# Sampled-data control: stabilization, tracking and disturbance rejection

submitted by

Zhenqing Ke

for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

August 2008



Signature of Author ..............................................................

Zhenqing Ke

## Summary

We study a number of issues in sampled-data control of linear systems. We investigate the relationship between the continuous-time finite-dimensional feedback systems and the corresponding sampled-data systems obtained by sample-and-hold operations with a constant sampling period. Using state-space methods, we show that the sampled-data system recovers the state stability of the continuous-time feedback system as the sampling period goes to zero. State feedback systems and dynamic output feedback systems are considered. We explore sampled-data feedback systems with time-varying sampling period. It is shown that, applying an adaptive law for adjusting the sampling period, we can achieve the stability of the sampled-data feedback systems. State feedback, static and dynamic output feedback are considered. We solve tracking and disturbance rejection problems for stable infinite-dimensional systems, using a simple low-gain discrete-time controller suggested by the internal model principle, with reference signals which are finite sums of sinusoids, and disturbance signals which are asymptotic to finite sums of sinusoids. The results are given for both input-output systems and state space systems. We present adaptive low-gain control strategies for tracking constant reference signals for infinite-dimensional, well-posed, exponentially stable, linear systems.

# Contents

**List of symbols.**

| | |
|---|---|
| $\mathbb{N}$ | Set of natural numbers $\{1, 2, \ldots\}$. |
| $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ | Ring of integers, fields of real numbers and complex numbers. |
| $\mathbb{Z}_+$ | Non-negative integers $\{0\} \cup \mathbb{N}$. |
| $\operatorname{Re} z$, $\operatorname{Im} z$ | Real and imaginary parts of a complex number $z$. |
| $\mathbb{R}_+$ | $\{s \in \mathbb{R} \colon s \geq 0\}$. |
| $\mathbb{C}_\alpha$ | Open right half-plane $\{s \in \mathbb{C} \colon \operatorname{Re} s > \alpha\}$ for $\alpha \in \mathbb{R}$. |
| $\mathbb{C}_-$ | Open left half-plane $\{s \in \mathbb{C} \colon \operatorname{Re} s < 0\}$. |
| $\underline{N}$ | $\{1, \ldots, N\}$ for $N \in \mathbb{N}$. |
| $\mathbb{B}(\lambda, r)$ | $\{s \in \mathbb{C} \colon \lvert s - \lambda \rvert < r\}$. |
| $\mathbb{E}_\rho$ | $\{s \in \mathbb{C} \colon \lvert s \rvert > \rho\}$. |
| $\operatorname{cl}(U)$ | The closure of $U \subset \mathbb{C}$. |
| $\overline{\lambda}$ | The complex of conjugate of $\lambda \in \mathbb{C}$. |
| $\mathscr{L}$ | Laplace transform. |
| $\mathscr{Z}$ | Z-transform. |

Let $X, Y$ be Banach spaces.

| | |
|---|---|
| $\mathcal{B}(X, Y)$ | The set of all bounded linear operators from $X$ to $Y$. |
| $\mathcal{B}(X)$ | The set of all bounded linear operators from space $X$ to $X$. |
| $A^*$ | Self-adjoint operator of $A \in \mathcal{B}(X)$. |
| $\sigma(A)$, $\varrho(A)$ | Spectrum and resolvent of $A \in \mathcal{B}(X)$. |
| $r(A)$ | Spectral radius of $A \in \mathcal{B}(X)$. |
| $L_b(\mathbb{R}_+, X)$ | The set of bounded $X$-valued Lebesgue measurable functions with the sup-norm $\lVert \cdot \rVert_\infty$. |
| $H^\infty(\Omega, X)$ | $\{f \colon \Omega \to X \mid f$ is holomorphic and bounded$\}$, where $\Omega \subset \mathbb{C}$ is open. |
| $H^\infty_<(\mathbb{E}_1, X)$ | $\displaystyle\bigcup_{0<\gamma<1} H^\infty(\mathbb{E}_\gamma, X)$. |
| $H^2(\mathbb{C}_\alpha, X)$ | $\{f \colon \mathbb{C}_\alpha \to X \mid f$ is holomorphic and $\displaystyle\sup_{x>\alpha} \int_{-\infty}^{\infty} \lVert f(x + i\sigma) \rVert^2 d\sigma < \infty\}$. |
| $\ell^1_\alpha(\mathbb{Z}_+, X)$ | Weighted $\ell^1$-space $\{v \colon \mathbb{Z}_+ \to X \mid (v(k)\alpha^{-k})_{k \in \mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, X)\}$ for $\alpha > 0$. |
| $\hat{\ell}^1_\alpha(X)$ | $\{\mathscr{Z}(g) \colon g \in \ell^1_\alpha(\mathbb{Z}_+, X)\} \subset H^\infty(\mathbb{E}_\alpha, X)$. |
| $L^q_\beta(\mathbb{R}_+, X)$ | Exponentially weighted $L^q$-space $\{f \in L^q_{\mathrm{loc}}(\mathbb{R}_+, X) \colon f(\cdot)e^{-\beta \cdot} \in L^q(\mathbb{R}_+, X)\}$ for $1 \leq q < \infty$ and $\beta \in \mathbb{R}$. |

# Chapter 1

# Introduction

Since digital computing equipment offers many benefits, such as accuracy, speed, small size and low price, it has been used more and more to implement feedback controls. Therefore, the analysis and synthesis of sampled-data systems (or digital control systems) have been of continuing interest for several decades (see [1], [2], [12], [17], [84], [85], to name just a few references).

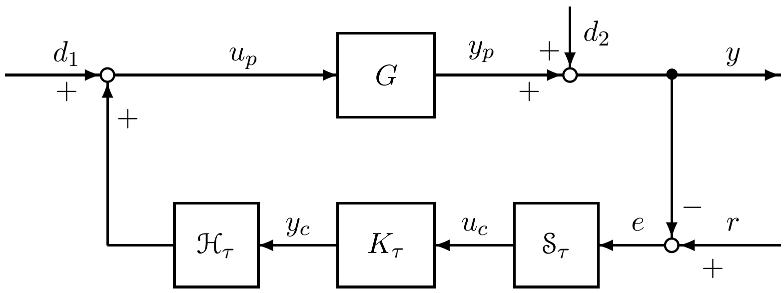


Figure 1-1: Sampled-data systems.

We use the specific class of linear feedback systems shown in Figure 1-1 to give a short introduction to sampled-data systems. In a sampled-data system, a continuous-time plant $G$ is controlled, via sample and hold operations, by a discrete-time controller $K_\tau$ which is normally a microprocessor or the central processing unit of a digital computer. Sampled-data systems are *hybrid* systems which involve both continuous-time and discrete-time signals.

The *sampling operator* $\mathcal{S}_\tau$ is the mathematical model of a digital-to-analog (D/A) convertor which converts a continuous-time signal to a discrete-time signal. In many practical situations, $\mathcal{S}_\tau$ is the ideal sampling operator which samples periodically at *sampling points* $0, \tau, 2\tau, 3\tau, \ldots$, i.e.,

$$u_c(k) = (\mathcal{S}_\tau e)(k) := e(k\tau)\,, \quad \forall k \in \mathbb{Z}_+\,,$$

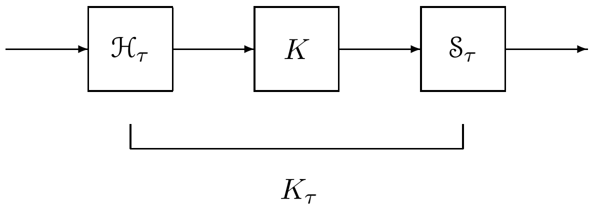


Figure 1-2: Sample-hold discretization.

where $\tau > 0$ is the so-called *sampling period*. The operator $\mathcal{H}_\tau$ is the mathematical model for an analog-to-digital (A/D) convertor. In this thesis, $\mathcal{H}_\tau$ is the zero-order *hold operator*: it produces a piecewise constant signal by holding a discrete-time signal constant during the *sampling intervals*, that is,

$$(\mathcal{H}_\tau y_c)(t) := y_c(k)\,, \quad \forall t \in [k\tau, (k+1)\tau)\,.$$

As for the exogenous signals in Figure 1-1, $r$ denotes the reference signal, $d_1$ denotes a plant input disturbance and $d_2$ denotes a plant output disturbance.

This thesis explores how to design a discrete-time controller for a given continuous-time plant such that one or several of the following goals are achieved for the sampled-data feedback system:

- exponential stability,
- input-output stability,
- the output $y$ of the closed-loop system (approximately) tracks certain reference signals $r$ and (approximately) rejects certain external disturbances $d_1$ and $d_2$.

One approach to discrete-time controller design for sampled-data systems, called *indirect sampled-data control*, is to first design a continuous-time controller $K$ to achieve specific performance goals (for example, stability of the state, input-output stability) using continuous-time design methods. A discrete-time controller is obtained by discretization of $K$. There are many methods for this purpose (see, for example, Kowalczuk [30]). One commonly used method is the so-called sample-hold discretization illustrated in Figure 1-2. Here the discretization $K_\tau$ of $K$ is given by $K_\tau := \mathcal{S}_\tau K \mathcal{H}_\tau$. A natural and important question in indirect sampled-data control is whether continuous-time stability and/or performance is recovered as $\tau \to 0$.

In practice, there are several potential technical disadvantages to indirect sampled-data control. For example, the use of indirect sampled-data control may lead to very small sampling period, so that practical implementation may be too expensive or may not even be feasible. Another approach to sampled-data controller design, called *direct sampled-data control*, is to design the discrete-time controller $K_\tau$ directly (see [2], [10], [19] and [57], to name a few references). The obvious advantage is that it solves the problem without approximation. The disadvantage is that this approach is more difficult since sampled-data systems are time-varying .

In this thesis, we focus on a number of issues in indirect sampled-data control: stabilization of linear finite-dimensional systems (Chapters 3-5) and (approximate) tracking and disturbance rejection for stable infinite-dimensional systems using low-gain controllers (Chapters 6 and 7).

This thesis is organised as follows: Chapter 2 contains some preliminaries used throughout the thesis. In Chapter 3, we study the sampled-data systems obtained from state feedback controlled continuous-time systems by sample-hold discretization. We discuss the relationship between exponential growths, transient bounds and trajectories of the continuous-time state feedback system and the corresponding sampled-data system.

In Chapter 4, we extend the results in Chapter 3 to dynamic output feedback systems. We also use state-space method to show that, for an exponentially stable dynamic output feedback system, if the sampling period $\tau$ is sufficiently small, then the corresponding sampled-data system is exponentially stable and input-output stable in the sense that the $L^p$-norm of the output is bounded by the sum of the $W^{1,p}$-norm of the input for $1 \leq p \leq \infty$ and the Euclidean norm of the initial data.

In Chapter 5, we study sampled-data state feedback systems with time-varying sampling period. We develop an approach, which is based on an adaptive law for adjusting the sampling period, to achieve the stability of the state feedback sampled-data systems. This adaptive approach is extended to static and dynamic output feedback.

In Chapter 6, we first show that, for power stable infinite-dimensional discrete-time systems, the application of a certain discrete-time low-gain controller (depending on only one gain parameter) leads to a stable closed-loop system which asymptotically tracks reference signal $r$ of the form $r(k) = \sum_{j=1}^{N} \lambda_j^k \mathfrak{r}_j$ where $\mathfrak{r}_j \in \mathbb{C}^p$ and $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$ for $j = 1, \ldots, N$. The closed-loop system also rejects disturbance signals which are asymptotically equal to signals of this form. The discrete-time results are used to derive results on approximate tracking and disturbance rejection for a large class of infinite-dimensional systems, using sampled-data control. The reference signals are finite sums of sinusoids, and disturbance signals are asymptotic to finite sums of sinusoids. The results are given for both input-output systems and state space systems. For purpose of illustration, a number of examples and simulations are included.

One of the main issues in low-gain control as developed in Chapter 6 is the tuning of the gain parameter. For the case of integral control, this issue is addressed in Chapter 7: after a detailed analysis of adaptive discrete-time low-gain integral control of infinite-dimensional, multivariable, discrete-time, power-stable systems, the discrete-time results are applied in the development of adaptive sampled-data low-gain control for infinite-dimensional, multivariable, well-posed, exponentially stable systems.

Finally, we mention that some of the results from Chapters 6 and 7 have been submitted for publication ([24]-[27]). Another manuscript containing results from Chapters 4 and 5 is in preparation [20].

**Acknowledgments:**

I thank my supervisor Professor Hartmut Logemann for his valuable advice and criticism. I also thank Professor Achim Ilchmann, Professor Richard Rebarber and Professor Eugene Ryan, for their fruitful discussions. I express my gratitude to my colleagues James, Ray, Zhivko, CF, Phil and all the staff in the Department of Mathematical Sciences at University of Bath, and to the financial support of the Overseas Research Scholarship (ORS). Finally, special thanks to Diana and Ainley, my parents and Xuan for their support over my five years' stay in the U.K..

# Chapter 2

# Preliminaries

In this chapter, we collect a number of preliminary results used in this thesis.

## 2.1 Exponential rates, exponential growth and transient bounds

We consider the following continuous-time linear system

$$\dot{x}(t) = Ax(t)\,, \quad \forall t \geq 0\,; \quad x(0) = x^0 \in \mathbb{R}^n\,, \tag{2.1}$$

where $A \in \mathbb{R}^{n \times n}$.

**Definition 2.1.1.** A number $\alpha \in \mathbb{R}$ is said to be an *exponential rate* of system (2.1) (or of $A$) if there exists $M \geq 1$ such that

$$\|e^{At}\| \leq Me^{\alpha t}\,, \quad \forall t \geq 0\,. \tag{2.2}$$

We define the *exponential growth* $\omega$ of system (2.1) (or of $A$) by

$$\omega := \inf\{\alpha : \alpha \text{ is an exponential rate of system (2.1) (or of } A)\}\,.$$

We say that system (2.1) is *exponentially stable* if and only if $\omega < 0$. $\diamond$

It is well known that

$$\omega = \max\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}\,,$$

where $\sigma(A)$ is the spectrum of $A$.

Trajectories of an exponentially stable linear system may temporarily move a long way from the origin before approaching it as $t \to \infty$. From a practical point of view, if the "state excursions" are very large, the stable system actually behaves like an unstable one. We use the concept of *transient bound* as to quantify the transient behaviour. This concept was introduced by Hinrichsen and Pritchard (see [18, Section 5.5, p. 648]).

**Definition 2.1.2.** If $\alpha$ is an exponential rate of system (2.1), then

$$M(\alpha) := \inf\{M \geq 1 : (2.2) \text{ holds}\}$$

is the *transient bound* of system (2.1), associated with the exponential rate $\alpha$. $\diamond$

It is clear that

$$\omega < \alpha \leq \beta \quad \Longrightarrow \quad M(\beta) \leq M(\alpha) . \tag{2.3}$$

**Remark 2.1.3.** Hinrichsen and Pritchard also discussed the interplay between exponential rates and the associated transient bounds. They showed that if $\omega < \alpha < \beta$ and $M(\beta) > 1$, then $M(\beta) < M(\alpha)$ (see [18], p. 650). $\diamond$

The following proposition seems to be new.

**Proposition 2.1.4.** *Let $\omega$ be the exponential growth of system* (2.1)*. The function*

$$(\omega, \infty) \to [1, \infty) , \quad \alpha \mapsto M(\alpha)$$

*is continuous.*

*Proof.* Let $x(\,\cdot\,; x^0)$ denote the solution of system (2.1) and let $\alpha > \omega$. First we show left continuity. There exists $\delta_1 > 0$ such that $\alpha - 2\delta_1 > \omega$. By the definition of the transient bound, we have

$$\|x(t; x^0)\| = \|e^{At}x^0\| \leq M(\alpha - 2\delta_1)\, e^{-\delta_1 t} e^{(\alpha-\delta_1)t}\|x^0\| , \quad \forall t \geq 0 , \ \forall x^0 \in \mathbb{R}^n . \tag{2.4}$$

By (2.3), $M(\alpha) \leq M(\alpha - 2\delta_1)$. Trivially, there exists $T \geq 0$ such that

$$M(\alpha - 2\delta_1)e^{-\delta_1 t} \leq M(\alpha) , \quad \forall t \geq T .$$

Hence, it follows from (2.4) that

$$\|x(t; x^0)\| \leq M(\alpha)e^{(\alpha-\delta_1)t}\|x^0\| , \quad \forall t \geq T , \ \forall x^0 \in \mathbb{R}^n . \tag{2.5}$$

For every $\varepsilon > 0$, there exists $\delta_2 \in (0, \delta_1)$ such that, if $\delta \in (0, \delta_2)$, then

$$M(\alpha)e^{\delta T} \leq M(\alpha) + \varepsilon .$$

Hence, for every $\delta \in (0, \delta_2)$,

$$\begin{aligned} \|x(t; x^0)\| &\leq M(\alpha)\, e^{\delta t} e^{(\alpha-\delta)t}\|x^0\| \\ &\leq M(\alpha)e^{\delta T} e^{(\alpha-\delta)t}\|x^0\| \\ &\leq (M(\alpha) + \varepsilon)e^{(\alpha-\delta)t}\|x^0\| , \quad \forall t \in [0, T] , \ \forall x^0 \in \mathbb{R}^n . \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we see that if $\delta \in (0, \delta_2)$, then

$$\|x(t; x^0)\| \leq (M(\alpha) + \varepsilon)e^{(\alpha-\delta)t}\|x^0\| , \quad \forall t \geq 0 , \ \forall x^0 \in \mathbb{R}^n .$$

Therefore, we conclude that, for every $\varepsilon > 0$, there exists $\delta_2 > 0$ such that if $\delta \in (0, \delta_2)$, then

$$M(\alpha) \leq M(\alpha - \delta) \leq M(\alpha) + \varepsilon\,,$$

where the first inequality follows from (2.3). Hence the mapping $\alpha \mapsto M(\alpha)$ is left continuous.

It remains to show right continuity. We consider two cases.

*Case 1:* $M(\alpha) = 1$.

Then we have

$$\|x(t; x^0)\| \leq e^{\alpha t} \|x^0\|\,, \quad \forall t \geq 0\,,\ \forall x^0 \in \mathbb{R}^n\,.$$

It is clear that

$$\|x(t; x^0)\| \leq e^{(\alpha+\delta)t} \|x^0\|\,, \quad \forall t \geq 0\,,\ \forall x^0 \in \mathbb{R}^n\,,\ \forall \delta > 0\,.$$

Thus $M(\alpha + \delta) = M(\alpha) = 1$ for all $\delta > 0$, showing the right continuity in this case.

*Case 2:* $M(\alpha) > 1$.

Seeking a contradiction, suppose that the function $\alpha \mapsto M(\alpha)$ is not right continuous. Then, by (2.3), there exist $M < M(\alpha)$ and a sequence $(\delta_k)_{k \in \mathbb{Z}_+} \subset (0, \infty)$ satisfying $\lim_{k\to\infty} \delta_k = 0$, such that $M(\alpha + \delta_k) \leq M$ for all $k \in \mathbb{Z}_+$. By the definition of the transient bound, we have

$$\|x(t; x^0) e^{-\delta_k t}\| \leq M(\alpha + \delta_k) e^{\alpha t} \|x^0\| \leq M e^{\alpha t} \|x^0\|\,, \quad \forall t \geq 0\,,\ \forall x^0 \in \mathbb{R}^n\,,\ \forall k \in \mathbb{Z}_+\,.$$

Letting $k \to \infty$, we see that

$$\|x(t; x^0)\| \leq M e^{\alpha t} \|x^0\|\,, \quad \forall t \geq 0\,,\ \forall x^0 \in \mathbb{R}^n\,,$$

contradicting the definition of $M(\alpha)$. This proves the right continuity in this case. $\square$

## 2.2 Power stability, power rates and power growth

Let $X$ be a Banach space and let $A \in \mathcal{B}(X)$, a bounded linear operator on $X$. It is well known that

$$r(A) = \lim_{k\to\infty} \|A^k\|^{1/k}\,, \tag{2.7}$$

where $r(A)$ denotes the spectral radius of $A$ (see, for example, [66, Theorem 18.9, p. 360]). We say that $A$ is *power stable* if and only if $\lim_{k\to\infty} A^k = 0$.

The following theorem is well-known. We provide a proof for completeness.

**Proposition 2.2.1.** *The following statements are equivalent:*

(1) $A \in \mathcal{B}(X)$ *is power stable;*

(2) *there exist* $\rho \in (0,1)$ *and* $M \geq 1$ *such that* $\|A^k\| \leq M\rho^k$ *for all* $k \in \mathbb{Z}_+$*;*

(3) $r(A) < 1$*;*

(4) $z \mapsto (zI - A)^{-1} \in H^\infty_<(\mathbb{E}_1, \mathcal{B}(X))$.

*Proof.* We first show (1) $\Rightarrow$ (3). If $\lim_{k\to\infty} A^k = 0$, then there exists $k_0 \in \mathbb{Z}_+$ such that $\|A^{k_0}\| =: q \in [0,1)$. Hence

$$\|A^{k_0 n}\|^{1/(k_0 n)} \leq (\|A^{k_0}\|^n)^{1/(k_0 n)} = \|A^{k_0}\|^{1/k_0} = q^{1/k_0} < 1\,, \quad \forall n \in \mathbb{Z}_+\,.$$

Letting $n \to \infty$ and using (2.7) proves that $r(A) < 1$.

We next show (3) $\Rightarrow$ (2). Assume that $r(A) < 1$. Let $\rho \in (r(A), 1)$. By (2.7), there exists $k_0 \in \mathbb{Z}_+$ such that $\|A^k\|^{1/k} \leq \rho$ if $k \geq k_0$. Choosing $M \geq 1$ such that $\|A^k\|^{1/k} \leq M^{1/k}\rho$ if $k = 0, \dots, k_0$, we conclude that Statement (2) holds.

Trivially, (2) $\Rightarrow$ (1). Finally, we show that (3) $\Leftrightarrow$ (4). Assume that $r(A) < 1$. There exists $\alpha \in (r(A), 1)$ such that $\mathrm{cl}(\mathbb{E}_\alpha)$ is contained in the resolvent set of $A$. Note that $\lim_{z\to\infty}(zI - A)^{-1} = 0$. Hence $(zI - A)^{-1}$ is bounded for all $z \in \mathbb{E}_\beta$ for some $\beta > 1$. Clearly, $z \mapsto (zI - A)^{-1}$ is bounded on the compact annulus $\mathrm{cl}(\mathbb{E}_\alpha) \setminus \mathbb{E}_\beta$, showing that $z \mapsto (zI - A)^{-1}$ is bounded on $\mathbb{E}_\alpha$. Moreover, it is a standard result that $z \mapsto (zI - A)^{-1}$ is holomorphic at every point of the resolvent set of $A$ (see, for example, [29], p. 389, Theorem 7.5-2) so that $z \mapsto (zI - A)^{-1}$ is holomorphic on $\mathbb{E}_\alpha$. We conclude that $z \mapsto (zI - A)^{-1} \in H^\infty(\mathbb{E}_\alpha, \mathcal{B}(X))$. The proof for (4) $\Rightarrow$ (3) can be found in [31, Lemma 1]. □

**Remark 2.2.2.** Logemann showed that if $z \mapsto (zI - A)^{-1} \in H^\infty(\mathbb{E}_1, \mathcal{B}(X))$, then $r(A) < 1$ (see [31, Lemma 1]). This assumption is weaker than Statement (4). ◇

Consider the discrete-time system

$$x(k+1) = Ax(k)\,, \quad \forall k \in \mathbb{Z}_+\,; \quad x(0) = x^0 \in X\,. \tag{2.8}$$

**Definition 2.2.3.** A number $\rho > 0$ is said to be a *power rate* of system (2.8) (or of $A$) if there exists $M \geq 1$ such that

$$\|A^k\| \leq M\rho^k\,, \quad \forall k \in \mathbb{Z}_+\,.$$

The *power growth* of system (2.8) (or of $A$) is defined to be

$$\inf\{\rho : \rho \text{ is a power rate of system (2.8) (or of } A)\}\,.$$

We say that system (2.8) is *power stable* if and only if $A$ is power stable. ◇

Invoking (2.7) and a routine argument, it can be shown that $r(A)$ is equal to the power growth of system (2.8).

## 2.3 Properties of sampling and hold operators

In this section, we discuss the boundedness properties of the sampling and hold operators $\mathcal{S}_\tau$ and $\mathcal{H}_\tau$, which are fundamental in the study of sampled-data systems.

Let $F(\mathbb{R}_+, \mathbb{C}^n)$ and $F(\mathbb{Z}_+, \mathbb{C}^n)$ denote the spaces of all $\mathbb{C}^n$-valued functions defined on $\mathbb{R}_+$ and $\mathbb{Z}_+$, respectively.

**Definition 2.3.1.** Let $\tau > 0$ denote the sampling period. We define the *ideal sampling operator* $\mathcal{S}_\tau \colon F(\mathbb{R}_+, \mathbb{C}^n) \to F(\mathbb{Z}_+, \mathbb{C}^n)$ by

$$(\mathcal{S}_\tau u)(k) := u(k\tau)\,, \quad \forall k \in \mathbb{Z}_+\,.$$

The (zero-order) *hold operator* $\mathcal{H}_\tau \colon F(\mathbb{Z}_+, \mathbb{C}^n) \to F(\mathbb{R}_+, \mathbb{C}^n)$ is defined by

$$(\mathcal{H}_\tau v)(t) := v(k)\,, \quad \forall t \in [k\tau, (k+1)\tau)\,,$$

that is, $\mathcal{H}_\tau$ converts the discrete-time signal $v$ into a piecewise continuous function by holding it constant over the sampling intervals. $\diamond$

Let $L_b(\mathbb{R}_+, \mathbb{C}^n)$ denote the space of bounded Lebesgue measurable functions with the sup-norm $\|\cdot\|_\infty$ on $\mathbb{R}_+$ and let $C(\mathbb{R}_+, \mathbb{C}^n)$ denote the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{C}^n$. The following proposition shows that $\mathcal{H}_\tau$ has the nice property that, by suitable scaling, it is norm preserving from $\ell^p(\mathbb{Z}_+, \mathbb{C}^n)$ to $L^p(\mathbb{R}_+, \mathbb{C}^n)$ for all $1 \le p \le \infty$.

**Proposition 2.3.2.** *For $1 \le p \le \infty$, $\mathcal{H}_\tau \colon \ell^p(\mathbb{Z}_+, \mathbb{C}^n) \to L^p(\mathbb{R}_+, \mathbb{C}^n)$ is bounded. Moreover,*

$$\|\mathcal{H}_\tau v\|_{L^p} = \tau^{1/p}\|v\|_{\ell^p}\,, \quad \forall v \in \ell^p(\mathbb{Z}_+, \mathbb{C}^n)\,,\ \forall 1 \le p < \infty\,,$$

*and*

$$\|\mathcal{H}_\tau v\|_{L^\infty} = \|v\|_{\ell^\infty}\,, \quad \forall v \in \ell^\infty(\mathbb{Z}_+, \mathbb{C}^n)\,.$$

*Thus $\mathcal{H}_\tau \colon \ell^\infty(\mathbb{Z}_+, \mathbb{C}^n) \to L^\infty(\mathbb{R}_+, \mathbb{C}^n)$ is an isometry.*

*The sampling operator $\mathcal{S}_\tau \colon L_b(\mathbb{R}_+, \mathbb{C}^n) \to \ell^\infty(\mathbb{Z}_+, \mathbb{C}^n)$ is bounded and of norm 1.*

The proof is simple and can be found in [2, Theorem 9.3.1, p. 211]. A counter-example given in the proof of [2, Theorem 9.3.1] shows that the sampling operator $\mathcal{S}_\tau$ is not well defined from $L^p(\mathbb{R}_+, \mathbb{C}^n) \cap C(\mathbb{R}_+, \mathbb{C}^n)$ to $\ell^p(\mathbb{Z}_+, \mathbb{C}^n)$ for $1 \le p < \infty$:

**Example 2.3.3.** Define $u \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$u(t) := \sum_{k=1}^{\infty} v_k(t)\,, \quad \text{where} \quad v_k(t) := \begin{cases} 1 - \dfrac{2k^2}{\tau}|t - k\tau|\,, & \text{if } |t - k\tau| < \dfrac{\tau}{2k^2} \\ 0\,, & \text{otherwise} \end{cases}\,,$$

as shown in Figure 2-1. It is clear that $u \in L^p(\mathbb{R}_+, \mathbb{R}) \cap C(\mathbb{R}_+, \mathbb{R})$ for $1 \le p < \infty$. However, $\mathcal{S}_\tau u \notin \ell^p$ since $(\mathcal{S}_\tau u)(k) = 1$ for all $k \in \mathbb{N}$.

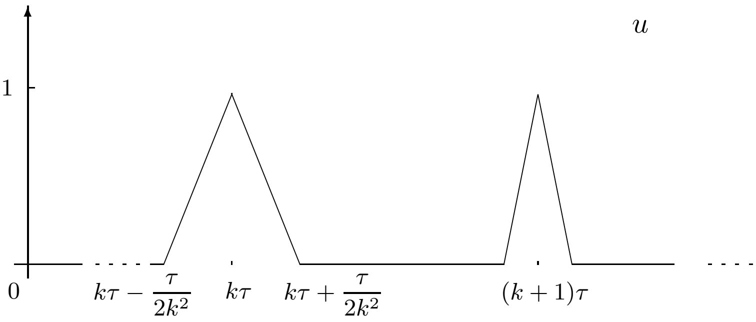


Figure 2-1: Function u.

We next show that $\mathcal{S}_\tau$ is bounded from the Sobolev spaces $W^{1,p}(\mathbb{R}_+,\mathbb{C}^n)$ to $\ell^p(\mathbb{Z}_+,\mathbb{C}^n)$ for $1 \le p \le \infty$.

**Definition 2.3.4.** For $p \in [1,\infty]$, we define

$$W^{1,p}(\mathbb{R}_+,\mathbb{C}^n) := \{u : \mathbb{R}_+ \to \mathbb{C}^n \mid u \text{ is absolutely continuous }, u, \dot{u} \in L^p(\mathbb{R}_+,\mathbb{C}^n)\}\,,$$

where $\dot{u}$ denotes the classical derivative of $u$ (which exists almost everywhere). The $W^{1,p}$-norm is defined by

$$\begin{aligned} \|u\|_{W^{1,p}} &:= \left(\int_0^\infty \|u(s)\|^p ds + \int_0^\infty \|\dot{u}(s)\|^p ds\right)^{1/p}, \quad \text{for } p \in [1,\infty)\,, \\ \|u\|_{W^{1,\infty}} &:= \max\{\|u\|_{L^\infty}, \|\dot{u}\|_{L^\infty}\}\,. \end{aligned}$$

◇

**Theorem 2.3.5.** *The sampling operator $\mathcal{S}_\tau$ is bounded from $W^{1,p}(\mathbb{R}_+,\mathbb{C}^n)$ to $\ell^p(\mathbb{Z}_+,\mathbb{C}^n)$ for every $1 \le p \le \infty$. In particular,*

$$\|\mathcal{S}_\tau u\|_{\ell^p} \le M(p,n,\tau)\|u\|_{W^{1,p}}\,, \quad \forall u \in W^{1,p}(\mathbb{R}_+,\mathbb{C}^n)\,,$$

*where*

$$M(p,n,\tau) = \begin{cases} 2^{1-\frac{1}{p}}\sqrt{n}(\tau^{p-1}+\tau^{-1})^{\frac{1}{p}}\,, & p \in [1,\infty) \\ 1\,, & p = \infty \end{cases}.$$

*Proof.* If $p = \infty$, then, by Proposition 2.3.2,

$$\|\mathcal{S}_\tau u\|_{\ell^\infty} \le \|u\|_{L^\infty} \le \|u\|_{W^{1,\infty}}\,, \quad \forall u \in W^{1,\infty}(\mathbb{R}_+,\mathbb{C}^n)\,.$$

Assume that $p \in [1,\infty)$ and let $u \in W^{1,p}(\mathbb{R}_+,\mathbb{C}^n)$. Writing $u = (u_1,\ldots,u_n)^T$, it follows that $u_j \in W^{1,p}(\mathbb{R}_+,\mathbb{C})$ for $j = 1,\ldots,n$. Define a mapping $f\colon \mathbb{R}_+^n \to \mathbb{C}^n$ by

$$f((t_1,\ldots,t_n)^T) := (u_1(t_1),\ldots,u_n(t_n))^T\,.$$

By the continuity of $|u_j|$ on $\mathbb{R}_+$ and the mean-value theorem of integration, there exist

$\xi_{j,k} \in [k\tau, (k+1)\tau]$ such that

$$|u_j(\xi_{j,k})| = \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} |u_j(s)| ds \,, \quad \forall k \in \mathbb{Z}_+ \,, \; \forall j = 1, \ldots, n \,. \tag{2.9}$$

Trivially,

$$(\alpha_1^2 + \ldots + \alpha_n^2)^{1/2} \leq \alpha_1 + \ldots + \alpha_n \,, \quad \forall \alpha_1, \ldots, \alpha_n \geq 0 \,. \tag{2.10}$$

Moreover, a routine application of the Cauchy-Schwarz inequality (in $\mathbb{R}^n$) yields that

$$\sum_{j=1}^{n} \alpha_j \leq \sqrt{n} \left( \sum_{j=1}^{n} \alpha_j^2 \right)^{1/2} \,, \quad \forall \alpha_1, \ldots, \alpha_n \geq 0 \,. \tag{2.11}$$

Setting $\xi_k := (\xi_{1,k}, \ldots, \xi_{n,k})^T$ for $k \in \mathbb{Z}_+$, by (2.9)–(2.11), we have

$$\begin{aligned} \|f(\xi_k)\| = \left( \sum_{j=1}^{n} |u_j(\xi_{j,k})|^2 \right)^{1/2} &\leq \sum_{j=1}^{n} |u_j(\xi_{j,k})| \\ &= \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} \left( \sum_{j=1}^{n} |u_j(s)| \right) ds \\ &\leq \frac{\sqrt{n}}{\tau} \int_{k\tau}^{(k+1)\tau} \|u(s)\| ds \,, \quad \forall k \in \mathbb{Z}_+ \,. \end{aligned} \tag{2.12}$$

The fundamental theorem of calculus for absolutely continuous functions shows,

$$u_j(\xi_{j,k}) - u_j(k\tau) = \int_{k\tau}^{\xi_{j,k}} \dot{u}_j(s) ds \,, \quad \forall k \in \mathbb{Z}_+ \,, \; \forall j = 1, \ldots, n \,. \tag{2.13}$$

Consequently, by (2.10), (2.11) and (2.13),

$$\begin{aligned} \|f(\xi_k) - u(k\tau)\| &= \left( \sum_{j=1}^{n} |u_j(\xi_{j,k}) - u_j(k\tau)|^2 \right)^{1/2} \\ &\leq \sum_{j=1}^{n} |u_j(\xi_{j,k}) - u_j(k\tau)| \\ &\leq \sum_{j=1}^{n} \int_{k\tau}^{\xi_{j,k}} |\dot{u}_j(s)| ds \\ &\leq \int_{k\tau}^{(k+1)\tau} \sum_{j=1}^{n} |\dot{u}_j(s)| ds \\ &\leq \sqrt{n} \int_{k\tau}^{(k+1)\tau} \|\dot{u}(s)\| ds \,, \quad \forall k \in \mathbb{Z}_+ \,. \end{aligned} \tag{2.14}$$

Since $p \in [1, \infty)$, the function $\mathbb{R}_+ \to \mathbb{R}_+ : x \mapsto x^p$ is convex. Hence

$$\left(\frac{\alpha + \beta}{2}\right)^p \leq \frac{\alpha^p + \beta^p}{2}, \quad \forall \alpha, \beta \geq 0,$$

that is,

$$(\alpha + \beta)^p \leq 2^{p-1}(\alpha^p + \beta^p), \quad \forall \alpha, \beta \geq 0. \tag{2.15}$$

Using (2.12), (2.14), (2.15) and the Hölder's inequality, we obtain that, for $k \in \mathbb{Z}_+$

$$\begin{aligned}
\|u(k\tau)\|^p &\leq (\|f(\xi_k) - u(k\tau)\| + \|f(\xi_k)\|)^p \\
&\leq 2^{p-1}(\|f(\xi_k) - u(k\tau)\|^p + \|f(\xi_k)\|^p) \\
&\leq 2^{p-1}\left[n^{p/2}\left(\int_{k\tau}^{(k+1)\tau} \|\dot{u}(s)\| ds\right)^p + \tau^{-p} n^{p/2}\left(\int_{k\tau}^{(k+1)\tau} \|u(s)\| ds\right)^p\right] \\
&\leq 2^{p-1} n^{p/2}\left[\tau^{p-1}\int_{k\tau}^{(k+1)\tau} \|\dot{u}(s)\|^p ds + \tau^{-1}\int_{k\tau}^{(k+1)\tau} \|u(s)\|^p ds\right] \\
&\leq 2^{p-1} n^{p/2}(\tau^{p-1} + \tau^{-1})\left(\int_{k\tau}^{(k+1)\tau} \|\dot{u}(s)\|^p ds + \int_{k\tau}^{(k+1)\tau} \|u(s)\|^p ds\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathcal{S}_\tau u\|_{\ell^p}^p &= \sum_{k=0}^{\infty} \|u(k\tau)\|^p \\
&\leq 2^{p-1} n^{p/2}(\tau^{p-1} + \tau^{-1}) \sum_{k=0}^{\infty}\left(\int_{k\tau}^{(k+1)\tau} \|\dot{u}(s)\|^p ds + \int_{k\tau}^{(k+1)\tau} \|u(s)\|^p ds\right) \\
&= 2^{p-1} n^{p/2}(\tau^{p-1} + \tau^{-1})\|u\|_{W^{1,p}}^p,
\end{aligned}$$

showing that $\|\mathcal{S}_\tau u\|_{\ell^p} \leq 2^{1-(1/p)}\sqrt{n}(\tau^{p-1} + \tau^{-1})^{1/p}\|u\|_{W^{1,p}}$. □

## 2.4 Infinite-dimensional well-posed systems

In this section, we recall briefly some facts about admissible control and observation operators, infinite-dimensional well-posed linear systems, their input-output operators and transfer functions, which will be useful in Chapter 6 and 7. For the details, we refer to Salamon [67, 68], Staffans [72, 74, 75], Staffans and Weiss [76], or Weiss [80, 81, 83, 82]. These papers offer equivalent definitions of well-posed systems but formulate them quite differently.

The class of well-posed systems captures the systems-theoretic properties of linearity, time-invariance, and causality together with natural continuity properties. It is the largest class of infinite-dimensional systems for which there exists a well-developed state-space and frequency-domain theory. It includes many distributed parameter systems and all time-delay systems which are of interest in applications. Every well-posed system has a well-defined transfer function.

Throughout this section, we consider a well-posed system $\Sigma$ with state-space $X$, input space $U$, and output space $Y$ (all Hilbert spaces), generating operators $(A, B, C)$, input-output operator $G$ and transfer function $\mathbf{G}$. Here $A$ is the generator of a strongly continuous semigroup ($C_0$-semigroup) $\mathbf{T}$ on $X$, $B \in \mathcal{B}(U, X_{-1})$, and $C \in \mathcal{B}(X_1, Y)$, where $X_1$ denotes the domain of $A$, as an operator defined on $X$, endowed with the graph norm $\|x\|_1 := \|x\| + \|Ax\|$, and $X_{-1}$ denotes the completion of $X$ with respect to the norm $\|x\|_{-1} := \|(\beta I - A)^{-1}x\|$. The number $\beta$ is in the resolvent set $\varrho(A)$ of $A$. It can be verified that different choices of $\beta$ lead to equivalent norms. We have $X_1 \hookrightarrow X \hookrightarrow X_{-1}$. It is known that $\mathbf{T}$ restricts to a $C_0$-semigroup on $X_1$ and extends to a $C_0$-semigroup on $X_{-1}$ with the exponential growth constant being the same on all three spaces $X_1$, $X$ and $X_{-1}$. The generator of the restricted (extended) semigroup is a restriction (extension) of $A$. The restricted/extended semigroups and their generators will be denoted by the same symbols $\mathbf{T}$ and $A$, respectively.

The control operator $B$ is *admissible*, that is, for every $t \geq 0$, there exists $b_t \geq 0$ such that

$$\left\| \int_0^t \mathbf{T}(t-s)Bu(s)ds \right\| \leq b_t \|u\|_{L^2}\,, \quad \forall u \in L^2([0,t], U)\,, \tag{2.16}$$

and the observation operator $C$ is also *admissible*, that is, for every $t \geq 0$, there exists $c_t \geq 0$ such that

$$\int_0^t \|C\mathbf{T}(t)z\|^2 dt \leq c_t \|z\|^2\,, \quad \forall z \in X_1\,.$$

It follows from (2.16) that $(sI - A)^{-1}B \in \mathcal{B}(U, X)$ for all $s \in \varrho(A)$. The control operator $B$ is said to be *bounded* if $B \in \mathcal{B}(U, X)$ (and *unbounded* otherwise), whilst $C$ is called *bounded* if it can be extended such that $C \in \mathcal{B}(X, Y)$ (and *unbounded* otherwise).

The so-called $\Lambda$-*extension* of $C$ is defined by

$$C_\Lambda z := \lim_{\lambda \to \infty\,,\, \lambda \in \mathbb{R}} C\lambda(\lambda I - A)^{-1} z\,, \quad \forall z \in \mathrm{dom}(C_\Lambda)\,,$$

where $\mathrm{dom}(C_\Lambda)$ is the set of all $z \in X$ for which the above limit exists. Clearly, $X_1 \subset \mathrm{dom}(C_\Lambda)$. For each $z \in X$, $\mathbf{T}(t)z \in \mathrm{dom}(C_\Lambda)$ for almost all $t \geq 0$ and $C_\Lambda \mathbf{T} z \in L^2_\alpha(\mathbb{R}_+, Y)$ for all $\alpha > \omega(\mathbf{T})$, where

$$\omega(\mathbf{T}) := \lim_{t \to \infty} \frac{1}{t} \ln \|\mathbf{T}(t)\|$$

denotes the exponential growth constant of $\mathbf{T}$. The transfer function $\mathbf{G}$ satisfies

$$\frac{\mathbf{G}(s) - \mathbf{G}(\eta)}{s - \eta} = -C(sI - A)^{-1}(\eta I - A)^{-1}B\,, \quad \forall s, \eta \in \mathbb{C}_{\omega(\mathbf{T})}\,,\; s \neq \eta\,, \tag{2.17}$$

and $\mathbf{G} \in H^\infty(\mathbb{C}_\alpha, \mathcal{B}(U, Y))$ for every $\alpha > \omega(\mathbf{T})$. Moreover, the input-output operator $G : L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ is continuous and shift-invariant; for every $\alpha > \omega(\mathbf{T})$,

$G \in \mathcal{B}(L^2_\alpha(\mathbb{R}_+, U), L^2_\alpha(\mathbb{R}_+, Y))$ and

$$(\mathscr{L}(Gu))(s) = \mathbf{G}(s)(\mathscr{L}(u))(s)\,, \quad \forall s \in \mathbb{C}_\alpha\,,\ \forall u \in L^2_\alpha(\mathbb{R}_+, U)\,.$$

where $\mathscr{L}(u)$ denotes the Laplace transform of function $u$.

For $x^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, let $x$ and $y$ denote the state and output functions of a well-posed system $\Sigma$, respectively, corresponding to the initial condition $x(0) = x^0 \in X$ and the input function $u$. Then

$$x(t) = \mathbf{T}(t)x^0 + \int_0^t \mathbf{T}(t-s)Bu(s)ds\,, \quad \forall t \geq 0\,,$$

and $x(t) - (\eta I - A)^{-1}Bu(t) \in \mathrm{dom}(C_\Lambda)$ for almost all $t \geq 0$. Moreover,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,; \quad x(0) = x^0 \in X\,, \quad \text{for a.a. } t \geq 0\,, && \text{(2.18a)} \\ y(t) &= C_\Lambda[x(t) - (\eta I - A)^{-1}Bu(t)] + \mathbf{G}(\eta)u(t)\,, \quad \text{for a.a. } t \geq 0\,, && \text{(2.18b)} \end{aligned}$$

where $\eta \in \mathbb{C}_{\omega(\mathbf{T})}$ is arbitrary. The differential equation (2.18a) has to be interpreted in $X_{-1}$. We identify $\Sigma$ and (2.18) and refer to (2.18) as a well-posed system. We say that (2.18) is *exponentially stable* if $\mathbf{T}$ is exponentially stable, i.e., $\omega(\mathbf{T}) < 0$.

The well-posed system (2.18) is said to be *regular* if there exists a linear operator $D$ such that

$$\lim_{s\to\infty, s\in\mathbb{R}} \mathbf{G}(s)\mathfrak{u} = D\mathfrak{u}\,, \quad \forall \mathfrak{u} \in U\,.$$

In this case, by the uniform boundedness theorem, $D \in \mathcal{B}(U, Y)$, and $D$ is called the *feedthrough* operator of (2.18b). Moreover, $x(t) \in \mathrm{dom}(C_\Lambda)$ for almost all $t \geq 0$, the output equation (2.18b) can be simplified as

$$y(t) = C_\Lambda x(t) + Du(t)\,, \quad \text{for a.a. } t \geq 0\,,$$

$\mathrm{im}[(sI - A)^{-1}B] \subset \mathrm{dom}(C_\Lambda)$ for all $s \in \varrho(A)$, and

$$\mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B + D\,, \quad \forall s \in \mathbb{C}_{\omega(\mathbf{T})}\,.$$

## 2.5 Notes and references

While Theorem 2.3.5 should be well known, we could not find it in the literature. Our proof here is elementary and seems to be new. Kannai and Weiss showed that $\mathcal{S}_\tau$ is bounded from $W^{s,2}(\mathbb{R}, \mathbb{C})$ to $\ell^2(\mathbb{Z}, \mathbb{C})$ for all $s > 1/2$ (see [23, Proposition 2.1]). Note that in their result, $s$ is allowed to take non-integer values. In this respect, their result is more general than Theorem 2.3.5. On the other hand, in Theorem 2.3.5, the domain of $\mathcal{S}_\tau$ is $W^{1,p}$, where $p$ is allowed to be in the interval $[1, \infty]$, not just for $p = 2$. Closely related to Theorem 2.3.5 is a result by Chen and Francis: they proved that the sampling operator preceded by a filter $F$, i.e., $\mathcal{S}_\tau F$, is bounded from $L^p(\mathbb{R}_+, \mathbb{C}^n)$ to $\ell^p(\mathbb{Z}_+, \mathbb{C}^n)$ for all $1 \leq p \leq \infty$ (see [1, Theorem 1] or [2, Theorem 9.3.2, p. 212]).

# Chapter 3

# Indirect sampled-data control: state feedback

Consider the finite-dimensional continuous-time state feedback system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,; \quad x(0) = x^0 \in \mathbb{R}^n\,, && (3.1a) \\ u(t) &= Fx(t)\,, && (3.1b) \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $F \in \mathbb{R}^{m\times n}$.

Let $\tau > 0$ be the sampling period. Using sampling and hold in (3.1b), we obtain the corresponding sampled-data feedback system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,; \quad x(0) = x^0 \in \mathbb{R}^n\,, && (3.2a) \\ u(t) &= Fx(k\tau)\,, \quad \forall t \in [k\tau, (k+1)\tau)\,. && (3.2b) \end{aligned}$$

In this chapter, we introduce the concepts of exponential rate, exponential growth and transient bound associated with a particular exponential rate for the sampled-data feedback system (3.2). In Section 3.1, it is shown that the exponential growth of system (3.2) approaches the exponential growth of system (3.1) as $\tau \to 0$. In Section 3.2, we derive that the limit superior (as $\tau \to 0$) of the transient bound of system (3.2) associated with the exponential rate $\alpha$ of (3.2) is less than the transient bound of (3.1) associated with $\alpha$. Section 3.3 deals with the convergence of the solution of the sampled-data system (3.2) as $\tau \to 0$.

Throughout this chapter, let $x(\cdot\,; x^0, \tau)$ denote the unique solution of the sampled-data system (3.2).

## 3.1 Exponential growth

First we generalize Definition 2.1.1 to the sampled-data system (3.2).

**Definition 3.1.1.** A number $\alpha \in \mathbb{R}$ is said to be an *exponential rate* of system (3.2) if there exists $M \geq 1$ such that

$$\|x(t; x^0, \tau)\| \leq M e^{\alpha t} \|x^0\|, \quad \forall t \geq 0, \, \forall x^0 \in \mathbb{R}^n. \tag{3.3}$$

We define the *exponential growth* $\omega_s(\tau)$ of system (3.2) by

$$\omega_s(\tau) = \inf\{\alpha : \alpha \text{ is an exponential rate of system (3.2)}\}.$$

We say that system (3.2) is *exponentially stable* if and only if $\omega_s(\tau) < 0$. ◇

By the variation-of-parameters formula, the solution $x(\,\cdot\,; x^0, \tau)$ of (3.2) satisfies

$$x(k\tau + \theta; x^0, \tau) = \left(e^{A\theta} + \int_0^\theta e^{As} ds BF\right) x(k\tau; x^0, \tau), \quad \forall \theta \in [0, \tau], \, \forall k \in \mathbb{Z}_+. \tag{3.4}$$

Define $x_k := x(k\tau; x^0, \tau)$ for all $k \in \mathbb{Z}_+$. It follows from (3.4) with $\theta = \tau$ that

$$x_{k+1} = \Delta_\tau x_k = \Delta_\tau^{k+1} x^0, \quad \forall k \in \mathbb{Z}_+, \tag{3.5}$$

where

$$\Delta_\tau := e^{A\tau} + \int_0^\tau e^{As} ds BF. \tag{3.6}$$

We know that the spectral radius $r(\Delta_\tau)$ of $\Delta_\tau$ is the power growth of system (3.5).

**Theorem 3.1.2.** *The number $\rho > 0$ is a power rate of system* (3.5) *if and only if* $(\ln \rho)/\tau$ *is an exponential rate of system* (3.2)*. Consequently,*

$$\omega_s(\tau) = \frac{1}{\tau} \ln(r(\Delta_\tau)).$$

*Proof.* If $\rho$ is a power rate of system (3.5), then there exists $M_1 \geq 1$ such that

$$\|\Delta_\tau^k\| \leq M_1 \rho^k, \quad \forall k \in \mathbb{Z}_+.$$

Setting $M_2 := \max_{\theta \in [0,\tau]} \|e^{A\theta} + \int_0^\theta e^{As} ds BF\|$, it follows from (3.4) and (3.5) that

$$\|x(k\tau + \theta; x^0, \tau)\| \leq \left\|e^{A\theta} + \int_0^\theta e^{As} ds BF\right\| \|x_k\| \leq M_2 \|\Delta_\tau^k\| \|x^0\| \leq M_1 M_2 \rho^k \|x^0\|.$$

*Case 1*: $\rho \geq 1$.

Then, since $\rho^{\theta/\tau} \geq 1$,

$$\begin{aligned} \|x(k\tau + \theta; x^0, \tau)\| \;\leq\; & M_1 M_2 \rho^k \rho^{\theta/\tau} \|x^0\| = M_1 M_2 e^{((\ln \rho)/\tau)(k\tau + \theta)} \|x^0\|, \\ & \forall \theta \in [0, \tau), \, \forall k \in \mathbb{Z}_+, \, \forall x^0 \in \mathbb{R}^n. \end{aligned}$$

*Case 2*: $0 < \rho < 1$.

Then, since $\rho^{(\theta/\tau)-1} \geq 1$,

$$\begin{aligned}\|x(k\tau+\theta;x^0,\tau)\| &\leq M_1 M_2 \rho^{-1} \rho^{(k\tau+\theta)/\tau} \|x^0\| = M_1 M_2 \rho^{-1} e^{((\ln\rho)/\tau)(k\tau+\theta)} \|x^0\| \,, \\ &\qquad \forall \theta \in [0,\tau)\,,\ \forall k \in \mathbb{Z}_+\,,\ \forall x^0 \in \mathbb{R}^n\,,\end{aligned}$$

Combining the above two cases, we conclude that $(\ln\rho)/\tau$ is an exponential rate of system (3.2).

Conversely, if $(\ln\rho)/\tau$ is an exponential rate of system (3.2), then, by Definition 3.1.1 and (3.5), there exists $M \geq 1$ such that

$$\|\Delta_\tau^k x^0\| = \|x(k\tau;x^0,\tau)\| \leq M\rho^k \|x^0\|\,, \quad \forall k \in \mathbb{Z}_+\,,\ \forall x^0 \in \mathbb{R}^n\,,$$

showing that $\|\Delta_\tau^k\| \leq M\rho^k$ for all $k \in \mathbb{Z}_+$. This proves that $\rho$ is a power rate of system (3.5). Taking infima, we have

$$\omega_s(\tau) = \ln(r(\Delta_\tau))/\tau\,. \qquad \square$$

**Remark 3.1.3.** It is clear that $\alpha$ is an exponential rate of (3.2) if and only if $e^{\alpha\tau}$ is a power rate of (3.5). $\diamond$

**Corollary 3.1.4.** *We have*

$$\lim_{t\to\infty} x(t;x^0,\tau) = 0\,, \quad \forall x^0 \in \mathbb{R}^n\,,$$

*if and only if* $\omega_s(\tau) < 0$.

*Proof.* The sufficiency is obvious. For the necessity, assume that $\lim\limits_{t\to\infty} x(t;x^0,\tau) = 0$ for all $x^0 \in \mathbb{R}^n$. Hence, by (3.5),

$$\lim_{k\to\infty} x(k\tau;x^0,\tau) = \lim_{k\to\infty} \Delta_\tau^k x^0 = 0\,, \quad \forall x^0 \in \mathbb{R}^n\,,$$

showing that $\lim_{k\to\infty} \Delta_\tau^k = 0$. Therefore, it follows from Theorem 2.2.1 that $r(\Delta_\tau) < 1$. By Theorem 3.1.2, it follows that $\omega_s(\tau) < 0$. $\square$

Let $\omega_c$ denote the exponential growth of (3.1). Observe that when $\tau$ tends to 0, system (3.2) seems to approximate system (3.1). Does $\omega_s(\tau)$ tend to $\omega_c$ as $\tau \to 0$? The following theorem shows that the answer is yes.

**Theorem 3.1.5.** *Let $\omega_c$ and $\omega_s(\tau)$ denote the exponential growths of state feedback system* (3.1) *and corresponding sampled-data system* (3.2)*, respectively. Then*

$$\lim_{\tau\to 0} \omega_s(\tau) = \lim_{\tau\to 0} \frac{1}{\tau} \ln(r(\Delta_\tau)) = \omega_c\,,$$

*where $\Delta_\tau$ is defined in* (3.6).

*Proof.* We show first that

$$\limsup_{\tau \to 0} \frac{1}{\tau} \ln(r(\Delta_\tau)) \leq \omega_c \,. \tag{3.7}$$

Using the power series expansion of $e^{At}$, we obtain

$$\Delta_\tau = e^{A\tau} + \int_0^\tau e^{As} ds BF = I + \tau(A + BF + P(\tau)) \,, \tag{3.8}$$

where

$$P(\tau) = \frac{\tau}{2} A(A+BF) + \frac{\tau^2}{3!} A^2(A+BF) + \ldots + \frac{\tau^{j-1}}{j!} A^{j-1}(A+BF) + \ldots \,.$$

Let $\lambda_\tau \in \sigma(\Delta_\tau)$. By (3.8), we see that $\lambda_\tau$ is of the form $\lambda_\tau = 1 + \tau\mu_\tau$, where $\mu_\tau \in \sigma(A + BF + P(\tau))$. Hence,

$$|\lambda_\tau|^2 = |1 + \tau\mu_\tau|^2 = (1 + \tau \mathrm{Re}\, \mu_\tau)^2 + (\tau \mathrm{Im}\, \mu_\tau)^2 = 1 + \tau(2\mathrm{Re}\, \mu_\tau + \tau|\mu_\tau|^2) \,. \tag{3.9}$$

By the definition of $\omega_c$, $\sigma(A+BF) \subset \{s \in \mathbb{C} : \mathrm{Re}\, s \leq \omega_c\}$. Note that $\lim_{\tau \to 0} P(\tau) = 0$. By perturbation theory, the mapping $A \mapsto \sigma(A)$ is continuous in the sense of [18] (see [18, Corollary 4.2.1, p. 399]). In particular, for every $\varepsilon > 0$, there exists $\tau_1 = \tau_1(\varepsilon) > 0$ such that

$$\sigma(A + BF + P(\tau)) \subset \left\{ s \in \mathbb{C} : \mathrm{Re}\, s < \omega_c + \frac{\varepsilon}{4} \right\} , \quad \forall \tau \in (0, \tau_1) \,.$$

Hence, $2\mathrm{Re}\, \mu_\tau < 2\omega_c + \varepsilon/2$ for all $\tau \in (0, \tau_1)$. Consequently, there exists $\tau_2 \in (0, \tau_1)$ such that

$$2\mathrm{Re}\, \mu_\tau + \tau|\mu_\tau|^2 < 2\omega_c + \varepsilon \,, \quad \forall \tau \in (0, \tau_2) \,.$$

Thus, by (3.9), $|\lambda_\tau|^2 < 1 + \tau(2\omega_c + \varepsilon)$ for all $\tau \in (0, \tau_2)$. Hence

$$r(\Delta_\tau)^2 < 1 + \tau(2\omega_c + \varepsilon) \,, \quad \forall \tau \in (0, \tau_2) \,.$$

Then

$$\frac{\ln(r(\Delta_\tau))}{\tau} = \frac{\ln(r(\Delta_\tau)^2)}{2\tau} < \frac{\ln(1 + \tau(2\omega_c + \varepsilon))}{2\tau} = \omega_c + \frac{\varepsilon}{2} + E(\tau) \,, \quad \forall \tau \in (0, \tau_2)$$

where $\lim_{\tau \to 0} E(\tau) = 0$. Therefore, for every $\varepsilon > 0$, there exists $\tau_3 \in (0, \tau_2)$ such that

$$\frac{\ln r(\Delta_\tau)}{\tau} < \omega_c + \varepsilon \,, \quad \forall \tau \in (0, \tau_3) \,,$$

showing that (3.7) is true.

Furthermore, we claim that

$$\liminf_{\tau \to 0} \frac{1}{\tau} \ln(r(\Delta_\tau)) \geq \omega_c \,. \tag{3.10}$$

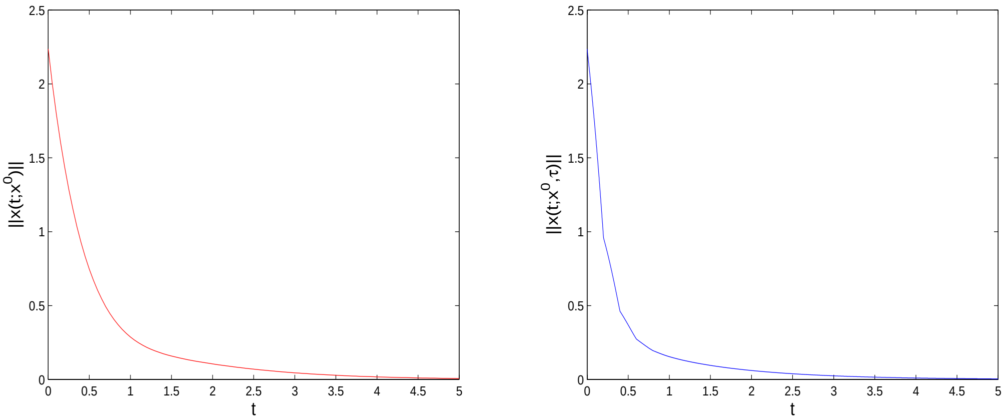


Figure 3-1: Indirect sampled-data state feedback control.

It follows from perturbation theory that, for every $\varepsilon > 0$, there exist $\tau_4 = \tau_4(\varepsilon) > 0$ and $\tilde{\mu}_\tau \in \sigma(A + BF + P(\tau))$ such that

$$\mathrm{Re}\,\tilde{\mu}_\tau > \omega_c - \frac{\varepsilon}{2}\,, \quad \forall \tau \in (0, \tau_4)\,.$$

Let $\tilde{\lambda}_\tau := 1 + \tau\tilde{\mu}_\tau$. By (3.8), we know that $\tilde{\lambda}_\tau \in \sigma(\Delta_\tau)$. By (3.9),

$$r(\Delta_\tau)^2 \geq |\tilde{\lambda}_\tau|^2 > 1 + 2\tau\mathrm{Re}\,\tilde{\mu}_\tau > 1 + \tau(2\omega_c - \varepsilon)\,, \quad \forall \tau \in (0, \tau_4)\,.$$

Using the same argument as above, for every $\varepsilon > 0$, there exists $\tau_5 \in (0, \tau_4)$ such that

$$\frac{\ln r(\Delta_\tau)}{\tau} > \omega_c - \varepsilon\,, \quad \forall \tau \in (0, \tau_5)\,.$$

showing that (3.10) is true. Combining (3.7) and (3.10), we have $\lim_{\tau\to 0} \ln(r(\Delta_\tau))/\tau = \omega_c$. Invoking Theorem 3.1.2 completes the proof. □

The following corollary is a direct consequence of Theorem 3.1.5.

**Corollary 3.1.6.** *Assume that the state feedback system* (3.1) *is exponentially stable. Then there exists $\tau^* > 0$ such that, for all $\tau \in (0, \tau^*)$, the sampled-data feedback system* (3.2) *is exponentially stable.*

We give a simple example to illustrate Corollary 3.1.6.

**Example 3.1.7.** The continuous-time state feedback system is given by

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)\,; \quad x(0) = x^0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \\ u(t) &= (2, -5)x(t)\,. \end{aligned}$$

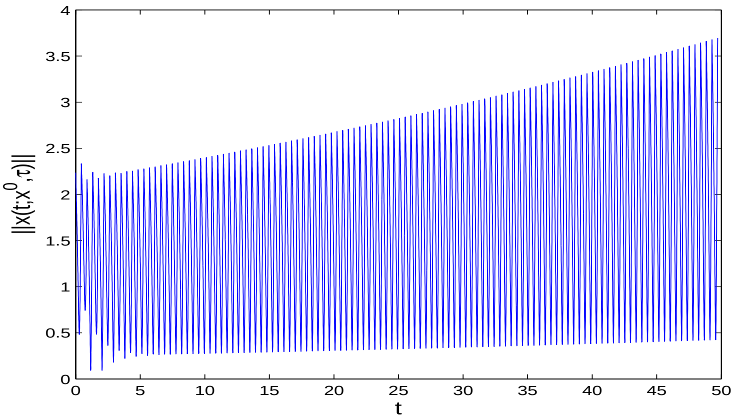


Figure 3-2: Instability occurs when $\tau = 0.45$.

It is easy to check that the above system is exponentially stable. Let $x(\,\cdot\,; x^0)$ and $x(\,\cdot\,; x^0, \tau)$ denote the solutions of the above system and the corresponding sampled-data system, respectively. Figure 3-1 illustrates the effect of sampling with constant sampling period $\tau = 0.2$. Simulations shows that, at $\tau \approx 0.44$, the sampled-data system is not exponentially stable, as shown in Figure 3-2.

## 3.2 Transient bounds

In this section, we will discuss the relation between the transient bounds of the state feedback system (3.1) and the transient bounds of the corresponding sampled-data feedback system (3.2).

**Definition 3.2.1.** If $\alpha$ is an exponential rate of system (3.2), then the number

$$M_s(\alpha, \tau) := \inf\{M \geq 1 : (3.3) \text{ holds}\}$$

is said to be the *transient bound* of (3.2), associated with the exponential rate $\alpha$ and the sampling period $\tau$. $\diamond$

It is clear that

$$\omega_c < \alpha < \beta \Longrightarrow M_s(\beta, \tau) \leq M_s(\alpha, \tau)\,. \tag{3.11}$$

Using the same argument as in the proof of Proposition 2.1.4, we can show that for fixed sampling period $\tau$, the function $(\omega_s(\tau), \infty) \to [1, \infty), \alpha \mapsto M_s(\alpha, \tau)$ is continuous.

Let $M_c(\alpha)$ denote the transient bound of (3.1), associated with the exponential rate $\alpha$ of (3.1). The following theorem is the main result of this section.

**Theorem 3.2.2.** *For every $\beta > \omega_c$ and every $M > M_c(\beta)$, there exists $\tau^* = \tau^*(\beta, M)$ $> 0$ such that, for all $\tau \in (0, \tau^*)$, $\beta$ is an exponential rate of the sampled-data feedback system* (3.2) *and $M_s(\beta, \tau) < M$.*

We need two more lemmas to prove this theorem.

**Lemma 3.2.3.** *For $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $F \in \mathbb{R}^{m\times n}$,*

$$e^{(A+BF)t} = e^{At} + \int_0^t e^{As}BFe^{(A+BF)(t-s)}ds\,, \quad \forall t \geq 0\,.$$

*Proof.* Let $x^0 \in \mathbb{R}^n$ and set $x(t) := e^{(A+BF)t}x^0$ for all $t \geq 0$. Then

$$\dot{x} = (A+BF)x = Ax + BFx\,.$$

Regarding $BFx$ as a forcing term and using the variation-of-parameters formula, we have

$$x(t) = e^{At}x^0 + \int_0^t e^{As}BFx(t-s)ds\,, \quad \forall t \geq 0\,,$$

showing that

$$e^{(A+BF)t}x^0 = e^{At}x^0 + \int_0^t e^{As}BFe^{(A+BF)(t-s)}ds\,x^0\,, \quad \forall t \geq 0\,.$$

This holds for all $x^0 \in \mathbb{R}^n$ and thus the claim follows. □

The following lemma is a version of the discrete-time Gronwall Lemma.

**Lemma 3.2.4.** *Let $a \in \mathbb{R}$ and $b \geq 0$. If the sequence $f : \mathbb{Z}_+ \to \mathbb{R}$ satisfies*

$$f(k) \leq a + b\sum_{j=0}^{k-1} f(j)\,, \quad \forall k \in \mathbb{N}\,,$$

*then*

$$f(k) \leq (1+b)^{k-1}(a + bf(0))\,, \quad \forall k \in \mathbb{N}\,.$$

*Proof.* We use strong induction. When $k = 1$, we have $f(1) \leq a + bf(0)$ by our hypothesis, thus the claim is true for $k = 1$. For $N \geq 2$, assume the claim is true for $k = 1, 2, \ldots, N-1$. Since $b \geq 0$,

$$\begin{aligned}
f(N) &\leq a + b\sum_{j=0}^{N-1} f(j) \\
&\leq a + bf(0) + b(a+bf(0))\sum_{j=1}^{N-1}(1+b)^{j-1} \\
&= a + bf(0) + (a+bf(0))[(1+b)^{N-1} - 1] \\
&= (1+b)^{N-1}(a+bf(0))\,.
\end{aligned}$$

Therefore the claim holds for $k = N$. □

*Proof of Theorem 3.2.2.* Define $V_\tau, W_\tau \in \mathbb{R}^{n\times n}$ by

$$\begin{aligned} V_\tau &:= e^{(A+BF)\tau}\,, \\ W_\tau &:= \int_0^\tau e^{As}BF\left(I - e^{(A+BF)(\tau-s)}\right) ds = \int_0^\tau e^{A(\tau-s)}BF\left(I - e^{(A+BF)s}\right) ds\,. \end{aligned}$$

Define $x_k := x(k\tau; x^0, \tau)$ for all $k \in \mathbb{Z}_+$, where $x(\,\cdot\,; x^0, \tau)$ denotes the solution of (3.2). By Lemma 3.2.3 and (3.5), we obtain

$$x_{k+1} = V_\tau x_k + W_\tau x_k\,, \quad \forall k \in \mathbb{Z}_+\,. \tag{3.12}$$

Considering $k \mapsto W_\tau x_k$ as a forcing term, it follows from the discrete-time variation-of-parameters formula that

$$x_k = V_\tau^k x^0 + \sum_{j=0}^{k-1} V_\tau^{k-j-1} W_\tau x_j\,, \quad \forall k \in \mathbb{N}\,. \tag{3.13}$$

Let $\beta > \omega_c$ and $M > M_c(\beta)$. By Proposition 2.1.4, there exists $\alpha \in (\omega_c, \beta)$ such that $M > M_c(\alpha)$. Since $\alpha$ is an exponential rate of system (3.1),

$$\|V_\tau^k\| = \|e^{(A+BF)k\tau}\| \le M_c(\alpha)e^{\alpha k\tau}\,, \quad \forall k \in \mathbb{Z}_+\,.$$

Consequently, by (3.13),

$$\|x_k\| \le M_c(\alpha)e^{\alpha\tau k}\|x^0\| + M_c(\alpha)\|W_\tau\| \sum_{j=0}^{k-1} e^{\alpha\tau(k-j-1)}\|x_j\|\,, \quad \forall k \in \mathbb{N}\,,$$

or, equivalently,

$$e^{-\alpha\tau k}\|x_k\| \le M_c(\alpha)\|x^0\| + M_c(\alpha)e^{-\alpha\tau}\|W_\tau\| \sum_{j=0}^{k-1} e^{-\alpha\tau j}\|x_j\|\,, \quad \forall k \in \mathbb{N}\,.$$

Set $a := M_c(\alpha)\|x^0\|$, $b := M_c(\alpha)e^{-\alpha\tau}\|W_\tau\|$ and $f(k) := e^{-\alpha\tau k}\|x_k\|$ for $k \in \mathbb{Z}_+$. Applying Lemma 3.2.4 yields

$$e^{-\alpha\tau k}\|x_k\| \le (1+b)^{k-1}(a + bf(0)) \le e^{bk}(a + b\|x^0\|)\,, \quad \forall k \in \mathbb{N}\,,$$

since $1 + b \le e^b$. Hence,

$$\|x_k\| \le e^{(\alpha\tau+b)k} M_c(\alpha)(1 + \|W_\tau\|e^{-\alpha\tau})\|x^0\|\,, \quad \forall k \in \mathbb{Z}_+\,. \tag{3.14}$$

Let $\tau_0 > 0$. Note that there exists $N \ge 0$ such that

$$\|e^{A(\tau-s)}BF\|\|I - e^{(A+BF)s}\| \le 2Ns\,, \quad \forall \tau, s \in [0, \tau_0]\,.$$

Therefore

$$\|W_\tau\| \le \int_0^\tau 2Nsds \le N\tau^2\,, \quad \forall \tau \in (0,\tau_0]\,. \tag{3.15}$$

Defining $\hat{\alpha}(\tau) := \alpha + NM_c(\alpha)e^{-\alpha\tau}\tau$, we have

$$\alpha\tau + b = \alpha\tau + M_c(\alpha)e^{-\alpha\tau}\|W_\tau\| \le \alpha\tau + NM_c(\alpha)e^{-\alpha\tau}\tau^2 = \hat{\alpha}(\tau)\tau\,, \quad \forall \tau \in (0,\tau_0]\,. \tag{3.16}$$

By (3.4) and (3.14)–(3.16),

$$\begin{aligned}\|x(k\tau+\theta;x^0,\tau)\| &\le \left\| e^{A\theta} + \int_0^\theta e^{As}dsBF \right\| \|x_k\| \\ &\le e^{\|A\|\tau}(1+\|BF\|\tau)e^{\hat{\alpha}(\tau)(k\tau+\theta)}e^{-\hat{\alpha}(\tau)\theta}M_c(\alpha)(1+Ne^{-\alpha\tau}\tau^2)\|x^0\|\,, \\ &\qquad \forall \theta \in [0,\tau)\,,\ \forall \tau \in (0,\tau_0]\,,\ \forall k \in \mathbb{Z}_+\,.\end{aligned}$$

Since $\hat{\alpha}(\tau) \ge \alpha$ for all $\tau > 0$, it is clear that $e^{-\hat{\alpha}(\tau)\theta} \le e^{|\alpha|\tau}$ for all $\tau > 0$ and all $\theta \in [0,\tau)$. Setting

$$C(\tau) := M_c(\alpha)(1+\|BF\|\tau)(1+Ne^{-\alpha\tau}\tau^2)e^{(\|A\|+|\alpha|)\tau}\,,$$

we conclude that

$$\|x(t;x^0,\tau)\| \le C(\tau)e^{\hat{\alpha}(\tau)t}\|x^0\|\,, \quad \forall t \ge 0\,,\ \forall \tau \in (0,\tau_0]\,,\ \forall x^0 \in \mathbb{R}^n\,. \tag{3.17}$$

By our choice of $\alpha$ and $\beta$, we have $\omega_c < \alpha < \beta$. By Theorem 3.1.5 and the definition of $\hat{\alpha}(\tau)$ , there exists $\tau_1 \in (0,\tau_0)$ such that if $\tau \in (0,\tau_1)$, then

$$\omega_s(\tau) < \alpha < \hat{\alpha}(\tau) < \beta\,, \quad \forall \tau \in (0,\tau_1)\,, \tag{3.18}$$

showing that $\beta$ is an exponential rate of system (3.2). Note that $C(\tau) > M_c(\alpha)$ for all $\tau > 0$ and $\lim_{\tau\to 0} C(\tau) = M_c(\alpha)$. Hence, since $M_c(\alpha) < M$, there exists $\tau^* \in (0,\tau_1)$ such that if $\tau \in (0,\tau^*)$, then $C(\tau) < M$. It follows from (3.17) that

$$M_s(\hat{\alpha}(\tau),\tau) \le C(\tau) < M\,, \quad \forall \tau \in (0,\tau^*)\,.$$

By (3.11) and (3.18), we obtain that

$$M_s(\beta,\tau) \le M_s(\hat{\alpha}(\tau),\tau) < M\,, \quad \forall \tau \in (0,\tau^*)\,.$$ □

## 3.3 Convergence as the sampling period tends to 0

Let $x(\,\cdot\,;x^0)$ denote the solution of the continuous-time state feedback system (3.1). By Theorem 3.1.5, we know that the exponential growth $\omega_s(\tau)$ of the sampled-data system (3.2) converges to the exponential growth $\omega_c$ of (3.1) as $\tau \to 0$. In this context, it is natural to ask: Does $x(\,\cdot\,;x^0,\tau)$ converge to $x(\,\cdot\,;x^0)$ as $\tau \to 0$? The answer is positive.

**Theorem 3.3.1.** *Let $\Omega \subset \mathbb{R}^n$ be bounded. Then, for every $T > 0$,*

$$\lim_{\tau \to 0} \sup_{\substack{t \in [0,T] \\ x^0 \in \Omega}} \|x(t; x^0, \tau) - x(t; x^0)\| = 0 \,. \tag{3.19}$$

*Moreover, if $\omega_c < 0$, then*

$$\lim_{\tau \to 0} \sup_{\substack{t \geq 0 \\ x^0 \in \Omega}} \|x(t; x^0, \tau) - x(t; x^0)\| = 0 \,. \tag{3.20}$$

*Proof.* Let $\Omega \subset \mathbb{R}^n$ be bounded. We first prove that (3.19) holds for every $T > 0$. Define $V, W \colon [0, \tau] \to \mathbb{R}^{n \times n}$ by

$$\begin{aligned} V(\theta) &:= e^{(A+BF)\theta} \,, \\ W(\theta) &:= \int_0^\theta e^{As} BF \left( I - e^{(A+BF)(\theta - s)} \right) ds \,. \end{aligned}$$

Trivially,

$$x(k\tau + \theta; x^0) = e^{(A+BF)(k\tau+\theta)} x^0 = V(\theta) V(\tau)^k x^0 \,, \quad \forall \theta \in [0, \tau) \,, \ \forall k \in \mathbb{Z}_+ \,.$$

By (3.4), (3.5) and Lemma 3.2.3, we have

$$\begin{aligned} x(k\tau + \theta; x^0, \tau) &= [V(\theta) + W(\theta)] x(k\tau; x^0, \tau) \\ &= [V(\theta) + W(\theta)][V(\tau) + W(\tau)]^k x^0 \\ &= [V(\theta) + W(\theta)][V(\tau)^k + Z(\tau)] x^0 \,, \quad \forall \theta \in [0, \tau) \,, \ \forall k \in \mathbb{Z}_+ \,, \end{aligned}$$

where $Z(\tau) := [V(\tau) + W(\tau)]^k - V(\tau)^k$. It follows that

$$\begin{aligned} &\|x(k\tau + \theta; x^0, \tau) - x(k\tau + \theta; x^0)\| \\ &\quad \leq \|V(\theta) + W(\theta)\| \|Z(\tau)\| \|x^0\| + \|W(\theta)\| \|V(\tau)\|^k \|x^0\| \\ &\quad \leq (\|V(\theta)\| + \|W(\theta)\|) \left( \sum_{j=0}^{k-1} \binom{k}{j} \|V(\tau)\|^j \|W(\tau)\|^{k-j} \right) \|x^0\| \\ &\qquad\qquad + \|W(\theta)\| \|V(\tau)\|^k \|x^0\| \,, \quad \forall \theta \in [0, \tau) \,, \ \forall k \in \mathbb{Z}_+ \,. \end{aligned} \tag{3.21}$$

Let $\tau \in (0, \tau_0)$ for some $\tau_0 > 0$. By an argument identical to that leading to (3.15), we conclude that there exists $M_1 \geq 0$ such that

$$\|W(\theta)\| \leq M_1 \theta^2 \leq M_1 \tau^2 \,, \quad \forall \theta \in [0, \tau] \,. \tag{3.22}$$

Fix $T > 0$ and let $t \in [0, T]$. Then $t := k\tau + \theta$ for some $k \in \mathbb{Z}_+$ and some $\theta \in [0, \tau)$. Trivially, $k\tau \leq T$. Setting $M_2 := e^{\|A+BF\|T}$, we have

$$\|V(\theta)\|^j \leq e^{\|A+BF\| j\theta} \leq e^{\|A+BF\| k\tau} \leq M_2 \,, \quad \forall j = 0, \ldots, k \,, \ \forall \theta \in [0, \tau] \,. \tag{3.23}$$

Consequently, by (3.22) and (3.23),

$$\begin{aligned}
\sum_{j=0}^{k-1}\binom{k}{j}\|V(\tau)\|^j\|W(\tau)\|^{k-j} &\le \sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}M_2M_1^{k-j}\tau^{2(k-j)} \\
&\le M_2\sum_{j=0}^{k-1}\frac{(k\tau)^{(k-j)}}{(k-j)!}(M_1\tau)^{(k-j)} \\
&\le M_2\sum_{j=0}^{k-1}\frac{(TM_1\tau)^{(k-j)}}{(k-j)!} \\
&= M_2\sum_{j=1}^{k}\frac{(TM_1\tau)^j}{j!} \\
&\le M_2(e^{TM_1\tau}-1)\,. \qquad (3.24)
\end{aligned}$$

Therefore, combining (3.21)–(3.24), we conclude that there exists $M_3 \ge 0$ such that

$$\begin{aligned}
\|x(t;x^0,\tau)-x(t;x^0)\| &\le [M_2^2(e^{TM_1\tau}-1)+M_1M_2\tau^2e^{TM_1\tau}]\|x^0\| \\
&\le M_3\tau\|x^0\|\,, \quad \forall \tau\in(0,\tau_0]\,,\ \forall t\in[0,T]\,.
\end{aligned}$$

As a consequence, (3.19) follows.

Finally assume that $\omega_c < 0$. Let $\alpha \in (\omega_c, 0)$. By Theorem 3.1.5, there exists $\tau_1 > 0$ such that if $\tau \in (0, \tau_1)$, then $\omega_s(\tau) < \alpha$. Let $\tau \in (0, \tau_1)$. It follows from the boundedness of $\Omega$ that there exists $M \ge 0$ such that

$$\|x(t;x^0)\| \le Me^{\alpha t}\,, \quad \|x(t;x^0,\tau)\| \le Me^{\alpha t}\,, \qquad \forall t\ge 0\,,\ \forall x^0\in\Omega\,.$$

Since $\alpha < 0$, for every $\varepsilon > 0$, there exists $T > 0$ such that

$$\|x(t;x^0)\| \le \frac{\varepsilon}{2}\,, \quad \|x(t;x^0,\tau)\| \le \frac{\varepsilon}{2}\,, \qquad \forall t\ge T\,,\ \forall x^0\in\Omega\,.$$

Hence,

$$\sup_{\substack{t\ge T\\ x^0\in\Omega}}\|x(t;x^0,\tau)-x(t;x^0)\| \le \varepsilon\,.$$

Invoking (3.19), we conclude that, for every $\varepsilon > 0$, there exists $\tau_2 \in (0, \tau_1)$ such that

$$\sup_{\substack{t\ge 0\\ x^0\in\Omega}}\|x(t;x^0,\tau)-x(t;x^0)\| \le \varepsilon\,, \quad \forall \tau\in(0,\tau_2)\,.$$

showing that (3.20) is true. □

The following simple example shows that (3.20) is not true if $\omega_c \ge 0$.

**Example 3.3.2.** Consider

$$\begin{aligned} \dot{x}(t) &= x(t) + u(t)\,; \quad x(0) = x^0 \in \mathbb{R}\,, \\ u(t) &= x(t)\,. \end{aligned}$$

Obviously, $x(t;x^0) = e^{2t}x^0$ for $t \geq 0$ and

$$x(k\tau + \theta; x^0, \tau) = (2e^\theta - 1)(2e^\tau - 1)^k x^0\,, \quad \forall\theta \in [0,\tau)\,,\ \forall k \in \mathbb{Z}_+\,.$$

Then

$$|x(k\tau;x^0) - x(k\tau;x^0,\tau)| \;=\; |e^{2k\tau} - (2e^\tau - 1)^k||x^0|\,, \quad \forall k \in \mathbb{Z}_+\,.$$

We see that, for all $\tau > 0$ and all $x^0 \in \mathbb{R}$ with $x^0 \neq 0$, $|x(k\tau;x^0) - x(k\tau;x^0,\tau)|$ goes to $\infty$ as $k \to \infty$.

## 3.4 Notes and references

We remark that Corollary 3.1.6 can be obtained as a consequence of [9, Theorem 1]. However, we found the proof given in [9] is difficult to penetrate. We emphasize that Theorem 3.1.5 does not follow from results in [9]. To the best of our knowledge, Theorem 3.1.5 is new.

Theorem 3.2.2 is implicitly contained in [9, Theorem 1]. Whilst we found the proof in [9] difficult to penetrate, it partly inspired the above proof of Theorem 3.2.2.

Logemann, Rebarber and Townley [39] showed that Corollary 3.1.6 is still true in the infinite-dimensional case if one of the following assumptions hold:

- the input operator $B$ is bounded, i.e., $B \in \mathcal{B}(U,X)$, and $F \in \mathcal{B}(U,X)$ is compact, where $X$ is the state space and $U$ is the input space (both Hilbert spaces) (see [39, Theorem 3.1]).
- the input operator $B$ is unbounded, $F \in \mathcal{B}(U,X)$ is compact and the semigroups generated by $A$ is analytic (see [39, Theorem 4.8]).

Whether Theorem 3.1.5 extends to infinite-dimensional systems is an open problem. We note that the proof of Theorem 3.1.5 relies on certain properties of the spectrum of a matrix $A$ and on the convergence of the Taylor series of $e^{At}$: it certainly does not generalize to interesting infinite-dimensional situations in a straight forward way.

# Chapter 4

# Indirect sampled-data control: dynamic output feedback

In this chapter, we extend the results in Chapter 3 to dynamic output feedback. Moreover, using state-space methods, the input-output stability of sampled-data systems is investigated.

## 4.1 Exponential growth and transient bounds

Consider the continuous-time dynamic feedback system shown in Figure 4-1. The plant $\Sigma_p$ is given by

$$\dot{x}_p(t) = A_p x_p(t) + B_p u_p(t)\,; \quad x_p(0) = x_p^0 \in \mathbb{R}^{n_p}\,, \tag{4.1a}$$
$$y_p(t) = C_p x_p(t) + D_p u_p(t)\,, \tag{4.1b}$$

where $A_p \in \mathbb{R}^{n_p \times n_p}, B_p \in \mathbb{R}^{n_p \times m}, C_p \in \mathbb{R}^{p \times n_p}$ and $D_p \in \mathbb{R}^{p \times m}$. The controller $\Sigma_c$ is of the form

$$\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t)\,; \quad x_c(0) = x_c^0 \in \mathbb{R}^{n_c}\,, \tag{4.2a}$$
$$y_c(t) = C_c x_c(t) + D_c u_c(t)\,, \tag{4.2b}$$

where $A_c \in \mathbb{R}^{n_c \times n_c}, B_c \in \mathbb{R}^{n_c \times p}, C_c \in \mathbb{R}^{m \times n_c}$ and $D_c \in \mathbb{R}^{m \times p}$. We use the output $y_p$ of $\Sigma_p$ as the input for $\Sigma_c$, and the output $y_c$ of $\Sigma_c$ as the input for $\Sigma_p$, i.e.,

$$u_c = y_p\,, \quad u_p = y_c\,, \tag{4.3}$$

to obtain a feedback interconnection of (4.1) and (4.2).

In order for the closed-loop system to be well-posed, we assume that the matrix $I - D_c D_p$ is invertible. Then $I - D_p D_c$ is also invertible, with

$$(I - D_p D_c)^{-1} = I + D_p (I - D_c D_p)^{-1} D_c\,.$$

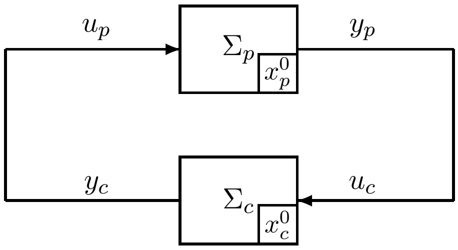


Figure 4-1: Continuous-time dynamic feedback system

Set

$$E_p := (I - D_c D_p)^{-1}\,, \quad E_c := (I - D_p D_c)^{-1}\,. \tag{4.4}$$

It is convenient to define

$$x := \begin{pmatrix} x_p \\ x_c \end{pmatrix}, \qquad A := \begin{pmatrix} A_p & 0 \\ 0 & A_c \end{pmatrix}, \qquad B := \begin{pmatrix} B_p & 0 \\ 0 & B_c \end{pmatrix},$$
$$C := \begin{pmatrix} C_p & 0 \\ 0 & C_c \end{pmatrix}, \qquad D := \begin{pmatrix} D_c & I \\ I & D_p \end{pmatrix}, \qquad E := \begin{pmatrix} E_p & 0 \\ 0 & E_c \end{pmatrix}.$$

Consequently, by a routine calculation†, the continuous-time dynamic output feedback system given by (4.1)–(4.3) can be written as

$$\dot{x}(t) = (A + BEDC)x(t)\,; \quad x(0) = x^0 := \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} \in \mathbb{R}^{n_p + n_c}\,. \tag{4.5}$$

We now consider the sample-hold discretization of $\Sigma_c$. Let $\tau > 0$ be the sampling period and let the input $u_c$ in (4.2a) be given by $u_c = \mathcal{H}_\tau v$, where $v$ is a function: $\mathbb{Z}_+ \to \mathbb{R}^m$. By the variation-of-parameters formula, we obtain that

$$x_c((k+1)\tau) \;=\; e^{A_c\tau}x_c(k\tau) + \int_0^\tau e^{A_c s}ds\, B_c v(k)\,; \quad x_c(0) = x_c^0 \in \mathbb{R}^{n_c}\,, \tag{4.6a}$$
$$y_c(k\tau) \;=\; C_c x_c(k\tau) + D_c v(k)\,. \tag{4.6b}$$

We use the discrete-time system (4.6) to control the continuous-time system (4.1) by sampled-data feedback, i.e., we consider the feedback interconnection of (4.1) and (4.6) given by

$$v(k) = y_p(k\tau)\,, \quad u_p(k\tau + \theta) = y_c(k\tau)\,, \quad \forall \theta \in [0, \tau)\,,\ \forall k \in \mathbb{Z}_+\,. \tag{4.7}$$

The sampled-data feedback system given by (4.1), (4.6) and (4.7) has a unique solution

†See Appendix A.4.1.

which will be denoted by

$$\begin{pmatrix} x_p(k\tau+\theta;x^0,\tau) \\ x_c(k\tau;x^0,\tau) \end{pmatrix}, \quad \forall\theta\in[0,\tau)\,,\ \forall k\in\mathbb{Z}_+\,.$$

For convenience, sometimes we write $x_p(\cdot) := x_p(\,\cdot\,;x^0,\tau)$ and $x_c(k\tau) := x_c(k\tau;x^0,\tau)$. Consequently, a routine calculation† yields the following sampled-data feedback equations

$$x_p(k\tau+\theta) = e^{A_p\theta}x_p(k\tau) + \int_0^\theta e^{A_ps}dsB_pE_p[D_cC_px_p(k\tau)+C_cx_c(k\tau)]\,, \tag{4.8a}$$
$$\forall\theta\in[0,\tau)\,,\ \forall k\in\mathbb{Z}_+\,;\quad x_p(0)=x_p^0\in\mathbb{R}^{n_p}\,,$$
$$x_c((k+1)\tau) = e^{A_c\tau}x_c(k\tau) + \int_0^\tau e^{A_cs}dsB_cE_c[C_px(k\tau)+D_pC_cx_c(k\tau)]\,, \tag{4.8b}$$
$$\forall k\in\mathbb{Z}_+\,;\quad x_c(0)=x_c^0\in\mathbb{R}^{n_c}\,.$$

**Definition 4.1.1.** A number $\alpha\in\mathbb{R}$ is said to be an *exponential rate* of system (4.8) if there exists $M\geq 1$ such that

$$\left\|\begin{pmatrix} x_p(k\tau+\theta;x^0,\tau) \\ x_c(k\tau;x^0,\tau) \end{pmatrix}\right\| \leq Me^{\alpha(k\tau+\theta)}\|x^0\|\,, \quad \forall\theta\in[0,\tau)\,,\ \forall k\in\mathbb{Z}_+\,,\ \forall x^0\in\mathbb{R}^{n_p+n_c}\,. \tag{4.9}$$

We define the *exponential growth* $\omega_d(\tau)$ of system (4.8) by

$$\omega_d(\tau) := \inf\{\alpha : \alpha \text{ is an exponential rate of system (4.8)}\}\,.$$

System (4.8) is said to be *exponentially stable* if and only if $\omega_d(\tau)<0$. ◇

Letting $\theta\nearrow\tau$ in (4.8), it follows from the continuity of the terms depending on $\theta$ in (4.8) that

$$\begin{pmatrix} x_p((k+1)\tau) \\ x_c((k+1)\tau) \end{pmatrix} = \Delta_\tau\begin{pmatrix} x_p(k\tau) \\ x_c(k\tau) \end{pmatrix} = \Delta_\tau^{k+1}x^0\,, \quad \forall k\in\mathbb{Z}_+\,. \tag{4.10}$$

where $\Delta_\tau\in\mathbb{R}^{(n_p+n_c)\times(n_p+n_c)}$ is defined by

$$\Delta_\tau := e^{A\tau} + \int_0^\tau e^{As}ds\,BEDC\,. \tag{4.11}$$

**Theorem 4.1.2.** *A number $\rho$ is a power rate of* (4.10) *if and only if* $(\ln\rho)/\tau$ *is an exponential rate of* (4.8)*. Consequently,*

$$\omega_d(\tau) = \frac{1}{\tau}\ln(r(\Delta_\tau))\,.$$

†See Appendix A.4.3 with $\sigma=-1$, $\varepsilon=1$, $r=0$, $d=0$, $A_c$ replaced by $e^{A_c\tau}$ and $B_c$ replaced by $\int_0^\tau e^{A_cs}ds$.

*Proof.* By (4.8), we have

$$\begin{pmatrix} x_p(k\tau+\theta) \\ x_c(k\tau) \end{pmatrix} = Q(\theta) \begin{pmatrix} x_p(k\tau) \\ x_c(k\tau) \end{pmatrix} , \quad \forall\theta \in [0,\tau) \, , \; \forall k \in \mathbb{Z}_+ \, , \tag{4.12}$$

where $Q \colon [0,\tau) \to \mathbb{R}^{(n_p+n_c)\times(n_p+n_c)}$ is defined by

$$Q(\theta) := \begin{pmatrix} e^{A_p\theta} + \int_0^\theta e^{A_p s} ds B_p E_p D_c C_p & \int_0^\theta e^{A_p s} ds B_p E_p C_c \\ 0 & I \end{pmatrix} . \tag{4.13}$$

Assume that $\rho$ is a power rate of (4.10), so that there exists $M \geq 1$ such that

$$\|\Delta_\tau^k\| \leq M\rho^k \, , \quad \forall k \in \mathbb{Z}_+ \, .$$

By (4.10) and (4.12),

$$\left\| \begin{pmatrix} x_p(k\tau+\theta) \\ x_c(k\tau) \end{pmatrix} \right\| \leq M_1 \|\Delta_\tau^k\| \|x^0\| \leq M_1 M \rho^k \|x^0\| \, , \quad \forall k \in \mathbb{Z}_+ \, , \; \forall\theta \in [0,\tau) \, ,$$

where $M_1 := \max_{\theta\in[0,\tau]} \|Q(\theta)\|$.

*Case 1*: $\rho \geq 1$.

$$\begin{aligned} \left\| \begin{pmatrix} x_p(k\tau+\theta; x^0, \tau) \\ x_c(k\tau; x^0, \tau) \end{pmatrix} \right\| \; &\leq \; M_1 M \rho^{(k\tau+\theta)/\tau} \|x^0\| = M_1 M e^{((\ln\rho)/\tau)(k\tau+\theta)} \|x^0\| \, , \\ &\qquad \forall\theta \in [0,\tau) \, , \; \forall k \in \mathbb{Z}_+ \, , \; \forall x^0 \in \mathbb{R}^{n_p+n_c} \, . \end{aligned}$$

*Case 2*: $0 < \rho < 1$.

$$\begin{aligned} \left\| \begin{pmatrix} x_p(k\tau+\theta; x^0, \tau) \\ x_c(k\tau; x^0, \tau) \end{pmatrix} \right\| \; &\leq \; M_1 M \rho^{-1} \rho^{(k\tau+\theta)/\tau} \|x^0\| = M_1 M \rho^{-1} e^{((\ln\rho)/\tau)(k\tau+\theta)} \|x^0\| \, , \\ &\qquad \forall\theta \in [0,\tau) \, , \; \forall k \in \mathbb{Z}_+ \, , \; \forall x^0 \in \mathbb{R}^{n_p+n_c} \, . \end{aligned}$$

Combining the two cases above, we conclude that $(\ln\rho)/\tau$ is an exponential rate of system (4.8).

Conversely, assume that $(\ln\rho)/\tau$ is an exponential rate of (4.8). It follows from (4.9) with $\theta = 0$ that there exists $M_2 \geq 1$ such that

$$\left\| \begin{pmatrix} x_p(k\tau; x^0, \tau) \\ x_c(k\tau; x^0, \tau) \end{pmatrix} \right\| \leq M_2 e^{((\ln\rho)/\tau)k\tau} \|x^0\| = M_2 \rho^k \|x^0\| \, , \quad \forall k \in \mathbb{Z}_+ \, ; \; \forall x^0 \in \mathbb{R}^{n_p+n_c} \, .$$

Consequently, by (4.10),

$$\|\Delta_\tau^k x^0\| \leq M_2 \rho^k \|x^0\|\,, \quad \forall k \in \mathbb{Z}_+\,,\ \forall x^0 \in \mathbb{R}^{n_p+n_c}\,,$$

showing that $\|\Delta_\tau^k\| \leq M_2 \rho^k$ for all $k \in \mathbb{Z}_+$. Thus $\rho$ is a power rate of (4.10).

Clearly, $r(\Delta_\tau)$ is the power growth of system (4.10). Taking infima, we have $\omega_d(\tau) = \ln(r(\Delta_\tau))/\tau$. □

**Corollary 4.1.3.** *For system* (4.8),

$$\lim_{k\to\infty} \begin{pmatrix} x_p(k\tau+\theta; x^0, \tau) \\ x_c(k\tau; x^0, \tau) \end{pmatrix} = 0\,, \quad \forall \theta \in [0,\tau)\,,\ \forall x^0 = \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} \in \mathbb{R}^{n_p+n_c}\,, \tag{4.14}$$

*if and only if* $\omega_d(\tau) < 0$.

*Proof.* The sufficiency is obvious. For the necessity, assume that (4.14) holds. By (4.10), it is clear that

$$\lim_{k\to\infty} \Delta_\tau^k x^0 = \lim_{k\to\infty} \begin{pmatrix} x_p(k\tau; x^0, \tau) \\ x_c(k\tau; x^0, \tau) \end{pmatrix} = 0\,, \quad \forall x^0 \in \mathbb{R}^{n_p+n_c}\,.$$

Since this holds for all $x^0 \in \mathbb{R}^{n_p+n_c}$, $\lim_{k\to\infty} \Delta_\tau^k = 0$. It follows from Theorem 2.2.1 that $r(\Delta_\tau) < 1$. By Theorem 4.1.2, we know that $\omega_d(\tau) < 0$. Therefore (4.8) is exponentially stable. □

Let $M_c(\alpha)$ denote the transient bound of system (4.5) associated with the exponential rate $\alpha$ of (4.5). If $\alpha$ is an exponential rate of (4.8), then the number

$$M_d(\alpha,\tau) := \inf\{M \geq 1 : (4.9) \text{ holds}\}$$

is said to be the *transient bound* of (4.8), associated with the exponential rate $\alpha$ and the sampling period $\tau$. The theorems on exponential growth and transient bounds in the state feedback case (see Theorem 3.1.5 and Theorem 3.2.2) still hold in the dynamic output feedback case.

**Theorem 4.1.4.** *Let $\omega_c$ and $\omega_d(\tau)$ denote the exponential growths of system* (4.5) *and system* (4.8)*, respectively. Then,*

$$\lim_{\tau\to 0} \omega_d(\tau) = \lim_{\tau\to 0} \frac{1}{\tau} \ln(r(\Delta_\tau)) = \omega_c\,. \tag{4.15}$$

*where $\Delta_\tau$ is given by* (4.11)*.*

*For every $\alpha > \omega_c$ and every $M > M_c(\alpha)$, there exists $\tau^* = \tau^*(\alpha, M)$ such that, for all $\tau \in (0,\tau^*)$, $\alpha$ is an exponential rate of system* (4.8)*, and $M_d(\alpha,\tau) < M$.*

*Proof.* Note that (4.5) can be written as a state feedback system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,; \quad x(0) = x^0 \in \mathbb{R}^{n_p+n_c}\,, && (4.16a)\\ u(t) &= Fx(t)\,, && (4.16b) \end{aligned}$$

where $F := EDC$. Applying sampling and hold in (4.16b), we obtain the corresponding sampled-data state feedback system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,; \quad x(0) = x^0 \in \mathbb{R}^{n_p+n_c}\,, && (4.17a)\\ u(t) &= Fx(k\tau)\,, \quad \forall t \in [k\tau, (k+1)\tau)\,;\ \forall k \in \mathbb{Z}_+\,. && (4.17b) \end{aligned}$$

Let $x(\cdot\,; x^0, \tau)$ denote the solution of (4.17). Using the variation-of-parameters formula, we obtain the discrete-time system

$$x((k+1)\tau; x^0, \tau) = \Delta_\tau x(k\tau; x^0, \tau)\,, \quad \forall k \in \mathbb{Z}_+\,; \quad x(0; x^0, \tau) = x^0\,, \tag{4.18}$$

where $\Delta_\tau$ is defined in (4.11). It follows from Theorem 3.1.5 that

$$\lim_{\tau\to 0} \frac{1}{\tau} \ln(r(\Delta_\tau)) = \omega_c\,.$$

Invoking Theorem 4.1.2 proves that (4.15) holds.

Let $\alpha > \omega_c$ and $M > M_c(\alpha)$. By (4.15) and Theorem 3.2.2, there exists $\tau_1 = \tau_1(\alpha, M) > 0$ such that if $\tau \in (0, \tau_1)$, then $\alpha$ is an exponential rate of system (4.8) and system (4.17), and $M_s(\alpha, \tau) < M$, where $M_s(\alpha, \tau)$ is the transient bound of (4.17) associated with $\alpha$. Let $\tau \in (0, \tau_1)$. Note that the dynamics of discrete-time systems (4.10) and (4.18) are both governed by $\Delta_\tau$. Thus,

$$\left\| \begin{pmatrix} x_p(k\tau; x^0, \tau) \\ x_c(k\tau; x^0, \tau) \end{pmatrix} \right\| = \|x(k\tau; x^0, \tau)\| \le M_s(\alpha, \tau) e^{\alpha k\tau} \|x^0\|\,, \quad \forall k \in \mathbb{Z}_+\,,\ \forall x^0 \in \mathbb{R}^{n_p+n_c}\,. \tag{4.19}$$

By (4.12) and (4.19), we have

$$\begin{aligned} \left\| \begin{pmatrix} x_p(k\tau + \theta; x^0, \tau) \\ x_c(k\tau; x^0, \tau) \end{pmatrix} \right\| &\le \|Q(\theta)\| \left\| \begin{pmatrix} x_p(k\tau; x^0, \tau) \\ x_c(k\tau; x^0, \tau) \end{pmatrix} \right\| \\ &\le \|Q(\theta)\| M_s(\alpha, \tau) e^{\alpha k\tau} \|x^0\| \\ &\le \|Q(\theta)\| e^{|\alpha|\tau} M_s(\alpha, \tau) e^{\alpha(k\tau+\theta)} \|x^0\|\,, \\ &\qquad \forall \theta \in [0, \tau)\,,\ \forall k \in \mathbb{Z}_+\,,\ \forall x^0 \in \mathbb{R}^{n_p+n_c}\,, \end{aligned}$$

where $Q(\theta)$ is defined in (4.13). Therefore,

$$M_d(\alpha, \tau) \le \|Q(\theta)\| e^{|\alpha|\tau} M_s(\alpha, \tau)\,, \quad \forall \theta \in [0, \tau)\,.$$

Noting that $\lim_{\tau\to 0} \|Q(\theta)\| e^{|\alpha|\tau} = 1$ for all $\theta \in [0, \tau)$ and $M_s(\alpha, \tau) < M$, it follows that

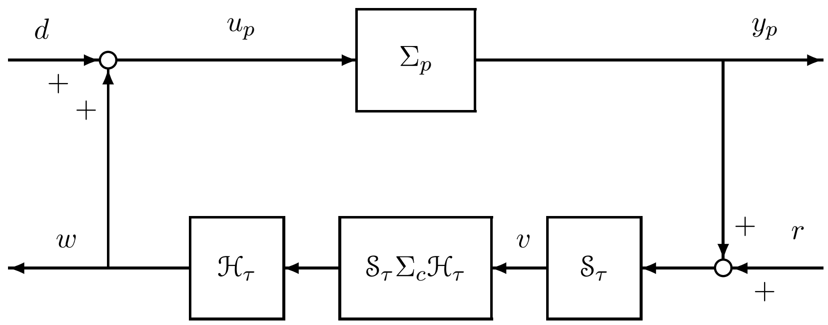


Figure 4-2: Indirect sampled-data control with exogenous inputs.

there exists $\tau^* \in (0, \tau_1)$ such that if $\tau \in (0, \tau^*)$, then $M_d(\alpha, \tau) < M$. □

The following corollary follows immediately from (4.15) in Theorem 4.1.4.

**Corollary 4.1.5.** *If the continuous-time dynamic feedback system* (4.5) *is exponentially stable, then there exists* $\tau^* > 0$ *such that, for all* $\tau \in (0, \tau^*)$*, the sampled-data system* (4.8) *is exponentially stable.*

## 4.2 Input-output stability

In this section, we discuss the input-output properties of sampled-data systems. Consider the sampled-data feedback interconnection of continuous-time plant (4.1) and discrete-time controller (4.6) given by

$$\begin{aligned} u_p(k\tau + \theta) &= y_c(k\tau) + d(k\tau + \theta), \quad \forall \theta \in [0, \tau), \ \forall k \in \mathbb{Z}_+, & \text{(4.20a)} \\ v(k) &= y_p(k\tau) + r(k\tau), \quad \forall k \in \mathbb{Z}_+, & \text{(4.20b)} \end{aligned}$$

as shown in Figure 4-2. Throughout this section, we assume that

$$d \in W^{1,q}(\mathbb{R}_+, \mathbb{R}^m), \quad r \in W^{1,q}(\mathbb{R}_+, \mathbb{R}^p),$$

for some $1 \le q < \infty$, or

$$d \in L_b(\mathbb{R}_+, \mathbb{R}^m), \quad r \in L_b(\mathbb{R}_+, \mathbb{R}^p),$$

where $L_b(\mathbb{R}_+, \mathbb{R}^m)$ denotes the space of $\mathbb{R}^m$-valued bounded Lebesgue measurable functions equipped with the sup-norm $\|\cdot\|_\infty$.

The sampled-data feedback system given by (4.1), (4.6) and (4.20a) has a unique solution which will be denoted by

$$\begin{pmatrix} x_p(k\tau + \theta; x^0, d, r, \tau) \\ x_c(k\tau; x^0, d, r, \tau) \end{pmatrix}, \quad \forall \theta \in [0, \tau), \ \forall k \in \mathbb{Z}_+, \quad \text{where } x^0 := \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix}.$$

For convenience, we write

$$x_p(\cdot) := x_p(\,\cdot\,; x^0, d, r, \tau)\,, \quad x_c(k\tau) := x_c(k\tau; x^0, d, r, \tau)\,.$$

By a routine calculation†, $x_p$ and $x_c$ satisfy the following sampled-data feedback equations

$$\begin{aligned}
&x_p(k\tau+\theta) \\
&\quad = e^{A_p\theta}x_p(k\tau) + \int_0^\theta e^{A_p s}ds B_p E_p[D_c C_p x_p(k\tau) + C_c x_c(k\tau)] \\
&\qquad + \int_{k\tau}^{k\tau+\theta} e^{A_p(k\tau+\theta-s)}B_p d(s)\,ds + \int_0^\theta e^{A_p s}ds B_p E_p D_c[D_p d(k\tau) + r(k\tau)]\,, \\
&\qquad\qquad \forall\theta\in[0,\tau)\,,\ \forall k\in\mathbb{Z}_+\,; \quad x_p(0) = x_p^0 \in \mathbb{R}^n\,, \qquad (4.21a) \\
&x_c((k+1)\tau) \\
&\quad = e^{A_c\tau}x_c(k\tau) + \int_0^\tau e^{A_c s}ds B_c E_c[C_p x_p(k\tau) + D_p C_c x_c(k\tau)] \\
&\qquad + \int_0^\tau e^{A_c s}ds B_c E_c[D_p d(k\tau) + r(k\tau)]\,,\ \forall k\in\mathbb{Z}_+\,;\ x_c(0) = x_c^0\in\mathbb{R}^{n_c}\,, \qquad (4.21b)
\end{aligned}$$

where $E_p$ and $E_c$ are defined in (4.4). Setting $R\colon \mathbb{Z}_+ \to \mathbb{R}^{n_p+n_c}$ by

$$R(k) := \begin{pmatrix} \displaystyle\int_{k\tau}^{(k+1)\tau} e^{A_p[(k+1)\tau-s]}B_p d(s)\,ds + \int_0^\tau e^{A_p s}ds B_p E_p D_c[D_p d(k\tau) + r(k\tau)] \\ \displaystyle\int_0^\tau e^{A_c s}ds B_c E_c[D_p d(k\tau) + r(k\tau)] \end{pmatrix}, \qquad (4.22)$$

a routine calculation† gives

$$\begin{pmatrix} x_p(k\tau) \\ x_c(k\tau) \end{pmatrix} = \Delta_\tau^k x^0 + \sum_{j=0}^{k-1}\Delta_\tau^{k-j-1}R(j)\,, \quad \forall k\in\mathbb{N}\,. \qquad (4.23)$$

where $\Delta_\tau$ is defined in (4.11).

The following theorem is the main result of this section.

**Theorem 4.2.1.** *Assume the continuous-time feedback system* (4.5) *is exponentially stable. There exists $\tau_0 > 0$ such that if $\tau\in(0,\tau_0)$, then the sampled-data system, given by* (4.1), (4.6) *and* (4.20a) *(as shown in Figure 4-2), is input-output stable in the sense that, for all $\tau\in(0,\tau_0)$, there exists $M\geq 0$ such that*

$$\begin{aligned}
\left\|\begin{pmatrix} y_p \\ w \end{pmatrix}\right\|_{L^q} &\leq M(\|x^0\| + \|d\|_{W^{1,q}} + \|r\|_{W^{1,q}})\,, \\
&\qquad \forall x^0\in\mathbb{R}^{n_p+n_c}\,,\ \forall d\in W^{1,q}(\mathbb{R}_+,\mathbb{R}^m)\,,\ \forall r\in W^{1,q}(\mathbb{R}_+,\mathbb{R}^p)\,,
\end{aligned}$$

†See Appendix A.4.3 with $\sigma=-1$, $\varepsilon=1$, $A_c$ replaced by $e^{A_c\tau}$ and $B_c$ replaced by $\int_0^\tau e^{A_c s}ds$.

*for* $1 \le q < \infty$*, and*

$$\left\| \begin{pmatrix} y_p \\ w \end{pmatrix} \right\|_\infty \le M(\|x^0\| + \|d\|_\infty + \|r\|_\infty)\,,$$
$$\forall x^0 \in \mathbb{R}^{n_p+n_c}\,,\ \forall d \in L_b(\mathbb{R}_+, \mathbb{R}^m)\,,\ \forall r \in L_b(\mathbb{R}_+, \mathbb{R}^p)\,,$$

*where* $y_p$ *and* $w$ *are the outputs of the sampled-data system shown in Figure 4-2.*

*Proof.* Note that $w = \mathcal{H}_\tau \mathcal{S}_\tau y_c$. By a routine calculation†, we have

$$\begin{aligned}\begin{pmatrix} y_p(k\tau+\theta) \\ w(k\tau+\theta) \end{pmatrix} &= \begin{pmatrix} C_p x_p(k\tau+\theta) + D_p u_p(k\tau+\theta) \\ C_c x_c(k\tau) + D_c v(k) \end{pmatrix} \\ &= Q(\theta) \begin{pmatrix} x_p(k\tau) \\ x_c(k\tau) \end{pmatrix} + \begin{pmatrix} G(k,\theta) \\ D_c E_c[D_p d(k\tau) + r(k\tau)] \end{pmatrix}\,, \\ &\qquad \forall \theta \in [0,\tau)\,,\ \forall k \in \mathbb{Z}_+\,,\end{aligned} \tag{4.24}$$

where $Q\colon [0,\tau) \to \mathbb{R}^{(p+m)\times(n_p+n_c)}$ is defined by

$$Q(\theta) := \begin{pmatrix} C_p e^{A_p\theta} + \left(C_p \int_0^\theta e^{A_p s} ds B_p + D_p\right) E_p D_c C_p & \left(C_p \int_0^\theta e^{A_p s} ds B_p + D_p\right) E_p C_c \\ D_c E_c C_p & C_c + D_c E_c D_p C_c \end{pmatrix} \tag{4.25}$$

and

$$\begin{aligned} G(k,\theta) := C_p \int_{k\tau}^{k\tau+\theta} e^{A_p(k\tau+\theta-s)} B_p d(s) ds + C_p \int_0^\theta e^{A_p s} ds B_p E_p D_c [D_p d(k\tau) + r(k\tau)] \\ + D_p d(k\tau+\theta) + D_p E_p D_c [D_p d(k\tau) + r(k\tau)]\,. \end{aligned} \tag{4.26}$$

Recall that $\Delta_\tau$ and $R(j)$ are defined in (4.11) and (4.22), respectively, and define $\Pi_1, \Pi_2, \Pi_3\colon \mathbb{R}_+ \to \mathbb{R}^{m+p}$ by

$$\Pi_1(k\tau+\theta) := Q(\theta)\Delta_\tau^k x^0\,, \tag{4.27a}$$

$$\Pi_2(k\tau+\theta) := \begin{cases} 0\,, & k = 0 \\ Q(\theta) \sum_{j=0}^{k-1} \Delta_\tau^{k-j-1} R(j)\,, & k \in \mathbb{N} \end{cases}\,, \tag{4.27b}$$

$$\Pi_3(k\tau+\theta) := \begin{pmatrix} G(k,\theta) \\ D_c E_c [D_p d(k\tau) + r(k\tau)] \end{pmatrix}\,. \tag{4.27c}$$

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†See Appendix A.4.3 with $\sigma = -1$, $\varepsilon = 1$, $A_c$ replaced by $e^{A_c\tau}$ and $B_c$ replaced by $\int_0^\tau e^{A_c s} ds$.

Combining (4.23) and (4.24), we obtain that

$$\begin{pmatrix} y_p(t) \\ w(t) \end{pmatrix} = \Pi_1(t) + \Pi_2(t) + \Pi_3(t)\,, \quad \forall t \in \mathbb{R}_+\,. \tag{4.28}$$

It follows from Theorem 4.1.4 and the exponential stability of system (4.5) that there exists $\tau_0 > 0$ such that, for all $\tau \in (0, \tau_0)$, $\Delta_\tau$ is power stable. Hence, setting $\Delta := (\Delta_\tau^k)_{k\in\mathbb{Z}_+}$, we conclude that $\Delta \in \ell^q(\mathbb{Z}_+, \mathbb{R}^{(n_p+n_c)\times(n_p+n_c)})$ for all $1 \le q \le \infty$.

Let $1 \le q < \infty$ and $\tau \in (0, \tau_0)$. Clearly,

$$\begin{aligned} \|\Pi_1\|_{L^q} &\le \left( \sum_{k=0}^{\infty} \int_{k\tau}^{(k+1)\tau} M_1^q \, \|\Delta_\tau^k\|^q \, \|x^0\|^q \, dt \right)^{1/q} \\ &= M_1 \tau_0^{1/q} \|\Delta\|_{\ell^q} \|x^0\|\,, \quad \forall x^0 \in \mathbb{R}^{n_p+n_c}\,, \end{aligned} \tag{4.29}$$

where $M_1 := \max_{\theta\in[0,\tau_0]} \|Q(\theta)\|$. Let $d \in W^{1,q}(\mathbb{R}_+, \mathbb{R}^m)$ and $r \in W^{1,q}(\mathbb{R}_+, \mathbb{R}^p)$. To estimate $\Pi_2$, define $S\colon \mathbb{Z}_+ \to \mathbb{R}^{n_p}$ and $M_2 \ge 0$ by

$$S(k) := \int_{k\tau}^{(k+1)\tau} e^{A_p[(k+1)\tau - s]} B_p d(s) ds \quad \text{and} \quad M_2 := \max_{t\in[0,\tau_0]} \|e^{A_p t} B_p\|\,.$$

It follows from the Hölder's inequality that

$$\begin{aligned} \|S\|_{\ell^q} &= \left( \sum_{k=0}^{\infty} \left\| \int_{k\tau}^{(k+1)\tau} e^{A_p((k+1)\tau - s)} B_p d(s) ds \right\|^q \right)^{1/q} \\ &\le \left( M_2^q \tau^{q-1} \sum_{k=0}^{\infty} \int_{k\tau}^{(k+1)\tau} \|d(s)\|^q ds \right)^{1/q} \\ &= M_2 \tau_0^{1-1/q} \|d\|_{L^q} \le M_2 \tau_0^{1-1/q} \|d\|_{W^{1,q}}\,. \end{aligned} \tag{4.30}$$

Setting

$$M_3 := \max\{M_2\|E_p D_c\|, \max_{t\in[0,\tau_0]} \|e^{A_c t} B_c E_c\|\}\,,$$

by (4.22), we see that

$$\begin{aligned} \|R(k)\| &\le \|S(k)\| + \left\| \int_0^\tau e^{A_p s} ds B_p E_p D_c [D_p d(k\tau) + r(k\tau)] \right\| \\ &\qquad + \left\| \int_0^\tau e^{A_c s} ds B_c E_c (D_p d(k\tau) + r(k\tau)) \right\| \\ &\le \|S(k)\| + 2\tau_0 M_3 (\|D_p\| \|d(k\tau)\| + \|r(k\tau)\|)\,, \quad \forall k \in \mathbb{Z}_+\,. \end{aligned} \tag{4.31}$$

Moreover, by Theorem 2.3.5, there exists $M_4 \ge 0$ such that

$$\|\mathcal{S}_\tau d\|_{\ell^q} \le M_4 \|d\|_{W^{1,q}}\,, \quad \|\mathcal{S}_\tau r\|_{\ell^q} \le M_4 \|r\|_{W^{1,q}}\,. \tag{4.32}$$

Consequently, combining (4.30) and (4.31) yields

$$\|R\|_{\ell^q} \leq \|S\|_{\ell^q} + 2\tau_0 M_3(\|D_p\|\|\mathcal{S}_\tau d\|_{\ell^q} + \|\mathcal{S}_\tau r\|_{\ell^q}) \leq M_5(\|d\|_{W^{1,q}} + \|r\|_{W^{1,q}})\,,$$

where $M_5 := \max\{M_2\tau_0^{1-1/q} + 2\tau_0 M_3 M_4\|D_p\|, 2\tau_0 M_3 M_4\}$. It follows from the power stability of $\Delta_\tau$ that

$$\|\Delta \star R\|_{\ell^q} \leq \|\Delta\|_{\ell^1}\|R\|_{\ell^q} \leq M_5\|\Delta\|_{\ell^1}(\|d\|_{W^{1,q}} + \|r\|_{W^{1,q}})\,. \tag{4.33}$$

Therefore, by (4.27b) and (4.33),

$$\begin{aligned}\|\Pi_2\|_{L^q} &\leq M_1\left(\sum_{k=0}^{\infty}\int_{(k+1)\tau}^{(k+2)\tau}\|(\Delta\star R)(k)\|^q dt\right)^{1/q} \\ &= M_1\tau^{1/q}\|\Delta\star R\|_{\ell^q} \leq M_1 M_5\tau_0^{1/q}\|\Delta\|_{\ell^1}(\|d\|_{W^{1,q}} + \|r\|_{W^{1,q}})\,. \end{aligned}\tag{4.34}$$

Finally, by (4.26) and (4.27c), for all $\theta \in [0,\tau)$ and for all $k \in \mathbb{Z}_+$,

$$\begin{aligned}&\|\Pi_3(k\tau+\theta)\| \\ &\leq \left\|C_p\int_{k\tau}^{k\tau+\theta} e^{A_p(k\tau+\theta-s)}B_p d(s)ds\right\| + \|D_p\|\|d(k\tau+\theta)\| + \\ &\qquad \left(\left\|C_p\int_0^\theta e^{A_p s}ds B_p E_p D_c\right\| + \|D_p E_p D_c\| + \|D_c E_c\|\right)(\|D_p\|\|d(k\tau)\| + \|r(k\tau)\|) \\ &\leq M_2\|C_p\|\int_{k\tau}^{(k+1)\tau}\|d(s)\|ds + \|D_p\|\|d(k\tau+\theta)\| + \\ &\qquad (\tau_0 M_3\|C_p\| + \|D_p E_p D_c\| + \|D_c E_c\|)(\|D_p\|\|d(k\tau)\| + \|r(k\tau)\|)\,. \end{aligned}\tag{4.35}$$

By the Hölder's inequality,

$$\begin{aligned}\left[\sum_{k=0}^{\infty}\left(\int_{k\tau}^{(k+1)\tau}\|d(s)\|ds\right)^q\right]^{1/q} &\leq \tau^{q-1}\left(\sum_{k=0}^{\infty}\int_{k\tau}^{(k+1)\tau}\|d(s)\|^q ds\right)^{1/q} \\ &\leq \tau_0^{q-1}\|d\|_{W^{1,q}}\,. \end{aligned}\tag{4.36}$$

Hence, by (4.32), (4.35), (4.36) and a routine argument, there exists $M_6 \geq 0$ such that

$$\|\Pi_3\|_{L^q} \leq M_6(\|d\|_{W^{1,q}} + \|r\|_{W^{1,q}})\,. \tag{4.37}$$

Combining (4.28), (4.29), (4.34) and (4.37), we conclude that there exists $M \geq 0$ such that if $\tau \in (0,\tau_0)$, then

$$\begin{aligned}\left\|\begin{pmatrix} y_p \\ w\end{pmatrix}\right\|_{L^q} &\leq \|\Pi_1\|_{L^q} + \|\Pi_2\|_{L^q} + \|\Pi_3\|_{L^q} \leq M(\|x^0\| + \|d\|_{W^{1,q}} + \|r\|_{W^{1,q}}) \\ &\qquad \forall x^0 \in \mathbb{R}^{n_p+n_c}\,,\ \forall d \in W^{1,q}(\mathbb{R}_+,\mathbb{R}^m)\,,\ \forall r \in W^{1,q}(\mathbb{R}_+,\mathbb{R}^p)\,. \end{aligned}$$

Using a similar argument, we can show that

$$\left\| \begin{pmatrix} y_p \\ w \end{pmatrix} \right\|_{L^\infty} \leq M(\|x^0\| + \|d\|_\infty + \|r\|_\infty)\,,$$
$$\forall x^0 \in \mathbb{R}^{n_p+n_c}\,,\ \forall d \in L_b(\mathbb{R}_+, \mathbb{R}^m)\,,\ \forall r \in L_b(\mathbb{R}_+, \mathbb{R}^p)\,.$$

□

**Remark 4.2.2.** (1) The constant $M$ does not depend on $\tau \in (0, \tau_0)$.

(2) If $D_p = 0$, i.e., if the plant (4.1) is strictly causal, then we only need to assume $d \in L^q(\mathbb{R}_+, \mathbb{R}^m)$, $r \in W^{1,q}(\mathbb{R}_+, \mathbb{R}^p)$ $(1 \leq q < \infty)$ for Theorem 4.2.1 to hold (with $\|d\|_{W^{1,q}}$ replaced by $\|d\|_{L^q}$). ◇

**Corollary 4.2.3.** *Assume the continuous-time dynamic feedback system* (4.5) *is exponentially stable. Consider the sampled-data system given by* (4.1)*,* (4.6) *and* (4.20a)*, as shown in Figure 4-2. If $d \in W^{1,q}(\mathbb{R}_+, \mathbb{R}^m)$ and $r \in W^{1,q}(\mathbb{R}_+, \mathbb{R}^p)$, $1 \leq q < \infty$, then, for sufficiently small $\tau > 0$,*

$$\lim_{t\to 0} \begin{pmatrix} y_p(t) \\ w(t) \end{pmatrix} = 0\,, \quad \forall x^0 \in \mathbb{R}^{n_p+n_c}\,,$$

*where $y_p, w$ are the outputs of the sampled-data system, as shown in Figure 4-2.*

*Proof.* By (4.28), we have

$$\left\| \begin{pmatrix} y_p(t) \\ w(t) \end{pmatrix} \right\| \leq \|\Pi_1(t)\| + \|\Pi_2(t)\| + \|\Pi_3(t)\|\,, \quad \forall t \geq 0\,,$$

where $\Pi_1$, $\Pi_2$ and $\Pi_3$ are defined in (4.27). It is sufficient to show that $\lim_{t\to\infty} \Pi_j(t) = 0$ for $j = 1, 2, 3$ if $\tau$ is sufficiently small.

It follows from Theorem 4.1.4 and the exponential stability of system (4.5) that there exists $\tau_0 > 0$ such that, for all $\tau \in (0, \tau_0)$, $\Delta_\tau$ is power stable, where $\Delta_\tau$ is defined in (4.11). Let $\tau \in (0, \tau_0)$ and set $M_1 := \max_{\theta\in[0,\tau_0]} \|Q(\theta)\|$, where $Q$ is defined in (4.25). By (4.27a),

$$\|\Pi_1(k\tau + \theta)\| \leq M_1 \|\Delta_\tau^k\| \|x^0\|\,, \quad \forall \theta \in [0, \tau)\,,\ \forall k \in \mathbb{Z}_+\,,$$

showing that $\lim_{t\to\infty} \|\Pi_1(t)\| = 0$ for all $x^0 \in \mathbb{R}^{n_p+n_c}$.

Let $1 \leq q < \infty$ and let $d \in W^{1,q}(\mathbb{R}_+, \mathbb{R}^m)$ and $r \in W^{1,q}(\mathbb{R}_+, \mathbb{R}^p)$. By (4.33), $(\Delta \star R) \in \ell^q(\mathbb{Z}_+, \mathbb{R}^{n_p+n_c})$, where $\Delta := (\Delta_\tau^k)_{k\in\mathbb{Z}_+}$ and $R$ is defined in (4.22). By (4.27b),

$$\|\Pi_2(k\tau + \theta)\| \leq M \|(\Delta \star R)(k - 1)\|\,, \quad \forall \theta \in [0, \tau)\,,\ \forall k \in \mathbb{N}\,,$$

showing that $\lim_{t\to\infty} \|\Pi_2(t)\| = 0$.

Finally,

$$\int_{k\tau}^{k\tau+\theta} \|e^{A_p(k\tau+\theta-s)}B_p d(s)\| ds \leq M_2 \int_{k\tau}^{(k+1)\tau} \|d(s)\| ds\,, \quad \forall \theta \in [0,\tau)\,,\ \forall k \in \mathbb{Z}_+\,.$$

where $M_2 := \max_t \in [0,\tau_0] \|e^{A_p t}B_p\|$. It follows from (4.36) that

$$\left(\int_{k\tau}^{(k+1)\tau} \|d(s)\| ds\right)_{k\in\mathbb{Z}_+} \in \ell^q(\mathbb{Z}_+,\mathbb{R})\,,$$

showing that

$$\int_{k\tau}^{k\tau+\theta} \|e^{A_p(k\tau+\theta-s)}B_p d(s)\| ds \to 0 \quad \text{as} \quad k\to\infty\,, \quad \forall\theta\in[0,\tau)\,.$$

Moreover, since $d \in W^{1,q}(\mathbb{R}_+,\mathbb{R}^m)$ and $r \in W^{1,q}(\mathbb{R}_+,\mathbb{R}^p)$, for $1 \leq q < \infty$, we have

$$\lim_{t\to\infty} d(t) = \lim_{k\to\infty} d(k\tau) = \lim_{k\to\infty} r(k\tau) = 0\,, \tag{4.38}$$

(see Theorem A.3.1 in the Appendix A.3). Hence, by (4.35) and (4.38), we conclude that $\lim_{t\to\infty} \Pi_3(t) = 0$. □

## 4.3 Notes and references

It was shown in Logemann *et al.* [39] that Corollary 4.1.5 can be extended to infinite-dimensional systems under certain conditions (see [39, Theorem 6.1]). Similar to the remarks of last chapter, whether Theorem 4.1.4 extends to infinite-dimensional systems is an open problem, since in the proof of Theorem 4.1.4, we use Theorem 3.1.5 which relies on certain properties of the spectrum of a matrix $A$ and on the convergence of the Taylor series of $e^{At}$: it certainly does not generalize to interesting infinite-dimensional situations in a straight forward way.

We have proved Theorem 4.2.1 using a state-space approach. Chen and Francis proved a similar result (see [1, Theorem 4] and [2, Theorem 9.4.1, p. 219]) using input-output methods, where it is assumed that the exogenous inputs are in $L^p(\mathbb{R}_+,\mathbb{R}^n)$ instead of $W^{1,p}(\mathbb{R}_+,\mathbb{R}^n)$ for $1 \leq p < \infty$. However, since the sampling operator does not necessarily map $L^p(\mathbb{R}_+,\mathbb{R}^n)$ to $\ell^p(\mathbb{R}_+,\mathbb{R}^n)$ for $1 \leq p < \infty$, Chen and Francis used a filter $F$ such that the composition $\mathcal{S}_\tau \circ F$ is bounded from $L^p(\mathbb{R}_+,\mathbb{R}^n)$ to $\ell^p(\mathbb{R}_+,\mathbb{R}^n)$. By Theorem 2.3.5, $\mathcal{S}_\tau$ is bounded from $W^{1,p}(\mathbb{R}_+,\mathbb{R}^n)$ to $\ell^p(\mathbb{R}_+,\mathbb{R}^n)$ for $1 \leq p < \infty$. In the light of this, the conditions on exogenous inputs in [1] (and [2]) and in Theorem 4.2.1 are the same. More importantly, in [1] and [2] it is required that the plant and controller are strictly causal (in particular ruling out static output feedback), whilst we allow for feedthroughs $D$ and $D_c$ in the plant and controller.

# Chapter 5

# Indirect sampled-data control with variable sampling period

In this chapter, we study indirect sampled-data control with variable sampling period: we consider (pre-specified) time-varying sampling period as well as sampling period which is updated on the basis of an adaptation rule.

## 5.1 Time-varying sampling period

In this section, we consider indirect sampled-data control with time-varying sampling period. This is relevant in some practical digital control applications, where computer overloading, networks, communication errors, etc. may cause delays and sampling period jitter. Moreover, the analysis of sampled-data systems with time-varying sampling period is important in the context of problems where the sampling period is determined by an adaptive feedback mechanism.

Consider the continuous-time state feedback system again

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t)\,; \quad x(0) = x^0 \in \mathbb{R}^n\,, && (5.1a)\\
u(t) &= Fx(t)\,, && (5.1b)
\end{aligned}
$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $F \in \mathbb{R}^{m\times n}$.

Let $\boldsymbol{t} := (t_k)_{k\in\mathbb{Z}_+}$ be a sequence of sampling points, where

$$
t_0 = 0\,, \qquad t_{k+1} > t_k\,, \quad \forall k \in \mathbb{Z}_+\,, \qquad t_k \to \infty \ \text{ as } \ k \to \infty\,. \tag{5.2}
$$

The sampling period $\tau_k := t_{k+1} - t_k$ is not constant anymore, and becomes *time-varying*. We use sampling and hold in (5.1b) to obtain the corresponding sampled-data feedback system

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t)\,; \quad x(0) = x^0 \in \mathbb{R}^n\,, && (5.3a)\\
u(t) &= Fx(t_k)\,, \quad \forall t \in [t_k, t_{k+1})\,. && (5.3b)
\end{aligned}
$$

Define

$$\underline{\tau} := \liminf_{k\to\infty}\{\tau_k\}\,, \quad \overline{\tau} := \limsup_{k\to\infty}\{\tau_k\}\,.$$

Let $x(\,\cdot\,;x^0,\boldsymbol{t})$ denote the solution of (5.3). The variation-of-parameters formula yields

$$x(t_k+\theta;x^0,\boldsymbol{t}) = \left(e^{A\theta} + \int_0^\theta e^{As}ds BF\right) x(t_k;x^0,\boldsymbol{t})\,, \quad \forall\theta\in[0,\tau_k)\,,\forall k\in\mathbb{Z}_+\,. \tag{5.4}$$

Set $x_k := x(t_k;x^0,\boldsymbol{t})$ for all $k\in\mathbb{Z}_+$. It follows from (5.4) with $\theta=\tau_k$ that

$$x_{k+1} = \Delta_k x_k\,, \quad \forall k\in\mathbb{Z}_+\,, \tag{5.5}$$

where

$$\Delta_k := e^{A\tau_k} + \int_0^{\tau_k} e^{As}ds BF\,.$$

Set $D := \{(k,l)\in\mathbb{Z}_+\times\mathbb{Z}_+ : k\geq l\}$. The matrices $(\Delta_k)_{k\in\mathbb{Z}_+}$ generate an evolution operator $\Phi(\cdot,\cdot)\colon D\to\mathbb{R}^{n\times n}$, satisfying

$$\Phi(k,l) = \begin{cases} I_n\,, & \text{if } k=l \\ \Delta_{k-1}\Delta_{k-2}\cdots\Delta_l\,, & \text{if } k>l \end{cases}\,. \tag{5.6}$$

For every $l\in\mathbb{Z}_+$ and every $z\in\mathbb{R}^n$, the solution of the initial value problem

$$x_{k+1} = \Delta_k x_k\,, \quad k\geq l\,; \qquad x_l = z$$

is given by

$$k\mapsto \Phi(k,l)z\,, \quad k\geq l\,.$$

**Definition 5.1.1.** A number $\rho>0$ is said to be a *power rate* of (5.5), if for every $l\in\mathbb{Z}_+$, there exists $M_l\geq 1$ such that

$$\|\Phi(k,l)\| \leq M_l\rho^{k-l}\,, \quad \forall k\geq l\,.$$

$\diamond$

Let $\omega_c$ denote the exponential growth of (5.1).

**Theorem 5.1.2.** *Let $\alpha>\omega_c$. Assume that $\sup_{k\in\mathbb{Z}_+}\{\tau_k\}<\infty$ and $\overline{\tau}>0$. There exists $\tau^*>0$ such that, if $\overline{\tau}\in(0,\tau^*)$, then the following statements hold:*

(1) *If $\omega_c\geq 0$, then $\rho=e^{\alpha\overline{\tau}}$ is a power rate of system (5.5).*

(2) *If $\omega_c<0$, then $\rho=e^{\alpha\underline{\tau}}$ is a power rate of system (5.5).*

*Proof.* Let $\alpha>\omega_c$ and $\beta\in(\omega_c,\alpha)$, where we choose $\alpha<0$ if $\omega_c<0$. We know that $A+BF-\beta I$ is exponentially stable. There exists $P=P^T$ with $P>0$ such that

$$(A+BF-\beta I)^T P + P(A+BF-\beta I) = -I\,,$$

(see, for example, [71, Theorem 18, p. 231]). Thus

$$(A+BF)^T P + P(A+BF) = 2\beta P - I\,. \tag{5.7}$$

Define a new norm $\|\cdot\|_P$ on $\mathbb{R}^n$ by setting

$$\|z\|_P^2 := \langle z, Pz\rangle\,, \quad \forall z \in \mathbb{R}^n\,.$$

Note that $\|\cdot\|_P$ is equivalent to the Euclidean norm $\|\cdot\|$, i.e, there exist $\lambda, \Lambda > 0$ such that

$$\lambda\|z\|^2 \le \|z\|_P^2 \le \Lambda\|z\|^2\,, \quad \forall z \in \mathbb{R}^n\,. \tag{5.8}$$

(Possible choices for $\lambda$ and $\Lambda$ are the smallest and largest eigenvalues of $P$, respectively.) To prove that $\rho$ is a power rate of system (5.5), we need to show that for every $l \in \mathbb{Z}_+$, there exists $M_l \ge 1$ such that

$$\|\rho^{-(k-l)}\Phi(k,l)\|_P \le M_l\,, \quad \forall k \ge l\,.$$

It is sufficient to prove that there exists $N_l \ge l$ such that $(\|\rho^{-(k-l)}\Phi(k,l)\|_P)_{k\ge N_l}$ is non-increasing, i.e.,

$$\|\rho^{-(k+1-l)}\Phi(k+1,l)\|_P \le \|\rho^{-(k-l)}\Phi(k,l)\|_P\,, \quad \forall k \ge N_l\,.$$

To this end, by (5.6) and since $\rho > 0$, it is sufficient to prove that $\|\rho^{-1}\Delta_k\|_P \le 1$ for almost all $k \in \mathbb{Z}_+$ (a.a. $k \in \mathbb{Z}_+$), that is,

$$\|\rho^{-1}\Delta_k z\|_P^2 - \|z\|_P^2 \le 0\,, \quad \text{a.a. } k \in \mathbb{Z}_+\,,\ \forall z \in \mathbb{R}^n\,.$$

Using the power series expansion of $e^{At}$,

$$\Delta_k = I + \tau_k(A+BF) + \tau_k^2\Gamma(\tau_k)\,, \quad \forall k \in \mathbb{Z}_+\,, \tag{5.9}$$

where

$$\Gamma(\tau_k) := \frac{1}{2}A(A+BF) + \frac{\tau_k}{3!}A^2(A+BF) + \ldots + \frac{\tau_k^j}{(j+2)!}A^{j+1}(A+BF) + \ldots\,.$$

Fix $\tau^0 > 0$ and only consider sequences $(\tau_k)_{k\in\mathbb{Z}_+}$ such that $\sup_{k\in\mathbb{Z}_+}\tau_k \le \tau^0$. There exists $M \ge 0$ such that $\|\Gamma(\tau_k)\| \le M$ for all $k \in \mathbb{Z}_+$. Let $z \in \mathbb{R}^n$. By (5.7), (5.9) and the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\|\rho^{-1}\Delta_k z\|_P^2 - \|z\|_P^2 \\
&\quad = \rho^{-2}\langle\Delta_k z, P\Delta_k z\rangle - \langle z, Pz\rangle \\
&\quad \le \rho^{-2}\langle[I+\tau_k(A+BF)]z, P[I+\tau_k(A+BF)]z\rangle - \langle z, Pz\rangle + O(\tau_k^2)\|z\|^2 \\
&\quad \le (\rho^{-2}-1)\langle z, Pz\rangle + \rho^{-2}\tau_k\langle z, [(A+BF)^T P + P(A+BF)]z\rangle + O(\tau_k^2)\|z\|^2 \\
&\quad = (\rho^{-2}-1)\|z\|_P^2 + \rho^{-2}\tau_k(2\beta\|z\|_P^2 - \|z\|^2) + O(\tau_k^2)\|z\|^2 \\
&\quad = [\rho^{-2}(1+2\beta\tau_k)-1]\|z\|_P^2 - \rho^{-2}\tau_k\|z\|^2 + O(\tau_k^2)\|z\|^2\,, \quad \forall k \in \mathbb{Z}_+\,.
\end{aligned} \tag{5.10}$$

We first prove Statement (1). Assume that $\omega_c \geq 0$. There exists $\mu > 1$ such that $0 < \beta\mu \leq \alpha$. Setting $\rho = e^{\alpha\overline{\tau}}$, we have

$$\begin{aligned}
\rho^{-2}(1+2\beta\tau_k) - 1 &= e^{-2\alpha\overline{\tau}}(1+2\beta\tau_k) - 1 \\
&= [1 - 2\alpha\overline{\tau} + O(\overline{\tau}^2)](1+2\beta\tau_k) - 1 \\
&\leq [1 - 2\beta\mu\overline{\tau} + O(\overline{\tau}^2)](1+2\beta\tau_k) - 1 \\
&\leq (-2\beta)(\mu\overline{\tau} - \tau_k) - 4\beta^2\mu\tau_k\overline{\tau} + O(\overline{\tau}^2)\,, \quad \forall k \in \mathbb{Z}_+\,. \quad (5.11)
\end{aligned}$$

By the definition of $\overline{\tau}$, $\tau_k \leq \mu\overline{\tau}$ for almost all $k \in \mathbb{Z}_+$. Consequently, by (5.10) and (5.11), we have

$$\begin{aligned}
\|e^{-\alpha\overline{\tau}}\Delta_k z\|_P^2 - \|z\|_P^2 &\leq (-2\beta)(\mu\overline{\tau} - \tau_k)\|z\|_P^2 + O(\overline{\tau}^2)\|z\|_P^2 - e^{-2\alpha\overline{\tau}}\tau_k\|z\|^2 \\
&\quad + O(\tau_k^2)\|z\|^2\,, \quad \text{a.a. } k \in \mathbb{Z}_+\,. \quad (5.12)
\end{aligned}$$

Define

$$K_1 := \{k \in \mathbb{Z}_+ : (5.12) \text{ holds}, \tau_k \leq \overline{\tau}\}\,, \quad K_2 := \{k \in \mathbb{Z}_+ : (5.12) \text{ holds}, \overline{\tau} < \tau_k \leq \mu\overline{\tau}\}\,.$$

Then $K_1 \cup K_2$ is *co-finite* in $\mathbb{Z}_+$ (i.e., $\mathbb{Z}_+ \setminus (K_1 \cup K_2)$ is finite).
*Case 1:* $k \in K_1$.
It follows from (5.12) and the positivity of $\beta$ that

$$\|e^{-\alpha\overline{\tau}}\Delta_k z\|_P^2 - \|z\|_P^2 \leq [(-2\beta)(\mu - 1)\overline{\tau} + O(\overline{\tau}^2)]\|z\|_P^2 + [-e^{-2\alpha\overline{\tau}}\tau_k + O(\tau_k^2)]\|z\|^2\,.$$

Since $\beta > 0$ and $\mu > 1$, there exists $\tau^* \in (0, \tau^0)$ such that if $\overline{\tau} \in (0, \tau^*)$ (and hence $\tau_k \in (0, \tau^*)$), then $\|e^{-\alpha\overline{\tau}}\Delta_k z\|_P^2 - \|z\|_P^2 \leq 0$ for all $k \in K_1$.
*Case 2:* $k \in K_2$.
By (5.8) and (5.12),

$$\begin{aligned}
&\|e^{-\alpha\overline{\tau}}\Delta_k z\|_P^2 - \|z\|_P^2 \\
&\leq (-2\beta)(\mu\overline{\tau} - \tau_k)\|z\|_P^2 + \left[-\frac{1}{2}e^{-2\alpha\overline{\tau}}\tau_k + \Lambda O(\overline{\tau}^2)\right]\|z\|^2 + \left[-\frac{1}{2}e^{-2\alpha\overline{\tau}}\tau_k + O(\tau_k^2)\right]\|z\|^2 \\
&\leq (-2\beta)(\mu\overline{\tau} - \tau_k)\|z\|_P^2 + \left[-\frac{1}{2}e^{-2\alpha\overline{\tau}}\overline{\tau} + O(\overline{\tau}^2)\right]\|z\|^2 + \left[-\frac{1}{2}e^{-2\alpha\overline{\tau}}\tau_k + O(\tau_k^2)\right]\|z\|^2\,.
\end{aligned}$$

Consequently, there exists $\tau^{**} \in (0, \tau^*)$ such that if $\overline{\tau} \in (0, \tau^{**})$ (and hence $\tau_k \in (0, \mu\tau^{**})$), then $\|e^{-\alpha\overline{\tau}}\Delta_k z\|_P^2 - \|z\|_P^2 \leq 0$ for all $k \in K_2$. Combining the above two cases, if $\overline{\tau} \in (0, \tau^{**})$, then $\|e^{-\alpha\overline{\tau}}\Delta_k z\|_P^2 - \|z\|_P^2 \leq 0$ for almost all $k \in \mathbb{Z}_+$. Hence $e^{\alpha\overline{\tau}}$ is a power rate of (5.5) if $\overline{\tau}$ is sufficiently small. This finishes the proof of Statement (1).

To prove Statement (2), assume that $\omega_c < 0$, so that $\omega_c < \beta < \alpha < 0$. There exists $\theta \in (0, 1)$ such that $\beta\theta \leq \alpha$. Setting $\rho = e^{\alpha\underline{\tau}}$, a straightforward calculation shows that

$$e^{-2\alpha\underline{\tau}}(1+2\beta\tau_k) - 1 \leq (-2\beta)(\theta\underline{\tau} - \tau_k) - 4\beta^2\theta\tau_k\underline{\tau} + O(\underline{\tau}^2)\,, \quad \forall k \in \mathbb{Z}_+\,. \quad (5.13)$$

By the definition of $\overline{\tau}$ and $\underline{\tau}$,

$$\theta\underline{\tau} \leq \tau_k \leq \frac{3}{2}\overline{\tau}\,, \quad \text{a.a. } k \in \mathbb{Z}_+\,.$$

Consequently, by (5.10) and (5.13), we have

$$\begin{aligned}
&\|e^{-\alpha\underline{\tau}}\Delta_k z\|_P^2 - \|z\|_P^2 \\
&\qquad \leq \quad (-2\beta)(\theta\underline{\tau} - \tau_k)\|z\|_P^2 + O(\underline{\tau}^2)\|z\|_P^2 - e^{-2\alpha\underline{\tau}}\tau_k\|z\|^2 + O(\tau_k^2)\|z\|^2 \\
&\qquad \leq \quad (-2\beta)(\theta\underline{\tau} - \tau_k)\|z\|_P^2 - e^{-2\alpha\underline{\tau}}\tau_k\|z\|^2 + O(\tau_k^2)\|z\|^2\,, \quad \text{a.a. } k \in \mathbb{Z}_+\,.
\end{aligned}$$

Since $\beta < 0$, we conclude that there exists $\tau^* > 0$ such that if $\overline{\tau} \in (0, \tau^*)$ (and hence $\tau_k \in (0, (3/2)\tau^*)$), we have

$$\|e^{-\alpha\underline{\tau}}\Delta_k z\|_P^2 - \|z\|_P^2 \leq 0\,, \quad \text{a.a. } k \in \mathbb{Z}_+\,,$$

showing that $e^{\alpha\underline{\tau}}$ is a power rate of (5.5) if $\overline{\tau}$ is sufficiently small. □

Alternatively, we have a second proof for Theorem 5.1.2.

*Alternative proof of Theorem 5.1.2.* Let $\alpha > \omega_c$ and $\beta \in (\omega_c, \alpha)$, where we choose $\alpha < 0$ if $\omega_c < 0$. There exists $M \geq 1$ such that

$$\|e^{(A+BF)t}z\| \leq Me^{\beta t}\|z\|\,, \quad \forall t \geq 0\,,\ \forall z \in \mathbb{R}^n\,.$$

We introduce a new norm $|\cdot|$ on $\mathbb{R}^n$ by setting

$$|z| := \sup_{t\geq 0}\|e^{-\beta t}e^{(A+BF)t}z\|\,, \quad \forall z \in \mathbb{R}^n\,.$$

It is clear that

$$\|z\| \leq |z| \leq M\|z\|\,, \quad \forall z \in \mathbb{R}^n\,, \tag{5.14}$$

showing that $|\cdot|$ is equivalent to the Euclidean norm $\|\cdot\|$. Moreover,

$$\begin{aligned}
|e^{(A+BF)t}z| &= \sup_{s\geq 0}\|e^{-\beta s}e^{(A+BF)(t+s)}z\| \\
&= e^{\beta t}\sup_{s\geq 0}\|e^{-\beta(t+s)}e^{(A+BF)(t+s)}z\| \\
&\leq e^{\beta t}\sup_{s\geq 0}\|e^{-\beta s}e^{(A+BF)s}z\| \\
&= e^{\beta t}|z|\,, \quad \forall z \in \mathbb{R}^n\,,\ \forall t \geq 0\,.
\end{aligned} \tag{5.15}$$

Similar to the first proof of Theorem 5.1.2, if we can show that there exists $\rho > 0$ such that

$$|\rho^{-1}\Delta_k z| - |z| \leq 0\,, \quad \text{a.a. } k \in \mathbb{Z}_+\,,\ \forall z \in \mathbb{R}^n\,,$$

for sufficiently small $\overline{\tau}$, then $\rho$ a power rate of (5.5). To this end, define $W\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ by

$$W(t) := \int_0^t e^{As} BF \left( I - e^{(A+BF)(t-s)} \right) ds\,.$$

Fix $\tau^0 > 0$ and only consider $(\tau_k)_{k\in\mathbb{Z}_+}$ such that $\sup_{k\to\infty} \tau_k \le \tau^0$. Then there exists $M_1 \ge 1$ such that

$$\|e^{As}BF\|\|I - e^{(A+BF)(\tau_k - s)}\| \le 2M_1(\tau_k - s)\,, \quad \forall s \in [0, \tau_k]\,,\ \forall k \in \mathbb{Z}_+\,.$$

Therefore,

$$\|W(\tau_k)\| \le \int_0^{\tau_k} 2M_1(\tau_k - s)ds \le M_1\tau_k^2\,, \quad \forall k \in \mathbb{Z}_+\,. \tag{5.16}$$

By Lemma 3.2.3, we know that

$$\Delta_k = e^{(A+BF)\tau_k} + W(\tau_k)\,, \quad \forall k \in \mathbb{Z}_+\,. \tag{5.17}$$

First, we assume that $\omega_c \ge 0$. By (5.14)–(5.17), we obtain

$$\begin{aligned} |e^{-\alpha\overline{\tau}}\Delta_k z| - |z| &\le e^{-\alpha\overline{\tau}}|e^{(A+BF)\tau_k}z| + e^{-\alpha\overline{\tau}}|W(\tau_k)z| - |z| \\ &\le (e^{\beta\tau_k - \alpha\overline{\tau}} + e^{-\alpha\overline{\tau}}M\|W(\tau_k)\| - 1)|z| \\ &\le (e^{-(\alpha\overline{\tau} - \beta\tau_k)} + e^{-\alpha\overline{\tau}}MM_1\tau_k^2 - 1)|z|\,, \quad \forall k \in \mathbb{Z}_+\,. \end{aligned} \tag{5.18}$$

Choose $\gamma_1 > 0$ such that $\beta + \gamma_1 < \alpha$. By the definition of $\overline{\tau}$,

$$\tau_k \le \frac{\alpha}{\beta + \gamma_1}\overline{\tau}\,, \quad \text{a.a. } k \in \mathbb{Z}_+\,.$$

Consequently,

$$\beta\tau_k \le (\beta + \gamma_1)\tau_k \le \alpha\overline{\tau}\,, \quad \text{a.a. } k \in \mathbb{Z}_+\,.$$

Invoking the fact that $e^{-t} \le 1 - te^{-t}$ for all $t \ge 0$, it follows from (5.18) that

$$\begin{aligned} &|e^{-\alpha\overline{\tau}}\Delta_k z| - |z| \\ &\quad\le \left( -(\alpha\overline{\tau} - \beta\tau_k)e^{-(\alpha\overline{\tau} - \beta\tau_k)} + e^{-\alpha\overline{\tau}}MM_1\tau_k^2 \right)|z| \\ &\quad= \left( -\left[\alpha\overline{\tau} - (\beta + \gamma_1)\tau_k\right] e^{-(\alpha\overline{\tau} - \beta\tau_k)} - \gamma_1 e^{-(\alpha\overline{\tau} - \beta\tau_k)}\tau_k + e^{-\alpha\overline{\tau}}MM_1\tau_k^2 \right)|z| \\ &\quad\le \left( -\left[\alpha\overline{\tau} - (\beta + \gamma_1)\tau_k\right] e^{-(\alpha\overline{\tau} - \beta\tau_k)} - \gamma_1 e^{-\alpha\tau^0}\tau_k + MM_1\tau_k^2 \right)|z|\,, \quad \text{a.a. } k \in \mathbb{Z}_+\,. \end{aligned}$$

Hence if $\overline{\tau}$ is sufficiently small, then $|e^{-\alpha\overline{\tau}}\Delta_k z| - |z| \le 0$ for almost all $k \in \mathbb{Z}_+$. This completes the proof of Statement (1).

Next we assume that $\omega_c < 0$, so that $\omega_c < \beta < \alpha < 0$. Invoking (5.14)–(5.17) and the same argument used to obtain (5.18), we conclude that

$$|e^{-\alpha\underline{\tau}}\Delta_k z| - |z| \le (e^{-(\alpha\underline{\tau} - \beta\tau_k)} + e^{-\alpha\underline{\tau}}MM_1\tau_k^2 - 1)|z|\,, \quad \forall k \in \mathbb{Z}_+\,.$$

Choose $\gamma_2 > 0$ such that $\beta + \gamma_2 < \alpha$. By the definition of $\overline{\tau}$ and $\underline{\tau}$,

$$\frac{\alpha}{\beta+\gamma_2}\,\underline{\tau} \,\le\, \tau_k \,\le\, \frac{3}{2}\,\overline{\tau}\,, \quad \text{a.a. } k \in \mathbb{Z}_+\,.$$

Consequently,

$$\beta\tau_k \le (\beta+\gamma_2)\tau_k \le \alpha\underline{\tau}\,, \quad \text{a.a. } k \in \mathbb{Z}_+\,.$$

Since $e^{-t} \le 1 - te^{-t}$ for all $t \ge 0$, it follows that

$$\begin{aligned}
&|e^{-\alpha\underline{\tau}}\Delta_k z| - |z| \\
&\quad \le \;\; \left(-(\alpha\underline{\tau} - \beta\tau_k)e^{-(\alpha\underline{\tau}-\beta\tau_k)} + e^{-\alpha\underline{\tau}}MM_1\tau_k^2\right)|z| \\
&\quad = \;\; \left(-\left[\alpha\underline{\tau} - (\beta+\gamma_2)\tau_k\right]e^{-(\alpha\underline{\tau}-\beta\tau_k)} - \gamma_2 e^{-(\alpha\underline{\tau}-\beta\tau_k)}\tau_k + e^{-\alpha\underline{\tau}}MM_1\tau_k^2\right)|z| \\
&\quad \le \;\; \left(-\left[\alpha\underline{\tau} - (\beta+\gamma_2)\tau_k\right]e^{-(\alpha\underline{\tau}-\beta\tau_k)} - \gamma_2 e^{\beta\tau^0}\tau_k + e^{-\alpha\underline{\tau}}MM_1\tau_k^2\right)|z|\,, \quad \text{a.a. } k \in \mathbb{Z}_+\,.
\end{aligned}$$

Hence, if $\overline{\tau}$ is sufficiently small, then $|e^{-\alpha\underline{\tau}}\Delta_k z| - |z| \le 0$ for almost all $k \in \mathbb{Z}_+$. This proves Statement (2). □

**Definition 5.1.3.** A number $\alpha$ is said to be an *exponential rate* of (5.3) if there exists a constant $M \ge 1$ such that

$$\|x(t;x^0,\boldsymbol{t})\| \le Me^{\alpha t}\|x^0\|\,, \quad \forall t \ge 0\,,\ \forall x^0 \in \mathbb{R}^n\,.$$

We say that system (5.3) is *exponentially stable* if it has a negative exponential rate. ◇

The following theorem shows that an exponential rate of the continuous-time feedback system (5.1) is also an exponential rate of the sampled-data system (5.3) under certain conditions.

**Theorem 5.1.4.** *Let $\alpha > \omega_c$. Assume that $\lim_{k\to\infty}\tau_k = \tau_\infty \in (0,\infty)$. If $\tau_\infty$ is sufficiently small, then $\alpha$ is an exponential rate of* (5.3)*.*

*Proof.* By assumption, $\overline{\tau} = \underline{\tau} = \tau_\infty$. Let $\alpha > \omega_c$, where we choose $\alpha < 0$ if $\omega_c < 0$. It follows from Theorem 5.1.2 that there exists $\tau^* > 0$ such that if $\tau_\infty \in (0,\tau^*)$, then $e^{\alpha\tau_\infty}$ is a power rate of (5.5). Assume that $\tau_\infty \in (0,\tau^*)$.

We first assume that $\omega_c < 0$, so that $\alpha \in (\omega_c, 0)$. For sufficiently small $\varepsilon > 0$, $e^{\alpha(\tau_\infty+\varepsilon)}$ is also an power rate of (5.5). Hence there exists $M \ge 1$ such that

$$\|x(t_k;x^0,\boldsymbol{t})\| = \|\Phi(k,0)x^0\| \le Me^{\alpha(\tau_\infty+\varepsilon)k}\|x^0\|\,, \quad \forall k \in \mathbb{Z}_+\,,\ \forall x^0 \in \mathbb{R}^n\,. \tag{5.19}$$

where $\Phi(\cdot,\cdot)$ is defined in (5.6). There exists $N \in \mathbb{N}$ such that $\tau_k \le \tau_\infty + \varepsilon$ for all $k \ge N$, that is, $t_{k+1} - t_k \le \tau_\infty + \varepsilon$ for all $k \ge N$. Summing over $k$, we obtain

$$t_k - t_N \le (k-N)(\tau_\infty+\varepsilon)\,, \quad \forall k \ge N\,. \tag{5.20}$$

Since $\alpha < 0$, by (5.19) and (5.20),

$$\begin{aligned} \|x(t_k; x^0, \boldsymbol{t})\| &\leq Me^{\alpha(\tau_\infty+\varepsilon)N} e^{\alpha(\tau_\infty+\varepsilon)(k-N)} \|x^0\| \\ &\leq Me^{\alpha(\tau_\infty+\varepsilon)N} e^{\alpha(t_k - t_N)} \|x^0\| \\ &= Me^{\alpha[(\tau_\infty+\varepsilon)N - t_N]} e^{\alpha t_k} \|x^0\|, \quad \forall k \geq N, \ \forall x^0 \in \mathbb{R}^n. \end{aligned} \tag{5.21}$$

Set

$$\gamma := \sup_{k\in\mathbb{Z}_+} \{\tau_k\} \quad \text{and} \quad M_1 := \max_{t\in[0,\gamma]} \left\| e^{At} + \int_0^t e^{As} ds BF \right\|. \tag{5.22}$$

*Case 1:* $t \geq t_N$.

Then $t \in [t_k, t_{k+1})$ for some $k \geq N$. By (5.4) and (5.21), we obtain that

$$\begin{aligned} \|x(t; x^0, \boldsymbol{t})\| &\leq M_1 \|x(t_k; x^0, \boldsymbol{t})\| \\ &\leq M_1 M e^{\alpha[(\tau_\infty+\varepsilon)N - t_N]} e^{-\alpha(t-t_k)} e^{\alpha t} \|x^0\| \\ &\leq M_2 e^{\alpha t} \|x^0\|, \quad \forall x^0 \in \mathbb{R}^n, \end{aligned}$$

where $M_2 := MM_1 e^{\alpha[(\tau_\infty+\varepsilon)N - t_N]} e^{-\alpha\gamma}$.

*Case 2:* $t \in [0, t_N)$.

Then $t \in [t_k, t_{k+1})$ for some $k \in \{0, 1, \dots, N-1\}$. Set

$$M_3 := M_1 e^{-\alpha t_N} \max\{\|\Phi(k, 0)\| : k \in \{0, 1, \dots, N-1\}\}.$$

By (5.4), we have

$$\begin{aligned} \|x(t; x^0, \boldsymbol{t})\| \leq M_1 \|x(t_k; x^0, \boldsymbol{t})\| &\leq M_1 \|\Phi(k, 0)\| \|x^0\| \\ &\leq M_1 \|\Phi(k, 0)\| e^{-\alpha t_N} e^{\alpha t} \|x^0\| \\ &\leq M_3 e^{\alpha t} \|x^0\|, \quad \forall x^0 \in \mathbb{R}^n. \end{aligned}$$

By the two cases above, we see that $\alpha$ is an exponential rate of (5.3) if $\tau_\infty$ is sufficiently small.

If $\omega_c \geq 0$, then for sufficiently small $\varepsilon > 0$, $e^{\alpha(\tau_\infty - \varepsilon)}$ is also a power rate of (5.5). There exists $\tilde{N} \in \mathbb{N}$ such that $\tau_k \geq \tau_\infty - \varepsilon$ for all $k \geq \tilde{N}$, that is, $t_{k+1} - t_k \geq \tau_\infty - \varepsilon$ for all $k \geq \tilde{N}$. Summing over $k$, we obtain

$$(k - \tilde{N})(\tau_\infty - \varepsilon) \leq t_k - t_{\tilde{N}}, \quad \forall k \geq \tilde{N}.$$

Consequently, since $\alpha > 0$ and $e^{\alpha(\tau_\infty - \varepsilon)}$ is also a power rate of (5.5), there exists $\tilde{M} \geq 1$ such that

$$\begin{aligned} \|x(t_k; x^0, \boldsymbol{t})\| &\leq \tilde{M} e^{\alpha(\tau_\infty-\varepsilon)\tilde{N}} e^{\alpha(\tau_\infty-\varepsilon)(k-\tilde{N})} \|x^0\| \\ &\leq \tilde{M} e^{\alpha(\tau_\infty-\varepsilon)\tilde{N}} e^{\alpha(t_k - t_{\tilde{N}})} \|x^0\| \\ &= \tilde{M} e^{\alpha[(\tau_\infty-\varepsilon)\tilde{N} - t_{\tilde{N}}]} e^{\alpha t_k} \|x^0\|, \quad \forall k \geq \tilde{N}, \ \forall x^0 \in \mathbb{R}^n. \end{aligned} \tag{5.23}$$

*Case 1:* $t \geq t_{\tilde{N}}$.

Then $t \in [t_k, t_{k+1})$ for some $k \geq \tilde{N}$. Set $\tilde{M}_2 := \tilde{M} M_1 e^{\alpha[(\tau_\infty - \varepsilon)\tilde{N} - t_{\tilde{N}}]} e^{\alpha\gamma}$, where $\gamma$ and $M_1$ are defined in (5.22). By (5.4) and (5.23), we obtain that

$$\begin{aligned}\|x(t; x^0, \boldsymbol{t})\| &\leq M_1 \|x(t_k; x^0, \boldsymbol{t})\| \\ &\leq M_1 \tilde{M} e^{\alpha[(\tau_\infty - \varepsilon)\tilde{N} - t_{\tilde{N}}]} e^{-\alpha(t - t_k)} e^{\alpha t} \|x^0\| \\ &\leq \tilde{M}_2 e^{\alpha t} \|x^0\|, \quad \forall x^0 \in \mathbb{R}^n.\end{aligned}$$

*Case 2:* $t \in [0, t_{\tilde{N}})$.

Then $t \in [t_k, t_{k+1})$ for some $k \in \{0, 1, \ldots, \tilde{N} - 1\}$. Set

$$\tilde{M}_3 := M_1 \max\{\|\Phi(k, 0)\| : k \in \{0, 1, \ldots, \tilde{N} - 1\}\}.$$

By (5.4), we have

$$\begin{aligned}\|x(t; x^0, \boldsymbol{t})\| \leq M_1 \|x(t_k; x^0, \boldsymbol{t})\| &\leq M_1 \|\Phi(k, 0)\| \|x^0\| \\ &\leq M_1 \|\Phi(k, 0)\| e^{\alpha t} \|x^0\| \\ &\leq \tilde{M}_3 e^{\alpha t} \|x^0\|, \quad \forall x^0 \in \mathbb{R}^n.\end{aligned}$$

By the two cases above, we see that $\alpha$ is an exponential rate of (5.3) if $\tau_\infty$ is sufficiently small. □

We state and prove the following theorem, which is crucial to the proof of Theorem 5.2.2, the main result of next section.

**Theorem 5.1.5.** *Assume that the continuous-time feedback system* (5.1) *is exponentially stable. If* $(\tau_k)_{k \in \mathbb{Z}_+}$ *satisfies the conditions*

$$\lim_{k \to \infty} \tau_k = 0 \quad \textit{and} \quad \inf_{k \in \mathbb{N}} \{\tau_k k^\alpha\} > 0 \quad \textit{for some } \alpha \in (0, 1), \tag{5.24}$$

*then the solution* $(x_k)_{k \in \mathbb{Z}_+}$ *of* (5.5) *is in* $\ell^1(\mathbb{Z}_+, \mathbb{R}^n)$.

*Proof.* It follows from the exponential stability of (5.1) that there exists $P = P^T$, $P > 0$ such that

$$(A + BF)^T P + P(A + BF) = -I \tag{5.25}$$

(see, for example, [71, Theorem 18, p. 231]). We define a norm $\|\cdot\|_P$ on $\mathbb{R}^n$ by setting $\|z\|_P^2 := \langle z, Pz \rangle$, as in the proof of Theorem 5.1.2. Thus there exist $\lambda, \Lambda > 0$ such that

$$\lambda \|z\|^2 \leq \|z\|_P^2 \leq \Lambda \|z\|^2, \quad \forall z \in \mathbb{R}^n. \tag{5.26}$$

Using the power series expansion of $e^{At}$,

$$\Delta_k = I + \tau_k (A + BF) + \tau_k^2 \Gamma(\tau_k), \quad \forall k \in \mathbb{Z}_+,$$

where

$$\Gamma(\tau_k) := \frac{1}{2}A(A+BF) + \frac{\tau_k}{3!}A^2(A+BF) + \ldots + \frac{\tau_k^j}{(j+2)!}A^{j+1}(A+BF) + \ldots .$$

The convergence of $(\tau_k)_{k\in\mathbb{Z}_+}$ implies $(\tau_k)_{k\in\mathbb{Z}_+}$ is bounded. Hence, the sequence $(\|\Gamma(\tau_k)\|)_{k\in\mathbb{Z}_+}$ is bounded. Therefore, by the Cauchy-Schwarz inequality and (5.25), there exists $M_1 \geq 0$ such that

$$\begin{aligned} \|x_{k+1}\|_P^2 - \|x_k\|_P^2 &= \langle \Delta_k x_k, P\Delta_k x_k\rangle - \langle x_k, Px_k\rangle \\ &\leq \tau_k \langle x_k, \left[(A+BF)^T P + P(A+BF)\right] x_k\rangle + M_1\tau_k^2\|x_k\|^2 \\ &\leq -\tau_k\|x_k\|^2 + M_1\tau_k^2\|x_k\|^2 , \quad \forall k \in \mathbb{Z}_+ . \end{aligned} \tag{5.27}$$

Since $\lim_{k\to\infty}\tau_k = 0$, there exists $N \in \mathbb{Z}_+$ such that

$$-\tau_k\|x_k\|^2 + M_1\tau_k^2\|x_k\|^2 \leq -\frac{\tau_k}{2}\|x_k\|^2 , \quad \forall k \geq N .$$

Consequently, it follows from (5.26) and (5.27) that

$$\|x_{k+1}\|_P^2 \leq \|x_k\|_P^2 - \frac{\tau_k}{2}\|x_k\|^2 \leq \left(1 - \frac{\tau_k}{2\Lambda}\right)\|x_k\|_P^2 , \quad \forall k \geq N . \tag{5.28}$$

Hence

$$\|x_k\|_P^2 \leq \left[\prod_{j=N}^{k-1}\left(1 - \frac{\tau_j}{2\Lambda}\right)\right]\|x_N\|_P^2 , \quad \forall k \geq N+1 . \tag{5.29}$$

If $x_{k_0} = 0$ for some $k_0 \geq N$, then it follows from (5.28) that $x_k = 0$ for all $k \geq k_0$. Thus $(x_k)_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^n)$. Assume now that $x_k \neq 0$ for all $k \geq N$. By (5.28), we see that $1 - \tau_k/(2\Lambda) > 0$ for all $k \geq N$. Moreover, since $M := \inf_{k\in\mathbb{N}}\{\tau_k k^\alpha\} > 0$, $\tau_k \geq M/k^\alpha$ for all $k \in \mathbb{N}$. Thus

$$0 < 1 - \frac{\tau_k}{2\Lambda} \leq 1 - \frac{M}{2\Lambda k^\alpha} , \quad \forall k \geq N .$$

Therefore, by (5.29), we obtain

$$\|x_k\|_P \leq \left[\prod_{j=N}^{k-1}\left(1 - \frac{M}{2\Lambda j^\alpha}\right)^{\frac{1}{2}}\right]\|x_N\|_P , \quad \forall k \geq N+1 . \tag{5.30}$$

Define $v\colon \mathbb{Z}_+ \to \mathbb{R}_+$ by

$$v(k) := \prod_{j=N}^{N+k}\left(1 - \frac{M}{2\Lambda j^\alpha}\right)^{\frac{1}{2}} .$$

By (5.26) and (5.30), in order to show that $(x_k)_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^n)$, it suffices to show that $v \in \ell^1(\mathbb{Z}_+, \mathbb{R})$. Invoking the inequality $1 - t \le e^{-t}$ for $t \in \mathbb{R}$, we have

$$
\begin{aligned}
\sum_{k=0}^{l} v(k) \;&=\; \left(1 - \frac{M}{2\Lambda N^\alpha}\right)^{\frac{1}{2}} + \left(1 - \frac{M}{2\Lambda N^\alpha}\right)^{\frac{1}{2}} \left(1 - \frac{M}{2\Lambda (N+1)^\alpha}\right)^{\frac{1}{2}} \\
&\qquad + \ldots + \prod_{j=N}^{N+l} \left(1 - \frac{M}{2\Lambda j^\alpha}\right)^{\frac{1}{2}} \\
&\le\; \exp\left(-\frac{M}{4\Lambda N^\alpha}\right) + \exp\left[-\frac{M}{4\Lambda}\left(\frac{1}{N^\alpha} + \frac{1}{(N+1)^\alpha}\right)\right] \\
&\qquad + \ldots + \exp\left[-\frac{M}{4\Lambda}\sum_{j=N}^{N+l}\left(\frac{1}{j^\alpha}\right)\right] \\
&\le\; \exp\left(-\frac{M}{4\Lambda N^\alpha}\right) + \exp\left(-\frac{2M}{4\Lambda (N+1)^\alpha}\right) \\
&\qquad + \ldots + \exp\left(-\frac{(l+1)M}{4\Lambda (N+l)^\alpha}\right), \quad \forall l \in \mathbb{Z}_+ \,.
\end{aligned} \tag{5.31}
$$

Since $\alpha \in (0, 1)$, it follows that

$$
\exp\left(-\frac{M(l+1)}{4\Lambda (N+l)^\alpha}\right) \le 1/l^2 \,,
$$

for sufficiently large $l$. Hence, the right-hand side of (5.31) converges to a finite limit as $l \to \infty$, showing that $v \in \ell^1(\mathbb{Z}_+, \mathbb{R})$. □

## 5.2 Adaptation of the sampling period

By Corollary 3.1.6, we know that if the continuous-time state feedback system is exponentially stable, then the corresponding sampled-data system with constant sampling period is also exponentially stable, provided that the sampling period $\tau > 0$ is sufficiently small. The problem is that it is difficult to estimate how small the sampling period has to be in order to achieve stability of the sampled-data system. In this section, we first develop an approach for state feedback systems which is based on an adaptive law for adjusting the sampling period. We then extend this approach to static output feedback systems and dynamic output feedback systems.

Throughout this section, let $(\beta_k)_{k \in \mathbb{Z}_+}$ be such that

$$
\beta_0 = 0\,, \quad \beta_{k+1} > \beta_k\,,\ \forall k \in \mathbb{Z}_+\,, \quad \beta_k \to \infty \text{ as } k \to \infty\,.
$$

Let $(\delta_k)_{k \in \mathbb{Z}_+}$ be a positive sequence, i.e., $\delta_k > 0$ for all $k \in \mathbb{Z}_+$. Define the function

$$
\varphi\colon \mathbb{R}_+ \to \mathbb{Z}_+\,,\ z \mapsto \max\{j \in \mathbb{Z}_+ : \beta_j \le z\}\,.
$$

Trivially, $z \in [\beta_{\varphi(z)}, \beta_{\varphi(z)+1})$ for all $z \in \mathbb{R}_+$.

### 5.2.1 State feedback

Consider the continuous-time state feedback system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,; \quad x(0) = x^0 \in \mathbb{R}^n\,, && (5.32a)\\ u(t) &= Fx(t)\,, && (5.32b) \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $F \in \mathbb{R}^{m\times n}$. Let $\boldsymbol{t} := (t_k)_{k\in\mathbb{Z}_+}$ be the sampling points satisfying (5.2). Set $\tau_k := t_{k+1} - t_k$ for $k \in \mathbb{Z}_+$. We use sampling and hold in (5.32b) to obtain that

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,; \quad x(0) = x^0 \in \mathbb{R}^n\,, && (5.33a)\\ u(t) &= Fx(t_k)\,, \quad \forall t \in [t_k, t_{k+1})\,. && (5.33b) \end{aligned}$$

We generate the sampling points $(t_k)_{k\in\mathbb{Z}_+}$ by the following recursive adaptive law

$$\left.\begin{array}{l} \quad t_0 := 0\,,\ k_0 := 0\,,\ \sigma_0 := \|x^0\|\,,\\ \quad \sigma_{j+1} := \sigma_j + \|x(t_{j+1})\|\,,\\ \quad t_j := t_{k_l} + (j - k_l)\delta_l\,, \quad j = k_l + 1, \ldots, k_{l+1}\,,\\ \text{where } k_{l+1} \geq k_l + 1 \text{ is such that}\\ \quad \sigma_{k_{l+1}-1} < \beta_{\varphi(\sigma_{k_l})+1}\,, \quad \sigma_{k_{l+1}} \geq \beta_{\varphi(\sigma_{k_l})+1}\,.\\ \text{Or, if such a } k_{l+1} \text{ does not exist, then}\\ \quad t_j := t_{k_l} + (j - k_l)\delta_l\,, \quad j > k_l\,. \end{array}\right\} \text{Adaptive strategy } \textbf{(AS1)}.$$

In the latter case, we set

$$k_{l+j} := k_l\,, \quad \forall j \in \mathbb{N}\,,$$

and say that the sequence $(k_j)_{j\in\mathbb{Z}_+}$ is *ultimately constant.* The adaptation of the sampling period terminates in finite time if and only if $(k_j)_{j\in\mathbb{Z}_+}$ is ultimately constant.

The idea of (AS) is that the $\beta_k$ are thresholds and the $\delta_k$ are possible sampling periods. Note that $(\sigma_j)_{j\in\mathbb{Z}_+}$ is non-decreasing and each $\sigma_j$ lies in an interval given by two consecutive thresholds: if $\sigma_{j+1}$ lies in an interval different from the interval containing $\sigma_j$, then the algorithm changes the sampling period; otherwise, the same sampling period is used.

System (5.33) combined with the adaptive strategy (AS1) has a unique solution, denoted by $x(\,\cdot\,; x^0)$. Of course, $x(\,\cdot\,; x^0)$ depends on $(\beta_k)_{k\in\mathbb{Z}_+}$ and $(\delta_k)_{k\in\mathbb{Z}_+}$, but since these two sequences are fixed, we do not indicate the dependence of $x(\cdot; x^0)$ on $(\beta_k)_{k\in\mathbb{Z}_+}$ and $(\delta_k)_{k\in\mathbb{Z}_+}$. Note that

$$\sigma_k = \sum_{j=0}^{k} \|x(t_j; x^0)\|\,, \quad \forall k \in \mathbb{Z}_+\,. \tag{5.34}$$

In Figure 5-1, which illustrates strategy (AS1) for the adaptation of the sampling points,

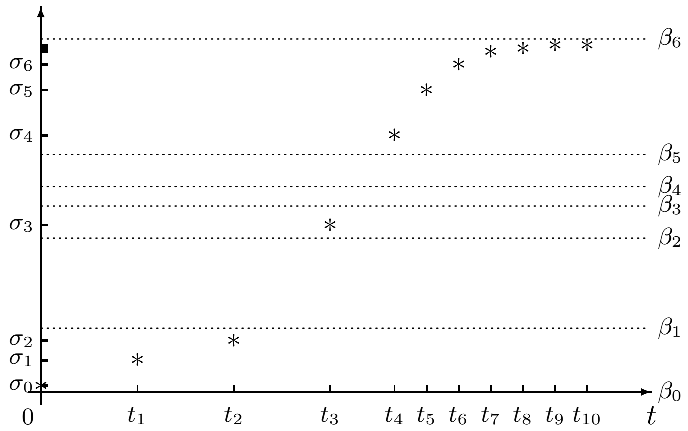


Figure 5-1: Illustration of adaptive strategy (AS1).

we have

$$k_0 = 0\,, \quad k_1 = 3\,, \quad k_j = 4\,,\ \forall j \geq 2\,;$$
$$t_j = j\delta_0\,, \quad \forall j = 1,2,3\,; \qquad t_4 = t_3 + \delta_1\,; \qquad t_j = t_4 + (j-4)\delta_2\,, \quad \forall j \geq 5\,.$$

**Lemma 5.2.1.** *The sequence $(k_j)_{j\in\mathbb{Z}_+}$ is ultimately constant if and only if the sequence $(\sigma_j)_{j\in\mathbb{Z}_+}$ is bounded.*

*Proof.* By (AS1), the sequence $(k_j)_{j\in\mathbb{Z}_+}$ is ultimately constant if and only if there exists $l \in \mathbb{Z}_+$, such that

$$\sigma_{k_l+j} \in [\beta_{\varphi(\sigma_{k_l})}, \beta_{\varphi(\sigma_{k_l})+1})\,, \quad \forall j \in \mathbb{Z}_+\,.$$

The existence of such a number $l$ is equivalent to the boundedness of $(\sigma_j)_{j\in\mathbb{Z}_+}$. □

**Theorem 5.2.2.** *Assume that the continuous-time state feedback system* (5.32) *is exponentially stable and that the positive sequence $(\delta_k)_{k\in\mathbb{Z}_+}$ satisfies*

$$\lim_{k\to\infty} \delta_k = 0 \quad \text{and} \quad \inf_{k\in\mathbb{N}} \{\delta_k k^\alpha\} > 0 \quad \text{for some } \alpha \in (0,1)\,. \tag{5.35}$$

*Then, for any $x^0 \in \mathbb{R}^n$, the following statements hold for the closed-loop system given by* (5.33) *and (AS1):*

(1) *the adaptation of the sampling period terminates in finite time;*

(2) $\lim_{t\to\infty} x(t;x^0) = 0$, *$x(\,\cdot\,;x^0) \in L^1(\mathbb{R}_+,\mathbb{R}^n)$ and $(x(t_k;x^0))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+,\mathbb{R}^n)$, where $x(\,\cdot\,;x^0)$ is the solution of the adaptive system given by* (5.33) *and (AS1).*

*Proof.* We first show that $(\sigma_j)_{j\in\mathbb{Z}_+}$ is bounded. Seeking a contradiction, suppose $(\sigma_j)_{j\in\mathbb{Z}_+}$ is unbounded. By Lemma 5.2.1, we know that $(k_j)_{j\in\mathbb{Z}_+}$ is not ultimately constant, so that $\lim_{j\to\infty} k_j = \infty$. By (AS1), the sampling period $\tau_j := t_{j+1} - t_j$ satisfies

$$\tau_j = \delta_l\,, \quad \forall j = k_l, \ldots, k_{l+1} - 1\,,\ \forall l \in \mathbb{Z}_+\,,$$

and $k_l \geq l$ for all $l \in \mathbb{Z}_+$. Thus,

$$\tau_j j^\alpha = \delta_l j^\alpha \geq \delta_l l^\alpha\,, \quad \forall j = k_l, \ldots, k_{l+1} - 1\,,\ \forall l \in \mathbb{Z}_+\,,$$

showing that

$$\inf_{k\in\mathbb{N}} \{\tau_k k^\alpha\} \geq \inf_{k\in\mathbb{N}} \{\delta_k k^\alpha\} > 0\,, \quad \forall l \in \mathbb{Z}_+\,.$$

Moreover, since $\lim_{k\to\infty} \delta_k = 0$, it is easy to see that $\lim_{k\to\infty} \tau_k = 0$. Therefore, it follows from Theorem 5.1.5 that $(x(t_k; x^0))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^n)$. By (5.34), we see that $(\sigma_j)_{j\in\mathbb{Z}_+}$ is bounded, contradicting our supposition. Consequently, the supposition is wrong, showing that $(\sigma_j)_{j\in\mathbb{Z}_+}$ is bounded. Invoking Lemma 5.2.1 completes the proof of Statement (1).

To prove Statement (2), we set

$$\bar{\delta} := \sup_{k\in\mathbb{Z}_+} \delta_k < \infty \quad \text{and} \quad M := \sup_{t\in[0,\bar{\delta}]} \left\| e^{At} + \int_0^t e^{As} BF ds \right\|\,.$$

Since $(\sigma_j)_{j\in\mathbb{Z}_+}$ is bounded, by (5.34), we conclude that $(x(t_k; x^0))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^n)$. Thus $\lim_{k\to\infty} x(t_k; x^0) = 0$ for all $x^0 \in \mathbb{R}^n$. By the variation-of-parameters formula, we have

$$\|x(t_k + \theta; x^0)\| \leq M \|x(t_k; x^0)\|\,, \quad \forall \theta \in [0, \tau_k)\,, \quad \forall x^0 \in \mathbb{R}^n\,.$$

Consequently,

$$\lim_{t\to\infty} x(t; x^0) = 0\,, \quad \forall x^0 \in \mathbb{R}^n\,.$$

Moreover,

$$\|x\|_{L^1} = \sum_{k=0}^\infty \int_{t_k}^{t_{k+1}} \|x(t; x^0)\| dt \leq M\bar{\delta} \sum_{k=0}^\infty \|x(t_k; x^0)\| = M\bar{\delta} \|(x(t_k; x^0))_{k\in\mathbb{Z}_+}\|_{\ell^1} < \infty\,,$$

showing that $x \in L^1(\mathbb{R}_+, \mathbb{R}^n)$. □

To illustrate Theorem 5.2.2, we present two numerical simulations.

**Example 5.2.3.** Let the matrices $A, B$ and $F$ in system (5.32) be given by

$$A = \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix}\,, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,, \quad F = \begin{pmatrix} 2 & -5 \end{pmatrix}\,. \tag{5.36}$$

Then

$$\sigma(A) = \{1 \pm \sqrt{2}\}\,, \quad \sigma(A + BF) = \{-1, -2\}\,,$$

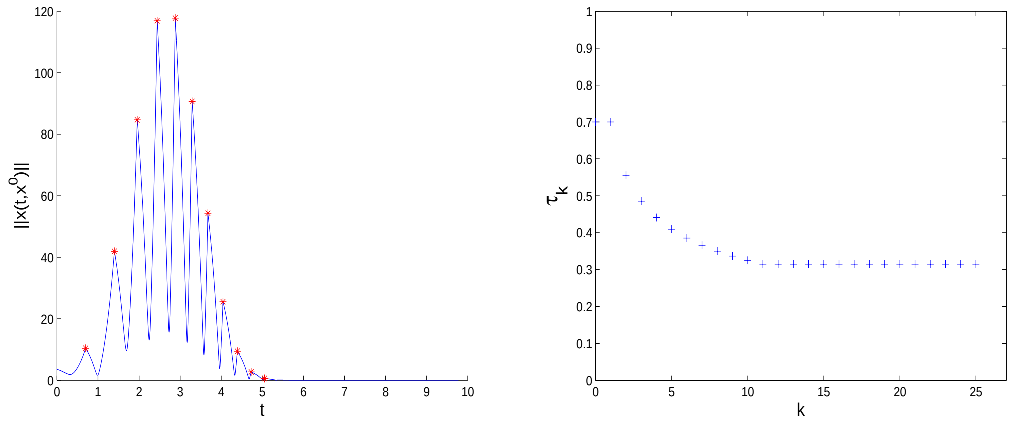


Figure 5-2: Sampled-data control with adaptive sampling period for Example 5.2.3 (at starred points, the sampling period is adapted).

showing that $A$ is unstable, whilst $F$ stabilizes the pair $(A, B)$. Set

$$\beta_k := 2k\,, \quad \delta_k := \frac{0.7}{(k+1)^{1/3}}\,; \quad \forall k \in \mathbb{Z}_+\,,$$

and initial condition $x^0 := (2,3)^T$. Clearly, $(\delta_k)_{k\in\mathbb{Z}_+}$ satisfies condition (5.35) with $\alpha = 1/2$. Hence, the conclusions of Theorem 5.2.2 are true. Figure 5-2 shows $\|x(t;x^0)\|$ and $\tau_j = t_{j+1} - t_j$ for $j \in \mathbb{Z}_+$ when the adaptive sampled-data strategy (AS1) applied to system (5.33), with $A$, $B$ and $F$ given by (5.36).

Next we consider a 2-input/2-output linear system.

**Example 5.2.4.** Assume that $A, B$ and $F$ in system (5.32) are given by

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 2 & -1 & -2 \\ -2 & 0 & 0.5 \end{pmatrix}.$$

Then

$$\sigma(A) = \{1, 1, -2\}\,, \quad \sigma(A+BF) = \{-1, -1 \pm i\}\,.$$

Set

$$\beta_k := 100k\,, \quad \delta_k := \frac{1.5}{(k+1)^{1/2}}\,; \quad \forall k \in \mathbb{Z}_+\,,$$

and the initial condition $x^0 := (1,2,3)^T$. Clearly, $(\delta_k)_{k\in\mathbb{Z}_+}$ satisfies condition (5.35) with $\alpha = 1/2$. Hence, the conclusions of Theorem 5.2.2 are true. The result is illustrated by Figure 5-3.

**Remark 5.2.5.** If the assumptions of Theorem 5.2.2 hold, then there exists $N \in \mathbb{Z}_+$ such that the adaptation of the sampling period terminates after $t_{k_N}$ and the constant

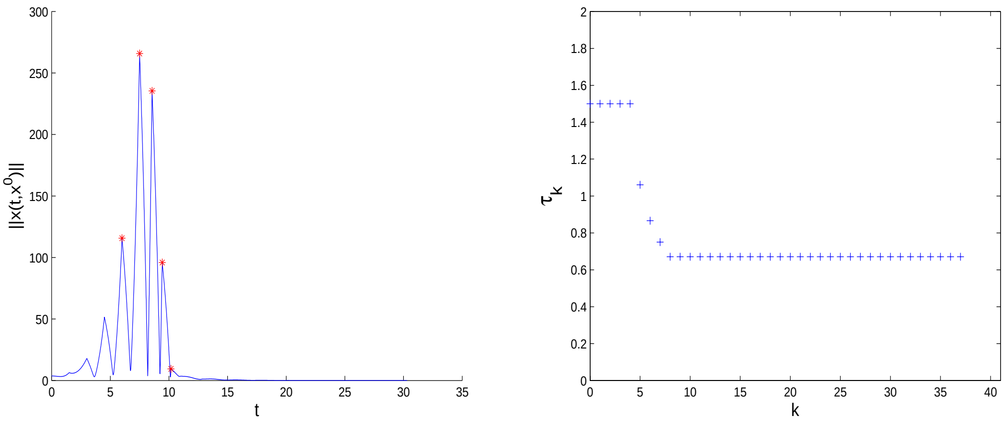


Figure 5-3: Sampled-data control with adaptive sampling period for Example 5.2.4 (at starred points, the sampling period is adapted).

sampling period $\delta_N$ is used from $t = t_{k_N}$ onwards, i.e.,

$$t_j = t_{k_N} + (j - k_N)\delta_N\,, \quad \forall j \geq k_N\,.$$

Moreover, by Theorem 5.2.2,

$$\Delta_{k_N}^{j-k_N} x(t_N; x^0) = x(t_j; x^0) \to 0 \quad \text{as} \quad j \to \infty\,.$$

However, $\Delta_{k_N}$ may not be power stable, since $N$ depends on $x^0$ and thus $\Delta_{k_N}^j$ may not go to 0 as $j \to \infty$. To see this, consider Example 5.2.3, where we still set $\beta_k := 2k$ and $\delta_k := 0.7/(k+1)^{1/3}$. The two eigenvalues $\lambda_1, \lambda_2$ of $\Delta_0 = e^{A\delta_0} + \int_0^{\delta_0} e^{As} BF$ are

$$\lambda_1 \approx 0.5493\,, \quad \lambda_2 \approx -3.9359\,.$$

Let $v$ denote a normalized eigenvector of $\Delta_0$ corresponding to $\lambda_1$. With $x^0 := v$, it follows trivially that

$$\|\Delta_0^j x^0\| = \|\lambda_1^j v\| \in (\beta_0, \beta_1) = (0, 2)\,, \quad \forall j \in \mathbb{Z}_+\,,$$

showing that no adaptation takes place, implying that $N = 0$, i.e., $k_N = k_0 = 0$. Whilst

$$\lim_{j\to\infty} x(t_j; x^0) = \lim_{j\to\infty} \Delta_0^j x^0 = \lim_{j\to\infty} \lambda_1^j v = 0\,,$$

$\Delta_0$ is not power stable. $\diamond$

### 5.2.2 Static output feedback

Consider the static output feedback system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,; \quad x(0) = x^0 \in \mathbb{R}^n\,, && (5.37a)\\ y(t) &= Cx(t)\,, && (5.37b)\\ u(t) &= Ky(t)\,, && (5.37c) \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $K \in \mathbb{R}^{m\times p}$. The system (5.37) is exponentially stable if and only if $A + BKC$ is exponentially stable.

Let $\boldsymbol{t} := (t_k)_{k\in\mathbb{Z}_+}$ be sampling points satisfying (5.2). Using sampling and hold in (5.37c), we obtain

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,; \quad x(0) = x^0 \in \mathbb{R}^n\,, && (5.38a)\\ y(t) &= Cx(t)\,, && (5.38b)\\ u(t) &= Ky(t_k)\,, \quad \forall t \in [t_k, t_{k+1})\,. && (5.38c) \end{aligned}$$

Let $x(\,\cdot\,; x^0, \boldsymbol{t})$ denote the solution of system (5.38) and set $\tau_k := t_{k+1} - t_k$ for $k \in \mathbb{Z}_+$. The variation-of-parameters formula yields

$$x(t_k + \theta; x^0, \boldsymbol{t}) = \left(e^{A\theta} + \int_0^\theta e^{As} ds BKC\right) x(t_k; x^0, \boldsymbol{t})\,, \quad \forall \theta \in [0, \tau_k]\,. \tag{5.39}$$

It is easy to see that

$$x(t_{k+1}; x^0, \boldsymbol{t}) = \left(e^{A\tau_k} + \int_0^{\tau_k} e^{As} ds BKC\right) x(t_k; x^0, \boldsymbol{t})\,, \quad \forall k \in \mathbb{Z}_+\,. \tag{5.40}$$

Note that (5.37), (5.38) and (5.40) are special cases of (5.1), (5.3) and (5.5), respectively, with $F = KC$. Hence, the lemma below follows immediately from Theorem 5.1.5.

**Lemma 5.2.6.** *Assume that the static output feedback system* (5.37) *is exponentially stable. If* $(\tau_k)_{k\in\mathbb{Z}_+}$ *satisfies* (5.24)*, then* $(x(t_k; x^0, \boldsymbol{t}))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^n)$.

We now develop an adaptive strategy for the generation of the sampling points $(t_k)_{k\in\mathbb{Z}_+}$. Instead of using state information as in (AS1), we now use output information, i.e., we replace (AS1) by

$$\left.\begin{array}{l} \quad t_0 := 0\,,\ k_0 := 0\,,\ \sigma_0 := \|y(0)\| = \|Cx^0\|\,,\\ \sigma_{j+1} := \sigma_j + \|y(t_{j+1})\|\,,\\ \quad t_j := t_{k_l} + (j - k_l)\delta_l\,, \quad j = k_l + 1, \ldots, k_{l+1}\,,\\ \text{where } k_{l+1} \ge k_l + 1 \text{ is such that}\\ \quad \sigma_{k_{l+1}-1} < \beta_{\varphi(\sigma_{k_l})+1}\,, \quad \sigma_{k_{l+1}} \ge \beta_{\varphi(\sigma_{k_l})+1}\,.\\ \text{Or, if such a } k_{l+1} \text{ does not exist, then}\\ \quad t_j := t_{k_l} + (j - k_l)\delta_l\,, \quad j > k_l\,. \end{array}\right\} \text{Adaptive strategy } \textbf{(AS2)}.$$

System (5.38) combined with the adaptive strategy (AS2) has a unique solution, denoted by $x(\,\cdot\,;x^0)$. Since $(\beta_k)_{k\in\mathbb{Z}_+}$ and $(\delta_k)_{k\in\mathbb{Z}_+}$ are fixed, we do not indicate the dependence of $x(\,\cdot\,;x^0)$ on these two sequences. Note that

$$\sigma_k = \sum_{j=0}^{k} \|y(t_j)\|\,, \quad \forall k \in \mathbb{Z}_+\,. \tag{5.41}$$

By (AS2), it is clear that Lemma 5.2.1 still holds, i.e., the sequence $(k_j)_{j\in\mathbb{Z}_+}$ is ultimately constant if and only if the sequence $(\sigma_j)_{j\in\mathbb{Z}_+}$ is bounded.

**Definition 5.2.7.** A number $\tau > 0$ is said to be *pathological* relative to $A \in \mathbb{R}^{n\times n}$ if there exist $k \in \mathbb{Z}\setminus\{0\}$ and $\lambda, \mu \in \sigma(A)\cap \mathrm{cl}(\mathbb{C}_0)$ such that

$$\tau(\lambda - \mu) = 2k\pi i\,.$$

Otherwise, $\tau$ is said to be *non-pathological* relative to $A$. We say a positive sequence $(\tau_j)_{j\in\mathbb{Z}_+}$ is *pathological* relative to $A$ if there exists $j \in \mathbb{Z}_+$ such that $\tau_j$ is pathological relative to $A$. On the other hand, $(\tau_j)_{j\in\mathbb{Z}_+}$ is said to be *non-pathological* relative to $A$, if $\tau_j$ is non-pathological relative to $A$ for all $j \in \mathbb{Z}_+$. $\diamond$

**Theorem 5.2.8.** *Assume that the continuous-time static output feedback system* (5.37) *is exponentially stable and that the positive sequence* $(\delta_k)_{k\in\mathbb{Z}_+}$ *satisfies*

$$\lim_{k\to\infty} \delta_k = 0 \quad \textit{and} \quad \inf_{k\in\mathbb{N}}\{\delta_k k^\alpha\} > 0 \quad \textit{for some } \alpha \in (0,1)\,.$$

*For all $x^0 \in \mathbb{R}^n$, the following statements hold for the closed-loop system given by* (5.38) *and (AS2):*

(1) *the adaptation of the sampling period terminates in finite time;*

(2) *if $(\delta_k)_{k\in\mathbb{Z}_+}$ is non-pathological relative to $A$, the $\lim_{t\to\infty} x(t;x^0) = 0$, $x(\,\cdot\,;x^0) \in L^1(\mathbb{R}_+,\mathbb{R}^n)$ and $(x(t_k;x^0))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+,\mathbb{R}^n)$, where $x(\,\cdot\,;x^0)$ is the solution of the adaptive system given by* (5.38) *and (AS2).*

*Proof.* We first show that $(\sigma_j)_{j\in\mathbb{Z}_+}$ is bounded. Seeking a contradiction, suppose $(\sigma_j)_{j\in\mathbb{Z}_+}$ is unbounded. By Lemma 5.2.1, we know that $(k_j)_{j\in\mathbb{Z}_+}$ is not ultimately constant, so that $\lim_{j\to\infty} k_j = \infty$. By (AS2), the sampling period $\tau_j := t_{j+1} - t_j$ satisfies

$$\tau_j = \delta_l\,, \quad \forall j = k_l, \ldots, k_{l+1}-1\,;\ \forall l \in \mathbb{Z}_+\,,$$

and $k_l \geq l$ for all $l \in \mathbb{Z}_+$. Thus,

$$\tau_j j^\alpha = \delta_l j^\alpha \geq \delta_l l^\alpha\,, \quad \forall j = k_l, \ldots, k_{l+1}-1\,;\ \forall l \in \mathbb{Z}_+\,,$$

showing that

$$\inf_{k\in\mathbb{N}}\{\tau_k k^\alpha\} \geq \inf_{k\in\mathbb{N}}\{\delta_k k^\alpha\} > 0\,, \quad \forall l \in \mathbb{Z}_+\,.$$

Moreover, since $\lim_{k\to\infty} \delta_k = 0$, it is easy to see that $\lim_{k\to\infty} \tau_k = 0$. Therefore, by Lemma 5.2.6, $(x(t_k; x^0))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^n)$. It follows from (5.38b) that $(y(t_k))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^p)$. Clearly, by (5.41), $(\sigma_j)_{j\in\mathbb{Z}_+}$ is bounded, contradicting our supposition. Consequently, the supposition is wrong, showing that $(\sigma_j)_{j\in\mathbb{Z}_+}$ is bounded. Invoking Lemma 5.2.1 completes the proof of Statement (1).

Statement (1) implies there exists $N \in \mathbb{Z}_+$ such that the sampling period $\delta_N =: \tau$ is used from $t = t_{k_N}$ onwards, i.e.,

$$t_j = t_{k_N} + (j - k_N)\tau\,, \quad \forall j \geq k_N\,.$$

Since system (5.37) is exponentially stable, the pair $(C, A)$ is detectable. By assumption, $\tau$ is non-pathological relative to $A$. Therefore the pair $(C, e^{A\tau})$ is discrete-time detectable (see Theorem A.2.2 in the Appendix). Hence there exists $H \in \mathbb{R}^{n\times p}$ such that $e^{A\tau} + HC$ is power stable. By (5.40), we obtain that

$$\begin{aligned} x(t_{j+1}; x^0) &= e^{A\tau}x(t_j; x^0) + B_\tau KCx(t_j; x^0) \\ &= (e^{A\tau} + HC)x(t_j; x^0) + (B_\tau K - H)y(t_j)\,, \quad \forall j \geq k_N\,, \end{aligned} \tag{5.42}$$

where $B_\tau = \int_0^\tau e^{As}ds B$. Since $(\sigma_j)_{j\in\mathbb{Z}_+}$ is bounded, by (5.41), $(y(t_k))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^p)$. It follows from (5.42) and the power stability of $e^{A\tau} + HC$ that $(x(t_k; x^0))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^n)$. Invoking (5.39), it follows from an argument identical to that used in the proof of Theorem 5.2.2 that $x \in L^1(\mathbb{R}_+, \mathbb{R}^n)$ and $\lim_{t\to\infty} x(t; x^0) = 0$. □

**Remark 5.2.9.** For $(\delta_k)_{k\in\mathbb{Z}_+}$, define $\mathcal{D} \subset \mathbb{R}^{n\times n}$ by

$$\mathcal{D} := \{A \in \mathbb{R}^{n\times n} : (\delta_k)_{k\in\mathbb{Z}_+} \text{ is non-pathological relative to } A\}\,.$$

By Theorem A.1.1 and Corollary A.1.2 in the Appendix, we know that $\mathcal{D}$ is dense in $\mathbb{R}^{n\times n}$, and that additionally if $\lim_{k\to\infty} \delta_k = 0$, then $\mathcal{D}$ is also open. Hence, given a positive sequence $(\delta_k)_{k\in\mathbb{Z}_+}$ with $\lim_{k\to\infty} \delta_k = 0$, the probability that a randomly chosen matrix $A \in \mathbb{R}^{n\times n}$ has the property that $(\delta_k)_{k\in\mathbb{Z}_+}$ is pathological relative to $A$ is zero. ◇

### 5.2.3 Dynamic output feedback

Finally, consider the dynamic output feedback system, where the plant is given by

$$\dot{x}_p = A_p x_p + B_p u_p\,; \quad x_p(0) = x_p^0 \in \mathbb{R}^{n_p}\,, \tag{5.43a}$$

$$y_p = C_p x_p + D_p u_p\,, \tag{5.43b}$$

where $A_p \in \mathbb{R}^{n_p\times n_p}, B_p \in \mathbb{R}^{n_p\times m}, C_p \in \mathbb{R}^{p\times n_p}$ and $D_p \in \mathbb{R}^{p\times m}$. The controller is of the form

$$\dot{x}_c = A_c x_c + B_c u_c\,; \quad x_c(0) = x_c^0 \in \mathbb{R}^{n_c}\,, \tag{5.44a}$$

$$y_c = C_c x_c + D_c u_c\,, \tag{5.44b}$$

where $A_c \in \mathbb{R}^{n_c \times n_c}, B_c \in \mathbb{R}^{n_c \times p}, C_c \in \mathbb{R}^{m \times n_c}$ and $D_c \in \mathbb{R}^{m \times p}$. We use the output $y_p$ of (5.43) as the input for (5.44), and the output $y_c$ of (5.44) as the input for (5.43), i.e.,

$$u_c = y_p\,, \quad u_p = y_c\,, \tag{5.45}$$

to obtain the feedback interconnection of (5.43) and (5.44). In order for the feedback system to be well-posed, we assume that the matrix $I - D_cD_p$ is invertible. Then $I - D_pD_c$ is also invertible, with

$$(I - D_pD_c)^{-1} = I + D_p(I - D_cD_p)^{-1}D_c\,.$$

For convenience, set

$$E_p := (I - D_cD_p)^{-1}\,, \quad E_c := (I - D_pD_c)^{-1}\,.$$

and

$$x := \begin{pmatrix} x_p \\ x_c \end{pmatrix}, \qquad A := \begin{pmatrix} A_p & 0 \\ 0 & A_c \end{pmatrix}, \qquad B := \begin{pmatrix} B_p & 0 \\ 0 & B_c \end{pmatrix},$$
$$C := \begin{pmatrix} C_p & 0 \\ 0 & C_c \end{pmatrix}, \qquad D := \begin{pmatrix} D_c & I \\ I & D_p \end{pmatrix}, \qquad E := \begin{pmatrix} E_p & 0 \\ 0 & E_c \end{pmatrix}.$$

Then, by a routine calculation†, the continuous-time dynamic feedback system given by (5.43)–(5.45) can be written as

$$\dot{x} = (A + BEDC)x\,; \quad x(0) = x^0 = \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} \in \mathbb{R}^{n_p+n_c}\,. \tag{5.46}$$

Let $\boldsymbol{t} := (t_k)_{k \in \mathbb{Z}_+}$ be sampling points satisfying (5.2). Set $\tau_k := t_{k+1} - t_k$ for $k \in \mathbb{Z}_+$. Let the input $u_c$ in (5.44a) be given by

$$u_c(t) = v(k)\,, \quad t \in [t_k, t_{k+1})\,,$$

where $v$ is a function $\mathbb{Z}_+ \to \mathbb{R}^p$. By the variation-of-parameters formula, we obtain from (5.44) that

$$\begin{aligned} x_c(t_{k+1}) &= e^{A_c\tau_k}x_c(t_k) + \int_0^{\tau_k} e^{A_c s}ds B_c v(k)\,; \quad x_c(0) = x_c^0 \in \mathbb{R}^{n_c}\,, && (5.47a) \\ y_c(t_k) &= C_c x_c(t_k) + D_c v(k)\,, \quad \forall k \in \mathbb{Z}_+\,. && (5.47b) \end{aligned}$$

We consider the sampled-data feedback interconnection given by

$$v(k) = y_p(t_k)\,, \quad u_p(t_k + \theta) = y_c(t_k)\,, \quad \forall \theta \in [0, \tau_k)\,,\ \forall k \in \mathbb{Z}_+\,. \tag{5.48}$$

†See Appendix A.4.1.

The sampled-data feedback system given by (5.43), (5.47) and (5.48) has a unique solution which will be denoted by

$$\begin{pmatrix} x_p(t_k+\theta; x^0, \boldsymbol{t}) \\ x_c(t_k; x^0, \boldsymbol{t}) \end{pmatrix}, \quad \forall \theta \in [0, \tau_k)\,,\ \forall k \in \mathbb{Z}_+\,.$$

For convenience, we write

$$x_p(\cdot) := x_p(\cdot; x^0, \boldsymbol{t}) \quad \text{and} \quad x_c(t_k) := x_c(t_k; x^0, \boldsymbol{t})\,.$$

By the variation-of-parameters formula and a routine calculation†, we obtain that

$$\begin{pmatrix} x_p(t_k+\theta) \\ x_c(t_{k+1}) \end{pmatrix} = \left[ \begin{pmatrix} e^{A_p\theta} & 0 \\ 0 & e^{A_c\tau_k} \end{pmatrix} + \begin{pmatrix} \int_0^\theta e^{A_p s} ds & 0 \\ 0 & \int_0^{\tau_k} e^{A_c s} ds \end{pmatrix} BEDC \right] \begin{pmatrix} x_p(t_k) \\ x_c(t_k) \end{pmatrix}, \tag{5.49}$$

$$\forall \theta \in [0, \tau_k]\,,\ \forall k \in \mathbb{Z}_+\,; \qquad \begin{pmatrix} x_p(0) \\ x_c(0) \end{pmatrix} = x^0 \in \mathbb{R}^{n_p+n_c}\,.$$

Setting

$$\Delta_k := e^{A\tau_k} + \int_0^{\tau_k} e^{As} ds BEDC\,,$$

it follows from (5.49) with $\theta = \tau_k$ that

$$\begin{pmatrix} x_p(t_{k+1}) \\ x_c(t_{k+1}) \end{pmatrix} = \Delta_k \begin{pmatrix} x_p(t_k) \\ x_c(t_k) \end{pmatrix}, \quad \forall k \in \mathbb{Z}_+\,. \tag{5.50}$$

A simple calculation yields

$$\begin{pmatrix} y_c(t_k) \\ y_p(t_k) \end{pmatrix} = \begin{pmatrix} E_pD_cC_p & E_pC_c \\ E_cC_p & E_cD_pC_c \end{pmatrix} \begin{pmatrix} x_p(t_k) \\ x_c(t_k) \end{pmatrix} = EDC \begin{pmatrix} x_p(t_k) \\ x_c(t_k) \end{pmatrix}. \tag{5.51}$$

**Corollary 5.2.10.** *Assume that the continuous-time system* (5.46) *is exponentially stable. If* $(\tau_k)_{k\in\mathbb{Z}_+}$ *satisfies* (5.24)*, then*

$$\left( \begin{pmatrix} x_p(t_k) \\ x_c(t_k) \end{pmatrix} \right)_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^{n_p+n_c})\,.$$

†See Appendix A.4.3 with $\sigma = -1$, $\varepsilon = 1$, $r = 0$, $d = 0$, $A_c$ replaced by $e^{A_c\tau}$ and $B_c$ replaced by $\int_0^\tau e^{A_c s} ds$.

*Proof.* Note that (5.46) can be written as the state feedback system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,; \quad x(0) = x^0 \in \mathbb{R}^{n_p+n_c}\,, && (5.52a)\\ u(t) &= EDCx(t)\,. && (5.52b) \end{aligned}$$

With sampling points $\boldsymbol{t} = (t_k)_{k\in\mathbb{Z}_+}$, we apply sampling and hold in (5.52b) to obtain the sampled-data system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,; \quad x(0) = x^0 \in \mathbb{R}^{n_p+n_c}\,,\\ u(t) &= EDCx(t_k)\,, \quad \forall t \in [t_k, t_{k+1})\,. \end{aligned}$$

Let $x(\cdot; x^0, \boldsymbol{t})$ denote its solution. Using the variation-of-parameters formula, we obtain

$$x(t_{k+1}; x^0, \boldsymbol{t}) = \Delta_k x(t_k; x^0, \boldsymbol{t})\,, \quad \forall k \in \mathbb{Z}_+\,; \quad x(0; x^0, \boldsymbol{t}) = x^0\,. \tag{5.53}$$

By Theorem 5.1.5, we know that $(x(t_k; x^0, \boldsymbol{t}))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^{n_p+n_c})$. Noting the dynamics of systems (5.50) and (5.53) are both governed by $\Delta_k$, it follows that

$$x(t_k; x^0, \boldsymbol{t}) = \begin{pmatrix} x_p(t_k) \\ x_c(t_k) \end{pmatrix}, \quad \forall k \in \mathbb{Z}_+\,,$$

showing that $\left(\begin{pmatrix} x_p(t_k) \\ x_c(t_k) \end{pmatrix}\right)_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^{n_p+n_c})$. ☐

Similar to (AS1), we consider the following adaptive strategy

$$\left.\begin{array}{l} \quad t_0 := 0\,,\ k_0 := 0\,,\ \sigma_0 := \|y_p(0)\| + \|y_c(0)\|\,, \\ \quad \sigma_{j+1} := \sigma_j + \|y_p(t_{j+1})\| + \|y_c(t_{j+1})\|\,, \\ \quad t_j := t_{k_l} + (j - k_l)\delta_l\,, \quad j = k_l + 1, \dots, k_{l+1}\,, \\ \text{where } k_{l+1} \geq k_l + 1 \text{ is such that} \\ \quad \sigma_{k_{l+1}-1} < \beta_{\varphi(\sigma_{k_l})+1}\,, \quad \sigma_{k_{l+1}} \geq \beta_{\varphi(\sigma_{k_l})+1}\,. \\ \text{Or, if such a } k_{l+1} \text{ does not exist, then} \\ \quad t_j := t_{k_l} + (j - k_l)\delta_l\,, \quad j > k_l\,. \end{array}\right\} \text{Adaptive strategy } \textbf{(AS3)}.$$

System (5.49) combined with the adaptive strategy (AS3) has a unique solution, denoted by

$$\begin{pmatrix} x_p(t_k + \theta; x^0) \\ x_c(t_k; x^0) \end{pmatrix}, \quad \forall \theta \in [0, \tau_k)\,, \quad k \in \mathbb{Z}_+\,.$$

Since $(\beta_k)_{k\in\mathbb{Z}_+}$ and $(\delta_k)_{k\in\mathbb{Z}_+}$ are fixed, we do not indicate the dependence on these two sequences. Note that

$$\sigma_k = \sum_{j=0}^{k} \left( \|y_p(t_j)\| + \|y_c(t_j)\| \right), \quad \forall k \in \mathbb{Z}_+ . \tag{5.54}$$

By (AS3), it is clear that Lemma 5.2.1 still holds, i.e., the sequence $(k_j)_{j\in\mathbb{Z}_+}$ is ultimately constant if and only if the sequence $(\sigma_j)_{j\in\mathbb{Z}_+}$ is bounded.

**Theorem 5.2.11.** *Assume that the continuous-time dynamic output feedback system* (5.46) *is exponentially stable and that the positive sequence* $(\delta_k)_{k\in\mathbb{Z}_+}$ *satisfies*

$$\lim_{k\to\infty} \delta_k = 0 \quad \textit{and} \quad \inf_{k\in\mathbb{N}} \{\delta_k k^\alpha\} > 0 \quad \textit{for some } \alpha \in (0,1) .$$

*For all initial condition* $x^0 \in \mathbb{R}^{n_p+n_c}$*, the following statements hold for the closed-loop system given by* (5.49) *and (AS3):*

(1) *the adaptation of the sampling period terminates in finite time;*

(2) *if* $(\delta_k)_{k\in\mathbb{Z}_+}$ *is non-pathological relative to* $A = \mathrm{diag}(A_p, A_c)$*, then*

$$\lim_{t\to\infty} x_p(t; x^0) = 0 \quad \textit{and} \quad \lim_{k\to\infty} x_c(t_k; x^0) = 0 ,$$

*where*

$$\begin{pmatrix} x_p(t_k+\theta; x^0) \\ x_c(t_k; x^0) \end{pmatrix}, \quad \forall \theta \in [0, \tau_k) , \quad k \in \mathbb{Z}_+$$

*is the solution of the adaptive system given by* (5.49) *and (AS3). Moreover,* $x_p \in L^1(\mathbb{R}_+, \mathbb{R}^{n_p})$*,* $(x_p(t_k; x^0))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^{n_p})$ *and* $(x_c(t_k; x^0))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^{n_c})$.

*Proof.* We first show that $(\sigma_j)_{j\in\mathbb{Z}_+}$ is bounded. Seeking a contradiction, suppose $(\sigma_j)_{j\in\mathbb{Z}_+}$ is unbounded. By Lemma 5.2.1, we know that $(k_j)_{j\in\mathbb{Z}_+}$ is not ultimately constant, so that $\lim_{j\to\infty} k_j = \infty$. By (AS2), the sampling period $\tau_j := t_{j+1} - t_j$ satisfies

$$\tau_j = \delta_l , \quad \forall j = k_l, \ldots, k_{l+1} - 1 ; \ \forall l \in \mathbb{Z}_+ ,$$

and $k_l \geq l$ for all $l \in \mathbb{Z}_+$. Thus,

$$\tau_j j^\alpha = \delta_l j^\alpha \geq \delta_l l^\alpha , \quad \forall j = k_l, \ldots, k_{l+1} - 1 ; \ \forall l \in \mathbb{Z}_+ ,$$

showing that

$$\inf_{k\in\mathbb{N}} \{\tau_k k^\alpha\} \geq \inf_{k\in\mathbb{N}} \{\delta_k k^\alpha\} > 0 , \quad \forall l \in \mathbb{Z}_+ .$$

Moreover, since $\lim_{k\to\infty} \delta_k = 0$, it is easy to see that $\lim_{k\to\infty} \tau_k = 0$. Therefore, by Theorem 5.2.10,

$$\left( \begin{pmatrix} x_p(t_k; x^0) \\ x_c(t_k; x^0) \end{pmatrix} \right)_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^{n_p+n_c}) .$$

It follows from (5.51) that

$$\left( \begin{pmatrix} y_p(t_k) \\ y_c(t_k) \end{pmatrix} \right)_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^{p+m}) \, .$$

Clearly, by (5.54), $(\sigma_j)_{j\in\mathbb{Z}_+}$ is bounded, contradicting our supposition. Consequently, the supposition is wrong, showing that $(\sigma_j)_{j\in\mathbb{Z}_+}$ is bounded. Invoking Lemma 5.2.1 completes the proof of Statement (1).

Now Statement (1) implies that there exists $N \in \mathbb{Z}_+$ such that the sampling period $\delta_N =: \tau$ is used from time $t = t_{k_N}$ onwards, i.e.,

$$t_j = t_{k_N} + (j - k_N)\tau \, , \quad \forall j \geq k_N \, .$$

By (5.54) and boundedness of $(\sigma_j)_{j\in\mathbb{Z}_+}$, we know that

$$(y_p(t_k))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^p) \, , \quad (y_c(t_k))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^m) \, . \tag{5.55}$$

Since system (5.46) is exponentially stable, i.e., $A + BEDC$ is exponentially stable, the pair $(EDC, A)$ is detectable. By assumption, $\tau$ is non-pathological relative to $A$. Therefore the pair $(EDC, e^{A\tau})$ is discrete-time detectable (see Theorem A.2.2 in the Appendix). Hence there exists $H \in \mathbb{R}^{(n_p+n_c)\times(m+p)}$ such that $e^{A\tau} + HEDC$ is power stable. By (5.50) and (5.51),

$$\begin{aligned} \begin{pmatrix} x_p(t_{k+1}; x^0) \\ x_c(t_{k+1}; x^0) \end{pmatrix} &= e^{A\tau} \begin{pmatrix} x_p(t_k; x^0) \\ x_c(t_k; x^0) \end{pmatrix} + B_\tau EDC \begin{pmatrix} x_p(t_k; x^0) \\ x_c(t_k; x^0) \end{pmatrix} \\ &= (e^{A\tau} + HEDC) \begin{pmatrix} x_p(t_k; x^0) \\ x_c(t_k; x^0) \end{pmatrix} + (B_\tau - H) \begin{pmatrix} y_c(t_k) \\ y_p(t_k) \end{pmatrix} \, , \quad \forall k \geq k_N \, , \end{aligned}$$

where $B_\tau = \int_0^\tau e^{As} ds B$. By (5.55) and the power stability of $e^{A\tau} + HEDC$, we see that

$$(x_p(t_k; x^0))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^{n_p}) \, , \quad (x_c(t_k; x^0))_{k\in\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^{n_c}) \, ,$$

showing that $\lim_{k\to\infty} x_p(t_k; x^0) = \lim_{k\to\infty} x_c(t_k; x^0) = 0$. Invoking (5.49) and using an argument similar to that in the proof of Theorem 5.2.2, we conclude that that $x_p \in L^1(\mathbb{R}_+, \mathbb{R}^n)$ and $\lim_{t\to\infty} x_p(t; x^0) = 0$. $\square$

## 5.3 Notes and references

Sampled-data control with time-varying sampling period arises when the output is not available at equidistant sampling points due to errors. Moreover, the analysis of sampled-data systems with time-varying sampling period is useful in the context of problems where the sampling period is determined by an adaptive feedback machanism. In the literature, it was also considered in the following situations:

- For sampled-data high-gain stabilization problems, Owens [58] gave an example to show that using constant sampling period, implementing a high-gain discrete-time adaptive controller to a high-gain stabilizable continuous-time plant does not lead to the stability of the closed-loop system. He presented an adaptive law to adjust the sampling period. Sampled-data control of high-gain stabilizable systems using adaptive sampling period is also studied by Ilchmann and Townley [21].

- Motivated by the high-gain results in [21] and [58], Özdemir and Townley [59], in the context of infinite-dimensional systems, analysed a low-gain sampled-data integral control scheme involving adaptation of the sampling period.

- If a continuous-time system is controllable/observable, then the discrete-time system obtained by sample-hold discretization is also controllable/observable provided that the sampling period $\tau$ satisfies the so-called "Kalman-Ho-Narenda" criterion (see, for example, [2, Theorem 3.2.1, p. 41]). But if the uncertainty of the system is large, then a very small $\tau$ may have to be chosen to satisfy this condition: too small to be practically feasible. Kreisselmeier [28] showed that applying sample-hold discretization with a periodic sampling pattern to a controllable/observable continuous-time system leads to a controllable/observable discrete-time system, provided certain mild conditions are satisfied.

To the best of our knowledge, all the results in this chapter are new. We remark that Theorem 5.1.5 is crucial to the development of Section 5.2. A journal publication containing the main results of Section 5.2 is in preparation [20].

# Chapter 6

# Low-gain tracking and disturbance rejection for infinite-dimensional systems by sampled-data control

There has been much interest in low-gain integral control over the last thirty years. The following principle has become well established: an application of the integrator $(\varepsilon/s)I$ to an asymptotically stable, finite-dimensional continuous-time plant, with square transfer function matrix $\mathbf{G}(s)$, leads to an asymptotically stable closed-loop system which achieves asymptotic tracking of arbitrary constant reference signals, provided that the gain parameter $\varepsilon > 0$ is sufficiently small and the eigenvalues of the steady-state matrix $\mathbf{G}(0)$ have positive real parts. This result has been proved by Davison [8] and Lunze [48] using state-space methods and by Grosdidier *et al.* [14] and Morari [55] using frequency-domain methods (see also the book by Lunze [49, Chapter 10] and the textbook by Morari and Zafiriou [56, Theorem 14.3-2, p. 362]). The low-gain integral control approach has been successfully applied to industrial control problems (see, for example, Coppus *et al.* [5]).

The above tuning integrator result has been extended by Hämäläinen and Pohjolainen [15], Logemann *et al.* [33], Logemann and Owens [38], Logemann and Townley [44] Pohjolainen [61, 62], Pohjolainen and Lätti [63] and Rebarber and Weiss [65] to various classes of (abstract) infinite-dimensional continuous-time systems. Furthermore, in [15] and [65], the tuning integrator has been further developed into a tuning regulator which achieves asymptotic tracking and disturbance rejection of signals of the form

$$\sum_{j=1}^{N} e^{i\omega_j t}\mathfrak{v}_j\,, \quad \omega_j \in \mathbb{R}\,,\ \mathfrak{v}_j \in \mathbb{C}^m\,,$$

for large classes of stable infinite-dimensional systems.

The aim of this chapter is to solve tracking and disturbance rejection problems for a certain class of stable infinite-dimensional systems using low-gain sampled-data control

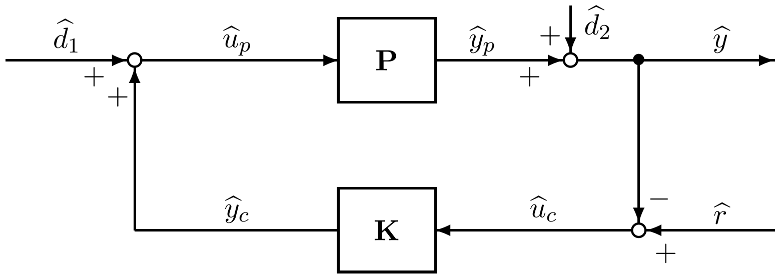


Figure 6-1: Discrete-time closed-loop system.

(that is, to obtain sampled-data versions of the tuning regulator results in [15] and [65]). In Section 6.1, we first design a simple discrete-time low-gain controller (depending on only one gain parameter) for a power stable infinite-dimensional discrete-time plant such that the closed-loop system is power stable and the output of the closed-loop system tracks the reference signal $r$ of the form $r(k) = \sum_{j=1}^{N} \lambda_j^k \mathfrak{r}_j$ where $\mathfrak{r}_j \in \mathbb{C}^p$ and $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$ for $j \in \underline{N}$, and rejects disturbance signal $d$ satisfies $\lim_{k\to\infty}(d(k) - \sum_{j=1}^{N} \lambda_j^k \mathfrak{d}_j) = 0$, where $\mathfrak{d}_j \in \mathbb{C}^m$. The discrete-time results are used in Section 6.2 to derive results on approximate tracking and disturbance rejection for a large class of infinite-dimensional systems with impulse responses given by Borel measures. The reference signals are finite sums of sinusoids, and disturbance signals are asymptotic to finite sums of sinusoids. In Section 6.3, we conclude the chapter by extending the results in Section 6.2 to exponentially stable well-posed systems with transfer functions which are holomorphic and bounded in some half plane $\{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$, where $\alpha < 0$, by using suitable low-pass filters.

## 6.1 Low-gain control of discrete-time systems

### 6.1.1 Preliminaries

Consider the following discrete-time closed-loop feedback system

$$\widehat{y}_p = \mathbf{P}(\widehat{d}_1 + \widehat{y}_c)\,, \quad \widehat{y}_c = \mathbf{K}(\widehat{r} - \widehat{y})\,, \quad \widehat{y} = \widehat{y}_p + \widehat{d}_2\,, \tag{6.1}$$

as shown in Figure 6-1, where $\widehat{y}$ denotes the Z-transform of $y$. Let $\Omega \subset \mathbb{C}$ be open and let $\mathcal{Q}_\Omega$ denote the quotient field of $H^\infty(\Omega, \mathbb{C})$, i.e., $\mathcal{Q}_\Omega = \{n/d : n, d \in H^\infty(\Omega, \mathbb{C}), d \neq 0\}$. For $(\mathbf{P}, \mathbf{K}) \in \mathcal{Q}_\Omega^{p\times m} \times \mathcal{Q}_\Omega^{m\times p}$ such that $\det(I + \mathbf{PK}) \neq 0$, we set

$$F(\mathbf{P}, \mathbf{K}) := \begin{pmatrix} (I + \mathbf{PK})^{-1} & \mathbf{P}(I + \mathbf{KP})^{-1} \\ \mathbf{K}(I + \mathbf{PK})^{-1} & (I + \mathbf{KP})^{-1} \end{pmatrix}. \tag{6.2}$$

The feedback system (6.1) is called $\ell^q$-*stable* for $1 \leq q \leq \infty$ if there exists $M \geq 0$ such that

$$\|y_p\|_{\ell^q} + \|y_c\|_{\ell^q} \le M(\|r\|_{\ell^q} + \|d_1\|_{\ell^q} + \|d_2\|_{\ell^q}) , \quad \forall r, d_2 \in \ell^q(\mathbb{Z}_+, \mathbb{C}^p) , \; \forall d_1 \in \ell^q(\mathbb{Z}_+, \mathbb{C}^m) .$$

We observe that system (6.1) is $\ell^q$-stable if $F(\mathbf{P}, \mathbf{K}) \in \hat{\ell}^1(\mathbb{C}^{(m+p)\times(m+p)})$. It is a standard result that (6.1) is $\ell^2$-stable if and only if $F(\mathbf{P}, \mathbf{K}) \in H^\infty(\mathbb{E}_1, \mathbb{C}^{(m+p)\times(m+p)})$ (see [60, Theorem 3.2.1, p. 45]).

**Definition 6.1.1.** A *left-coprime factorization* of $\mathbf{P} \in \mathfrak{Q}_\Omega^{p\times m}$ (over $H^\infty(\Omega, \mathbb{C})$) is a pair $(\mathbf{D}, \mathbf{N}) \in H^\infty(\Omega, \mathbb{C}^{p\times p}) \times H^\infty(\Omega, \mathbb{C}^{p\times m})$ such that

(1) $\det \mathbf{D} \not\equiv 0$,

(2) $\mathbf{P} = \mathbf{D}^{-1}\mathbf{N}$,

(3) $\mathbf{D}, \mathbf{N}$ are left coprime, i.e., there exist $\mathbf{X} \in H^\infty(\Omega, \mathbb{C}^{p\times p})$, $\mathbf{Y} \in H^\infty(\Omega, \mathbb{C}^{m\times p})$ satisfying $\mathbf{DX} + \mathbf{NY} = I$.

A *right-coprime factorization* of $\mathbf{P} \in \mathfrak{Q}_\Omega^{p\times m}$ (over $H^\infty(\Omega, \mathbb{C})$) is a pair $(\mathbf{N}, \mathbf{D}) \in H^\infty(\Omega, \mathbb{C}^{p\times m}) \times H^\infty(\Omega, \mathbb{C}^{m\times m})$ such that

(1) $\det \mathbf{D} \not\equiv 0$,

(2) $\mathbf{P} = \mathbf{N}\mathbf{D}^{-1}$,

(3) $\mathbf{N}, \mathbf{D}$ are right coprime, i.e., there exist $\mathbf{X} \in H^\infty(\Omega, \mathbb{C}^{m\times p})$, $\mathbf{Y} \in H^\infty(\Omega, \mathbb{C}^{m\times m})$ satisfying $\mathbf{XN} + \mathbf{YD} = I$. $\diamond$

**Remark 6.1.2.** It follows from [70, Theorem 1] that $\mathbf{P} \in \mathfrak{Q}_{\mathbb{E}_1}^{p\times m}$ and $\mathbf{K} \in \mathfrak{Q}_{\mathbb{E}_1}^{m\times p}$ admit left and right coprime factorizations (over $H^\infty(\mathbb{E}_1, \mathbb{C})$) if system (6.1) is $\ell^2$-stable. $\diamond$

An application of a standard result in fractional representation theory (see [79, Lemma 3.1]) gives the following necessary and sufficient algebraic condition for closed-loop stability in terms of coprime factors.

**Proposition 6.1.3.** *Let $\mathbf{P} \in \mathfrak{Q}_\Omega^{p\times m}$ and $\mathbf{K} \in \mathfrak{Q}_\Omega^{m\times p}$. Assume that there exist a left-coprime factorization $(\mathbf{D_P}, \mathbf{N_P})$ of $\mathbf{P}$ and a right-coprime factorization $(\mathbf{N_K}, \mathbf{D_K})$ of $\mathbf{K}$ (both over $H^\infty(\Omega, \mathbb{C})$). Then $F(\mathbf{P}, \mathbf{K}) \in H^\infty(\Omega, \mathbb{C}^{(m+p)\times(m+p)})$ if and only if the matrix $\mathbf{N_P N_K} + \mathbf{D_P D_K}$ has an inverse in $H^\infty(\Omega, \mathbb{C}^{p\times p})$, i.e., if and only if*

$$\inf_{z\in\Omega} |\det[\mathbf{N_P}(z)\mathbf{N_K}(z) + \mathbf{D_P}(z)\mathbf{D_K}(z)]| > 0 .$$

**Proposition 6.1.4** ([3, Lemma 3.1])**.** *Assume that $\mathbf{G} \in \hat{\ell}^1(\mathbb{C}^{m\times m})$. Then $\mathbf{G}$ has an inverse in $\hat{\ell}^1(\mathbb{C}^{m\times m})$ if and only if*

$$\inf_{z\in\mathbb{E}_1} |\det \mathbf{G}(z)| > 0 .$$

The next result will be an important tool in the proof of Theorem 6.1.9, the main theorem of this section, and it is also interesting in its own right.

**Proposition 6.1.5.** *Assume that the feedback system* (6.1) *is $\ell^2$-stable. Let* $(\mathbf{D_P}, \mathbf{N_P})$ *be a left-coprime factorization of* $\mathbf{P} \in \mathfrak{Q}_{\mathbb{E}_1}^{p\times m}$ *and let* $(\mathbf{N_K}, \mathbf{D_K})$ *be a right-coprime factorization of* $\mathbf{K} \in \mathfrak{Q}_{\mathbb{E}_1}^{m\times p}$ *(both over* $H^\infty(\mathbb{E}_1, \mathbb{C})$*). Assume that* $\mathbf{D_P}, \mathbf{D_K} \in \hat{\ell}^1(\mathbb{C}^{p\times p})$*,* $\mathbf{N_P} \in \hat{\ell}^1(\mathbb{C}^{p\times m})$ *and* $\mathbf{N_K} \in \hat{\ell}^1(\mathbb{C}^{m\times p})$*. Then* $F(\mathbf{P}, \mathbf{K}) \in \hat{\ell}^1(\mathbb{C}^{(m+p)\times(m+p)})$*. In particular,* (6.1) *is $\ell^q$-stable for* $1 \le q \le \infty$*.*

*Proof.* By hypothesis, it is clear that $\mathbf{N_P N_K} + \mathbf{D_P D_K} \in \hat{\ell}^1(\mathbb{C}^{p\times p})$. Since system (6.1) is $\ell^2$-stable, i.e., $F(\mathbf{P}, \mathbf{K}) \in H^\infty(\mathbb{E}_1, \mathbb{C}^{(m+p)\times(m+p)})$, by Proposition 6.1.3,

$$\inf_{z\in\mathbb{E}_1} |\det[\mathbf{N_P}(z)\mathbf{N_K}(z) + \mathbf{D_P}(z)\mathbf{D_K}(z)]| > 0\,.$$

Therefore, it follows from Proposition 6.1.4 that $(\mathbf{N_P N_K} + \mathbf{D_P D_K})^{-1} \in \hat{\ell}^1(\mathbb{C}^{p\times p})$. It is easy to see that

$$(I + \mathbf{PK})^{-1} = \mathbf{D_K}(\mathbf{N_P N_K} + \mathbf{D_P D_K})^{-1}\mathbf{D_P}\,,$$

so that $(I + \mathbf{PK})^{-1} \in \hat{\ell}^1(\mathbb{C}^{p\times p})$. By simple calculations, we obtain

$$\begin{aligned}
\mathbf{K}(I + \mathbf{PK})^{-1} &= \mathbf{N_K}(\mathbf{N_P N_K} + \mathbf{D_P D_K})^{-1}\mathbf{D_P} \in \hat{\ell}^1(\mathbb{C}^{m\times p})\,,\\
\mathbf{P}(I + \mathbf{KP})^{-1} &= (I + \mathbf{PK})^{-1}\mathbf{P} = \mathbf{D_K}(\mathbf{N_P N_K} + \mathbf{D_P D_K})^{-1}\mathbf{N_P} \in \hat{\ell}^1(\mathbb{C}^{p\times m})\,,\\
(I + \mathbf{KP})^{-1} &= I - \mathbf{K}(I + \mathbf{PK})^{-1}\mathbf{P}\\
&= I - \mathbf{N_K}(\mathbf{N_P N_K} + \mathbf{D_P D_K})^{-1}\mathbf{N_P} \in \hat{\ell}^1(\mathbb{C}^{m\times m})\,.
\end{aligned}$$

Hence $F(\mathbf{P}, \mathbf{K}) \in \hat{\ell}^1(\mathbb{C}^{(m+p)\times(m+p)})$. □

The following frequency-response result for transfer functions in $\hat{\ell}^1(\mathbb{C}^{p\times m})$ will be useful for understanding the asymptotic behaviour of the closed-loop system.

**Lemma 6.1.6.** *Let $G$ be a discrete-time input-output operator with impulse response $g$ and transfer function $\mathbf{G}$ and let $u$ be a function:* $\mathbb{Z}_+ \to \mathbb{C}^m$*,* $\lambda \in \mathrm{cl}(\mathbb{E}_1)$*,* $\mathfrak{v} \in \mathbb{C}^m$*.*

(1) *If* $g \in \ell^1(\mathbb{Z}_+, \mathbb{C}^{p\times m})$ *and* $\lim_{n\to\infty}(u(n) - \lambda^n\mathfrak{v}) = 0$*, then*

$$\lim_{n\to\infty}[(Gu)(n) - \lambda^n\mathbf{G}(\lambda)\mathfrak{v}] = 0\,.$$

(2) *If there exist* $\beta \in (0, 1)$ *and* $M \ge 0$ *such that* $g \in \ell^1_\beta(\mathbb{Z}_+, \mathbb{C}^{p\times m})$ *and*

$$\|u(n) - \lambda^n\mathfrak{v}\| \le M\beta^n\,, \quad \forall n \in \mathbb{Z}_+\,,$$

*then there exists* $L \ge 0$ *such that*

$$\|(Gu)(n) - \mathbf{G}(\lambda)\lambda^n\mathfrak{v}\| \le L\beta^n\,, \quad \forall n \in \mathbb{Z}_+\,.$$

*Proof.* Since $g \in \ell^1(\mathbb{Z}_+, \mathbb{C}^{p\times m})$,

$$\|\mathbf{G}(z)\| = \left\| \sum_{k=0}^{\infty} g(k) z^{-k} \right\| \le \sum_{k=0}^{\infty} \|g(k)\| |z|^{-k} \le \sum_{k=0}^{\infty} \|g(k)\| < \infty\,, \quad \forall z \in \mathrm{cl}(\mathbb{E}_1)\,,$$

so that $\mathbf{G}(z)$ is well defined for $z \in \mathrm{cl}(\mathbb{E}_1)$. Define $v \colon \mathbb{Z}_+ \to \mathbb{C}^m$ by $v(k) := \lambda^k \mathfrak{v}$. Since $\lambda \in \mathrm{cl}(\mathbb{E}_1)$, $|\lambda|^{-k} \le 1$ for all $k \in \mathbb{Z}_+$. Therefore,

$$\begin{aligned}
\|(Gu)(n) - \lambda^n \mathbf{G}(\lambda)\mathfrak{v}\| &= \left\| \sum_{k=0}^{n} g(k) u(n-k) - \sum_{k=0}^{\infty} \lambda^{n-k} g(k) \mathfrak{v} \right\| \\
&\le \left\| \sum_{k=0}^{n} g(k)(u(n-k) - v(n-k)) \right\| + \|\mathfrak{v}\| \sum_{k=n+1}^{\infty} |\lambda|^{n-k} \|g(k)\| \\
&\le \|(G(u-v))(n)\| + \|\mathfrak{v}\| \sum_{k=n}^{\infty} \|g(k)\|\,, \quad \forall n \in \mathbb{Z}_+\,. 
\end{aligned} \tag{6.3}$$

We proceed to prove Statement (1). By hypothesis, $\lim_{k\to\infty}(u(k) - v(k)) = 0$ and $g \in \ell^1(\mathbb{Z}_+, \mathbb{C}^{p\times m})$. There exists $M_1 \ge 0$ such that $\|u(k) - v(k)\| \le M_1$ for all $k \in \mathbb{Z}_+$. Moreover, for $\varepsilon > 0$, there exists $k_0 \in \mathbb{Z}_+$ such that

$$\|u(k) - v(k)\| \le \frac{\varepsilon}{2\|g\|_{\ell^1}}\,, \qquad \sum_{j=k}^{\infty} \|g(j)\| \le \frac{\varepsilon}{2M_1}\,; \quad \forall k \ge k_0\,,$$

Then, for $n \ge 2k_0$,

$$\begin{aligned}
\|(G(u-v))(n)\| &\le \sum_{k=0}^{k_0} \|g(k)\| \|(u-v)(n-k)\| + \sum_{k=k_0+1}^{n} \|g(k)\| \|(u-v)(n-k)\| \\
&\le \frac{\varepsilon}{2\|g\|_{\ell^1}} \sum_{k=0}^{k_0} \|g(k)\| + M_1 \sum_{k=k_0+1}^{n} \|g(k)\| \\
&\le \varepsilon\,,
\end{aligned}$$

showing that

$$\lim_{n\to\infty} \|G(u-v)(n)\| = 0\,. \tag{6.4}$$

A combination of (6.3), (6.4) and the fact that $\lim_{n\to\infty} \sum_{k=n}^{\infty} \|g(k)\| = 0$ yields Statement (1).

To prove Statement (2), we set $M_2 := \sum_{k=0}^{\infty} \beta^{-k} \|g(k)\| < \infty$. By hypothesis, there exists $M \ge 0$ such that

$$\|(u-v)(n)\| \le M\beta^n\,, \quad \forall n \in \mathbb{Z}_+\,.$$

Since $\beta \in (0,1)$ and by (6.3), we have

$$\begin{aligned}
\beta^{-n}\|(Gu)(n) - \mathbf{G}(\lambda)\lambda^n \mathfrak{v}\| &\le \beta^{-n}\sum_{k=0}^{n}\|g(k)\|\|(u-v)(n-k)\| + \beta^{-n}\|\mathfrak{v}\|\sum_{k=n}^{\infty}\|g(k)\| \\
&\le \beta^{-n}\sum_{k=0}^{n}\|g(k)\|M\beta^{n-k} + \|\mathfrak{v}\|\sum_{k=n}^{\infty}\beta^{-k}\|g(k)\| \\
&\le M\sum_{k=0}^{\infty}\beta^{-k}\|g(k)\| + \|\mathfrak{v}\|\sum_{k=0}^{\infty}\beta^{-k}\|g(k)\| \\
&\le MM_2 + \|\mathfrak{v}\|M_2\,, \quad \forall n \in \mathbb{Z}_+\,.
\end{aligned}$$

Hence $\|(Gu)(n) - \mathbf{G}(\lambda)\lambda^n\mathfrak{v}\| \le M_2(M + \|\mathfrak{v}\|)\beta^n$ for all $n \in \mathbb{Z}_+$. □

The following result shows that Lemma 6.1.6 applies in particular to input-output operators with transfer functions in $H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m})$.

**Proposition 6.1.7.** *For* $0 < \alpha < \beta$, $H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{p\times m}) \subset \hat{\ell}^1_\beta(\mathbb{C}^{p\times m})$.

*Proof.* Let $0 < \alpha < \beta$ and $f \in H^\infty(\mathbb{E}_\alpha, \mathbb{C})$. To prove $H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{p\times m}) \subset \hat{\ell}^1_\beta(\mathbb{C}^{p\times m})$, it is sufficient to show that $f \in \hat{\ell}^1_\beta(\mathbb{C})$. Since $f$ is holomorphic and bounded on $\mathbb{E}_\alpha$, $f$ can be written as

$$f(z) = \sum_{k=0}^{\infty} a_k z^{-k}\,, \quad \forall z \in \mathbb{E}_\alpha\,,$$

where $a_k \in \mathbb{C}$. Taking $\rho \in (\alpha, \beta)$, we have

$$f(\rho e^{i\theta}) = \sum_{k=0}^{\infty} a_k \rho^{-k} e^{-ik\theta}\,, \quad \forall \theta \in [0, 2\pi)\,.$$

By Parseval's formula (see, for example, [66, Theorem 10.22, p. 211]), we have

$$\sum_{k=0}^{\infty} |a_k|^2 \rho^{-2k} = \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|^2 d\theta \le \left(\sup_{\theta\in[0,2\pi]} |f(\rho e^{i\theta})|\right)^2 < \infty\,.$$

By the Hölder's inequality,

$$\sum_{k=0}^{\infty} |a_k|\beta^{-k} \le \left(\sum_{k=0}^{\infty} |a_k|^2\rho^{-2k}\right)^{1/2} \left(\sum_{k=0}^{\infty}(\rho/\beta)^{2k}\right)^{1/2} < \infty\,,$$

showing that $f \in \hat{\ell}^1_\beta(\mathbb{C})$. □

**Remark 6.1.8.** Consider a discrete-time state-space system

$$x_p(k+1) = A_p x_p(k) + B_p u_p(k)\,, \tag{6.5a}$$
$$y_p(k) = C_p x_p(k) + D_p u_p(k)\,, \tag{6.5b}$$

evolving on a Banach space $X$, where $A_p \in \mathcal{B}(X)$, $B_p \in \mathcal{B}(\mathbb{C}^m, X)$, $C_p \in \mathcal{B}(X, \mathbb{C}^p)$ and $D_p \in \mathcal{B}(\mathbb{C}^m, \mathbb{C}^p)$. The transfer function $\mathbf{P}$ of (6.5) is given by

$$\mathbf{P}(z) = C_p(zI - A_p)^{-1}B_p + D_p \, .$$

System (6.5) is called *power stable* if $A_p$ is power stable. By Proposition 2.2.1, the power stability of (6.5) implies that $\mathbf{P} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m})$. Hence, it follows from Proposition 6.1.7 that Lemma 6.1.6 applies to power stable systems of the form (6.5). $\diamond$

### 6.1.2 Main results

The following asymptotic tracking theorem is the main result of this section. It is the discrete-time counterpart of a continuous-time result due to Rebarber and Weiss [65], which is a partial extension of the main results in Hämäläinen and Pohjolainen [15].

**Theorem 6.1.9.** *Consider the feedback system* (6.1) *with* $\mathbf{K}$ *replaced by* $\mathbf{K}_\varepsilon$ *defined in* (6.6)*. Let* $N \in \mathbb{N}$*. For* $j \in \underline{N}$*, let* $\lambda_j \in \mathbb{C}$*,* $|\lambda_j| = 1$ *be such that* $\lambda_j \neq \lambda_k$ *for* $j, k \in \underline{N}$*,* $j \neq k$*. Assume that* $\mathbf{P} \in \hat{\ell}^1(\mathbb{C}^{p\times m})$ *and* $\mathbf{K}_\varepsilon$ *is given by*

$$\mathbf{K}_\varepsilon(z) := \varepsilon \left( \mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{K_j}{z - \lambda_j} \right) , \tag{6.6}$$

*where* $\mathbf{K}^0 \in \hat{\ell}^1(\mathbb{C}^{m\times p})$ *and* $K_j \in \mathbb{C}^{m\times p}$*. If*

$$\sigma[\bar{\lambda}_j \mathbf{P}(\lambda_j) K_j] \subset \mathbb{C}_0 \, , \quad \forall j \in \underline{N} \, , \tag{6.7}$$

*and*

$$\limsup_{z\to\lambda_j,\, z\in\mathbb{E}_1} \left\| \frac{\mathbf{P}(z) - \mathbf{P}(\lambda_j)}{z - \lambda_j} \right\| < \infty \, , \quad \forall j \in \underline{N} \, , \tag{6.8}$$

*then there exists* $\varepsilon^* > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon^*)$*,* $F(\mathbf{P}, \mathbf{K}_\varepsilon) \in \hat{\ell}^1(\mathbb{C}^{(m+p)\times(m+p)})$*, where* $F(\mathbf{P}, \mathbf{K}_\varepsilon)$ *is given by* (6.2)*, with* $\mathbf{K}$ *replaced with* $\mathbf{K}_\varepsilon$*.*

*Moreover, if the reference signal* $r$ *is given by*

$$r(k) := \sum_{j=1}^{N} \lambda_j^k \mathfrak{r}_j \, , \; \mathfrak{r}_j \in \mathbb{C}^p \, , \quad \forall k \in \mathbb{Z}_+ \, , \tag{6.9}$$

*and the disturbance signals* $d_1, d_2$ *satisfy*

$$\lim_{k\to\infty} (d_1(k) - \sum_{j=1}^{N} \lambda_j^k \mathfrak{d}_{1j}) = 0 \, , \; \lim_{k\to\infty} (d_2(k) - \sum_{j=1}^{N} \lambda_j^k \mathfrak{d}_{2j}) = 0 \, , \quad \mathfrak{d}_{1j} \in \mathbb{C}^m \, , \; \mathfrak{d}_{2j} \in \mathbb{C}^p \, , \tag{6.10}$$

*then, for every* $\varepsilon \in (0, \varepsilon^*)$*, the output* $y$ *of the closed-loop system asymptotically tracks* $r$*, in the presence of* $d_1, d_2$*, that is* $\lim_{k\to\infty}(y(k) - r(k)) = 0$*.*

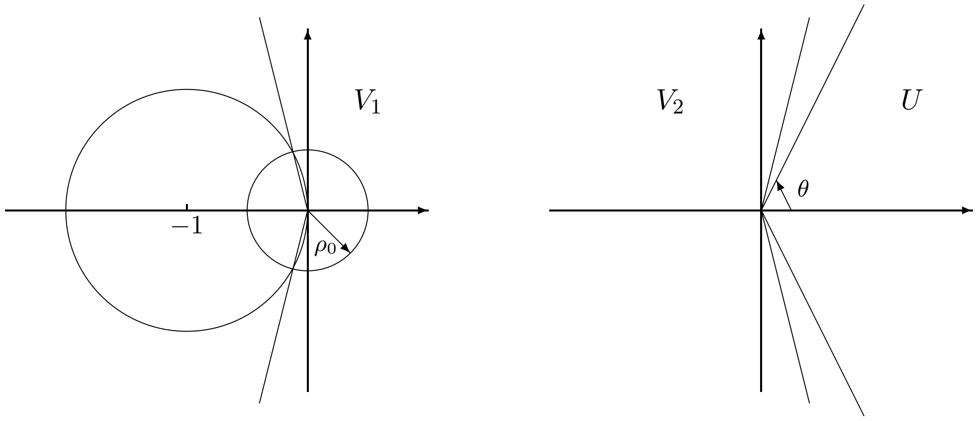


Figure 6-2: An illustration of the sets $U$, $V_1$ and $V_2$.

For the proof of Theorem 6.1.9, a key step is to show that the transfer function $(I+\mathbf{P}\mathbf{K}_\varepsilon)^{-1}$, the so-called sensitivity function, is in $H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ for sufficiently small $\varepsilon > 0$.

**Lemma 6.1.10.** *Let $N \in \mathbb{N}$ and let $\lambda_j \in \mathbb{C}$, $|\lambda_j| = 1$ be such that $\lambda_j \neq \lambda_k$ for $j, k \in \underline{N}$, $j \neq k$. Let $\mathbf{P} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times m})$ be such that the limit $\mathbf{P}(\lambda_j) := \lim_{z\to\lambda_j,\, z\in\mathbb{E}_1} \mathbf{P}(z)$ exists for every $j \in \underline{N}$. Let $\mathbf{K}_\varepsilon$ be given by (6.6), where $\mathbf{K}^0 \in H^\infty(\mathbb{E}_1, \mathbb{C}^{m\times p})$ and $K_j \in \mathbb{C}^{m\times p}$. Assume that (6.7) and (6.8) hold. Then there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$, $(I+\mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$. Moreover, if the additional assumptions that $\mathbf{P} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m})$ and $\mathbf{K}^0 \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$ are satisfied, then, for every $\varepsilon \in (0, \varepsilon^*)$, $(I+\mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times p})$.*

*Proof.* Since $\sigma[\bar{\lambda}_j \mathbf{P}(\lambda_j) K_j] \subset \mathbb{C}_0$ for all $j \in \underline{N}$, there exists $\theta \in (0, \pi/2)$ such that

$$\bigcup_{j=1}^{N} \sigma[\bar{\lambda}_j \mathbf{P}(\lambda_j) K_j] \subset \{z \in \mathbb{C}\setminus\{0\} : \arg z \in (-\theta, \theta)\} =: U\,. \tag{6.11}$$

Let $\rho \in (0,1)$ and consider Figure 6-2. The circles $\{z \in \mathbb{C} : |z| = \rho\}$ and $\{z \in \mathbb{C} : |z+1| = 1\}$ intersect at two points, denoted by $\rho e^{i\phi(\rho)}$ and $\rho e^{-i\phi(\rho)}$, where $\phi(\rho) \in (\pi/2, \pi)$. Note that $\phi(\rho) \to \pi/2$ monotonically as $\rho \to 0$. Hence there exists $\rho_0 \in (0,1)$ such that $\pi - \phi(\rho) > \theta$ for all $\rho \in (0, \rho_0]$. Set

$$V_1 := \{z \in \mathbb{C}\setminus\{0\} : \arg z \in (-\phi(\rho_0), \phi(\rho_0))\}\,.$$

and

$$V_2 := -V_1 = \{z \in \mathbb{C}\setminus\{0\} : \arg z \in (\pi - \phi(\rho_0), \pi + \phi(\rho_0))\}\,,$$

Clearly,

$$U \cap \mathrm{cl}(V_2) = \emptyset\,. \tag{6.12}$$

There exists $\rho_1 \in (0, \rho_0]$ such that $|\lambda_j - \lambda_k| > 2\rho_1$ for all $j, k \in \underline{N}$, $j \neq k$. Defining

$$\Omega_j := \mathbb{E}_1 \bigcap \{z \in \mathbb{C} : |z - \lambda_j| < \rho_1\},$$

we have that $\Omega_j \cap \Omega_k = \emptyset$ for $j, k \in \underline{N}$, $j \neq k$. Moreover, set $\Omega := \mathbb{E}_1 \setminus \bigcup_{j=1}^{N} \Omega_j$. Assume that $\mathbf{P} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times m})$ and $\mathbf{K}^0 \in H^\infty(\mathbb{E}_1, \mathbb{C}^{m\times p})$. It is clear that

$$\sup_{z\in\Omega} \left\| \mathbf{P}(z) \left( \mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{K_j}{z - \lambda_j} \right) \right\| < \infty .$$

Therefore, there exists $\varepsilon_\infty > 0$ such that

$$\mathbf{S}(z) := [I + \mathbf{P}(z)\mathbf{K}_\varepsilon(z)]^{-1} = \left[ I + \varepsilon \mathbf{P}(z) \left( \mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{K_j}{z - \lambda_j} \right) \right]^{-1}$$

is uniformly bounded for all $z \in \Omega$ and for all $\varepsilon \in (0, \varepsilon_\infty)$. Fix $j \in \underline{N}$. To analyze $\mathbf{S}$ on $\Omega_j$, we define

$$\mathbf{S}_j(z) := \left( I + \frac{\varepsilon \mathbf{P}(\lambda_j) K_j}{z - \lambda_j} \right)^{-1} = \left( I + \frac{\varepsilon \bar{\lambda}_j \mathbf{P}(\lambda_j) K_j}{\bar{\lambda}_j z - 1} \right)^{-1},$$

and

$$\mathbf{Q}_j(z) := \frac{\mathbf{P}(z) - \mathbf{P}(\lambda_j)}{z - \lambda_j} K_j + \mathbf{P}(z)\mathbf{K}^0(z) + \sum_{k\in\underline{N},\, k\neq j} \frac{\mathbf{P}(z) K_k}{z - \lambda_k} .$$

By (6.8), we see that $\mathbf{Q}_j$ is bounded on $\Omega_j$, with a bound that is independent of $\varepsilon$. For convenience, we set $\Gamma_j := \bar{\lambda}_j \mathbf{P}(\lambda_j) K_j$. Noticing that $\bar{\lambda}_j \Omega_j - 1 \subset V_1$ and if $w \in V_1$, then $\gamma w \in V_1$ for all $\gamma \geq 0$, we have

$$\begin{aligned}
\sup_{z\in\Omega_j} \|\mathbf{S}_j(z)\| &= \sup \left\{ \left\| \left( I + \varepsilon \frac{\Gamma_j}{w} \right)^{-1} \right\| : w \in \bar{\lambda}_j \Omega_j - 1 \right\} \\
&\leq \sup_{s\in V_1} \|s(sI + \Gamma_j)^{-1}\| \\
&= \sup_{s\in V_2} \|s(sI - \Gamma_j)^{-1}\| .
\end{aligned}$$

By (6.11) and (6.12), the function $s \mapsto s(sI - \Gamma_j)^{-1}$ is holomorphic on an open set $W \supset \mathrm{cl}(V_2)$. Moreover,

$$\lim_{|s|\to\infty} s(sI - \Gamma_j)^{-1} = I .$$

Hence $s \mapsto s(sI - \Gamma_j)^{-1}$ is a bounded on $\mathrm{cl}(V_2)$. Therefore, $\mathbf{S}_j$ is bounded on $\Omega_j$ with a bound independent of $\varepsilon$. We have $\mathbf{S}^{-1} - \mathbf{S}_j^{-1} = \varepsilon \mathbf{Q}_j$, so that

$$\mathbf{S}(z) = \mathbf{S}_j(z)(I + \varepsilon \mathbf{Q}_j(z)\mathbf{S}_j(z))^{-1} .$$

Hence there exists $\varepsilon_j \in (0, \varepsilon_\infty)$ such that $\mathbf{S}$ is uniformly bounded on $\Omega_j$ for all $\varepsilon \in (0, \varepsilon_j)$. Setting

$$\varepsilon^* := \min\{\varepsilon_j : j \in \underline{N}\},$$

it follows that

$$(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p}), \quad \forall \varepsilon \in (0, \varepsilon^*). \tag{6.13}$$

Finally, let $\varepsilon \in (0, \varepsilon^*)$ and assume that $\mathbf{P} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m})$ and $\mathbf{K}^0 \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$. It is clear that $(I+\mathbf{P}\mathbf{K}_\varepsilon)^{-1}$ is meromorphic on $\mathbb{E}_\gamma$ for some $\gamma \in (0,1)$. Letting $\beta \in (\gamma, 1)$, it follows that $(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}$ has at most finitely many poles in the compact annulus $\mathrm{cl}(\mathbb{E}_\beta) \setminus \mathbb{E}_1$. By (6.13), $(I+\mathbf{P}\mathbf{K}_\varepsilon)^{-1}$ does not have any poles on the unit circle $\partial\mathbb{E}_1$ and so there exists $\alpha \in (\beta, 1)$ such that $(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{p\times p})$. □

We are now in the position to prove Theorem 6.1.9.

*Proof of Theorem 6.1.9.* By Lemma 6.1.10, we know that there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$, $(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$. Let $\varepsilon \in (0, \varepsilon^*)$.

We first show that the other block entries of $F(\mathbf{P}, \mathbf{K}_\varepsilon)$ are also $H^\infty$-functions. Due to the stability of $\mathbf{P}$, it suffices to show that $\mathbf{K}_\varepsilon(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{m\times p})$. In the following of the proof, when we write $z \to \lambda_j$, it is assumed that $z \in \mathbb{E}_1$. By assumption, $\lambda_j \neq \lambda_k$ for $j, k \in \underline{N}$, $j \neq k$. Note that, by (6.7), $\mathbf{P}(\lambda_j)K_j$ is invertible. Consequently,

$$\begin{aligned}
&\lim_{z\to\lambda_j} \frac{1}{z - \lambda_j}(I + \mathbf{P}(z)\mathbf{K}_\varepsilon(z))^{-1} \\
&= \lim_{z\to\lambda_j} \left[\varepsilon\mathbf{P}(z)K_j + (z - \lambda_j)\left(I + \varepsilon\mathbf{P}(z)\mathbf{K}^0(z) + \varepsilon \sum_{k\in\underline{N},\, k\neq j} \frac{\mathbf{P}(z)K_k}{z - \lambda_k}\right)\right]^{-1} \\
&= (\varepsilon\mathbf{P}(\lambda_j)K_j)^{-1}, \quad \forall j \in \underline{N}.
\end{aligned} \tag{6.14}$$

By (6.6) and (6.14), we conclude that $\mathbf{K}_\varepsilon(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}$ has a finite limit at $\lambda_j$, so that $\mathbf{K}_\varepsilon(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}$ is bounded on $\mathbb{E}_1 \cap \Lambda$, where $\Lambda$ is a neighbourhood of the set $\{\lambda_j : j \in \underline{N}\}$. Since $(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ and $\mathbf{K}_\varepsilon$ is bounded on $\mathbb{E}_1 \setminus \Lambda$, it follows that $\mathbf{K}_\varepsilon(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{m\times p})$. Consequently, $F(\mathbf{P}, \mathbf{K}_\varepsilon) \in H^\infty(\mathbb{E}_1, \mathbb{C}^{(m+p)\times(m+p)})$.

To prove that $F(\mathbf{P}, \mathbf{K}_\varepsilon) \in \hat{\ell}^1(\mathbb{C}^{(m+p)\times(m+p)})$, we set

$$\mathbf{K}^1(z) := \sum_{j=1}^{N} \frac{K_j}{z - \lambda_j}.$$

We see that $\mathbf{K}^1$ is a (strictly proper) rational matrix function. Let $\mathcal{R}_\mathrm{s}$ denote the ring of discrete-time stable proper complex rational functions, i.e., rational functions with complex coefficients which are bounded at infinity and have all their poles in $\{z \in \mathbb{C} : |z| < 1\}$. By a standard result (see [78, Theorem 4.1.43, p. 75]), $\mathbf{K}^1$ has a right-coprime factorization over $\mathcal{R}_{\mathrm{s}}$, i.e., $\mathbf{K}^1 = \mathbf{N}\mathbf{D}^{-1}$, where $\mathbf{N} \in \mathcal{R}_{\mathrm{s}}^{m\times p}$, $\mathbf{D} \in \mathcal{R}_{\mathrm{s}}^{p\times p}$ and there exist $\mathbf{X} \in \mathcal{R}_{\mathrm{s}}^{p\times m}$, $\mathbf{Y} \in \mathcal{R}_{\mathrm{s}}^{p\times p}$ such that $\mathbf{XN} + \mathbf{YD} = I$ . Therefore,

$$\mathbf{K}_\varepsilon = \varepsilon(\mathbf{K}^0 + \mathbf{K}^1) = \varepsilon(\mathbf{K}^0\mathbf{D} + \mathbf{N})\mathbf{D}^{-1}\,,$$

showing that $\mathbf{K}_\varepsilon$ has right coprime factorization $(\varepsilon(\mathbf{K}^0\mathbf{D} + \mathbf{N}), \mathbf{D})$, since

$$(\varepsilon^{-1}\mathbf{X})\varepsilon(\mathbf{K}^0\mathbf{D} + \mathbf{N}) + (\mathbf{Y} - \mathbf{X}\mathbf{K}^0)\mathbf{D} = \mathbf{XN} + \mathbf{YD} = I\,.$$

Since $\mathbf{K}^0, \mathbf{N} \in \hat{\ell}^1(\mathbb{C}^{m\times p})$ and $\mathbf{D} \in \hat{\ell}^1(\mathbb{C}^{p\times p})$, we have that $\mathbf{K}^0\mathbf{D} + \mathbf{N} \in \hat{\ell}^1(\mathbb{C}^{m\times p})$. Therefore, invoking Proposition 6.1.5 and the assumption that $\mathbf{P} \in \hat{\ell}^1(\mathbb{C}^{m\times p})$, we have $F(\mathbf{P}, \mathbf{K}_\varepsilon) \in \hat{\ell}^1(\mathbb{C}^{(m+p)\times(m+p)})$.

To prove tracking and disturbance rejection, we note first that, since $\mathbf{P}(\lambda_j)K_j$ is invertible,

$$(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}(\lambda_j) = \lim_{z\to\lambda_j} (I + \mathbf{P}(z)\mathbf{K}_\varepsilon(z))^{-1} = 0\,, \quad \forall j \in \underline{N}\,, \tag{6.15}$$

and

$$((I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}\mathbf{P})(\lambda_j) = \lim_{z\to\lambda_j} (I + \mathbf{P}(z)\mathbf{K}_\varepsilon(z))^{-1}\mathbf{P}(z) = 0\,, \quad \forall j \in \underline{N}\,. \tag{6.16}$$

Let $r$ be given by (6.9) and let $d_1, d_2$ satisfy (6.10). For $j \in \underline{N}$, define $a_j : \mathbb{Z}_+ \to \mathbb{C}^p$, $b_j : \mathbb{Z}_+ \to \mathbb{C}^m$ by

$$a_j(k) := \lambda_j^k \mathfrak{r}_j\,, \quad b_j(k) := \lambda_j^k \mathfrak{d}_{1j}\,.$$

Obviously, $r = \sum_{j=1}^N a_j$ and $\lim_{k\to\infty} d_1(k) - \sum_{j=1}^N b_j(k) = 0$. Let $\mathscr{Z}^{-1}$ denote the inverse Z-transform. Then, by Lemma 6.1.6, (6.15) and (6.16), we obtain

$$\begin{aligned}
&\lim_{k\to\infty} [\mathscr{Z}^{-1}((I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}) \star r](k) \\
&\quad = \sum_{j=1}^N \lim_{k\to\infty} \{[\mathscr{Z}^{-1}((I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}) \star a_j)](k) - ((I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1})(\lambda_j)\lambda_j^k \mathfrak{r}_j\} \\
&\quad = 0\,,
\end{aligned} \tag{6.17}$$

and

$$\begin{aligned}
&\lim_{k\to\infty} [\mathscr{Z}^{-1}((I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}\mathbf{P}) \star d_1](k) \\
&\quad = \sum_{j=1}^N \lim_{k\to\infty} \{[\mathscr{Z}^{-1}((I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}\mathbf{P}) \star b_j](k) - ((I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}\mathbf{P})(\lambda_j)\lambda_j^k \mathfrak{d}_{1j}\} \\
&\qquad\quad + \lim_{k\to\infty} [\mathscr{Z}^{-1}((I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}\mathbf{P}) \star (d_1 - \sum_{j=1}^N b_j)](k) \\
&\quad = 0\,.
\end{aligned} \tag{6.18}$$

Similarly, by Lemma 6.1.6 and (6.15),

$$\lim_{k\to\infty} [\mathscr{Z}^{-1}((I+\mathbf{P}\mathbf{K}_\varepsilon)^{-1}) \star d_2](k) = 0\,. \tag{6.19}$$

It follows from system (6.1) (with $\mathbf{K}$ replaced by $\mathbf{K}_\varepsilon$) that

$$\widehat{r} - \widehat{y} = (I+\mathbf{P}\mathbf{K}_\varepsilon)^{-1}(\widehat{r} - \widehat{d}_2) - (I+\mathbf{P}\mathbf{K}_\varepsilon)^{-1}\mathbf{P}\widehat{d}_1\,. \tag{6.20}$$

Therefore, by (6.17)–(6.20),

$$\begin{aligned}\lim_{k\to\infty}(r-y)(k) &= \lim_{k\to\infty}\{[\mathscr{Z}^{-1}((I+\mathbf{P}\mathbf{K}_\varepsilon)^{-1}) \star (r-d_2)](k)\\ &\qquad - [\mathscr{Z}^{-1}((I+\mathbf{P}\mathbf{K}_\varepsilon)^{-1}\mathbf{P}) \star d_1](k)\}\\ &= 0\,.\end{aligned}$$

□

**Remark 6.1.11.** (1) In (6.6), the term $\mathbf{K}^0$ may be employed to satisfy additional design requirements, for example, to improve robustness properties or to speed up the transient response. The existence of matrices $K_j$ such that (6.7) holds is guaranteed if and only if $\operatorname{rk}\mathbf{P}(\lambda_j) = p$ for all $j \in \underline{N}$, in which case, $K_j = \lambda_j \mathbf{P}^*(\lambda_j)[\mathbf{P}(\lambda_j)\mathbf{P}^*(\lambda_j)]^{-1}$ is a possible choice.

(2) The lim sup condition (6.8) is not very restrictive. It is trivially satisfied if $\mathbf{P} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m})$.

(3) If, in Theorem 6.1.9, we replace the controller $\mathbf{K}_\varepsilon$ given in (6.6) by

$$\widetilde{\mathbf{K}}_\varepsilon(z) := \varepsilon\left(\widetilde{\mathbf{K}}^0(z) + \sum_{j=1}^N \frac{z\widetilde{K}_j}{z-\lambda_j}\right)\,,$$

where $\widetilde{\mathbf{K}}^0 \in \hat{\ell}^1(\mathbb{C}^{m\times p})$ and $\widetilde{K}_j \in \mathbb{C}^{m\times p}$, and condition (6.7) by

$$\sigma(\mathbf{P}(\lambda_j)\widetilde{K}_j) \subset \mathbb{C}_0\,, \quad \forall j \in \underline{N}\,,$$

whilst all the other conditions in the theorem remain the same, then the conclusions on stability, tracking and disturbance rejection in Theorem 6.1.9 are still valid. This follows directly from Theorem 6.1.9, since

$$\widetilde{\mathbf{K}}_\varepsilon(z) = \varepsilon\left(\widetilde{\mathbf{K}}^0(z) + \sum_{j=1}^N \widetilde{K}_j + \sum_{j=1}^N \frac{\lambda_j\widetilde{K}_j}{z-\lambda_j}\right)$$

is of the form (6.6) with

$$\mathbf{K}^0(z) := \widetilde{\mathbf{K}}^0(z) + \sum_{j=1}^N \widetilde{K}_j\,, \quad K_j := \lambda_j\widetilde{K}_j\,,$$

and $\sigma(\bar{\lambda}_j\mathbf{P}(\lambda_j)K_j) = \sigma(\mathbf{P}(\lambda_j)\widetilde{K}_j) \subset \mathbb{C}_0$. ◇

Next we show that, under a mild extra assumption on $\mathbf{P}$, $\mathbf{K}^0$, $d_1$ and $d_2$, the convergence of $y(k)$ to $r(k)$ as $k \to \infty$ is exponentially fast.

**Theorem 6.1.12.** *Let $N \in \mathbb{N}$ and let $\lambda_j \in \mathbb{C}$, $|\lambda_j| = 1$ be such that $\lambda_j \neq \lambda_k$ for $j, k \in \underline{N}$, $j \neq k$. Consider the feedback system (6.1) with $\mathbf{P} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m})$ and $\mathbf{K}_\varepsilon$ given by (6.6), where $\mathbf{K}^0 \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$ and $K_j \in \mathbb{C}^{m\times p}$ for $j \in \underline{N}$. If (6.7) holds, then there exists $\varepsilon^* > 0$ such that, for every $\varepsilon \in (0, \varepsilon^*)$, $F(\mathbf{P}, \mathbf{K}_\varepsilon) \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{(m+p)\times(m+p)})$. Moreover, if the reference signal $r$ is given by (6.9) and there exist $M \geq 0$ and $\rho \in (0, 1)$ such that the disturbance signals $d_1, d_2$ satisfy*

$$\|d_1(k) - \sum_{j=1}^{N} \lambda_j^k \mathfrak{d}_{1j}\| \leq M\rho^k\,,\ \|d_2(k) - \sum_{j=1}^{N} \lambda_j^k \mathfrak{d}_{2j}\| \leq M\rho^k\,, \quad \mathfrak{d}_{1j} \in \mathbb{C}^m\,,\ \mathfrak{d}_{2j} \in \mathbb{C}^p\,, \tag{6.21}$$

*then, for every $\varepsilon \in (0, \varepsilon^*)$, there exist $L \geq 0$ and $\beta \in (\rho, 1)$ such that*

$$\|y(k) - r(k)\| \leq L\beta^k\,, \quad \forall k \in \mathbb{Z}_+\,.$$

*Proof.* By the hypotheses on $\mathbf{P}$, $\mathbf{K}^0$, and Lemma 6.1.10, we know that there exists $\varepsilon^* > 0$ such that, for every $\varepsilon \in (0, \varepsilon^*)$, there exists $\alpha \in (\rho, 1)$ such that

$$(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{p\times p})\,,\ \mathbf{P} \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{p\times m})\,,\ \mathbf{K}^0 \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{m\times p})\,.$$

To prove that $F(\mathbf{P}, \mathbf{K}_\varepsilon) \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{(m+p)\times(m+p)})$, it suffices to show that $\mathbf{K}_\varepsilon(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{m\times p})$. By (6.14), we conclude that $\mathbf{K}_\varepsilon(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}$ has a finite limit as $z \to \lambda_j$ for every $j \in \underline{N}$, so that $\mathbf{K}_\varepsilon(I+\mathbf{P}\mathbf{K}_\varepsilon)^{-1}$ is bounded on a neighbourhood $\Lambda$ of the set $\{\lambda_j : j \in \underline{N}\}$. Since $(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{p\times p})$ and $\mathbf{K}_\varepsilon$ is bounded on $\mathbb{E}_\alpha \setminus \Lambda$, it follows that

$$\mathbf{K}_\varepsilon(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{m\times p})\,.$$

Hence $F(\mathbf{P}, \mathbf{K}_\varepsilon) \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{(m+p)\times(m+p)})$. Therefore, it follows from Proposition 6.1.7 that, for every $\beta \in (\alpha, 1)$, we have

$$(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in \hat{\ell}^1_\beta(\mathbb{C}^{p\times p})\,, \quad (I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}\mathbf{P} \in \hat{\ell}^1_\beta(\mathbb{C}^{p\times m})\,.$$

Finally, invoking Lemma 6.1.6, (6.9), (6.15), (6.16) and (6.21), we conclude that there exists $M_1 \geq 0$ such that

$$\begin{aligned}
\|[\mathscr{Z}^{-1}((I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}) \star r](k)\| &\leq M_1\beta^k\,, \quad \forall k \in \mathbb{Z}_+\,,\\
\|[\mathscr{Z}^{-1}((I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}\mathbf{P}) \star d_1](k)\| &\leq M_1\beta^k\,, \quad \forall k \in \mathbb{Z}_+\,,\\
\|[\mathscr{Z}^{-1}((I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}) \star d_2](k)\| &\leq M_1\beta^k\,, \quad \forall k \in \mathbb{Z}_+\,.
\end{aligned}$$

Consequently, by (6.20), we have

$$\|y(k) - r(k)\| \leq 3M_1\beta^k\,, \quad \forall k \in \mathbb{Z}_+\,.$$

□

### 6.1.3 Application to state-space systems

We now apply Theorem 6.1.9 to obtain tracking results for discrete-time state-space systems. Let $X$ be a Banach space and let the plant $\Sigma_p$ be given by

$$\begin{aligned} x_p(k+1) &= A_p x_p(k) + B_p u_p(k)\,; \quad x_p(0) = x_p^0 \in X\,, && \text{(6.22a)} \\ y_p(k) &= C_p x_p(k) + D_p u_p(k)\,, && \text{(6.22b)} \end{aligned}$$

where $A_p \in \mathcal{B}(X)$, $B_p \in \mathcal{B}(\mathbb{C}^m, X)$, $C_p \in \mathcal{B}(X, \mathbb{C}^p)$ and $D_p \in \mathcal{B}(\mathbb{C}^m, \mathbb{C}^p)$. The transfer function $\mathbf{P}$ of $\Sigma_p$ is given by

$$\mathbf{P}(z) = C_p(zI - A_p)^{-1}B_p + D_p\,.$$

We say that system (6.22) is *power stable* if $A_p$ is power stable.

Next we construct a state-space realization of the controller transfer function (6.6). Let $\mathbf{K}^0$ be a discrete-time stable proper complex rational function and let $(A_0, B_0, C_0, D_0) \in \mathbb{C}^{n_0\times n_0} \times \mathbb{C}^{n_0\times p} \times \mathbb{C}^{m\times n_0} \times \mathbb{C}^{m\times p}$ be a stabilizable and detectable realization of $\mathbf{K}^0$, i.e., $\mathbf{K}^0(z) = C_0(zI - A_0)^{-1}B_0 + D_0$, $(A_0, B_0)$ is stabilizable and $(C_0, A_0)$ is detectable. Since $\mathbf{K}^0 \in H^\infty(\mathbb{E}_1, \mathbb{C}^{m\times p})$, $A_0$ is power stable. Let $K_j \in \mathbb{C}^{m\times p}$ and let $\lambda_j \in \mathbb{C}$, $|\lambda_j| = 1$ be such that $\lambda_j \neq \lambda_k$ for $j, k \in \underline{N}$, $j \neq k$. Define $A_c \in \mathbb{C}^{(Np+n_0)\times(Np+n_0)}$, $B_c \in \mathbb{C}^{(Np+n_0)\times p}$, $C_c \in \mathbb{C}^{m\times(Np+n_0)}$ and $D_c \in \mathbb{C}^{m\times p}$ by

$$A_c := \begin{pmatrix} A_0 & 0 & \dots & 0 \\ 0 & \lambda_1 I_p & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N I_p \end{pmatrix}, \; B_c := \begin{pmatrix} B_0 \\ I_p \\ \vdots \\ I_p \end{pmatrix}, \; C_c := (C_0, K_1, \dots, K_N)\,, \; D_c := D_0\,, \tag{6.23}$$

where $I_p$ is the $p \times p$ identity matrix. Let $\varepsilon > 0$ and we define the controller $\Sigma_c$ by

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + B_c u_c(k)\,; \quad x_c(0) = x_c^0 \in \mathbb{C}^{Np+n_0}\,, && \text{(6.24a)} \\ y_c(k) &= \varepsilon C_c x_c(k) + \varepsilon D_c u_c(k)\,. && \text{(6.24b)} \end{aligned}$$

Obviously, the transfer function $\mathbf{K}_\varepsilon$ of $\Sigma_c$ is given by

$$\mathbf{K}_\varepsilon(z) = \varepsilon C_c(zI - A_c)^{-1}B_c + \varepsilon D_c = \varepsilon\left(\mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{K_j}{z - \lambda_j}\right).$$

Consider the feedback interconnection of (6.22) and (6.24) given by

$$u_c = r - y_p - d_2\,, \quad u_p = y_c + d_1\,, \tag{6.25}$$

where $r$ is a reference signal and $d_1$ and $d_2$ are disturbance signals. Let $\mathcal{F}(\Sigma_p, \Sigma_c)$ denote the feedback system given by (6.22), (6.24) and (6.25). The state-space system

$\mathcal{F}(\Sigma_p, \Sigma_c)$ is a state-space realization of the system (6.1) (with $\mathbf{K}$ replaced by $\mathbf{K}_\varepsilon$). It is clear that $\mathcal{F}(\Sigma_p, \Sigma_c)$ has a unique solution which will be denoted by

$$\begin{pmatrix} x_p(\cdot\,; x_p^0, x_c^0, \varepsilon, r, d_1, d_2) \\ x_c(\cdot\,; x_p^0, x_c^0, \varepsilon, r, d_1, d_2) \end{pmatrix} .$$

**Remark 6.1.13.** Note that the plant $\Sigma_p$ is infinite-dimensional, but the discrete-time controller $\Sigma_c$ is finite-dimensional and hence can be implemented with on-line digital computers. The order of the discrete-time controller depends on the number of frequencies of the reference and disturbance signals. $\diamond$

**Theorem 6.1.14.** *Assume that* (6.22) *is power stable and that*

$$\sigma(\bar{\lambda}_j \mathbf{P}(\lambda_j) K_j) \subset \mathbb{C}_0\,, \quad \forall j \in \underline{N}\,. \tag{6.26}$$

*Then there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$, the following statements hold:*

(1) *$\mathcal{F}(\Sigma_p, \Sigma_c)$ is power stable. Moreover, $\mathcal{F}(\Sigma_p, \Sigma_c)$ is input-to-state stable in the sense that there exist $M_1 \geq 1$ and $\gamma \in (0,1)$ such that*

$$\left\| \begin{pmatrix} x_p(k; x_p^0, x_c^0, \varepsilon, r, d_1, d_2) \\ x_c(k; x_p^0, x_c^0, \varepsilon, r, d_1, d_2) \end{pmatrix} \right\| \leq M_1 \left( \gamma^k \left\| \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} \right\| + \|r\|_{\ell^\infty} + \|d_1\|_{\ell^\infty} + \|d_2\|_{\ell^\infty} \right), \tag{6.27}$$

$$\forall x_p^0 \in X\,,\ \forall x_c^0 \in \mathbb{C}^{Np+n_0}\,,\ \forall d_1 \in \ell^\infty(\mathbb{Z}_+, \mathbb{C}^m)\,,$$
$$\forall r, d_2 \in \ell^\infty(\mathbb{Z}_+, \mathbb{C}^p)\,.$$

(2) *If $r$ is given by* (6.9) *and $d_1, d_2$ satisfy* (6.10)*, then for all initial conditions $x_p^0 \in X$ and $x_c^0 \in \mathbb{C}^{Np+n_0}$, the output $y = y_p + d_2$ asymptotically tracks $r$, that is $\lim_{k\to\infty}(y(k) - r(k)) = 0$. Additionally, if* (6.21) *holds with $M \geq 0$ and $\rho \in (0,1)$, then the convergence is exponentially fast.*

*Proof.* There exists $\varepsilon_1 > 0$ such that $(I + \varepsilon D_c D_p)$ is invertible for all $\varepsilon \in (0, \varepsilon_1)$. Then $I + \varepsilon D_p D_c$ is also invertible for $\varepsilon \in (0, \varepsilon_1)$, with $(I + \varepsilon D_p D_c)^{-1} = I - D_p(I + \varepsilon D_c D_p)^{-1} D_c$. It is clear that $\mathbf{K}^0 \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$. It follows from Proposition 2.2.1 and the power stability of system (6.22) that $\mathbf{P} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m})$. Since (6.26) holds, Theorem 6.1.12 shows that, there exists $\varepsilon^* \in (0, \varepsilon_1)$ such that, for all $\varepsilon \in (0, \varepsilon^*)$,

$$F(\mathbf{P}, \mathbf{K}_\varepsilon) \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{(m+p)\times(m+p)})\,, \tag{6.28}$$

where $F(\mathbf{P}, \mathbf{K}_\varepsilon)$ is defined in (6.2) with $\mathbf{K}$ replaced by $\mathbf{K}_\varepsilon$. Let $\varepsilon \in (0, \varepsilon^*)$. It follows from Proposition 6.1.7 that $F(\mathbf{P}, \mathbf{K}_\varepsilon) \in \hat{\ell}^1(\mathbb{Z}_+, \mathbb{C}^{(m+p)\times(m+p)})$, i.e., $\mathcal{F}(\Sigma_p, \Sigma_c)$ is $\ell^q$-stable for $1 \leq q \leq \infty$.

To prove the power stability of $\mathcal{F}(\Sigma_p, \Sigma_c)$, we first show that $\Sigma_c$ given by (6.24) is detectable and stabilizable. By (6.26), we see that $\operatorname{rk}(\mathbf{P}(\lambda_j) K_j) = p$. It follows that

$\operatorname{rk} K_j = p$ (in particular $p \leq m$). Moreover, the power stability of $A_0$ and Proposition 2.2.1 implies that $\sigma(A_0) \subset \{z \in \mathbb{C} : |z| < 1\}$. Note that $\lambda_j \neq \lambda_k$ for all $j, k \in \underline{N}$, $j \neq k$. Therefore,

$$\begin{aligned}
\operatorname{rk} \begin{pmatrix} zI - A_c \\ C_c \end{pmatrix} &= \operatorname{rk} \begin{pmatrix} zI - A_0 & 0 & \dots & 0 \\ 0 & (z-\lambda_1)I_p & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (z-\lambda_N)I_p \\ C_0 & K_1 & \dots & K_N \end{pmatrix} \\
&= Np + n_0\,, \quad \forall z \in \operatorname{cl}(\mathbb{E}_1)\,,
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{rk}(zI - A_c, B_c) &= \operatorname{rk} \begin{pmatrix} zI - A_0 & 0 & \dots & 0 & B_0 \\ 0 & (z-\lambda_1)I_p & \dots & 0 & I_p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & (z-\lambda_N)I_p & I_p \end{pmatrix} \\
&= Np + n_0\,, \quad \forall z \in \operatorname{cl}(\mathbb{E}_1)\,.
\end{aligned}$$

Hence, by the Hautus criterion, $\Sigma_c$ is detectable stabilizable. Since $A_p$ is power stable, $\Sigma_p$ given by (6.22) is detectable and stabilizable. Therefore, $\mathcal{F}(\Sigma_p, \Sigma_c)$ is stabilizable and detectable. Since $\mathcal{F}(\Sigma_p, \Sigma_c)$ is $\ell^2$-stable, it follows that $\mathcal{F}(\Sigma_p, \Sigma_c)$ is power stable (see [31, Theorem 2]).

To show that $\mathcal{F}(\Sigma_p, \Sigma_c)$ is input-to-state stable, we set $E_p := (I + \varepsilon D_c D_p)^{-1}$, $E_c := (I + \varepsilon D_p D_c)^{-1}$ and define

$$\Delta := \begin{pmatrix} A_p & 0 \\ 0 & A_c \end{pmatrix} + \begin{pmatrix} B_p & 0 \\ 0 & B_c \end{pmatrix} \begin{pmatrix} E_p & 0 \\ 0 & E_c \end{pmatrix} \begin{pmatrix} -\varepsilon D_c & \varepsilon I \\ -I & -\varepsilon D_p \end{pmatrix} \begin{pmatrix} C_p & 0 \\ 0 & C_c \end{pmatrix}.$$

For convenience, we write

$$x_p(\cdot) := x_p(\,\cdot\,; x_p^0, x_c^0, \varepsilon, r, d_1, d_2)\,, \quad x_c(\cdot) := x_c(\,\cdot\,; x_p^0, x_c^0, \varepsilon, r, d_1, d_2)\,.$$

By a routine calculation†, we obtain that

$$\begin{pmatrix} x_p(k+1) \\ x_c(k+1) \end{pmatrix} = \Delta \begin{pmatrix} x_p(k) \\ x_c(k) \end{pmatrix} + \begin{pmatrix} B_p E_p[d_1(k) + \varepsilon D_c(r(k) - d_2(k))] \\ B_c E_c[-D_p d_1(k) + r(k) - d_2(k)] \end{pmatrix}, \quad \forall k \in \mathbb{Z}_+\,.$$

By power stability of $\Delta$, it follows from the discrete-time variation-of-parameters formula that there exist $M_1 \geq 1$ and $\gamma \in (0,1)$ such that (6.27) holds. This completes the

†See Appendix A.4.2 with $r$ replaced by $r - d_2$.

proof of Statement (1).

Set $y = y_p + d_2$. To prove asymptotic tracking and disturbance rejection, we note that it follows from a routine calculation† that

$$\begin{pmatrix} y(k) \\ y_c(k) \end{pmatrix} = \begin{pmatrix} E_c C_p & \varepsilon D_p E_p C_c \\ -\varepsilon D_c E_c C_p & \varepsilon E_p C_c \end{pmatrix} \Delta^k \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} + \begin{pmatrix} y^{\mathrm{io}}(k) \\ y_c^{\mathrm{io}}(k) \end{pmatrix},$$

where $y^{\mathrm{io}}$ and $y_c^{\mathrm{io}}$ satisfy

$$\widehat{y^{\mathrm{io}}} = \mathbf{P}\widehat{y_c^{\mathrm{io}}} + \mathbf{P}\widehat{d}_1 + \widehat{d}_2\,, \quad \widehat{y_c^{\mathrm{io}}} = \mathbf{K}_\varepsilon(\widehat{r} - \widehat{y^{\mathrm{io}}})\,. \tag{6.29}$$

An application of Theorem 6.1.9 to the system (6.29) shows that $\lim_{k\to\infty}(y^{\mathrm{io}}(k) - r(k)) = 0$. Since $\Delta$ is power stable, it is easy to see that $\lim_{k\to\infty}(y(k) - r(k)) = 0$.

Finally, assume that there exist $M \geq 0$ and $\rho \in (0,1)$ such that (6.21) holds. Applying Theorem 6.1.12 to the system (6.29) and invoking the fact that $\Delta$ is power stable completes the proof of Statement (2). □

**Remark 6.1.15.** Note that the matrices $(A_c, B_c, C_c, D_c)$ of the controller $\Sigma_c$ defined in (6.23) are generally complex. Assume that the non-real numbers in $\{\lambda_j \in \mathbb{C} : |\lambda_j| = 1, j \in \underline{N}\}$ occur in complex conjugate pairs. Without loss of generality, we write

$$\lambda_{2j-1} = \alpha_j + i\beta_j\,,\ \lambda_{2j} = \alpha_j - i\beta_j\,,\ \forall j \in \underline{n_1}\,; \quad \lambda_{2n_1+j} = 1 \text{ or } -1\,,\ \forall j \in \underline{n_2}\,,$$

where

$$\underline{0} := \emptyset\,; \quad 2n_1 + n_2 = N\,; \quad \alpha_j, \beta_j \in \mathbb{R}\,,\ \beta_j \neq 0\,,\ \forall j \in \underline{n_1}\,,$$

We now design a real state-space realization of the controller with transfer function of the form (6.6). Let $\mathbf{K}^0$ be a discrete-time proper stable real rational function matrix and let $(A_0, B_0, C_0, D_0) \in \mathbb{R}^{n_0\times n_0} \times \mathbb{R}^{n_0\times p} \times \mathbb{R}^{m\times n_0} \times \mathbb{R}^{m\times p}$ be a realization of $\mathbf{K}^0$ such that $(A_0, B_0)$ is stabilizable and $(C_0, A_0)$ is detectable. Let $K_j \in \mathbb{C}^{m\times p}$ for $j \in \underline{n_1}$, and set

$$C_{2j-1} := \frac{K_j + \bar{K}_j}{2} - \frac{K_j - \bar{K}_j}{2i}\,,\ C_{2j} := \frac{K_j + \bar{K}_j}{2} + \frac{K_j - \bar{K}_j}{2i}\,, \quad \forall j \in \underline{n_1}\,.$$

Note that $C_j \in \mathbb{R}^{m\times p}$ for all $j \in \underline{2n_1}$. Let $K_{2n_1+j} \in \mathbb{R}^{m\times p}$ for $j \in \underline{n_2}$. We define $A_c \in \mathbb{R}^{(Np+n_0)\times(Np+n_0)}$, $B_c \in \mathbb{R}^{(Np+n_0)\times p}$, $C_c \in \mathbb{R}^{m\times(Np+n_0)}$ and $D_c \in \mathbb{R}^{m\times p}$ by

$$A_c := \operatorname{diag}(A_0, \Lambda_1, \ldots, \Lambda_{n_1}, \lambda_{2n_1+1}I_p, \ldots, \lambda_N I_p)\,, \quad B_c := \begin{pmatrix} B_0 \\ I_p \\ \vdots \\ I_p \end{pmatrix}, \tag{6.30a}$$

$$C_c := (C_0, C_1, \ldots, C_{2n_1}, K_{2n_1+1}, \ldots, K_N)\,, \quad D_c := D_0\,, \tag{6.30b}$$

where

$$\Lambda_j := \begin{pmatrix} \alpha_j I_p & \beta_j I_p \\ -\beta_j I_p & \alpha_j I_p \end{pmatrix}, \quad \forall j \in \underline{n_1}.$$

It is easy to check that

$$C_c(zI - A_c)^{-1}B_c + D_c = \mathbf{K}^0(z) + \sum_{j=1}^{n_1} \left( \frac{K_j}{z - \alpha_j - i\beta_j} + \frac{\bar{K}_j}{z - \alpha_j + i\beta_j} \right) + \sum_{j=2n_1+1}^{N} \frac{K_j}{z - \lambda_j}.$$

Moreover, if $\operatorname{rk} K_j = p$ for $j = 1, \ldots, n_1, 2n_1 + 1, \ldots, N$, then, using Hautus Criterion, it can be shown that $(A_c, B_c)$ is stabilizable and $(C_c, A_c)$ is detectable.

If, in Theorem 6.1.14, $(A_c, B_c, C_c, D_c)$ in the controller (6.24) is given by (6.30), $\mathbf{P}$ is real, i.e., $\overline{\mathbf{P}(z)} = \mathbf{P}(\bar{z})$ for $z \in \mathrm{cl}(\mathbb{E}_1)$, and

$$\sigma(\overline{\lambda_{2j-1}}\mathbf{P}(\lambda_{2j-1})K_j) \subset \mathbb{C}_0, \ \forall j \in \underline{n_1}; \quad \sigma(\bar{\lambda}_j \mathbf{P}(\lambda_j)K_j) \subset \mathbb{C}_0, \ \forall j = 2n_1 + 1, \ldots, N,$$

whilst the other conditions in the theorem remain the same, then the conclusions of Theorem 6.1.14 are still valid. This follows from the fact that

$$\sigma(\overline{\lambda_{2j}}\mathbf{P}(\lambda_{2j})\bar{K}_j) = \sigma(\overline{\overline{\lambda_{2j-1}}\mathbf{P}(\lambda_{2j-1})K_j}) = \overline{\sigma(\overline{\lambda_{2j-1}}\mathbf{P}(\lambda_{2j-1})K_j)} \subset \mathbb{C}_0, \quad \forall j \in \underline{n_1},$$

so that (6.26) is satisfied. ◇

### 6.1.4 The case of positive transfer functions

If we know that the plant $\mathbf{P}$ is positive (see definition below), then we can design a simple controller which achieves the control objective (tracking and disturbance rejection), but does not require low gain. In the following, for $K \in \mathbb{C}^{m \times m}$, set $\operatorname{Re} K := (1/2)(K + K^*)$.

**Definition 6.1.16.** Let $A, B \in \mathbb{C}^{m \times m}$. We say that $A > 0$ if $\langle Au, u \rangle > 0$ for all $u \in \mathbb{C}^m \setminus \{0\}$, and $A \geq 0$ if $\langle Au, u \rangle \geq 0$ for all $u \in \mathbb{C}^m$. We say $A \geq B$ if $A - B \geq 0$.

**Remark 6.1.17.** If $A \geq 0$, then $A = A^*$ (see [29, Theorem 3.10-3, p. 203]). ◇

We say $\mathbf{P} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{m \times m})$ is a *positive* transfer function if

$$\operatorname{Re} \mathbf{P}(z) = \frac{1}{2}[\mathbf{P}(z) + (\mathbf{P}(z))^*] \geq 0, \quad \forall z \in \mathbb{E}_1. \qquad \diamond$$

The proof of the following lemma can be found in [65] (see [65, Lemma 3.3]).

**Lemma 6.1.18.** *Let $K \in \mathbb{C}^{m \times m}$. Then $\operatorname{Re} K \geq I/2$ if and only if there exists $Q \in \mathbb{C}^{m \times m}$ with $\|Q\| \leq 1$ such that $K = (I - Q)^{-1}$. Furthermore, for such $Q$,*

$$\operatorname{Re} Q \leq \left(1 - \frac{1}{2\|K\|^2}\right) I.$$

**Lemma 6.1.19.** *Let $K \in \mathbb{C}^{m\times m}$. If $\operatorname{Re} K \geq \alpha I$ for some $\alpha \geq 0$, then*

$$\|Ku\| \geq \alpha\|u\|, \quad \forall u \in \mathbb{C}^m.$$

*Proof.* Let $u \in \mathbb{C}^m$. It follows from the Cauchy-Schwarz inequality that

$$\|Ku\|\|u\| \geq |\langle Ku, u\rangle| \geq \operatorname{Re}\langle Ku, u\rangle = \langle(\operatorname{Re} K)u, u\rangle \geq \alpha\|u\|^2,$$

showing that $\|Ku\| \geq \alpha\|u\|$. □

The following theorem is the discrete-time counterpart of [65, Theorem 3.4].

**Theorem 6.1.20.** *Consider the closed-loop feedback system* (6.1). *Let $N \in \mathbb{N}$ and let $\lambda_j \in \mathbb{C}$, $|\lambda_j| = 1$ be such that $\lambda_j \neq \lambda_k$ for $j, k \in \underline{N}$, $j \neq k$. Assume that $\mathbf{P} \in \hat{\ell}^1(\mathbb{C}^{m\times m})$, $\mathbf{P}$ is a positive transfer function and $\operatorname{Re}\mathbf{P}(\lambda_j)$ is invertible for all $j \in \underline{N}$. Let $\mathbf{K}$ be given by*

$$\mathbf{K}(z) := \mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{zK_j}{z-\lambda_j}, \tag{6.31}$$

*where $\mathbf{K}^0 \in \hat{\ell}^1(\mathbb{C}^{m\times m})$,*

$$\operatorname{Re}\mathbf{K}^0(z) \geq \frac{1}{2}I, \quad \forall z \in \mathbb{E}_1, \tag{6.32}$$

*and $K_j \in \mathbb{C}^{m\times m}$ with $K_j > 0$ for every $j \in \underline{N}$. Then $F(\mathbf{P}, \mathbf{K}) \in \hat{\ell}^1(\mathbb{C}^{2m\times 2m})$, where $F(\mathbf{P}, \mathbf{K})$ is given by* (6.2). *Moreover, the output $y$ of closed-loop system* (6.1) *asymptotically tracks the reference signal $r$ given by* (6.9)*, in the presence of the disturbance $d_1$, $d_2$ satisfying* (6.10)*, that is, $\lim_{k\to\infty}(y(k) - r(k)) = 0$.*

*Proof.* Since $\operatorname{Re}\mathbf{P}(\lambda_j)$ is invertible for every $j \in \underline{N}$ and $\operatorname{Re}\mathbf{P}(z) \geq 0$ for all $z \in \mathbb{E}_1$, we conclude that $\operatorname{Re}\mathbf{P}(\lambda_j) > 0$ for every $j \in \underline{N}$. Hence, there exist $\alpha > 0$ and $\delta_j > 0$ such that

$$\operatorname{Re}\mathbf{P}(z) \geq \alpha I, \quad \forall z \in \Omega_j := \{z \in \mathbb{E}_1 : |z - \lambda_j| < \delta_j\}, \ \forall j \in \underline{N}. \tag{6.33}$$

By Remark 6.1.17, $K_j > 0$ implies that $K_j = K_j^*$ for every $j \in \underline{N}$. Therefore, for every $j \in \underline{N}$,

$$\operatorname{Re}\left(\frac{zK_j}{z-\lambda_j}\right) = \frac{z\bar{z} - \operatorname{Re}(z\bar{\lambda}_j)}{|z-\lambda_j|^2}K_j \geq \frac{|z|^2 - |z\bar{\lambda}_j|}{|z-\lambda_j|^2}K_j = \frac{|z|(|z|-1)}{|z-\lambda_j|^2}K_j \geq 0, \quad \forall z \in \mathbb{E}_1.$$

It follows from (6.31) and (6.32) that

$$\operatorname{Re}\mathbf{K}(z) \geq \frac{1}{2}I, \quad \forall z \in \mathbb{E}_1, \tag{6.34}$$

and thus, by Lemma 6.1.19,

$$\|\mathbf{K}(z)u\| \geq \frac{1}{2}\|u\|, \quad \forall u \in \mathbb{C}^m, \ \forall z \in \mathbb{E}_1. \tag{6.35}$$

Invoking Lemma 6.1.18 and (6.34), there exists $\mathbf{Q}\colon \mathbb{E}_1 \to \mathbb{C}^{m\times m}$ such that

$$\mathbf{K} = (I - \mathbf{Q})^{-1}\,,$$

where

$$\operatorname{Re}\mathbf{Q}(z) \leq \left(1 - \frac{1}{2\|\mathbf{K}(z)\|^2}\right) I \leq I\,, \quad \forall z \in \mathbb{E}_1\,. \tag{6.36}$$

Consequently, setting $\mathbf{S} := (I + \mathbf{PK})^{-1}$, we obtain that

$$\mathbf{S}^{-1} = I + \mathbf{P}(I - \mathbf{Q})^{-1} = (I - \mathbf{Q} + \mathbf{P})(I - \mathbf{Q})^{-1} = (I - \mathbf{Q} + \mathbf{P})\mathbf{K}\,. \tag{6.37}$$

We first show that $\mathbf{S}^{-1}$ is bounded from below on $\bigcup_{j=1}^N \Omega_j$. By (6.33) and (6.36), we have

$$\operatorname{Re}\,[I - \mathbf{Q}(z) + \mathbf{P}(z)] \geq \alpha I\,, \quad \forall z \in \bigcup_{j=1}^N \Omega_j\,.$$

It follows from Lemma 6.1.19 that

$$\|[I - \mathbf{Q}(z) + \mathbf{P}(z)]u\| \geq \alpha\|u\|\,, \quad \forall u \in \mathbb{C}^m\,,\ \forall z \in \bigcup_{j=1}^N \Omega_j\,.$$

Using this, together with (6.35) and (6.37), we obtain

$$\|\mathbf{S}^{-1}(z)u\| \geq \frac{\alpha}{2}\|u\|\,, \quad \forall u \in \mathbb{C}^m\,,\ \forall z \in \bigcup_{j=1}^N \Omega_j\,. \tag{6.38}$$

Setting $\Omega := \mathbb{E}_1 \setminus \bigcup_{j=1}^N \Omega_j$, we next show that $\mathbf{S}^{-1}$ is bounded from below on $\Omega$. It is clear that $\mathbf{K}$ is bounded on $\Omega$, i.e., there exists $\beta > 0$ such that

$$\|\mathbf{K}(z)\| \leq \frac{1}{\sqrt{2\beta}}\,, \quad \forall z \in \Omega\,.$$

Hence, by (6.36),

$$\operatorname{Re}\mathbf{Q}(z) \leq \left(1 - \frac{1}{2\|\mathbf{K}(z)\|^2}\right) I \leq (1-\beta)I\,, \quad \forall z \in \Omega\,.$$

Consequently, using the assumption that $\operatorname{Re}\mathbf{P}(z) \geq 0$ for all $z \in \mathbb{E}_1$, we obtain that

$$\operatorname{Re}\,[I - \mathbf{Q}(z) + \mathbf{P}(z)] \geq \beta I\,, \quad \forall z \in \Omega\,.$$

Invoking Lemma 6.1.19 again, we have

$$\|[I - \mathbf{Q}(z) + \mathbf{P}(z)]u\| \geq \beta\|u\|\,, \quad \forall u \in \mathbb{C}^m\,,\ \forall z \in \Omega\,. \tag{6.39}$$

It follows from (6.35), (6.37) and (6.39) that

$$\|\mathbf{S}^{-1}(z)u\| \geq \frac{\beta}{2}\|u\|, \quad \forall u \in \mathbb{C}^m, \ \forall z \in \Omega. \tag{6.40}$$

Combining (6.38) and (6.40), we see that $\mathbf{S}^{-1}$ is bounded from below on $\mathbb{E}_1$, showing that $\mathbf{S}$ is bounded on $\mathbb{E}_1$. Therefore, $\mathbf{S} = (I + \mathbf{PK})^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{m\times m})$.

Since $\operatorname{Re}\mathbf{P}(\lambda_j) > 0$ for every $j \in \underline{N}$, we see that, for all $u \in \mathbb{C}^m \setminus \{0\}$,

$$\|\mathbf{P}(\lambda_j)u\|\|u\| \geq |\langle \mathbf{P}(\lambda_j)u, u\rangle| \geq \operatorname{Re}\langle \mathbf{P}(\lambda_j)u, u\rangle = \langle (\operatorname{Re}\mathbf{P}(\lambda_j))u, u\rangle > 0,$$

showing that $\mathbf{P}(\lambda_j)$ is invertible for all $j \in \underline{N}$. Moreover, $K_j > 0$ implies that $K_j$ is invertible for all $j \in \underline{N}$. Therefore, $\mathbf{P}(\lambda_j)K_j$ is invertible for all $j \in \underline{N}$. Invoking arguments identical to those used in the proof of Theorem 6.1.9, we conclude that $F(\mathbf{P}, \mathbf{K}) \in \hat{\ell}^1(\mathbb{C}^{2m\times 2m})$ and $\lim_{k\to\infty}(y(k) - r(k)) = 0$, where $r$ is given by (6.9), in the presence of the disturbance $d_1$, $d_2$ satisfying (6.10). □

**Remark 6.1.21.** A simple choice for $\mathbf{K}^0$ and $K_j$ is $\mathbf{K}^0 = \alpha I_m$, where $\alpha \geq 1/2$, and $K_j = \beta I_m$, where $\beta > 0$. ◇

## 6.2 Low-gain sampled-data control of systems with measure impulse responses

### 6.2.1 Preliminaries

Let $\mathfrak{B}(\mathbb{R}_+)$ denote the Borel-$\sigma$-algebra on $\mathbb{R}_+$. For a $\mathbb{C}^{p\times m}$-valued Borel measure $\mu$ on $\mathbb{R}_+$, the total variation $|\mu| : \mathfrak{B}(\mathbb{R}_+) \to [0, \infty]$ of $\mu$ is defined by

$$|\mu|(E) := \sup\left\{\sum_{j=1}^{\infty}\|\mu(E_j)\| : E_j \in \mathfrak{B}(\mathbb{R}_+),\ E_j \cap E_k = \emptyset \text{ if } j \neq k,\ E = \bigcup_{j=1}^{\infty} E_j\right\}.$$

It is clear that

$$\|\mu(E)\| \leq |\mu|(E), \quad \forall E \in \mathfrak{B}(\mathbb{R}_+).$$

The following proposition shows that a $\mathbb{C}^{p\times m}$-valued Borel measure is necessarily bounded.

**Proposition 6.2.1.** *The total variation $|\mu|$ of a $\mathbb{C}^{p\times m}$-valued Borel measure $\mu$ on $\mathbb{R}_+$ is a finite non-negative Borel measure on $\mathbb{R}_+$.*

*Proof.* For the scalar-valued case (i.e., $m = p = 1$), it follows from [66, Theorem 6.2] that $|\mu|$ is a non-negative Borel measure on $\mathbb{R}_+$. An inspection of the proof of Theorem 6.2 in [66] shows that it carries over to the matrix-valued case.

To prove the finiteness of $|\mu|$, we define a complex-valued Borel measure $\mu_{ij}$ by

$$\mu_{ij}(E) := (\mu(E))_{ij}\,, \quad \forall i \in \underline{p}\,,\ \forall j \in \underline{m}\,, \quad \forall E \in \mathfrak{B}(\mathbb{R}_+)\,.$$

It follows from [66, Theorem 6.4] that there exists $M \geq 0$ such that

$$|\mu_{ij}|(\mathbb{R}_+) \leq M\,, \quad \forall i \in \underline{p}\,,\ \forall j \in \underline{m}\,.$$

Let $(E_k)_{k\in\mathbb{Z}_+}$ be a partition of $\mathbb{R}_+$, where $E_k \in \mathfrak{B}(\mathbb{R}_+)$ for all $k \in \mathbb{Z}_+$. Then there exists $\alpha \geq 0$ such that

$$\begin{aligned}
\sum_{k=1}^{\infty} \|\mu(E_k)\| \leq \alpha \sum_{k=1}^{\infty}\sum_{i=1}^{p}\sum_{j=1}^{m} |(\mu(E_k))_{ij}| &= \alpha \sum_{i=1}^{p}\sum_{j=1}^{m}\sum_{k=1}^{\infty} |\mu_{ij}(E_k)| \\
&\leq \alpha \sum_{i=1}^{p}\sum_{j=1}^{m} |\mu_{ij}|(\mathbb{R}_+) \\
&\leq \alpha M p m\,.
\end{aligned}$$

It now follows from the definition of $|\mu|$ that $|\mu|(\mathbb{R}_+) \leq \alpha M p m < \infty$. $\square$

The following technical result is used later.

**Proposition 6.2.2.** *Let $\mu$ be a $\mathbb{C}^{p\times m}$-valued Borel measure on $\mathbb{R}_+$. Then*

$$\lim_{t\to\infty} \int_t^{\infty} |\mu|(ds) = 0\,.$$

*Proof.* Define $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f(t) := |\mu|([k, k+1))\,, \quad \forall t \in [k, k+1)\,,\ \forall k \in \mathbb{Z}_+\,.$$

Obviously, by Proposition 6.2.1,

$$\int_0^{\infty} f(s)ds = \sum_{k=0}^{\infty} \int_k^{k+1} f(s)ds = \sum_{k=0}^{\infty} |\mu|([k, k+1)) = |\mu|(\mathbb{R}_+) < \infty\,,$$

showing that $f \in L^1(\mathbb{R}_+, \mathbb{R})$. Hence, for every $\varepsilon > 0$, there exists $T \in \mathbb{N}$ such that

$$\int_T^{\infty} f(s)ds < \varepsilon\,.$$

Consequently,

$$\int_t^{\infty} |\mu|(ds) \leq \int_T^{\infty} |\mu|(ds) = \sum_{k=T}^{\infty} \int_k^{k+1} f(s)ds = \int_T^{\infty} f(s)ds < \varepsilon\,, \quad \forall t \geq T\,,$$

showing that $\lim_{t\to\infty} \int_t^{\infty} |\mu|(ds) = 0$. $\square$

Let $\mu$ be a $\mathbb{C}^{p\times m}$-valued Borel measure on $\mathbb{R}_+$. Define the continuous-time input-output operator $G$ by

$$(Gu)(t) := (\mu \star u)(t) = \int_0^t \mu(ds)u(t-s)\,, \quad t \geq 0\,,\ u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^m)\,, \tag{6.41}$$

It is well-known that $G \in \mathcal{B}(L^q(\mathbb{R}_+, \mathbb{C}^m), L^q(\mathbb{R}_+, \mathbb{C}^m))$ for $1 \leq q \leq \infty$ (see [13, Proposition 8.49, p. 271]). The transfer function $\mathbf{G}$ of $G$ is given by

$$\mathbf{G}(s) = \int_{\mathbb{R}_+} e^{-st}\mu(dt)\,, \quad \forall s \in \mathrm{cl}(\mathbb{C}_0)\,, \tag{6.42}$$

Trivially, by Proposition 6.2.1, $\|\mathbf{G}(s)\| \leq \int_0^\infty |\mu|(dt) < \infty$ for all $s \in \mathrm{cl}(\mathbb{C}_0)$. It follows in particular that $\mathbf{G} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times m})$.

Recall that $L_b(\mathbb{R}_+, \mathbb{C}^m)$ denotes the set of bounded $\mathbb{C}^m$-valued Lebesgue measurable functions with the sup-norm $\|\cdot\|_\infty$.

**Lemma 6.2.3.** *Let the operator $G$ be given by* (6.41)*, where $\mu$ is a $\mathbb{C}^{p\times m}$-valued Borel measure on $\mathbb{R}_+$ and let $u \in L_b(\mathbb{R}_+, \mathbb{C}^m)$. Then*

$$\limsup_{t\to\infty} \|(Gu)(t)\| \leq |\mu|(\mathbb{R}_+) \limsup_{t\to\infty} \|u(t)\|\,. \tag{6.43}$$

*In particular, if $\lim_{t\to\infty} u(t) = 0$, then $\lim_{t\to\infty}(Gu)(t) = 0$.*

*Proof.* Let $\varepsilon > 0$ and $u \in L_b(\mathbb{R}_+, \mathbb{C}^m)$. Set $M := |\mu|(\mathbb{R}_+)$ and $\sigma := \limsup_{t\to\infty} \|u(t)\|$. By Proposition 6.2.2, there exists $T > 0$ such that

$$\int_T^\infty |\mu|(ds) \leq \frac{\varepsilon}{2\|u\|_\infty} \quad \text{and} \quad \|u(t)\| \leq \sigma + \frac{\varepsilon}{2M}\,, \quad \forall t \geq T\,.$$

Hence, for $t \geq 2T$,

$$\begin{aligned}
\|(Gu)(t)\| &\leq \int_0^{t/2} \|u(t-s)\||\mu|(ds) + \int_{t/2}^t \|u(t-s)\||\mu|(ds) \\
&\leq (\sigma + \frac{\varepsilon}{2M}) \int_0^{t/2} |\mu|(ds) + \|u\|_\infty \int_{t/2}^t |\mu|(ds) \\
&\leq (\sigma + \frac{\varepsilon}{2M}) \int_0^\infty |\mu|(ds) + \|u\|_\infty \int_T^\infty |\mu|(ds) \\
&\leq (\sigma + \frac{\varepsilon}{2M})M + \|u\|_\infty \frac{\varepsilon}{2\|u\|_\infty} \\
&\leq M\sigma + \varepsilon\,.
\end{aligned}$$

Since this holds for all $\varepsilon > 0$, the (6.43) follows. Consequently, if $\lim_{t\to\infty} u(t) = 0$, then $\lim_{t\to\infty}(Gu)(t) = 0$. ☐

**Lemma 6.2.4.** *Let $\xi \in \mathrm{cl}(\mathbb{C}_0)$, $\mathfrak{v} \in \mathbb{C}^m$, $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^m)$ and let $G$ be given by (6.41), where $\mu$ is a $\mathbb{C}^{p\times m}$-valued Borel measure on $\mathbb{R}_+$.*

(1) *If $\lim_{t\to\infty}(u(t) - e^{\xi t}\mathfrak{v}) = 0$, then*

$$\lim_{t\to\infty}[(Gu)(t) - e^{\xi t}\mathbf{G}(\xi)\mathfrak{v}] = 0\,.$$

(2) *If there exist $\alpha < 0$ and $M \geq 0$ such that*

$$\int_0^\infty e^{-\alpha s}|\mu|(ds) < \infty \quad \textit{and} \quad \|u(t) - e^{\xi t}\mathfrak{v}\| \leq Me^{\alpha t}\,, \quad \forall t \geq 0\,,$$

*then there exists $L \geq 0$ such that*

$$\|(Gu)(t) - e^{\xi t}\mathbf{G}(\xi)\mathfrak{v}\| \leq Le^{\alpha t}\,, \quad \forall t \geq 0\,.$$

*Proof.* Define $v\colon \mathbb{R}_+ \to \mathbb{C}^m$ by $v(t) := e^{\xi t}\mathfrak{v}$. By (6.41) and (6.42), using $\xi \in \mathrm{cl}(\mathbb{C}_0)$, we have

$$\begin{aligned}
\|(Gu)(t) - e^{\xi t}\mathbf{G}(\xi)\mathfrak{v}\| &= \left\|\int_0^t \mu(ds)u(t-s) - \int_0^\infty e^{\xi(t-s)}\mu(ds)\mathfrak{v}\right\| \\
&\leq \left\|\int_0^t \mu(ds)(u(t-s) - e^{\xi(t-s)}\mathfrak{v})\right\| + \|\mathfrak{v}\|\int_t^\infty |e^{\xi(t-s)}||\mu|(ds) \\
&= \|(G(u-v))(t)\| + \|\mathfrak{v}\|\int_t^\infty |\mu|(ds)\,, \quad \forall t \geq 0\,.
\end{aligned} \tag{6.44}$$

By hypothesis, $\lim_{t\to\infty}(u(t) - v(t)) = 0$, and hence, by Lemma 6.2.3,

$$\lim_{t\to\infty}(G(u-v))(t) = 0\,. \tag{6.45}$$

Moreover, it follows from Proposition 6.2.2 that $\lim_{t\to\infty}\int_t^\infty |\mu|(ds) = 0$. Hence, invoking (6.44) and (6.45) completes the proof of Statement (1).

To prove Statement (2), assume that there exist $\alpha < 0$ and $M \geq 0$ such that

$$M_1 := \int_0^\infty e^{-\alpha s}|\mu|(ds) < \infty \quad \text{and} \quad \|u(t) - e^{\xi t}\mathfrak{v}\| \leq Me^{\alpha t}\,, \quad \forall t \geq 0\,.$$

Since $\alpha < 0$, it follows from (6.44) that

$$\begin{aligned}
e^{-\alpha t}\|(Gu)(t) - e^{\xi t}\mathbf{G}(\xi)\mathfrak{v}\| &\leq e^{-\alpha t}\int_0^t \|(u-v)(t-s)\||\mu|(ds) + \|\mathfrak{v}\|e^{-\alpha t}\int_t^\infty |\mu|(ds) \\
&\leq M\int_0^t e^{-\alpha s}|\mu|(ds) + \|\mathfrak{v}\|\int_t^\infty e^{-\alpha s}|\mu|(ds) \\
&\leq MM_1 + \|\mathfrak{v}\|M_1\,, \quad \forall t \geq 0\,.
\end{aligned}$$

Hence $\|(Gu)(t) - e^{\xi t}\mathbf{G}(\xi)\mathfrak{v}\| \leq M_1(M + \|\mathfrak{v}\|)e^{\alpha t}$ for all $t \geq 0$. ∎

Let $\tau > 0$ be the sampling period. Define the sample-hold discretization $G_\tau$ of $G$ by

$$G_\tau := \mathcal{S}_\tau G \mathcal{H}_\tau \,, \tag{6.46}$$

where $\mathcal{S}_\tau$ and $\mathcal{H}_\tau$ are the sampling and hold operators, respectively.

Define $g_\tau \colon \mathbb{Z}_+ \to \mathbb{C}^{p\times m}$ by

$$g_\tau(k) := \mu(E_k)\,, \qquad \text{where} \qquad E_k := \begin{cases} \{0\}\,, & k = 0 \\ ((k-1)\tau, k\tau]\,, & k \in \mathbb{N} \end{cases} \,. \tag{6.47}$$

Let $\mathbf{G}_\tau$ denote the transfer function of $G_\tau$.

**Proposition 6.2.5.** *Assume that $G$ is given by (6.41), where $\mu$ is a $\mathbb{C}^{p\times m}$-valued Borel measure on $\mathbb{R}_+$ such that $\int_0^\infty e^{-\alpha t}|\mu|(dt) < \infty$ for some $\alpha \le 0$. Then $g_\tau$ defined by (6.47) is in $\ell^1_\rho(\mathbb{Z}_+, \mathbb{C}^{p\times m})$, where $\rho := e^{\alpha\tau}$, and is the impulse response of $G_\tau$. Consequently, $G_\tau \in \mathcal{B}(\ell^q(\mathbb{Z}_+, \mathbb{C}^m), \ell^q(\mathbb{Z}_+, \mathbb{C}^p))$ for $1 \le q \le \infty$, and $\mathbf{G}_\tau \in \hat{\ell}^1_\rho(\mathbb{C}^{p\times m}) \subset H^\infty(\mathbb{E}_\rho, \mathbb{C}^{p\times m})$. Moreover,*

$$\lim_{\tau\to 0} \mathbf{G}_\tau(e^{\xi\tau}) = \mathbf{G}(\xi)\,, \quad \forall \xi \in \mathrm{cl}(\mathbb{C}_0)\,.$$

*Proof.* Clearly,

$$\begin{aligned}
\sum_{k=0}^\infty \|g_\tau(k)\|\rho^{-k} &\le \sum_{k=0}^\infty \int_{k\tau}^{(k+1)\tau} e^{-\alpha\tau(k+1)}|\mu|(dt) \\
&\le e^{-\alpha\tau}\sum_{k=0}^\infty \int_{k\tau}^{(k+1)\tau} e^{-\alpha t}|\mu|(dt) \\
&= e^{-\alpha\tau}\int_0^\infty e^{-\alpha t}|\mu|(dt) < \infty\,.
\end{aligned}$$

showing that $g_\tau \in \ell^1_\rho(\mathbb{Z}_+, \mathbb{C}^{p\times m})$. For any discrete-time input $v \colon \mathbb{Z}_+ \to \mathbb{C}^m$, we have

$$\begin{aligned}
(G_\tau v)(k) = ((\mathcal{S}_\tau G\mathcal{H}_\tau)v)(k) = (G(\mathcal{H}_\tau v))(k\tau) &= \int_0^{k\tau} \mu(ds)(\mathcal{H}_\tau v)(k\tau - s) \\
&= \sum_{j=0}^k \int_{E_j} \mu(ds) v(k-j) \\
&= \sum_{j=0}^k g_\tau(k) v(k-j)\,, \quad \forall k \in \mathbb{Z}_+\,.
\end{aligned}$$

Hence, the impulse response of $G_\tau$ is $g_\tau$, so that $G_\tau \in \mathcal{B}(\ell^q(\mathbb{Z}_+, \mathbb{C}^m), \ell^q(\mathbb{Z}_+, \mathbb{C}^p))$ for

$1 \le q \le \infty$, and $\mathbf{G}_\tau \in \hat{\ell}^1_\rho(\mathbb{C}^{p\times m}) \subset H^\infty(\mathbb{E}_\rho, \mathbb{C}^{p\times m})$. Moreover, it is clear that

$$\mathbf{G}_\tau(e^{\xi\tau}) = \sum_{k=0}^{\infty} g_\tau(k) e^{-\xi\tau k} = \sum_{k=0}^{\infty} \int_{k\tau}^{(k+1)\tau} e^{-\xi\tau(k+1)} \mu(dt)\,,$$

and

$$\mathbf{G}(\xi) = \int_{\mathbb{R}_+} e^{-\xi t}\mu(dt) = \sum_{k=0}^{\infty} \int_{k\tau}^{(k+1)\tau} e^{-\xi t}\mu(dt)\,.$$

Using $\xi \in \mathrm{cl}(\mathbb{C}_0)$, we obtain

$$\begin{aligned} \|\mathbf{G}_\tau(e^{\xi\tau}) - \mathbf{G}(\xi)\| &= \left\| \sum_{k=0}^{\infty} \int_{k\tau}^{(k+1)\tau} \left(e^{-\xi\tau(k+1)} - e^{-\xi t}\right) \mu(dt) \right\| \\ &\le \sum_{k=0}^{\infty} \int_{k\tau}^{(k+1)\tau} |e^{-\xi\tau(k+1)}||1 - e^{-\xi[t-(k+1)\tau]}||\mu|(dt) \\ &\le \sum_{k=0}^{\infty} \int_{k\tau}^{(k+1)\tau} |1 - e^{\xi[(k+1)\tau - t]}||\mu|(dt) \\ &\le \sup_{t\in[0,\tau]} |1 - e^{\xi t}||\mu|(\mathbb{R}_+)\,. \end{aligned}$$

Since $\lim_{\tau\to 0}(\sup_{t\in[0,\tau]}|1 - e^{\xi t}|) = 0$ and $|\mu|(\mathbb{R}_+)$ is finite, the claim follows. □

**Remark 6.2.6.** The convergence of $\mathbf{G}_\tau(e^{\xi\tau})$ to $\mathbf{G}(\xi)$ as $\tau \to 0$ is uniform for all $\xi \in U$, where $U \subset \mathrm{cl}(\mathbb{C}_0)$ is compact. Moreover, it follows from the above that $\mathbf{G}_\tau(1) = \mathbf{G}(0)$ for all $\tau > 0$. ◇

### 6.2.2 Main result

Consider the sampled-data system shown in Figure 6-3, where $G$ is the input-output operator of the continuous-time plant, $K_{\tau,\varepsilon}$ is the input-output operator of the discrete-time controller, $r$ is a reference signal, and $d_1$ and $d_2$ are disturbance signals. Mathematically, Figure 6-3 can be expressed as

$$y_p = G(\mathcal{H}_\tau y_c + d_1)\,, \quad y = y_p + d_2\,, \quad y_c = K_{\tau,\varepsilon}\mathcal{S}_\tau(r - y)\,. \tag{6.48}$$

The following theorem is the main result of this section.

**Theorem 6.2.7.** *Let $N \in \mathbb{N}$ and let $\xi_j \in i\mathbb{R}$ for all $j \in \underline{N}$ be such that $\xi_j \neq \xi_k$ for $j, k \in \underline{N}$, $j \neq k$. Let $G$ be given by (6.41), where $\mu$ is a $\mathbb{C}^{p\times m}$-valued Borel measure on $\mathbb{R}_+$ such that $\int_0^\infty e^{-\alpha t}|\mu|(dt) < \infty$ for some $\alpha < 0$, Let the discrete-time controller $K_{\tau,\varepsilon}$ be such that its transfer function $\mathbf{K}_{\tau,\varepsilon}$ is given by*

$$\mathbf{K}_{\tau,\varepsilon}(z) = \varepsilon \left( \mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{K_j}{z - e^{\xi_j \tau}} \right)\,, \tag{6.49}$$

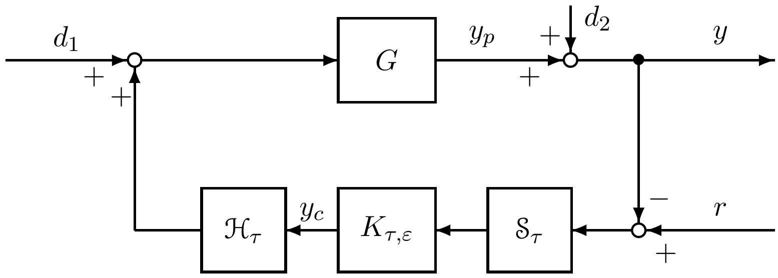


Figure 6-3: Sampled-data low-gain control.

*where* $\mathbf{K}^0 \in \hat{\ell}^1(\mathbb{C}^{m\times p})$ *and* $K_j \in \mathbb{C}^{m\times p}$*. Assume that*

$$\sigma(\mathbf{G}(\xi_j)K_j) \subset \mathbb{C}_0\,, \quad \forall j \in \underline{N}\,. \tag{6.50}$$

*The following statements hold for the output* $y$ *of the sampled-data system* (6.48)*:*

(1) *There exists* $\tau^* > 0$ *such that, for every sampling period* $\tau \in (0, \tau^*)$*, there exists* $\varepsilon_\tau > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_\tau)$*, the feedback system is* $L^\infty$*-stable, in the sense that there exists* $N_1 \geq 0$ *such that*

$$\|y\|_\infty \leq N_1(\|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty)\,, \quad \forall r, d_2 \in L_b(\mathbb{R}_+, \mathbb{C}^p)\,,\ \forall d_1 \in L_b(\mathbb{R}_+, \mathbb{C}^m)\,.$$

(2) *If the reference signal* $r\colon \mathbb{R}_+ \to \mathbb{C}^p$ *is given by*

$$r(t) := \sum_{j=1}^{N} e^{\xi_j t}\mathfrak{r}_j\,, \quad \mathfrak{r}_j \in \mathbb{C}^p\,, \tag{6.51}$$

*the disturbance signals* $d_1 \in L_b(\mathbb{R}_+, \mathbb{C}^m)$ *and* $d_2 \in L_b(\mathbb{R}_+, \mathbb{C}^p)$ *satisfy*

$$\lim_{t\to\infty}(d_1(t) - \sum_{j=1}^{N} e^{\xi_j t}\mathfrak{d}_{1j}) = 0\,,\ \lim_{t\to\infty}(d_2(t) - \sum_{j=1}^{N} e^{\xi_j t}\mathfrak{d}_{2j}) = 0\,, \quad \mathfrak{d}_{1j} \in \mathbb{C}^m\,,\ \mathfrak{d}_{2j} \in \mathbb{C}^p\,, \tag{6.52}$$

*then, for every* $\delta > 0$*, there exists* $\tau_\delta > 0$ *such that, for every sampling period* $\tau \in (0, \tau_\delta)$*, there exists* $\varepsilon_\tau > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_\tau)$*,*

$$\limsup_{t\to\infty} \|y(t) - r(t)\| \leq \delta\,. \tag{6.53}$$

(3) *Under the additional assumptions that* $\mathbf{K}^0 \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$ *and that there exist* $\gamma \in (\alpha, 0)$ *and* $N_2 \geq 0$ *such that*

$$\|d_1(t) - \sum_{j=1}^{N} e^{\xi_j t}\mathfrak{d}_{1j}\| \leq N_2 e^{\gamma t}\,, \quad \|d_2(t) - \sum_{j=1}^{N} e^{\xi_j t}\mathfrak{d}_{2j}\| \leq N_2 e^{\gamma t}\,, \quad \forall t \geq 0\,, \tag{6.54}$$

(6.53) *can be replaced by*

$$\|y(t) - r(t)\| \leq \delta + N_3 e^{\beta t}\,, \quad \forall t \geq 0\,,$$

*for some* $\beta \in (\gamma, 0)$ *and* $N_3 \geq 0$ *(both depending on* $\tau$ *and* $\varepsilon$*).*

*Proof.* Setting $\tau_0 := 2\pi/\sup\{|\xi_j - \xi_k| : j,k \in \underline{N},\ j \neq k\}$, we know that if $\tau \in (0, \tau_0)$, then $e^{\xi_j \tau} \neq e^{\xi_k \tau}$ for all $j,k \in \underline{N}$, $j \neq k$. Let $G_\tau$ be the sample-hold discretization of $G$ defined by (6.46) and $\mathbf{G}_\tau$ be the transfer function of $G_\tau$. It follows from Proposition 6.2.5 that

$$\lim_{\tau \to 0} e^{\bar{\xi}_j \tau} \mathbf{G}_\tau(e^{\xi_j \tau}) K_j = \mathbf{G}(\xi_j) K_j\,, \quad \forall j \in \underline{N}\,.$$

Hence, by hypothesis (6.50), there exists $\tau^* \in (0, \tau_0)$ such that if $\tau \in (0, \tau^*)$, then

$$\sigma(e^{\bar{\xi}_j \tau} \mathbf{G}_\tau(e^{\xi_j \tau}) K_j) \subset \mathbb{C}_0\,, \quad \forall j \in \underline{N}\,. \tag{6.55}$$

By assumption, there exists $\alpha < 0$ such that $\int_0^\infty e^{-\alpha t}|\mu|(dt) < \infty$. Therefore, by Proposition 6.2.5, $\mathbf{G}_\tau \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m}) \subset \hat{\ell}^1(\mathbb{C}^{p\times m})$. Moreover, by assumption, $\mathbf{K}^0 \in \hat{\ell}^1(\mathbb{C}^{m\times p})$. It follows from Theorem 6.1.9 that, for every $\tau \in (0, \tau^*)$, there exists $\varepsilon_\tau > 0$ such that

$$\mathbf{K}_{\tau,\varepsilon}(I + \mathbf{G}_\tau \mathbf{K}_{\tau,\varepsilon})^{-1} \in \hat{\ell}^1(\mathbb{C}^{m\times p})\,, \quad \forall \varepsilon \in (0, \varepsilon_\tau)\,,$$

showing that $K_{\tau,\varepsilon}(I + G_\tau K_{\tau,\varepsilon})^{-1}$ is a convolution operator with impulse response in $\ell^1(\mathbb{Z}_+, \mathbb{C}^{m\times p})$. Let $\tau \in (0, \tau^*)$ and $\varepsilon \in (0, \varepsilon_\tau)$. Set

$$M := |\mu|(\mathbb{R}_+) \quad \text{and} \quad M_1 := \|K_{\tau,\varepsilon}(I + G_\tau K_{\tau,\varepsilon})^{-1}\|\,. \tag{6.56}$$

Let $d_1 \in L_b(\mathbb{R}_+, \mathbb{C}^m)$ and $d_2, r \in L_b(\mathbb{R}_+, \mathbb{C}^p)$. It is well-known that $\|Gd_1\|_\infty \leq M\|d_1\|_\infty$. Furthermore, set

$$d := Gd_1 + d_2\,. \tag{6.57}$$

Trivially,

$$\|\mathcal{S}_\tau d\|_{\ell^\infty} \leq \|d\|_\infty \leq M\|d_1\|_\infty + \|d_2\|_\infty \quad \text{and} \quad \|\mathcal{S}_\tau r\|_{\ell^\infty} \leq \|r\|_\infty\,. \tag{6.58}$$

The output $y_c$ of the discrete-time controller (see (6.48)) is given by

$$y_c = K_{\tau,\varepsilon}\mathcal{S}_\tau[r - (G\mathcal{H}_\tau y_c + d)] = K_{\tau,\varepsilon}[\mathcal{S}_\tau r - (G_\tau y_c + \mathcal{S}_\tau d)]\,.$$

It follows that

$$y_c = K_{\tau,\varepsilon}(I + G_\tau K_{\tau,\varepsilon})^{-1}(\mathcal{S}_\tau r - \mathcal{S}_\tau d)\,. \tag{6.59}$$

Invoking (6.56) and (6.58), we have

$$\|y_c\|_{\ell^\infty} \leq M_1(\|\mathcal{S}_\tau r\|_{\ell^\infty} + \|\mathcal{S}_\tau d\|_{\ell^\infty}) \leq M_1(\|r\|_\infty + M\|d_1\|_\infty + \|d_2\|_\infty)\,. \tag{6.60}$$

Clearly, the output $y$ of the closed-loop sampled-data system (6.48) satisfies

$$y = G\mathcal{H}_\tau y_c + Gd_1 + d_2 = G\mathcal{H}_\tau y_c + d\,. \tag{6.61}$$

Since $\|\mathcal{H}_\tau y_c\|_\infty = \|y_c\|_{\ell^\infty}$, it follows from (6.60) and (6.61) that

$$\begin{aligned}
\|y\|_\infty &\leq \|G\mathcal{H}_\tau y_c\|_\infty + \|Gd_1\|_\infty + \|d_2\|_\infty \\
&\leq M\|\mathcal{H}_\tau y_c\|_\infty + M\|d_1\|_\infty + \|d_2\|_\infty \\
&= M\|y_c\|_{\ell^\infty} + M\|d_1\|_\infty + \|d_2\|_\infty \\
&\leq MM_1(\|r\|_\infty + M\|d_1\|_\infty + \|d_2\|_\infty) + M\|d_1\|_\infty + \|d_2\|_\infty \\
&\leq N_1(\|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty)\,,
\end{aligned}$$

with $N_1 := (M+1)(MM_1+1)$. This completes the proof of Statement (1).

To prove Statement (2), note that, by (6.55), $\mathbf{G}_\tau(e^{\xi_j\tau})K_j$ is invertible for every $j \in \underline{N}$. In the following, we take limits as $z \to e^{\xi_j\tau}$ for $z \in \mathbb{E}_1$. It is easy to calculate that

$$\lim_{z\to e^{\xi_j\tau}} (I + \mathbf{G}_\tau(z)\mathbf{K}_{\tau,\varepsilon}(z))^{-1} = 0\,, \quad \forall j \in \underline{N}\,,$$

and

$$\begin{aligned}
&\lim_{z\to e^{\xi_j\tau}} \frac{1}{z - e^{\xi_j\tau}}\,(I + \mathbf{G}_\tau(z)\mathbf{K}_{\tau,\varepsilon}(z))^{-1} \\
&= \lim_{z\to e^{\xi_j\tau}} \left[\varepsilon\mathbf{G}_\tau(z)K_j + (z - e^{\xi_j\tau})\left(I + \varepsilon\mathbf{G}_\tau(z)\mathbf{K}^0(z) + \varepsilon\sum_{k\in\underline{N},\, k\neq j}\frac{\mathbf{G}_\tau(z)K_k}{z - e^{\xi_k\tau}}\right)\right]^{-1} \\
&= (\varepsilon\mathbf{G}_\tau(e^{\xi_j\tau})K_j)^{-1}\,, \quad \forall j \in \underline{N}\,.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&(\mathbf{K}_{\tau,\varepsilon}(I + \mathbf{G}_\tau\mathbf{K}_{\tau,\varepsilon})^{-1})(e^{\xi_j\tau}) \\
&= \lim_{z\to e^{\xi_j\tau}} \varepsilon\mathbf{K}^0(z)(I + \mathbf{G}_\tau(z)\mathbf{K}_{\tau,\varepsilon}(z))^{-1} + \lim_{z\to e^{\xi_j\tau}} \sum_{k\in\underline{N}}\left(\frac{\varepsilon K_k}{z - e^{\xi_k\tau}}(I + \mathbf{G}_\tau(z)\mathbf{K}_{\tau,\varepsilon}(z))^{-1}\right) \\
&= \lim_{z\to e^{\xi_j\tau}} \frac{\varepsilon K_j}{z - e^{\xi_j\tau}}(I + \mathbf{G}_\tau(z)\mathbf{K}_{\tau,\varepsilon}(z))^{-1} \\
&= K_j(\mathbf{G}_\tau(e^{\xi_j\tau})K_j)^{-1}\,, \quad \forall j \in \underline{N}\,.
\end{aligned} \tag{6.62}$$

Set

$$\mathfrak{d}_j := \mathbf{G}(\xi_j)\mathfrak{d}_{1j} + \mathfrak{d}_{2j}\,, \quad \forall j \in \underline{N}\,. \tag{6.63}$$

By Lemma 6.2.4 and (6.52), we obtain that

$$\begin{aligned}\lim_{t\to\infty}(d(t)-\sum_{j=1}^{N}e^{\xi_j t}\mathfrak{d}_j) &= \lim_{t\to\infty}[(Gd_1)(t)-\sum_{j=1}^{N}e^{\xi_j t}\mathbf{G}(\xi_j)\mathfrak{d}_{1j}] + \lim_{t\to\infty}(d_2(t)-\sum_{j=1}^{N}e^{\xi_j t}\mathfrak{d}_{2j})\\ &= 0\,, \end{aligned}\tag{6.64}$$

where $d$ is defined in (6.57).

It follows trivially from (6.51) and (6.64) that

$$(\mathcal{S}_\tau r)(k) = \sum_{j=1}^{N} e^{\xi_j k\tau}\mathfrak{r}_j\,,\ \forall k\in\mathbb{Z}_+ \ \text{ and } \ \lim_{k\to\infty}[(\mathcal{S}_\tau d)(k) - \sum_{j=1}^{N} e^{\xi_j k\tau}\mathfrak{d}_j] = 0\,. \tag{6.65}$$

Define $a_\tau, b_\tau\colon \mathbb{Z}_+ \to \mathbb{C}^m$ by

$$a_\tau(k) := \sum_{j=1}^{N} e^{\xi_j\tau k}K_j(\mathbf{G}_\tau(e^{\xi_j\tau})K_j)^{-1}\mathfrak{r}_j\,, \quad b_\tau(k) := \sum_{j=1}^{N} e^{\xi_j\tau k}K_j(\mathbf{G}_\tau(e^{\xi_j\tau})K_j)^{-1}\mathfrak{d}_j\,. \tag{6.66}$$

It follows from Lemma 6.1.6, (6.59), (6.62) and (6.65) that

$$\lim_{k\to\infty}[y_c(k) - a_\tau(k) + b_\tau(k)] = 0\,. \tag{6.67}$$

By (6.50), $\mathbf{G}(\xi_j)K_j$ is invertible for every $j\in\underline{N}$. Define $v_1, v_2\colon \mathbb{R}_+\to\mathbb{C}^m$ by

$$v_1(t) := \sum_{j=1}^{N} e^{\xi_j t}K_j(\mathbf{G}(\xi_j)K_j)^{-1}\mathfrak{r}_j\,, \quad v_2(t) := \sum_{j=1}^{N} e^{\xi_j t}K_j(\mathbf{G}(\xi_j)K_j)^{-1}\mathfrak{d}_j\,. \tag{6.68}$$

We conclude from Lemma 6.2.4, (6.51) and (6.64) that

$$\lim_{t\to\infty}[(Gv_1)(t) - r(t)] = 0\,, \quad \lim_{t\to\infty}[(Gv_2)(t) - d(t)] = 0\,. \tag{6.69}$$

Let $\delta > 0$. Invoking Proposition 6.2.5 and the fact that the fact that $\xi_j \in i\mathbb{R}$, there exists $\tau_\delta \in (0, \tau^*)$ such that if $\tau \in (0, \tau_\delta)$, then then

$$
\begin{aligned}
&\sup_{t\in[k\tau,(k+1)\tau)} \|v_1(t) - (\mathcal{H}_\tau a_\tau)(t)\| \\
&\quad= \sup_{t\in[k\tau,(k+1)\tau)} \left\| \sum_{j=1}^{N} e^{\xi_j t} K_j (\mathbf{G}(\xi_j) K_j)^{-1} \mathfrak{r}_j - \sum_{j=1}^{N} e^{\xi_j \tau k} K_j (\mathbf{G}_\tau(e^{\xi_j \tau}) K_j)^{-1} \mathfrak{r}_j \right\| \\
&\quad\leq \sup_{t\in[k\tau,(k+1)\tau)} \sum_{j=1}^{N} |e^{\xi_j (t-k\tau)} - 1| \|K_j (\mathbf{G}(\xi_j) K_j)^{-1} \mathfrak{r}_j\| \\
&\qquad\quad + \sum_{j=1}^{N} \|(\mathbf{G}(\xi_j) K_j)^{-1} - (\mathbf{G}_\tau(e^{\xi_j \tau}) K_j)^{-1}\| \|K_j\| \|\mathfrak{r}_j\| \\
&\quad\leq \frac{\delta}{2M}, \quad \forall k \in \mathbb{Z}_+,
\end{aligned}
$$

where $M$ is defined in (6.56). Hence,

$$
\|v_1(t) - (\mathcal{H}_\tau a_\tau)(t)\| \leq \frac{\delta}{2M}, \quad \forall t \geq 0, \tag{6.70}
$$

and, similarly,

$$
\|v_2(t) - (\mathcal{H}_\tau b_\tau)(t)\| \leq \frac{\delta}{2M}, \quad \forall t \geq 0, \tag{6.71}
$$

Let $\tau \in (0, \tau_\delta)$ and $\varepsilon \in (0, \varepsilon_\tau)$. By (6.67), (6.70) and (6.71), we obtain

$$
\begin{aligned}
&\limsup_{t\to\infty} \|(\mathcal{H}_\tau y_c)(t) - v_1(t) + v_2(t)\| \\
&\qquad\leq \limsup_{t\to\infty} \|(\mathcal{H}_\tau(y_c - a_\tau + b_\tau))(t)\| + \limsup_{t\to\infty} \|(\mathcal{H}_\tau a_\tau)(t) - v_1(t)\| \\
&\qquad\qquad + \limsup_{t\to\infty} \|v_2(t) - (\mathcal{H}_\tau b_\tau)(t)\| \\
&\qquad\leq \frac{\delta}{M}.
\end{aligned} \tag{6.72}
$$

By (6.61) and (6.69), it follows that

$$
\begin{aligned}
\limsup_{t\to\infty} \|y(t) - r(t)\| &\leq \limsup_{t\to\infty} \|(G(\mathcal{H}_\tau y_c - v_1 + v_2))(t)\| + \limsup_{t\to\infty} \|d(t) - (Gv_2)(t)\| \\
&\qquad + \limsup_{t\to\infty} \|(Gv_1)(t) - r(t)\| \\
&= \limsup_{t\to\infty} \|(G(\mathcal{H}_\tau y_c - v_1 + v_2))(t)\|.
\end{aligned}
$$

Finally, $\mathcal{H}_\tau y_c - v_1 + v_2$ is bounded and thus, by Lemma 6.2.3 and (6.72),

$$
\limsup_{t\to\infty} \|y(t) - r(t)\| \leq M \limsup_{t\to\infty} \|(\mathcal{H}_\tau y_c)(t) - v_1(t) + v_2(t)\| \leq \delta.
$$

This completes the proof of Statement (2).

To prove Statement (3), assume that $\mathbf{K}^0 \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$ and that there exist $N_2 \geq 0$ and $\gamma \in (\alpha, 0)$ such that (6.54) holds. Therefore, by Theorem 6.1.12, $\mathbf{K}_{\tau,\varepsilon}(I + \mathbf{G}_\tau \mathbf{K}_{\tau,\varepsilon})^{-1} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$. Hence, by Proposition 6.1.7, there exists $\rho \in (e^{\gamma\tau}, 1)$ such that

$$\mathbf{K}_{\tau,\varepsilon}(I + \mathbf{G}_\tau \mathbf{K}_{\tau,\varepsilon})^{-1} \in \hat{\ell}^1_\rho(\mathbb{C}^{m\times p})\,.$$

By Lemma 6.2.4 and (6.54), there exists $M_2 \geq 0$ such that

$$\|(Gd_1)(t) - \sum_{j=1}^N e^{\xi_j t}\mathbf{G}(\xi_j)\mathfrak{d}_{1j}\| \leq M_2 e^{\gamma t}\,, \quad \forall t \geq 0\,.$$

Invoking (6.54), it follows that

$$\begin{aligned} \|d(t) - \sum_{j=1}^N e^{\xi_j t}\mathfrak{d}_j\| &\leq \|(Gd_1)(t) - \sum_{j=1}^N e^{\xi_j t}\mathbf{G}(\xi_j)\mathfrak{d}_{1j}\| + \|d_2(t) - \sum_{j=1}^N e^{\xi_j t}\mathfrak{d}_{2j}\| \\ &\leq (M_2 + N_2)e^{\gamma t}\,, \quad \forall t \geq 0\,, \end{aligned} \tag{6.73}$$

where $d$ and $\mathfrak{d}_j$ are defined in (6.57) and (6.63), respectively. Trivially,

$$\|(\mathcal{S}_\tau d)(k) - \sum_{j=1}^N e^{\xi_j k\tau}\mathfrak{d}_j\| \leq (M_2 + N_2)(e^{\gamma\tau})^k \leq (M_2 + N_2)\rho^k\,, \quad \forall k \in \mathbb{Z}_+\,.$$

It follows from (6.62) and Lemma 6.1.6 that there exists $M_3 \geq 0$ such that

$$\|y_c(k) - a_\tau(k) + b_\tau(k)\| \leq M_3\rho^k\,, \quad \forall k \in \mathbb{Z}_+\,, \tag{6.74}$$

where $y_c$ is given by (6.59) and $a_\tau$, $b_\tau$ are defined in (6.66). We conclude from Lemma 6.2.4, (6.51) and (6.73) that there exists $M_4 \geq 0$ such that

$$\|(Gv_1)(t) - r(t)\| \leq M_4 e^{\gamma t}\,, \quad \|(Gv_2)(t) - d(t)\| \leq M_4 e^{\gamma t}\,; \quad \forall t \geq 0\,, \tag{6.75}$$

where $v_1$ and $v_2$ are defined in (6.68). Since $\rho \in (0,1)$, we have

$$\rho^k \leq \rho^{-1}\rho^{(k\tau+\theta)/\tau} = \rho^{-1}e^{\beta(k\tau+\theta)}\,, \quad \forall \theta \in [0,\tau)\,,\ \forall k \in \mathbb{Z}_+\,,$$

where $\beta := (\ln\rho)/\tau$. Consequently, by (6.70), (6.71) and (6.74),

$$\begin{aligned} &\|(\mathcal{H}_\tau y_c)(t) - v_1(t) + v_2(t)\| \\ &\quad\leq \|(\mathcal{H}_\tau y_c - \mathcal{H}_\tau a_\tau + \mathcal{H}_\tau b_\tau)(t)\| + \|(\mathcal{H}_\tau a_\tau)(t) - v_1(t)\| + \|v_2(t) - (\mathcal{H}_\tau b_\tau)(t)\| \\ &\quad\leq M_3\rho^{-1}e^{\beta t} + \frac{\delta}{M}\,, \quad \forall t \geq 0\,. \end{aligned}$$

Since $\rho \in (e^{\gamma\tau}, 1)$, we have that $\beta \in (\gamma, 0) \subset (\alpha, 0)$ and hence,

$$\begin{aligned}
\|(G(\mathcal{H}_\tau y_c - v_1 + v_2))(t)\| &\leq \int_0^t \|(\mathcal{H}_\tau y_c - v_1 + v_2)(t-s)\||\mu|(ds) \\
&\leq \int_0^t M_3\rho^{-1}e^{\beta(t-s)}|\mu|(ds) + \frac{\delta}{M}\int_0^\infty |\mu|(ds) \\
&\leq M_3\rho^{-1}e^{\beta t}\int_0^\infty e^{-\beta s}|\mu|(ds) + \delta \\
&\leq M_3M_5\rho^{-1}e^{\beta t} + \delta\,, \quad \forall t \geq 0\,,
\end{aligned}$$

where $M_5 := \int_0^\infty e^{-\beta s}|\mu|(ds) \leq \int_0^\infty e^{-\alpha s}|\mu|(ds) < \infty$. Therefore, by (6.61) and (6.75), it follows that

$$\begin{aligned}
\|y(t) - r(t)\| &\leq \|(G(\mathcal{H}_\tau y_c - v_1 + v_2))(t)\| + \|(d(t) - Gv_2(t))\| + \|(Gv_1)(t) - r(t)\| \\
&\leq \|(G(\mathcal{H}_\tau y_c - v_1 + v_2))(t)\| + 2M_4e^{\gamma t} \\
&\leq \delta + (M_3M_5\rho^{-1} + 2M_4)e^{\beta t}\,, \quad \forall t \geq 0\,.
\end{aligned}$$

This completes the proof of Statement (3). □

**Remark 6.2.8.** (1) By [65], the low-gain continuous-time controller which achieves tracking and disturbance rejection for systems given by (6.41) is of the form $\varepsilon\sum_{j=1}^N [K_j/(s-\xi_j)]$. That is, the impulse response of the controller is given by $\varepsilon\sum_{j=1}^N e^{\xi_j t}K_j$. Applying sample-hold discretization to this continuous-time controller and invoking arguments similar to those used in the proof of Proposition 6.2.5, it can be shown that the corresponding discrete-time controller has transfer function given by

$$\varepsilon\sum_{j=1}^N \frac{\alpha_j K_j}{z - e^{\xi_j\tau}}\,, \quad \text{where} \quad \alpha_j := \begin{cases} \dfrac{(e^{\xi_j\tau} - 1)}{\xi_j}\,, & \text{if } \xi \neq 0 \\ \tau\,, & \text{if } \xi = 0 \end{cases}.$$

Apart from the coefficients $\alpha_j$, this low-gain discrete-time controller obtained by sample-hold discretization is the same as (6.49) (where, for the thesis discussion, we ignore the term $\mathbf{K}^0$). In this respect, Theorem 6.2.7 can be regarded as a result on indirect sampled-data control.

(2) In (6.49), the term $\mathbf{K}^0$ may be employed to satisfy additional design requirements, for example, to improve robustness properties or to speed up the transient response. The existence of matrices $K_j$ such that (6.50) holds is guaranteed if and only if $\operatorname{rk}\mathbf{G}(\xi_j) = p$ for all $j \in \underline{N}$, in which case, $K_j = \mathbf{G}^*(\xi_j)[\mathbf{G}(\xi_j)\mathbf{G}^*(\xi_j)]^{-1}$ is a possible choice.

(3) The proof of Theorem 6.2.7 shows that, for fixed $\{\xi_j : j \in \underline{N}\}$, $\tau_\delta$ and $\varepsilon_\tau$ can be chosen to be uniform for all signals $r, d_1$ and $d_2$ with $\mathfrak{r}_j$, $\mathfrak{d}_{1j}$ and $\mathfrak{d}_{2j}$, $j \in \underline{N}$, satisfying a pre-specified bound. ◇

### 6.2.3 Application to state-space systems

In the following, we consider sampled-data systems in the state-space form: we apply the input-output results to a class of infinite-dimensional state-space systems.

Let $X$ be a Hilbert space and assume that the plant is given by

$$\begin{aligned} \dot{x}_p(t) &= A_p x_p(t) + B_p u_p(t)\,; \quad x_p(0) = x_p^0 \in X\,, & (6.76a)\\ y_p(t) &= C_p x_p(t) + D_p u_p(t)\,, & (6.76b) \end{aligned}$$

where $A_p : D(A) \to X$ is the generator of a strongly continuous semigroup $\mathbf{T}(t)$ on $X$, $B_p \in \mathcal{B}(\mathbb{C}^m, X_{-1})$ is the control operator, $C_p \in \mathcal{B}(X, \mathbb{C}^p)$ is the (bounded) observation operator and $D_p \in \mathbb{C}^{p\times m}$ is the feedthrough matrix. Here $X_{-1}$ is the completion of $X$ with respect to the norm $\|x\|_{-1} := \|(\beta I - A_p)^{-1}x\|_X$, where $\beta$ is in the resolvent set of $A_p$. We assume that $B_p$ is admissible, that is, for every $t \geq 0$, there exists $b_t \geq 0$ such that

$$\left\| \int_0^t \mathbf{T}(t-s) B_p u_p(s) \right\|_X \leq b_t \|u_p\|_{L^2}\,, \quad \forall u_p \in L^2([0,t], \mathbb{C}^m)\,.$$

The admissibility assumption implies, in particular, that system (6.76) is regular (see Section 2.4 and the references within for more details on admissible control operators and regular systems). For $u_p \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^m)$, the mild solution $x_p$ of (6.76a), given by

$$x_p(t) = \mathbf{T}(t)x_p^0 + \int_0^t \mathbf{T}(t-s) B_p u_p(s) ds\,, \tag{6.77}$$

is a continuous $X$-valued function, satisfying the differential equation (6.76a) in $X_{-1}$ for almost every $t \in \mathbb{R}_+$. The transfer function $\mathbf{G}$ of (6.76) is given by

$$\mathbf{G}(s) = C_p(sI - A_p)^{-1}B_p + D_p\,, \quad \forall s \in \mathbb{C}_{\omega(\mathbf{T})}\,,$$

where

$$\omega(\mathbf{T}) := \lim_{t\to\infty} \frac{1}{t} \ln \|\mathbf{T}(t)\|$$

denotes the exponential constant of $\mathbf{T}$. We say that (6.76) is *exponentially stable* if $\omega(\mathbf{T}) < 0$.

Next we construct a state-space realization of the controller transfer function (6.49). Let $\mathbf{K}^0$ be a discrete-time stable proper complex rational function and let $(A_0, B_0, C_0, D_0) \in \mathbb{C}^{n_0\times n_0} \times \mathbb{C}^{n_0\times p} \times \mathbb{C}^{m\times n_0} \times \mathbb{C}^{m\times p}$ be a stabilizable and detectable realization of $\mathbf{K}^0$, i.e., $\mathbf{K}^0(z) = C_0(zI - A_0)^{-1}B_0 + D_0$, $(A_0, B_0)$ is stabilizable and $(C_0, A_0)$ is detectable. Since $\mathbf{K}^0$ is $\ell^2$-stable, it follows that $A_0$ is power stable. Let $A_c \in \mathbb{C}^{(Np+n_0)\times(Np+n_0)}$, $B_c \in \mathbb{C}^{(Np+n_0)\times p}$, $C_c \in \mathbb{C}^{m\times(Np+n_0)}$ and $D_c \in \mathbb{C}^{m\times p}$ are given by (6.23) with $\lambda_j = e^{\xi_j \tau}$, where $\xi_j \in i\mathbb{R}$ for all $j \in \underline{N}$ and $\xi_j \neq \xi_k$ for $j \in \underline{N}$, $j \neq k$. If the non-zero numbers in $\{\xi_j \in i\mathbb{R} : j \in \underline{N}\}$ occur in complex conjugate pairs, then we can design $(A_c, B_c, C_c, D_c)$

to be real matrices (see Remark 6.1.15). The controller is defined by

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + B_c u_c(k)\,; \quad x_c(0) = x_c^0 \in \mathbb{C}^{Np+n_0}\,, && \text{(6.78a)} \\ y_c(k) &= \varepsilon C_c x_c(k) + \varepsilon D_c u_c(k)\,. && \text{(6.78b)} \end{aligned}$$

The transfer function $\mathbf{K}_{\tau,\varepsilon}$ of (6.78) is given by

$$\mathbf{K}_{\tau,\varepsilon}(z) = \varepsilon\left(\mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{K_j}{z - e^{\xi_j \tau}}\right).$$

The interconnection of (6.76) and (6.78) is given by

$$u_p = \mathcal{H}_\tau y_c + d_1\,, \quad y = y_p + d_2\,, \quad u_c = \mathcal{S}_\tau(r - y)\,, \tag{6.79}$$

where $r$ is a reference signal and $d_1$ and $d_2$ are disturbance signals. The state-space sampled-data feedback system given by (6.76), (6.78) and (6.79) is a state-space realization of the system (6.48), and has a unique solution which will be denoted by

$$\begin{pmatrix} x_p(\,\cdot\,; x_p^0, x_c^0, \tau, \varepsilon, r, d_1, d_2) \\ x_c(\,\cdot\,; x_p^0, x_c^0, \tau, \varepsilon, r, d_1, d_2) \end{pmatrix}.$$

**Remark 6.2.9.** Note that the plant $\Sigma_p$ is infinite-dimensional, but the discrete-time controller $\Sigma_c$ is finite-dimensional and hence can be implemented with on-line digital computers. The order of the discrete-time controller depends on the number of frequencies of the reference and disturbance signals. ◇

**Theorem 6.2.10.** *Consider the sampled-data state-space system given by* (6.76), (6.78) *and* (6.79). *Assume that* (6.76) *is exponentially stable and* $\sigma(\mathbf{G}(\xi_j)K_j) \subset \mathbb{C}_0$ *for all* $j = \underline{N}$. *The following statements hold:*

(1) *There exists* $\tau^* > 0$ *such that, for every sampling period* $\tau \in (0, \tau^*)$, *there exists* $\varepsilon_\tau > 0$ *such that if* $\varepsilon \in (0, \varepsilon_\tau)$, *then the sampled-data system is exponentially stable, i.e., for every* $\varepsilon \in (0, \varepsilon_\tau)$, *there exist* $N_1 \geq 0$ *and* $\beta < 0$ *such that*

$$\begin{aligned} &\left\| \begin{pmatrix} x_p(k\tau + \theta; x_p^0, x_c^0, \tau, \varepsilon, r, d_1, d_2) \\ x_c(k; x_p^0, x_c^0, \tau, \varepsilon, r, d_1, d_2) \end{pmatrix} \right\| \\ &\qquad \leq N_1 \left( e^{\beta(k\tau+\theta)} \left\| \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} \right\| + \|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty \right), \\ &\qquad\qquad \forall \theta \in [0, \tau)\,,\ \forall k \in \mathbb{Z}_+\,,\ \forall x_p^0 \in X\,,\ \forall x_c^0 \in \mathbb{C}^{Np+n_0}\,, \\ &\qquad\qquad \forall r, d_2 \in L_b(\mathbb{R}_+, \mathbb{C}^p)\,,\ \forall d_1 \in L_b(\mathbb{R}_+, \mathbb{C}^m)\,. \end{aligned}$$

(2) *If* $r$ *is of the form* (6.51) *and* $d_1 \in L_b(\mathbb{R}_+, \mathbb{C}^m)$, $d_2 \in L_b(\mathbb{R}_+, \mathbb{C}^p)$ *satisfy* (6.52), *then, for every* $\delta > 0$, *there exists* $\tau_\delta > 0$ *such that, for every sampling period*

*$\tau \in (0,\tau_\delta)$, there exists $\varepsilon_\tau > 0$, such that, for all $\varepsilon \in (0,\varepsilon_\tau)$, all $x_p^0 \in X$ and all $x_c^0 \in \mathbb{C}^{Np+n_0}$,*

$$\limsup_{t\to\infty} \|y(t) - r(t)\| \le \delta\,.$$

*Moreover, if (6.54) holds for some $\gamma < 0$ and $N_2 \ge 0$, then, for every $\varepsilon \in (0,\varepsilon_\tau)$, there exist $\eta \in (\gamma, 0)$ and $N_3 \ge 0$ such that, for all $x_p^0 \in X$ and all $x_c^0 \in \mathbb{C}^{Np+n_0}$,*

$$\|y(t) - r(t)\| \le \delta + N_3 e^{\eta t}\,, \quad \forall t \ge 0\,.$$

*Proof.* The sample-hold discretization of (6.76) is given by the quadruple

$$\left(\mathbf{T}(\tau)\,,\ \int_0^\tau \mathbf{T}(s)B_p ds\,,\ C_p\,,\ D_p\right). \tag{6.80}$$

Clearly, since $\mathbf{T}(t)$ is exponentially stable, $\mathbf{T}(\tau)$ is power stable. Since $A_p^{-1}B_p \in \mathcal{B}(\mathbb{C}^m, X)$ and

$$\int_0^\tau \mathbf{T}(s)B_p ds\, v = (\mathbf{T}(\tau) - I)A_p^{-1}B_p\, v\,, \quad \forall v \in \mathbb{C}^m\,,$$

we see that $\int_0^\tau \mathbf{T}(s)B_p ds \in \mathcal{B}(\mathbb{C}^m, X)$ for every $\tau > 0$. Define

$$E_p := (I + \varepsilon D_c D_p)^{-1}\,, \quad E_c := (I + \varepsilon D_p D_c)^{-1}\,,$$

and $\Delta\colon [0,\tau] \to \mathcal{B}(X \times \mathbb{C}^{Np+n_0})$ by

$$\Delta(\theta) :=$$

$$\begin{pmatrix} \mathbf{T}(\theta) & 0 \\ 0 & A_c \end{pmatrix} + \begin{pmatrix} \int_0^\theta \mathbf{T}(s)B_p ds & 0 \\ 0 & B_c \end{pmatrix} \begin{pmatrix} E_p & 0 \\ 0 & E_c \end{pmatrix} \begin{pmatrix} -\varepsilon D_c & \varepsilon I \\ -I & -\varepsilon D_p \end{pmatrix} \begin{pmatrix} C_p & 0 \\ 0 & C_c \end{pmatrix}.$$

For $\theta \in [0,\tau]$ and $k \in \mathbb{Z}_+$, define $R(k,\theta) : L_b(\mathbb{R}_+, \mathbb{C}^m) \times L_b(\mathbb{R}_+, \mathbb{C}^p) \times L_b(\mathbb{R}_+, \mathbb{C}^p) \to X \times \mathbb{C}^{Np+n_0}$ by

$$R(k,\theta) \begin{pmatrix} d_1 \\ d_2 \\ r \end{pmatrix} :=$$

$$\begin{bmatrix} \int_{k\tau}^{k\tau+\theta} \mathbf{T}(k\tau + \theta - s)B_p d_1(s) ds + \varepsilon \int_0^\theta \mathbf{T}(s) ds B_p E_p D_c[-D_p d_1(k\tau) + r(k\tau) - d_2(k\tau)] \\ B_c E_c[-D_p d_1(k\tau) + r(k\tau) - d_2(k\tau)] \end{bmatrix}.$$

For convenience, we write

$$x_p(\cdot) := x_p(\,\cdot\,; x_p^0, x_c^0, \tau, \varepsilon, r, d_1, d_2)\,, \quad x_c(\cdot) := x_c(\,\cdot\,; x_p^0, x_c^0, \tau, \varepsilon, r, d_1, d_2)\,.$$

By (6.77)–(6.79) and a routine calculation†, we obtain

$$\begin{pmatrix} x_p(k\tau+\theta) \\ x_c(k+1) \end{pmatrix} = \Delta(\theta) \begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} + R(k,\theta) \begin{pmatrix} d_1 \\ d_2 \\ r \end{pmatrix}, \quad \forall k \in \mathbb{Z}_+ \,, \ \theta \in [0,\tau) \,. \tag{6.81}$$

It follows from (6.81) with $\theta = \tau$ that

$$\begin{pmatrix} x_p((k+1)\tau) \\ x_c(k+1) \end{pmatrix} = \Delta(\tau) \begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} + R(k,\tau) \begin{pmatrix} d_1 \\ d_2 \\ r \end{pmatrix}, \quad \forall k \in \mathbb{Z}_+ \,. \tag{6.82}$$

The transfer function of (6.80) is denoted by $\mathbf{G}_\tau$. By Proposition 6.2.5 and the assumption that $\sigma(\mathbf{G}(\xi_j)K_j) \subset \mathbb{C}_0$, there exists $\tau^* > 0$ such that if $\tau \in (0,\tau^*)$, then $e^{\xi_j\tau} \neq e^{\xi_k\tau}$ for all $j,k \in \underline{N}$, $j \neq k$, and

$$\sigma(e^{\bar{\xi}_j\tau}\mathbf{G}_\tau(e^{\xi_j\tau})K_j) \subset \mathbb{C}_0 \,, \quad \forall j \in \underline{N} \,. \tag{6.83}$$

Applying Statement (1) of Theorem 6.1.14 to the feedback inter-connection of discrete-time systems (6.80) and (6.78) (the free dynamics of which is governed by $\Delta(\tau)$), we conclude that, for every $\tau \in (0,\tau^*)$, there exists $\varepsilon_\tau > 0$ such that, for every $\varepsilon \in (0,\varepsilon_\tau)$, $\Delta(\tau)$ is power stable.

Let $\tau \in (0,\tau^*)$, $\varepsilon \in (0,\varepsilon_\tau)$, $d_1 \in L_b(\mathbb{R}_+,\mathbb{C}^m)$ and $r, d_2 \in L_b(\mathbb{R}_+,\mathbb{C}^p)$. By the admissibility of $B_p$, there exists $M_1 \geq 0$ such that

$$\left\| \int_{k\tau}^{k\tau+\theta} \mathbf{T}(k\tau+\theta-s)B_p d_1(s)ds \right\|_X = M_1 \|d_1\|_{L^2((k\tau,k\tau+\theta),\mathbb{C}^m)} \leq M_1\sqrt{\tau^*}\|d_1\|_\infty \,, \quad \forall k \in \mathbb{Z}_+ \,, \ \forall \theta \in [0,\tau] \,.$$

Therefore, there exists $M_2 \geq 0$ such that

$$\left\| R(k,\theta) \begin{pmatrix} d_1 \\ d_2 \\ r \end{pmatrix} \right\| \leq M_2(\|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty) \,, \quad \forall k \in \mathbb{Z}_+ \,, \ \forall \theta \in [0,\tau] \,. \tag{6.84}$$

Let $x_p^0 \in X$ and $x_c^0 \in \mathbb{C}^{Np+n_0}$. It follows from the discrete-time variation-of-parameters formula, the power stability of $\Delta(\tau)$, (6.82) and (6.84) that there exist $M_3 \geq 1$ and $\rho \in (0,1)$ such that

$$\left\| \begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} \right\| \leq M_3 \left( \rho^k \left\| \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} \right\| + \|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty \right), \quad \forall k \in \mathbb{Z}_+ \,. \tag{6.85}$$

† See Appendix A.4.3 with $\sigma = 1$ and $r$ replaced by $r - d_2$.

By (6.81), we obtain that

$$\begin{pmatrix} x_p(k\tau+\theta) \\ x_c(k) \end{pmatrix} = Q_1(\theta) \begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} + \begin{pmatrix} R_1(k,\theta) \\ 0 \end{pmatrix} \tag{6.86}$$

where

$$Q_1(\theta) := \begin{pmatrix} \mathbf{T}(\theta) - \varepsilon \int_0^\theta \mathbf{T}(s)dsB_pE_pD_cC_p & \varepsilon \int_0^\theta \mathbf{T}(s)dsB_pE_pD_c \\ 0 & I \end{pmatrix},$$

and

$$R_1(k,\theta) := \\ \int_{k\tau}^{k\tau+\theta} \mathbf{T}(k\tau+\theta-s)B_pd_1(s)ds + \varepsilon \int_0^\theta \mathbf{T}(s)dsB_pE_pD_c[-D_pd_1(k\tau) + r(k\tau) - d_2(k\tau)] .$$

Consequently, setting $M_4 := \max_{\theta\in[0,\tau^*]} \|Q_1(\theta)\|$, it follows from (6.84), (6.85) and (6.86) that

$$\begin{aligned} \left\| \begin{pmatrix} x_p(k\tau+\theta) \\ x_c(k) \end{pmatrix} \right\| &\le M_4 \left\| \begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} \right\| + M_2(\|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty) \\ &\le M_3M_4\rho^k \left\| \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} \right\| + (M_2 + M_3M_4)(\|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty) \\ &\le N_1 \left( e^{\beta(k\tau+\theta)} \left\| \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} \right\| + \|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty \right), \\ &\qquad \forall \theta \in [0,\tau),\ \forall k \in \mathbb{Z}_+, \end{aligned}$$

where $\beta := (\ln \rho)/\tau < 0$ and $N_1 := \max\{M_3M_4\rho^{-1}, M_2 + M_3M_4\}$. This completes the proof of Statement (1).

To prove the approximate tracking and disturbance rejection result claimed in Statement (2), note that, by exponential stability of (6.76) and boundedness of $C$, the impulse response of (6.76) is a $\mathbb{C}^{p\times m}$-valued Borel measure $\mu$ of the form $\mu(ds) = g(s)ds + D\delta_0(ds)$, where $g \in L^1_\alpha(\mathbb{R}_+, \mathbb{C}^{p\times m})$ for some $\alpha < 0$, and $\delta_0$ is the Dirac measure (see [41, Lemma 2.3]). By (6.76)–(6.79) and a routine calculation†, we obtain

$$\begin{pmatrix} y(k\tau+\theta) \\ y_c(k) \end{pmatrix} = Q_2(\theta)\Delta(\tau)^k \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} + \begin{pmatrix} y^{\mathrm{io}}(k\tau+\theta) \\ y_c^{\mathrm{io}}(k) \end{pmatrix}, \quad \forall \theta \in [0,\tau),\ \forall k \in \mathbb{Z}_+, \tag{6.87}$$

†See Appendix A.4.3 with $\sigma = 1$ and $r$ replaced by $r - d_2$.

where

$$Q_2(\theta) :=$$
$$\begin{pmatrix} C\mathbf{T}(\theta) - \varepsilon C \int_0^\theta \mathbf{T}(s)BdsED_cC - \varepsilon DED_cC & \varepsilon C \int_0^\theta \mathbf{T}(s)BdsED_c + \varepsilon DEC_c \\ -\varepsilon D_cE_cC & \varepsilon C_c - \varepsilon^2 D_cE_cDC_c \end{pmatrix},$$

and $y^{\text{io}}$, $y_c^{\text{io}}$ satisfy

$$y^{\text{io}} = G(d_1 + \mathcal{H}_\tau y_c^{\text{io}}) + d_2\,, \quad y_c^{\text{io}} = K_{\tau,\varepsilon}\mathcal{S}_\tau(r - y^{\text{io}})\,. \tag{6.88}$$

An application of Theorem 6.2.7 to system (6.88) with $r$ and $d_1$, $d_2$ given by (6.51) and (6.52), respectively, shows that, for every $\delta > 0$, there exists $\tau_\delta \in (0, \tau^*)$ such that, for every sampling period $\tau \in (0, \tau_\delta)$, there exists $\varepsilon_\tau > 0$, such that, for every $\varepsilon \in (0, \varepsilon_\tau)$,

$$\limsup_{t\to\infty} \|y^{\text{io}}(t) - r(t)\| \leq \delta\,.$$

Therefore, by power stability of $\Delta(\tau)$ and (6.87),

$$\limsup_{t\to\infty} \|y(t) - r(t)\| \leq \delta\,.$$

Moreover, an application of Theorem 6.2.7 to system (6.88) with $r$ given by (6.51) and $d_1$, $d_2$ satisfying (6.54) for some $\gamma < 0$ and $N_2 \geq 0$, shows that, for every $\delta > 0$, there exists $\tau_\delta \in (0, \tau^*)$ such that, for every sampling period $\tau \in (0, \tau_\delta)$, there exists $\varepsilon_\tau > 0$, such that, for every $\varepsilon \in (0, \varepsilon_\tau)$,

$$\|y^{\text{io}}(t) - r(t)\| \leq \delta + M_5 e^{\eta_1 t}\,.$$

for some $\eta_1 \in (\gamma, 0)$ and $M_5 \geq 0$. Therefore, by power stability of $\Delta(\tau)$ and (6.87), there exist $\eta \in (\eta_1, 0)$ and $N_3 \geq M_5$ such that

$$\|y(t) - r(t)\| \leq \delta + N_3 e^{\eta t}\,. \qquad \square$$

**Example 6.2.11.** For purpose of illustration, we consider the problem of heating a bar of length 1. We keep both endpoints at zero temperature and inject heat of magnitude $u_p$ at the point $\eta_1 \in (0, 1)$. The temperature measurement is generated by a spatial averaging of the state over an $\sigma$-neighbourhood of a point $\eta_2 \in (\eta_1, 1)$. The system to be controlled can be formulated as follows

$$\begin{aligned} z_t(\eta, t) &= z_{\eta\eta}(\eta, t) + \delta(\eta - \eta_1)u_p(t)\,, \\ y_p(t) &= \frac{1}{2\sigma}\int_{\eta_2-\sigma}^{\eta_2+\sigma} z(s, t)ds\,, \end{aligned}$$

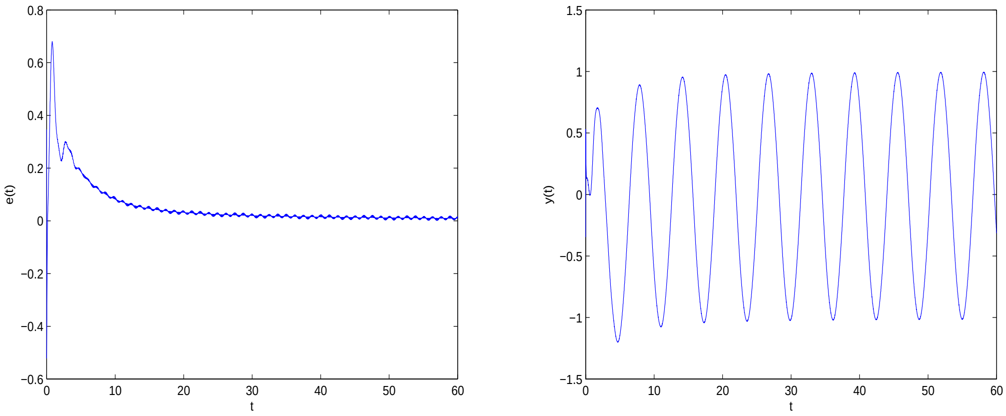

Figure 6-4: Error signal $e$ and output $y$.

with boundary conditions

$$z(0,t) = z(1,t) = 0\,, \quad \forall t > 0\,.$$

For simplicity, we assume zero initial condition

$$z(\eta,0) = 0\,, \quad \forall \eta \in [0,1]\,.$$

Analog low-gain integral control of this system (in the presence of input hysteresis) was studied in [37].

With input $u_p$ and output $y_p$, it can be shown that this system is a regular linear system with state space $X = L^2((0,1),\mathbb{R})$ and bounded observation. In particular, the semigroup $\mathbf{T}(t)$ given by

$$(\mathbf{T}(t)z)(\eta) = \sum_{n=1}^{\infty} 2\exp(-n^2\pi^2 t)\sin(n\pi\eta)\int_0^1 \sin(n\pi\lambda)x(\lambda)d\lambda\,.$$

is exponentially stable. The transfer function $\mathbf{G}$ is given by

$$\mathbf{G}(s) = \frac{\sinh(\sigma\sqrt{s})\sinh(\eta_1\sqrt{s})\sinh((1-\eta_2)\sqrt{s})}{\sigma s\sinh(\sqrt{s})}\,.$$

The aim is to design a robust controller such that the closed-loop system approximately tracks the reference signal $r(t) = \sin t$, in the presence of disturbance signals $d_1, d_2$ given by

$$d_1(t) = \frac{1}{5}\cos(5t) + \frac{1}{t+1}\,, \quad d_2(t) = \frac{1}{5}\sin(5t) - \frac{1}{2}\ln\left(1 + \frac{1}{t+1}\right)\,, \quad t \geq 0\,.$$

Set

$$K_1 := 1/\mathbf{G}(i)\,, \quad K_2 := \overline{K_1}\,, \quad K_3 := 1/\mathbf{G}(5i)\,, \quad K_4 := \overline{K_3}\,,$$

and $\mathbf{K}^0(z) \equiv 10$, so that the transfer function $\mathbf{K}_{\tau,\varepsilon}$ of the controller $K_{\tau,\varepsilon}$ (see (6.49)) is given by

$$\begin{aligned} \mathbf{K}_{\tau,\varepsilon}(z) &:= \varepsilon\left(10 + \frac{K_1}{z-e^{i\tau}} + \frac{K_2}{z-e^{-i\tau}} + \frac{K_3}{z-e^{5i\tau}} + \frac{K_4}{z-e^{-5i\tau}}\right) \\ &= \varepsilon\left(10 + \frac{2\mathrm{Re}\,(K_1)z - 2\mathrm{Re}\,(K_1e^{-i\tau})}{z^2 - 2\cos\tau z + 1} + \frac{2\mathrm{Re}\,(K_3)z - 2\mathrm{Re}\,(K_3e^{-5i\tau})}{z^2 - 2\cos(5\tau)z + 1}\right)\,. \end{aligned}$$

Since all the relevant hypotheses are satisfied, the conclusions of Theorem 6.2.7 are valid. In Figures 6-4, simulations are shown for the specific values

$$\eta_1 = 0.2\,, \quad \eta_2 = 0.6\,, \quad \sigma = 0.01\,, \quad \tau = 0.1\,, \quad \varepsilon = 0.1\,,$$

with zero initial conditions for the controller. The error signal $e = r - y_p - d_2$ and the output $y = y_p + d_2$ of the sampled-data system are shown in Figure 6-4. Asymptotically, the error is bounded by 0.0028, that is, $\limsup_{t\geq 0}|e(t)| \leq 0.0028$. Simulations show that, for the sampling period $\tau = 0.1$, instability occurs at $\varepsilon \approx 0.22$.

## 6.3 Low-gain sampled-data control of exponentially stable well-posed systems

We extend our results in Section 6.2 to exponentially stable well-posed systems with transfer functions in $H^\infty(\mathbb{C}_\alpha, \mathbb{C}^{p\times m})$, where $\alpha < 0$, by using suitable low-pass filters. A (finite-dimensional) *filter* is an exponentially stable, strictly causal, finite-dimensional system. In particular, a filter has impulse response of the form $t \mapsto Ce^{At}B$, where $A \in \mathbb{C}^{\ell\times\ell}$, $B \in \mathbb{C}^{\ell\times m}$ and $C \in \mathbb{C}^{p\times\ell}$, and every eigenvalue of $A$ has negative real part.

**Lemma 6.3.1.** *Let $G$ is a continuous-time input-output operator with transfer function $\mathbf{G} \in H^\infty(\mathbb{C}_\alpha, \mathbb{C})$ for some $\alpha < 0$, and let $F$ be a single-input-single-output filter. Then there exists $\beta \in (\alpha, 0)$ such that the impulse response of $GF$ is in $L^1_\beta(\mathbb{R}_+, \mathbb{C})$.*

*Proof.* Let $\mathbf{F}$ denote the transfer function of $F$, which is a strictly proper stable rational function. Hence there exists $\gamma \in (\alpha, 0)$ such that $\mathbf{F} \in H^2(\mathbb{C}_\gamma, \mathbb{C})$, so that $\mathbf{GF}$ is in $H^2(\mathbb{C}_\gamma, \mathbb{C})$. Let $h$ denote the impulse response of $GF$. By the Paley-Wiener Theorem, $h \in L^2_\gamma(\mathbb{R}_+, \mathbb{C})$. Therefore, it follows easily from the Hölder's inequality that $h \in L^1_\beta(\mathbb{R}_+, \mathbb{C})$ for every $\beta \in (\gamma, 0)$. □

The following Lemma will be useful in the proof of Theorem 6.3.4, the main result of this section. It is essentially the same as the first claim of Theorem 6.1.12, but we provide an alternative proof.

**Lemma 6.3.2.** *Let $N \in \mathbb{N}$ and let $\lambda_j \in \mathbb{C}$, $|\lambda_j| = 1$ for all $j \in \underline{N}$ be such that $\lambda_j \neq \lambda_k$ for all $j, k \in \underline{N}$, $j \neq k$. Assume that $\mathbf{P} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m})$ and that there exist $K_j \in \mathbb{C}^{m\times p}$ such that*

$$\sigma(\bar{\lambda}_j \mathbf{P}(\lambda_j) K_j) \subset \mathbb{C}_0\,, \quad \forall j \in \underline{N}\,.$$

*Let $\mathbf{K}^0 \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$ and set*

$$\mathbf{K}_\varepsilon(z) := \varepsilon \left( \mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{K_j}{z - \lambda_j} \right) .$$

*Then there exists $\varepsilon^* > 0$ such that*

$$\mathbf{K}_\varepsilon (I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})\,, \quad \forall \varepsilon \in (0, \varepsilon^*)\,.$$

It is convenient to first state and prove the following lemma which will facilitate the proof of Lemma 6.3.2.

**Lemma 6.3.3.** *For $\rho > 0$, set $B_\rho := \mathbb{B}(1, \rho) \cap \mathbb{E}_1$ and let $U \supset \mathrm{cl}(B_\rho)$ be open. Let $\mathbf{Q} \in H^\infty(U, \mathbb{C}^{p\times m})$, $\mathbf{H} \in H^\infty(U, \mathbb{C}^{m\times p})$ and $K \in \mathbb{C}^{m\times p}$. If*

$$\sigma(\mathbf{Q}(1)K) \subset \mathbb{C}_0\,, \tag{6.89}$$

*then there exists $\varepsilon^* > 0$ such that*

$$z \mapsto \left( I + \varepsilon \mathbf{Q}(z) \left( \mathbf{H}(z) + \frac{K}{z-1} \right) \right)^{-1} \in H^\infty(B_\rho, \mathbb{C}^{p\times p})\,, \quad \forall \varepsilon \in (0, \varepsilon^*)\,.$$

*Proof.* Note that, by (6.89), $\mathrm{rk}\, K = p$, so that $K^*K$ is invertible. Setting

$$\mathbf{D}(z) := \frac{z-1}{z} I_p\,, \quad \mathbf{N}(z) := \mathbf{H}(z)\mathbf{D}(z) + \frac{1}{z} K\,,$$

we conclude that $(\mathbf{N}, \mathbf{D})$ is a right coprime factorization of $\mathbf{H}(z) + K/(z-1)$ over $H^\infty(B_\rho)$, since $\mathbf{N}(z)\mathbf{D}^{-1}(z) = \mathbf{H}(z) + K/(z-1)$ and

$$(K^*K)^{-1}K^*\mathbf{N}(z) + [I_p - (K^*K)^{-1}K^*\mathbf{H}(z)]\mathbf{D}(z) = \frac{1}{z} I_p + \frac{z-1}{z} I_p = I_p\,.$$

By Proposition 6.1.3, it is sufficient to show that there exists $\varepsilon^* > 0$ such that

$$\inf_{z \in B_\rho} |\det[\varepsilon \mathbf{Q}(z)\mathbf{N}(z) + \mathbf{D}(z)]| > 0\,, \quad \forall \varepsilon \in (0, \varepsilon^*)\,.$$

Seeking a contradiction, suppose such an $\varepsilon^*$ does not exist. Then there exists a sequence $\varepsilon_n \downarrow 0$ such that

$$\inf_{z \in B_\rho} \left| \det \left[ \varepsilon_n \mathbf{Q}(z)\mathbf{H}(z)\mathbf{D}(z) + \frac{\varepsilon_n}{z} \mathbf{Q}(z)K + \frac{z-1}{z} I_p \right] \right| = 0\,, \quad \forall n \in \mathbb{Z}_+\,.$$

It follows that there exists a number $z_n \in \mathrm{cl}(B_\rho)$ such that

$$\det\left[\frac{\varepsilon_n(z_n-1)}{z_n}\mathbf{Q}(z_n)\mathbf{H}(z_n)+\frac{\varepsilon_n}{z_n}\mathbf{Q}(z_n)K+\frac{z_n-1}{z_n}I_p\right]=0\,,\quad \forall n\in\mathbb{Z}_+\,.$$

showing that

$$\det[\varepsilon_n(z_n-1)\mathbf{Q}(z_n)\mathbf{H}(z_n)+\varepsilon_n\mathbf{Q}(z_n)K+(z_n-1)I_p]=0\,,\quad \forall n\in\mathbb{Z}_+\,. \tag{6.90}$$

Since $\lim_{n\to\infty}\varepsilon_n=0$, we conclude from (6.90) that

$$\lim_{n\to\infty} z_n = 1\,. \tag{6.91}$$

Moreover, we obtain from (6.90) that

$$\frac{1-z_n}{\varepsilon_n}\in\sigma[(z_n-1)\mathbf{Q}(z_n)\mathbf{H}(z_n)+\mathbf{Q}(z_n)K]\,,\quad \forall n\in\mathbb{Z}_+\,.$$

Consequently, by (6.89) and (6.91), there exists $\beta>0$ and $n_0\in\mathbb{N}$ such that

$$\frac{1-z_n}{\varepsilon_n}\in\mathbb{C}_\beta\,,\quad \forall n\geq n_0\,.$$

Setting $z_n' := 1+i\mathrm{Im}\,z_n$ for $n\in\mathbb{Z}_+$, and invoking an argument identical to that used in the proof of Theorem 2.5 in [45], it can be shown that

$$\liminf_{n\to\infty}\left(\mathrm{Re}\,\frac{1-z_n'}{\varepsilon_n}\right)\geq\beta>0\,.$$

This is in contradiction to the trivial fact that $\mathrm{Re}\,((1-z_n')/\varepsilon_n)=0$ for all $n\in\mathbb{Z}_+$. $\square$

We are now in the position to prove Lemma 6.3.2.

*Proof of Lemma 6.3.2.* We first show that $(I+\mathbf{PK}_\varepsilon)^{-1}\in H^\infty(\mathbb{E}_1,\mathbb{C}^{p\times p})$ for sufficiently small $\varepsilon$. Since $\lambda_j\neq\lambda_k$ for all $j,k\in\underline{N}$, $j\neq k$, we can choose $\rho>0$ sufficiently small such that

$$\mathrm{cl}(\mathbb{B}(\lambda_j,\rho))\cap\mathrm{cl}(\mathbb{B}(\lambda_k,\rho))=\emptyset\,,\quad \forall j,k\in\underline{N}\,,\ j\neq k\,.$$

Setting $\Omega_j := \mathbb{E}_1\cap\mathbb{B}(\lambda_j,\rho)$ and $\Omega := \bigcup_{j=1}^N\Omega_j$, it is clear that

$$z\mapsto\mathbf{P}(z)\left(\mathbf{K}^0(z)+\sum_{j=1}^N\frac{K_j}{z-\lambda_j}\right)$$

is bounded on $\mathbb{E}_1\setminus\Omega$. Thus, exists $\varepsilon^\infty>0$ such that

$$(I+\mathbf{PK}_\varepsilon)^{-1}\in H^\infty(\mathbb{E}_1\setminus\Omega,\mathbb{C}^{p\times p})\,,\quad \forall\varepsilon\in(0,\varepsilon^\infty)\,. \tag{6.92}$$

Fix $j \in \underline{N}$ and set

$$\mathbf{H}(z) := \mathbf{K}^0(z) + \sum_{k \in \underline{N},\, k \neq j} \frac{K_k}{z - \lambda_k}\,.$$

Then there exists an open set $V_j \supset \mathrm{cl}(\Omega_j)$ such that $\mathbf{H} \in H^\infty(V_j, \mathbb{C}^{m\times p})$ and, furthermore,

$$\mathbf{P}(z)\mathbf{K}_\varepsilon(z) = \varepsilon\mathbf{P}(z)\left(\mathbf{H}(z) + \frac{K_j}{z - \lambda_j}\right) = \varepsilon\mathbf{P}(\lambda_j w)\left(\mathbf{H}(\lambda_j w) + \frac{\bar{\lambda}_j K_j}{w - 1}\right)\,,$$

where $w := \bar{\lambda}_j z$. Setting

$$\widetilde{\mathbf{H}}(w) := \mathbf{H}(\lambda_j w)\,, \quad \mathbf{Q}(w) := \mathbf{P}(\lambda_j w)\,, \quad \widetilde{K}_j := \bar{\lambda}_j K_j\,,$$

it follows that

$$\mathbf{P}(z)\mathbf{K}_\varepsilon(z) = \varepsilon\mathbf{Q}(w)\left(\tilde{\mathbf{H}}(w) + \frac{\tilde{K}_j}{w - 1}\right)\,.$$

Since $\mathbf{P} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m})$ and $\mathbf{K}^0 \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$, we see that $\mathbf{Q} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m})$ and $\widetilde{\mathbf{H}} \in H^\infty(U_j, \mathbb{C}^{m\times p})$, where

$$U_j := \bar{\lambda}_j V_j \supset \bar{\lambda}_j\, \mathrm{cl}(\Omega_j) = \mathrm{cl}(\mathbb{E}_1 \cap \mathbb{B}(1,\rho)) = \mathrm{cl}(B_\rho)\,.$$

Moreover,

$$\sigma(\mathbf{Q}(1)\widetilde{K}_j) = \sigma(\bar{\lambda}_j \mathbf{P}(\lambda_j) K_j) \subset \mathbb{C}_0\,.$$

It follows from the Lemma 6.3.3 that there exists $\varepsilon_j \in (0, \varepsilon^\infty)$ such that

$$w \mapsto \left[I + \varepsilon\mathbf{Q}(w)\left(\widetilde{\mathbf{H}}(w) + \frac{\widetilde{K}_j}{w - 1}\right)\right]^{-1} \in H^\infty(B_\rho, \mathbb{C}^{p\times p})\,, \quad \forall \varepsilon \in (0, \varepsilon_j)\,.$$

Hence,

$$(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\Omega_j, \mathbb{C}^{p\times p})\,, \quad \forall \varepsilon \in (0, \varepsilon_j)\,. \tag{6.93}$$

Letting $\varepsilon^* := \min\{\varepsilon_j : j \in \underline{N}\}$ and invoking (6.92) and (6.93), we conclude that

$$(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})\,, \quad \forall \varepsilon \in (0, \varepsilon^*)\,. \tag{6.94}$$

Invoking arguments identical to those used in the proof of the second claim of Lemma 6.1.10, it can be shown that $(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times p})$ for all $\varepsilon \in (0, \varepsilon^*)$. Furthermore, invoking arguments identical to those used in the proof of the first claim of Theorem 6.1.12, we conclude that $\mathbf{K}_\varepsilon(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$ for all $\varepsilon \in (0, \varepsilon^*)$. □

Consider the sampled-data system shown in Figure 6-5, where $G$ is the input-output operator of the continuous-time plant, $K_{\tau,\varepsilon}$ is the input-output operator of the discrete-

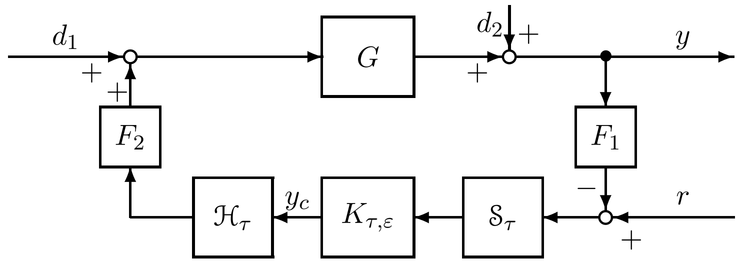


Figure 6-5: Sampled-data low-gain control with filters.

time controller, $F_1$ and $F_2$ are filters, $r$ is a reference signal, and $d_1$ and $d_2$ are disturbance signals. Mathematically, Figure 6-5 can be expressed as

$$y = G(F_2\mathcal{H}_\tau y_c + d_1) + d_2\,, \quad y_c = K_{\tau,\varepsilon}\mathcal{S}_\tau(r - F_1 y)\,. \tag{6.95}$$

The following theorem is the main result of this section.

**Theorem 6.3.4.** *Let $N \in \mathbb{N}$ and let $\xi_j \in i\mathbb{R}$ for all $j \in \underline{N}$ be such that $\xi_j \neq \xi_k$ for $j, k \in \underline{N}$, $j \neq k$. Assume that the transfer function $\mathbf{G}$ of $G$ is in $H^\infty(\mathbb{C}_\alpha, \mathbb{C}^{p\times m})$ for some $\alpha < 0$ and there exist $K_j$ such that*

$$\sigma(\mathbf{G}(\xi_j)K_j) \subset \mathbb{C}_0\,, \quad \forall j \in \underline{N}\,. \tag{6.96}$$

*Let $\tau > 0$ be the sampling period and let the transfer function $\mathbf{K}_{\tau,\varepsilon}$ of $K_{\tau,\varepsilon}$ be given by*

$$\mathbf{K}_{\tau,\varepsilon}(z) := \varepsilon\left(\mathbf{K}^0(z) + \sum_{j=1}^{N}\frac{K_j}{z - e^{\xi_j\tau}}\right)\,, \tag{6.97}$$

*where $\mathbf{K}^0 \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$. Assume that the transfer functions $\mathbf{F}_1$ of $F_1$ and $\mathbf{F}_2$ of $F_2$ satisfy*

$$\mathbf{F}_1(\xi_j) = I_p\,, \quad \mathbf{F}_2(\xi_j) = I_m\,, \qquad \forall j \in \underline{N}\,. \tag{6.98}$$

*If $r$ is given by $r(t) := \sum_{j=1}^{N} e^{\xi_j t}\mathfrak{r}_j$, $\mathfrak{r}_j \in \mathbb{C}^p$, and $d_1$, $d_2$ are given by*

$$d_1(t) := \sum_{j=1}^{N} e^{\xi_j t}\mathfrak{d}_{1j} + p_1(t)\,,\ d_2(t) := \sum_{j=1}^{N} e^{\xi_j t}\mathfrak{d}_{2j} + p_{21}(t) + p_{22}(t)\,,\ \mathfrak{d}_{1j} \in \mathbb{C}^m\,,\ \mathfrak{d}_{2j} \in \mathbb{C}^p\,, \tag{6.99}$$

*where $p_1 \in L^2_\gamma(\mathbb{R}_+, \mathbb{C}^m)$, $p_{21} \in L^2_\gamma(\mathbb{R}_+, \mathbb{C}^p)$ for some $\gamma \in (\alpha, 0)$, and $p_{22} \in L^1_{loc}(\mathbb{R}_+, \mathbb{C}^p)$ with $\lim_{t\to\infty} p_{22}(t) = 0$, then, for every $\delta > 0$, there exists $\tau_\delta > 0$ such that, for every sampling period $\tau \in (0, \tau_\delta)$, there exists $\varepsilon_\tau > 0$ such that, for every $\varepsilon \in (0, \varepsilon_\tau)$, the output $y$ of the sampled-data feedback system* (6.95) *can be decomposed as $y = y_1 + y_2$, where $y_1 \in L^2_\gamma(\mathbb{R}_+, \mathbb{C}^p)$ and $y_2$ satisfies*

$$\limsup_{t\to\infty} \|y_2(t) - r(t)\| \leq \delta\,.$$

*Proof.* Setting $\tau_0 := 2\pi/\sup\{|\xi_j - \xi_k| : j,k \in \underline{N}, j \neq k\}$, we know that if $\tau \in (0,\tau_0)$, then $e^{\xi_j \tau} \neq e^{\xi_k \tau}$ for all $j,k \in \underline{N}$, $j \neq k$. Define

$$H := F_1 G F_2\,, \quad H_\tau := \mathcal{S}_\tau H \mathcal{H}_\tau = \mathcal{S}_\tau F_1 G F_2 \mathcal{H}_\tau\,.$$

The transfer functions of $H$ and $H_\tau$ are denoted by $\mathbf{H}$ and $\mathbf{H}_\tau$, respectively. By Lemma 6.3.1, there exists $\beta \in (\alpha, 0)$ such that the impulse responses of $H$, $F_1G$ and $GF_2$ are in $L^1_\beta(\mathbb{R}_+, \mathbb{C}^{p\times m})$. Hence, by Proposition 6.2.5 and (6.98), $\mathbf{H}_\tau \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m})$ and

$$\lim_{\tau\to 0} \mathbf{H}_\tau(e^{\xi_j\tau}) = \mathbf{H}(\xi_j) = \mathbf{G}(\xi_j)\,, \quad \forall j \in \underline{N}\,. \tag{6.100}$$

By (6.96) and (6.100), there exists $\tau^* \in (0,\tau_0)$ such that if $\tau \in (0,\tau^*)$, then

$$\sigma(e^{\bar{\xi}_j\tau}\mathbf{H}_\tau(e^{\xi_j\tau})K_j) \subset \mathbb{C}_0\,, \quad \forall j \in \underline{N}\,. \tag{6.101}$$

Let $\tau \in (0,\tau^*)$. Invoking Theorem 6.1.12, we conclude that there exists $\varepsilon_\tau > 0$ such that

$$\mathbf{K}_{\tau,\varepsilon}(I + \mathbf{H}_\tau \mathbf{K}_{\tau,\varepsilon})^{-1} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})\,, \quad \forall \varepsilon \in (0,\varepsilon_\tau)\,. \tag{6.102}$$

By (6.101), $\mathbf{H}_\tau(e^{\xi_j\tau})K_j$ is invertible, and thus, we calculate that

$$(\mathbf{K}_{\tau,\varepsilon}(I + \mathbf{H}_\tau \mathbf{K}_{\tau,\varepsilon})^{-1})(e^{\xi_j\tau}) = K_j(\mathbf{H}_\tau(e^{\xi_j\tau})K_j)^{-1}\,, \quad \forall j \in \underline{N}\,. \tag{6.103}$$

The output $y_c$ of the discrete-time controller (see (6.95)) is given by

$$\begin{aligned} y_c &= K_{\tau,\varepsilon}\mathcal{S}_\tau[r - F_1(GF_2\mathcal{H}_\tau y_c + Gd_1 + d_2)] \\ &= K_{\tau,\varepsilon}\mathcal{S}_\tau r - K_{\tau,\varepsilon}H_\tau y_c - K_{\tau,\varepsilon}\mathcal{S}_\tau F_1 G d_1 - K_{\tau,\varepsilon}\mathcal{S}_\tau F_1 d_2\,, \end{aligned}$$

so that,

$$y_c = K_{\tau,\varepsilon}(I + H_\tau K_{\tau,\varepsilon})^{-1}(\mathcal{S}_\tau r - \mathcal{S}_\tau F_1 G d_1 - \mathcal{S}_\tau F_1 d_2)\,. \tag{6.104}$$

Since the impulse reponses of $F_1G$ and $F_1$, $p_1$ and $p_{21}$ are $L^2$-functions,

$$\lim_{t\to\infty}(F_1 G p_1)(t) = 0\,, \quad \lim_{t\to\infty}(F_1 p_{21})(t) = 0\,. \tag{6.105}$$

Invoking the fact that the impulse responses of $F_1G$ and $F_1$ are $L^1$-functions, together with Lemma 6.2.4, (6.98) and (6.105), we obtain

$$\lim_{t\to\infty}[(F_1Gd_1)(t) - \sum_{j=1}^N e^{\xi_j t}\mathbf{G}(\xi_j)\mathfrak{d}_{1j}] = 0\,, \quad \lim_{t\to\infty}[(F_1 d_2)(t) - \sum_{j=1}^N e^{\xi_j t}\mathfrak{d}_{2j}] = 0\,,$$

showing that

$$\lim_{k\to\infty}[(\mathcal{S}_\tau F_1 G d_1)(k) - \sum_{j=1}^{N} e^{\xi_j k\tau}\mathbf{G}(\xi_j)\mathfrak{d}_{1j}] = 0\,, \quad \lim_{k\to\infty}[(\mathcal{S}_\tau F_1 d_2)(k) - \sum_{j=1}^{N} e^{\xi_j k\tau}\mathfrak{d}_{2j}] = 0\,. \tag{6.106}$$

Define $a_\tau, b_\tau, c_\tau \colon \mathbb{Z}_+ \to \mathbb{C}^m$ by

$$\begin{aligned}
a_\tau(k) &:= \sum_{j=1}^{N} e^{\xi_j\tau k} K_j(\mathbf{H}_\tau(e^{\xi_j\tau})K_j)^{-1}\mathfrak{r}_j\,,\\
b_\tau(k) &:= \sum_{j=1}^{N} e^{\xi_j\tau k} K_j(\mathbf{H}_\tau(e^{\xi_j\tau})K_j)^{-1}\mathbf{G}(\xi_j)\mathfrak{d}_{1j}\,,\\
c_\tau(k) &:= \sum_{j=1}^{N} e^{\xi_j\tau k} K_j(\mathbf{H}_\tau(e^{\xi_j\tau})K_j)^{-1}\mathfrak{d}_{2j}\,.
\end{aligned}$$

By (6.102), the impulse response of $K_{\tau,\varepsilon}(I + H_\tau K_{\tau,\varepsilon})^{-1}$ is in $\ell^1(\mathbb{Z}_+, \mathbb{C}^{m\times p})$. It follows from Lemma 6.1.6, (6.103), (6.104) and (6.106) that

$$\lim_{k\to\infty}(y_c - a_\tau + b_\tau + c_\tau)(k) = 0\,. \tag{6.107}$$

By (6.96), $\mathbf{G}(\xi_j)K_j$ is invertible for every $j \in \underline{N}$. Define functions $v_1, v_2$ and $v_3$ on $\mathbb{R}_+$ by

$$\begin{aligned}
v_1(t) &:= \sum_{j=1}^{N} e^{\xi_j t} K_j(\mathbf{G}(\xi_j)K_j)^{-1}\mathfrak{r}_j\,,\\
v_2(t) &:= \sum_{j=1}^{N} e^{\xi_j t} K_j(\mathbf{G}(\xi_j)K_j)^{-1}\mathbf{G}(\xi_j)\mathfrak{d}_{1j}\,,\\
v_3(t) &:= \sum_{j=1}^{N} e^{\xi_j t} K_j(\mathbf{G}(\xi_j)K_j)^{-1}\mathfrak{d}_{2j}\,.
\end{aligned}$$

By (6.98), $\mathbf{G}(\xi_j) = \mathbf{H}(\xi_j)$ for every $j \in \underline{N}$. Since $\xi_j \in i\mathbb{R}$ for $j \in \underline{N}$, we have, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned}
&\sup_{t\in[k\tau,(k+1)\tau)} \|v_1(t) - (\mathcal{H}_\tau a_\tau)(t)\|\\
&\quad\leq \sum_{j=1}^{N} \|(\mathbf{H}(\xi_j)K_j)^{-1} - (\mathbf{H}_\tau(e^{\xi_j\tau})K_j)^{-1}\|\|K_j\|\|\mathfrak{r}_j\| +\\
&\qquad\qquad \sup_{t\in[k\tau,(k+1)\tau)} \sum_{j=1}^{N} |e^{\xi_j(t-k\tau)} - 1|\|K_j(\mathbf{G}(\xi_j)K_j)^{-1}\mathfrak{r}_j\|\,.
\end{aligned} \tag{6.108}$$

Let $\delta > 0$. By (6.100) and (6.108), there exists $\tau_\delta \in (0, \tau^*)$ such that if $\tau \in (0, \tau_\delta)$, then

$$\|v_1(t) - (\mathcal{H}_\tau a_\tau)(t)\| \leq \frac{\delta}{3M}, \quad \forall t \geq 0, \tag{6.109}$$

where $M$ denotes the $L^1$-norm of the impulse response of $GF_2$. Similarly,

$$\|v_2(t) - (\mathcal{H}_\tau b_\tau)(t)\| \leq \frac{\delta}{3M}, \quad \|v_3(t) - (\mathcal{H}_\tau c_\tau)(t)\| \leq \frac{\delta}{3M}, \quad \forall t \geq 0. \tag{6.110}$$

Let $\tau \in (0, \tau_\delta)$ and $\varepsilon \in (0, \varepsilon_\tau)$. Then, by (6.107), (6.109) and (6.110), we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \|(\mathcal{H}_\tau y_c)(t) - v_1(t) + v_2(t) + v_3(t)\| \\
&\quad \leq \limsup_{t\to\infty} \|(\mathcal{H}_\tau(y_c - a_\tau + b_\tau + c_\tau))(t)\| + \limsup_{t\to\infty} \|(\mathcal{H}_\tau a_\tau)(t) - v_1(t)\| \\
&\qquad + \limsup_{t\to\infty} \|v_2(t) - (\mathcal{H}_\tau b_\tau)(t)\| + \limsup_{t\to\infty} \|v_3(t) - (\mathcal{H}_\tau c_\tau)(t)\| \\
&\quad \leq \frac{\delta}{M}.
\end{aligned} \tag{6.111}$$

Moreover, we conclude from Lemma 6.2.4 and (6.98) that

$$\lim_{t\to\infty} [(GF_2 v_1)(t) - r(t)] = 0, \quad \lim_{t\to\infty} [(GF_2 v_2)(t) - \sum_{j=1}^{N} e^{\xi_j t}\mathbf{G}(\xi_j)\mathfrak{d}_{1j}] = 0, \tag{6.112}$$

and

$$\begin{aligned}
&\lim_{t\to\infty} [(GF_2 v_3)(t) - d_2(t) + p_{21}(t)] \\
&\quad = \sum_{j=1}^{N} \lim_{t\to\infty} [(GF_2(e^{\xi_j \cdot} K_j(\mathbf{G}(\xi_j)K_j)^{-1}\mathfrak{d}_{2j}))(t) - e^{\xi_j t}\mathfrak{d}_{2j}] - \lim_{t\to\infty} p_{22}(t) \\
&\quad = 0.
\end{aligned} \tag{6.113}$$

Setting

$$y_1(t) := (Gd_1)(t) - \sum_{j=1}^{N} \mathbf{G}(\xi_j)e^{\xi_j t}\mathfrak{d}_{1j} + p_{21}(t),$$

and

$$y_2(t) := (GF_2\mathcal{H}_\tau y_c)(t) + \sum_{j=1}^{N} \mathbf{G}(\xi_j)e^{\xi_j t}\mathfrak{d}_{1j} + d_2(t) - p_{21}(t),$$

it follows that $y = y_1 + y_2$. Denoting the Laplace transform by $\mathscr{L}$ and invoking (6.99), we obtain that

$$(\mathscr{L}(y_1))(s) = \sum_{j=1}^{N} \frac{[\mathbf{G}(s) - \mathbf{G}(\xi_j)]\mathfrak{d}_{1j}}{s - \xi_j} + \mathbf{G}(s)(\mathscr{L}(p_1))(s) + (\mathscr{L}(p_{21}))(s).$$

Since $\mathbf{G} \in H^\infty(\mathbb{C}_\alpha, \mathbb{C}^{p\times m})$, $p_1 \in L^2_\gamma(\mathbb{R}_+, \mathbb{C}^m)$ and $p_{21} \in L^2_\gamma(\mathbb{R}_+, \mathbb{C}^p)$ with $\alpha < \gamma < 0$, it follows that $\mathscr{L}(y_1) \in H^2(\mathbb{C}_\gamma, \mathbb{C}^p)$. Hence, the Paley-Wiener Theorem implies that $y_1 \in L^2_\gamma(\mathbb{R}_+, \mathbb{C}^p)$. Furthermore, since

$$\begin{aligned}
\|y_2(t) - r(t)\| \quad &\leq \quad \|(GF_2(\mathcal{H}_\tau y_c - v_1 + v_2 + v_3))(t)\| + \|(GF_2 v_1)(t) - r(t)\| \\
&\quad + \left\| \sum_{j=1}^N \mathbf{G}(\xi_j) e^{\xi_j t} \mathfrak{d}_{1j} - (GF_2 v_2)(t) \right\| \\
&\quad + \|d_2(t) - p_{21}(t) - (GF_2 v_3)(t)\| \,, \quad \forall t \geq 0 \,,
\end{aligned}$$

it follows from (6.112) and (6.113) that

$$\limsup_{t\to\infty} \|y_2(t) - r(t)\| \leq \limsup_{t\to\infty} \|(GF_2(\mathcal{H}_\tau y_c - v_1 + v_2 + v_3))(t)\| \,.$$

Finally, $\mathcal{H}_\tau y_c - v_1 + v_2 + v_3$ is bounded and thus, by Lemma 6.2.3 and (6.111),

$$\limsup_{t\to\infty} \|y_2(t) - r(t)\| \leq M \limsup_{t\to\infty} \|(\mathcal{H}_\tau y_c)(t) - v_1(t) + v_2(t)\| \leq \delta \,. \qquad \square$$

**Remark 6.3.5.** (1) Let $N \in \mathbb{N}$ and let $\xi_j \in i\mathbb{R}$ be such that $\xi_j \neq \xi_j$ for all $j \in \underline{N}$, $j \neq k$. A filter with transfer function $\mathbf{F}$ satisfying $\mathbf{F}(\xi_j) = I$ for all $j \in \underline{N}$ can be constructed in the following way:

$$\mathbf{F}(s) := \frac{1}{h(s)} \sum_{j=1}^N \left[ h(\xi_j) \prod_{k \in \underline{N},\, k\neq j} \frac{(s - \xi_k)}{\xi_j - \xi_k} \right] I \,,$$

where $h(s)$ is a real Hurwitz polynomial of degree $N$. It is cleat that $\mathbf{F}$ is a strictly proper stable rational function. Moreover, if the numbers in $\{\xi_j \in i\mathbb{R} : j \in \underline{N}\} \setminus \{0\}$ occur in complex conjugate pairs, then it is easy to see that $\mathbf{F}$ has real coefficients.

(2) Theorem 6.3.4 implies that for every $\delta > 0$, there exists $\tau_\delta > 0$ such that, for every sampling period $\tau \in (0, \tau_\delta)$, there exists $\varepsilon_\tau > 0$ such that, for every $\varepsilon \in (0, \varepsilon_\tau)$, the output $y$ of system (6.95) satisfies

$$\lim_{T\to\infty} \mu_L(\{t \geq T : \|r(t) - y(t)\| \geq \delta\}) = 0 \,,$$

where $\mu_L$ denotes the Lebesgue measure on $\mathbb{R}_+$. $\diamond$

**Example 6.3.6.** For purpose of illustration, we consider the problem of heating a bar of length 1. We keep both endpoints at zero temperature and inject heat of magnitude $u_j$ at the point $\eta_j \in (0, 1)$, $j = 1, 2$. Temperature measurements are taken at the points $\eta_1, \eta_2 \in (0, 1)$. The system to be controlled can be formulated as follows

$$\begin{aligned}
z_t(\xi, t) \quad &= \quad z_{\xi\xi}(\xi, t) + \delta(\xi - \xi_1) u_1(t) + \delta(\xi - \xi_2) u_2(t) \,, \quad \forall \xi \in (0, 1) \,,\ \forall t > 0 \,, \\
y_{p1}(t) \quad &= \quad z(\eta_1, t) \,, \quad y_{p2}(t) = z(\eta_2, t) \,; \quad \forall t > 0 \,,
\end{aligned}$$

with boundary conditions

$$z(0,t) = z(1,t) = 0\,, \quad \forall t > 0\,.$$

For simplicity, we assume zero initial condition

$$z(\xi, 0) = 0\,, \quad \forall \xi \in [0,1]\,.$$

Continuous-time low-gain integral control of this system was studied in [33].

It can be shown that this system can be formulated as a well-posed system with the state space $X = L^2((0,1),\mathbb{R})$. In particular, the semigroup $\mathbf{T}(t)$ given by

$$(\mathbf{T}(t)z)(\xi) = \sum_{n=1}^{\infty} 2\exp(-n^2\pi^2 t)\sin(n\pi\xi)\int_0^1 \sin(n\pi\lambda)x(\lambda)d\lambda\,.$$

is exponentially stable. Assuming that

$$0 < \xi_1 \leq \eta_1 \leq \xi_2 \leq \eta_2 < 1\,,$$

the transfer function $\mathbf{G}(s)$ is given by

$$\mathbf{G}(s) = \frac{1}{\sqrt{s}\sinh(\sqrt{s})}\begin{pmatrix} \sinh((1-\eta_1)\sqrt{s})\sinh(\xi_1\sqrt{s}) & \sinh((1-\xi_2)\sqrt{s})\sinh(\eta_1\sqrt{s}) \\ \sinh((1-\eta_2)\sqrt{s})\sinh(\xi_1\sqrt{s}) & \sinh((1-\eta_2)\sqrt{s})\sinh(\xi_2\sqrt{s}) \end{pmatrix}\,.$$

The aim is to determine $\varepsilon$ $\mathbf{K}^0$, and $K_j$ such that the controller (6.97) leads to a sampled-data feedback system such that the output $y = (y_{p1}, y_{p2})^T + d_2$ of this system approximately tracks the reference signal

$$r(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\,, \quad \forall t \geq 0\,,$$

in the presence of disturbance signals $d_1, d_2$ given by

$$d_1(t) = \frac{1}{5}\begin{pmatrix} \sin(2t) + e^{-t} \\ \cos(2t) + \dfrac{1}{t+1} \end{pmatrix}\,, \quad d_2(t) = \frac{1}{5}\begin{pmatrix} \cos(2t) \\ \sin(2t) - \ln\left(1 + \dfrac{1}{t+1}\right) \end{pmatrix}\,, \quad t \geq 0\,.$$

It can be shown that, if $\xi_2 > \eta_1$, then $\mathbf{G}(s)$ is invertible for $s \in \mathrm{cl}(\mathbb{C}_0)$. Set

$$K_1 := \mathbf{G}^{-1}(i)\,, \quad K_2 := K_1^*\,, \quad K_3 := \mathbf{G}^{-1}(2i)\,, \quad K_4 := K_3^*\,,$$

and $\mathbf{K}^0(z) \equiv 1$, so that the transfer function $\mathbf{K}_{\tau,\varepsilon}$ of the controller $K_{\tau,\varepsilon}$ (see (6.97)) is

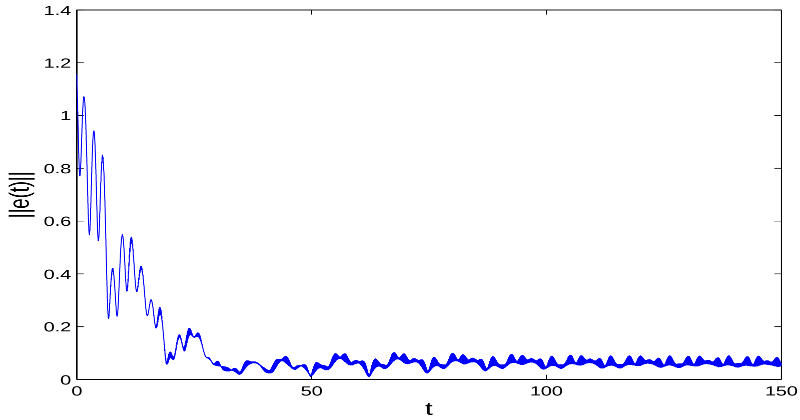

Figure 6-6: Norm of the error signal $e$.

given by

$$\begin{aligned}\mathbf{K}_{\tau,\varepsilon}(z) &:= \varepsilon\left(1+\frac{K_1}{z-e^{i\tau}}+\frac{K_2}{z-e^{-i\tau}}+\frac{K_3}{z-e^{2i\tau}}+\frac{K_4}{z-e^{-2i\tau}}\right)\\ &= \varepsilon\left(1+\frac{2z\mathrm{Re}\,(K_1)-2\mathrm{Re}\,(e^{-i\tau}K_1)}{z^2-2\cos(\tau)z+1}+\frac{2z\mathrm{Re}\,(K_3)-2\mathrm{Re}\,(e^{-2i\tau}K_3)}{z^2-2\cos(2\tau)z+1}\right),\end{aligned}$$

where $\mathrm{Re}\,K_j = (1/2)(K_j + K_j^*)$. Define the transfer functions $\mathbf{F}_1$ and $\mathbf{F}_2$ of filters $F_1$ and $F_2$, respectively, by

$$\mathbf{F}_1(s) = \mathbf{F}_2(s) := \frac{1}{3(s+1)^4}[-4(s^2+4)+(12s+7)(s^2+1)]I_2\,.$$

It is easy to compute that $\mathbf{F}_j(\pm i) = \mathbf{F}_j(\pm 2i) = I_2$ for $j = 1, 2$. Since all the relevant hypotheses are satisfied, the conclusions of Theorem 6.3.4 are valid. (For this example, it can be shown that $y_1(t)$ in Theorem 6.3.4 goes to 0 as $t \to \infty$.) Simulations are shown for the specific values

$$\xi_1 = 0.2\,,\ \xi_2 = 0.6\,,\ \eta_1 = 0.4\,,\ \eta_2 = 0.8\,,\ \tau = 0.1\,,\ \varepsilon = 0.01\,,$$

with zero initial conditions for the controller and the filters. The norm of the error signal $e = r-y$ is shown in Figure 6-6, and the output $y = (y^1, y^2)^T$ of the sampled-data system is shown in Figure 6-7. Asymptotically, the error $e$ is bounded by 0.088, that is, $\limsup_{t\geq 0}\|e(t)\| \leq 0.0882$. Simulations show that, for the sampling period $\tau = 0.1$, instability occurs at $\varepsilon \approx 0.013$.

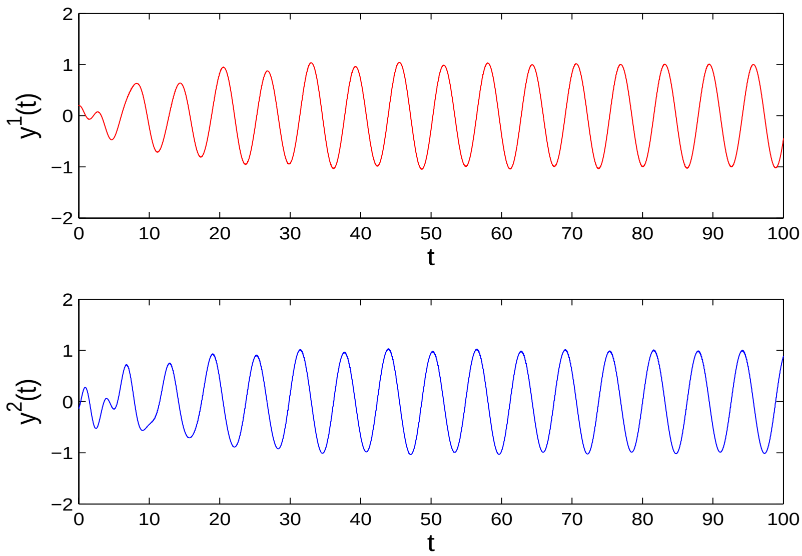


Figure 6-7: Output $y = (y^1, y^2)^T$.

## 6.4 Notes and references

Whilst the main results of Sections 6.1 and 6.2 are contained in Ke, Logemann and Rebarber [24], the results in Section 6.3 are contained in [26] by Ke, Logemann and Rebarber. To the best of our knowledge, Theorem 6.1.9 and Theorem 6.2.7 are new even for finite-dimensional systems. Theorem 6.1.9 and Theorem 6.1.20 are discrete-time counterparts of the continuous-time results in Rebarber and Weiss [65] (see Theorem 3.2 and Theorem 3.4 in [65]).

In Section 6.1, we make use of fractional representation theory which is a significant tool in the analysis and synthesis of feedback systems. This theory has been extensively developed and there is a wealth of literature, see, for example, [7], [32], [52], [73], [78], [79].

The important feature of condition (6.50) is that the only plant information needed is $\mathbf{G}(\xi_j)$, the transfer function evaluated at the frequencies of the reference and disturbance signals. In principle, $\mathbf{G}(\xi_j)$ can be calculated by performing frequency-response experiments on the plant. Moreover, the values of $\mathbf{G}(\xi_j)$ do not need to be known precisely, since condition (6.50) is robust with respect to small changes of $\mathbf{G}(\xi_j)$. If the impulse response of the plant is a Borel measure, then Lemma 6.2.4 can be used to estimate the value of $\mathbf{G}(\xi_j)$. If the plant is an exponentially stable regular system, then using suitable modifications of the input $t \mapsto e^{\xi t}$, $\mathbf{G}(\xi_j)$ can still be estimated by input-output experiments (see [16, Theorem 10]).

# Chapter 7

# Adaptive low-gain integral control of infinite-dimensional systems

An important issue in low-gain sampled-data control (as developed in Chapter 6) is the tuning of the gain parameter $\varepsilon$. In this chapter, we address this issue in the context of low-gain integral control (that is, the reference and disturbance signals are constants). In Section 7.1, an universal adaptive discrete-time low-gain control strategy is presented for tracking constant reference signals and rejecting constant disturbance signals for infinite-dimensional, discrete-time, power-stable, linear systems. The discrete-time results are applied in Section 7.2 in the development of universal adaptive sampled-data low-gain control for infinite-dimensional, well-posed, exponentially stable, linear systems. By "universal" we mean that the controllers are not based on system identification or plant parameter estimation algorithms. Our results considerably extend, improve and simplify previous work by Logemann and Townley [45].

## 7.1 Adaptive discrete-time low-gain control

Let $X$, $U$ and $Y$ be Hilbert spaces. Consider the discrete-time system

$$x(k+1) = Ax(k) + Bu(k)\,; \quad x(0) = x^0 \in X\,, \tag{7.1a}$$
$$y(k) = Cx(k) + Du(k)\,, \tag{7.1b}$$

where $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(U,X)$, $C \in \mathcal{B}(X,Y)$ and $D \in \mathcal{B}(U,Y)$. The transfer function of (7.1), $\mathbf{P} : \mathbb{C} \to \mathcal{B}(U,Y)$, is given by

$$\mathbf{P}(z) := C(zI - A)^{-1}B + D\,.$$

System (7.1) is called *power stable* if $A$ is power stable.

The aim of this section is to find an adaptive control law which achieves setpoint tracking in the presence of constant disturbances. To this end, consider the adaptive controller given by

$$\begin{aligned} u(k) &= Kw(k) + d\,, && (7.2a)\\ w(k+1) &= w(k) + \gamma^{-q}(k)(r - y(k))\,;\ w(0) = w^0, && (7.2b)\\ \gamma(k+1) &= \gamma(k) + \|r - y(k)\|^2\,;\ \gamma(0) = \gamma^0\,, && (7.2c) \end{aligned}$$

where $r \in Y$ is the reference vector, $d \in U$ is the disturbance vector, $K \in \mathcal{B}(Y, U)$ and $q \in (0, 1]$.

The following theorem is the main result of this section. It forms the discrete-time counterpart of the continuous-time result in [47].

**Theorem 7.1.1.** *Assume that* (7.1) *is power stable and that there exists* $K \in \mathcal{B}(Y, U)$ *such that*

$$\sigma(\mathbf{P}(1)K) \subset \mathbb{C}_0\,.$$

*Let* $q \in (0, 1]$. *Then, for all* $(x^0, w^0) \in X \times Y$, *all* $\gamma^0 > 0$, *all* $r \in Y$ *and all* $d \in U$, *the closed-loop system given by* (7.1) *and* (7.2) *has the following properties:*

(1) $r - y \in \ell^2(\mathbb{Z}_+, Y)$, *so in particular* $\lim_{k\to\infty} y(k) = r$;

(2) $\lim_{k\to\infty} \gamma(k) = \gamma^\infty < \infty$;

(3) $u - u^\infty \in \ell^2(\mathbb{Z}_+, U)$, *where* $u^\infty := K(\mathbf{P}(1)K)^{-1}[r - \mathbf{P}(1)d] + d$;

(4) $x - x^\infty \in \ell^2(\mathbb{Z}_+, X)$, *where* $x^\infty := (I - A)^{-1}Bu^\infty$.

*Proof.* We use a change of coordinates. Define

$$\begin{aligned} z(k) &:= x(k) - (I - A)^{-1}B[Kw(k) + d]\,, \quad \forall k \in \mathbb{Z}_+\,, && (7.3a)\\ v(k) &:= w(k) - (\mathbf{P}(1)K)^{-1}[r - \mathbf{P}(1)d]\,, \quad \forall k \in \mathbb{Z}_+\,. && (7.3b) \end{aligned}$$

Invoking the identity $A(I - A)^{-1} + I = (I - A)^{-1}$ together with (7.1)–(7.3), a routine calculation gives

$$\begin{aligned} z(k+1) &= x(k+1) - (I - A)^{-1}B[Kw(k+1) + d]\\ &= Ax(k) + Bu(k) - (I - A)^{-1}B[Kw(k+1) + d]\\ &= Az(k) + [A(I - A)^{-1} + I]B[Kw(k) + d] - (I - A)^{-1}B[Kw(k+1) + d]\\ &= Az(k) + (I - A)^{-1}BK[w(k) - w(k+1)]\\ &= Az(k) - \gamma^{-q}(k)\Gamma e(k)\,, \quad \forall k \in \mathbb{Z}_+\,, && (7.4) \end{aligned}$$

where $\Gamma := (I - A)^{-1}BK$ and $e := r - y$, and

$$\begin{aligned} v(k+1) &= w(k+1) - (\mathbf{P}(1)K)^{-1}[r - \mathbf{P}(1)d]\\ &= w(k) - (\mathbf{P}(1)K)^{-1}[r - \mathbf{P}(1)d] + \gamma^{-q}(k)e(k)\\ &= v(k) + \gamma^{-q}(k)e(k)\,, \quad \forall k \in \mathbb{Z}_+\,. && (7.5) \end{aligned}$$

Moreover, noting that $\mathbf{P}(1) = C(I-A)^{-1}B + D$, we have

$$\begin{aligned} e(k) = r - y(k) &= r - Cx(k) - Du(k) \\ &= r - Cz(k) - C(I-A)^{-1}B(Kw(k)+d) - D(Kw(k)+d) \\ &= -Cz(k) - \mathbf{P}(1)K[w(k) - (\mathbf{P}(1)K)^{-1}(r - \mathbf{P}(1)d)] \\ &= -[Cz(k) + \mathbf{P}(1)Kv(k)], \quad \forall k \in \mathbb{Z}_+ . \end{aligned} \tag{7.6}$$

Since $A$ is power stable and $\sigma(\mathbf{P}(1)K) \subset \mathbb{C}_0$, there exist $P \in \mathcal{B}(X)$, $P = P^*$, $P > 0$ and $Q \in \mathcal{B}(Y)$, $Q = Q^*$, $Q > 0$ such that

$$A^*PA - P = -I, \quad (\mathbf{P}(1)K)^*Q + Q(\mathbf{P}(1)K) = I, \tag{7.7}$$

(see [64, Proposition 5] and [71, Theorem 18, p. 231]). It follows from (7.4)–(7.7) and the Cauchy-Schwarz inequality that there exists $M_1 \geq 0$ such that, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned} &\langle z(k+1), Pz(k+1)\rangle - \langle z(k), Pz(k)\rangle \\ &\quad = \langle Az(k) - \gamma^{-q}(k)\Gamma e(k), P[Az(k) - \gamma^{-q}(k)\Gamma e(k)]\rangle - \langle z(k), Pz(k)\rangle \\ &\quad \leq \langle z(k), (A^*PA - P)z(k)\rangle + 2\gamma^{-q}(k)|\langle Az(k), P\Gamma e(k)\rangle| + M_1\gamma^{-2q}(k)\|e(k)\|^2 \\ &\quad \leq -\|z(k)\|^2 + 2\gamma^{-q}(k)|\langle Az(k), P\Gamma Cz(k)\rangle| + 2\gamma^{-q}(k)|\langle Az(k), P\Gamma\mathbf{P}(1)Kv(k)\rangle| \\ &\qquad\quad + M_1\gamma^{-2q}(k)\|e(k)\|^2 \\ &\quad \leq -\|z(k)\|^2 + M_1\gamma^{-q}(k)\|z(k)\|^2 + M_1\gamma^{-q}(k)\|z(k)\|\|v(k)\| + M_1\gamma^{-2q}(k)\|e(k)\|^2 \\ &\quad \leq \left[-1 + M_1\gamma^{-q}(k)\left(1 + \frac{\alpha}{2}\right)\right]\|z(k)\|^2 + \frac{M_1\gamma^{-q}(k)}{2\alpha}\|v(k)\|^2 + M_1\gamma^{-2q}(k)\|e(k)\|^2 , \end{aligned}$$

and

$$\begin{aligned} &\langle v(k+1), Qv(k+1)\rangle - \langle v(k), Qv(k)\rangle \\ &\quad = \langle v(k) + \gamma^{-q}(k)e(k), Q[v(k) + \gamma^{-q}(k)e(k)]\rangle - \langle v(k), Qv(k)\rangle \\ &\quad \leq -\gamma^{-q}(k)\langle v(k), [(\mathbf{P}(1)K)^*Q + Q(\mathbf{P}(1)K)]v(k)\rangle + 2\gamma^{-q}(k)|\langle v(k), QCz(k)\rangle| \\ &\qquad\quad + M_1\gamma^{-2q}(k)\|e(k)\|^2 \\ &\quad \leq -\gamma^{-q}(k)\|v(k)\|^2 + M_1\gamma^{-q}(k)\|z(k)\|\|v(k)\| + M_1\gamma^{-2q}(k)\|e(k)\|^2 \\ &\quad \leq \frac{M_1\gamma^{-q}(k)\alpha}{2}\|z(k)\|^2 + \left(-1 + \frac{M_1}{2\alpha}\right)\gamma^{-q}(k)\|v(k)\|^2 + M_1\gamma^{-2q}(k)\|e(k)\|^2 , \end{aligned}$$

where $\alpha > 0$ is arbitrary. Defining $V\colon \mathbb{Z}_+ \to \mathbb{R}$ by

$$V(k) := \langle z(k), Pz(k)\rangle + \langle v(k), Qv(k)\rangle ,$$

it follows that

$$\begin{aligned} V(k+1) - V(k) \;\leq\; & [-1 + M_1(1+\alpha)\gamma^{-q}(k)]\|z(k)\|^2 + (-1 + \frac{M_1}{\alpha})\gamma^{-q}(k)\|v(k)\|^2 \\ & + 2M_1\gamma^{-2q}(k)\|e(k)\|^2 , \quad \forall k \in \mathbb{Z}_+ . \end{aligned} \tag{7.8}$$

We first prove $\lim_{k\to\infty}\gamma(k) = \gamma^\infty < \infty$. To this end, it is sufficient to show that $\gamma$ is bounded since, by (7.2c), $\gamma$ is non-decreasing. Seeking a contradiction, suppose that $\gamma$ is not bounded. Then, since $q > 0$, $k \mapsto \gamma^{-q}(k)$ is monotonically decreasing and converging to 0. Hence, there exists $N_1 \in \mathbb{Z}_+$ such that

$$\gamma^{-q}(k) \leq \frac{1}{2M_1(1+2M_1)}\,, \quad \forall k \geq N_1\,.$$

Choosing $\alpha = 2M_1$, it follows from (7.8) that

$$V(k+1) - V(k) \leq -\frac{1}{2}(\|z(k)\|^2 + \gamma^{-q}(k)\|v(k)\|^2) + 2M_1\gamma^{-2q}(k)\|e(k)\|^2\,, \quad \forall k \geq N_1\,.$$

Note from (7.6) that

$$\|e(k)\|^2 = \|Cz(k) + \mathbf{P}(1)Kv(k)\|^2 \leq 2(\|Cz(k)\|^2 + \|\mathbf{P}(1)Kv(k)\|^2)\,, \quad \forall k \in \mathbb{Z}_+\,.$$

Consequently, there exists $M_2 \geq 0$ such that

$$\begin{aligned} V(k+1) - V(k) &\leq -4M_2\gamma^{-q}(k)(\|Cz(k)\|^2 + \|\mathbf{P}(1)Kv(k)\|^2) + 2M_1\gamma^{-2q}(k)\|e(k)\|^2 \\ &\leq [-2M_2 + 2M_1\gamma^{-q}(k)]\gamma^{-q}(k)\|e(k)\|^2\,, \quad \forall k \geq N_1\,. \end{aligned}$$

By the fact that $k \mapsto \gamma^{-q}(k)$ is monotonically decreasing and converging to 0 and (7.2c), there exists $N_2 \geq N_1$ such that

$$V(k+1) - V(k) \leq -M_2\gamma^{-q}(k)\|e(k)\|^2 = -M_2\gamma^{-q}(k)[\gamma(k+1) - \gamma(k)]\,, \quad \forall k \geq N_2\,.$$

Summing up over $k$, we obtain

$$V(k) - V(N_2) \leq -M_2 \sum_{j=N_2}^{k-1} \gamma^{-q}(j)[\gamma(j+1) - \gamma(j)]\,, \quad \forall k \geq N_2 + 1\,.$$

Since $s \mapsto s^{-q}$ is positive monotonically decreasing for $s > 0$ and the fact that $V$ is non-negative, it follows that

$$\begin{aligned} \int_{\gamma(N_2)}^{\gamma(k)} s^{-q} ds = \sum_{j=N_2}^{k-1} \int_{\gamma(j)}^{\gamma(j+1)} s^{-q} ds &\leq \sum_{j=N_2}^{k-1} \gamma^{-q}(j)[\gamma(j+1) - \gamma(j)] \\ &\leq \frac{V(N_2) - V(k)}{M_2} \\ &\leq \frac{V(N_2)}{M_2}\,, \quad \forall k \geq N_2 + 1\,. \end{aligned}$$

By the assumption that $q \in (0, 1]$, we conclude that $\gamma$ is bounded, contradicting our assumption. Hence $\gamma$ is bounded. This proves Statement (2). It follows immediately from (7.2c) that

$$r - y = e \in \ell^2(\mathbb{Z}_+, Y)\,. \tag{7.9}$$

In particular, $\lim_{k\to\infty} y(k) = r$. Thus Statement (1) is true. Since $A$ is power stable, statement (2) together with (7.4) and (7.9) imply

$$z \in \ell^2(\mathbb{Z}_+, X)\,, \tag{7.10}$$

so that $Cz \in \ell^2(\mathbb{Z}_+, Y)$. It follows from (7.6), (7.9) and the invertibility of $\mathbf{P}(1)K$ that $v \in \ell^2(\mathbb{Z}_+, Y)$. Invoking (7.2a) and (7.3b),

$$Kv = u - K(\mathbf{P}(1)K)^{-1}[r - \mathbf{P}(1)d] - d = u - u^\infty \in \ell^2(\mathbb{Z}_+, U)\,. \tag{7.11}$$

This completes the proof of statement (3). By (7.3a),

$$x - (I-A)^{-1}Bu^\infty = z + (I-A)^{-1}B(u - u^\infty)\,.$$

Then Statement (4) follows from (7.10) and (7.11). □

## 7.2 Adaptive sampled-data low-gain control

Consider a well-posed system with state-space $X$, input space $U$, and output space $Y$ (all Hilbert spaces), generating operators $(A, B, C)$, input-output operator $G$ and transfer function $\mathbf{G}$. For $x^0 \in X$ and $v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, the state $x$ and output $y$ corresponding to the initial condition $x(0) = x^0 \in X$ and the input function $v$ satisfy

$$\dot{x}(t) = Ax(t) + Bv(t)\,; \quad x(0) = x^0 \in X\,, \quad \text{for a.a. } t \geq 0\,, \tag{7.12a}$$

$$y(t) = C_\Lambda[x(t) - (\eta I - A)^{-1}Bv(t)] + \mathbf{G}(\eta)v(t)\,, \tag{7.12b}$$

and

$$x(t) = \mathbf{T}(t - t_0)x(t_0) + \int_{t_0}^{t} \mathbf{T}(t-s)Bv(s)ds\,, \quad \forall t_0 \geq 0\,,\ \forall t \geq t_0\,. \tag{7.13}$$

Let $\tau > 0$ be the sampling period and let $a \in L^2([0,\tau], \mathbb{R})$ be such that

$$\text{(i)} \int_0^\tau a(t)dt = 1\,, \qquad \text{(ii)} \int_0^\tau a(t)\mathbf{T}(t)z\,dt \in X_1\,, \quad \forall z \in X\,. \tag{7.14}$$

Whilst the above condition (ii) is difficult to check for general $a$, it can be shown by using integration by parts that (ii) holds if there exists a partition $0 = t_0 < t_1 < \cdots < t_m = \tau$ such that $a|_{(t_{j-1},t_j)} \in W^{1,1}((t_{j-1}, t_j), \mathbb{R})$ for $j = 1, 2, \ldots, m$. A simple example of $a$ satisfying (7.14) is that $a(t) \equiv 1/\tau$ for $t \in [0, \tau]$.

Define $L\colon X \to X_1$ by

$$Lz := \int_0^\tau a(t)\mathbf{T}(t)zdt\,. \tag{7.15}$$

**Lemma 7.2.1.** *Let $a \in L^2([0,\tau],\mathbb{R})$ satisfy (7.14). Then $L$ given by (7.15) is in $\mathcal{B}(X, X_1)$.*

*Proof.* We first show that $L$ is a closed linear operator. Let $(z_n)_{n\in\mathbb{Z}_+} \subset X_1$, $z \in X$ and $y \in X_1$ be such that $z_n \to z$ in $X$ and $Lz_n \to y$ in $X_1$ as $n \to \infty$. We need to prove that $Lz = y$. To this end, note that there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|\mathbf{T}(t)(z_n - z)\| \leq Me^{\omega t}\|z_n - z\|$. Therefore, by the Hölder's inequality,

$$\|Lz_n - Lz\| \leq M\int_0^\tau |a(t)|e^{\omega t}dt\,\|z_n - z\| \leq \frac{M\sqrt{e^{2\omega\tau}-1}\|a\|_{L^2}}{\sqrt{2\omega}}\|z_n - z\|\,.$$

showing that $Lz_n \to Lz$ in $X$ as $n \to \infty$. Since $Lz_n \to y$ in $X_1$ as $n \to \infty$, $Lz_n \to y$ in $X$ as $n \to \infty$. Hence, $Lz = y$. It follows from the closed-graph theorem that $L$ is in $\mathcal{B}(X, X_1)$. $\square$

We define a *generalized sampling operator* $\mathcal{S}\colon L^2_{\text{loc}}(\mathbb{R}_+, Y) \to F(\mathbb{Z}_+, Y)$ by

$$(\mathcal{S}y)(k) := \int_0^\tau a(t)y(k\tau + t)dt\,, \quad \forall k \in \mathbb{Z}_+\,, \tag{7.16}$$

where $F(\mathbb{Z}_+, Y)$ denotes the $Y$-valued functions defined on $\mathbb{Z}_+$, and define

$$\begin{pmatrix} A_\tau & B_\tau \\ C_\tau & D_\tau \end{pmatrix} := \begin{pmatrix} \mathbf{T}(\tau) & \int_0^\tau \mathbf{T}(s)dsB \\ CL & CLA^{-1}B + \mathbf{G}(0) \end{pmatrix}. \tag{7.17}$$

Trivially, $A_\tau \in \mathcal{B}(X)$. Moreover, $B_\tau = (\mathbf{T}(\tau) - I)A^{-1}B \in \mathcal{B}(U, X)$, and, by Lemma 7.2.1, $C_\tau \in \mathcal{B}(X, Y)$ and $D_\tau \in \mathcal{B}(U, Y)$.

**Proposition 7.2.2.** *Assume that (7.12) is exponentially stable and consider (7.12) with $v = \mathcal{H}_\tau u$, where $u$ is a function $\mathbb{Z}_+ \to U$ and $\mathcal{H}_\tau$ is the hold operator. Then*

$$x((k+1)\tau) = A_\tau x(k\tau) + B_\tau u(k)\,, \tag{7.18a}$$

$$(\mathcal{S}y)(k) = C_\tau x(k\tau) + D_\tau u(k)\,, \tag{7.18b}$$

*where $\mathcal{S}$ is the generalized sampling operator defined in (7.16). Moreover, $A_\tau$ is power stable and*

$$\mathbf{G}_\tau(1) = C_\tau(I - A_\tau)^{-1}B_\tau + D_\tau = \mathbf{G}(0)\,,$$

*where $\mathbf{G}_\tau$ denotes the transfer function of the discrete-time system (7.18).*

*Proof.* The equation (7.18a) follows easily from (7.13). To prove (7.18b), let $z \in X$. There exists $(z_n)_{n\in\mathbb{Z}_+} \subset X_1$ such that $z_n \to z$ as $n \to \infty$. Since $C \in \mathcal{B}(X_1, Y)$,

$$CLz_n = C\int_0^\tau a(t)\mathbf{T}(t)z_n dt = \int_0^\tau a(t)C\mathbf{T}(t)z_n dt\,, \quad \forall n \in \mathbb{Z}_+\,. \tag{7.19}$$

Using the admissibility of $C$, we have

$$\lim_{n\to\infty}\int_0^\tau \|C_\Lambda \mathbf{T}(t)(z_n - z)\|^2 dt = 0\,,$$

so that, letting $n \to \infty$ in (7.19), we conclude that

$$CLz = \int_0^\tau a(t) C_\Lambda \mathbf{T}(t) z dt\,. \tag{7.20}$$

Without loss of generality, we may choose $\eta = 0$ in (7.12b). By (7.13), we obtain that

$$\begin{aligned} y(k\tau + t) &= C_\Lambda \left[\mathbf{T}(t)x(k\tau) + \int_0^t \mathbf{T}(s)Bu(k)ds + A^{-1}Bu(k)\right] + \mathbf{G}(0)u(k) \\ &= C_\Lambda[\mathbf{T}(t)x(k\tau) + \mathbf{T}(t)A^{-1}Bu(k)] + \mathbf{G}(0)u(k)\,, \quad \forall k \in \mathbb{Z}_+\,,\ \forall t \in [0,\tau)\,. \end{aligned}$$

Consequently, it follows from (7.20) that,

$$\begin{aligned} (\mathcal{S}y)(k) &= \int_0^\tau a(t) y(k\tau + t) dt \\ &= \int_0^\tau a(t) C_\Lambda \mathbf{T}(t)[x(k\tau) + A^{-1}Bu(k)]dt + \mathbf{G}(0)u(k) \\ &= CLx(k\tau) + CLA^{-1}Bu(k) + \mathbf{G}(0)u(k) \\ &= C_\tau x(k\tau) + D_\tau u(k)\,, \quad \forall k \in \mathbb{Z}_+\,, \end{aligned}$$

showing that (7.18b) is true. Moreover, $A_\tau = \mathbf{T}(\tau)$ is power stable since $\mathbf{T}(t)$ is exponentially stable. Finally, since $B_\tau = (\mathbf{T}(\tau) - I)A^{-1}B$, it follows that

$$\mathbf{G}_\tau(1) = C_\tau(I - A_\tau)^{-1}B_\tau + D_\tau = -CLA^{-1}B + CLA^{-1}B + \mathbf{G}(0) = \mathbf{G}(0)\,. \qquad \square$$

We seek an adaptive controller which achieves setpoint tracking. To this end, consider the adaptive control law given by

$$\begin{aligned} v(t) &= (\mathcal{H}_\tau(Kw))(t) + d\,, && (7.21a) \\ w(k+1) &= w(k) + \gamma^{-q}(k)(r - (\mathcal{S}y)(k))\,; \quad w(0) = w^0, && (7.21b) \\ \gamma(k+1) &= \gamma(k) + \|r - (\mathcal{S}y)(k)\|^2\,; \quad \gamma(0) = \gamma^0\,, && (7.21c) \end{aligned}$$

where $(\mathcal{S}y)(k)$ is defined in (7.16), $r \in Y$ is the reference vector, $d \in U$ is the disturbance vector, $K \in \mathcal{B}(Y,U)$ and $q \in (0,1]$.

**Remark 7.2.3.** (1) We emphasize that for well-posed systems, ideal sampling of the output $y$ is in general not well-defined due to the potentially high irregularity of $y$ and therefore generalized sampling is unavoidable.

(2) Note that the control law (7.21) is "causal", in the sense that, in order to compute $v(t)$ for $t \in [k\tau, (k+1)\tau)$, we need to know $(\mathcal{S}y)(k-1)$, which is available at time $t = k\tau$. $\diamond$

**Theorem 7.2.4.** *Assume that the well-posed system (7.12) is exponentially stable, there exists $K \in \mathcal{B}(Y, U)$ such that*

$$\sigma(\mathbf{G}(0)K) \subset \mathbb{C}_0 \,, \tag{7.22}$$

*and $q \in (0, 1]$. Then, for all $(x^0, w^0) \in X \times Y$, all $\gamma^0 > 0$, all $\tau > 0$, all $r \in Y$ and all $d \in U$, the closed-loop sampled-data system given by (7.12) and (7.21) has the following properties:*

(1) $\lim_{k\to\infty} \gamma(k) = \gamma^\infty < \infty$;

(2) $\lim_{t\to\infty} v(t) = v^\infty$ *and* $v - v^\infty \in L^2(\mathbb{R}_+, U)$, *where* $v^\infty := K(\mathbf{G}(0)K)^{-1}(r - \mathbf{G}(0)d) + d$;

(3) $\lim_{t\to\infty} x(t) = x^\infty := -A^{-1}Bv^\infty$ *and* $x - x^\infty \in L^2(\mathbb{R}_+, X)$;

(4) *the error signal $e := r - y$ can be decomposed as $e = e_1 + e_2$, where*

$$\lim_{t\to\infty} e_1(t) = 0 \quad \textit{and} \quad e_2 \in L^2(\mathbb{R}_+, Y)\,;$$

(5) *under the additional assumption that*

$$\lim_{t\to\infty} (Gf)(t) = 0\,, \quad \forall f \in PC(\mathbb{R}_+, U) \cap L^2(\mathbb{R}_+, U) \textit{ with } \lim_{t\to\infty} f(t) = 0\,, \tag{7.23}$$

*where $PC(\mathbb{R}_+, U)$ denotes the set of piecewise continuous functions defined on $\mathbb{R}_+$ with value in $U$, the error signal $e = r - y$ can be decomposed as $e = e_1 + e_2$, where*

$$\lim_{t\to\infty} e_1(t) = 0 \quad \textit{and} \quad e_2 \in L^2_\alpha(\mathbb{R}_+, Y)\,, \quad \forall \alpha > \omega(\mathbf{T})\,.$$

*Furthermore, if (7.23) holds and $\mathbf{T}(t_0)(Ax^0 + BKw^0 + Bd) \in X$ for some $t_0 \geq 0$, then $\lim_{t\to\infty} e(t) = 0$.*

(6) *under the additional assumption that $U$ and $Y$ are finite-dimensional, the impulse response of $G$ is a (matrix-valued) Borel measure on $\mathbb{R}_+$ and $\mathbf{T}(t_0)x^0 \in X_1$ for some $t_0 \geq 0$, we have $\lim_{t\to\infty} e(t) = 0$.*

*Proof.* Let $(x^0, w^0) \in X \times Y$ and $\gamma^0 > 0$. Defining $u : \mathbb{Z}_+ \to U$ by

$$u(k) := Kw(k) + d\,, \tag{7.24}$$

it follows from (7.21a) that $v = \mathcal{H}_\tau u$. We obtain $x$, $y$, $(u(k))_{k\in\mathbb{Z}_+}$ and $(\gamma(k))_{k\in\mathbb{Z}_+}$ by applying (7.21) to (7.12). Set

$$x_k := x(k\tau)\,, \qquad y_k := (\mathcal{S}y)(k)\,, \quad \forall k \in \mathbb{Z}_+$$

where $\mathcal{S}$ is defined in (7.16). By assumption, (7.12) is exponentially stable. It follows from Proposition 7.2.2 that $(x_k)_{k\in\mathbb{Z}_+}$, $(u(k))_{k\in\mathbb{Z}_+}$ and $(y_k)_{k\in\mathbb{Z}_+}$ satisfy (7.18) with

$(A_\tau, B_\tau, C_\tau, D_\tau)$ given by (7.17), $A_\tau$ is power stable and, by (7.22),

$$\sigma(\mathbf{G}_\tau(1)K) = \sigma(\mathbf{G}(0)K) \subset \mathbb{C}_0 \,,$$

where $\mathbf{G}_\tau$ denotes the transfer function of the discrete-time system (7.18). Therefore, applying Theorem 7.1.1 to the discrete-time system (7.18) and the discrete-time controller given by (7.21b), (7.21c) and (7.24), we conclude that $\lim_{k\to\infty} \gamma(k) = \gamma^\infty$. This proves Statement (1). Moreover, setting

$$v^\infty := K(\mathbf{G}(0)K)^{-1}(r - \mathbf{G}(0)d) + d = K(\mathbf{G}_\tau(1)K)^{-1}(r - \mathbf{G}_\tau(1)d) + d \,,$$

we have

$$u - v^\infty \in \ell^2(\mathbb{Z}_+, U) \,, \quad \triangle u \in \ell^2(\mathbb{Z}_+, U) \,, \tag{7.25}$$

where $\triangle u : \mathbb{Z}_+ \to U$ is defined by $(\triangle u)(k) := u(k+1) - u(k)$. Hence, it is clear that $v - v^\infty = \mathcal{H}_\tau(u - v^\infty) \in L^2(\mathbb{R}_+, U)$ and

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} (\mathcal{H}_\tau u)(t) = v^\infty \,.$$

This completes the proof of Statement (2). To prove Statement (3), note that, for each $k \in \mathbb{N}$ and $t \in [k\tau, (k+1)\tau)$,

$$\begin{aligned} x(t) = \mathbf{T}(t)x^0 &+ \mathbf{T}(t - k\tau)\sum_{j=0}^{k-1}\int_{j\tau}^{(j+1)\tau} \mathbf{T}(k\tau - s)Bu(j)ds + \int_{k\tau}^{t} \mathbf{T}(t-s)Bu(k)ds \\ &- \int_0^t \mathbf{T}(t-s)Bv^\infty ds + \int_0^t \mathbf{T}(t-s)Bv^\infty ds \\ = \mathbf{T}(t)x^0 &+ \mathbf{T}(t-k\tau)[\mathbf{T}(\tau) - I]\sum_{j=0}^{k-1} \mathbf{T}((k-j-1)\tau)A^{-1}B(u(j) - v^\infty) \\ &+ [\mathbf{T}(t-k\tau) - I]A^{-1}B(u(k) - v^\infty) + [\mathbf{T}(t) - I]A^{-1}Bv^\infty \,. \end{aligned}$$

Consequently,

$$\begin{aligned} \|x(t) - x^\infty\| \quad \le \quad & \|\mathbf{T}(t)\|\|x^0\| + M\|A^{-1}B\|\|\mathbf{T}(\tau) - I\| \sum_{j=0}^{k-1} \|\mathbf{T}(k-1-j)\|\|u(j) - v^\infty\| \\ & + (M+1)\|A^{-1}B\|\|u(k) - v^\infty\| + \|\mathbf{T}(t)\|\|x^\infty\| \,, \\ & \forall t \in [k\tau, (k+1)\tau) \,,\ \forall k \in \mathbb{N} \,, \end{aligned}$$

where $M := \max_{t\in[0,\tau]} \|\mathbf{T}(t)\|$. Therefore Statement (3) follows from the exponential stability of $\mathbf{T}$ and the fact that $u - v^\infty \in \ell^2(\mathbb{Z}_+, U)$.

To prove Statement (4), define the integral operator $J$ by

$$(Jv)(t) := \int_0^t v(s)ds \,, \quad \forall v \in L^1_{\mathrm{loc}}(\mathbb{R}_+, U) \,,\ \forall t \in \mathbb{R}_+ \,,$$

and define the function $\theta\colon \mathbb{R}_+ \to \mathbb{R}$ by $\theta(t) := 1$ for all $t \in \mathbb{R}_+$. For every $t \in \mathbb{R}_+$, let $k_t \in \mathbb{Z}_+$ be such that $t \in [k_t\tau, (k_t+1)\tau)$. Then,

$$\begin{aligned}(J\mathcal{H}_\tau(\triangle u))(t) &= \sum_{j=0}^{k_t-1}\int_{j\tau}^{(j+1)\tau}(\mathcal{H}_\tau(\triangle u))(s)ds + \int_{k_t\tau}^{t}(\mathcal{H}_\tau(\triangle u))(s)ds \\ &= \tau\sum_{j=0}^{k_t-1}[u(j+1)-u(j)] + (t-k_t\tau)(\mathcal{H}_\tau(\triangle u))(t) \\ &= \tau(\mathcal{H}_\tau u)(t) - \tau\theta(t)u(0) + h(t)\,, \quad \forall t \geq 0\,, \end{aligned} \tag{7.26}$$

where $h(t) := (t-k_t\tau)(\mathcal{H}_\tau(\triangle u))(t)$ for all $t \geq 0$. It follows from (7.26) that

$$\begin{aligned}GJ\mathcal{H}_\tau(\triangle u) - \mathbf{G}(0)J\mathcal{H}_\tau(\triangle u) = \tau G(\mathcal{H}_\tau u) - \tau\mathbf{G}(0)(\mathcal{H}_\tau u) - \tau G(\theta u(0)) \\ + \tau\mathbf{G}(0)\theta u(0) + Gh - \mathbf{G}(0)h\,.\end{aligned}$$

Consequently, setting

$$e_1 := -\frac{1}{\tau}(GJ - \mathbf{G}(0)J)\mathcal{H}_\tau(\triangle u) - \frac{1}{\tau}\mathbf{G}(0)h + r - \mathbf{G}(0)\mathcal{H}_\tau u\,, \tag{7.27}$$

and

$$e_2 := -C_\Lambda\mathbf{T}(t)x^0 - [G(\theta u(0)) - \mathbf{G}(0)\theta u(0)] + \frac{1}{\tau}Gh\,, \tag{7.28}$$

it follows that

$$e = r - y = r - C_\Lambda\mathbf{T}(t)x^0 - G(\mathcal{H}_\tau u) = e_1 + e_2\,.$$

We first prove that $\lim_{t\to\infty} e_1(t) = 0$. Noting that

$$s \mapsto [\mathscr{L}(GJ - \mathbf{G}(0)J)](s) = s \mapsto \frac{1}{s}(\mathbf{G}(s) - \mathbf{G}(0)) \in H^\infty(\mathbb{C}_0, \mathcal{B}(U,Y))\,,$$

it follows that $GJ - \mathbf{G}(0)J \in \mathcal{B}(L^2(\mathbb{R}_+,U), L^2(\mathbb{R}_+,Y))$. By (7.25), we see that $\mathcal{H}_\tau(\triangle u) \in L^2(\mathbb{R}_+,U)$. Hence

$$(GJ - \mathbf{G}(0)J)\mathcal{H}_\tau(\triangle u) \in L^2(\mathbb{R}_+,Y)\,. \tag{7.29}$$

Moreover, since, by shift-invariance, $G$ and $J$ commute,

$$[(GJ - \mathbf{G}(0)J)\mathcal{H}_\tau(\triangle u)]' = (G - \mathbf{G}(0))\mathcal{H}_\tau(\triangle u) \in L^2(\mathbb{R}_+,Y)\,. \tag{7.30}$$

As a consequence of (7.29) and (7.30), we obtain

$$\lim_{t\to\infty}[(GJ - \mathbf{G}(0)J)\mathcal{H}_\tau(\triangle u)](t) = 0\,. \tag{7.31}$$

Moreover, (7.25) implies that

$$h \in L^2(\mathbb{R}_+,U) \cap PC(\mathbb{R}_+,U)\,, \quad \lim_{t\to\infty} h(t) = 0\,, \tag{7.32}$$

and

$$\lim_{t\to\infty} \mathbf{G}(0)(\mathcal{H}_\tau u)(t) = \mathbf{G}(0)v^\infty = r\,. \tag{7.33}$$

Combining (7.27), (7.31)–(7.33) gives $\lim_{t\to\infty} e_1(t) = 0$. We proceed to prove that $e_2 \in L^2(\mathbb{R}_+, Y)$. Obviously,

$$C_\Lambda \mathbf{T} x^0 \in L^2_\alpha(\mathbb{R}_+, Y)\,, \quad \forall \alpha > \omega(\mathbf{T})\,,\ \forall x^0 \in X\,. \tag{7.34}$$

Now

$$[\mathscr{L}(G(\theta u(0)) - \mathbf{G}(0)\theta u(0))](s) = \frac{1}{s}[\mathbf{G}(s) - \mathbf{G}(0)]u(0)\,,$$

and we see that

$$\mathscr{L}(G(\theta u(0)) - \mathbf{G}(0)\theta u(0)) \in H^2(\mathbb{C}_\alpha, U)\,, \quad \forall \alpha > \omega(\mathbf{T})\,.$$

Hence, by the Paley-Wiener theorem,

$$G(\theta u(0)) - \mathbf{G}(0)\theta u(0) \in L^2_\alpha(\mathbb{R}_+, U)\,, \quad \forall \alpha > \omega(\mathbf{T})\,. \tag{7.35}$$

Using $G \in \mathcal{B}(L^2(\mathbb{R}_+, U), L^2(\mathbb{R}_+, Y))$ and $h \in L^2(\mathbb{R}_+, U)$ (see (7.32)), we have $Gh \in L^2(\mathbb{R}_+, Y)$. Combining this with (7.28), (7.34), (7.35) and the exponential stability of $\mathbf{T}$, yields that $e_2 \in L^2(\mathbb{R}_+, Y)$. This completes the proof of Statement (4).

To prove the first claim of Statement (5), we assume that $(Gf)(t) \to 0$ as $t \to 0$ for all $f \in PC(\mathbb{R}_+, U) \cap L^2(\mathbb{R}_+, U)$ with $\lim_{t\to\infty} f(t) = 0$. Then, by (7.32), we have

$$\lim_{t\to\infty} (Gh)(t) = 0\,. \tag{7.36}$$

Writing $e = \tilde{e}_1 + \tilde{e}_2$, where

$$\tilde{e}_1 := \frac{1}{\tau} Gh - \frac{1}{\tau}(GJ - \mathbf{G}(0)J)\mathcal{H}_\tau(\triangle u) - \frac{1}{\tau}\mathbf{G}(0)h + r - \mathbf{G}(0)\mathcal{H}_\tau u\,,$$

and

$$\tilde{e}_2 := -C_\Lambda \mathbf{T}(t)x^0 - [G(\theta u(0)) - \mathbf{G}(0)\theta u(0)]\,, \tag{7.37}$$

it follows from (7.31)–(7.33) and (7.36) that $\lim_{t\to\infty} \tilde{e}_1(t) = 0$, and from (7.34) and (7.35) that $\tilde{e}_2 \in L^2_\alpha(\mathbb{R}_+, Y)$ for all $\alpha > \omega(\mathbf{T})$. This proves the first claim of Statement (5). To prove the second claim of Statement (5), it suffices to show that $\lim_{t\to\infty} \tilde{e}_2(t) = 0$ under the extra assumption that such that $\mathbf{T}(t_0)(Ax^0 + BKw^0 + Bd) \in X$ for some $t_0 \geq 0$. Laplace transform of (7.37) gives

$$(\mathscr{L}(\tilde{e}_2))(s) = -C(sI - A)^{-1}x^0 - \frac{1}{s}[\mathbf{G}(s) - \mathbf{G}(0)]u(0)\,.$$

It follows from (2.17) with $\eta = 0$ that

$$\frac{1}{s}[\mathbf{G}(s) - \mathbf{G}(0)] = C(sI - A)^{-1}A^{-1}B\,,$$

so that

$$(\mathscr{L}(\tilde{e}_2))(s) = -C(sI-A)^{-1}A^{-1}[Ax^0 + B(Kw^0+d)]\,.$$

Since $\mathbf{T}(t_0)[Ax^0 + B(Kw^0+d)] \in X$,

$$\begin{aligned}\tilde{e}_2(t) &= -C_\Lambda \mathbf{T}(t)A^{-1}[Ax^0 + B(Kw^0+d)] \\ &= -CA^{-1}\mathbf{T}(t-t_0)\mathbf{T}(t_0)[Ax^0 + B(Kw^0+d)]\,, \quad t \geq t_0\,.\end{aligned}$$

By the exponential stability of $\mathbf{T}$, $\lim_{t\to\infty} \tilde{e}_2(t) = 0$. This completes the proof of Statement (5).

Finally, assume that $U$ and $Y$ are finite-dimensional, the impulse response of $G$ is a (matrix-valued) Borel measure on $\mathbb{R}_+$ and $\mathbf{T}(t_0)x^0 \in X_1$ for some $t_0 \geq 0$. Using Lemma 6.2.4, we know that $\lim_{t\to\infty}(Gv)(t) = \mathbf{G}(0)v^\infty = r$, so that

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}\big(r - (Gv)(t) - C\mathbf{T}(t-t_0)\mathbf{T}(t_0)x^0\big) = 0\,. \qquad \square$$

**Remark 7.2.5.** (1) The proof for Statement (4) is inspired by the proof of [6, Proposition 7.3.4, p. 131].

(2) Statement (4) (first claim of Statement (5), respectively) in Theorem 7.2.4 shows that the error signal $e$ becomes small in the sense that $e = e_1 + e_2$, where $e_1 \to 0$ as $t \to \infty$ and $e_2 \in L^2(\mathbb{R}_+, Y)$ ($e_2 \in L^2_\alpha(\mathbb{R}_+, Y)$ for $\alpha > \omega(\mathbf{T})$, respectively). This implies, in particular, we have "tracking in measure", i.e., for every $\varepsilon > 0$,

$$\lim_{T\to\infty} \mu_L(\{t \geq T : \|e(t)\| \geq \varepsilon\}) = 0\,,$$

where $\mu_L$ denotes the Lebesgue measure on $\mathbb{R}_+$. The second claim of Statement (5) and Statement (6) show that "asymptotic tracking" (i.e., $\lim_{t\to\infty} e(t) = 0$) is guaranteed under certain conditions.

(3) If $\mathbf{T}$ is analytic, then

$$\mathbf{T}(t_0)[Ax^0 + B(Kw^0+d)] = A\mathbf{T}(t_0)[x^0 + A^{-1}B(Kw^0+d)] \in X$$

for all $t_0 > 0$, all $x^0 \in X$, all $w^0 \in Y$ and all $d \in U$, since $\mathbf{T}(t)$ maps $X$ into $X_1$ for all $t > 0$. $\diamond$

**Example 7.2.6.** For purpose of illustration, we consider the problem of heating a bar of length 1. We keep both endpoints at temperature 0 and inject heat of magnitude $v_j(t)$ at the point $\xi_j \in (0,1)$, $j = 1,2$. Temperature measurements are taken at the points $\eta_1, \eta_2 \in (0,1)$. The system to be controlled can be formulated as follows

$$z_t(\xi,t) = \kappa z_{\xi\xi}(\xi,t) + \delta(\xi-\xi_1)v_1(t) + \delta(\xi-\xi_2)v_2(t)\,, \quad \forall \xi \in (0,1)\,,\ \forall t > 0\,, \tag{7.38a}$$

$$y_1(t) = z(\eta_1,t)\,, \quad y_2(t) = z(\eta_2,t)\,; \quad \forall t > 0\,, \tag{7.38b}$$

$$z(0,t) = z(1,t) = 0\,, \quad \forall t \geq 0\,; \quad z(\xi,0) = z^0(\xi)\,, \quad \forall \xi \in (0,1)\,. \tag{7.38c}$$

Here $\kappa$ is a positive constant and $\delta(\cdot)$ denotes the Dirac delta function. Non-adaptive continuous-time low-gain integral control of this system was studied in [33].

System (7.38) can be formulated as a well-posed system with the state space $X = L^2((0,1),\mathbb{R})$. In particular, the semigroup $\mathbf{T}(t)$, given by

$$(\mathbf{T}(t)z^0)(\xi) = \sum_{n=1}^{\infty} 2\exp(-\kappa n^2\pi^2 t)\sin(n\pi\xi)\int_0^1 \sin(n\pi\lambda)z^0(\lambda)d\lambda\,,$$

is exponentially stable. Assuming that

$$0 < \xi_1 \leq \eta_1 \leq \xi_2 \leq \eta_2 < 1\,,$$

the transfer function $\mathbf{G}(s)$ is given by

$$\mathbf{G}(s) = \begin{pmatrix} \dfrac{\sinh((1-\eta_1)\sqrt{s/\kappa})\sinh(\xi_1\sqrt{s/\kappa})}{\sqrt{s\kappa}\sinh(\sqrt{s/\kappa})} & \dfrac{\sinh((1-\xi_2)\sqrt{s/\kappa})\sinh(\eta_1\sqrt{s/\kappa})}{\sqrt{s\kappa}\sinh(\sqrt{s/\kappa})} \\ \dfrac{\sinh((1-\eta_2)\sqrt{s/\kappa})\sinh(\xi_1\sqrt{s/\kappa})}{\sqrt{s\kappa}\sinh(\sqrt{s/\kappa})} & \dfrac{\sinh((1-\eta_2)\sqrt{s/\kappa})\sinh(\xi_2\sqrt{s/\kappa})}{\sqrt{s\kappa}\sinh(\sqrt{s/\kappa})} \end{pmatrix}.$$

It is then easy to see that

$$\mathbf{G}(0) = \frac{1}{\kappa}\begin{pmatrix} (1-\eta_1)\xi_1 & (1-\xi_2)\eta_1 \\ (1-\eta_2)\xi_1 & (1-\eta_2)\xi_2 \end{pmatrix}.$$

As a consequence, the characteristic polynomial of $\mathbf{G}(0)$ is given by

$$\det(\lambda I - \mathbf{G}(0)) = \lambda^2 - \kappa^{-1}[(1-\eta_1)\xi_1 + (1-\eta_2)\xi_2]\lambda + \kappa^{-2}\xi_1(1-\eta_2)(\xi_2-\eta_1)\,.$$

Since $\xi_1, \xi_2, \eta_1, \eta_2 \in (0,1)$, it follows that $\sigma(\mathbf{G}(0)) \subset \mathbb{C}_0$ if and only if $\xi_2 > \eta_1$. We sample the output using the simple averaging sampling operation defined by

$$(\mathcal{S}y)(k) = \frac{1}{\tau}\int_0^\tau y(k\tau + t)dt\,, \quad \text{(i.e., } a(t) \equiv 1/\tau \text{ in (7.16))}\,.$$

To be specific, we set

$$\xi_1 = 0.2\,,\ \xi_2 = 0.6\,,\ \eta_1 = 0.4\,,\ \eta_2 = 0.8\,,\ \tau = 1\,,\ K = I\,,\ \kappa = 0.1\,,$$

$$z^0(\xi) = \sin(\pi\xi)\,,\ r = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\,,\ d = \begin{pmatrix} -1 \\ 0 \end{pmatrix}\,,\ q = 0.55\,,\ \gamma^0 = 2\,,\ w^0 = 0\,.$$

MATLAB simulations of the closed-loop system given by (7.38) and (7.21) (with $v = (v_1, v_2)^T$ and $y = (y_1, y_2)^T$) are shown in Figures 7-1 to 7-3. By Theorem 7.2.4, we

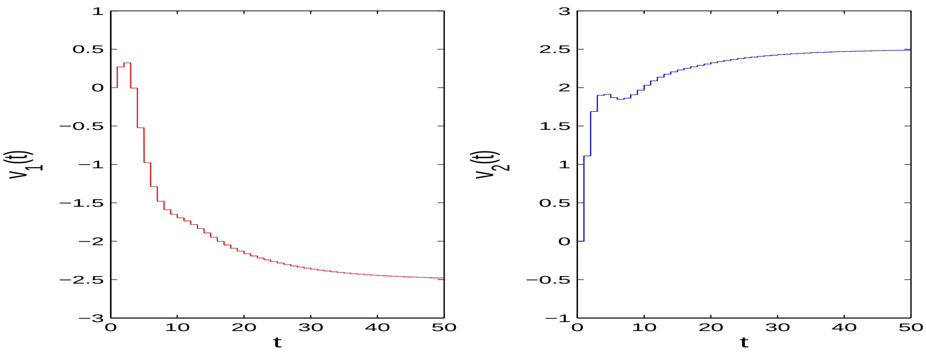


Figure 7-1: Input signals $v_1$, $v_2$.

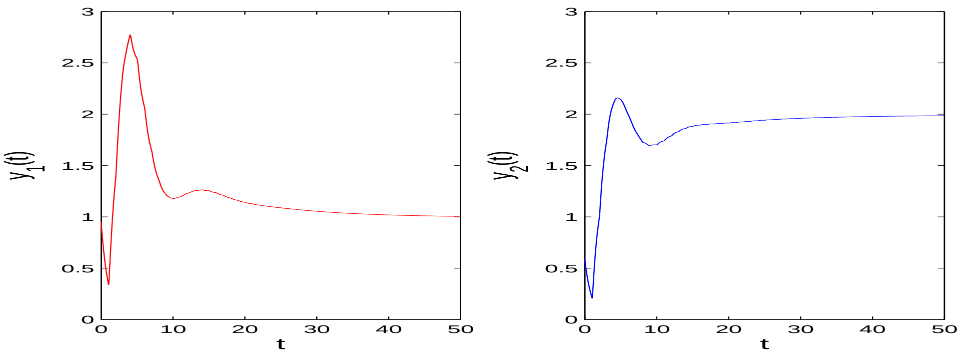


Figure 7-2: Temperature measurements $y_1$, $y_2$.

know that

$$\lim_{t\to\infty} v(t) = (\mathbf{G}(0))^{-1} r = \begin{pmatrix} -2.5 \\ 2.5 \end{pmatrix},$$

as is illustrated by Figure 7-1. It can be shown that the impulse response of $\mathbf{G}$ is in $L^1(\mathbb{R}_+, \mathbb{R}^{2\times 2})$ (see [51, Appendix 6]). It follows from Statement (6) in Theorem 7.2.4 that

$$\lim_{t\to\infty} y(t) = r = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

as is illustrated by Figure 7-2. The sequence $\gamma$ and the evolution of the temperature profile are shown in Figure 7-3 and Figure 7-4, respectively.

## 7.3 Notes and references

For continuous-time low-gain integral control of continuous-time systems, there have been two basic approaches to the tuning of $\varepsilon$ - either steady-state data from the plant is used off-line to determine suitable ranges for the gain $\varepsilon$ (see, for example, Davison [8],

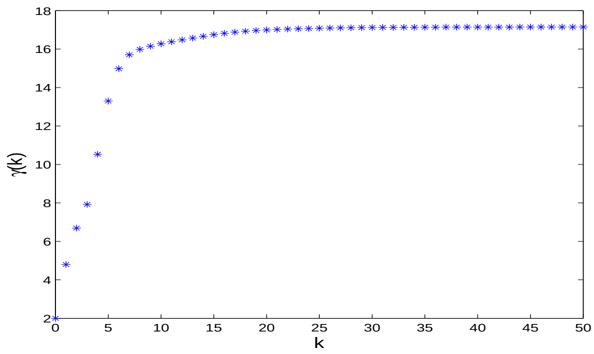


Figure 7-3: Sequence $\gamma$.

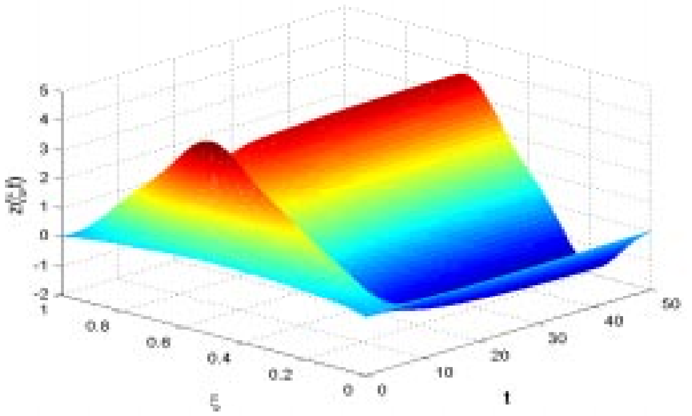


Figure 7-4: Temperature $z(\xi, t)$.

Logemann, Ryan and Townley [43], or Lunze [49]), or simple on-line universal adaptive tuning of $\varepsilon$ is used (see Cook [4], Miller and Davison [53, 54] in the finite-dimensional case and Logemann and Ryan [40, 41], Logemann, Ryan and Townley [42], Logemann and Townley [44, 46, 47] in the infinite-dimensional case).

Whilst universal adaptive continuous-time control of infinite-dimensional systems has developed to some extent (see, for example, Logemann and Ilchmann [34], Logemann and Mårtensson [35], Logemann and Townley [44] and Townley [77]), to the best of our knowledge, the only only result on universal adaptive discrete-time control of discrete-time infinite-dimensional systems is contained in the note by Logemann and Mårtensson [36] which is an extension of an earlier finite-dimensional result by Mårtensson [50], and in Logemann and Townley [45].

Most of the results in this chapter are contained in Ke, Logemann and Townley [27]. The main results Theorem 7.1.1 and Theorem 7.2.4 are new. The coordinates changing technique plays a key role in the proof of Theorem 7.1.1. It has also been used in [42] and [47] (see the proofs of Theorem 3.3 in [42] and Theorem 3.1 in [47]).

Theorem 7.1.1 improves [45, Theorem 3.2] on adaptive low-gain control of discrete-time systems in the following aspects:

- In [45], it is assumed that the steady-state gain matrix $\mathbf{P}(1)$ is symmetric and positive definite, where $\mathbf{P}$ is the transfer of the discrete-time plant. This symmetry assumption is restrictive and highly nonrobust, essentially limiting the applications of the above result to single-input single-output systems. Theorem 7.1.1 replace this assumption by the considerably weaker (and essentially necessary) assumption that all the eigenvalues of $\mathbf{P}(1)$ have positive real parts.

- The range of the parameter $q$ is $(0, 1]$ in (7.2) instead of $(0, 1/2)$ in [45].

- In comparing the analysis presented in the proof of Theorem 7.1.1 to that in [45], we use a change of coordinates technique which is the discrete-time counterpart to that used in [47], leading to a dramatic simplification of the proof.

- We allow for a constant input disturbance which is not considered in [45].

Our results in Theorem 7.2.4 are extensions and improvements of those in [45] with respect to the following aspects:

- The continuous-time plant is assumed to belong to the class of exponentially stable well-posed systems, which is considerably more general than the class of exponentially stable regular systems considered in [45].

- In [45], it is assumed that $\mathbf{G}(0)$ is symmetric and positive definite, where $\mathbf{G}$ denotes the transfer of the continuous-time plant. As discussed above, this assumption is restrictive and highly nonrobust. In Theorem 7.2.4, we only assume that the eigenvalues of $\mathbf{G}(0)$ have positive real parts.

- The simple averaging sampling operator used in [45] is a special case of the generalized sampling operator $\mathcal{S}$ defined in (7.16).

- The range of the parameter $q$ is $(0, 1]$ instead of $(0, 1/2)$ in [45].

- The analysis of the behaviour of the tracking error has been considerably improved, see Statements (4)-(6) of Theorem 7.2.4.

- We allow for a constant input disturbance which is not considered in [45].

# Appendix

## A.1 Pathological and non-pathological sampling periods

The concept of pathological and non-pathological sequences (relative to a given square matrix) has been defined in Definition 5.2.7. Given a positive sequence $(\tau_j)_{j\in\mathbb{Z}_+}$, define $\mathcal{D} \subset \mathbb{R}^{n\times n}$ by

$$\mathcal{D} := \{A \in \mathbb{R}^{n\times n} : (\tau_j)_{j\in\mathbb{Z}_+} \text{ is non-pathological relative to } A\}\,.$$

Set $\sigma^+(A) := \sigma(A) \cap \mathrm{cl}(\mathbb{C}_0)$.

**Theorem A.1.1.** *The follows statements hold for $\mathcal{D}$:*

(1) *$\mathcal{D}$ is non-empty;*

(2) *$\mathcal{D}$ is dense in $\mathbb{R}^{n\times n}$;*

(3) *if the set $\{k/\tau_j : k \in \mathbb{Z}\setminus\{0\}\,,\ j \in \mathbb{Z}_+\}$ has no accumulation points in $\mathbb{R}$, then $\mathcal{D}$ is open.*

*Proof.* Define

$$\mathrm{dist}(u, V) := \inf\{|u - v| : v \in V\}\,,$$

the distance between a point $u \in \mathbb{C}$ and a set $V \subset \mathbb{C}$, and

$$\mathrm{dist}(U, V) := \inf\{|u - v| : u \in U\,,\ v \in V\}\,,$$

the distance between two sets $U, V \subset \mathbb{C}$.

We proceed in three steps.

*Step 1: Non-emptiness of $\mathcal{D}$.*

Obviously, $\{A \in \mathbb{R}^{n\times n} : \sigma^+(A) = \emptyset\} \subset \mathcal{D}$. Alternatively, if all elements in $\sigma^+(A)$ are real, then $A \in \mathcal{D}$, i.e., $\{A \in \mathbb{R}^{n\times n} : \sigma^+(A) \subset \mathbb{R}\} \subset \mathcal{D}$. This proves Statement (1).

*Step 2: Density of $\mathcal{D}$.*

Let $A \in \mathbb{R}^{n\times n} \setminus \mathcal{D}$. Assume that $A$ has $m$ real eigenvalues $\lambda_1, \ldots, \lambda_m$ and $2\ell$ non-real eigenvalues $\alpha_1 \pm i\beta_1, \ldots, \alpha_\ell \pm i\beta_\ell$, counting multiplicities in each case, so that $m+2\ell = n$.

There exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that $A = T^{-1}JT$, where $J$ is in the real Jordan canonical form (see [69, p. 159]), i.e., $J = \mathrm{diag}(R, C)$, where $R$ and $C$ are of the form:

$$R = \begin{pmatrix} \lambda_1 & \gamma_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & \lambda_{m-1} & \gamma_{m-1} \\ 0 & 0 & \cdots & 0 & \lambda_m \end{pmatrix},$$

and

$$C = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_m & 0 & \cdots & 0 & 0 \\ -\beta_1 & \alpha_1 & 0 & \gamma_m & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_{\ell-1} & \beta_{\ell-1} & \gamma_{m+\ell-2} & 0 \\ 0 & 0 & \cdots & -\beta_{\ell-1} & \alpha_{\ell-1} & 0 & \gamma_{m+\ell-2} \\ 0 & 0 & \cdots & 0 & 0 & \alpha_\ell & \beta_\ell \\ 0 & 0 & \cdots & 0 & 0 & -\beta_\ell & \alpha_\ell \end{pmatrix}.$$

Here $\gamma_j$ ($j \in \underline{m+\ell-2}$) takes value of either 0 or 1 (depending on $A$). Define

$$V := \left\{ \frac{2k\pi i}{\tau_j} : k \in \mathbb{Z} \setminus \{0\} \,,\; j \in \mathbb{Z}_+ \right\}.$$

For $j \in \underline{\ell}$, choose $(\xi_{j,k})_{k\in\mathbb{N}} \subset \mathbb{R}$ such that

$$2(\beta_j + \xi_{j,k})i \notin V \,, \quad \forall k \in \mathbb{N} \quad \text{and} \quad \lim_{k\to\infty} \xi_{j,k} = 0 \,. \tag{A.1}$$

To prove the density of $\mathcal{D}$, it is sufficient to show that there exist $(\Delta_k)_{k\in\mathbb{N}} \subset \mathbb{R}^{n\times n}$ such that $A + \Delta_k \in \mathcal{D}$ for sufficiently large $k$ and $\lim_{k\to\infty} \Delta_k = 0$. For $k \in \mathbb{N}$, define $P_k \in \mathbb{R}^{n\times n}$ by $P_k := \mathrm{diag}(1/k, \ldots, 1/(mk), P_{k,1}, \ldots, P_{k,\ell})$, where

$$P_{k,j} := \begin{pmatrix} \dfrac{1}{(m+j)k} & \xi_{j,k} \\ -\xi_{j,k} & \dfrac{1}{(m+j)k} \end{pmatrix}, \quad \forall j \in \underline{\ell},$$

and let $\Delta_k := T^{-1}P_kT$. Note that $\lim_{k\to\infty} \Delta_k = 0$, since $\lim_{k\to\infty} P_k = 0$. Moreover, a simple calculation yields

$$\sigma(A + \Delta_k) = \left\{ \lambda_j + \frac{1}{jk} : j \in \underline{m} \right\} \bigcup \left\{ \left( \alpha_j + \frac{1}{(m+j)k} \right) \pm i(\beta_j + \xi_{j,k}) : j \in \underline{\ell} \right\}.$$

Let $\lambda, \mu \in \sigma(A + \Delta_k)$.

*Case 1:* $\lambda \neq \bar{\mu}$.

We see that $\mathrm{Re}\,\lambda \neq \mathrm{Re}\,\mu$ for sufficiently large $k$. Hence $\lambda - \mu \notin V$ for sufficiently large $k$.

*Case 2:* $\lambda = \bar{\mu}$.

By (A.1), it is clear that

$$\lambda - \mu = 2\mathrm{Im}\,\lambda \notin V\,, \quad \forall k \in \mathbb{N}\,.$$

Combining the above two cases completes the proof of Statement (2).

*Step 3: Openness of* $\mathcal{D}$.

Let $A \in \mathcal{D}$. We consider two cases: $\sigma^+(A) \neq \emptyset$ and $\sigma^+(A) = \emptyset$.

*Case 1:* $\sigma^+(A) \neq \emptyset$.

Define

$$U := \{\lambda - \mu : \lambda, \mu \in \sigma^+(A)\}\,.$$

By assumption, $\{k/\tau_j : k \in \mathbb{Z} \setminus \{0\}\,,\ j \in \mathbb{Z}_+\}$ has no accumulation points in $\mathbb{R}$. Hence, $V$ has no accumulation points in $\mathbb{C}$, and thus $V$ is closed. It follows immediately from the closedness of $V$, the fact that $U \cap V = \emptyset$ and the finiteness of $U$ that

$$d_1 := \mathrm{dist}(U, V) > 0\,.$$

Recall that $\mathbb{C}_- := \{s \in \mathbb{C} : \mathrm{Re}\,s < 0\}$. We set

$$d := \begin{cases} \min\{d_1, \min\{|\mathrm{Re}\,\lambda| : \lambda \in \sigma(A) \cap \mathbb{C}_-\}\}\,, & \text{if } \sigma(A) \cap \mathbb{C}_- \neq \emptyset \\ d_1\,, & \text{if } \sigma(A) \cap \mathbb{C}_- = \emptyset \end{cases}\,.$$

By perturbation theory, the mapping $A \mapsto \sigma(A)$ is continuous in the sense of [18] (see [18, Corollary 4.2.1, p. 399]). Therefore, there exists $\delta > 0$ such that, for every $\Delta \in \mathbb{R}^{n\times n}$ with $\|\Delta\| \leq \delta$,

$$\sigma(A + \Delta) \subset \bigcup_{\lambda \in \sigma(A)} \mathbb{B}(\lambda, d/4)\,, \tag{A.2}$$

where $\mathbb{B}(\lambda, d/4)$ denotes the open disk centered at $\lambda$ with radius $d/4$. Let $\Delta \in \mathbb{R}^{n\times n}$ with $\|\Delta\| \leq \delta$. We claim that $A + \Delta \in \mathcal{D}$. Assume that $\sigma^+(A + \Delta) \neq \emptyset$ (otherwise there is nothing to show). If $\sigma(A) \cap \mathbb{C}_- = \emptyset$, then $\sigma(A) = \sigma^+(A)$. It is clear from (A.2) that

$$\sigma^+(A + \Delta) \subset \bigcup_{\lambda \in \sigma^+(A)} \mathbb{B}(\lambda, d/4)\,. \tag{A.3}$$

If $\sigma(A) \cap \mathbb{C}_- \neq \emptyset$, then let $\lambda \in \sigma(A) \cap \mathbb{C}_-$ and $\lambda' \in \sigma^+(A + \Delta)$. We have

$$|\lambda - \lambda'|^2 = (|\mathrm{Re}\,\lambda| + \mathrm{Re}\,\lambda')^2 + |\mathrm{Im}\,(\lambda - \lambda')|^2 \geq |\mathrm{Re}\,\lambda|^2 \geq d^2\,,$$

showing that $\lambda' \notin \mathbb{B}(\lambda, d/4)$. Combining this with (A.2), we see that again (A.3) holds. Let $\lambda', \mu' \in \sigma^+(A + \Delta)$ and set $u' := \lambda' - \mu'$. By (A.3), there exist $\lambda, \mu \in \sigma^+(A)$ such that

$$|\lambda - \lambda'| \leq \frac{d}{4}, \quad |\mu - \mu'| \leq \frac{d}{4}.$$

Setting $u := \lambda - \mu \in U$, we have

$$|u' - u| \leq |\lambda' - \lambda| + |\mu' - \mu| \leq \frac{d}{2}.$$

Then, for any $v \in V$,

$$|u' - v| = |(u - v) - (u - u')| \geq |u - v| - |u' - u| \geq d - \frac{d}{2} = \frac{d}{2} > 0.$$

Hence $u' \notin V$. Consequently $A + \Delta \in \mathcal{D}$.

*Case 2:* $\sigma^+(A) = \emptyset$, or equivalently, $\sigma(A) \subset \mathbb{C}_-$.

Again, by perturbation theory, we know that there exists $\delta > 0$ such that, for every $\Delta \in \mathbb{R}^{n\times n}$ with $\|\Delta\| \leq \delta$, we have $\sigma(A+\Delta) \subset \mathbb{C}_-$. Hence $A+\Delta \in \mathcal{D}$ for every $\Delta$ with $\|\Delta\| \leq \delta$.

Combining the above two cases, we conclude that $\mathcal{D}$ is open. □

Alternatively, we can use the pole-shifting theorem to prove the denseness of $\mathcal{D}$ in $\mathbb{R}^{n\times n}$.

*Alternative proof of Statement (2) of Theorem A.1.1.* Define

$$\mathcal{C} := \{(A, b) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n : (A, b) \text{ is controllable}\},$$

and

$$V := \left\{ \frac{2k\pi i}{\tau_j} : k \in \mathbb{Z} \setminus \{0\},\ j \in \mathbb{Z}_+ \right\}.$$

Let $A \in \mathbb{R}^{n\times n} \setminus \mathcal{D}$. The aim is to show that, for every $\delta > 0$, there exists $\Delta \in \mathbb{R}^{n\times n}$ with $\|\Delta\| \leq \delta$ such that $A + \Delta \in \mathcal{D}$.

*Case 1:* There exists $b \in \mathbb{R}^n$ such that $(A, b) \in \mathcal{C}$.

Write the characteristic polynomial of $A$ as

$$p(s) = \det(sI - A) = \prod_{j=1}^{m} (s - \lambda_j) \prod_{j=1}^{\ell} [s - (\alpha_j \pm i\beta_j)],$$

where

$$\lambda_j \in \mathbb{R},\ \forall j \in \underline{m}; \quad \alpha_j, \beta_j \in \mathbb{R},\ \beta_j \neq 0,\ \forall j \in \underline{\ell}; \quad m + 2\ell = n.$$

For $j \in \underline{\ell}$, choose $(\xi_{j,k})_{k\in\mathbb{N}} \subset \mathbb{R}$ such that

$$2(\beta_j + \xi_{j,k})i \notin V\,, \quad \forall k \in \mathbb{N} \quad \text{and} \quad \lim_{k\to\infty} \xi_{j,k} = 0\,.$$

Define

$$p_k(s) := \prod_{j=1}^{m} \left[ s - \left( \lambda_j + \frac{1}{kj} \right) \right] \prod_{j=1}^{\ell} \left[ s - \left( \alpha_j + \frac{1}{(m+j)k} \pm i(\beta_j + \xi_{j,k}) \right) \right]\,.$$

Since $(A, b)$ is controllable, by the pole-shifting Theorem (see, for example, [71, Theorem 13, p. 186]), there exists $f_k \in \mathbb{R}^n$ such that

$$\det(sI - A - bf_k^T) = p_k(s)\,, \quad \forall k \in \mathbb{N}\,,$$

that is, for all $k \in \mathbb{N}$,

$$\sigma(A + bf_k) = \left\{ \lambda_j + \frac{1}{jk} : j \in \underline{m} \right\} \bigcup \left\{ \left( \alpha_j + \frac{1}{(m+j)k} \right) \pm i(\beta_j + \xi_{j,k}) : j \in \underline{\ell} \right\}\,.$$

Let $\lambda, \mu \in \sigma(A + bf_k^T)$.

If $\lambda \neq \bar{\mu}$, then we see that $\operatorname{Re}\lambda \neq \operatorname{Re}\mu$ for sufficiently large $k$. Hence $\lambda - \mu \notin V$ for sufficiently large $k$. If $\lambda = \bar{\mu}$, then it is clear that

$$\lambda - \mu = 2\operatorname{Im}\lambda \notin V\,, \quad \forall k \in \mathbb{N}\,.$$

Hence, we conclude that $A + bf_k^T \in \mathcal{D}$ for sufficiently large $k$. It is now sufficient to prove that $\lim_{j\to\infty} bf_k^T = 0$. Note that the coefficients of $p_k(s)$ converge to the corresponding coefficients of $p(s)$. By the proof of the the pole-shifting theorem (see [71, p. 186]), or by Ackermann's formula (see [71, Exercise 5.1.12, p. 188]) combined with the Cayley-Hamilton theorem, we conclude that

$$\lim_{j\to\infty} f_k = 0\,,$$

showing that $\lim_{j\to\infty} bf_k^T = 0$.

*Case 2:* There does not exist $b \in \mathbb{R}^n$ such that $(A, b) \in \mathcal{C}$.

Let $\delta > 0$. Since $\mathcal{C}$ is open and dense in $\mathbb{R}^{n\times n} \times \mathbb{R}^n$ (see [71, Proposition 3.3.12, p. 97]), there exists $(A_1, b_1) \in \mathcal{C}$ such that $\|A_1 - A\| \leq \delta/2$. If $A_1 \in \mathcal{D}$, then there nothing to show. If $A_1 \notin \mathcal{D}$, then, by Case 1, there exists $f \in \mathbb{R}^n$ such that

$$A_1 + b_1 f^T \in \mathcal{D} \quad \text{and} \quad \|b_1 f^T\| \leq \frac{\delta}{2}\,.$$

Therefore

$$\|A_1 + b_1 f^T - A\| \leq \|A_1 - A\| + \|b_1 f^T\| \leq \delta\,. \qquad \square$$

**Corollary A.1.2.** *If* $\lim_{j\to\infty} \tau_j = 0$*, then* $\mathcal{D}$ *is open.*

*Proof.* If $\lim_{j\to\infty} \tau_j = 0$, then $k/\tau_j \to \pm\infty$ as $k \to \pm\infty$ and $j \to \infty$, showing that $\{k/\tau_j : k \in \mathbb{Z} \setminus \{0\}\,,\ j \in \mathbb{Z}_+\}$ has no accumulation points in $\mathbb{R}$. Invoking Theorem A.1.1 completes the proof. $\square$

In the following, we give conditions on $(\tau_j)_{j\in\mathbb{Z}_+}$ in terms of spectral data of $A$, which guarantee that $(\tau_j)_{j\in\mathbb{Z}_+}$ is non-pathological relative to $A$.

Trivially, if $\sigma^+(A) = \emptyset$, every positive sequence $(\tau_j)_{j\in\mathbb{Z}_+}$ is non-pathological relative to $A$. Assume that $\sigma^+(A) \neq \emptyset$ and set

$$\mathrm{Im}\,(\sigma^+(A)) := \{\mathrm{Im}\,\lambda : \lambda \in \sigma^+(A)\}\,, \quad \mathrm{Im}\,(\sigma(A)) := \{\mathrm{Im}\,\lambda : \lambda \in \sigma(A)\}\,,$$

and

$$\omega^+ := \max\left\{\frac{|\mathrm{Im}\,\lambda|}{2\pi} : \lambda \in \sigma^+(A)\right\}\,, \quad \omega := \max\left\{\frac{|\mathrm{Im}\,\lambda|}{2\pi} : \lambda \in \sigma(A)\right\}\,.$$

We call $\omega$ to be the *maximum frequency* of $A$. The *diameter* of a bounded set $U \subset \mathbb{C}$ is defined as

$$\mathrm{diam}(U) := \sup\{|u - v| : u, v \in U\}\,.$$

It is easy to check that

$$4\pi\omega^+ = \mathrm{diam}(\mathrm{Im}\,(\sigma^+(A)))\,, \quad 4\pi\omega = \mathrm{diam}(\mathrm{Im}\,(\sigma(A)))\,,$$

since $\sigma^+(A)$ and $\sigma(A)$ are symmetric with respect to the real line.

**Proposition A.1.3.** *Let* $A \in \mathbb{R}^{n\times n}$ *and assume that* $\sigma^+(A) \neq \emptyset$*. If one of the following conditions is satisfied,*

(1) $\tau_j < \dfrac{2\pi}{\mathrm{diam}(\mathrm{Im}\,(\sigma^+(A)))} = \dfrac{1}{2\omega^+}, \quad \forall j \in \mathbb{Z}_+,$

(2) $\tau_j < \dfrac{2\pi}{\mathrm{diam}(\mathrm{Im}\,\sigma(A))} = \dfrac{1}{2\omega}, \quad \forall j \in \mathbb{Z}_+,$

(3) $\tau_j < \dfrac{2\pi}{\mathrm{diam}(\sigma^+(A))}, \quad \forall j \in \mathbb{Z}_+,$

(4) $\tau_j < \dfrac{2\pi}{\mathrm{diam}(\sigma(A))}, \quad \forall j \in \mathbb{Z}_+,$

(5) $\tau_j < \dfrac{\pi}{r(A)}, \quad \forall j \in \mathbb{Z}_+,$

*then* $(\tau_j)_{j\in\mathbb{Z}_+}$ *is non-pathological relative to* $A$*.*

*Proof.* If condition (1) is satisfied, then

$$\tau_j < \frac{2k\pi}{|\mathrm{Im}\,\lambda - \mathrm{Im}\,\mu|}, \quad \forall \lambda, \mu \in \sigma^+(A), \ \forall k \in \mathbb{N}, \ \forall j \in \mathbb{Z}_+ .$$

Hence $(\tau_j)_{j\in\mathbb{Z}_+}$ is non-pathological relative to $A$.
For $\lambda, \mu \in \sigma(A)$, we have

$$|\mathrm{Im}\,\lambda - \mathrm{Im}\,\mu| \le |\lambda - \mu| \le |\lambda| + |\mu| .$$

Hence

$$\mathrm{diam}(\mathrm{Im}\,(\sigma^+(A))) \le \mathrm{diam}(\mathrm{Im}\,(\sigma(A))) = 4\pi\omega \le \mathrm{diam}(\sigma(A)) \le 2r(A) ,$$

showing that

$$\frac{\pi}{r(A)} \le \frac{2\pi}{\mathrm{diam}(\sigma(A))} \le \frac{2\pi}{\mathrm{diam}(\mathrm{Im}\,(\sigma(A)))} = \frac{1}{2\omega} \le \frac{2\pi}{\mathrm{diam}(\mathrm{Im}\,(\sigma^+(A)))} .$$

Moreover, it is easy to see that

$$\frac{2\pi}{\mathrm{diam}(\sigma^+(A))} \le \frac{2\pi}{\mathrm{diam}(\mathrm{Im}\,(\sigma^+(A)))} .$$

Therefore, if one of the conditions (2)-(5) is satisfied, then condition (1) is satisfied, so that $(\tau_j)_{j\in\mathbb{Z}_+}$ is non-pathological relative to $A$. □

## A.2 Stabilizability and detectability under sampling

Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function, i.e., a function analytic on the entire complex plane. Then the power series expansion of $f$ around 0,

$$f(z) = \sum_{j=0}^{\infty} a_j z^j ,$$

converges for every $z \in \mathbb{C}$. For a matrix $A \in \mathbb{R}^{n\times n}$, we define $f(A)$ by

$$f(A) := \sum_{j=0}^{\infty} a_j A^j .$$

This series converges absolutely for every $A \in \mathbb{R}^{n\times n}$.

**Theorem A.2.1** (Spectral mapping theorem)**.** *Assume that $f$ is an entire function. Then, for $A \in \mathbb{R}^{n\times n}$,*

$$\sigma(f(A)) = \{f(\lambda) : \lambda \in \sigma(A)\} .$$

For a proof of the spectral mapping theorem, see, for example, [71, Appendix A.3, p. 454].

Let $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}$ and let $\tau > 0$. Set

$$A_\tau := e^{A\tau}\,, \quad B_\tau := \int_0^\tau e^{As} ds B \quad \text{and} \quad \sigma^+(A) := \sigma(A) \cap \mathrm{cl}(\mathbb{C}_0)\,.$$

**Theorem A.2.2.** *Assume that $(A, B)$ is stabilizable and $(C, A)$ is detectable. If $\tau$ is non-pathological relative to $A$, then $(A_\tau, B_\tau)$ is discrete-time stabilizable and $(C, A_\tau)$ is discrete-time detectable, i.e., there exist $F \in \mathbb{R}^{m\times n}$ and $H \in \mathbb{R}^{n\times p}$ such that $A_\tau + B_\tau F$ and $A_\tau + HC$ are power stable.*

*Proof.* By the spectral mapping theorem,

$$\sigma(A_\tau) = \{e^{\lambda\tau} : \lambda \in \sigma(A)\}\,.$$

Hence

$$\{s \in \sigma(A_\tau) : |s| \geq 1\} = \{e^{\lambda\tau} : \lambda \in \sigma^+(A)\}\,. \tag{A.4}$$

If $\sigma^+(A) = \emptyset$, then, by (A.4), $\{s \in \sigma(A_\tau) : |s| \geq 1\} = \emptyset$. Consequently,

$$\mathrm{rk}\,(sI - A_\tau, B_\tau) = \mathrm{rk}\begin{pmatrix} sI - A_\tau \\ C \end{pmatrix} = n\,, \quad \forall s \in \mathbb{C}\,,\ |s| \geq 1\,.$$

By the Hautus criterion, $(A_\tau, B_\tau)$ is discrete-time stabilizable and $(C, A_\tau)$ is discrete-time detectable. In the following, we assume that $\sigma^+(A) \neq \emptyset$. By the Hautus criterion and (A.4), discrete-time stabilizability of $(A_\tau, B_\tau)$ and discrete-time detectability of $(C, A_\tau)$ is equivalent to

$$\mathrm{rk}\,(e^{\lambda\tau} I - A_\tau, B_\tau) = \mathrm{rk}\begin{pmatrix} e^{\lambda\tau} I - A_\tau \\ C \end{pmatrix} = n\,, \quad \forall \lambda \in \sigma^+(A)\,.$$

Let $\lambda \in \sigma^+(A)$ be arbitrary. Define $f, g : \mathbb{C} \to \mathbb{C}$ by

$$f(s) := \begin{cases} \dfrac{e^{s\tau} - 1}{s}\,, & s \neq 0 \\ \tau\,, & s = 0 \end{cases}\,, \quad g(s) := \begin{cases} \dfrac{e^{s\tau} - e^{\lambda\tau}}{s - \lambda}\,, & s \neq \lambda \\ \tau e^{\lambda\tau}\,, & s = \lambda \end{cases}\,.$$

Note that $f, g$ are entire functions. Furthermore,

$$Af(A) = f(A)A\,, \quad Ag(A) = g(A)A\,.$$

By assumption, $\tau$ is non-pathological relative to $A$, and thus,

$$\frac{2k\pi i}{\tau} \notin \sigma(A)\,, \quad \lambda + \frac{2k\pi i}{\tau} \notin \sigma(A)\,, \quad \forall k \in \mathbb{Z} \setminus \{0\}\,.$$

It follows from the spectral mapping theorem that $0 \notin \sigma(f(A))$ and $0 \notin \sigma(g(A))$. Thus $f(A)$ and $g(A)$ are invertible. Let $g_1(s) := e^{s\tau} - e^{\lambda\tau}$ and $g_2(s) := s - \lambda$. We have $g_1(s) = g(s)g_2(s)$. Since $g$ and $g_2$ are entire functions, we conclude that $g_1(A) = g(A)g_2(A)$, i.e.,

$$e^{\lambda\tau}I - A_\tau = g(A)(\lambda I - A)\,.$$

It is easy to see that $B_\tau = f(A)B$. Hence

$$\begin{aligned}(e^{\lambda\tau}I - A_\tau, B_\tau) &= (g(A)(\lambda I - A), f(A)B)\\ &= f(A)(\lambda I - A, B)\begin{pmatrix}(f(A))^{-1}g(A) & 0\\ 0 & I\end{pmatrix},\end{aligned} \tag{A.5}$$

and

$$\begin{pmatrix}e^{\lambda\tau}I - A_\tau\\ C\end{pmatrix} = \begin{pmatrix}g(A)(\lambda I - A)\\ C\end{pmatrix} = \begin{pmatrix}g(A) & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}\lambda I - A\\ C\end{pmatrix}. \tag{A.6}$$

Since $f(A), g(A)$ are invertible, the matrices

$$f(A)\,,\quad \begin{pmatrix}(f(A))^{-1}g(A) & 0\\ 0 & I\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}g(A) & 0\\ 0 & I\end{pmatrix}$$

have full ranks. Moreover, by assumption, $(A, B)$ is stabilizable and $(C, A)$ is detectable. By the Hautus criterion,

$$\mathrm{rk}\,(\lambda I - A, B) = \mathrm{rk}\begin{pmatrix}\lambda I - A\\ C\end{pmatrix} = n\,.$$

Therefore, by (A.5) and (A.6),

$$\mathrm{rk}\,(e^{\lambda\tau}I - A_\tau, B_\tau) = \mathrm{rk}\begin{pmatrix}e^{\lambda\tau}I - A_\tau\\ C\end{pmatrix} = n\,. \qquad \square$$

**Remark A.2.3.** (1) The above proof is essentially due to [11, Lemma 8]. In [11], the discrete-time stabilizability of $(A_\tau, B_\tau)$ was proved, and it was mentioned without proof that the discrete-time detectability of $(C, A_\tau)$ can be shown in a similar way.

(2) It is a standard result that if $(A, B)$ is controllable, $(C, A)$ is observable and

$$\tau(\lambda - \mu) \neq 2k\pi i\,, \quad \forall \lambda, \mu \in \sigma(A)\,,\ \forall k \in \mathbb{Z}_+ \setminus \{0\}\,, \tag{A.7}$$

then $(A_\tau, B_\tau)$ is discrete-time controllable and $(C, A_\tau)$ is discrete-time observable. The proof can be found in [22, Theorem 12] or [2, Theorem 3.2.1, p. 41]. Condition (A.7) is sometimes called the "Kalman-Ho-Narenda" criterion. $\diamond$

## A.3 Asymptotic behaviour of functions in $W^{1,p}(\mathbb{R}_+, \mathbb{R}^n)$

For $p \in [1, \infty]$, the Sobolev space $W^{1,p}(\mathbb{R}_+, \mathbb{R}^n)$ has been defined in Definition 2.3.4.

**Proposition A.3.1.** *Let $1 \leq p < \infty$. Then*

$$\lim_{t \to \infty} u(t) = 0\,, \quad \forall u \in W^{1,p}(\mathbb{R}_+, \mathbb{R})\,.$$

*Proof.* First we consider $p = 1$. Assume $u \in W^{1,1}(\mathbb{R}_+, \mathbb{R})$. It follows from the fundamental theorem of calculus for absolutely continuous functions that

$$u(t) = u(0) + \int_0^t \dot{u}(s) ds\,, \quad \forall t \geq 0\,.$$

Letting $t \to \infty$, the right-hand side converges since $\dot{u} \in L^1(\mathbb{R}_+, \mathbb{R})$. Thus $\lim_{t \to 0} u(t) = a$ for some $a \in \mathbb{R}$. Since $u \in L^1(\mathbb{R}_+, \mathbb{R})$, it is clear that $a = 0$.

Next, consider $p \in (1, \infty)$. Let $u \in W^{1,p}(\mathbb{R}_+, \mathbb{R})$. Define

$$\Omega := \{t \in \mathbb{R}_+ : u \text{ is differentiable at } t\}\,.$$

Note that the set $\mathbb{R}_+ \setminus \Omega$ has zero measure, since $u$ is differentiable almost everywhere. We define

$$\Omega_0 := \{t \in \Omega : u(t) = 0, \dot{u}(t) \neq 0\} \subset \Omega\,.$$

Setting $v := |u|$, it follows from the absolute continuity of $u$ and the triangle inequality that $v$ is absolutely continuous. Hence $v$ is differentiable almost everywhere. We want to show that $v$ is not differentiable in $\Omega_0$, but is differentiable in $\Omega \setminus \Omega_0$. To this end, let $t_0 \in \Omega_0$. Since $v(t_0) = 0$, we have

$$\frac{v(t_0 + h) - v(t_0)}{h} = \frac{|u(t_0 + h)|}{h} = \operatorname{sgn}(h) \left| \frac{u(t_0 + h)}{h} \right|\,,$$

where sgn denotes the sign function. Hence

$$\lim_{h \uparrow 0} \frac{v(t_0 + h) - v(t_0)}{h} = -|\dot{u}(t_0)| \neq |\dot{u}(t_0)| = \lim_{h \downarrow 0} \frac{v(t_0 + h) - v(t_0)}{h}\,,$$

showing that $v$ is not differentiable at $t_0 \in \Omega_0$. Consequently, $\Omega_0$ has zero measure, since $v$ is differentiable almost everywhere. Next let $t_1 \in \Omega \setminus \Omega_0$.

*Case 1:* $u(t_1) = \dot{u}(t_1) = 0$.

Then

$$\frac{dv}{dt}(t_1) = \lim_{h \to 0} \frac{|u(t_1 + h)| - |u(t_1)|}{h} = \lim_{h \to 0} \left( \operatorname{sgn}(h) \left| \frac{u(t_1 + h)}{h} \right| \right) = 0\,.$$

*Case 2:* $u(t_1) \neq 0$.

Then for sufficiently small $h$, $u(t_1+h)$ and $u(t_1)$ have the same sign. Hence

$$\begin{aligned}\frac{dv}{dt}(t_1) = \lim_{h\to 0}\frac{|u(t_1+h)|-|u(t_1)|}{h} &= \operatorname{sgn}(u(t_1))\lim_{h\to 0}\frac{u(t_1+h)-u(t_1)}{h}\\ &= \operatorname{sgn}(u(t_1))\,\dot u(t_1)\,.\end{aligned}$$

Consequently,

$$f_u(t) := \begin{cases}\operatorname{sgn}(u(t))\,\dot u(t)\,, & t\in\Omega\setminus\Omega_0\\ 0\,, & \text{elsewhere}\end{cases}$$

it follows that

$$\frac{dv}{dt}(t) = f_u(t)\,,\quad \text{a.e. } t\in\mathbb{R}_+\,.$$

Since $\mathbb{R}_+\setminus\Omega$ and $\Omega_0$ have zero measure, $\mathbb{R}_+\setminus(\Omega\setminus\Omega_0) = (\mathbb{R}_+\setminus\Omega)\cup\Omega_0$ has zero measure and thus

$$\frac{d(v^p)}{dt}(t) = pv^{p-1}(t)f_u(t)\,,\quad \text{a.e. } t\in\mathbb{R}_+\,.$$

Hence, by the fundamental theorem of calculus for absolutely continuous functions

$$|u(t)|^p = |u(0)|^p + p\int_0^t |u(s)|^{p-1}f_u(s)ds\,,\quad \forall t\in\mathbb{R}_+\,. \tag{A.8}$$

Let $q$ be such that $(1/p)+(1/q)=1$. The Hölder inequality yields

$$\begin{aligned}\int_0^t |u(s)|^{p-1}|f_u(s)|ds &\le \int_0^\infty |u(s)|^{p-1}|\dot u(s)|ds\\ &\le \left(\int_0^\infty |u(s)|^{(p-1)q}ds\right)^{1/q}\left(\int_0^\infty |\dot u(s)|^p ds\right)^{1/p}\\ &= \|u\|_{L^p}^{p/q}\|\dot u\|_{L^p} = \|u\|_{L^p}^{p-1}\|\dot u\|_{L^p}\,,\quad \forall t\in\mathbb{R}_+\,,\end{aligned} \tag{A.9}$$

showing that $|u|^{p-1}f_u\in L^1(\mathbb{R}_+,\mathbb{R})$. Therefore, the right-hand side of (A.8) converges as $t\to\infty$. Thus $u(t)$ has a limit as $t\to\infty$ and this limit must be 0, since $u\in L^p(\mathbb{R}_+,\mathbb{R})$. □

## A.4 Routine calculations for dynamic output feedback systems

### A.4.1 Continuous-time systems

Consider the continuous-time closed-loop system, where the plant is given by

$$\dot x_p(t) = A_p x_p(t) + B_p u_p(t)\,;\quad x_p(0) = x_p^0\in\mathbb{R}^{n_p}\,, \tag{A.10a}$$

$$y_p(t) = C_p x_p(t) + D_p u_p(t)\,, \tag{A.10b}$$

where $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_c \in \mathbb{R}^{n_p \times m}$, $C_c \in \mathbb{R}^{p \times n_p}$ and $D_c \in \mathbb{R}^{p \times m}$. The discrete-time controller is given by

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c u_c(t) ; \quad x_c(0) = x_c^0 \in \mathbb{R}^{n_c} , && \text{(A.11a)} \\ y_c(t) &= C_c x_c(t) + D_c u_c(t) , && \text{(A.11b)} \end{aligned}$$

where $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times p}$, $C_c \in \mathbb{R}^{m \times n_c}$ and $D_c \in \mathbb{R}^{m \times p}$.

The interconnection of (A.10) and (A.11) is given by

$$u_p = y_c , \quad u_c = y_p . \tag{A.12}$$

By (A.10)–(A.12),

$$u_p = y_c = C_c x_c + D_c u_c = C_c x_c + D_c y_p = C_c x_c + D_c (C_p x_p + D_p u_p) ,$$

and

$$u_c = y_p = C_p x_p + D_p u_p = C_p x_p + D_p y_c = C_p x_p + D_p (C_c x_c + D_c u_c) .$$

Setting $E_p := (I - D_c D_p)^{-1}$ and $E_c := (I - D_p D_c)^{-1}$, it follows that

$$u_p = E_p (D_c C_p x_p + C_c x_c) , \quad u_c = E_c (C_p x_p + D_p C_c x_c) .$$

Consequently, (A.10a) and (A.11a) can be written as

$$\begin{pmatrix} \dot{x}_p \\ \dot{x}_c \end{pmatrix} = \left[ \begin{pmatrix} A_p & 0 \\ 0 & A_c \end{pmatrix} + \begin{pmatrix} B_p & 0 \\ 0 & B_c \end{pmatrix} \begin{pmatrix} E_p & 0 \\ 0 & E_c \end{pmatrix} \begin{pmatrix} D_c & I \\ I & D_p \end{pmatrix} \begin{pmatrix} C_p & 0 \\ 0 & C_c \end{pmatrix} \right] \begin{pmatrix} x_p \\ x_c \end{pmatrix} .$$

### A.4.2 Discrete-time systems

Consider the discrete-time closed-loop system, where the plant is given by

$$\begin{aligned} x_p(k+1) &= A_p x_p(k) + B_p u_p(k) ; \quad x_p(0) = x_p^0 \in \mathbb{R}^{n_p} , && \text{(A.13a)} \\ y_p(k) &= C_p x_p(k) + D_p u_p(k) , && \text{(A.13b)} \end{aligned}$$

where $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_c \in \mathbb{R}^{n_p \times m}$, $C_c \in \mathbb{R}^{p \times n_p}$ and $D_c \in \mathbb{R}^{p \times m}$. Let $\varepsilon > 0$ be a parameter. The discrete-time controller is given by

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + B_c u_c(k) ; \quad x_c(0) = x_c^0 \in \mathbb{R}^{n_c} , && \text{(A.14a)} \\ y_c(k) &= \varepsilon C_c x_c(k) + \varepsilon D_c u_c(k) , && \text{(A.14b)} \end{aligned}$$

where $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times p}$, $C_c \in \mathbb{R}^{m \times n_c}$ and $D_c \in \mathbb{R}^{m \times p}$. The interconnection of (A.13) and (A.14) is given by

$$u_p = d + y_c , \quad u_c = r - y_p , \tag{A.15}$$

where $r$ is a reference signal and $d$ is a disturbance signal. Set $E_p := (I + \varepsilon D_c D_p)^{-1}$ and $E_c := (I + \varepsilon D_p D_c)^{-1}$. By (A.13)–(A.15),

$$\begin{aligned} u_p \;=\; d + y_c &= d + \varepsilon C_c x_c + \varepsilon D_c u_c \\ &= d + \varepsilon C_c x_c + \varepsilon D_c (r - y_p) \\ &= d + \varepsilon C_c x_c + \varepsilon D_c r - \varepsilon D_c (C_p x_p + D_p u_p)\,. \end{aligned}$$

It follows that

$$u_p = E_p(-\varepsilon D_c C_p x_p + \varepsilon C_c x_c + d + \varepsilon D_c r)\,. \tag{A.16}$$

Consequently, by (A.14a),

$$x_p(k+1) = (A_p - \varepsilon B_p E_p D_c C_p) x_p(k) + \varepsilon B_p E_p C_c x_c(k) + B_p E_p[d(k) + \varepsilon D_c r(k)]\,. \tag{A.17}$$

On the other hand,

$$\begin{aligned} u_c = r - y_p = r - (C_p x_p + D_p u_p) &= r - C_p x_p - D_p(d + y_c) \\ &= r - C_p x_p - D_p d - \varepsilon D_p (C_c x_c + D_c u_c)\,, \end{aligned}$$

showing that

$$u_c = E_c(-C_p x_p - \varepsilon D_p C_c x_c - D_p d + r)\,. \tag{A.18}$$

Hence, by (A.14a),

$$x_c(k+1) = -B_c E_c C_p x_p(k) + (A_c - \varepsilon B_c E_c D_p C_c) x_c(k) + B_c E_c[-D_p d(k) + r(k)]\,. \tag{A.19}$$

Define $\Delta \in \mathbb{R}^{(n_p+n_c)\times(n_p+n_c)}$ by

$$\Delta := \begin{pmatrix} A_p & 0 \\ 0 & A_c \end{pmatrix} + \begin{pmatrix} B_p & 0 \\ 0 & B_c \end{pmatrix} \begin{pmatrix} E_p & 0 \\ 0 & E_c \end{pmatrix} \begin{pmatrix} -\varepsilon D_c & \varepsilon I \\ -I & -\varepsilon D_p \end{pmatrix} \begin{pmatrix} C_p & 0 \\ 0 & C_c \end{pmatrix}.$$

It follows from (A.17) and (A.19) that

$$\begin{pmatrix} x_p(k+1) \\ x_c(k+1) \end{pmatrix} = \Delta \begin{pmatrix} x_p(k) \\ x_c(k) \end{pmatrix} + \begin{pmatrix} B_p E_p[d(k) + \varepsilon D_c r(k)] \\ B_c E_c[-D_p d(k) + r(k)] \end{pmatrix}, \quad \forall k \in \mathbb{Z}_+\,.$$

Consequently, by the discrete-time variation-of-parameters formula,

$$\begin{pmatrix} x_p(k) \\ x_c(k) \end{pmatrix} = \Delta^k \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} + \sum_{j=0}^{k-1} \Delta^{k-1-j} \begin{pmatrix} B_p E_p[d(j) + \varepsilon D_c r(j)] \\ B_c E_c[-D_p d(j) + r(j)] \end{pmatrix}, \quad \forall k \in \mathbb{N}\,. \tag{A.20}$$

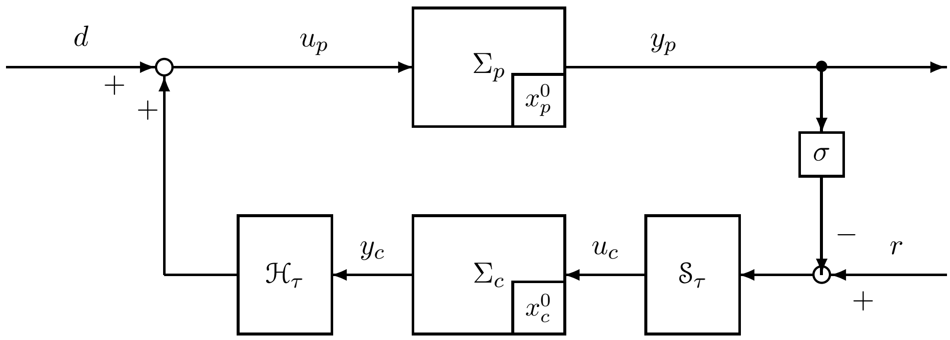


Figure A-1: Sampled-data closed-loop system.

Note that $I - \varepsilon D_p E_p D_c = E_p$ and $I - \varepsilon D_c E_c D_p = E_c$. By (A.16), (A.18) and (A.20),

$$\begin{aligned}
\begin{pmatrix} y_p(k) \\ y_c(k) \end{pmatrix} &= \begin{pmatrix} C_p x_p(k) \\ \varepsilon C_c x_c(k) \end{pmatrix} + \begin{pmatrix} D_p u_p(k) \\ \varepsilon D_c u_c(k) \end{pmatrix} \\
&= \begin{pmatrix} C_p - \varepsilon D_p E_p D_c C_p & \varepsilon D_p E_p C_c \\ -\varepsilon D_c E_c C_p & \varepsilon C_c - \varepsilon^2 D_c E_c D_p C_c \end{pmatrix} \begin{pmatrix} x_p(k) \\ x_c(k) \end{pmatrix} \\
&\quad + \begin{pmatrix} D_p E_p [d(k) + \varepsilon D_c r(k)] \\ \varepsilon D_c E_c [-D_p d(k) + r(k)] \end{pmatrix} \\
&= \begin{pmatrix} E_c C_p & \varepsilon D_p E_p C_c \\ -\varepsilon D_c E_c C_p & \varepsilon E_p C_c \end{pmatrix} \Delta^k \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} + \begin{pmatrix} y_p^{\mathrm{io}}(k) \\ y_c^{\mathrm{io}}(k) \end{pmatrix}, \quad \forall k \in \mathbb{Z}_+ ,
\end{aligned}$$

where $y_p^{\mathrm{io}}$, $y_c^{\mathrm{io}}$ satisfy

$$y_p^{\mathrm{io}} = G(d + y_c^{\mathrm{io}}), \quad y_c^{\mathrm{io}} = K_\varepsilon(r - y_p^{\mathrm{io}}),$$

where $G$ and $K_\varepsilon$ are the input-output operators of (A.13) and (A.14), respectively.

### A.4.3 Sampled-data systems

Consider the sampled-data closed-loop system shown in Figure A-1. The continuous-time plant is given by

$$\dot{x}_p(t) = A_p x_p(t) + B_p u_p(t); \quad x_p(0) = x_p^0 \in \mathbb{R}^{n_p}, \tag{A.21a}$$

$$y_p(t) = C_p x_p(t) + D_p u_p(t), \tag{A.21b}$$

where $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_c \in \mathbb{R}^{n_p \times m}$, $C_c \in \mathbb{R}^{p \times n_p}$ and $D_c \in \mathbb{R}^{p \times m}$. Let $\varepsilon > 0$ be a parameter. The discrete-time controller is given by

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + B_c u_c(k)\,; \quad x_c(0) = x_c^0 \in \mathbb{R}^{n_c}\,, && \text{(A.22a)} \\ y_c(k) &= \varepsilon C_c x_c(k) + \varepsilon D_c u_c(k)\,, && \text{(A.22b)} \end{aligned}$$

where $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times p}$, $C_c \in \mathbb{R}^{m \times n_c}$ and $D_c \in \mathbb{R}^{m \times p}$.

The interconnection of (A.21) and (A.22) is given by

$$u_p = \mathcal{H}_\tau y_c + d\,, \quad u_c = \mathcal{S}_\tau(r - \sigma y_p)\,, \tag{A.23}$$

where $r$ is a reference signal, $d$ is a disturbance signal and $\sigma \in \{-1, 1\}$. Set

$$E_p := (I + \varepsilon\sigma D_c D_p)^{-1} \quad \text{and} \quad E_c := (I + \varepsilon\sigma D_p D_c)^{-1}\,.$$

By (A.21)–(A.23), it is clear that, for $\theta \in [0, \tau)$,

$$\begin{aligned} u_p(k\tau + \theta) &= d(k\tau + \theta) + y_c(k) \\ &= d(k\tau + \theta) + \varepsilon C_c x_c(k) + \varepsilon D_c u_c(k) \\ &= d(k\tau + \theta) + \varepsilon C_c x_c(k) + \varepsilon D_c[r(k\tau) - \sigma y_p(k\tau)] \\ &= d(k\tau + \theta) + \varepsilon C_c x_c(k) + \varepsilon D_c r(k\tau) - \varepsilon\sigma D_c[C_p x_p(k\tau) + D_p u_p(k\tau)]\,. \end{aligned} \tag{A.24}$$

For $\theta = 0$, it follows that

$$u_p(k\tau) = d(k\tau) + \varepsilon C_c x_c(k) + \varepsilon D_c r(k\tau) - \varepsilon\sigma D_c[C_p x_p(k\tau) + D_p u_p(k\tau)]\,,$$

showing that

$$u_p(k\tau) = E_p[d(k\tau) + \varepsilon C_c x_c(k) + \varepsilon D_c r(k\tau) - \varepsilon\sigma D_c C_p x_p(k\tau)]\,. \tag{A.25}$$

Note that

$$I - \varepsilon\sigma D_c D_p E_p = E_p \quad \text{and} \quad D_c D_p E_p = E_p D_c D_p\,.$$

Substituting (A.25) into (A.24), we obtain

$$\begin{aligned} u_p(k\tau + \theta) &= d(k\tau + \theta) - \varepsilon\sigma D_c D_p E_p d(k\tau) + (I - \varepsilon\sigma D_c D_p E_p)[\varepsilon C_c x_c(k) \\ &\qquad + \varepsilon D_c r(k\tau) - \varepsilon\sigma D_c C_p x_p(k\tau)] \\ &= d(k\tau + \theta) + \varepsilon E_p[-\sigma D_c D_p d(k\tau) + C_c x_c(k) + D_c r(k\tau) \\ &\qquad - \sigma D_c C_p x_p(k\tau)]\,. \end{aligned} \tag{A.26}$$

Consequently, by the variation-of-parameters formula,

$$\begin{aligned}
x_p(k\tau+\theta) &= e^{A_p\theta}x_p(k\tau) + \int_{k\tau}^{k\tau+\theta} e^{A_p(k\tau+\theta-s)}B_p u_p(s)ds \\
&= \left(e^{A_p\theta} - \varepsilon\sigma\int_0^\theta e^{A_p s}dsB_pE_pD_cC_p\right)x_p(k\tau) + \varepsilon\int_0^\theta e^{A_p s}dsB_pE_pC_cx_c(k) \\
&\qquad + \int_{k\tau}^{k\tau+\theta} e^{A_p(k\tau+\theta-s)}B_p d(s)ds \\
&\qquad + \varepsilon\int_0^\theta e^{A_p s}dsB_pE_pD_c[-\sigma D_p d(k\tau) + r(k\tau)]\,.
\end{aligned} \tag{A.27}$$

On the other hand,

$$\begin{aligned}
u_c(k) &= r(k\tau) - \sigma y_p(k\tau) \\
&= r(k\tau) - \sigma[C_p x_p(k\tau) + D_p u_p(k\tau)] \\
&= r(k\tau) - \sigma C_p x_p(k\tau) - \sigma D_p[d(k\tau) + y_c(k)] \\
&= r(k\tau) - \sigma C_p x_p(k\tau) - \sigma D_p d(k\tau) - \varepsilon\sigma D_p[C_c x_c(k) + D_c u_c(k)]\,, \quad \forall k \in \mathbb{Z}_+\,,
\end{aligned}$$

showing that

$$u_c(k) = E_c[-\sigma C_p x_p(k\tau) - \varepsilon\sigma D_p C_c x_c(k) - \sigma D_p d(k\tau) + r(k\tau)]\,, \quad \forall k \in \mathbb{Z}_+\,. \tag{A.28}$$

Hence, by (A.22a),

$$x_c(k+1) = -\sigma B_c E_c C_p x_p(k\tau) + (A_c - \varepsilon\sigma B_c E_c D_p C_c)x_c(k) + B_c E_c[-\sigma D_p d(k\tau) + r(k\tau)]\,. \tag{A.29}$$

Define $\Delta\colon [0,\tau] \to \mathbb{R}^{(n_p+n_c)\times(n_p+n_c)}$ by

$$\Delta(\theta) := \\
\begin{pmatrix} e^{A_p\theta} & 0 \\ 0 & A_c \end{pmatrix} + \begin{pmatrix} \int_0^\theta e^{A_p s}dsB_p & 0 \\ 0 & B_c \end{pmatrix} \begin{pmatrix} E_p & 0 \\ 0 & E_c \end{pmatrix} \begin{pmatrix} -\varepsilon\sigma D_c & \varepsilon I \\ -\sigma I & -\varepsilon\sigma D_p \end{pmatrix} \begin{pmatrix} C_p & 0 \\ 0 & C_c \end{pmatrix}, \tag{A.30}$$

and, for $\theta \in [0,\tau]$ and $k \in \mathbb{Z}_+$, define $R(k,\theta)\colon L_b(\mathbb{R}_+, \mathbb{R}^m) \times L_b(\mathbb{R}_+, \mathbb{R}^p) \to \mathbb{R}^{n_p+n_c}$ by

$$R(k,\theta)\begin{pmatrix} d \\ r \end{pmatrix} \\
:= \begin{pmatrix} \int_{k\tau}^{k\tau+\theta} e^{A_p(k\tau+\theta-s)}B_p d(s)ds + \varepsilon\int_0^\theta e^{A_p s}dsB_pE_pD_c[-\sigma D_p d(k\tau) + r(k\tau)] \\ B_c E_c[-\sigma D_p d(k\tau) + r(k\tau)] \end{pmatrix}.$$

It follows from (A.27) and (A.29) that

$$\begin{pmatrix} x_p(k\tau+\theta) \\ x_c(k+1) \end{pmatrix} = \Delta(\theta) \begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} + R(k,\theta) \begin{pmatrix} d \\ r \end{pmatrix}, \quad \forall k \in \mathbb{Z}_+ ,\ \theta \in [0,\tau)\,. \tag{A.31}$$

Letting $\theta \nearrow \tau$, it follows from the continuity of the terms depending on $\theta$ that

$$\begin{pmatrix} x_p((k+1)\tau) \\ x_c(k+1) \end{pmatrix} = \Delta(\tau) \begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} + R(k,\tau) \begin{pmatrix} d \\ r \end{pmatrix}, \quad \forall k \in \mathbb{Z}_+ \,.$$

Consequently, by the discrete-time variation-of-parameters formula,

$$\begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} = \Delta(\tau)^k \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} + \sum_{j=0}^{k-1} \Delta(\tau)^{k-j-1} R(j,\tau) \begin{pmatrix} d \\ r \end{pmatrix}, \quad \forall k \in \mathbb{N}\,. \tag{A.32}$$

Hence, by (A.26)–(A.28) and (A.32),

$$\begin{aligned}
&\begin{pmatrix} y_p(k\tau+\theta) \\ y_c(k) \end{pmatrix} \\
&= \begin{pmatrix} C_p x_p(k\tau+\theta) \\ \varepsilon C_c x_c(k) \end{pmatrix} + \begin{pmatrix} D_p & 0 \\ 0 & \varepsilon D_c \end{pmatrix} \begin{pmatrix} u_p(k\tau+\theta) \\ u_c(k) \end{pmatrix} \\
&= \begin{pmatrix} C_p e^{A_p\theta} - \varepsilon\sigma C_p \int_0^\theta e^{A_p s} ds B_p E_p D_c C_p & \varepsilon C_p \int_0^\theta e^{A_p s} ds B_p E_p C_c \\ 0 & \varepsilon C_c \end{pmatrix} \begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} \\
&\quad + \begin{pmatrix} C_p \int_{k\tau}^{k\tau+\theta} e^{A_p(k\tau+\theta-s)} B_p d(s) ds + \varepsilon C_p \int_0^\theta e^{A_p s} ds B_p E_p D_c[-\sigma D_p d(k\tau) + r(k\tau)] \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} D_p & 0 \\ 0 & \varepsilon D_c \end{pmatrix} \begin{pmatrix} -\varepsilon\sigma E_p D_c C_p & \varepsilon E_p C_c \\ -\sigma E_c C_p & -\varepsilon\sigma E_c D_p C_c \end{pmatrix} \begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} \\
&\quad + \begin{pmatrix} D_p & 0 \\ 0 & \varepsilon D_c \end{pmatrix} \begin{pmatrix} d(k\tau+\theta) + \varepsilon E_p D_c[-\sigma D_p d(k\tau) + r(k\tau)] \\ E_c[-\sigma D_p d(k\tau) + r(k\tau)] \end{pmatrix} \\
&= Q(\theta) \begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} + \begin{pmatrix} G(k,\theta) \\ \varepsilon D_c E_c[-\sigma D_p d(k\tau) + r(k\tau)] \end{pmatrix} \\
&= Q(\theta)\Delta(\tau)^k \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} + \begin{pmatrix} y_p^{\mathrm{io}}(k\tau+\theta) \\ y_c^{\mathrm{io}}(k) \end{pmatrix}, \quad \forall \theta \in [0,\tau)\,,\ \forall k \in \mathbb{Z}_+\,,
\end{aligned}$$

where

$$Q(\theta) \quad := \quad \begin{pmatrix} C_p e^{A_p\theta} - \varepsilon\sigma C_p \int_0^\theta e^{A_p s} ds B_p E_p D_c C_p & \varepsilon C_p \int_0^\theta e^{A_p s} ds B_p E_p C_c \\ 0 & \varepsilon C_c \end{pmatrix}$$
$$+ \begin{pmatrix} -\varepsilon\sigma D_p E_p D_c C_p & \varepsilon D_p E_p C_c \\ -\varepsilon\sigma D_c E_c C_p & -\varepsilon^2 \sigma D_c E_c D_p C_c \end{pmatrix},$$

$$G(k,\theta) := C_p \int_{k\tau}^{k\tau+\theta} e^{A_p(k\tau+\theta-s)} B_p d(s) ds + C_p \int_0^\theta e^{A_p s} ds B_p E_p D_c[-\sigma D_p d(k\tau) + r(k\tau)]$$
$$+ D_p d(k\tau+\theta) + \varepsilon D_p E_p D_c[-\sigma D_p d(k\tau) + r(k\tau)],$$

and $y_p^{\mathrm{io}}$, $y_c^{\mathrm{io}}$ satisfy

$$y_p^{\mathrm{io}} = G(d + \mathcal{H}_\tau y_c^{\mathrm{io}}), \quad y_c^{\mathrm{io}} = K_{\tau,\varepsilon} \mathcal{S}_\tau (r - y_p^{\mathrm{io}}),$$

where $G$ and $K_{\tau,\varepsilon}$ are the input-output operators of (A.21) and (A.22), respectively.

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