability of input-output Lur'e systems

In this chapter we will put Theorem 11.4.8 to use in stability analysis of Lur'e systems arising from the linear input-output systems introduced in the previous chapter. We will initially consider input-output Lur'e systems

$$\sum_{j=0}^{k} P_j y(t+j) = \sum_{j=0}^{k} Q_j f(y(t+j)). \tag{12.1}$$

These arise when modelling digital filters, sampled systems and linear multistep methods from numerical analysis, see Coughlan, Hill and Logemann [11]. Using Theorem 11.4.8, we will be able to obtain absolute stability results that resemble the ones from Parts I and II.

The use of Theorem 11.4.8 will allow us to construct a state-space system (A, B, C, D) such that for each y, which satisfies (12.1), there exists x such that $(x, y) \in \mathcal{B}(A, B, C, D, f)$. We will use this idea to obtain a number of stability results for input-output Lur'e systems (12.1); these will be stated using only data from the input-output Lur'e system.

First, we will use this to obtain in Proposition 12.1.7 an input-output Aizerman version of the circle criterion: it will turn out that if the input-output Lur'e system (12.1) is globally asymptotically stable for *all complex linear* output feedback matrices F that satisfy the norm condition $\|F(\xi)\| < r\,\|\xi\|$ for some $r > 0$ and for all $\xi \in \mathbb{C}^p \setminus \{0\}$, then the input-output Lur'e system (12.1) is globally asymptotically stable for *all nonlinear* output feedback maps f that satisfy the same norm condition $\|f(\xi)\| < r\,\|\xi\|$. This result will then lead to corollaries in much the same way as in Parts I and II.

After absolute stability analysis of (12.1), we will turn our attention to forced input-output Lur'e systems

$$\sum_{j=0}^{k} P_j y(t+j) = \sum_{j=0}^{k} Q_j(f(y(t+j)) + d(t+j)). \tag{12.2}$$

Similarly as above, Theorem 11.4.8 will enable us to apply ISS results from Part I to obtain stability criteria guaranteeing an appropriate generalization of input-to-state stability to a setting without a notion of state. As in Parts I and II, this will lead to an Aizerman-like result, see Theorem 12.1.13.

Finally, in §12.2 we will briefly treat image input-output Lur'e systems in a similar way, see Proposition 12.2.3.

Stability properties of input-output feedback systems have been studied in Desoer and Vidyasagar [14] and Zames [64]. The results proved there depend on norm approximations and as a result establish input-to-output stability in the l^p sense ($1 \le p \le \infty$). These can then be used to obtain asymptotic or exponential stability results via the technique of exponential weighting. In contrast, the use of Theorem 11.4.8 allows us to analyse stability of trajectories satisfying (12.1) and (12.2) in a state-space setting, where Lyapunov methods are available.

12.1 Input-output Lur'e systems

In this section we will analyse input-output Lur'e systems (12.1) and (12.2). We will first analyse the existence and uniqueness of solutions of an initial value problem induced by (12.1) in §12.1.1. Then we will obtain absolute stability results for (12.1) in §12.1.2 and, after introducing an appropriate ISS counterpart in an input-output setting, stability results for (12.2) in §12.1.3.

Definition 12.1.1. Let $(P,Q) \in \mathrm{IO}(m,p;\mathbb{F})$; if P and Q are left coprime and f is a map $f : \mathbb{F}^p \to \mathbb{F}^m$, then we say that (P,Q,f) is an **input-output Lur'e system**.

We define the **behaviour** $\mathcal{B}(P,Q,f)$ **of the input-output Lur'e system** (P,Q,f) as

$$\mathcal{B}(P,Q,f) := \left\{ y \in (\mathbb{F}^p)^{\mathbb{N}_0} \ : \ (f \circ y, y) \in \mathcal{B}(P,Q) \right\}.$$

Set $P(z) = \sum_{j=0}^{k} P_j z^j$ and $Q(z) = \sum_{j=0}^{k} Q_j z^j$, where $k := \deg P$ and where we have defined $Q_j := 0$ for $\deg Q < j \le \deg P$ (recall that $\deg P \ge \deg Q$

by properness of $P^{-1}Q$). If $y \in \mathcal{B}(P, Q, f)$, then this simply means that

$$\sum_{j=0}^{k} P_j y(t+j) = \sum_{j=0}^{k} Q_j f(y((t+j)) \tag{12.1.1}$$

for all $t \in \mathbb{N}_0$.

Note that, for a general f, we cannot guarantee that the behaviour of an input-output Lur'e system is nonempty or, if it is nonempty, uniqueness of solutions given the first few values of $y(i)$. Hence we will briefly look at the initial value problem corresponding to equation (12.1.1) and exhibit a few simple conditions when we can guarantee the existence or uniqueness of its solutions.

12.1.1 Existence and uniqueness of initial value problems defined by input-output systems

Let (P, Q, f) be an input-output Lur'e system, where $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$. Set $k := \deg P$ and write $P(z) = \sum_{j=0}^{k} P_j z^j$ and $Q(z) = \sum_{j=0}^{k} Q_j z^j$, where we have defined $Q_j := 0$ for $\deg Q < j \le \deg P$ ($\deg P \ge \deg Q$ by the properness of $P^{-1}Q$). Now consider the related initial value problem

$$\sum_{j=0}^{k} P_j y(t+j) = \sum_{j=0}^{k} Q_j f(y(t+j)) \qquad \forall\, t \in \mathbb{N}_0, \quad y^k = \xi \in (\mathbb{F}^p)^k. \tag{IVP}$$

If we define $g : \mathbb{F}^p \to \mathbb{F}^p$ by $g(\xi) := P_k \xi - Q_k f(\xi)$, then a sufficient condition for existence of solutions of (IVP) for all initial conditions is surjectivity of g and a sufficient condition for existence of a unique solution is bijectivity of g. Note that pre-multiplication of both sides of the equation in (IVP) by an invertible matrix T does not change surjectivity and bijectivity properties of the corresponding map Tg - this will be used in Proposition 12.1.5.

We will now apply well-known results from Renardy and Rogers [49], to obtain sufficient conditions for surjectivity and bijectivity of g.

Definition 12.1.2. A function $h : \mathbb{F}^p \to \mathbb{F}^p$ is called **coercive** if

$$\frac{\operatorname{Re}\langle h(\xi), \xi\rangle}{\|\xi\|} \to \infty \qquad \text{as} \qquad \|\xi\| \to \infty.$$

A function $h : \mathbb{F}^p \to \mathbb{F}^p$ is called **monotone** if

$$\operatorname{Re}\langle h(\xi) - h(\mu), \xi - \mu\rangle \ge 0 \qquad \text{for all} \qquad \xi, \mu \in \mathbb{F}^p. \tag{12.1.2}$$

We say that h is **strictly monotone** if the inequality (12.1.2) is strict whenever $\xi \neq \mu$.

The real versions of the following two results are Theorems 10.40 and 10.37 from [49]. The extensions to the complex case are straightforward, so we omit them.

Theorem 12.1.3. *Let $h : \mathbb{F}^p \to \mathbb{F}^p$ be continuous and coercive. Then h is surjective.*

Theorem 12.1.4. *Let $h : \mathbb{F}^p \to \mathbb{F}^p$ be continuous, strictly monotone and coercive. Then h is bijective.*

As a corollary we can now provide sufficient conditions for the existence and uniqueness of solutions of the initial value problem (IVP).

Proposition 12.1.5. *Let (P, Q, f) be an input-output Lur'e system with $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$. Let $k := \deg P$, write $P(z) = \sum_{j=0}^{k} P_j z^j$ and $Q(z) = \sum_{j=0}^{k} Q_j z^j$, and consider the initial value problem (IVP). If there exists an invertible matrix $T \in \mathbb{F}^{p\times p}$ such that*

$$\frac{1}{\|\xi\|} \operatorname{Re} \langle TP_k\xi - TQ_k f(\xi), \xi\rangle \to \infty \qquad \text{as} \qquad \|\xi\| \to \infty,$$

then the initial value problem (IVP) has a solution $y \in \mathcal{B}(P, Q, f)$ for any initial conditions. If there exists an invertible matrix $T \in \mathbb{F}^{p\times p}$ such that

$$\operatorname{Re} \langle TP_k(\xi - \mu) - TQ_k(f(\xi) - f(\mu)), \xi - \mu\rangle > 0 \qquad \forall\, \xi, \mu \in \mathbb{F}^p,\ \xi \neq \mu,$$

then the initial value problem (IVP) has a unique solution $y \in \mathcal{B}(P, Q, f)$ for any initial conditions.

In what is to follow we will not care about the existence and uniqueness of solutions of the initial value problem (IVP). Instead, our stability results will hold for all trajectories $y \in \mathcal{B}(P, Q, f)$. However, Proposition 12.1.5 is a reassuring result to know and we will show in a remark after our main absolute stability result, Proposition 12.1.7, that - under its assumptions - existence of solutions of the initial value problem (IVP) is guaranteed.

12.1.2 Absolute stability

Recall two shorthands: for $y \in (\mathbb{F}^p)^{\mathbb{N}_0}$, we write $y^k = (y(0), y(1), \dots, y(k-1)) \in (\mathbb{F}^p)^k$ and, for $D \in \mathbb{F}^{p\times m}, K \in \mathbb{F}^{m\times p}$, we write $D^K = (I - DK)^{-1}D$.

We will be concerned with the following stability concepts.

Definition 12.1.6. Consider an input-output Lur'e system (P, Q, f) and set $k := \deg P$. We say that (P, Q, f) is

(a) **globally stable**, if there exists $c > 0$ such that

$$\|y(t)\| \leq c \left\| y^k \right\| \qquad \forall\, t \in \mathbb{N}_0 \ \ \forall\, y \in \mathcal{B}(P, Q, f);$$

(b) **globally asymptotically stable**, if it is globally stable and

$$\lim_{t\to\infty} y(t) = 0 \qquad \forall\, y \in \mathcal{B}(P,Q,f);$$

(c) **globally exponentially stable**, if there exist $c > 0$ and $a \in (0,1)$ such that

$$\|y(t)\| \le ca^t \left\| y^k \right\| \qquad \forall\, t \in \mathbb{N}_0 \;\; \forall\, y \in \mathcal{B}(P,Q,f).$$

Now the use of Theorem 11.4.8 and Proposition 4.2.1 gives us a short proof of the input-output Aizerman version of the circle criterion.

Proposition 12.1.7. *Consider an input-output Lur'e system (P,Q,f) with $(P,Q) \in \mathrm{IO}(m,p;\mathbb{F})$. Set $D := \lim_{|z|\to\infty}(P^{-1}Q)(z)$, let $K \in \mathbb{F}^{m\times p}$, $r > 0$ and assume that $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}\left(P^{-1}Q\right)$.*

(a) If $\left\|D^K\right\| < 1/r$ and

$$\|f(\xi) - K\xi\| \le r\,\|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p, \tag{12.1.3}$$

then the input-output Lur'e system (P,Q,f) is globally stable.

(b) If f is continuous, $\left\|D^K\right\| < 1/r$ and

$$\|f(\xi) - K\xi\| < r\,\|\xi\| \qquad \forall \xi \in \mathbb{F}^p \setminus \{0\}, \tag{12.1.4}$$

then the input-output Lur'e system (P,Q,f) is globally asymptotically stable.

(c) If there exists $\delta \in (0,r)$ such that

$$\|f(\xi) - K\xi\| \le (r-\delta)\,\|\xi\| \qquad \forall \xi \in \mathbb{F}^p, \tag{12.1.5}$$

then the input-output Lur'e system (P,Q,f) is globally exponentially stable.

Proof. Set $n := \deg\det P$ and $k := \deg P$; by Theorem 11.4.8 we know that there exists a controllable and observable realization $(A,B,C,D) \in \Sigma(m,n,p;\mathbb{F})$ of (P,Q). Also, for each $(u,y) \in \mathcal{B}(P,Q)$ there exists a unique $x \in (\mathbb{F}^n)^{\mathbb{N}_0}$ such that $(u,x,y) \in \mathcal{B}(A,B,C,0)$. Moreover, there exists a linear map $\phi\colon (\mathbb{F}^m)^k \times (\mathbb{F}^p)^k \to \mathbb{F}^n$ such that $x(0) = \phi(u^k, y^k)$.

Since $P^{-1}Q$ is the transfer function of (A,B,C,D), we can apply Proposition 4.2.9 to the Lur'e system (A,B,C,D,f).

(a) If f satisfies (12.1.3), then (A,B,C,D,f) is globally stable, so that, for some $c_1 > 0$ we have $\|y(t)\| \le c_1 \|x(0)\|$ for all $(x,y) \in \mathcal{B}(A,B,C,D,f)$.

(b) If f satisfies (12.1.4), then (A, B, C, D, f) is globally asymptotically stable, so it is globally stable and $\lim_{t\to\infty} y(t) = 0$ for all trajectories $(x, y) \in \mathcal{B}(A, B, C, D, f)$.

(c) If f satisfies (12.1.5), then (A, B, C, D, f) is globally exponentially stable, so there exists $a \in (0, 1)$ and c_2 such that $\|y(t)\| \le c_2 a^t \|x(0)\|$ for all $(x, y) \in \mathcal{B}(A, B, C, D, f)$.

Note that if $y \in \mathcal{B}(P, Q, f)$, then $(f \circ y, y) \in \mathcal{B}(P, Q)$, so that there exists $x \in (\mathbb{F}^n)^{\mathbb{N}_0}$ such that $(f \circ y, x, y) \in \mathcal{B}(A, B, C, D)$ or, equivalently, $(x, y) \in \mathcal{B}(A, B, C, D, f)$. Therefore, in view of the above application of Proposition 4.2.9, it suffices to show that there exists a positive c_3 such that

$$\|x(0)\| \le c_3 \left\| y^k \right\|$$

for all $y \in \mathcal{B}(P, Q, f)$ and for the corresponding x such that $(f \circ y, x, y) \in \mathcal{B}(A, B, C, D)$. This is straightforward. By linearity of ϕ, there exists a positive c_4 such that

$$\|x(0)\| \le c_4 \sum_{j=0}^{k-1} \left(\|f(y(j))\| + \|y(j)\|\right)$$

and the observation that $\|f(\xi)\| \le (r + \|K\|)\, \|\xi\|$ completes the proof. □

It is interesting to note that, under the assumptions of Proposition 12.1.7, the initial value problem (IVP) admits solutions for any initial conditions as long as P_k is invertible (here $k := \deg P$ and $P(z) = \sum_{j=0}^{k} P_j z^j$, $Q(z) = \sum_{j=0}^{k} Q_j z^j$). Note that, in all cases, the assumptions of Proposition 12.1.7 imply that there exists $\delta_1 > 0$ such that

$$\left\| D^K (f(\xi) - K\xi) \right\| \le (1 - \delta_1)\, \|\xi\| \tag{12.1.6}$$

for all $\xi \in \mathbb{F}^p$. Furthermore, by using $P_k D = Q_k$, we can check that, for $T \in \mathbb{F}^{p\times p}$, we have

$$\begin{aligned}
&\frac{1}{\|\xi\|} \operatorname{Re} \langle T P_k \xi - T Q_k f(\xi), \xi \rangle \\
&\qquad = \frac{1}{\|\xi\|} \operatorname{Re} \langle T(P_k - Q_k K)\xi - T Q_k (f(\xi) - K\xi), \xi \rangle \\
&\qquad = \frac{1}{\|\xi\|} \operatorname{Re} \langle T P_k \left[(I - DK)\xi - D(f(\xi) - K\xi) \right], \xi \rangle \\
&\qquad = \frac{1}{\|\xi\|} \operatorname{Re} \left\langle T P_k (I - DK) \left[\xi - D^K (f(\xi) - K\xi) \right], \xi \right\rangle .
\end{aligned}$$

If we now pick $T = (I - DK)^{-1} P_k^{-1}$, then the use of (12.1.6) and the Cauchy-Schwarz inequality gives us

$$\frac{1}{\|\xi\|} \operatorname{Re} \langle T P_k \xi - T Q_k f(\xi), \xi \rangle \geq \|\xi\| - (1 - \delta_1) \|\xi\| = \delta_1 \|\xi\|$$

for all $\xi \in \mathbb{F}^p$. Hence, by Proposition 12.1.5, the initial value problem (IVP) has a solution for all initial conditions.

Similarly, if we pick $T = (I - DK)^{-1} P_k^{-1}$, then we can check that

$$\begin{aligned}
&\frac{1}{\|\xi - \mu\|^2} \operatorname{Re} \langle T P_k(\xi - \mu) - T Q_k(f(\xi) - f(\mu)), \xi - \mu \rangle \\
&\qquad = 1 - \left\langle D^K [(f(\xi) - K\xi) - (f(\mu) - K\mu)], \xi - \mu \right\rangle \\
&\qquad \geq 1 - \frac{1}{r} \frac{\|(f(\xi) - K\xi) - (f(\mu) - K\mu)\|}{\|\xi - \mu\|}.
\end{aligned}$$

Hence, by Proposition 12.1.5, we see that the initial value problem (IVP) has a unique solution for all initial conditions as long as the map $\xi \mapsto f(\xi) - K\xi$ is Lipschitz continuous with Lipschitz constant strictly less than r.

By picking $K = 0$ in the input-output Aizerman version of the circle criterion and by using Lemma 3.2.7 to see that $\mathbb{B}_{\mathbb{C}}(0, r) \subseteq \mathbb{S}_{\mathbb{C}}\left(P^{-1} Q\right)$ is equivalent to $\left\|P^{-1} Q\right\|_{H^\infty} \leq \frac{1}{r}$, we obtain the following corollary, statement (c) of which is the small-gain theorem.

Corollary 12.1.8. *Consider an input-output Lur'e system (P, Q, f) with $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$ and set $D := \lim_{|z| \to \infty} (P^{-1} Q)(z)$.*

(a) If

$$\left\|P^{-1} Q\right\|_{H^\infty} \|f(\xi)\| \leq \|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p,$$

and if $\|D\| \sup_{\xi \in \mathbb{F}^p} \frac{f(\xi)}{\|\xi\|} < 1$, then the input-output Lur'e system (P, Q, f) is globally stable.

(b) If f is continuous,

$$\left\|P^{-1} Q\right\|_{H^\infty} \|f(\xi)\| < \|\xi\| \qquad \forall \xi \in \mathbb{F}^p \setminus \{0\},$$

and if $\|D\| \sup_{\xi \in \mathbb{F}^p} \frac{f(\xi)}{\|\xi\|} < 1$, then the input-output Lur'e system (P, Q, f) is globally asymptotically stable.

(c) If

$$\left\|P^{-1} Q\right\|_{H^\infty} \sup_{\xi \in \mathbb{F}^p} \frac{\|f(\xi)\|}{\xi} < 1$$

then the input-output Lur'e system (P, Q, f) is globally exponentially stable.

Note that if $\lim_{|z|\to\infty} P^{-1}Q(z) = 0$, then the inequalities involving D and D^K in Corollary 12.1.8 and Proposition 12.1.7 are automatically satisfied.

Example 12.1.9. We borrow an example from §1.2 in Hinrichsen and Pritchard [25], the Samuelson-Hicks multiplier-accelerator model. It models the evolution of total national income depending on consumer expenditure, investment and government expenditure. If we assume that the model has an equilibrium point, then the linearization around it is given by

$$y(t+2) = cy(t+1) + l(y(t+1) - y(t)), \tag{12.1.7}$$

where we are assuming, for simplicity, that the government expenditure is constant. Here c is assumed to be a constant between 0 and 1 related to consumer behaviour, while l is a parameter that a government might hope to influence. We will show that Corollary 12.1.8 can be used to guarantee stability for a range of parameter l values without finding explicit solutions. We model (12.1.7) by a SISO input-output Lur'e system (p, q, f), where $p(z) = z^2 - cz$, $q(z) = z - 1$ and $f(\xi) = l\xi$. Clearly,

$$\frac{q(z)}{p(z)} = \frac{z-1}{z(z-c)} = 1 - \frac{1-c}{z-c},$$

so that $\left\|\frac{q}{p}\right\|_{H^\infty} \geq 1 - \frac{1-c}{-1-c} = \frac{2}{1+c}$. Therefore, by Corollary 12.1.8 (a), we see that stable national income can be achieved by making sure that

$$|l| \leq \frac{1+c}{2}.$$

These values of l might be counterproductive if the government intends to increase the national income, but it is nevertheless interesting to know.

Similarly, one can obtain an input-output version of the MIMO circle criterion; for simplicity, we only state it for input-output Lur'e systems, where the underlying linear system has a strictly proper transfer function. Since the proof does not require new techniques, we omit it.

Proposition 12.1.10. *Consider an input-output Lur'e system (P, Q, f) with $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$ and define $G := P^{-1}Q$. Assume that G is strictly proper and that for some $K_1, K_2 \in \mathbb{F}^{m\times p}$ the rational function matrix $(I - K_2G)(I - K_1G)^{-1}$ is positive real.*

(a) If $\ker(K_1 - K_2) = \{0\}$ and if

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \leq 0 \qquad \forall\, \xi \in \mathbb{F}^p,$$

then the input-output Lur'e system (P, Q, f) is globally stable.

(b) If

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle < 0 \qquad \forall\, \xi \in \mathbb{F}^p \setminus \{0\},$$

then the input-output Lur'e system (P, Q, f) is globally asymptotically stable.

(c) If, for some positive δ, we have

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \leq -\delta \left\|\xi\right\|^2 \qquad \forall\, \xi \in \mathbb{F}^p,$$

then the input-output Lur'e system (P, Q, f) is globally exponentially stable.

12.1.3 Input-to-output stability

We will now introduce input-to-output stability, which captures the notion of input-to-state stability in our setting, where there is no state. Similarly as in §12.1.2, this will allow us to obtain stability results resembling the ones in Chapter 5.

Definition 12.1.11. Consider an input-output Lur'e system (P, Q, f) and let $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$. We define the **behaviour with disturbances** $\mathcal{B}_{\mathrm{d}}(P, Q, f)$ **of the input-output Lur'e system** (P, Q, f) as

$$\mathcal{B}_{\mathrm{d}}(P, Q, f) := \Big\{(d, y) \in (\mathbb{F}^m)^{\mathbb{N}_0} \times (\mathbb{F}^p)^{\mathbb{N}_0} \ : \ (f \circ y + d, y) \in \mathcal{B}(P, Q)\Big\}.$$

A notion of stability for systems with inputs and outputs has been explored by Sontag and Wang in [53], however they assume that there is a state as well and the initial state features in the definition of said stability notion instead of initial values of y. Since input-output Lur'e systems (P, Q, f) have no inherent notion of state, we prefer to use a stability notion defined in terms of only the data of (P, Q, f) and its behaviour with disturbances.

Definition 12.1.12. Consider an input-output Lur'e system (P, Q, f) with $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$ and set $k := \deg P$. We say that (P, Q, f) is **input-to-output stable** if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$\|y(t)\| \leq \beta\left(\left\|y^k\right\|, t\right) + \gamma(\|d\|_\infty) \qquad \forall\, t \in \mathbb{N}_0$$

and for all $(d, y) \in \mathcal{B}_{\mathrm{d}}(P, Q, f)$ with $d \in l^\infty(\mathbb{F}^m)$.

The combination of Theorem 11.4.8 and Proposition 5.3.5 leads us to the following result.

Theorem 12.1.13. *Consider an input-output Lur'e system* (P, Q, f) *with* $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$. *Assume that for some* $K \in \mathbb{F}^{m \times p}$, $r > 0$ *we have* $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}\left(P^{-1}Q\right)$ *and that there exists* $\alpha \in \mathcal{K}_\infty$ *such that*

$$\|f(\xi) - K\xi\| \leq r\,\|\xi\| - \alpha(\|\xi\|) \qquad \forall\, \xi \in \mathbb{F}^p. \tag{12.1.8}$$

Then the input-output Lur'e system (P, Q, f) *is input-to-output stable.*

Proof. Set $n := \deg\det P$; by Theorem 11.4.8, there exists a controllable and observable realization $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ of (P, Q). Also, for each $(u, y) \in \mathcal{B}(P, Q)$, there exists a unique $x \in (\mathbb{F}^n)^{\mathbb{N}_0}$ such that $(u, x, y) \in \mathcal{B}(A, B, C, D)$. Moreover, there exists a linear map $\phi\colon (\mathbb{F}^m)^k \times (\mathbb{F}^p)^k \to \mathbb{F}^n$ such that $x(0) = \phi(u^k, y^k)$.

Now as $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}((P^{-1}Q)(z)) = \mathbb{S}_{\mathbb{C}}(C(zI - A)^{-1}B + D)$, we can apply Proposition 5.3.5 to see that the Lur'e system (A, B, C, D, f) is input-to-state stable. Hence there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for each $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$ with $d \in l^\infty(\mathbb{F}^m)$ we have

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|d\|_\infty) \qquad \forall\, t \in \mathbb{N}_0.$$

By definition, $(d, x, y) \in \mathcal{B}_{\mathrm{d}}(A, B, C, D, f)$ if, and only if, $(f \circ y + d, x, y) \in \mathcal{B}(A, B, C, D)$. Thus we obtain

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|d\|_\infty) \tag{12.1.9}$$

for all $t \in \mathbb{N}_0$ and for all $(f \circ y + d, x, y) \in \mathcal{B}(A, B, C, D)$ with $d \in l^\infty(\mathbb{F}^m)$.

Now recall that if $(d, y) \in \mathcal{B}_{\mathrm{d}}(P, Q, f)$, then $(f \circ y + d, y) \in \mathcal{B}(P, Q)$, so that, there exists x such that $(f \circ y + d, x, y) \in \mathcal{B}(A, B, C, D)$ and $x(0) = \phi(f^k + d^k, y^k)$, where we have set $f^k := (f(y(0)), f(y(1)), \ldots, f(y(k-1))) \in (\mathbb{F}^p)^k$. Hence

$$\|x(t)\| \leq \beta\left(\left\|\phi(f^k + d^k, y^k)\right\|, t\right) + \gamma(\|d\|_\infty) \tag{12.1.10}$$

for all $t \in \mathbb{N}_0$ and for all $(d, y) \in \mathcal{B}_{\mathrm{d}}(P, Q, f)$, where x is such that $(f \circ y + d, x, y) \in \mathcal{B}(A, B, C, D)$. The use of (12.1.8) shows us that $\|f(\xi)\| \leq (\|K\| + r)\,\|\xi\|$ for all $\xi \in \mathbb{F}^p$, so that $\left\|f^k\right\| \leq (\|K\| + r)\left\|y^k\right\|$. Therefore, by the linearity of ϕ, there exists $c_1 > 0$ such that $\left\|\phi(f^k + d^k, y^k)\right\| \leq c_1(\left\|y^k\right\| + \left\|d^k\right\|)$. Since, for a fixed $t \in \mathbb{N}_0$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$ function, we can use (12.1.10) to see that

$$\begin{aligned}
\|x(t)\| &\leq \beta\left(\left\|\phi(f^k + d^k, y^k)\right\|, t\right) + \gamma(\|d\|_\infty) \\
&\leq \beta\left(c_1\left(\left\|y^k\right\| + \left\|d^k\right\|\right), t\right) + \gamma(\|d\|_\infty) \\
&\leq \beta\left(2c_1\left\|y^k\right\|, t\right) + \beta\left(2c_1\left\|d^k\right\|, t\right) + \gamma(\|d\|_\infty) \\
&\leq \beta\left(2c_1\left\|y^k\right\|, t\right) + \beta\left(2c_1 k\,\|d\|_\infty, 0\right) + \gamma(\|d\|_\infty)
\end{aligned} \tag{12.1.11}$$

for all $t \in \mathbb{N}_0$ and for all $(d, y) \in \mathcal{B}_{\mathrm{d}}(P, Q, f)$, where x is such that $(f \circ y + d, x, y) \in \mathcal{B}(A, B, C, D)$. Since $(s, t) \mapsto \beta(2c_1 s, t)$ is a $\mathcal{KL}$ function and $s \mapsto \beta(2c_1 ks, 0) + \gamma(s)$ is a $\mathcal{K}_\infty$ function, the proof of Theorem 12.1.13 will be complete if we can show that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\|y(t)\| \leq \alpha_1(\|x(t)\|) + \alpha_2(\|d\|_\infty)$ for all $(f \circ y + d, x, y) \in \mathcal{B}(A, B, C, D)$.

To prove this, we consider two cases. If $D = 0$, then

$$\|y(t)\| \leq \|C\| \, \|x(t)\| \qquad \forall\, t \in \mathbb{N}_0,$$

for all $(f \circ y + d, x, y) \in \mathcal{B}(A, B, C, D)$ and the desired conclusion follows. If however $D \neq 0$, then, by Lemma 3.1.4, $(f \circ y - Ky + d, x, y) \in \mathcal{B}(A_K, B_K, C_K, D_K)$, where the matrices are given by (3.1.3). Moreover, since $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$, by Lemma 3.2.7, we have $\|D_K\| \leq \frac{1}{r}$. Therefore, taking the norms of the output equation for a trajectory $(f \circ y - Ky + d, x, y) \in \mathcal{B}(A_K, B_K, C_K, D_K)$ and using the assumption (12.1.8) shows us that there exists a positive c_2 such that

$$\begin{aligned} \|y(t)\| &\leq \|C_K\| \, \|x(t)\| + \|D_K\| \, \|f(y(t)) - Ky(t)\| + \|D_K\| \, \|d(t)\| \\ &\leq c_2 \|x(t)\| + \frac{1}{r}(r \|y(t)\| - \alpha(\|y(t)\|)) + c_2 \|d\|_\infty \qquad \forall\, t \in \mathbb{N}_0. \end{aligned}$$

Hence we can estimate

$$\begin{aligned} \|y(t)\| &\leq \alpha^{-1}(rc_2 \|x(t)\| + rc_2 \|d\|_\infty) \\ &\leq \alpha^{-1}(2rc_2 \|x(t)\|) + \alpha^{-1}(2rc_2 \|d\|_\infty) \end{aligned}$$

for all $t \in \mathbb{N}_0$ and for all $(f \circ y + d, x, y) \in \mathcal{B}(A, B, C, D)$. Since $\alpha^{-1} \in \mathcal{K}_\infty$, this completes the proof. $\square$

Before illustrating this result with an example, we will obtain a corollary, which resembles the small-gain theorem. For this, we pick $K = 0$ and note that the ball condition $\mathbb{B}_{\mathbb{C}}(0, r) \subseteq \mathbb{S}_{\mathbb{C}}\left(P^{-1}Q\right)$ is equivalent to the inequality $\left\|P^{-1}Q\right\|_{H^\infty} \leq \frac{1}{r}$.

Corollary 12.1.14. *Consider an input-output Lur'e system (P, Q, f) with $(P, Q) \in \mathrm{IO}(m, p; \mathbb{F})$. If there exists $\alpha \in \mathcal{K}_\infty$ such that*

$$\left\|P^{-1}Q\right\|_{H^\infty} \|f(\xi)\| \leq \|\xi\| - \alpha(\|\xi\|) \qquad \forall \mathbb{F}^p,$$

then the input-output Lur'e system (P, Q, f) is input-to-output stable.

We also note a result, which resembles the classical SISO circle criterion for input-output systems, see Theorem 10 in Chapter 5 from [14]. We omit its proof as it follows in the same way as Corollary 5.3.4.

Corollary 12.1.15. *Consider a SISO input-output Lur'e system* (p, q, f) *with* $(p, q) \in \mathrm{IO}(1, 1; \mathbb{R})$*. Let* $k_1 < k_2$*, assume that* $k_1 \neq \lim_{|z|\to\infty} \frac{q(z)}{p(z)}$ *and that* $\frac{1-k_2 g}{1-k_1 g}$ *is positive real.*

If there exists $\alpha \in \mathcal{K}_\infty$ *such that*

$$k_1\xi^2 + \xi\alpha(|\xi|) \leq f(\xi)\xi \leq k_2\xi^2 - \xi\alpha(|\xi|) \qquad \forall\, \xi \in \mathbb{R}, \tag{12.1.12}$$

then the input-output Lur'e system (P, Q, f) *is input-to-output stable.*

Recall that the sector condition (12.1.12) can be visualized, see Figure 5.1.

12.2 Image input-output Lur'e systems

Recall Lemma 11.5.3, which related behaviours of image input-output systems to those of input-output systems. In this brief section we will introduce image input-output Lur'e systems and illustrate how one can use Lemma 11.5.3 to obtain stability results for them.

Definition 12.2.1. Let $(S, T) \in \mathrm{IO}_{\mathrm{im}}(m, p; \mathbb{F})$, assume that S and T are right coprime and let $f : \mathbb{F}^p \to \mathbb{F}^m$ be some map. We say that (S, T, f) is an **image input-output Lur'e system**.

We define the **behaviour** $\mathcal{B}(S, T, f)$ **of the image input-output Lur'e system** (S, T, f) as

$$\mathcal{B}(S, T, f) := \Big\{(y, v) \in (\mathbb{F}^p)^{\mathbb{N}_0} \times (\mathbb{F}^m) : (f \circ y, v, y) \in \mathcal{B}(S, T)\Big\}.$$

Consider an image input-output Lur'e system (S, T, f), where $(S, T) \in \mathrm{IO}_{\mathrm{im}}(m, p; \mathbb{F})$, and set $S(z) = \sum_{j=0}^{k} S_j z^j$ and $T(z) = \sum_{j=0}^{k} T_j z^j$, where $k := \deg S$ and where we have set $T_j = 0$ for $\deg T < j \leq \deg S$ ($\deg T \leq \deg S$ as TS^{-1} is proper). If $(y, v) \in \mathcal{B}(S, T, f)$, then this simply means that

$$f(y(t)) = \sum_{j=0}^{k} S_j v(t + j) \qquad \text{and} \qquad y(t) = \sum_{j=0}^{k} T_j v(t + j) \qquad \forall\, t \in \mathbb{N}_0.$$

We will be concerned with the following notions of stability.

Definition 12.2.2. We will say that an image input-output Lur'e system (S, T, f) with $(S, T) \in \mathrm{IO}_{\mathrm{im}}(m, p; \mathbb{F})$ is:

(a) **globally stable**, if there exist $c > 0$ and $k \in \mathbb{N}_0$ such that

$$\begin{aligned} \|y(t)\| &\leq c \left\| y^k \right\| \\ \|v(t)\| &\leq c \left\| y^k \right\| \qquad \forall\, t \in \mathbb{N}_0,\ \forall\, y \in \mathcal{B}(S, T, f); \end{aligned}$$

(b) **globally asymptotically stable**, if it is globally stable and

$$\lim_{t\to\infty} y(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} v(t) = 0 \qquad \forall\, (y,v) \in \mathcal{B}(S,T,f);$$

(c) **globally exponentially stable**, if there exist $c > 0$, $k \in \mathbb{N}_0$ and $a \in (0,1)$ such that

$$\begin{aligned} \|y(t)\| &\le ca^t \left\| y^k \right\| \\ \|v(t)\| &\le ca^t \left\| y^k \right\| \qquad \forall\, t \in \mathbb{N}_0 \ \ \forall\, y \in \mathcal{B}(S,T,f). \end{aligned}$$

Note that for an input-output Lur'e system (P,Q,f) the related stability concepts only involved y^k, where $k = \deg P$, and it would be desirable to have a similar condition here. Inspection of the proof of Proposition 12.2.3 reveals that this would be possible if, for the given $(S,T) \in \mathrm{IO}_{\mathrm{im}}(m,p;\mathbb{F})$, we could construct $(P,Q) \in \mathrm{IO}(m,p;\mathbb{F})$ such that $P^{-1}Q = TS^{-1}$, P and Q are left coprime and $\deg P \le \deg S$. While this seems plausible, the author has not been able to find a reference.

Proposition 12.2.3. *Let (S,T,f) be an image input-output Lur'e system with $(S,T) \in \mathrm{IO}_{\mathrm{im}}(m,p;\mathbb{F})$, and define $D := \lim_{|z|\to\infty}(TS^{-1})(z)$. Assume that, for some $K \in \mathbb{F}^{m\times p}$, $r > 0$, we have $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}\left(TS^{-1}\right)$.*

(a) If $\left\|D^K\right\| < 1/r$ and

$$\|f(\xi) - K\xi\| \le r\,\|\xi\| \qquad \forall\, \xi \in \mathbb{F}^p, \tag{12.2.1}$$

then the image input-output Lur'e system (S,T,f) is globally stable.

(b) If $\left\|D^K\right\| < 1/r$, f is continuous and

$$\|f(\xi) - K\xi\| < r\,\|\xi\| \qquad \forall \xi \in \mathbb{F}^p \setminus \{0\}, \tag{12.2.2}$$

then the image input-output Lur'e system (S,T,f) is globally asymptotically stable.

(c) If there exists $\delta \in (0,r)$ such that

$$\|f(\xi) - K\xi\| \le (r-\delta)\,\|\xi\| \qquad \forall \xi \in \mathbb{F}^p, \tag{12.2.3}$$

then the image input-output Lur'e system (S,T,f) is globally exponentially stable.

Proof. We prove only (a) as (b) and (c) can be proved in an almost identical manner. By Lemma 11.3.20, there exist left coprime polynomial matrices $P \in \mathbb{F}[z]^{p\times p}$ and $Q \in \mathbb{F}[z]^{p\times m}$ such that $P^{-1}Q = TS^{-1}$. Now $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq$

$\mathbb{S}_{\mathbb{C}}(TS^{-1}) = \mathbb{S}_{\mathbb{C}}(P^{-1}Q)$, so we can apply Proposition 12.1.7 to the input-output Lur'e system (P, Q, f) to see that it is globally stable. Hence, for $k := \deg P$, there exists $c > 0$ such that $\|w(t)\| \leq c \left\|w^k\right\|$ for all $w \in \mathcal{B}(P, Q, f)$ or, equivalently, for all $(f \circ w, w) \in \mathcal{B}(P, Q)$. By Lemma 11.5.2, if $(y, v) \in \mathcal{B}(S, T, f)$, then $(f \circ y, y) \in \mathcal{B}(P, Q)$, so that

$$\|y(t)\| \leq c \left\|y^k\right\|,$$

which is one of the two sought bounds. To obtain the other, we use Lemma 11.5.3 to see that there exist polynomial matrices X and Y such that if $(y, v) \in \mathcal{B}(S, T, f)$, then $v = X(\mathcal{L})(f \circ y) + Y(\mathcal{L})y$. Since $\|f(y(t))\| \leq (\|K\| + r) \|y(t)\| \leq c(\|K\| + r) \left\|y^k\right\|$, we conclude that there exists $c_1 > 0$ such that

$$\|v(t)\| \leq c_1 \left\|y^k\right\|$$

for all $(y, v) \in \mathcal{B}(S, T, f)$, which completes the proof. □

One can easily see that the technique employed in the proof of Proposition 12.2.3 can be extended to other results on input-output Lur'e systems, including the ones on input-to-output stability. However, their proofs require no new ideas and this thesis has seen enough repetition as it is, so we will omit them.

Chapter 13

Notes, references and future work

13.1 Notes and references

The main result in Chapter 11 is Theorem 11.4.8, which consists of three statements (a) to (c). Statement (a) is a standard result from realization theory and (b) is a known result from the theory of behaviours, see e.g. Theorems 2 and 3 from Willems [58]. However the proof of this result is difficult to locate as [58] omits it. Instead the proof is said to be similar to that of two results available in the literature. The first is Theorem 5.1 from Willems [57], where continuous-time systems are considered, however the proof there is short and difficult to penetrate. The second is from the PhD thesis van der Schaft [54], however its text seems to be unavailable. We also note that statement (c) from Theorem 11.4.8 does not seem to be available in the literature and it is crucial for our stability results for input-output Lur'e systems. By relating input-output trajectories to state-space trajectories, Theorem 11.4.8 allows us to essentially use Lyapunov techniques in analysing input-output Lur'e systems.

Lemma 11.5.3, which relates behaviours of input-output and image input-output systems, seems to be somewhat novel as it is not, in its present form, mentioned in Willems' papers on discrete-time behaviours: Willems [58, 59, 60]. At the same time, it is well-known that image input-output behaviours are related to linear, time-invariant differential systems, which are in turn related to input-output systems, see §6.6 from Willems [47] or Willems [61].

In Chapter 12 we apply Theorem 11.4.8 and Lemma 11.5.3 to obtain a number of novel stability results for input-output Lur'e systems and image input-output systems. As mentioned in the introduction to Part III, input-output stability properties of related input-output systems are studied in Desoer and Vidyasagar [14] and Zames [64]. However, their results revolve around the small-gain theorem and norm approximations, and they typically establish input-to-output stability in the l^p sense $(1 \leq p \leq \infty)$.

13.2 Future work

In §12 we obtained stability results for input-output Lur'e systems (P, Q, f) and a crucial assumption was that P and Q are left-coprime - this allowed us to conclude in Theorem 11.4.8 that the realization (A, B, C, D) of (P, Q) is controllable and observable. Therefore we were able to apply Proposition 4.2.1 in the stability analysis of (P, Q, f). Since Proposition 4.2.1 only requires the state-space system to be stabilizable and detectable, it seems the assumption that P and Q are left-coprime could be relaxed. The observer-form realization that we use in Theorem 11.4.8 is always going to be observable (see p. 417 from Kailath [34]), so it seems that an appropriately relaxed condition on the left-coprimeness of P and Q could result in a realization (A, B, C, D) that is only stabilizable. Recall the Hautus tests for controllability (namely, rank $\begin{pmatrix} zI - A & B \end{pmatrix}$ is full for all $z \in \mathbb{C}$) and stabilizability (namely, rank $\begin{pmatrix} zI - A & B \end{pmatrix}$ is full for all $z \in \mathbb{E}$). If we compare this with an alternative characterization of left coprimeness (see e.g. Lemma 6.3.6 from [34]): rank $\begin{pmatrix} P(z) & Q(z) \end{pmatrix}$ is full for all $z \in \mathbb{C}$, then we arrive at the following guess.

Conjecture: the results from §12.1 hold true for input-output Lur'e systems (P, Q, f) even if we replace the assumption that P and Q are left coprime with the assumption that rank $\begin{pmatrix} P(z) & Q(z) \end{pmatrix}$ is full for all $z \in \mathbb{E}$.

The statement of Proposition 11.4.9 hints at another possible extension of results from §12.1. It asserts that an input-output system (P, Q), where P and Q are not left coprime, admits the following decomposition of its behaviour. If R is the greatest common left divisor of P and Q and if X and Y are polynomial matrices that satisfy the (left) Bezout identity

$$PX + QY = R,$$

then there exist left coprime polynomial matrices P_1, Q_1 such that $RP_1 = P, RQ_1 = Q$ and such that

$$\mathcal{B}(P, Q) = \mathcal{B}(P_1, Q_1) \oplus [-Y(\mathcal{L}), X(\mathcal{L})] \ker R(\mathcal{L}).$$

Since $P_1^{-1}Q_1 = P^{-1}Q$, it suggests that we could be able to obtain the results from §12.1 by making sure that the trajectories in $[-Y(\mathcal{L}), X(\mathcal{L})] \ker R(\mathcal{L})$

satisfy appropriate stability criteria. A characterization of $\ker R(\mathcal{L})$ is provided by Theorem 11.1.2 and it is clear that all trajectories $y \in \ker R(\mathcal{L})$ will be such that $y(t) \to 0$ as $t \to \infty$ if, and only if, the zeros of $\det R$ all lie in $\mathbb{D}$. One can check that this condition is equivalent to P and Q being such that rank $\begin{pmatrix} P(z) & Q(z) \end{pmatrix}$ is full for all $z \in \mathbb{E}$. Hence we arrive at the same guess as above.

Conjecture: the results from §12.1 hold true for input-output Lur'e systems (P, Q, f) even if we replace the assumption that P and Q are left coprime with the assumption that rank $\begin{pmatrix} P(z) & Q(z) \end{pmatrix}$ is full for all $z \in \mathbb{E}$.

Interestingly, the two lines of thought that lead to this conjecture suggest completely different ways of proving it. The first one would require us to prove that the relaxed condition on P and Q provides us with a realization of (P, Q) that is stabilizable and observable, so that results from Part I can still be used. The second would use the decomposition of $\mathcal{B}(P, Q)$ from Proposition 11.4.9, namely that $\mathcal{B}(P, Q) = \mathcal{B}(P_1, Q_1) \oplus [-Y(\mathcal{L}), X(\mathcal{L})] \ker R(\mathcal{L})$. Here, P_1 and Q_1 are left coprime and such that $P_1^{-1}Q_1 = P^{-1}Q$; hence stability results for input-output Lur'e systems can be applied to $\mathcal{B}(P_1, Q_1)$. Therefore, we would have to prove that $\ker R(\mathcal{L})$ consists of y such that $y(t) \to 0$ as $t \to \infty$.

This latter approach could also translate to image input-output systems. If we could obtain a counterpart of Proposition 11.4.9 for image input-output systems, then the behaviour of $(S, T) \in \mathrm{IO}_{\mathrm{im}}(m, p; \mathbb{F})$ would probably admit a decomposition $\mathcal{B}(S, T) = \mathcal{B}(S_1, T_1) \oplus \mathcal{N}$, where S_1 and T_1 are right coprime and $\mathcal{N}$ is a linear subspace of $(\mathbb{F}^m)^{\mathbb{N}_0} \times (\mathbb{F}^p)^{\mathbb{N}_0}$ such that $(u, y) \in \mathcal{N}$ satisfy $u(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$. This leads us to the following.

Conjecture: the results from §12.1 hold true for image input-output Lur'e systems (S, T, f) even if we replace the assumption that S and T are right coprime with the assumption that rank $\begin{pmatrix} S(z) \\ T(z) \end{pmatrix}$ is full for all $z \in \mathbb{E}$.

Finally, we would like to note that Part III rests on realization theorems (Theorem 11.4.8 and Lemma 11.5.3) combined with stability results from Part I. Therefore, it seems likely that, if one developed appropriate counterparts of the realization theorems for continuous-time setting, much of the material in Part III could be obtained for continuous-time counterparts of input-output Lur'e systems and image input-output Lur'e systems.

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Appendices

Appendix A

Results involving the positive real lemma

A.1 In discrete-time

We note a consequence of the positive real property: if $G \in \mathbb{F}(z)^{m\times m}$ is positive real, then it cannot have poles or zeros in $\mathbb{E}$. In particular a positive real function is holomorphic on $\mathbb{E}$. We first note a useful lemma.

Lemma A.1.1. *Let $M \in \mathbb{C}^{m\times m}$.*

Then $\langle M\eta, \eta\rangle = 0\ \forall\, \eta \in \mathbb{C}^m$ if and only if $M = 0_{m\times m}$.

Proof. "If" part is trivial. To prove "Only if", let $\nu_1, \nu_2, \ldots, \nu_m$ be the standard basis for $\mathbb{C}^m$ and pick any $j, k \in \{1, 2, \ldots, m\}$. Let $r \geq 0$, $\theta \in [0, 2\pi)$ and set $\xi = r\nu_j + e^{i\theta}\nu_k$. Then by our assumption

$$0 = \langle M(r\nu_j + e^{i\theta}\nu_k), r\nu_j + e^{i\theta}\nu_k\rangle = r^2 M_{jj} + re^{-i\theta}M_{kj} + re^{i\theta}M_{jk} + M_{kk}.$$

It is easy to see that this can only be true for all $r \geq 0$, $\theta \in [0, 2\pi)$ if $M_{jj} = M_{kj} = M_{jk} = M_{kk} = 0$, which in turn completes the proof. □

Lemma A.1.2. *Consider a positive real $G \in \mathbb{F}(z)^{m\times m}$.*

Then all entries of $G(z)$ are analytic in $\mathbb{E}$, moreover $G(z)$ has no poles or zeros in $\mathbb{E}$.

Proof. Since the entries of G are rational functions, they are holomorphic at every point in the complex plane which is not a pole of G. Every function, holomorphic in an open set, is analytic (see e.g. [48] §14.9), so every entry of G is analytic in $\mathbb{C}$ except for poles of G. Hence it suffices to show that

G has no poles in $\mathbb{E}$ (as a rational function clearly cannot have an essential singularity).

To this end consider an arbitrary $z_0 \in \mathbb{E}$. If we define the Laurent series of a matrix function entrywise, then on some punctured disc $\mathbb{D}'(z_0, a)$ the matrix rational function G can be written as $G(z) = \sum_{j=k}^{\infty} M_j (z-z_0)^j$ where $M_j \in \mathbb{C}^{m\times m}$ for $j \geq k$ and $M_k \neq 0_{m\times m}$. Here we have used that $\mathbb{F} \subseteq \mathbb{C}$ and that $k > -\infty$ as G is a rational function matrix and hence cannot have an essential singularity. Note that $k < 0$ means that G has a pole at z_0, whereas $k > 0$ means that G has a zero at z_0; we will now rule out both of these possibilities.

Define

$$\begin{aligned} h : \mathbb{C} \times \mathbb{C}^m &\to \mathbb{C} \\ (z, \xi) &\mapsto \langle G(z)\xi, \xi\rangle \end{aligned}$$

Then in a neighbourhood of z_0, $h(z, \xi) = \big\langle (z - z_0)^k M_k \xi, \xi \big\rangle + o((z - z_0)^k)$, where $o\colon \mathbb{C} \to \mathbb{C}$ is such that $\lim_{s\to 0} o(s)/s = 0$. Since the number of poles of G is finite, they are isolated; also $\mathbb{E}$ is open, so for $r > 0$ small enough the punctured disc $\mathbb{D}'(z_0, 2r)$ contains no poles of G, is contained in $\mathbb{E}$ and has the Laurent series expansion as above (that is, $2r < a$). Now parametrize the points on the boundary of the disc $\mathbb{D}(z_0, r)$ $z(r, \theta) := z_0 + re^{i\theta}$ for $\theta \in [0, 2\pi)$. If we evaluate h on this circle, then

$$r^{-k} h(z(r, \theta), \xi) = e^{ik\theta} \langle M_k \xi, \xi\rangle + \frac{o(r^k e^{ik\theta})}{r^k}. \tag{A.1.1}$$

Since $M_k \neq 0_{m\times m}$, in view of Lemma A.1.1, there exists $\xi_0 \in \mathbb{C}^m$ such that $\langle M_k \xi_0, \xi_0\rangle \neq 0$. Suppose now that $k \neq 0$. Set $z_1 := \langle M_k \xi_0, \xi_0\rangle$ and pick $\theta_0 \in [0, 2\pi)$ such that $e^{ik\theta_0} = -\frac{\overline{z_1}}{|z_1|}$ (note that this is possible precisely because $k \neq 0$). Substitute this back in (A.1.1) to obtain

$$r^{-k} h(z(r, \theta_0), \xi_0) = -|z_1| + \frac{o(r^k e^{ik\theta_0})}{r^k}.$$

Since $\lim_{r\to 0} o(r^k e^{ik\theta_0})/r^k = 0$, there exists $r_0 > 0$ such that $h(z(r_0, \theta_0), \xi_0) + \overline{h(z(r_0, \theta_0), \xi_0)} < 0$. This however leads to a contradiction with $G(z) + (G(z))^* \geq 0$, because

$$\begin{aligned} 0 &> h(z(r_0, \theta_0), \xi_0) + \overline{h(z(r_0, \theta_0), \xi_0)} \\ &= \langle [G(z(r_0, \theta_0)) + G(z(r_0, \theta_0))^*]\xi_0, \xi_0\rangle \geq 0. \end{aligned}$$

Hence we must have $k = 0$ and this completes the proof. $\square$

We now prove the quadratic form estimate obtained from the positive real lemma.

Lemma 2.2.9. *Consider a controllable and observable linear state-space system $(A, B, C, D) \in \Sigma(m, n, m, \mathbb{F})$ and assume that its transfer function G is positive real.*

Then there exists a positive definite $P = P^* \in \mathbb{F}^{n\times n}$ such that the positive-definite function defined by $V(\xi) := \langle P\xi, \xi\rangle$ satisfies

$$V(x(t+1)) \leq V(x(t)) + \frac{1}{2}\left[\|u(t)+y(t)\|^2 - \|u(t)-y(t)\|^2\right]$$

for all $t \in \mathbb{N}_0$ and for all trajectories $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Proof. By the positive real lemma, there exists a positive definite $P = P^* \in \mathbb{F}^{n\times n}$ and matrices L and W such that:

$$A^*PA - P = -L^*L \tag{A.1.2a}$$

$$A^*PB - C^* = -L^*W \tag{A.1.2b}$$

$$D + D^* - B^*PB = W^*W. \tag{A.1.2c}$$

Let us now perform Lyapunov analysis with the positive definite function $V(\xi) := \langle P\xi, \xi\rangle$. The use of difference equations for the state and output as well as multiple uses of the equations (A.1.2) yield:

$$\begin{aligned}
V(x(t+1)) - V(x(t)) = \\
&= \langle P(Ax(t)+Bu(t)), Ax(t)+Bu(t)\rangle \\
&\quad - \langle Px(t), x(t)\rangle \\
&= \langle (A^*PA - P)x(t), x(t)\rangle + \langle A^*PBu(t), x(t)\rangle \\
&\quad + \langle x(t), A^*PBu(t)\rangle + \langle B^*PBu(t), u(t)\rangle \\
&= -\langle L^*Lx(t), x(t)\rangle + \langle (C^* - L^*W)u(t), x(t)\rangle \\
&\quad + \langle x(t), (C^* - L^*W)u(t)\rangle \\
&\quad + \langle (D + D^* - W^*W)u(t), u(t)\rangle \\
&= -\|Lx(t) + Wu(t)\|^2 + \langle u(t), Cx(t)\rangle \\
&\quad + \langle Cx(t), u(t)\rangle + \langle (D + D^*)u(t), u(t)\rangle \\
&= -\|Lx(t) + Wu(t)\|^2 + \langle y(t), u(t)\rangle \\
&\quad + \langle u(t), y(t)\rangle \\
&\leq \langle y(t), u(t)\rangle + \langle u(t), y(t)\rangle \\
&= \frac{1}{2}\left[\|u(t)+y(t)\|^2 - \|u(t)-y(t)\|^2\right]
\end{aligned}$$

for all $t \in \mathbb{N}_0$ and for all trajectories $(u, x, y) \in \mathcal{B}(A, B, C, D)$. □

Lemma 3.2.13. *Consider a controllable and observable state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{R})$, denote its transfer function by G and assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.*

Then there exists a positive-definite $P = P^* \in \mathbb{F}^{n\times n}$ *such that the function* $V : \mathbb{F}^n \to [0,\infty)$ *defined by* $V(\xi) := \langle P\xi, \xi\rangle$ *satisfies*

$$V(x(t+1)) - V(x(t)) \leq \|u(t) - Ky(t)\|^2 - r^2\,\|y(t)\|^2$$

for all $t \in \mathbb{N}_0$ *and for all trajectories* $(u,x,y) \in \mathcal{B}(A,B,C,D)$*.*

We will use Proposition 3.2.12 to relate the assumption $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ to the positive realness of $I + 2\lambda G^{\lambda I+K}$, which will then allow us to use the positive real lemma to define the sought quadratic form. For this, we first need a preliminary result about the trajectories of the state-space system that has $I + 2\lambda G^{\lambda I+K}$ as its transfer function.

Lemma A.1.3. *Consider a linear system* $(A,B,C,D) \in \Sigma(m,n,m;\mathbb{F})$ *and denote its transfer function by* G*. Let* $K, \lambda I + K \in \mathbb{A}_{\mathbb{C}}(D)$*,* $\lambda \in \mathbb{C}\setminus\{0\}$ *and, for* $N := \lambda I + K$*, define* (A_N, B_N, C_N, D_N) *as in equation* (3.1.3)*. Then*

(a) the transfer function of $(A_N, B_N, 2\lambda C_N, I + 2\lambda D_N)$ *is* $I + 2\lambda G^N$*,*

(b) if $(u,x,y) \in \mathcal{B}(A,B,C,D)$*, then* $(u - \lambda y - Ky, x, u + \lambda y - Ky) \in \mathcal{B}(A_N, B_N, 2\lambda C_N, I + 2\lambda D_N)$*,*

(c) if (A,B,C,D) *is controllable and observable, then* $(A_N, B_N, 2\lambda C_N, I + 2\lambda D_N)$ *is controllable and observable.*

Proof. By Lemma 3.1.4 (b), we know that the transfer function of the linear system (A_N, B_N, C_N, D_N) is G^N, that is $C_N(sI - A_N)^{-1}B_N + D_N = G^N$. Hence we obtain $2\lambda C_N(sI - A_N)^{-1}B_N + 2\lambda D_N + I = 2\lambda G^N + I$, which is exactly what we are after in (a).

By Lemma 3.1.4 (a), we know that if $(u,x,y) \in \mathcal{B}(A,B,C,D)$, then $(u - \lambda y - Ky, x, y) \in \mathcal{B}(A_N, B_N, C_N, D_N)$, which in turn means that

$$\begin{aligned} x(t+1) =& A_N x(t) + B_N[u(t) - \lambda y(t) - Ky(t)] \\ y(t) =& C_N x(t) + D_N[u(t) - \lambda y(t) - Ky(t)] \qquad \forall\, t \in \mathbb{N}_0. \end{aligned}$$

Multiply the output equation by 2λ and then add $u(t) - \lambda y(t) - Ky(t)$ to obtain

$$\begin{aligned} &u(t) + \lambda y(t) - Ky(t) \\ &\qquad = 2\lambda C_N x(t) + (I + 2\lambda D_N)(u(t) - \lambda y(t) - Ky(t)) \qquad \forall\, t \in \mathbb{N}_0, \end{aligned}$$

which in turn shows that $(u - \lambda y - Ky, x, u + \lambda y - Ky) \in \mathcal{B}(A_N, B_N, 2\lambda C_N, I + 2\lambda D_N)$ as required.

Finally, the last claim is a straightforward verification, which uses the Hautus test for controllability and observability alongside the fact that $\lambda \neq 0$. $\square$

Proof of Lemma 3.2.13. By Proposition 3.2.12, we know that there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = r$, $\lambda I + K \in \mathbb{A}_{\mathbb{C}}(D)$ and $I + 2\lambda G^{\lambda I + K}$ is positive real. Now by Lemma A.1.3, we know that $(A_N, B_N, 2\lambda C_N, I + 2\lambda D_N)$ is controllable and observable, and its transfer function is $I + 2\lambda G^{\lambda I+K}$. Hence Lemma 2.2.8 guarantees the existence of a positive-definite $P = P^* \in \mathbb{F}^{n\times n}$ such that the positive definite function $V\colon \mathbb{F}^n \to [0,\infty)$ defined by $V(\xi) := \langle P\xi, \xi\rangle$ satisfies

$$V(x(t+1)) \leq V(x(t)) + \frac{1}{2}\left[\|u(t)+y(t)\|^2 - \|u(t)-y(t)\|^2\right] \qquad \forall\, t \in \mathbb{N}_0$$

and for all $(u,x,y) \in \mathcal{B}(A_N, B_N, 2\lambda C_N, I + 2\lambda D_N)$. On the other hand, by Lemma A.1.3, we know that if $(u,x,y) \in \mathcal{B}(A,B,C,D)$, then $(u - \lambda y - Ky, x, u + \lambda y - Ky) \in \mathcal{B}(A_N, B_N, 2\lambda C_N, I + 2\lambda D_N)$. Thus we have

$$\begin{aligned} V(x(t+1)) - V(x(t)) &\leq \frac{1}{2}\left[\|2u(t) - 2Ky(t)\|^2 - \|-2\lambda y(t)\|^2\right] \\ &= 2\left[\|u(t) - Ky(t)\|^2 - r^2\|y(t)\|^2\right] \qquad \forall\, t\in\mathbb{N}_0 \end{aligned}$$

and for all $(u,x,y) \in \mathcal{B}(A,B,C,D)$. Thus $\frac{V}{2}$ has the required properties. $\square$

Recall the shorthand

$$\mathbb{M} := \{K \in \mathbb{C}^{m\times m} : K + K^* \text{ is negative definite}\}.$$

Lemma 3.2.19. *Consider $G \in \mathbb{F}(z)^{m\times m}$.*

$\mathbb{M} \subset \mathbb{S}_{\mathbb{C}}(G)$ if, and only if, G is positive-real.

Proof. First, we will show that $\mathbb{M} = \bigcup_{r>0} \mathbb{B}_{\mathbb{C}}(-rI, r)$.

$\underline{\subseteq}$: Consider $M \in \mathbb{M}$. Then there exists $c > 0$ such that $\langle (M+M^*)\xi, \xi\rangle \leq -c\|\xi\|^2$ for all $\xi \in \mathbb{F}^m$. Hence for all $r > 0$ and $\xi \in \mathbb{F}^m$ we have

$$\begin{aligned} \|(M + rI)\xi\|^2 - r^2\|\xi\|^2 &= \|M\xi\|^2 + r\langle (M+M^*)\xi, \xi\rangle \\ &\leq \left(\|M\|^2 - rc\right)\|\xi\|^2. \end{aligned}$$

Thus if we pick $r > \frac{\|M\|^2}{c}$, then $\|(M+rI)\xi\| < r\|\xi\|$ for all $\xi \in \mathbb{F}^m$, so that $\|M + rI\| < r$ and consequently $M \in \mathbb{B}_{\mathbb{C}}(-rI, r)$.

$\underline{\supseteq}$: Let $r > 0$ and consider $M \in \mathbb{B}_{\mathbb{C}}(-rI, r)$. Then $\|rI + M\| < r$ and hence, by the Cauchy-Schwarz inequality and the properties of the operator norm, we have

$$\begin{aligned} \langle (M+M^*)\xi, \xi\rangle &= \langle [(M+rI) + (M+rI)^*]\xi, \xi\rangle - 2r\|\xi\|^2 \\ &\leq 2\|M + rI\|\,\|\xi\|\,\|\xi\| - 2r\|\xi\|^2 \\ &< \|\xi\|^2(2r - 2r) = 0 \end{aligned}$$

for all $\xi \in \mathbb{F}^m \setminus \{0\}$. Hence $M \in \mathbb{M}$, which shows that $\mathbb{B}_{\mathbb{C}}(-rI, r) \subseteq \mathbb{M}$.

Secondly, we note that G is positive real if, and only if, $rG + I$ is positive real for all $r > 0$. Indeed, G being positive real clearly implies that $rG + I$ is positive real for any $r > 0$, so we only need to prove the other implication. To this end, suppose on the contrary, that $rG + I$ is positive real for all $r > 0$, but G is not positive real. Then there exists $z \in \mathbb{E}$ and $\xi \in \mathbb{F}^m \setminus \{0\}$ such that $\langle (G(z) + (G(z))^*)\xi, \xi \rangle < 0$ and set $c := \langle (G(z) + (G(z))^*)\xi, \xi \rangle$. Now pick r big enough so that $rc < -2 \|\xi\|^2$ to obtain $\langle (rG(z) + I + (rG(z) + I)^*)\xi, \xi \rangle = cr + 2 \|\xi\|^2 < 0$, which contradicts the positive realness of $rG + I$ and hence completes the proof of the claim.

With the above two equivalences in mind we can see that it now suffices to show that $\mathbb{B}_{\mathbb{C}}(-rI, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ if, and only if, $2rG + I$ is positive real. By Proposition 3.2.12, it is in turn sufficient to show that $2rG + I$ is positive real if, and only if, $\left\|G^{-rI}\right\|_{H^\infty} \leq \frac{1}{r}$ for all $r > 0$. We note that if $2rG + I$ is positive real, then $-1 \notin \sigma(2rG(z) + I)$ and hence, by Lemma 2.2.10, $2rG + I$ is positive real if, and only if,

$$\begin{aligned} 1 &\geq \left\|(I - (I + 2rG))(I + (I + 2rG))^{-1}\right\|_{H^\infty} \\ &= r \left\|G^{-rI}\right\|_{H^\infty}. \end{aligned}$$

□

Lemma 3.2.18. *Let $G \in \mathbb{F}(z)^{p \times m}$; then $\mathbb{S}_{\mathbb{C}}(G)$ is an open set.*

Proof. Let $K \in \mathbb{S}_{\mathbb{C}}(G)$, then $G^K \in H^\infty$. If we set $r := \left\|G^K\right\|_{H^\infty}^{-1}$, then Lemma 3.2.7 tells us that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$, which completes the proof. □

A.2 In continuous-time

Lemma 7.3.5. *Consider a controllable and observable linear state-space system $(A, B, C, D) \in \Sigma(m, n, m; \mathbb{F})$ and denote by G its transfer function.*

If G is positive real, then there exists a positive definite $P = P^* \in \mathbb{F}^{n \times n}$ such that the quadratic form $V : \mathbb{F}^n \to [0, \infty)$ defined by $V(\xi) := \langle P\xi, \xi \rangle$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq \frac{1}{2} \left[\|u(t) + y(t)\|^2 - \|u(t) - y(t)\|^2 \right] \qquad \text{a.e.}$$

for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

Proof. By Lemma 7.3.4, we know that there exist matrices L, W and a positive definite matrix $P^* = P > 0$ such that:

$$A^*P + PA = -L^*L, \tag{A.2.1a}$$
$$PB - C^* = -L^*W, \tag{A.2.1b}$$
$$D + D^* = W^*W. \tag{A.2.1c}$$

Consider the positive definite quadratic form $V(\xi) := \langle P\xi, \xi\rangle$ and pick an arbitrary trajectory $(u, x, y) \in \mathcal{B}(A, B, C, D)$. By Corollary 7.2.6, V is continuously differentiable, $V \circ x$ is absolutely continuous and $\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \operatorname{Re}\langle 2Px(t), \dot{x}(t)\rangle$ almost everywhere. We then use the positive real equations (A.2.1a) - (A.2.1c) and the technique of completing the square to obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) &= \operatorname{Re}\langle 2Px(t), Ax(t) + Bu(t)\rangle \\
&= \langle [A^*P + PA]x(t), x(t)\rangle + 2\langle x(t), PBu(t)\rangle \\
&= -\|Lx(t)\|^2 + \langle x(t), (C^* - L^*W)u(t)\rangle \\
&\quad + \langle (C^* - L^*W)u(t), x(t)\rangle \\
&= -\|Lx(t)\|^2 + \langle Cx(t), u(t)\rangle - \langle Lx(t), Wu(t)\rangle \\
&\quad + \langle u(t), Cx(t)\rangle - \langle Wu(t), Lx(t)\rangle \\
&= -\|Lx(t) + Wu(t)\|^2 \\
&\quad + \langle Cx(t) + Du(t), u(t)\rangle + \langle u(t), Cx(t) + Du(t)\rangle \\
&\le \langle y(t), u(t)\rangle + \langle u(t), y(t)\rangle \\
&= \frac{1}{2}\left[\|u(t) + y(t)\|^2 - \|u(t) - y(t)\|^2\right] \qquad \text{a.e.} \qquad \square
\end{aligned}$$

Lemma 7.6.15. *Consider a controllable and observable linear state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$, denote its transfer function by G and let $r > 0$, $K \in \mathbb{F}^{m\times p}$. Furthermore assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.*

Then there exists a positive definite $P = P^* \in \mathbb{F}^{n\times n}$ such that the function $V\colon \mathbb{F}^n \to [0, \infty)$ defined by $V(\xi) = \langle P\xi, \xi\rangle$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \le \|u(t) - Ky(t)\|^2 - r^2\|y(t)\|^2 \qquad \text{a.e.}$$

for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$.

We will use Proposition 7.6.14 to relate the assumption $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ to the positive realness of $I + 2\lambda G^{\lambda I+K}$, which will then allow us to use the positive real lemma to define the sought quadratic form. For this, we first need a preliminary result about the trajectories of the state-space system that has $I + 2\lambda G^{\lambda I+K}$ as its transfer function.

Lemma A.2.1. *Consider $(A, B, C, D) \in \Sigma(m, n, m; \mathbb{F})$, let $K \in \mathbb{A}_{\mathbb{C}}(D)$, $\lambda \in \mathbb{C} \setminus \{0\}$ and for $N := \lambda I + K$ define (A_N, B_N, C_N, D_N) as in equation (7.6.2).*

(a) The transfer function of $(A_N, B_N, 2\lambda C_N, I + 2\lambda D_N)$ is $I + 2\lambda G^{\lambda I+K}$.

(b) If $(u, x, y) \in \mathcal{B}(A, B, C, D)$, then $(u - \lambda y - Ky, x, u + \lambda y + Ky) \in \mathcal{B}(A_N, B_N, 2\lambda C_N, I + 2\lambda D_N)$.

(c) *If* (A, B, C, D) *is controllable and observable, then* $(A_N, B_N, 2\lambda C_N, I + 2\lambda D_N)$ *is controllable and observable.*

Proof. By Lemma 7.6.5 (b), we know that the transfer function of the linear system (A_N, B_N, C_N, D_N) is G^N, that is $C_N(sI - A_N)^{-1}B_N + D_N = G^N$. Hence we obtain $2\lambda C_N(sI - A_N)^{-1}B_N + 2\lambda D_N + I = 2\lambda G^N + I$, which is exactly what we are after in (a).

By Lemma 7.6.5 (a), we know that if $(u, x, y) \in \mathcal{B}(A, B, C, D)$, then $(u - \lambda y - Ky, x, y) \in \mathcal{B}(A_N, B_N, C_N, D_N)$, which in turn means that

$$\begin{aligned} \dot{x}(t) =& A_N x(t) + B_N(u(t) - \lambda y(t) - Ky(t)) \\ y(t) =& C_N x(t) + D_N(u(t) - \lambda y(t) - Ky(t)) \qquad \text{a.e.} \end{aligned}$$

Multiply the output equation by 2λ and then add $u(t) - \lambda y(t) - Ky(t)$ to obtain

$$\begin{aligned} &u(t) + \lambda y(t) - Ky(t) \\ &\qquad = 2\lambda C_N x(t) + (I + 2\lambda D_N)(u(t) - \lambda y(t) - Ky(t)) \qquad \text{a.e.} \end{aligned}$$

which in turn shows us that the trajectory $(u - \lambda y - Ky, x, u + \lambda y - Ky) \in \mathcal{B}(A_N, B_N, 2\lambda C_N, I + 2\lambda D_N)$ as required.

Finally, the last claim is a straightforward verification, which uses the Hautus test for controllability and observability alongside the fact that $\lambda \neq 0$. $\square$

Proof of Lemma 7.6.15. By Proposition 7.6.14, we know that there exists there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $|\lambda| = r$, $\lambda I + K \in \mathbb{A}_{\mathbb{C}}(D)$ and $I + 2\lambda G^{\lambda I + K}$ is positive real. By Lemma A.2.1 (c), we know that - for $N := \lambda I + K$ and (A_N, B_N, C_N, D_N) defined as in equation (7.6.2) - the state-space system $(A_N, B_N, 2\lambda C_N, I + 2\lambda D_N)$ is controllable and observable. Hence we can apply Lemma 7.3.5 to see that there exists a positive definite matrix $P = P^* \in \mathbb{F}^{n \times n}$ such that the function $V \colon \mathbb{F}^n \to [0, \infty)$ defined by $V(\xi) := \langle P\xi, \xi \rangle$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq \frac{1}{2} \left[\|u_1(t) + y_1(t)\|^2 - \|u_1(t) - y_1(t)\|^2 \right] \qquad \text{a.e.} \tag{A.2.2}$$

for all $(u_1, x_1, y_1) \in \mathcal{B}(A_N, B_N, 2\lambda C_N, I + 2\lambda D_N)$.

Now, by Lemma A.2.1 (a), we know that if $(u, x, y) \in \mathcal{B}(A, B, C, D)$, then $(u - \lambda y - Ky, x, u + \lambda y - Ky) \in \mathcal{B}(A_N, B_N, 2\lambda C_N, I + 2\lambda D_N)$ and hence the estimate (A.2.2) must hold, so that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq& \frac{1}{2} \left[\|2u(t) - 2Ky(t)\|^2 - \|2\lambda y(t)\|^2 \right] \\ =& 2 \left[\|u(t) - Ky(t)\|^2 - r^2 \|y(t)\|^2 \right] \qquad \text{a.e.} \end{aligned}$$

for all $(u, x, y) \in \mathcal{B}(A, B, C, D)$. Thus $\frac{1}{2}V$ has the required properties. $\square$

Appendix B

An alternative proof of Lemma 5.1.11

Lemma 5.1.11. $\alpha \in \mathcal{K}_\infty$ *satisfies the growth condition* (GC) *if, and only if, for each* $\varepsilon > 0$ *there exists* $\eta \in \mathcal{K}_\infty$ *such that*

$$\alpha(x-y) \leq \alpha\left((1+\varepsilon)x\right) - \eta(y) \qquad \forall\, x \geq y \geq 0.$$

Remark: we emphasize that η depends on ε.

Proof. "$\Longleftarrow$" part of the proof is identical to the one presented in the main body of the thesis, so we omit it.

$\underline{\Longrightarrow:}$

Let us set $\tilde{\eta}(y) := \inf_{x\in[y,\infty)}\{\alpha((1+\varepsilon)x) - \alpha(x-y)\}$; it then satisfies the required inequality and in the first proof of Lemma 5.1.11 we further showed that $\tilde{\eta}$ is continuous. The present proof does not use the continuity of $\tilde{\eta}$; instead we construct $\eta \in \mathcal{K}_\infty$ that is closely related to $\tilde{\eta}$ and that has the sought properties. Let us first note some properties of $\tilde{\eta}$.

Let us first show that $\tilde{\eta}(0) = 0$. Since α is a $\mathcal{K}_\infty$ function, we have $\alpha((1+\varepsilon)x) - \alpha(x) \geq 0$ for all $x, \varepsilon > 0$. Hence $\tilde{\eta}(0) \geq 0$. On the other hand $\alpha((1+\varepsilon)0) - \alpha(0) = 0$, so that $\tilde{\eta}(0) \leq 0$.

Now, we will show that $\lim_{y\to\infty}\tilde{\eta}(y) = \infty$. This follows from

$$\begin{aligned}\lim_{y\to\infty}\tilde{\eta}(y) &= \lim_{y\to\infty}\inf_{x\in[y,\infty)}\{\alpha((1+\varepsilon)x) - \alpha(x-y)\} \\ &\geq \lim_{y\to\infty}\inf_{x\in[y,\infty)}\{\alpha((1+\varepsilon)x) - \alpha(x)\} \\ &= \liminf_{x\to\infty}\{\alpha((1+\varepsilon)x) - \alpha(x)\}.\end{aligned}$$

However α satisfies the growth condition (GC), so that $\lim_{x\to\infty}\{\alpha((1+\varepsilon)x)-\alpha(x)\}=\infty$. It is a well known fact that if the limit exists, so does the limit inferior and moreover it is equal to the limit. Hence $\lim_{y\to\infty}\tilde{\eta}(y)=\infty$.

Let us now show that $\tilde{\eta}$ is strictly increasing. Since α satisfies the growth condition (GC), for each $y\geq 0$ there exists $k(y)\geq 0$ such that $\inf_{x\in[y,\infty)}\{\alpha((1+\varepsilon)x)-\alpha(x-y)\}=\inf_{x\in[y,k(y)]}\{\alpha((1+\varepsilon)x)-\alpha(x-y)\}$. Let us now fix $y\geq 0,\delta>0$ and set $a:=k(y+\delta)\vee k(y)$. Then

$$
\begin{aligned}
\tilde{\eta}(y+\delta) =: &\inf_{x\in[y+\delta,\infty)}\{\alpha((1+\varepsilon)x)-\alpha(x-y-\delta)\}\\
= &\inf_{x\in[y+\delta,a]}\{\alpha((1+\varepsilon)x)-\alpha(x-y)+\alpha(x-y)-\alpha(x-y-\delta)\}\\
\geq &\inf_{x\in[y+\delta,a]}\{\alpha((1+\varepsilon)x)-\alpha(x-y)\}\\
&+\inf_{x\in[y+\delta,a]}\{\alpha(x-y)-\alpha(x-y-\delta)\}\\
&[\text{as } \alpha \text{ is continuous, the infimum on } [y+\delta,a] \text{ is a minimum}]\\
= &\inf_{x\in[y+\delta,a]}\{\alpha((1+\varepsilon)x)-\alpha(x-y)\}\\
&+\min_{x\in[y+\delta,a]}\{\alpha(x-y)-\alpha(x-y-\delta)\}\\
\geq &\inf_{x\in[y,\infty)}\{\alpha((1+\varepsilon)x)-\alpha(x-y)\}\\
&+\min_{x\in[y+\delta,a]}\{\alpha(x-y)-\alpha(x-y-\delta)\}\\
&[\text{as } \alpha \text{ is continuous and strictly increasing}]\\
> &\tilde{\eta}(y)+0.
\end{aligned}
$$

As y and δ were arbitrary, $\tilde{\eta}$ is strictly increasing.

Now notice that $\alpha(x-y)\leq\alpha((1+\varepsilon)x)-\tilde{\eta}(y)$ for all $x\geq y\geq 0$. Hence it suffices to construct $\eta\in\mathcal{K}_\infty$ such that $\eta\leq\tilde{\eta}$.

What follows is inspired by [33], §16 A, B, where a decomposition of an increasing function is obtained: every (not necessarily strictly) increasing function can be written as a sum of a continuous function and a jump function which contains all its discontinuities. Unfortunately we want to work over the non-open interval $[0,\infty)$ and also find a continuous function that is smaller that $\tilde{\eta}$, but still goes to infinity as its argument goes to infinity, hence we cannot use the results from [33] directly. Since the concepts we will use here are only used in the present proof, we will - contrary to the usual practice - introduce some definitions mid-proof.

We say that $\tilde{\sigma}\colon[0,\infty)\to\mathbb{R}$ is an *elementary increasing jump function* if there exist real numbers $a\leq b\leq c$ with $a<b$ or $b<c$ and a discontinuity

point t such that for $x \in [0,\infty)$ we can write

$$\tilde{\sigma}(x) = \begin{cases} a & \text{if } x < t, \\ b & \text{if } x = t, \\ c & \text{if } x > t. \end{cases}$$

Note that this definition allows us to describe a discontinuity at 0. We further say that $\tilde{s}\colon [0,\infty) \to \mathbb{R}$ is an *increasing jump function* if there exist elementary increasing jump functions $\{\tilde{\sigma}_k\}_{k\in\mathbb{N}_0}$ such that

$$\tilde{s}(x) = \sum_{k\in\mathbb{N}_0} \tilde{\sigma}_k(x) \qquad \forall\, x \in [0,\infty).$$

Note that for a strictly increasing $f\colon [0,\infty) \to \mathbb{R}$ the following concepts are well-defined:

$$\begin{aligned} f(x+) &:= \lim_{y\to x, y>x} f(y), && \text{for } x \in [0,\infty) \\ f(x-) &:= \lim_{y\to x, y<x} f(y), && \text{for } x \in (0,\infty), \\ f(0-) &:= f(0). \end{aligned}$$

Moreover, it is easy to see that $f(x-) \le f(x) \le f(x+)$, with equalities if and only if f is continuous at x.

We will not prove the following claim as it is well-known, see e.g. [33].

Claim 1. *Let $f\colon [0,\infty) \to \mathbb{R}$ be increasing. Then f is discontinuous at a countable set of points.*

Claim 2. *Let $f\colon [0,\infty) \to \mathbb{R}$ be strictly increasing and assume that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$. Then there exists $g \in \mathcal{K}_\infty$ such that $g \le f$.*

Proof. We will first obtain a representation of an increasing function as a sum of a continuous function and a jump function, then we will modify the constructed functions to obtain a continuous function with the required properties. This decomposition is inspired by a similar approach in [33] §16 B p.525 - 526.

Let $\{t_i\}_{i\in\mathbb{N}_0}$ be the countable set of discontinuities of f (we know that it is countable by Claim 1) and set

$$\sigma_i(x) := \begin{cases} 0, & \text{for } x < t_i, \\ f(t_i) - f(t_i-) & \text{for } x = t_i, \\ f(t_i+) - f(t_i-) & \text{for } x > t_i. \end{cases}$$

Note that $f - \sigma_i$ is continuous at t_i and, by considering $(f - \sigma_i)|_{[0,t_i]}$ and $(f - \sigma_i)|_{[t_i,\infty)}$, we can easily see that $f - \sigma_i$ is strictly increasing. Hence we can show inductively that $f_n := f - \sum_{i=0}^{n} \sigma_i$ is strictly increasing and continuous at $t_0, t_1, \ldots, t_n$. Since $f_n(0) = 0$, this implies that $f_n(x) \geq 0$ for all $x \geq 0$ and hence $\sum_{i=0}^{n} \sigma_i(x) \leq f(x)$ for all $x \geq 0$ and for all $t \in \mathbb{N}_0$. As $\sigma_i \geq 0$, the following map

$$\begin{aligned} s \colon [0, \infty) &\to [0, \infty) \\ x &\mapsto \sum_{i \in \mathbb{N}_0} \sigma_i(x), \end{aligned}$$

is well-defined. Moreover $s(x) \leq f(x)$ for all $x \geq 0$, $s(0) = 0$ and as a pointwise limit of increasing functions, s is increasing as well.

Now let us set $\phi := f - s$. First, let us show that ϕ is indeed an increasing and continuous function.

ϕ is the pointwise limit of increasing functions f_n and hence it is increasing as well. Since ϕ is increasing, we know that $\phi(x+) \geq \phi(x-)$, so it suffices to show that $\phi(x+) \leq \phi(x-)$. If f is continuous at x, then for $y < x < z$ (as s is increasing) we have $s(z) \geq s(y)$ and $f(x+) = f(x-)$. Hence

$$\begin{aligned} \phi(x+) =& f(x+) - \lim_{z \to x, z > x} s(z) \\ =& f(x-) - \lim_{z \to x, z > x} s(z) \\ \leq& f(x-) - \lim_{y \to x, y < x} s(y) = \phi(x-). \end{aligned}$$

If, on the other hand, f is discontinuous at x, we can without loss of generality assume that $t_0 = x$ and hence for $z \geq x \geq y$,

$$s(z) - s(y) = \sum_{i \in \mathbb{N}_0} \sigma_i(z) - \sigma_i(y) \geq \sigma_0(z) - \sigma_0(y) = f(x+) - f(x-).$$

By taking limits as $z \to x$, $z > x$ and as $y \to x$, $y < x$, we obtain $\phi(x+) \leq \phi(x-)$. Thus for all cases $\phi(x+) = \phi(x-)$ and hence ϕ is continuous. Hence we have obtained a decomposition $f = \phi + s$, where s is an increasing jump function and ϕ is continuous and increasing.

We are now ready to finally construct $g \in \mathcal{K}_\infty$ such that $g \leq f$ on $[0, \infty)$. Set

$$\tilde{\sigma}_i(x) := \begin{cases} 0, & \text{for } x < t_i \\ [f(t_i+) - f(t_i-)]\xi & \text{for } x = t_i + \xi,\ \xi \in [0,1] \\ f(t_i+) - f(t_i-), & \text{for } x > t_i + 1, \end{cases}$$

and define $\tilde{s} := \sum_{i \in \mathbb{N}_0} \tilde{\sigma}_i$. Note that $\tilde{\sigma}_i$ is continuous and increasing for each i; moreover $\tilde{\sigma}_i \leq \sigma_i$, with $\tilde{\sigma}_i(x+1) \geq \sigma_i(x)$. This in turn implies that

$\tilde{s}(x+1) \geq s(x)$ for all $x \geq 0$. Finally we can also show that $\tilde{s}$ is continuous: on $[0,k]$ we have $\sum_{i\in\mathbb{N}_0} |\tilde{\sigma}_i(x)| \leq \sum_{i\in\mathbb{N}_0} |\sigma_i(x)| \leq f(k)$, so by the Weierstrass M-test $\tilde{s}|_{[0,k]}$ is continuous as the uniform limit of continuous functions. As k was arbitrary, we conclude that $\tilde{s}$ is a continuous function.

Let us now set $g := \phi + \tilde{s}$. Then it is continuous, $g(0) = 0$ and $g \leq f$ (recall that $f = \phi + s$). Moreover $g(x+1) = \phi(x+1) + \tilde{s}(x+1) \geq \phi(x) + s(x) = f(x)$, so that $\lim_{x\to\infty} g(x) = \infty$. Thus we only need to show that g is strictly increasing. Let $y > x$; then $g(y) - g(x) = \phi(y) - \phi(x) + \tilde{s}(y) - \tilde{s}(x)$. Both ϕ and $\tilde{s}$ are increasing, so we are done unless $\phi(y) = \phi(x)$ and $\tilde{s}(y) = \tilde{s}(x)$. But then, by definition of $\tilde{s}$ and s, f has no discontinuities on $[x,y]$ and hence $s|_{[x,y]}$ is a constant function, so that $\phi|_{[x,y]} = f|_{[x,y]} - s|_{[x,y]}$ is strictly increasing, as f is. This contradicts $\phi(y) = \phi(x)$ and hence g must be strictly increasing, which completes the proof of claim. $\square$

Now we obtain the required η by applying Claim 2 to $\tilde{\eta}$. This completes the proof of Lemma 5.1.11. $\square$

Appendix C

The bounded real lemma

Lemma 7.3.1. *Consider a stabilizable and detectable linear state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and assume that its transfer function G satisfies $\|G\|_{H^\infty} \leq 1$ and $\|D\| < 1$.*

Then there exist matrices L, W and a positive semi-definite $P = P^* \in \mathbb{F}^{n \times n}$ such that

$$\begin{aligned} A^*P + PA &= -C^*C - L^*L \\ PB &= -C^*D - L^*W \\ D^*D &= I - W^*W. \end{aligned}$$

We will prove this result in three steps: firstly we will prove it for controllable and observable systems with $\|G\|_{H^\infty} < 1$, then we will relax the assumptions on the linear system to stabilizability and detectability and finally we will extend the result to the case when $\|G\|_{H^\infty} = 1$.

We will use Riccati equation theory to prove this statement, so it is useful to record the following.

Lemma C.0.1. *Consider a linear system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ such that $\|D\| < 1$. The following statements are equivalent:*

(a) There exist matrices L, W and a positive definite (resp. semi-definite) $P = P^* \in \mathbb{F}^{n \times n}$ that solve the bounded real equations

$$A^*P + PA = -C^*C - L^*L \tag{C.0.1a}$$

$$PB = -C^*D - L^*W \tag{C.0.1b}$$

$$D^*D = I - W^*W. \tag{C.0.1c}$$

(b) *There exists a positive definite (resp. semi-definite)* $P = P^* \in \mathbb{F}^{n\times n}$ *that solves the algebraic Riccati equation*

$$A^*P + PA + C^*C + (PB + C^*D)(I - D^*D)^{-1}(B^*P + D^*C) = 0. \tag{C.0.2}$$

Proof. We first show that (a) implies (b). Since $\|D\| < 1$, we have $W^*W = I - D^*D > 0$ or equivalently $\|W\xi\| > 0$ for all $\xi \in \mathbb{F}^m \setminus \{0\}$. This shows that W is invertible, so that from equation (C.0.1b) we obtain $L^* = -(PB + C^*D)W^{-1}$. Equation (C.0.1a) now shows that the required Riccati equation has a solution.

We can show that (b) implies (a) in a very similar manner. Since $I - D^*D$ is positive definite and self-adjoint, it admits a positive definite self-adjoint square root, say W, so that $I - D^*D = W^*W$. Thus after setting $L := -(W^*)^{-1}(B^*P + D^*C)$ we now have shown that both (C.0.1b) and (C.0.1c) hold. The Riccati equation (C.0.2) then is precisely the last bounded real equation (C.0.1a). $\square$

We also record an easy, yet nonstandard result that will be useful later on.

Lemma C.0.2. *Let* X, Y, Z, U *be matrices whose dimensions are such that* $\begin{pmatrix} X & Y \\ Z & U \end{pmatrix}$ *is a block-matrix. Then* $\sigma\left(\begin{pmatrix} X & Y \\ Z & U \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} X & -Y \\ -Z & U \end{pmatrix}\right)$.

Proof. This follows from the observation

$$\begin{pmatrix} X & Y \\ Z & U \end{pmatrix}\begin{pmatrix} \xi \\ \mu \end{pmatrix} = \lambda \begin{pmatrix} \xi \\ \mu \end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{pmatrix} X & -Y \\ -Z & U \end{pmatrix}\begin{pmatrix} -\xi \\ \mu \end{pmatrix} = \lambda \begin{pmatrix} -\xi \\ \mu \end{pmatrix}.$$

$\square$

We now proceed to carefully proving the algebraic Riccati equation version of the bounded real lemma. The following version of it is easy to find over the real field, however over the complex field the author could only find Theorem 5.3.25 from [25], which was stated without proof (it was postponed to volume 2 of the book, which has not yet been published).

Lemma C.0.3. *Consider a controllable and detectable linear state-space system* $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{C})$ *and assume that its transfer function* G *satisfies* $\|G\|_{H^\infty} < 1$.

Then there exists a positive semi-definite $P = P^* \in \mathbb{F}^{n\times n}$ *such that*

$$A^*P + PA + C^*C + (PB + C^*D)(I - D^*D)^{-1}(B^*P + D^*C) = 0$$

and such that $\sigma\left(A + B(I - D^*D)^{-1}(D^*C + B^*P)\right) \subseteq \mathbb{C}_-$.

If (C, A) *is an observable pair, then* P *is positive definite.*

Proof. We modify the proof of Theorem 20.1.1 from [37] to systems with feedthrough.

By our assumption, $\|G\|_{H^\infty} < 1$, so that in particular $\|G(s)\| < 1$ for all $s \in i\mathbb{R}$. Set $H(s) := I - G(-\bar{s})^* G(s)$ and check that for $s \in i\mathbb{R}$ and for $\xi \in \mathbb{F}^m \setminus \{0\}$ we have $\langle H(s)\xi, \xi\rangle = \|\xi\|^2 - \|G(s)\xi\|^2 > 0$. In other words, $H > 0$ on $i\mathbb{R}$ and hence in particular has no zeros on $i\mathbb{R}$.

Now let us rewrite $H(s)$ as

$$H(s) = I - D^*D + \begin{pmatrix} -D^*C & B^* \end{pmatrix} \left[sI - \begin{pmatrix} A & 0 \\ C^*C & -A^* \end{pmatrix} \right]^{-1} \begin{pmatrix} B \\ C^*D \end{pmatrix}.$$

Using the easy observation that $R := I - D^*D > 0$ and equation 6.3.14 from [37] we can thus see that

$$H(s)^{-1} = R^{-1} + R^{-1} \begin{pmatrix} -D^*C & B^* \end{pmatrix} [sI - N]^{-1} \begin{pmatrix} B \\ C^*D \end{pmatrix} R^{-1},$$

where

$$N := \begin{pmatrix} A + BR^{-1}D^*C & -BR^{-1}B^* \\ C^*C + C^*DR^{-1}D^*C & -(A^* + C^*DR^{-1}B^*) \end{pmatrix}.$$

We will now verify that N has no eigenvalues on $i\mathbb{R}$. To this end suppose we have $s_0 \in i\mathbb{R}$ and $\xi, \mu \in \mathbb{F}^n$ such that $N \begin{pmatrix} \xi \\ \mu \end{pmatrix} = s_0 \begin{pmatrix} \xi \\ \mu \end{pmatrix}$. Consider

$$\begin{aligned}
&H(s)R^{-1} \begin{pmatrix} -D^*C & B^* \end{pmatrix} \begin{pmatrix} \xi \\ \mu \end{pmatrix} \\
&\quad = \left\{ R + \begin{pmatrix} -D^*C & B^* \end{pmatrix} \left[sI - \begin{pmatrix} A & 0 \\ C^*C & -A^* \end{pmatrix} \right]^{-1} \begin{pmatrix} B \\ C^*D \end{pmatrix} \right\} \\
&\qquad\qquad \times R^{-1} \begin{pmatrix} -D^*C & B^* \end{pmatrix} \begin{pmatrix} \xi \\ \mu \end{pmatrix} \\
&\quad = \begin{pmatrix} -D^*C & B^* \end{pmatrix} \left\{ I + \left[sI - \begin{pmatrix} A & 0 \\ C^*C & -A^* \end{pmatrix} \right]^{-1} \right. \\
&\qquad\qquad \left. \times \begin{pmatrix} -BR^{-1}D^*C & BR^{-1}B^* \\ -C^*DR^{-1}D^*C & C^*DR^{-1}B^* \end{pmatrix} \right\} \begin{pmatrix} \xi \\ \mu \end{pmatrix} \\
&\quad = \begin{pmatrix} -D^*C & B^* \end{pmatrix} \left[sI - \begin{pmatrix} A & 0 \\ C^*C & -A^* \end{pmatrix} \right]^{-1} (sI - N) \begin{pmatrix} \xi \\ \mu \end{pmatrix} \\
&\quad = (s - s_0) \begin{pmatrix} -D^*C & B^* \end{pmatrix} \left[sI - \begin{pmatrix} A & 0 \\ C^*C & -A^* \end{pmatrix} \right]^{-1} \begin{pmatrix} \xi \\ \mu \end{pmatrix},
\end{aligned}$$

so that $H(s_0)R^{-1}\begin{pmatrix}-D^*C & B^*\end{pmatrix}\begin{pmatrix}\xi\\ \mu\end{pmatrix} = 0$. However, since we have already established positivity of H on $i\mathbb{R}$, this means that $\begin{pmatrix}-D^*C & B^*\end{pmatrix}\begin{pmatrix}\xi\\ \mu\end{pmatrix} = 0$, or equivalently $D^*C\xi + B^*\mu = 0$. Therefore

$$s_0\begin{pmatrix}\xi\\ \mu\end{pmatrix} = N\begin{pmatrix}\xi\\ \mu\end{pmatrix} = \begin{pmatrix}A\xi\\ C^*C\xi - A^*\mu\end{pmatrix}.$$

By stabilizability and detectability and $G \in H^\infty$, we know that $\sigma(A) \subseteq \mathbb{C}_-$. Since $s_0 \in i\mathbb{R}$, this implies $\xi = 0$, which in turn implies $s_0\mu = -A^*\mu$. Thus as $\sigma(A^*) = \overline{\sigma(A)} \subseteq \mathbb{C}_-$, we infer $\mu = 0$, so that N indeed has no eigenvalues on $i\mathbb{R}$.

Now, by Theorem 7.6.1 from [37], we know that as long as the tuple $(A + BR^{-1}D^*C,\ BR^{-1}B^*)$ is a controllable pair, the algebraic Riccati equation

$$A^*X + XA + C^*C + (XB + C^*D)R^{-1}(B^*X + D^*C) = 0$$

admits a self-adjoint solution if and only if the partial multiplicities corresponding to the real eigenvalues of

$$M := i\begin{pmatrix}A + BR^{-1}D^*C & BR^{-1}B^*\\ -C^*C - C^*DR^{-1}D^*C & -(A^* + C^*DR^{-1}B^*)\end{pmatrix}$$

are all even (equivalently, if M has a real eigenvalue λ_0, then all the sizes of Jordan blocks corresponding to λ_0 are even). However we already know that N has no eigenvalues on $i\mathbb{R}$ and thus by Lemma C.0.2 we know that the matrix

$$\begin{pmatrix}A + BR^{-1}D^*C & BR^{-1}B^*\\ -C^*C - C^*DR^{-1}D^*C & -(A^* + C^*DR^{-1}B^*)\end{pmatrix}$$

has no eigenvalues on the imaginary axis, whence M has no eigenvalues on the real axis.

Let us now show that the pair $(A + BR^{-1}D^*C,\ BR^{-1}B^*)$ is controllable. To this end suppose that there exist $\xi \in \mathbb{F}^n$ and $s \in \mathbb{C}$ such that $\xi^*(sI - A - BR^{-1}D^*C) = 0$ and $\xi^*BR^{-1}B^* = 0$. Then post-multiplication of the latter equation by ξ gives us $\langle R^{-1}B^*\xi, B^*\xi\rangle = 0$, so that - by positive definiteness of R^{-1} - we have $\xi^*B = 0$. This in turn implies that $\xi^*(sI - A) = 0$, so that by the Hautus test of controllability (see e.g. Theorem 4.3.3 from [37]) $\xi = 0$, whence another application of the Hautus test of controllability shows that $(A + BR^{-1}D^*C,\ BR^{-1}B^*)$ is indeed a controllable pair.

Now an application of Theorem 7.5.1 from [37] shows that the minimal solution of (C.0.2) P_- satisfies $\sigma\left(A + BR^{-1}(D^*C + B^*P_-)\right) \subseteq \mathbb{C}_-$.

Thus we are only left with proving that P_- is positive semi-definite. Set $T := -C^*C - (C^*D + P_-B)R^{-1}(B^*P_- + D^*C)$, so that P_- is a solution of the Lyapunov equation $A^*P_- + P_-A = T$. By equation (5.3.3) from [37], P_- then satisfies $P_- = -\int_0^\infty e^{A^*t}Te^{At}\,\mathrm{d}t$ and hence P_- is easily seen to be positive semi-definite.

Moreover if (C, A) is observable, then by using $R^{-1} > 0$ we can see that $P_- = -\int_0^\infty e^{A^*t}Te^{At}\,\mathrm{d}t \geq \int_0^\infty e^{A^*t}C^*Ce^{At}\,\mathrm{d}t$. Hence for all $\xi \in \ker(P_-)$ we have $0 = \langle P_-\xi, \xi\rangle \geq \int_0^\infty \left\|Ce^{At}\xi\right\|^2 \mathrm{d}t \geq 0$, so that $Ce^{At}\xi = 0$ for all $t \in [0, \infty)$. Repeated evaluation at 0 and differentiation then shows that $CA^n\xi = 0$ for all $t \in \mathbb{N}_0$, which by observability of (C, A) implies that $\xi = 0$ and thus $P_- > 0$ as required. This completes the proof. □

Now we extend the above result to the case when the linear state-space system (A, B, C, D) is only stabilizable and detectable.

Lemma C.0.4. *Consider a stabilizable and detectable linear state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and assume that its transfer function G satisfies $\|G\|_{H^\infty} < 1$.*

Then there exists a positive semi-definite $P = P^* \in \mathbb{F}^{n\times n}$ such that

$$A^*P + PA + C^*C + (PB + C^*D)(I - D^*D)^{-1}(B^*P + D^*C) = 0$$

and such that $\sigma\left(A + B(I - D^*D)^{-1}(D^*C + B^*P)\right) \subseteq \mathbb{C}_-$.

Proof. For $\mathbb{F} = \mathbb{R}$, this is Theorem 3.7.1 from [19] (we use the fact that stabilizability, detectability and $G \in H^\infty$ imply $\sigma(A) \subset \mathbb{C}_-$).

We thus proceed with proving this for $\mathbb{F} = \mathbb{C}$.

It is well known (see e.g. Proposition 4.5.1 from [37]) that there exists an invertible matrix $T \in \mathbb{F}^{n\times n}$, such that

$$T^{-1}AT = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \qquad T^{-1}B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

where (A_1, B_1) is a controllable pair and - by Proposition 4.5.2 from [37] - we have $\sigma(A_3) \subset \mathbb{C}_-$.

Now suppose we have found the required solution P of the bounded real Riccati equation. Set $R := I - D^*D$ and write $CT = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$ and

$$T^*PT = \begin{pmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{pmatrix},$$

where the sizes of P_i's and C_i's are compatible with the sizes of A_i's and where the structure of P is determined by the fact that it is self-adjoint.

Then we can see that if we multiply the Riccati bounded real equation from the left by T^* and from the right by T, then it is equivalent to

$$\begin{pmatrix} A_1^* & 0 \\ A_2^* & A_3^* \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{pmatrix} + \begin{pmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} + \begin{pmatrix} C_1^* \\ C_2^* \end{pmatrix} \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$
$$+ \left[\begin{pmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{pmatrix} \begin{pmatrix} B_1 \\ 0 \end{pmatrix} + \begin{pmatrix} C_1^* \\ C_2^* \end{pmatrix} D \right] R^{-1}$$
$$\times \left[\begin{pmatrix} B_1^* & 0 \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{pmatrix} + D^* \begin{pmatrix} C_1 & C_2 \end{pmatrix} \right] = 0.$$

Thus we can see that finding a solution P to the required bounded real Riccati equation is equivalent to finding P_2 and self-adjoint P_1, P_3 that solve

$$A_1^* P_1 + P_1 A_1 + C_1^* C_1 + (P_1 B_1 + C_1^* D) R^{-1} (B_1^* P_1 + D^* C_1) = 0 \tag{C.0.3}$$
$$A_1^* P_2 + P_1 A_2 + P_2 A_3 + C_1^* C_2 + (P_1 B_1 + C_1^* D) R^{-1} (B_1^* P_2 + D^* C_2) = 0 \tag{C.0.4}$$
$$A_2^* P_2 + A_3^* P_3 + P_2^* A_2 + P_3 A_3 + C_2^* C_2 + (P_2^* B_1 + C_2^* D) R^{-1} (B_1^* P_2 + D^* C_2) = 0 \tag{C.0.5}$$

and such that $\begin{pmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{pmatrix}$ is positive semi-definite.

First we aim to use Lemma C.0.3 to solve equation (C.0.3). One can easily check that $C(sI - A)^{-1} B + D = C_1 (sI - A_1)^{-1} B_1 + D$ and by our choice of (A_1, B_1, C_1) we know that (A_1, B_1) is controllable, so we only need to check that (C_1, A_1) is detectable. This however follows from $\sigma(A_1) \cup \sigma(A_3) = \sigma(A) \subseteq \mathbb{C}_-$, as thus $sI - A_1$ has full rank for all $s \in \mathbb{C}_+$, so that by the Hautus test for detectability (C_1, A_1) is a detectable pair.

Hence we can now apply Lemma C.0.3 to infer that there exists a positive semi-definite $P_1 = P_1^* \in \mathbb{F}^{n_1 \times n_1}$ (here n_1 is the size of A_1) that solves (C.0.3) and such that $\sigma\left(A_1 + B_1 R^{-1} (D^* C_1 + B_1^* P_1)\right) \subseteq \mathbb{C}_-$.

Now equation (C.0.4) can be rewritten as

$$\left(A_1^* + (P_1 B_1 + C_1^* D) R^{-1} B_1^*\right) P_2 + P_2 A_3 + M_1 = 0,$$

where $M_1 := P_1 A_2 + C_1^* + (P_1 B_1 + C_1^* D) R^{-1} D^* C_2$. Since $\sigma(-A_3) \cap \sigma\left(A_1^* + (P_1 B_1 + C_1^* D) R^{-1} B_1^*\right) \subseteq \mathbb{C}_+ \cap \mathbb{C}_- = \emptyset$, standard Sylvester's equation theory (use e.g. Propositions 7.2.4 and 7.2.3 from [9]) says that there exists a solution P_2 of (C.0.4). Hence we are only left with solving

$$A_3^* P_3 + P_3 A_3 + M_2 = 0,$$

where $M_2 := A_2^* P_2 + P_2^* A_2 + C_2^* C_2 + (P_2^* B_1 + C_2^* D) R^{-1} (B_1^* P_2 + D^* C_2)$. This is a Lyapunov equation (note that $M_2^* = M_2$) and hence - as $\sigma(A_3) \subseteq \mathbb{C}_-$ - it admits a self-adjoint solution P_3.

We have now found solutions P_1, P_2, P_3 to equations (C.0.3), (C.0.4) and (C.0.5), so the self-adjoint matrix

$$P = (T^*)^{-1} \begin{pmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{pmatrix} T^{-1}$$

solves the algebraic Riccati equation

$$A^* P + PA + C^* C + (PB + C^* D) R^{-1} (B^* P + D^* C) = 0.$$

Thus if we set $M_3 := C^* C + (PB + C^* D) R^{-1} (B^* P + D^* C) \geq 0$, then P also solves the Lyapunov equation

$$A^* P + PA + M_3 = 0$$

and since M_3 is positive semi-definite, then so is P.

Finally we observe that a straightforward calculation gives us

$$\begin{aligned} &A + BR^{-1}(D^* C + B^* P) \\ &\qquad = T \begin{pmatrix} A_1 + B_1 R^{-1}(D^* C_1 + B_1^* P_1) & A_2 + B_1 R^{-1} B_1^* P_2 \\ 0 & A_3 \end{pmatrix} T^{-1}, \end{aligned}$$

so that $\sigma\left(A + BR^{-1}(D^* C + B^* P)\right) = \sigma\left(A_1 + B_1 R^{-1}(D^* C_1 + B_1^* P_1)\right) \cup \sigma\left(A_3\right) \subseteq \mathbb{C}_-$ thus completing the proof. □

Now we use perturbation theory of algebraic Riccati equations to extend the conclusions to the case when $\|G\|_{H^\infty} = 1$.

Lemma C.0.5. *Consider a stabilizable and detectable linear state-space system $(A, B, C, D) \in \Sigma(m, n, p; \mathbb{F})$ and assume that its transfer function G satisfies $\|G\|_{H^\infty} \leq 1$ and $\|D\| < 1$.*

Then there exists a positive semi-definite $P = P^* \in \mathbb{F}^{n \times n}$ such that

$$A^* P + PA + C^* C + (PB + C^* D)(I - D^* D)^{-1}(B^* P + D^* C) = 0 \quad \text{(C.0.6)}$$

and such that $\sigma\left(A + B(I - D^* D)^{-1}(D^* C + B^* P)\right) \subseteq \overline{\mathbb{C}_-}$.

Proof. Note that for $\rho < 1$, the stabilizable and detectable linear system $(A, B, \rho C, \rho D) \in \Sigma(m, n, p; \mathbb{F})$ is a realization of ρG and $\|\rho G\|_{H^\infty} = \rho < 1$.

We can set $R_\rho := I - \rho^2 D^* D > 0$, so that - by Lemma C.0.4 - for each $0 < \rho < 1$ there exists a positive semi-definite solution $P_\rho = P_\rho^* \in \mathbb{F}^{n \times n}$ of

$$P_\rho B R_\rho^{-1} B^* P_\rho - (A^* + \rho^2 D^* C R_\rho^{-1} B^*) P_\rho - P_\rho (A + \rho^2 B R_\rho^{-1} D^* C) \\ + \rho^4 C^* D R_\rho^{-1} D^* C = 0.$$

As $\|D\| < 1$, the matrix $I - D^* D$ is invertible and hence, by continuity of matrix inversion, $\lim_{\rho \to 1} R_\rho^{-1} = R_1^{-1} = (I - D^* D)^{-1}$. It is also easy to see that $\rho C \to C$ and $\rho D \to D$ as $\rho \to 1$.

Now consider the case $\mathbb{F} = \mathbb{C}$. By Theorem 11.1.1 from [37] equation (C.0.6) thus admits self-adjoint solutions as long as $(A + B R_1^{-1} D^* C,\ B R_1^{-1} B^*)$ is stabilizable tuple (Theorem 11.1.1 assumes "sign controllability" of this matrix pair, but it is easy to see from its definition on p. 155 of [37] that stabilizability implies sign controllability).

To this end we note that stabilizability of $(A + B R_1^{-1} D^* C,\ B R_1^{-1} B^*)$ is equivalent to $\operatorname{rank}(sI - A - B R_1^{-1} D^* C,\ B R_1^{-1} B^*) = n$ for all $s \in \overline{\mathbb{C}_+}$. Now suppose on the contrary, that there exists $s \in \overline{\mathbb{C}_+}$ and $\xi \in \mathbb{C}^n \setminus \{0\}$ such that $\xi^*(sI - A - B R_1^{-1} D^* C,\ B R_1^{-1} B^*) = 0$. Then $\xi^* B R_1^{-1} B^* = 0$, so that $\left\langle R_1^{-1} B^* \xi, B^* \xi \right\rangle = 0$. As $\|D\| < 1$, we see that $R_1^{-1} > 0$ and hence $B^* \xi = 0$ or equivalently $\xi^* B = 0$. Thus we obtain $0 = \xi^*(sI - A - B R_1^{-1} D^* C) = \xi^*(sI - A)$, so that $\xi^*(sI - A, B) = 0$ contradicting the stabilizability of (A, B) and in turn proving the stabilizability of $(A + B R_1^{-1} D^* C,\ B R_1^{-1} B^*)$.

Hence as mentioned above, we can apply Theorem 11.1.1 from [37] to infer that (C.0.6) has a self-adjoint solution P.

Now §11.4 from [37] says that Theorem 11.1.1 holds for the real case as well and thus, in exactly the same way as for $\mathbb{F} = \mathbb{C}$, we obtain a real symmetric solution Q of the Riccati equation (C.0.6). □

As a corollary we obtain Lemma 7.3.1.

Appendix D

On an initial value problem

In this section we will show that,as long as we assume that f and $(I-Df)^{-1}$ are locally Lipschitz, for a prescribed $d \in L^{\infty}_{\text{loc}}(\mathbb{F}^m)$, the initial value problem

$$\dot{x}(t) = Ax(t) + Bf \circ (I - Df)^{-1}(Cx(t) + Dd(t)) + Bd(t)$$
$$x(0) = \xi \in \mathbb{F}^n \quad \text{(D.0.1)}$$

admits a unique solution $x \in AC(\mathbb{F}^n)$, defined on some maximal interval $[0, \omega) \subseteq [0, \infty)$. This follows from an application of Theorem 54 from Appendix C in [51], so we only need check that the map $g\colon [0,\infty) \times \mathbb{F}^n \to \mathbb{F}^n$ defined by $g(t, \xi) := A\xi + Bf \circ (I - Df)^{-1}(C\xi + Dd(t)) + Bd(t)$ satisfies its conditions:

1. for each $\xi \in \mathbb{F}^n$ there is a real number $r > 0$ and a locally integrable $\alpha\colon [0, \infty) \to [0, \infty)$ such that

$$\|g(t, \xi_1) - g(t, \xi_2)\| \leq \alpha(t) \|\xi_1 - \xi_2\| \quad \text{(D.0.2)}$$

   for all $t \in [0, \infty)$ and for all $\xi_1, \xi_2 \in \mathbb{B}_{\mathbb{C}}(\xi, r)$,

2. for each fixed $\xi \in \mathbb{F}^n$ there is a locally integrable function $\beta : [0, \infty) \to [0, \infty)$ such that

$$\|g(t, \xi)\| \leq \beta(t) \qquad \text{a.e.} \quad \text{(D.0.3)}$$

Let us define a locally Lipschitz function $h(\mu) := Bf \circ (I - Df)^{-1}(\mu)$, so that $g(t, \xi) = A\xi + h(C\xi + Dd(t)) + Bd(t)$.

We will now check that condition 1. is satisfied. Let $\xi \in \mathbb{F}^n$ and pick any $r > 0$. Since h is locally Lipschitz, we can define $\alpha\colon [0, \infty) \to [0, \infty)$ by

$$\alpha(t) := \|A\| + \sup_{\xi_1, \xi_2 \in \mathbb{B}(\xi, r)} \frac{\|h(C\xi_1 + Dd(t)) - h(C\xi_2 + Dd(t))\|}{\|\xi_1 - \xi_2\|},$$

so that (D.0.2) is clearly satisfied. We will now verify that α is locally integrable. To this end, pick a compact $K \subseteq [0, \infty)$. Since $d \in L^\infty_{\mathrm{loc}}(\mathbb{F}^m)$, it is essentially bounded on K and hence there exist $\mu \in \mathbb{F}^p$ and $\rho > 0$ such that $Dd(t) \in \mathbb{B}(\mu, \rho)$ for almost all $t \in K$. As h is locally Lipschitz, there exists $l > 0$ such that $\|h(\mu_1) - h(\mu_2)\| \leq l\,\|\mu_1 - \mu_2\|$ for all $\mu_1, \mu_2 \in \mathbb{B}(\mu + C\xi, \rho + r)$. Hence we can estimate

$$\alpha(t) \leq \|A\| + l\,\|C\| \qquad \text{a.e. on } K$$

so that α is clearly integrable on K. Since K was an arbitrary compact set, we conclude that α is locally integrable as required.

To check condition 2., for a fixed $\xi \in \mathbb{F}^n$, we define $\beta(t) := \|A\|\,\|\xi\| + h(C\xi + Dd(t)) + Bd(t)$. Let $K \subseteq [0, \infty)$ be a compact set. Since $d \in L^\infty_{\mathrm{loc}}(\mathbb{F}^m)$, Dd is essentially bounded on K, so that there exists $r > 0$ such that $Dd(t) \in \mathbb{B}(0, r)$ and $Bd(t) \in \mathbb{B}(0, r)$ (note that the two balls lie in different spaces) for almost all $t \in K$. Thus, as h is locally Lipschitz, there exists $c > 0$ such that $\beta(t) \leq \|A\|\,\|\xi\| + c$ for almost all $t \in K$ and hence β is clearly integrable on K. Since K was an arbitrary compact set, we conclude that β is locally integrable as required.

Thus we can apply Theorem 54 from Appendix C in [51] and hence, for each $\xi \in \mathbb{F}^n$, there exists a solution x of (D.0.1), called the maximal solution, defined on some nonempty interval $[0, \omega) \subseteq [0, \infty)$ such that if x_1 is any other solution of (D.0.1), defined on $[0, \omega_1) \subseteq [0, \infty)$, then $[0, \omega_1) \subseteq [0, \omega)$ and $x = x_1$ on $[0, \omega_1)$. It is well-known that, if $x\colon [0, \omega) \to \mathbb{F}^n$ is a maximal solution of (D.0.1) and if $\omega < 0$, then

$$\lim_{t \to \omega} \|x(t)\| = \infty.$$