Nonnegative Feedback Systems in Population Ecology

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Summary

We develop and adapt absolute stability results for nonnegative Lur'e systems, that is, systems made up of linear part and a nonlinear feedback in which the state remains nonnegative for all time. This is done in both continuous and discrete time with an aim of applying these results to population modeling. Further to this, we consider forced nonnegative Lur'e systems, that is, Lur'e systems with an additional disturbance, and provide results on input-to-state stability (ISS), again in both continuous and discrete time. We provide necessary and sufficient conditions for a forced Lur'e system to have the converging-input converging-state (CICS) property in a general setting before specializing these results to nonnegative, single-input, single-output systems. Finally we apply integral control to nonnegative systems in order to control the output of the system with the key focus being on applications to population management.

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Chapter 1

Introduction

In this thesis we develop and adapt results about Lur'e systems. These are systems made up of two components, a linear system with a state x, an input u, an output y; and a nonlinear feedback $u = f(y)$. They exist in both continuous time and discrete time. In a continuous time setting the linear system is given by

$$\dot{x} = Ax + bu, \qquad x(0) = x^0 \in \mathbb{R}^n, \qquad y = c^T x,$$

where $A \in \mathbb{R}^{n \times n}$, $b, c \in \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}$. The resulting nonlinear feedback system is given by

$$\dot{x} = Ax + bf(c^T x), \qquad x(0) = x^0 \in \mathbb{R}^n.$$

In a similar fashion, in a discrete time setting, the linear system is given by

$$x(t+1) = Ax(t) + bu(t), \qquad x(0) = x^0 \in \mathbb{R}^n, \qquad y(t) = c^T x(t),$$

again where $A \in \mathbb{R}^{n \times n}$, $b, c \in \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}$. The resulting nonlinear feedback system is

$$x(t+1) = Ax(t) + bf(c^T x(t)), \qquad x(0) = x^0 \in \mathbb{R}^n.$$

See also Figure 1.1 for a block diagram representation of a Lur'e system.

Lur'e systems are a common class of nonlinear systems which are at the center of the classical subject of absolute stability theory. Absolute stability theory is a way of guaranteeing that a Lur'e system is stable, and the conditions for stability are usually stated in terms of the linear system and apply to a class of nonlinearities. We refer the reader to [55, 72, 86, 124, 146, 153] and the references within for more information on this subject.

We specialize absolute stability theory to nonnegative, single-input, single-output Lur'e systems with the aim of applying the results to model asymptotic

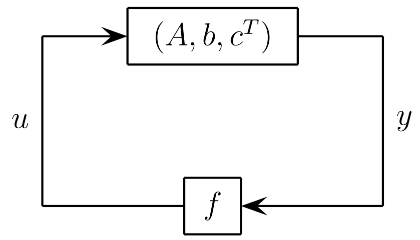


Figure 1.1: Block diagram of a Lur'e system.

behavior of populations. The inspiration for doing so is [143], in which a stability/instability trichotomy was established for nonnegative Lur'e systems. Through the use of absolute stability we are able to provide stronger stability properties, in particular exponential asymptotic stability, and include a larger class of nonlinearities, however as a consequence, the strict trichotomy is lost. We further develop the theory for continuous time systems along with the discrete time systems which were considered in [143].

Furthermore, we consider forced Lur'e systems. These systems take the form

$$\dot{x} = Ax + bf(c^T x) + d, \qquad x(0) = x^0 \in \mathbb{R}^n,$$

where $d : L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, in continuous time and

$$x(t+1) = Ax(t) + bf(c^T x(t)) + d(t), \qquad x(0) = x^0 \in \mathbb{R}^n,$$

where $d : \mathbb{N}_0 \to \mathbb{R}^n$, in discrete time. A block diagram of a forced Lur'e system is given in Figure 1.2.

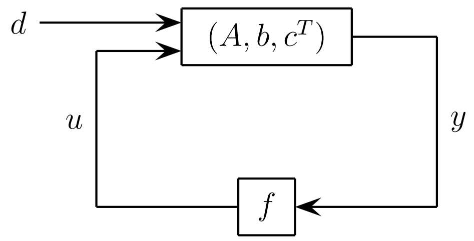


Figure 1.2: Block diagram of a forced Lur'e system.

The forcing term d goes by many names such as a disturbance, control or input, and will vary depending on its interpretation. While considering d to be a disturbance to a nonnegative, single-input, single-output system, we adapt input-to-state stability (ISS) results. Without going into detail here ISS means that the map $(x^0, d) \mapsto x(t)$ has nice boundedness and asymptotic properties. This is done in both a continuous and a discrete time setting. See [72, 124] for further details on ISS.

We consider d to be an input term which converges to a limit for a larger class of multi-input, multi-output, continuous time Lur'e systems. We provide necessary and sufficient conditions for the converging-input converging-state (CICS) property. A Lur'e system is said to have such a property if for every $d^\infty \in \mathbb{R}^n$, there exists $x^\infty \in \mathbb{R}^n$ such that $\lim_{t\to\infty} x(t) = x^\infty$ for all x^0 and all inputs d converging to d^∞. We also consider Lur'e systems of the form

$$\dot{x} = Ax + bf(c^T x - d), \qquad x(0) = x^0 \in \mathbb{R}^n,$$

for which the block diagram is given in Figure 1.3, and provide conditions for this system to have the CICS property.

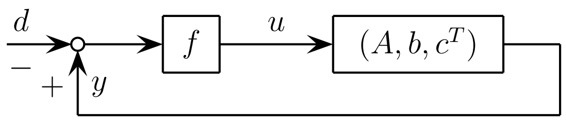


Figure 1.3: Block diagram of the controlled Lur'e system $\dot{x} = Ax+bf(c^T x-d)$.

The final type of system which we provide conditions for the CICS property are nonnegative, single-input, single-output Lur'e systems. This is the same type of system we considered when looking into ISS in continuous time.

We also look at integral control of nonnegative, single-input, single-output, discrete time Lur'e systems. We do this from a population management perspective. Population managers aim to regulate the population to a desired density in a way which is robust to parametric uncertainty and observation errors. The main problem which we consider is to design a method to restock a managed, but declining, population. This method should be implemented with only access to specified observations of the population and in a manner that is both independent of the initial population distribution and robust to model uncertainty. This means that the management action is to be taken at each time-step and is based on observations of the population. A scheme such as this is represented in Figure 1.4.

We progressively add features to our model which are necessitated by the specific demands of population modeling and illustrate theoretical concepts with ecological examples.

This thesis is structured as follows: Chapter 2 provides preliminary defini-

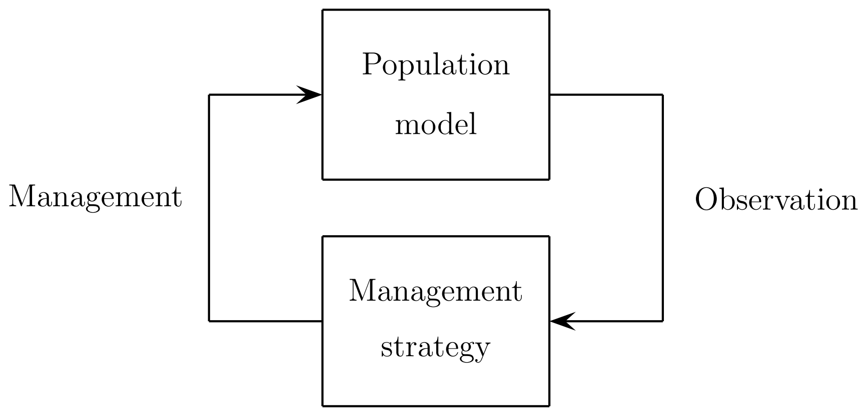


Figure 1.4: Feedback control for population management.

tions and results which are used throughout this thesis and contains material on nonnegative matrices, primitivity, Perron-Frobenius theory and Metzler matrices to name a few. Chapter 3 contains material on absolute stability and ISS of nonnegative single-input, single-output continuous time Lur'e systems, and is largely based on [10]. Chapter 4 are the results about the CICS property including the specialization to nonnegative single-input, single-output continuous time Lur'e systems and is based on [11]. Chapter 5 is the discrete time counterpart to Chapter 3 and contains the discrete time version of the absolute stability and ISS results presented in Chapter 3. Chapter 6 contains the material on integral control from a population management perspective and is based on [53]. This concludes the main text of this thesis. Also included are a list of main assumptions used in each of the chapters and an index of key terms which can be found beginning on pages 241 and 244 respectively. Finally on we provide a bibliography beginning on page 245.

Chapter 2

Preliminaries

In this chapter we introduce notation and concepts which will be used throughout this thesis. This chapter is arranged as follows. Section 2.1 introduces nonnegative matrices and contains results about irreducibility and primitivity. Section 2.2 introduces linear discrete time systems and how they can be used to model populations. Section 2.3 contains a definition of Metzler matrices and results about them. Section 2.4 introduces sector conditions and contains two examples of nonlinearities which fit these sector conditions. Finally Section 2.5 provides definitions of comparison functions.

2.1 Nonnegative Matrices

Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively. Denote the set of nonnegative real numbers by $\mathbb{R}_+$, that is

$$\mathbb{R}_+ := \{x \in \mathbb{R} \ : \ x \geq 0\}.$$

Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{N}_0$ be the set of nonnegative integers, that is

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

This is not the only way of denoting the set of nonnegative integers. An alternative notation sometimes used in the literature is to let $\mathbb{K}$ denote the set of integers, and $\mathbb{K}_+$ denote the set of nonnegative integers. Throughout this thesis we shall be using the $\mathbb{N}_0$ notation.

Definition 2.1.1. *Let $M = (m_{ij}) \in \mathbb{R}^{n\times p}$.*

(1) If $m_{ij} \geq 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq p$, M is said to be a nonnegative matrix. *This is often denoted as $M \in \mathbb{R}_+^{n\times p}$.*

(2) If $m_{ij} \geq 0$ *for all* $1 \leq i \leq n$ *and* $1 \leq j \leq p$ *and* $M \neq 0$ *then* M *is said to be nonnegative and nonzero which is often denoted as* $M \in \mathbb{R}_+^{n\times p}\backslash\{0\}$.

(3) If $m_{ij} > 0$ *for all* $1 \leq i \leq n$ *and* $1 \leq j \leq p$, M *is said to be a* positive matrix.

Often when considering matrices, it is convenient to use inequalities. These inequalities are slightly different than scalar inequalities and are defined below.

Definition 2.1.2. *Let* $M = (m_{ij}) \in \mathbb{R}^{n\times p}$ *and* $N = (n_{ij}) \in \mathbb{R}^{n\times p}$. *Write*

- $M \geq N$ *if* $m_{ij} \geq n_{ij}$ *for all* $1 \leq i \leq n$ *and* $1 \leq j \leq p$*;*
- $M > N$ *if* $M \geq N$ *and* $M \neq N$*;*
- $M \gg N$ *if* $m_{ij} > n_{ij}$ *for all* $1 \leq i \leq n$ *and* $1 \leq j \leq p$.

Using matrix inequalities we can express that a matrix M is nonnegative, nonnegative and nonzero, and positive by $M \geq 0$, $M > 0$ and $M \gg 0$ respectively.

2.1.1 Irreducibility of Matrices

Before defining irreducibility of matrices, we first must define what a nontrivial partition is.

Definition 2.1.3. *Let* $\mathcal{X}$ *be a set. The sets* $\mathcal{Y}$ *and* $\mathcal{Z}$ *form a* nontrivial partition *of* $\mathcal{X}$ *if* $\mathcal{Y} \cup \mathcal{Z} = \mathcal{X}$, $\mathcal{Y} \neq \emptyset$, $\mathcal{Z} \neq \emptyset$ *and* $\mathcal{Y} \cap \mathcal{Z} = \emptyset$.

We can now define what it means for a matrix to be irreducible. We do this by defining what it means for a matrix to be reducible and then negating it.

Definition 2.1.4. *A matrix* $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ *is said to be* reducible *if there exists a nontrivial partition* $\mathcal{I}$, $\mathcal{J}$ *of* $\mathcal{N} = \{1, \ldots, n\}$, *such that for all* $i \in \mathcal{I}$ *and all* $j \in \mathcal{J}$, $m_{ij} = 0$.

If a matrix is not reducible it is said to be irreducible.

Example 2.1.5. *1. Consider the matrix* $A = (a_{ij}) \in \mathbb{R}^{3\times 3}$ *given by*

$$A = \begin{pmatrix} -7 & 5 & 0 \\ 0 & 1 & -7 \\ 4 & 0 & 6 \end{pmatrix}.$$

We demonstrate that A *is irreducible by considering all* 6 *nontrivial partitions of the set* $\{1, 2, 3\}$.

- $\mathcal{I} = \{1\}$ *and* $\mathcal{J} = \{2, 3\}$. $a_{12} = 5 \neq 0$.
- $\mathcal{I} = \{2\}$ *and* $\mathcal{J} = \{1, 3\}$. $a_{23} = -7 \neq 0$.
- $\mathcal{I} = \{3\}$ *and* $\mathcal{J} = \{1, 2\}$. $a_{31} = 4 \neq 0$.
- $\mathcal{I} = \{1, 2\}$ *and* $\mathcal{J} = \{3\}$. $a_{23} = -7 \neq 0$.
- $\mathcal{I} = \{1, 3\}$ *and* $\mathcal{J} = \{2\}$. $a_{12} = 5 \neq 0$.
- $\mathcal{I} = \{2, 3\}$ *and* $\mathcal{J} = \{1\}$. $a_{31} = 4 \neq 0$.

Clearly, there does not exist a nontrivial partition $\mathcal{I}, \mathcal{J}$ *of* $\{1, 2, 3\}$ *such that* $a_{ij} = 0$ *for all* $i \in \mathcal{I}$ *and* $j \in \mathcal{J}$*, therefore,* A *is not reducible thus it is irreducible.*

2. *Consider the matrix* $B = (b_{ij}) \in \mathbb{R}^{4\times 4}$ *given by*

$$B = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 10 & 2 & 0 & 0 \\ 1 & -7 & -7 & -5 \\ -1 & 6 & 2 & -5 \end{pmatrix}.$$

We demonstrate that is is a reducible matrix. Let $\mathcal{I} = \{1, 2\}$ *and* $\mathcal{J} = \{3, 4\}$*. This is clearly a nontrivial partition of* $\{1, 2, 3, 4\}$*. Now* $b_{13} = b_{14} = b_{23} = b_{24} = 0$*, therefore* B *is a reducible matrix.*

We make a series of trivial remarks based on the definition of irreducibility.

Remark 2.1.6. *The lead diagonal entries of a matrix do not play a role in whether the matrix is reducible or irreducible. If* $\mathcal{I}, \mathcal{J}$ *form a nontrivial partition, then, for all* $i \in \mathcal{I}$ *and* $j \in \mathcal{J}$, $i \neq j$.

Remark 2.1.7. *A matrix is reducible if, ignoring the lead diagonal, it has a zero row or a zero column. This is easy to demonstrate. Let* $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ *and let the* i*-th row have all zero entries. Set* $\mathcal{I} = \{i\}$ *and* $\mathcal{J} = \mathcal{N} \backslash \{i\}$*. Then* $m_{ij} = 0$ *for all* $j \in \mathcal{J}$*, therefore,* M *is reducible.*

It is often convenient to test for irreducibility using a graphical approach. This method is also described in, for example, [100, Section 8.3].

The first step involves creating a *digraph* of the matrix. A digraph consists of n nodes, labeled $1, \ldots, n$. For all $1 \leq i, j \leq n$ with $i \neq j$ a directed line is drawn from node j to node i if the ij-th entry of the matrix is nonzero.

A digraph is is *strongly connected* if there is a cycle passing through every node. This means that every node can be reached from every other node following the directed lines.

Theorem 2.1.8. *A matrix is irreducible if, and only if, its digraph is strongly connected.*

For a proof of this theorem see [15, Theorem 3.2.1] We demonstrate this graphical method of testing for irreducibility in the following example.

Example 2.1.9. *1. We return to the matrix A in Example 2.1.5. The digraph of this matrix is given in Figure 2.1.*

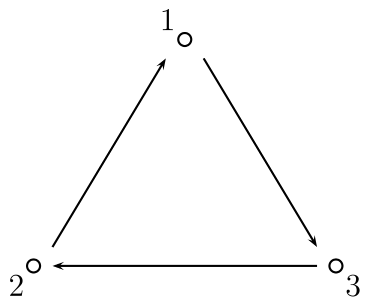


Figure 2.1: Digraph of the matrix A from Example 2.1.9.

It is easily verified that this digraph is strongly connected as there exists a cycle $1 \to 3 \to 2 \to 1$, which confirms that A is irreducible.

2. We now return to the matrix B in Example 2.1.5. The digraph of this matrix is given in Figure 2.2.

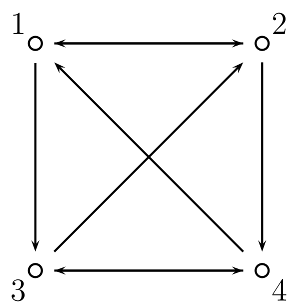


Figure 2.2: Digraph of the matrix B from Example 2.1.9.

This digraph is not strongly connected as the only cycles ending at node 1 is the cycle $1 \to 2 \to 1$, which clearly does not travel through nodes 3 or 4, therefore, the matrix is reducible (as it is not irreducible).

2.1.2 Primitivity of Matrices

Primitivity is a very useful property for matrices to have as it links nonnegativity and positivity. Before we define and speak about primitivity we first define the index of imprimitivity of a matrix.

Definition 2.1.10. *Given a nonnegative, irreducible matrix $M \in \mathbb{R}^{n \times n}$ with characteristic polynomial*

$$\det(\lambda I - M) = \lambda^n + a_1 \lambda^{n_1} + \ldots + a_m \lambda^{n_m}, \tag{2.1}$$

where $a_1, \ldots, a_m \neq 0$ and $n > n_1 > n_2 > \ldots > n_m$, define the index of imprimitivity *to be the greatest common divisor of the set*

$$\{n - n_1,\, n_1 - n_2,\, \ldots\, ,\, n_{m-1} - n_m\}.$$

We can now use this index of imprimitivity to define when a matrix is primitive.

Definition 2.1.11. *A matrix is said to be* primitive *if it is nonnegative, irreducible and its index of imprimitivity is equal to* 1*. Otherwise the matrix is said to be imprimitive.*

Example 2.1.12. *Consider the matrix*

$$A = \begin{pmatrix} 0 & 0 & 0 & 9 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \\ 1 & 2 & 7 & 0 \end{pmatrix}.$$

We demonstrate that this matrix is primitive. Firstly note that A is clearly nonnegative. To establish irreducibly we use the graphical approach. The digraph of A is given in Figure 2.3.

There is a path $1 \to 4 \to 3 \to 2 \to 4 \to 1$, therefore the digraph is strongly connected. This means that A is irreducible. Now

$$\det(\lambda I - A) = \lambda^4 - 65\lambda^2 - 64\lambda.$$

The index of imprimitivity is therefore the common divisor of the set

$$\{4 - 2, 2 - 1\} = \{2, 1\},$$

which clearly is 1*, therefore A is primitive.*

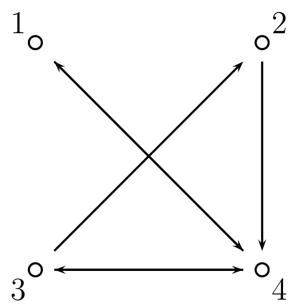


Figure 2.3: Digraph of the matrix A from Example 2.1.12.

Next we define what is meant by the trace of a matrix which will be required in the result which follows.

Definition 2.1.13. *Let* $M = (m_{ii}) \in \mathbb{R}^{n\times n}$. *The* trace *of the matrix* M, *denoted* $\mathrm{tr}(M)$ *is the sum of the lead diagonal entries of the matrix, that is*

$$\mathrm{tr}(M) = \sum_{i=1}^{n} m_{ii}.$$

We shall make use of the following lemma at various points throughout this thesis. It states that nonnegative, irreducible matrices with positive trace are primitive.

Lemma 2.1.14. *Let* $M \in \mathbb{R}^{n\times n}$ *be a nonnegative, irreducible matrix with* $\mathrm{tr}(M) > 0$. *Then* M *is a primitive matrix.*

Proof. Let $M \in \mathbb{R}_{+}^{n\times n}$ be an irreducible matrix with $\mathrm{tr}(M) > 0$. The characteristic polynomial of M takes the form

$$\lambda^n - \lambda^{n-1}\mathrm{tr}(M) + r(M),$$

where r is a polynomial of degree of at most $n-2$. Comparing to (2.1), the term n_1 would equal $n-1$ and therefore, the first difference is given by $n-(n-1) = 1$. This means that no mater the value of the other differences, the greatest common divisor will always be 1, thus the index of imprimitivity of M is 1. Therefore, M is primitive. □

Like for irreducible matrices we can use a graphical approach for testing if a matrix is primitive. We only need do this for nonnegative, irreducible matrices with zero trace, as we know that for a matrix to be primitive it must be nonnegative and irreducible and if additionally it has positive trace it is primitive.

Theorem 2.1.15. *Let $M \in \mathbb{R}_+^{n \times n}$ be irreducible. Let L_i denote the length of all cycles l_i passing through node i of the digraph of M. Denote*

$$h_i = \text{g.c.d.}_{l_i \in L_i}\{l_i\},$$

which is the greatest common divisor of these cycle lengths. Then, $h_1 = h_2 = \ldots = h_n = h$ and h is the index of imprimitivity of M.

A proof of this result can be found in [7, Theorem 2.2.30].

Example 2.1.16. *We return to the matrix A from Example 2.1.12 which has a digraph given in Figure 2.3. Clearly the matrix A is nonnegative and we established in Example 2.1.12 that A is irreducible. It is clear that A does not have a positive trace so we cannot apply Lemma 2.1.14, so we use the graphical approach to show that A is primitive.*

We look at the lengths of all cycles starting and ending at the first node. Listed below are just a few of these cycles:

- *Length 2:* $1 \to 4 \to 1$
- *Length 4:* $1 \to 4 \to 3 \to 4 \to 1$
- *Length 4:* $1 \to 4 \to 1 \to 4 \to 1$
- *Length 5:* $1 \to 4 \to 3 \to 2 \to 4 \to 1$

Clearly, the greatest common divisor of these lengths is 1, therefore the matrix A is primitive.

The following result is important as it links nonnegative and positive matrices. For a proof of this result see [46, Theorem 3.5.8].

Theorem 2.1.17. *Let M be a nonnegative matrix. M is primitive if, and only if, for some $k \geq 1$, $M^k \gg 0$.*

This not only is a key property of a primitive matrix which we will make use of several times in this thesis but also provides a method of testing if a matrix is primitive. It also results in the following corollary which follows immediately.

Corollary 2.1.18. *Let M be a nonnegative matrix. If, for some $k \geq 1$, $M^k \gg 0$, then M is an irreducible matrix.*

We demonstrate Theorem 2.1.17 as a method of testing for irreducibility.

Example 2.1.19. *We return the matrix A considered in Examples 2.1.12 and 2.1.16. We have already established that A is primitive. We demonstrate that $A^k \gg 0$ for some $k \geq 1$. We only need to compute up until the third power of A to demonstrate this as seen below.*

$$A^2 = \begin{pmatrix} 45 & 18 & 63 & 54 \\ 0 & 16 & 16 & 32 \\ 8 & 16 & 56 & 0 \\ 6 & 8 & 8 & 65 \end{pmatrix}, \qquad A^3 = \begin{pmatrix} 324 & 180 & 450 & 909 \\ 32 & 128 & 288 & 128 \\ 48 & 64 & 64 & 520 \\ 101 & 162 & 487 & 118 \end{pmatrix}.$$

The following remark, which can be found in [145, Section 2.2] relates the power k which a primitive matrix M must be raised to such that $M^k \gg 0$, and the lead diagonal components.

Remark 2.1.20. *Let $M = (m_{ij}) \in \mathbb{R}_+^{n\times n}$ be an irreducible matrix.*

- *If $m_{ii} > 0$ for all $i = 1, \ldots, n$ then, for every $k \geq n-1$, $M^k \gg 0$.*
- *If $m_{ii} > 0$ for some $i = 1, \ldots, n$ then, for every $k \geq 2n-2$, $M^k \gg 0$.*
- *If M is primitive then for every $k \geq n^2 - 2n + 2$, $M^k \gg 0$.*

2.1.3 Perron-Frobenius Theory

In this section we investigate spectral properties of nonnegative matrices. This topic is known as Perron-Frobenius theory as it has evolved from the contributions of Oskar Perron [108] and Ferdinand Georg Frobenius [45]. The original work of Perron pertained to positive matrices, however the contributions of Frobenius extended this to nonnegative matrices.

We begin by defining two important quantities.

Definition 2.1.21. *Let $M \in \mathbb{R}^{n\times n}$, with eigenvalues $\lambda_1, \ldots, \lambda_n$. The* spectrum *of M, denoted $\sigma(M)$, is the set of all eigenvalues, that is*

$$\sigma(M) = \{\lambda_1, \ldots, \lambda_n\}.$$

The spectral radius *of M, denoted $\rho(M)$, is the largest magnitude attained by any eigenvalue, that is*

$$\rho(M) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}.$$

We state the Perron-Frobenius theorem. This result can be found in [100, Section 8.3], along with a proof and discussion.

Theorem 2.1.22. *If $M \in \mathbb{R}_+^{n\times n}$ is irreducible, then each of the following is true:*

(1) $r = \rho(M) \in \sigma(M)$, $r > 0$ and r is simple.

(2) There exist unique vectors, $p, q \in \mathbb{R}^n$, satisfying

$$Mp = rp, \qquad q^T M = rq^T, \qquad p, q \gg 0 \quad \text{and} \quad \|p\|_1 = \|q\|_1 = 1.$$

p and q are called the right and left Perron vectors, respectively.

The next result, again stated and proved in [100, Section 8.3], concerns primitivity and Perron-Frobenius theory.

Theorem 2.1.23. *Let $M \in \mathbb{R}_+^{n\times n}$ be irreducible and set $r = \rho(M)$. Then the following statements are equivalent.*

(1) M is primitive.

(2) r is the only eigenvalue on the spectral circle of M, or equivalently, if $\lambda \in \sigma(M)$ such that $|\lambda| = r$, then $\lambda = r$.

(3) $\lim_{k\to\infty}(M/r)^k$ exists and

$$\lim_{k\to\infty}\left(\frac{M}{r}\right)^k = \frac{pq^T}{q^T p} \gg 0,$$

where p and q are the left and right Perron vectors of M.

The next result shows that the spectral radius of a nonnegative matrix has certain monotonicity properties.

Corollary 2.1.24. *Let $M, N \in \mathbb{R}^{n\times n}$.*

(1) If $M > N \geq 0$ and N is irreducible, then $\rho(M) > \rho(N)$.

(2) If $M \geq N \geq 0$, then $\rho(M) \geq \rho(N)$.

Proof. We begin by proving statement (1). Note that irreducibility of N implies irreducibility of M. By Theorem 2.1.22, there exists vectors $v, w \gg 0$ such that

$$w^T M = \rho(M) w^T, \qquad Nv = \rho(N)v.$$

Obviously, $w^T v > 0$ and, furthermore, $\rho(M)w^T v > \rho(N)w^T v$, showing $\rho(M) > \rho(N)$. Proceeding to prove statement (2), let $P \in \mathbb{R}^{n\times n}$ be a positive matrix and $\varepsilon > 0$. Then $M + 2\varepsilon P \gg N + \varepsilon P \gg 0$ and note that $N + \varepsilon P$ is an irreducible matrix as it is a positive matrix. By statement (1), $\rho(M + 2\varepsilon P) > \rho(N + \varepsilon P)$, and, letting $\varepsilon \to 0$, and using continuity of spectral radius (see [100, Chapter 7]) we conclude that $\rho(M) \geq \rho(N)$. □

2.2 Linear Discrete Time Systems and Population Projection Models

Let $t \in \mathbb{N}_0$ be time measured discretely. This could be something like months, years or seasons and is chosen at the modelers discretion.

The simplest form of population model involves $x(t) \in \mathbb{R}_+$ being the total population at time $t \in \mathbb{N}_0$. To model the population as time evolves, we use the system

$$x(t+1) = ax(t), \qquad x(0) = x^0 \in \mathbb{R}_+, \tag{2.2}$$

where $a \in \mathbb{R}_+$. Let $x(\,\cdot\,; x^0)$ denote the solution of the initial value problem (2.2). Note that populations can only take positive values, therefore we must restrict a and x^0 to the nonnegative real numbers to ensure $x(t) \in \mathbb{R}_+$ for all $t \in \mathbb{N}_0$.

The following theorem provides long term estimates for the state $x(t)$ for different values of a.

Theorem 2.2.1. *Consider the system* (2.2)*.*

(1) If $a = 0$ then $x(t; x^0) = 0$ for all $t \in \mathbb{N}$ and all $x^0 \in \mathbb{R}_+$.

(2) If $a \in (0, 1)$ then $x(t; x^0) \to 0$ as $t \to \infty$ for all $x^0 \in \mathbb{R}_+$.

(3) If $a = 1$ then $x(t; x^0) = x^0$ for all $t \in \mathbb{N}_0$ and all $x^0 \in \mathbb{R}_+$.

(4) If $a > 1$ then $x(t; x^0) \to \infty$ as $t \to \infty$ for all $x^0 \in \mathbb{R}_+$ with $x^0 > 0$.

(5) For all $a \in \mathbb{R}_+$, $x(t; 0) = 0$ for all $t \in \mathbb{N}_0$.

Proof. The proof of this result is trivial noting

$$x(t) = ax(t-1) = a^2x(t-2) = \ldots = a^t x^0.$$

□

The model given by (2.2) is a very simple model and can only be used to model total population. A more useful model involves replacing the scalar population by a vector population given by

$$x(t) := \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}.$$

Each of the x_i for $1 \leq i \leq n$ denote an age- or stage-class of the population and the total population is now given by $\|x(t)\|_1$. An age-class model involves individuals moving from one class to the next as each time interval passes. A stage-class model groups together individuals with similar dynamics which could represent life stages such as egg, larva, pupa and adult stages of an insect. In each time interval it could be the case that an individual remains in the same stage-class or moves into the next.

To model vector population dynamics we consider the system

$$x(t+1) = Ax(t), \qquad x(0) = x^0 \in \mathbb{R}^n_+, \tag{2.3}$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}_+$. This is equivalent to the n coupled systems

$$\begin{aligned} x_1(t+1) &= \sum_{j=1}^{n} a_{1j}x_j(t) \qquad x_1(0) = x_1^0 \in \mathbb{R}_+, \\ &\vdots \\ x_n(t+1) &= \sum_{j=1}^{n} a_{nj}x_j(t), \qquad x_n(0) = x_n^0 \in \mathbb{R}_+. \end{aligned}$$

The matrix A is commonly referred to as a *population projection matrix* or PPM for short. What happens to the state $x(t)$ as $t \to \infty$ now depends on the spectral radius of the matrix A.

The matrix A is said to be stable, or *Schur*, if $\rho(A) < 1$. In this case,

$$\lim_{t\to\infty} x(t+1) = \lim_{t\to\infty} Ax(t) = \lim_{t\to\infty} A^t x(0) = 0.$$

In Chapters 5 and 6, we consider discrete time systems and throughout most of those chapters, we only consider stable matrices.

In the following subsections we discuss three common structures of PPMs.

2.2.1 Leslie Matrices

Leslie matrices are commonly used in age-structured models. They take the form

$$L = \begin{pmatrix} b_1 & \cdots & \cdots & \cdots & b_n \\ g_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & g_{n-1} & 0 \end{pmatrix}, \tag{2.4}$$

where $b_i \geq 0$ for $1 \leq i \leq n$ and $0 < g_i \leq 1$ for $1 \leq i \leq n-1$. The b_i terms denote birth rates, that is the number of members entering the first age-class with parents in the i-th age-class each time step. The g_i terms represent a growth rate which is the proportion of individuals leaving age-class i and entering age-class $i+1$ each time step. It is assumed that if an individual in age-class i does not successfully enter age-class $i+1$ it dies.

An example of a Leslie matrix is

$$L_1 = \begin{pmatrix} 0 & 0 & 3.124 & 3.124 & 3.124 & 3.124 & 3.124 \\ 0.802 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.802 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.868 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.868 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.868 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.868 & 0 \end{pmatrix} \tag{2.5}$$

which is the PPM associated with the Tibetan Monkey (*Macaca thibetana*) which can be found in [106].

Another example of a Leslie matrix is that of black-footed ferret (*Mustela nigripes*) which can be found in [48] and is given by

$$L_2 = \begin{pmatrix} 0.73 & 1.25 & 1.25 & 1.25 & 0 \\ 0.39 & 0 & 0 & 0 & 0 \\ 0 & 0.67 & 0 & 0 & 0 \\ 0 & 0 & 0.67 & 0 & 0 \\ 0 & 0 & 0 & 0.67 & 0 \end{pmatrix}. \tag{2.6}$$

Due to the sparse structure of Leslie matrices we can characterize when a Leslie matrix is irreducible and when it is primitive. These results are difficult to find in the literature so are given in full here. We begin with irreducibility.

Theorem 2.2.2. *A Leslie matrix given by* (2.4) *is irreducible if, and only if,* $b_n \neq 0$.

Proof. Assume that $b_n \neq 0$. Figure 2.4 contains a minimal digraph of a Leslie matrix with $b_i = 0$ for $1 \leq i \leq n-1$

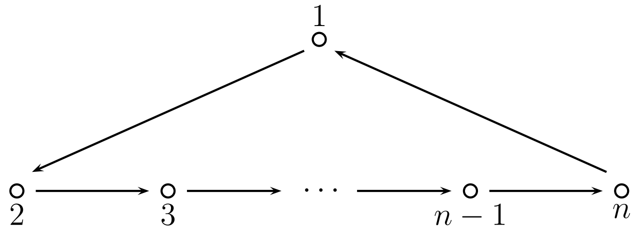


Figure 2.4: Digraph of a Leslie matrix with $b_n \neq 0$ and $b_i = 0$ for $1 \leq i \leq n-1$.

Clearly this graph is strongly connected as it will always contain cycle $1 \to n \to n-1 \to \cdots \to 2 \to 1$, regardless on the other values of b_i for $1 \le i \le n-1$, therefore the matrix is irreducible.

Now assume that a Leslie matrix is irreducible. By Remark 2.1.7 there must be at least one nonzero entry in each column and row, hence $b_n \neq 0$. □

By this theorem it follows immediately that the Leslie matrix L_1 given by (2.5) is irreducible and L_2 given by (2.6) is reducible.

What irreducibility means for Leslie matrices is that, individuals progress through the age-classes from smallest to larges at which time they reproduce and the newborn is in the first age-class and the cycle continuous.

We now consider primitivity of a Leslie matrix. To this end, define

$$\mathcal{I} := \{i \,:\, b_i \neq 0\}.$$

Theorem 2.2.3. *A Leslie matrix given by* (2.4) *is primitive if, and only if,* $b_n \neq 0$ *and the elements of the set* $\mathcal{I}$ *are coprime.*

Proof. The characteristic polynomial of a Leslie matrix is

$$\det(\lambda I - L) = \lambda^n - b_1\lambda^{n-1} - \sum_{i=2}^{n}\left(b_i \prod_{j=1}^{i-1} g_j\right)\lambda^{n-i},$$

hence defining $\beta_1 = b_1$ and

$$\beta_i = b_i \prod_{j=1}^{i-1} g_j, \qquad \forall\ 2 \le i \le n,$$

we have that

$$\det(\lambda I - L) = \lambda^n - \sum_{i=1}^{n} \beta_i \lambda^{n-i}$$

with $\beta_i \neq 0$ if, and only if, $i \in \mathcal{I}$.

Assume that L is primitive. This implies by Theorem 2.2.2 that L is irreducible, therefore, $b_n \neq 0$ and so $n \in \mathcal{I}$. Write $\mathcal{I} = \{i_1, \ldots, i_q\}$ with $i_1 < i_2 < \ldots < i_q = n$ and set

$$n_j = n - i_j, \qquad j = 1, \ldots, q \tag{2.7}$$

(and so $n_q = 0$). By primitivity

$$n - n_1,\ n_1 - n_2,\ \ldots,\ n_{q-1} - n_q$$

are coprime. Hence

$$i_1,\, i_2 - i_1,\, \dots,\, i_q - i_{q-1}$$

are coprime, and consequently, $i_1, \dots, i_q$ must be coprime.

Conversely, assume that $b_n \neq 0$ and the elements in $\mathcal{I}$ are coprime. By Theorem 2.2.2, the matrix L is irreducible. Defining n_j for $j = 1, \dots, q$, by (2.7), we need to show that

$$n - n_1,\, n_1 - n_2,\, \dots,\, n_{q-1} - n_q$$

are coprime. Now

$$\begin{aligned} n - n_1 &= i_1, \\ n_1 - n_2 &= i_2 - i_1, \\ &\vdots \\ n_{q-1} - n_q &= i_q - i_{p-1}. \end{aligned}$$

By hypothesis, $i_1, \dots, i_q$ are coprime. Let $p \in \mathbb{N}$ be such that p divides i_1, $i_2 - i_1$, $i_3 - i_2$, ... , $i_q - i_{q-1}$. Then we may conclude that p divides i_1, $i_2 = (i_2 - i_1) + i_1$, $i_3 = (i_3 - i_2) + i_2$, ... , i_q. Hence, $p = 1$, showing that $n - n_1$, $n_1 - n_2$, ... , $n_{q-1} - n_q$ are coprime. □

The following remark is a trivial consequence of Theorem 2.2.3 based on the elements of the set $\mathcal{I}$.

Remark 2.2.4. *Let L be an irreducible Leslie matrix.*

- *If $b_i \neq 0$ and $b_{i+1} \neq 0$ for some $1 \leq i \leq n-1$, then L is primitive.*
- *If n is prime, then if any $b_i \neq 0$ for $1 \leq i \leq n-1$, then L is primitive.*
- *If $b_1 \neq 0$, then L is primitive. (This also follows immediately from Lemma 2.1.14.)*

Returning to the Leslie matrix L_1 given by (2.5) which we saw earlier was irreducible, we can now say that it is also primitive by this remark noting that two consecutive birth rates are nonzero.

What it means biologically for a Leslie matrix, or a PPM in general, to be primitive is that after a certain amount of time has passed, an individual with have an ancestor and a descendent in age-class.

2.2.2 Leslie-Plus Matrices

A limitation of age-structured population projection models is that the maximum age of the species is n time steps. This means that for species which live for a long time, the PPM will become very large. One way of dealing with this is a hybrid system which consists of $n-1$ age-classes and 1 stage-class. The stage-class is the final class, that is the x_n term. The purpose of these models is to allow species with longer lives, where once they reach a certain age exhibit the same survival and birth rate, to be modeled without having to work with very large systems.

Leslie-Plus matrices can be used to model such species and take the form

$$L_+ = \begin{pmatrix} b_1 & \cdots & \cdots & \cdots & b_n \\ g_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \cdots & 0 & g_{n-1} & s \end{pmatrix}, \tag{2.8}$$

where, as with a Leslie matrix, $b_i \geq 0$ for $1 \leq i \leq n$, $0 < g_i \leq 1$ for $1 \leq i \leq n-1$ and now $0 < s \leq 1$. The only difference is the additional survival term, s which is the chance that an individual survives to the next time interval, once they are fully grown.

An example of a Leslie-Plus matrix is the PPM of a Wallaby (*Onychogalea fraenata*), which can be found in [42] and is

$$L_+ = \begin{pmatrix} 0 & 0 & 0 & 3.1 \\ 0.93 & 0 & 0 & 0 \\ 0 & 0.82 & 0 & 0 \\ 0 & 0 & 0.47 & 0.8 \end{pmatrix}. \tag{2.9}$$

Like Leslie matrices, we can characterize when a Leslie-Plus matrix is irreducible and primitive.

Theorem 2.2.5. *A Leslie-Plus matrix given by* (2.8) *is irreducible if, and only if,* $b_n \neq 0$.

Proof. The proof of this result follows immediately from Theorem 2.2.2 noting that the lead diagonal elements of a matrix play no role in if they are irreducible or not and that a Leslie-Plus matrix only differs from a Leslie matrix due to the presence of an addition entry on the lead diagonal. □

Theorem 2.2.6. *A Leslie-Plus matrix is primitive if, and only if, it is irreducible.*

Proof. Assume a Leslie-Plus matrix is primitive. From the definition of primitivity, it must be irreducible.

Now assume that a Leslie-Plus matrix, L_+, is irreducible. By Lemma 2.1.14 it follows immediately that it is also primitive, noting that $\mathrm{tr}(L_+) = b_1 + s \geq s > 0$. $\square$

By Theorem 2.2.5 we see that the Leslie-Plus matrix, (2.9) is irreducible and therefore, by Theorem 2.2.6, it is also primitive.

2.2.3 Growth Matrices

The final structure of a PPM we consider is that of a stage-class model and is known as a growth matrix. These matrices take the form

$$G = \begin{pmatrix} s_1 & b_2 & \cdots & \cdots & b_n \\ g_1 & s_2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g_{n-1} & s_n \end{pmatrix}, \tag{2.10}$$

where $b_i \geq 0$ for $2 \leq i \leq n$ denotes the birth rates of parents in stage-class i; $0 < g_1 \leq 1$ denotes the probability of an individual moving from the first stage class to the second; $0 < g_i < 1$ for $2 \leq i \leq n-1$ denotes the probability of an individual moving from the i-th stage class the $i+1$-th stage-class; $s_1 \geq 0$ is a combination of the probability of an individual remaining in the first stage-class and the birth rate of individuals born to parents in the first stage-class; $0 < s_i < 1$ for $2 \leq i \leq n-1$ is the probability that an individual remains in the i-th stage class; and $0 < s_n \leq 1$ is the probability that an individual remains in the final stage-class. An additional requirement is that $s_i + g_i \leq 1$ for $2 \leq i \leq n-1$, otherwise there is a probability that an individual could both progress to the next stage-class and remain in the same stage-class.

An example of a growth matrix is

$$G = \begin{pmatrix} 0 & 0 & 0 & 0.2488 & 40.5916 & 68.8415 \\ 0.4394 & 0.5704 & 0 & 0 & 0 & 0 \\ 0 & 0.0741 & 0.8413 & 0 & 0 & 0 \\ 0 & 0 & 0.0391 & 0.8405 & 0 & 0 \\ 0 & 0 & 0 & 0.0069 & 0.7782 & 0 \\ 0 & 0 & 0 & 0 & 0.17 & 0.9482 \end{pmatrix}, \tag{2.11}$$

which represents a green turtle (*Chelonia mydas*), which can be found in [20].

We characterize when a growth matrix is irreducible and primitive.

Theorem 2.2.7. *A growth matrix is irreducible if, and only if,* $b_n \neq 0$.

Proof. The proof of this result follows immediately from Theorem 2.2.2 noting that the lead diagonal elements of a matrix play no role in whether a matrix is irreducible or not and that a growth matrix only differs from a Leslie matrix due to the presence of additional terms on the lead diagonal. □

Theorem 2.2.8. *A growth matrix is primitive if, and only if, it is irreducible.*

Proof. Assume that a growth matrix is primitive. From the definition of primitivity, it must be irreducible.

Now assume that a growth matrix G is irreducible. Noting that $\mathrm{tr}(G) = \sum_{i=1}^{n} s_i > 0$, it follows immediately from Lemma 2.1.14 that G is primitive. □

The growth matrix associated with a green turtle in (2.11) is primitive noting that $n = 6$ and $b_6 = 68.8415$.

2.3 Metzler Matrices

In this section we gather from the literature results on Metzler matrices. Frequently, Metzler matrices go by other names such as essentially nonnegative matrices [6, p. 146] or quasi-positive matrices [131, p. 60]. In a dynamical systems context, they are the continuous-time analogue of nonnegative matrices which arise naturally in discrete-time nonnegative dynamical systems. We refer the reader to [6, 94, 145].

We begin by defining what a Metzler matrix is.

Definition 2.3.1. *A matrix $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ is said to be a* Metzler *matrix if all off-diagonal entries of M are nonnegative, that is, $m_{ij} \geq 0$ for all $1 \leq i, j \leq n$ with $i \neq j$.*

We also define a quantity which is useful when working with Metzler matrices.

Definition 2.3.2. *Let $M \in \mathbb{R}^{n\times n}$ be a Metzler matrix. Define*

$$\delta(M) := -\min_{1\leq i\leq n}(m_{ii}, 0) \geq 0,$$

which is the modulus of the most negative entry of M, or 0 if M has no negative entries.

The first result in this section links primitive matrices and Metzler matrices.

Lemma 2.3.3. *Let $M \in \mathbb{R}^{n\times n}$ be an irreducible Metzler matrix. If $\mu > \delta(M)$, then $\mu I + M$ is a primitive matrix.*

Proof. It is clear that $\mu I + M$ is a nonnegative, irreducible matrix with a positive trace, therefore by Lemma 2.1.14, $\mu I + M$ is a primitive matrix. □

The following well-known result demonstrates that the Metzler property characterizes linear flows which leave the nonnegative orthant invariant.

Lemma 2.3.4. *A matrix $M \in \mathbb{R}^{n\times n}$ is Metzler if, and only if, $e^{Mt} > 0$ for all $t \geq 0$. A matrix $M \in \mathbb{R}^{n\times n}$ is Metzler and irreducible if, and only if, $e^{Mt} \gg 0$ for all $t > 0$.*

Proof. The first claim is proved in, for example, [131, Section 3.1] or [128, Theorem 3]. The second claim may be found in [145, Theorem 8.2] or a more general version in [128, Proposition 1]. □

Definition 2.3.5. *A matrix $M \in \mathbb{R}^{n\times n}$ is said to be* Hurwitz *if all of its eigenvalues have negative real parts.*

Corollary 2.3.6. *If $M \in \mathbb{R}^{n\times n}$ is Metzler and Hurwitz, then $-M^{-1} > 0$. Further assume that M is irreducible, then $-M^{-1} \gg 0$.*

Proof. By the Hurwitz property of M, we have

$$\int_0^\infty e^{Mt} dt = -M^{-1}.$$

Since M is Metzler, Lemma 2.3.4 yields that

$$-M^{-1} = \int_0^\infty e^{Mt} dt > 0.$$

Now assume that M is irreducible. Again by Lemma 2.3.4

$$-M^{-1} = \int_0^\infty e^{Mt} dt \gg 0,$$

completing the proof. □

Classical Perron-Frobenius theory pertains to nonnegative matrices. Whilst a Metzler matrix M is in general not nonnegative, $\mu I + M$ is nonnegative for all $\mu \geq \delta(M)$. This observation, along with Lemma 2.3.3, enables applications of Perron-Frobenius theory to Metzler matrices. We first define the spectral abscissa of a matrix.

Definition 2.3.7. *Let $M \in \mathbb{R}^{n\times n}$. The* spectral abscissa $\alpha(M)$ *of M is defined by*

$$\alpha(M) = \max\{\mathrm{Re}\,\lambda \,:\, \lambda \in \sigma(M)\}.$$

Although all of the results in the following theorem are well known, they are scattered across the literature. For completeness and convenience of the reader, we provide a full proof.

Theorem 2.3.8. *Let $M \in \mathbb{R}^{n\times n}$ be an irreducible Metzler matrix, set $a := \alpha(M)$ and let $\mu > \delta(M)$. Then the following statements hold.*

(1) $a \in \sigma(M)$ and $a = \rho(\mu I + M) - \mu$.

(2) If $\lambda \in \sigma(M)$ and $\lambda \neq a$, then Re $\lambda < a$.

(3) a is simple.

(4) There exist unique vectors $v, w \in \mathbb{R}^n$ satisfying

$$v^T M = av^T, \qquad Mw = aw, \qquad v, w \gg 0, \qquad \|v\|_1 = \|w\|_1 = 1. \tag{2.12}$$

(5) The following convergence result holds:

$$\lim_{t\to\infty} e^{(M-aI)t} = \frac{1}{v^T w} wv^T \gg 0,$$

where v and w are the vectors satisfying (2.12).

Proof. Let $\mu > \delta(M)$ and set $r := \rho(\mu I + M)$. Since $\mu I + M$ is primitive, by Lemma 2.3.3, it follows from Theorem 2.1.22 that $r \in \sigma(\mu I + M)$, r is simple and $|z| < r$ for every $z \in \sigma(\mu I + M)$ such that $z \neq r$. Obviously,

$$\sigma(M) = \{z - \mu \ : \ z \in \sigma(\mu I + M)\},$$

and we have that $a = r - \mu \in \sigma(M)$ and a is simple. Moreover, for $z \in \sigma(\mu I + M)$, $z \neq r$, Re $z < r$ and therefore Re $\lambda < a$ for every $\lambda \in \sigma(M)$ such that $\lambda \neq a$, completing the proof of the first three statements.

We proceed to prove statement (4). Primitivity of $\mu I + M$, in combination with Theorem 2.1.23, shows that there exist unique $v, w \in \mathbb{R}^n$ such that $v, w \gg 0$, $\|v\|_1 = \|w\|_1 = 1$ and

$$v^T(\mu I + M) = rv^T, \qquad (\mu I + M)w = rw.$$

Therefore, by statement (1), $v^T M = av^T$ and $Mw = aw$, showing that statement (4) holds.

Finally, we prove statement (5). Let S be an invertible matrix such that

$$M = S^{-1} \begin{pmatrix} a & 0 \\ 0 & J \end{pmatrix} S,$$

where J is a Jordan matrix with $\sigma(J) = \sigma(M)\backslash\{a\}$. The matrix $\mu I + M$ is primitive, and, by statement (1), $\mu + a = \rho(\mu I + M)$. Therefore it follows from statement (2) of Theorem 2.1.23,

$$\mu + a = \rho(\mu I + M) > |\mu + \lambda| \qquad \forall \quad \lambda \in \sigma(J).$$

Consequently, $\rho((\mu + a)^{-1}(\mu I + J)) < 1$ and so

$$\begin{aligned}(\mu + a)^{-j}(\mu I + M)^j &= (\mu + a)^{-j} S^{-1} \begin{pmatrix} \mu + a & 0 \\ 0 & \mu I + J \end{pmatrix}^j S \\ &\to S \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S \quad \text{as} \quad j \to \infty.\end{aligned}$$

On the other hand, invoking statement (3) of Theorem 2.1.23, we have

$$(\mu + a)^{-j}(\mu I + M)^j \to \frac{1}{v^T w} w v^T \quad \text{as} \quad j \to \infty,$$

where v and w are the unique vectors satisfying (2.12). Hence,

$$S^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S = \frac{1}{v^T w} w v^T$$

and, furthermore,

$$e^{(M - aI)t} = e^{-at} e^{Mt} = S^{-1} \begin{pmatrix} 1 & 0 \\ 0 & e^{-at} e^{Jt} \end{pmatrix} S \to \frac{1}{v^T w} w v^T \quad \text{as} \quad t \to \infty,$$

where we have used that, by statement (3), $a > \operatorname{Re} \lambda$ for all $\lambda \in \sigma(J)$, to conclude that $e^{-at} e^{Jt} \to 0$ as $t \to \infty$. □

We shall also be making use of the following monotonicity property of the spectral abscissas of irreducible Metzler matrices.

Lemma 2.3.9. *Let $M, N \in \mathbb{R}^{n \times n}$ be Metzler matrices.*

(1) If $M > N$ and N is irreducible, then $\alpha(M) > \alpha(N)$.

(2) If $M \geq N$, then $\alpha(M) \geq \alpha(N)$.

Proof. Let $\mu > \delta(N)$ and assume that $M > N$ and N is irreducible. Then $\mu I + M > \mu I + N \geq 0$ and $\mu I + N$ is irreducible, and it follows from Corollary 2.1.24 that $\rho(\mu I + M) > \rho(\mu I + N)$. Hence, invoking statement (1) of Theorem 2.3.8,

$$\alpha(M) = \rho(\mu I + M) - \mu > \rho(\mu I + N) - \mu = \alpha(N),$$

completing the proof of statement (1). To prove statement (2), an argument similar to that in the proof of statement (2) of Corollary 2.1.24 can be used. □

2.3.1 Continuous Time Population Modeling

In this section we describe how Metzler matrices can be used to model in continuous times systems and act as a counterpart to Section 2.2. We refer the reader to [77] for further details on what are presented here along with a comparison to Leslie matrices.

As with discrete time models let $x(t) = (x_1(t), \ldots, x_n(t))^T$ denote a population divided into n age-classes at time t. Let $b_i \geq 0$ denote birth rates of the i-th age-class and $d_i > 0$ denote death rates of the i-th age-class. Finally let $m_i > 0$ for $1 \leq i \leq n-1$ denote the rate of movement from the i-th age-class to the $i+1$-th age-class. We can now write a set of differential equations to model the population. We have

$$\begin{aligned}
\dot{x}_1 &= b_1 x_1 + \ldots + b_n x_n - (d_1 + m_1)x_1, \\
\dot{x}_i &= m_{i-1}x_{i-1} - (d_i + m_i)x_i, \qquad i = 2, 3, \ldots, n-1, \\
\dot{x}_n &= m_{n-1}x_{n-1} - d_n x_n
\end{aligned}$$

We can combine these into vector form given by

$$\dot{x}(t) = \begin{pmatrix} b_1 - d_1 - m_1 & b_2 & b_3 & \cdots & b_n \\ m_1 & -d_2 - m_2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_{n-1} & -d_n \end{pmatrix} x(t) = Ax(t).$$

The matrix A is clearly a Metzler matrix and so we can apply the results in this section to such a system. Also note that the structure of A is the same as a growth matrix therefore, the matrix A is irreducible if and only if $b_n > 0$.

2.4 Sector Conditions

In this section we introduce two different types of sector conditions. Sector conditions are used at various stages throughout this thesis. In this section we will also introduce two nonlinearities which frequently appear in the literature for population modeling and mention how these fit in the different sector conditions.

2.4.1 Sectors Given by a Single Line

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$. We consider sectors of the form $f(y)/y < p$ for all $y > 0$, where $p > 0$. There are three different strengths of this type sector condition which we consider and are given by:

(S1) $f(y)/y \leq p$ for all $y > 0$,

(S2) $f(y)/y < p$ for all $y > 0$,

(S3) $\sup_{y>0} f(y)/y < p$.

For all three of these sector conditions, the graph of $f(y)$ lies in a sector bounded below by 0 and above by the line $l_1(y) = py$. For (S1), the graph of $f(y)$ need not lay strictly below the line $l_1(y)$, therefore, for some $y > 0$ it could be the case that $f(y) = py$. (S2) is a stronger condition in which we require $f(y) \neq py$ for any $y > 0$. (S3) says that there exists $q \in (0, p)$ such that f must satisfy $f(y)/y \leq q$ for all $y > 0$. That is the graph of $f(y)$ must lie in a smaller sector than in the other cases. It is obvious that (3) implies (2) which implies (1).

These three conditions are illustrated in Figure 2.5.

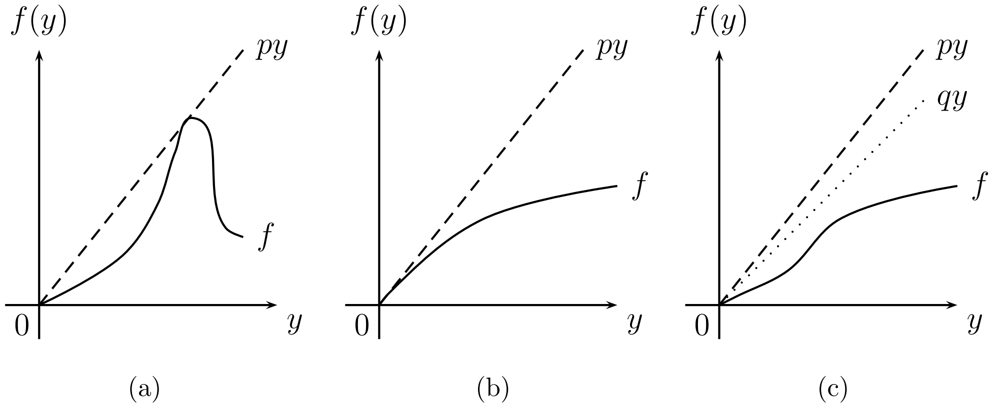


Figure 2.5: (a) A graph of a function f satisfying (S1). (b) A graph of a function f satisfying (S2). (c) A graph of a function f satisfying (S3).

2.4.2 Sectors Given by Two Lines

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$ and $p > 0$. Further assume there exists $y^* > 0$ such that $f(y^*) = py^*$. We consider two sectors, governed by the lines $l_1(y) = py$ and $l_2(y) = 2py^* - py$.

(S4) For $0 \leq y < y^*$, $py \leq f(y) \leq 2py^* - py$ and for $y > y^*$, $2py^* - py \leq f(y) \leq py$.

(S5) For $0 < y < y^*$, $py < f(y) < 2py^* - py$ and for $y > y^*$, $2py^* - py < f(y) < py$.

Sector (S4) could also be written as

$$\left|\frac{f(y) - f(y^*)}{y - y^*}\right| \leq p \qquad \forall \quad y \in \mathbb{R}_+ \backslash \{y^*\},$$

and sector (S5) could be written as

$$\left|\frac{f(y) - f(y^*)}{y - y^*}\right| < p \qquad \forall \quad y \in \mathbb{R}_+ \backslash \{0, y^*\}.$$

Both these sectors mean that the graph of $f(y)$ lies between the lines $l_1(y)$ and $l_2(y)$ for all $y > 0$. (S4) is a weaker condition as the graph of $f(y)$ could touch the lines $l_1(y)$ or $l_2(y)$ for some $y \neq y^*$, and (S5) is stronger as the possibility is ruled out. It is obvious that (S5) implies (S4).

We can strengthen sector (S5) by also imposing that f satisfies

$$\limsup_{y \to y^*} \left|\frac{f(y) - f(y^*)}{y - y^*}\right| < p \tag{2.13}$$

which means that the graph of $f(y)$ is not tangential to the lines $l_1(y)$ or $l_2(y)$ for $y = y^*$. We denote a sector given by (S5) which also satisfies (2.13) by (S6).

A way of strengthening sector (S6) would be to further assume that f satisfies

$$\limsup_{y \to \infty} \left|\frac{f(y) - f(y^*)}{y - y^*}\right| < p$$

which means that for sufficiently large y, the points $(y, f(y))$ lie in a sector defined by two straight lines of slope q and $-q$ where $0 < q < p$.

Sector conditions (S4), (S5) and (S6) are illustrated in Figure 2.6.

2.4.3 Examples of Nonlinearities

We look at two common nonlinearities appearing in population modeling.

Beverton-Holt Nonlinearity

Consider a Beverton-Holt nonlinearity [8] given by

$$f_1(y) = \frac{my}{k + y},$$

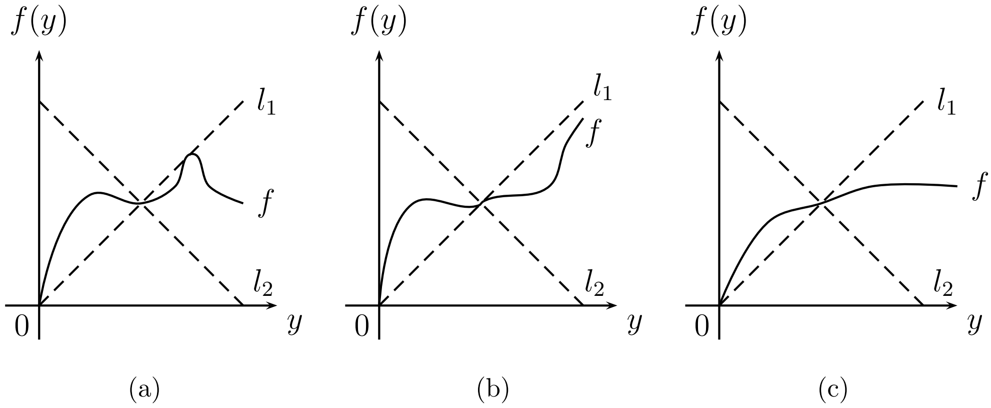


Figure 2.6: (a) A graph of a function f sector condition (S4). (b) A graph of a function f satisfying sector condition (S5). (c) A graph of a function f satisfying sector condition (S6).

where $m, k > 0$. It can be seen that m denotes a carrying capacity of the population as

$$\lim_{y \to \infty} f_1(y) = m.$$

This function will always satisfy a sector condition, and which it will satisfy depends on the constants and p.

- If $m/k < p$ then (S3) is satisfied.
- If $m/k = p$ then (S2) is satisfied.
- If $m/k > p$ then (S6) is satisfied.

We reach this conclusion noting that $f_1(y)$ is a strictly increasing function, and $f_1'(y)$ is a decreasing, yet positive function.

Example 2.4.1. *Consider the Beverton-Holt function given by*

$$f_1(y) = \frac{2y}{10 + y}. \tag{2.14}$$

The graph of this function is plotted in Figure 2.7 as a blue line. Note that $f_1(y) < 2$ *for all* $y \in \mathbb{R}_+$ *which is shown as a black dotted line in Figure 2.7. We consider* $p = 1/3, 1/5$ *and* $p = 1/20$ *such that (S2), (S3) and (S6) hold receptively. The lines* $l_1(y) = py$ *are plotted in Figure 2.7 for each of the values of* p *along with the line* $l_2(y)$ *for* $p = 1/20$ *to illustrate these sector conditions.*

Ricker Nonlinearity

Now consider a Ricker nonlinearity [117] given by

$$f_2(y) = ye^{-\beta y},$$

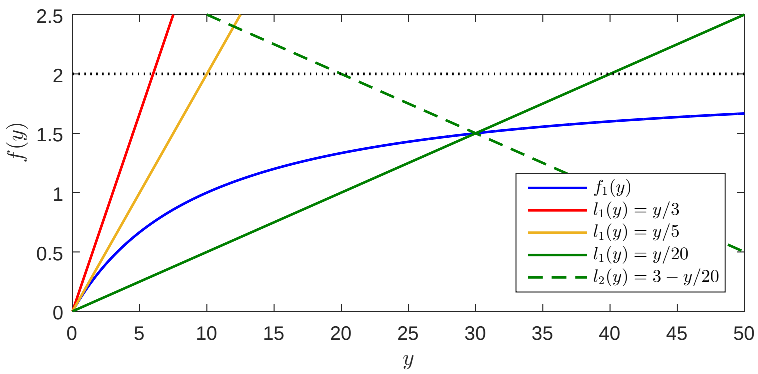


Figure 2.7: Graph for Example 2.4.1. Induces the graph of $f_1(y)$ given by (2.14) and the lines $l_1(y) = py$ for $p = 1/2, 1/5, 1/20$ and the line $l_2(y)$ for $p = 1/20$ to illustrate the sector conditions this function satisfies.

where $\beta > 0$. This will not always satisfy a sector condition depending on the value of p.

- If $p > 1$ then (S3) is satisfied.
- If $p = 1$ then (S2) is satisfied.
- If $e^{-2} \leq p < 1$ then (S6) is satisfied.
- If $p < e^{-2}$ then no sector condition is satisfied.

These claims can be easily verified by elementary calculus.

Example 2.4.2. *Consider the Ricker function given by*

$$f_2(y) = ye^{-0.2y}. \tag{2.15}$$

The graph of this function is plotted in Figure 2.8 as a blue line. We consider $p = 2, 1$ and $p = 1/2$ such that (S2), (S3) and (S6) hold respectively. We also consider $p = 1/10 < e^{-2}$ in which case no sector condition is satisfied. The lines $l_1(y) = py$ are plotted in Figure 2.7 for each of the values of p along with the line $l_2(y)$ for $p = 1/2$ and $p = 1/10$ to illustrate these sector conditions.

2.5 Comparison Functions

We define comparison functions which are used at several stages in this thesis.

Definition 2.5.1. • *Let $\mathcal{K}$ denote the set of all continuous functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(0) = 0$ and φ is strictly increasing.*

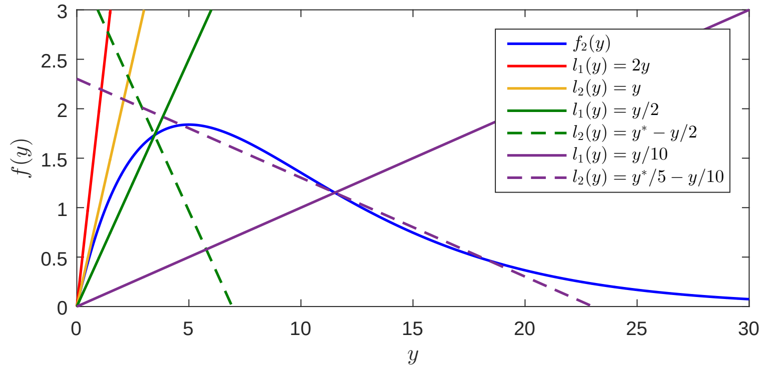


Figure 2.8: Graph for Example 2.4.2. Induces the graph of $f_2(y)$ given by (2.14) and the lines $l_1(y) = py$ for $p = 2, 1, 1/2, 1/10$ and the line $l_2(y)$ for $p = 1/2, 1/10$ to illustrate the sector conditions this function satisfies.

- *Moreover, define*

$$\mathcal{K}_\infty := \{\varphi \in \mathcal{K} \;:\; \varphi(s) \to \infty \text{ as } s \to \infty\}.$$

- *We denote by $\mathcal{KL}_D$ the set of functions $\psi : \mathbb{R}_+ \times \mathbb{N}_0 \to \mathbb{R}_+$ with the following properties:*
  - $\psi(\,\cdot\,, t) \in \mathcal{K}$ *for every* $t \in \mathbb{N}_0$
  - $\psi(s, \,\cdot\,)$ *is nonincreasing with*

$$\lim_{t\to\infty} \psi(s,t) = 0$$

for every $s \geq 0$.

- *We denote by $\mathcal{KL}$ the set of functions $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties:*
  - $\psi(\,\cdot\,, t) \in \mathcal{K}$ *for every* $t \in \mathbb{R}_+$
  - $\varphi(s, \,\cdot\,)$ *is nonincreasing with*

$$\lim_{t\to\infty} \psi(s,t) = 0$$

for every $s \geq 0$.

Chapter 3

Stability of Nonnegative Lur'e Systems in Continuous Time

This chapter is based on [10] and acts as a continuous time counterpart to Chapter 5.

3.1 Introduction

In mathematical control theory, much attention has been devoted to a class of nonlinear systems referred to as Lur'e systems [55, 72, 86, 96, 124, 146, 153]. These systems are comprised of two components: a linear system with state x, input u and output y, given by

$$\dot{x} = Ax + bu, \qquad x(0) = x^0, \qquad y = c^T x, \tag{3.1}$$

where $A \in \mathbb{R}^{n \times n}$, $b, c \in \mathbb{R}^n$, and a nonlinear feedback $u = f(y)$ where $f : \mathbb{R} \to \mathbb{R}$. The resulting nonlinear feedback system is given by

$$\dot{x} = Ax + bf(c^T x), \qquad x(0) = x^0. \tag{3.2}$$

Lur'e systems arise in various contexts in circuit, control and systems theory. Under the common assumption that the nonlinearity f satisfies $f(0) = 0$, it follows that 0 is an equilibrium of (3.2). In this chapter we not only deal with the case when $f(0) = 0$ as in [10], but we also consider the case where $f(0) > 0$.

The study of stability properties of the zero equilibrium of Lur'e systems is termed *absolute stability* and generally refers to the situation where the linear system (3.1) is known and the nonlinearity f is unknown, but usually sector bounded. Absolute stability is a well-studied and active area of research, and we refer the reader to [55, 72, 86, 124, 146, 153] and the references therein. A typical absolute stability result provides conditions on the linear components (either in time or frequency domain) which ensure that zero is globally asymptotically stable (GAS) for a class of sector bounded nonlinearities. Crucially, stability of the Lur'e system is determined by the sector bounds and not by the individual nonlinearity f itself. Such inherent robustness makes absolute stability results especially powerful. Furthermore, if the Lur'e system (3.2) is subject to an external additive time-dependent disturbance d, that is, if (3.2) is replaced by

$$\dot{x} = Ax + bf(c^T x) + d, \qquad x(0) = x^0, \tag{3.3}$$

then recent research [72, 124] shows that the conditions of a well-known classical absolute stability result, the so-called circle criterion, guarantee input-to-state stability (ISS) of the forced system (3.3), thereby adding to the inherent robustness properties of stable Lur'e systems. Without going into details here, we mention that ISS means that the map $\mathbb{R}^n \times L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}^n$, $(x^0, d) \mapsto x(t)$ has "nice" boundedness and asymptotic properties. For an overview of ISS theory we refer the reader to [25, 138].

Systems of type (3.2) or (3.3) also arise naturally in biology, ecology and chemistry, for example, in T-cell receptor signal transduction [98, 135]; enzyme synthesis [103, Section 7.2], [131, Chapter 4.2] and [144]; and population dynamics [51]. Lur'e type models for economic fluctuations have also been suggested, see [50]. In a population model, f captures density-dependence, for example, a carrying capacity. In a chemical reaction model, f may describe a nonlinear reaction rate between certain components. In these applied contexts, a common key feature is that the components of the state x of the model, which may represent population abundances, chemical concentrations, or economic quantities (such as prices) are, necessarily, nonnegative. In this case, the matrix A is Metzler, whilst b and c are nonnegative and f maps the interval $[0, \infty)$ into itself.

In the context of biological, ecological and chemical models the focus is often the existence and stability of nonzero equilibria which then correspond to the co-existence of populations or chemical compounds.

Small-gain techniques have been applied to Lur'e systems to develop stability/instability trichotomy results for classes of both finite-dimensional [143] and infinite-dimensional [112] discrete-time population models. For the situations considered in [112, 143], only one of three outcomes is possible: either zero is GAS, there is a stable nonzero equilibrium which attracts all nonzero solutions, or else all nonzero solutions diverge component-wise. Further trichotomies of stability for various classes of monotone discrete-time dynamical systems have been established in [82, Chapter 6] and [83] for finite-dimensional systems and in [64, 132] for infinite-dimensional systems. The paper [83] also contains a limit set trichotomy for a class of periodic continuous-time systems satisfying certain monotonicity conditions.

In this chapter we develop results reminiscent of a stability/instability trichotomy for continuous-time Lur'e systems using ideas from absolute stability theory (for instance, [72]). Our results cover asymptotic and exponential stability as well as ISS. We emphasize that the Lur'e systems considered in this chapter are in general not monotone and therefore results from the theory of monotone systems [131] do not apply.

This chapter is organized as follows. Section 3.2 collects material on absolute stability and input-to-state stability which we shall require. Section 3.3 introduces the concept of a nonnegative system along with some basic results. Section 3.4 contains the main results for unforced Lur'e systems and is split into three parts which resemble a stability/instability trichotomy. Section 3.5 contains the main results for forced Lur'e systems and is once again split into three parts which resemble a stability/instability trichotomy. We conclude this chapter with Section 3.6 which provides detailed discussions of two examples. The first example we consider is from a population modeling perspective and the second is about enzyme synthesis.

3.2 Stability of Continuous Time Lur'e Systems

Consider the continuous time Lur'e system

$$\dot{x} = Ax + bf(c^T x), \qquad x(0) = x^0 \in \mathbb{R}^n, \tag{3.4}$$

where $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz. Let $x(\,\cdot\,; x^0)$ denote the continuously differentiable unique maximally defined forward solution of the initial-value problem (3.4), the existence of which is guaranteed by [90, Theorem 4.22] or [134, Theorem 54], for example. If there exists an affine linear bound for the nonlinearity f, then $x(t; x^0)$ is defined for all $t \geq 0$ (see [90, Proposition 4.12]).

Application of linear output feedback of the form $f(y) = \kappa y$ to (3.4) leads to

$$\dot{x} = (A + \kappa bc^T)x, \tag{3.5}$$

where κ is a constant which is sometimes referred to as feedback gain. Define

$$\mathbb{S}(A, b, c^T) := \{\kappa \in \mathbb{C} \;:\; A + \kappa bc^T \text{ is Hurwitz}\},$$

which is the set of *complex* stabilizing output feedback gains for the linear system (A, b, c^T).

Definition 3.2.1. *Let $\mathbb{D}$ denote the complex disc centered at $k \in \mathbb{C}$ with radius $r > 0$, that is*

$$\mathbb{D}(k, r) := \{\kappa \in \mathbb{C} \ : \ |\kappa - k| < r\}.$$

There are many types of stability and appearing in the literature these types of stability go by different names. The following definition gives precise meaning to the types of stability appearing in this chapter.

Definition 3.2.2. *Consider the system* (3.4)*.*

1. *The equilibrium* 0 *is said to be* stable in the large *in the sense that there exists exists $g \geq 1$ such that, for every $x_0 \in \mathbb{R}$,*

$$\|x(t; x^0)\| \leq g\|x^0\| \qquad \forall \quad t \geq 0.$$

2. *The equilibrium* 0 *is said to be* globally asymptotically stable *if* 0 *is stable in the large and for every $x^0 \in \mathbb{R}^n$, $x(t; x^0) \to 0$ as $t \to \infty$.*

3. *The equilibrium* 0 *is said to be* globally exponentially stable *if there exists $\gamma > 0$ and $g \geq 1$ such that, for every $x^0 \in \mathbb{R}^n$,*

$$\|x(t; x^0)\| \leq ge^{-\gamma t}\|x^0\| \qquad \forall \quad t \geq 0.$$

The following result plays a key role in this chapter. For more information on this result the reader is referred to [72], where the result is developed and proved.

Theorem 3.2.3. *Let $A \in \mathbb{R}^{n \times n}$, $b, c \in \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz with $f(0) = 0$. Assume that*

$$\mathbb{D}(k, r) \subseteq \mathbb{S}(A, b, c^T), \tag{3.6}$$

where $k \in \mathbb{R}$ and $r > 0$.

(1) If

$$\frac{f(y)}{y} \in [k - r, k + r] \qquad \forall \quad y \in \mathbb{R}\backslash\{0\},$$

then there exists $g \geq 1$ such that

$$\|x(t; x^0)\| \leq g\|x^0\| \qquad \forall \quad t \geq 0, \ \forall \ x^0 \in \mathbb{R}^n. \tag{3.7}$$

In particular, the equilibrium 0 *of* (3.4) *is stable in the large.*

(2) If

$$\frac{f(y)}{y} \in (k-r, k+r) \qquad \forall \quad y \in \mathbb{R}\backslash\{0\}, \tag{3.8}$$

then the equilibrium 0 *of* (3.4) *is globally asymptotically stable, that is* 0 *is stable in the large and for all* $x^0 \in \mathbb{R}^n_+$, $x(t) \to 0$ *as* $t \to \infty$.

(3) If there exists $r_1 \in (0, r)$ *such that*

$$\frac{f(y)}{y} \in (k-r_1, k+r_1) \qquad \forall \quad y \in \mathbb{R}\backslash\{0\}, \tag{3.9}$$

then the equilibrium 0 *of* (3.4) *is globally exponentially stable, that is, there exists* $\gamma > 0$ *and* $g \geq 1$ *such that for all* $x^0 \in \mathbb{R}^n$

$$\|x(t;x^0)\| \leq ge^{-\gamma t}\|x^0\| \qquad \forall \quad t \geq 0, \quad \forall \quad x^0 \in \mathbb{R}^n.$$

The well known control theoretical circle criterion (see [146]) can be derived as a corollary to Theorem 3.2.3 (see [72]). Roughly speaking, statement (2) of Theorem 3.2.3 says that linear stability, namely $\mathbb{D}(k,r) \subseteq \mathbb{S}(A, b, c^T)$, implies global asymptotic stability for all nonlinearities $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\frac{f(y)}{y} \in (k-r, k+r), \qquad \forall \quad y \in \mathbb{R}\backslash\{0\}.$$

We emp hasize that stability of the linear feedback system (3.5) has to hold for all *complex* κ satisfying $|\kappa - k| < r$. It is easy to see that the conclusions in Theorem 3.2.3 remain true for *complex* nonlinearities $f : \mathbb{C} \to \mathbb{C}$, provided that, in statements (1)-(3) conditions (3.7)-(3.9) are replaced by

$$\frac{f(y)}{y} \in \overline{\mathbb{D}(k,r)}, \quad \frac{f(y)}{y} \in \mathbb{D}(k,r) \quad \text{and} \quad \frac{f(y)}{y} \in \mathbb{D}(k,r_1)$$

respectively, where the conditions hold for all $y \in \mathbb{C}\backslash\{0\}$ and $\overline{\mathbb{D}(k,r)}$ denotes the *closed* complex ball, centered at k with radius r.

We present a special case wherein the complex condition

$$\mathbb{D}(k,r) \subseteq \mathbb{S}(A, b, c^T)$$

can be replaced by its real counterpart

$$(k-r, k+r) \subseteq \mathbb{S}(A, b, c^T).$$

Corollary 3.2.4. *Let* $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, *let* $f : \mathbb{R} \to \mathbb{R}$ *be locally Lipschitz with* $f(0) = 0$ *and let* $k \in \mathbb{R}$ *and* $r > 0$. *Assume that* b *and* c *are nonnegative,*

$A + kbc^T$ is Metzler and

$$(k - r, k + r) \subset \mathbb{S}(A, b, c^T). \tag{3.10}$$

Under these conditions, statements (1)-(3) of Theorem 3.2.3 hold.

Proof. Set $A_k := A + kbc^T$ and define

$$r_{\mathbb{F}}(A_k; b, c^T) := \inf\{|\kappa| \; : \; \kappa \in \mathbb{F}, \; A_k + \kappa bc^T \text{ is not Hurwitz}\},$$

where $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, which is the stability radius of A_k with respect to perturbation structure given by b and c. Invoking (3.10), we see that $r \leq r_{\mathbb{R}}(A_k; b, c^T)$. By a stability radius result for nonnegative systems proved in [62],

$$r_{\mathbb{R}}(A_k; b, c^T) = r_{\mathbb{C}}(A_k; b, c^T)$$

and consequently, $\mathbb{D}(0, r) \subseteq \mathbb{S}(A_k, b, c^T)$, or equivalently, $\mathbb{D}(k, r) \subseteq \mathbb{S}(A, b, c^T)$. The claim now follows from Theorem 3.2.3. □

Let $\boldsymbol{G}$ denote the transfer function of (A, b, c^T) given by $\boldsymbol{G}(s) = c^T(sI - A)^{-1}b$. The next result considers a scenario wherein the Lur'e system (3.4) has an equilibrium $x^* \neq 0$ in addition to the zero equilibrium.

Theorem 3.2.5. *Consider the system* (3.4) *and assume that A is Hurwitz, $f(0) = 0$, $\|\boldsymbol{G}\|_{H^\infty} = |\boldsymbol{G}(0)| > 0$ and there exists $y^* \neq 0$ such that $y^* = \boldsymbol{G}(0)f(y^*)$. Then $x^* = -A^{-1}bf(y^*) \neq 0$ is an equilibrium of* (3.4) *and the following statements hold.*

(1) If

$$\left|\frac{f(y) - f(y^*)}{y - y^*}\right| \leq \frac{1}{|\boldsymbol{G}(0)|} \qquad \forall \quad y \in \mathbb{R}\backslash\{y^*\}, \tag{3.11}$$

then there exists $g \geq 1$ such that

$$\|x(t; x^0) - x^*\| \leq g\|x^0 - x^*\|$$

for all $x^0 \in \mathbb{R}^n$ and $t \geq 0$. In particular, the equilibrium x^* of (3.4) *is stable in the large.*

(2) If

$$\left|\frac{f(y) - f(y^*)}{y - y^*}\right| < \frac{1}{|\boldsymbol{G}(0)|} \qquad \forall \quad y \in \mathbb{R}\backslash\{0, y^*\}, \tag{3.12}$$

then, for every $x^0 \in \mathbb{R}^n$, we have that $x(t; x^0) \to x^*$ or $x(t; x^0) \to 0$ as $t \to \infty$.

Proof. Since $\boldsymbol{G}(0) \neq 0$ and $y^* \neq 0$, we have $f(y^*) \neq 0$ and $x^* \neq 0$. Noting that $c^T x^* = y^*$, we conclude that

$$Ax^* + bf(c^T x^*) = Ax^* + bf(y^*) = 0,$$

showing that x^* is an equilibrium of (3.4).

Let $x^0 \in \mathbb{R}^n$ and set $\tilde{x}(t) = x(t; x^0) - x^*$ for all $t \geq 0$. Furthermore, defining $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by $\tilde{f}(y) = f(y + y^*) - f(y^*)$ for all $y \in \mathbb{R}$, it follows that

$$\dot{\tilde{x}} = A\tilde{x} + b\tilde{f}(c^T \tilde{x}), \qquad \tilde{x}(0) = x^0 - x^*. \tag{3.13}$$

Setting $p := 1/|\boldsymbol{G}(0)|$, it follows by hypothesis that $p = 1/\|\boldsymbol{G}\|_{H^\infty}$ and thus, by elementary stability radius theory (see [61] or [62, Section 5.3])

$$\inf \left\{ |\kappa| \ : \ \kappa \in \mathbb{C}, \ A + \kappa b c^T \text{ is not Hurwitz} \right\} = p.$$

Consequently,

$$\mathbb{D}(0, p) \subseteq \mathbb{S}(A, b, c^T). \tag{3.14}$$

To prove statement (1), note that

$$\left| \frac{\tilde{f}(y)}{y} \right| \leq p \qquad \forall \quad y \in \mathbb{R} \backslash \{0\}. \tag{3.15}$$

Combining (3.13)-(3.15) with statement (1) of Theorem 3.2.3 yields the existence of a constant $g \geq 1$ such that, for every $x^0 \in \mathbb{R}^n$,

$$\|x(t; x^0) - x^*\| \leq g \|x^0 - x^*\|, \qquad \forall \quad t \geq 0.$$

We proceed to prove statement (2). To this end, observe that

$$\left| \frac{\tilde{f}(y)}{y} \right| < p \qquad \forall \quad y \in \mathbb{R} \backslash \{0, -y^*\}. \tag{3.16}$$

By [61, proof of Theorem 5.6.22], there exists a positive semi-definite $P = P^T \in \mathbb{R}^{n \times n}$ such that the quadratic form $V(z) = \langle Pz, z \rangle$ satisfies

$$\begin{aligned} V_d(z) : &= \left\langle (\nabla V)(z), Az + b\tilde{f}(c^T z) \right\rangle \\ &\leq \tilde{f}^2(c^T z) - p^2 (c^T z)^2 \qquad \forall \quad z \in \mathbb{R}^n, \end{aligned} \tag{3.17}$$

where the last inequality follows from (3.16).

By statement (1), using [90] for example, it follows that, $\tilde{x}$ is bounded and so its ω-limit set Ω is nonempty, compact, connected and invariant. Furthermore,

Ω is the smallest closed set with the property

$$\lim_{t\to\infty} \operatorname{dist}(\tilde{x}(t), \Omega) = 0.$$

As a consequence of LaSalle's invariance principle, $\Omega \subseteq V_d^{-1}(0)$. By (3.16) and (3.17),

$$V_d^{-1}(0) \subseteq \left\{ z \in \mathbb{R}^n \ : \ c^T z = 0 \text{ or } c^T z = -y^* \right\}.$$

Hence

$$\Omega \subseteq \ker c^T \cup (\ker c^T - x^*).$$

The sets $\ker c^T$ and $\ker c^T - x^*$ are closed and disjoint as $c^T x^* = y^* \neq 0$. Now Ω is connected and therefore,

$$\Omega \subset \ker c^T \quad \text{or} \quad \Omega \subseteq \ker c^T - x^*.$$

Consequently,

$$\lim_{t\to\infty} c^T \tilde{x}(t) = 0 \quad \text{or} \quad \lim_{t\to\infty} c^T \tilde{x}(t) = -y^*.$$

Hence by (3.13) and the Hurwitz property of A,

$$\lim_{t\to\infty} \tilde{x}(t) = 0 \quad \text{or} \quad \lim_{t\to\infty} \tilde{x}(t) = -A^{-1} b \tilde{f}(-y^*) = -x^*.$$

Thus

$$\lim_{t\to\infty} x(t; x^0) = x^* \quad \text{or} \quad \lim_{t\to\infty} x(t; x^0) = 0,$$

completing the proof. □

Note that (3.11) is a "sector" condition in the sense that the graph of f is "sandwiched" between the lines $l_1(y) = py$ and $l_2(y) = 2py^* - py^*$, where $p = 1/\boldsymbol{G}(0)$. See Figure 3.1 for an illustration of this. In the case of the "strict" sector condition (3.12), the graph of f "touches" these lines only at the points $(0,0)$ and $(y^*, f(y^*))$.

Let us now consider forced Lur'e systems of the form

$$\dot{x} = Ax + bf(c^T x) + d, \qquad x(0) = x^0 \in \mathbb{R}^n, \tag{3.18}$$

where $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$. Let $x(\,\cdot\,; x^0, d)$ denote the unique absolutely continuous maximally defined forward solution of the initial-value problem (3.18) (see [134, Theorem 54]).

The function d is an external disturbance, otherwise known as an input or forcing term. In most contexts d will be piecewise continuous, in which case $x(\,\cdot\,; x^0, d)$ is piecewise continuously differentiable. Furthermore, if the nonlinearity f satisfies an affine linear bound, then $x(t; x^0, d)$ is defined for all

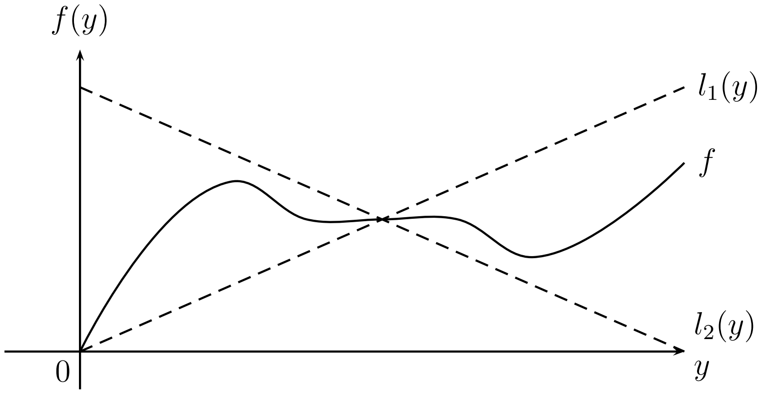


Figure 3.1: A graph of a function f satisfying a sector condition, that is it lies between the lines $l_1(y) = py$ and $l_2(y) = 2py^* - py$.

$t \geq 0$ (see [90, Proposition 4.12]).

We now introduce a different type of stability which relates to disturbed systems.

Definition 3.2.6. *Consider the system* (3.18)*.* 0 *is said to be* input-to-statte stable *(ISS) if there exists $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that, for all $x^0 \in \mathbb{R}^n$ and all $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$,*

$$\|x(t; x^0, d)\| \leq \psi(\|x^0\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \qquad \forall \quad t \geq 0.$$

Obviously, if $d = 0$ in (3.18), then 0 is an equilibrium of (3.18). The reader is referred to [25] and [138] for more details on ISS theory.

The proof of the following theorem can be found in [124].

Theorem 3.2.7. *Let $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, and let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz with $f(0) = 0$. Assume that*

$$\mathbb{D}(k, r) \subseteq \mathbb{S}(A, b, c^T), \tag{3.19}$$

where $k \in \mathbb{R}$ and $r > 0$. If there exists $\beta \in \mathcal{K}_\infty$ such that

$$|f(y) - ky| \leq r|y| - \beta(|y|) \quad \forall \quad y \in \mathbb{R}, \tag{3.20}$$

then there exists $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that, for all $x^0 \in \mathbb{R}^n$ and all $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$,

$$\|x(t; x^0, d)\| \leq \psi(\|x^0\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \quad \forall \quad t \geq 0. \tag{3.21}$$

Note that there exists $\beta \in \mathcal{K}_\infty$ such that (3.20) holds if, and only if, $|f(y) -$

$ky| < r|y|$ for all $y \neq 0$ and

$$r|y| - |f(y) - ky| \to \infty \quad \text{as} \quad |y| \to \infty.$$

Note that condition (3.8) from Theorem 3.2.3 can be rewritten in the form

$$|f(y) - ky| < r|y| \qquad \forall \quad y \in \mathbb{R}\backslash\{0\}, \tag{3.22}$$

thus it becomes apparent that Theorem 3.2.7 is structurally very similar to Theorem 3.2.3. In particular, Theorem 3.2.7 says that linear stability, namely $\mathbb{D}(k, r) \subseteq \mathbb{S}(A, b, c^T)$, implies ISS for all nonlinearities $f : \mathbb{R} \to \mathbb{R}$ satisfying (3.20).

It is easy to construct counterexamples to demonstrate that Theorem 3.2.7 does not remain valid if the condition $\beta \in \mathcal{K}_\infty$ is replaced by $\beta \in \mathcal{K}$ (see [124]). In particular, (3.19) together with (3.22) is not sufficient for ISS.

Finally, we mention that if there exists $r_1 \in (0, r)$ such that (3.9) holds, then (3.20) is satisfied with $\beta(s) = r_0 s$, where r_0 is an arbitrary constant satisfying $0 < r_0 < r - r_1$. In this case, assuming that the linear condition (3.19) holds, it can be shown that the ISS estimate (3.21) holds with ψ and φ given by $\psi(s, t) = c_1 e^{c_2 t} s$ and $\varphi(s) = c_3 s$ where c_1, c_2, c_3 are suitable positive constants.

3.3 Nonnegative Lur'e Systems in Continuous Time

In this section we introduce assumptions which ensure that the state $x(t)$ of a Lur'e system given by (3.4) remains nonnegative for all $t \in \mathbb{R}_+$. We then make a series of remarks and lemmas about these nonnegative Lur'e systems. To conclude this section we introduce a nonnegative Lur'e system which will be used as an example throughout this chapter and show that it satisfies the assumptions which have been introduced.

Firstly we make a trivial remark.

Remark 3.3.1. *Consider the system* (3.4). *If* $f(0) = 0$*, then* 0 *is an equilibrium of the system.*

We proceed to introduce assumptions which will be used throughout this chapter.

(A3.1) The matrix A is a Metzler matrix and $b, c \in \mathbb{R}^n_+$ are nonzero.

(A3.2) The matrix A is Hurwitz.

(A3.3) The matrix $A + bc^T$ is irreducible.

(A3.4) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz.

Note that if **(A3.1)** and **(A3.4)** hold, then, for every $x^0 \in \mathbb{R}^n_+$, the unique maximally defined forward solution $x(\,\cdot\,; x^0)$ of (3.4) satisfies $x(t; x^0) \in \mathbb{R}^n_+$ for all $t \geq 0$ for which the solution exists.

The following remark is a straightforward consequence of the results in Section 2.1.2.

Remark 3.3.2. *(1) If* **(A3.1)** *and* **(A3.3)** *are satisfied, then $A + kbc^T$ is irreducible for every $k > 0$.*

(2) If **(A3.1)** *and* **(A3.3)** *are satisfied, then for every $\mu > \delta(A)$ and every $k > 0$, $\mu I + A + kbc^T$ is primitive.*

(3) If there exist $\mu, k \geq 0$ such that $\mu I + A + kbc^T$ is primitive, then **(A3.3)** *holds.*

The following lemma plays an important role in this chapter. It demonstrates the nonnegativity of the steady-state gain of the linear system (A, b, c^T) and relates it to the H^∞-norm, under certain assumptions.

Lemma 3.3.3. *Assume that* **(A3.1)** *and* **(A3.2)** *are satisfied, then*

$$\|\boldsymbol{G}\|_{H^\infty} = |\boldsymbol{G}(0)| = \boldsymbol{G}(0) \geq 0.$$

If additionally **(A3.3)** *is satisfied, then $\boldsymbol{G}(0) > 0$.*

Proof. Assume that **(A3.1)** and **(A3.2)** are satisfied. Then, by Lemma 2.3.4, $c^T e^{At} b \geq 0$ for all $t \geq 0$, and hence, for all $s \in \mathbb{C}$ with Re $s \geq 0$,

$$|\boldsymbol{G}(s)| = \left| \int_0^\infty c^T e^{At} b e^{-st} dt \right| \leq \int_0^\infty |c^T e^{At} b| |e^{-st}| dt \leq \int_0^\infty c^T e^{At} b dt = \boldsymbol{G}(0),$$

showing that $\|\boldsymbol{G}\|_{H^\infty} = \boldsymbol{G}(0) \geq 0$.

Now assume that **(A3.1)**-**(A3.3)** are satisfied. It remains to show that $\boldsymbol{G}(0) > 0$ and to do so, we invoke a contradiction argument. To this end, suppose $\boldsymbol{G}(0) = 0$. Then

$$0 = \boldsymbol{G}(0) = \int_0^\infty c^T e^{At} b dt,$$

and noting that $c^T e^{At} b \geq 0$ for all $t \geq 0$ by **(A3.1)** and Lemma 2.3.4, we obtain

$$c^T e^{At} b = 0 \qquad \forall \quad t \geq 0.$$

Consequently,

$$c^T e^{(\mu I + A)t} b = e^{\mu t} c^T e^{At} b = 0 \qquad \forall \quad t \geq 0,$$

where $\mu > \delta(A)$. Repeated differentiation and evaluation at $t = 0$ yields

$$c^T (\mu I + A)^k b = 0 \qquad \forall \quad k \in \mathbb{N}_0.$$

As a consequence,

$$\sum_{k=0}^{\infty} c^T \frac{(\mu I + A + bc^T)^k}{k!} b = 0. \tag{3.23}$$

On the other hand, we know by part (2) of Remark 3.3.2, that $\mu I + A + bc^T$ is primitive, implying that the series in (3.23) has a positive sum. This provides the desired contradiction and therefore $\boldsymbol{G}(0) > 0$. □

Definition 3.3.4. *Define $p \in \mathbb{R}_+$ to be the inverse of the steady-state gain of (A, b, c^T), that is*

$$p := \frac{1}{\boldsymbol{G}(0)} = \frac{-1}{c^T A^{-1} b}.$$

For the remainder of this chapter p will always be defined by Definition 3.3.4.

Lemma 3.3.5. *Assume* **(A3.1)**-**(A3.3)** *hold and let $q > p$. Then*

$$0 = \alpha(A + pbc^T) < \alpha(A + qbc^T).$$

Proof. By Lemma 3.3.3 $p = 1/\|\boldsymbol{G}\|_{H^\infty}$ and from stability radius theory for nonnegative linear systems (see [62]) we know that the real and complex stability radii of A with respect to weightings b and c^T coincide and are equal to p. Moreover, p is the minimal destabilizing perturbation, implying in particular that $\alpha(A + pbc^T) = 0$.

Now, if $q > p$ then the Metzler matrices $A + qbc^T$ and $A + pbc^T$ satisfy $A + qbc^T > A + pbc^T$. By **(A3.3)** and part (1) of Remark 3.3.2, $A + pbc^T$ is irreducible, thus invoking Lemma 2.3.9, $\alpha(A + qbc^T) > \alpha(A + pbc^T)$. □

Throughout this chapter key results will be demonstrated by simulating data and plotting the time history of $x(t; x^0)$. For simplicity the linear system will remain unchanged with just the nonlinearity varying to fit the assumptions of the theorem which is being used. In the following example we introduce a linear system and verify that **(A3.1)**-**(A3.3)** are satisfied.

Example 3.3.6. *Consider the Lur'e system* (3.4) *with the following choice of* *linear system,*

$$A = \begin{pmatrix} -0.5 & 0 & 0 \\ 1 & -0.5 & 0 \\ 0 & 0.5 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}. \tag{3.24}$$

We demonstrate that **(A3.1)**-**(A3.3)** *are satisfied. Begin by noting that clearly A is a Metzler matrix as defined in Definition 2.3.1. Also it is clear that b, c are nonnegative and nonzero, therefore* **(A3.1)** *is satisfied. It can be easily show that all of the eigenvalues of A lie in the left half plane which means A is Hurwitz and so* **(A3.2)** *is satisfied. Note that*

$$A + bc^T = \begin{pmatrix} -0.5 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 0.5 & -1 \end{pmatrix}.$$

This matrix can easily be shown to be irreducible using the method described in Example 2.1.9, thus details are omitted here, however it verifies that **(A3.3)** *holds. Finally note that $p = -1/(c^T A^{-1} b) = 0.1$.*

We have demonstrated that this linear part of a Lur'e system satisfies the assumptions important for this chapter. We shall return to this example throughout this chapter to demonstrate some of the main results.

3.4 Absolute Stability of Nonnegative Lur'e Systems in Continuous Time

This section is split into three parts. The first looks into systems lacking a stable equilibrium and demonstrates that all solutions diverge. The second looks at systems with a unique equilibrium which is stable. In particular we look at two cases, when the system has a 0 equilibrium and when the system has a nonzero equilibrium. The final part looks at systems which have two equilibria, an equilibrium at 0 and a nonzero equilibrium which is stable.

3.4.1 Systems Without Stable Equilibria

We consider two cases where the system (3.4) lacks a stable equilibrium. The first occurs when $f(0) = 0$ and $\inf_{y>0} f(y)/y > p$ (see Figure 3.2(a)) in which case the system has a 0 equilibrium which is "strongly" unstable. What this means is that all entries of $x(t; x^0)$ diverge to ∞ as $t \to \infty$. The second occurs when $f(0) > 0$ and $\inf_{y \geq 0} f(y)/y > p$ (see Figure 3.2(b) in which case we have no equilibria and all solutions diverge.

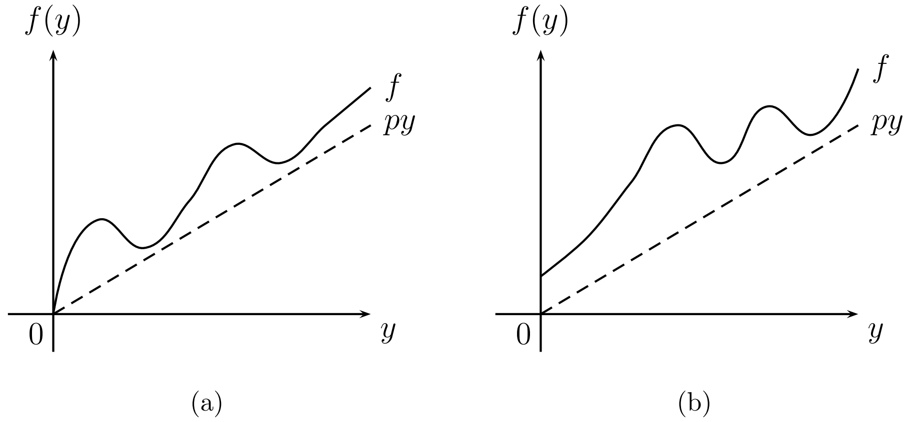


Figure 3.2: (a) A graph of a function f satisfying $f(0) = 0$ and $\inf_{y>0} f(y)/y > p$. (b) A graph of a function f satisfying $f(0) > 0$ and $\inf_{y\geq 0} f(y)/y > p$.

Theorem 3.4.1. *Consider the system* (3.4)*. Assume that* **(A3.1)**-**(A3.4)** *hold and that* f *satisfies* $f(0) = 0$ *and*

$$\inf_{y>0} \frac{f(y)}{y} > p.$$

If $x^0 \in \mathbb{R}^n_+$ *with* $x^0 \neq 0$*, is such that the solution* $x(t; x^0)$ *exists for every* $t \geq 0$*, then*

$$\lim_{t\to\infty} x_i(t; x^0) = \infty, \qquad \forall \quad i \in \{1, \ldots, n\},$$

where $x_i(t; x^0)$ *denotes the* i*-th component of* $x(t; x^0)$*.*

Proof. Let $x^0 \in \mathbb{R}^n_+$ with $x^0 \neq 0$, be such that the solution $x(t; x^0)$ exists for every $t \geq 0$. For simplicity write $x(t) := x(t; x^0)$ for all $t \geq 0$. By the hypothesis on f, there exists $q > p$ such that

$$f(y) \geq qy \qquad \forall \quad y \in \mathbb{R}_+.$$

By **(A3.1)**, **(A3.3)** and part (1) of Remark 3.3.2, $A + qbc^T$ is an irreducible Metzler matrix. Invoking statement (5) of Theorem 2.3.8 shows that there exists a positive matrix L such that

$$\lim_{t\to\infty} e^{(A+qbc^T-aI)t} = L \gg 0, \tag{3.25}$$

where $a := \alpha(A + qbc^T)$. Note that $a > 0$ which follows from Lemma 3.3.5. Moreover,

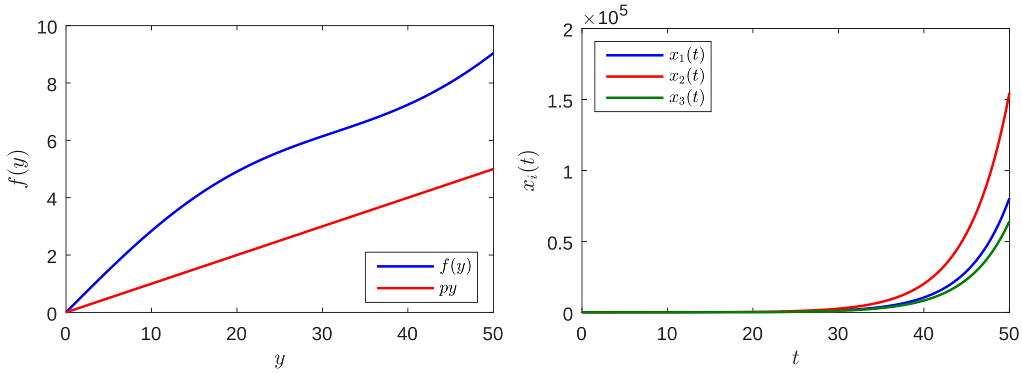
$$\dot{x} = (A + qbc^T)x + b(f(c^T x) - qc^T x)$$

and thus, by variation-of-parameters formula,

$$x(t) = e^{(A+qbc^T)t}x^0 + \int_0^t e^{(A+qbc^T)(t-s)} b(f(c^T x(s)) - qc^T x(s))ds$$
$$\geq e^{(A+qbc^T)t}x^0 \qquad \forall \quad t \geq 0. \tag{3.26}$$

Since $a > 0$, it follows from (3.25) that every component of $e^{(A+qbc^T)t}x^0$ diverges to ∞ as $t \to \infty$, completing the proof. □

Example 3.4.2. *Consider the Lur'e system* (3.4) *with linear part given by* (3.24). *Let* $f(y) = 0.2y + \sin(y/10)$ *and note that* **(A3.4)** *is satisfied. As demonstrated in Figure 3.3(a), which has* $f(y)$ *plotted in blue and the line* $py = 0.1y$ *in red, it can be easily seen that* $f(0) = 0$ *and* $\inf_{y>0} f(y)/y > p$. *Theorem 3.4.1 tells us that for all initial conditions* $x^0 \in \mathbb{R}^n_+$ *with* $x^0 \neq 0$, $\lim_{t\to\infty} x_i(t) = \infty$ *for each* $i = \{1, 2, 3\}$. *Figure 3.3(b) demonstrates this with an arbitrary initial condition.*



(a) (b)

Figure 3.3: Simulations of the system given in Example 3.4.2. (a) A plot of $f(y)$ and the line py. (b) Time history of the three components of $x(t)$.

Theorem 3.4.3. *Consider the system* (3.4). *Assume that* **(A3.1)**-**(A3.4)** *hold,* $f(0) > 0$ *and that* f *satisfies*

$$\inf_{y\geq 0} \frac{f(y)}{y} > p.$$

If $x^0 \in \mathbb{R}^n_+$ *is such that the solution* $x(t; x^0)$ *exists for every* $t \geq 0$, *then*

$$\lim_{t\to\infty} x_i(t; x^0) = \infty, \qquad \forall \quad i \in \{1, \ldots, n\},$$

where $x_i(t; x^0)$ *denotes the* i*-th component of* $x(t; x^0)$.

Proof. We consider two cases, when $x^0 \neq 0$ and when $x^0 = 0$. If $x^0 \neq 0$ then the proof is identical to the proof of Theorem 3.4.1 and thus there is nothing to show. When $x^0 = 0$, we are required to preform an extra step as (3.26) just tells us that $x(t; x^0) \geq 0$ for all $t \geq 0$, and not that $x(t; x^0)$ diverges.

Let $x^0 = 0$. Then

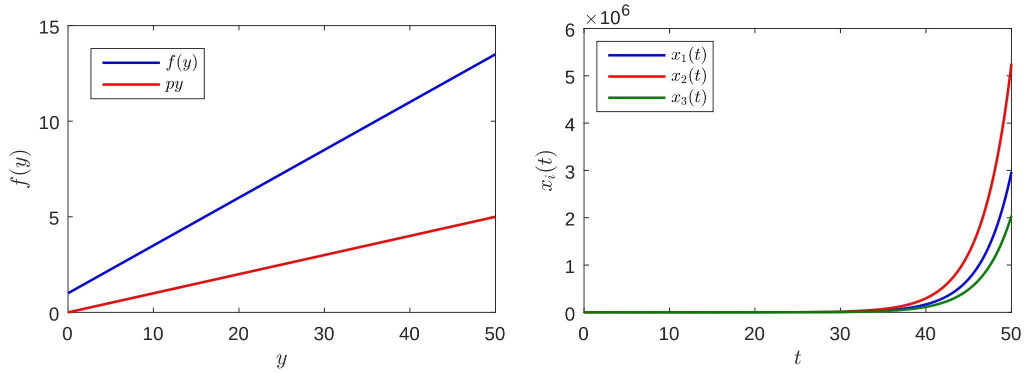
$$\dot{x}(0; 0) = Ax(0; 0) + bf(c^T x(0; 0)) = bf(0) > 0,$$

thus there exists $\tau > 0$, such that $x(\tau; 0) \neq 0$. Following the method of the proof of Theorem 3.4.1 we can reach the formula

$$\begin{aligned} x(t; 0) &= e^{(A+qbc^T)t} x(\tau; 0) + \int_\tau^t e^{(A+qbc^T)(t-s)} bf(c^T x(s; 0) - qc^T x(s; 0)) ds \\ &\geq e^{(A+qbc^T)t} x(\tau; 0) \qquad \forall \quad t \geq 0. \end{aligned}$$

The result now follows noting $x(\tau; 0) \neq 0$. □

Example 3.4.4. *Consider the Lur'e system* (3.4) *with linear part given by* (3.24). *Let* $f(y) = 0.1 + 0.25y$ *and note that* **(A3.4)** *is satisfied. Theorem 3.4.3 tells us that for all* $x^0 \in \mathbb{R}^n_+$, $\lim_{t\to\infty} x_i(t; x^0) = \infty$ *for all* $i = \{1, 2, 3\}$. *This is because* $f(y) > 0$ *and* $\inf_{y \geq 0} f(y)/y > p$, *which is illustrated in Figure 3.4(a), which contains the plot of* $f(y)$ *in blue and the line* $py = 0.1y$ *in red. Figure 3.4(b) shows the time history of* $x_i(t)$ *for* $i = \{1, 2, 3\}$ *and* $t \in [0, 100]$ *using the initial condition of* $x^0 = (0, 0, 0)^T$. *This initial value, which could take any value, has been chosen as it is where there is a major difference between Theorem 3.4.3 and Theorem 3.4.1. It is clear from this plot that* $x_i(t) \to \infty$ *as* $t \to \infty$ *for* $i = \{1, 2, 3\}$, *as expected.*



(a) (b)

Figure 3.4: Simulations of the system given in Example 3.4.4. (a) A plot of $f(y)$ and the line py. (b) Time history of the three components of $x(t)$.

3.4.2 Systems With A Unique Stable Equilibrium

In this section we consider systems with a unique equilibrium which exhibit somewhat nice stability properties. This equilibrium depends on the nonlinearity f and conditions which it must satisfy.

There are two cases which we consider. The first is if $f(0) = 0$ and $f(y)$ satisfies an inequality of the form $f(y)/y \leq p$ for all $y > 0$ as illustrated in Figure 3.5(a). In this case the equilibrium 0 of (3.4) will have certain stability properties. The second case relates to a nonnegative and nonzero equilibrium x^* which exists if f satisfies a sector condition as illustrated in 3.5(b).

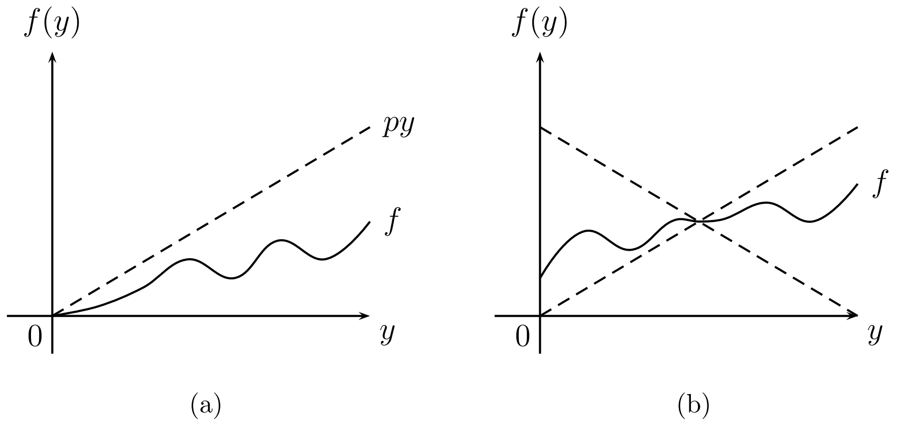


Figure 3.5: (a) A graph of a function f satisfying $f(0) = 0$ and $f(y)/y \leq p$ for all $y > 0$. (b) A graph of a function f satisfying a sector condition.

We begin by considering the simpler 0 equilibrium case.

Theorem 3.4.5. *Consider the system* (3.4) *and assume* **(A3.1)**-**(A3.4)** *hold and that* $f(0) = 0$.

(1) If $f(y)/y \leq p$ for all $y > 0$, then the equilibrium 0 is stable in the large in the sense that there exists $g \geq 1$ such that, for every $x^0 \in \mathbb{R}^n_+$,

$$\|x(t; x^0)\| \leq g\|x^0\| \qquad \forall \quad t \geq 0.$$

(2) If $f(y)/y < p$ for all $y > 0$, then the equilibrium 0 is globally asymptotically stable in the sense that 0 is stable in the large and, for every $x^0 \in \mathbb{R}^n_+$, $x(t; x^0) \to 0$ as $t \to \infty$.

(3) If $\sup_{y>0} f(y)/y < p$, then the equilibrium 0 is globally exponentially stable, that is, there exists $\gamma > 0$ and $g \geq 1$ such that, for every $x^0 \in \mathbb{R}^n_+$,

$$\|x(t; x^0)\| \leq ge^{-\gamma t}\|x^0\| \qquad \forall \quad t \geq 0.$$

Proof. By Lemma 3.3.3, $p = 1/\|\boldsymbol{G}\|_{H^\infty}$ and therefore, $\mathbb{D}(0, p) \subseteq \mathbb{S}(A, b, c^T)$. We seek to apply Theorem 3.2.3. We therefore extend f to the whole real line by defining an extension $\tilde{f} : \mathbb{R} \to \mathbb{R}$ as follows:

$$\tilde{f}(y) = \begin{cases} f(y) & \text{for } y > 0 \\ 0 & \text{for } y \leq 0. \end{cases} \tag{3.27}$$

Note that by linear boundedness of f and assumptions **(A3.1)** and **(A3.4)**, we have that, for every $x^0 \in \mathbb{R}^n_+$, $x(\,\cdot\,; x^0)$ is defined on $\mathbb{R}_+$ and $x(t; x^0) \in \mathbb{R}^n_+$ for all $t \geq 0$. Therefore, for every $x^0 \in \mathbb{R}^n_+$, $x(\,\cdot\,; x^0)$ is also the uniquely maximally defined forward solution of

$$\dot{x} = Ax + b\tilde{f}(c^T x), \qquad x(0) = x^0. \tag{3.28}$$

To prove statement (1), assume that $f(y)/y \leq p$ for all $y > 0$. Then, trivially,

$$\frac{\tilde{f}(y)}{y} \in [0, p] \qquad \forall \quad y \in \mathbb{R} \backslash \{0\},$$

and the claim follows from statement (1) of Theorem 3.2.3 applied to (3.28).

To prove statement (2), assume that $f(y)/y < p$ for all $y > 0$. Then,

$$\frac{\tilde{f}(y)}{y} \in (0, p), \qquad \forall \quad y \in \mathbb{R} \backslash \{0\},$$

and the claim follows from statement (2) of Theorem 3.2.3 applied to (3.28).

Finally, to prove statement (3), assume that $\sup_{y>0} f(y)/y < p$. Then there exists $q \in (0, p)$ such that

$$\frac{\tilde{f}(y)}{y} \in (0, q), \qquad \forall \quad y \in \mathbb{R} \backslash \{0\},$$

and the claim follows from statement (3) of Theorem 3.2.3 applied to (3.28).

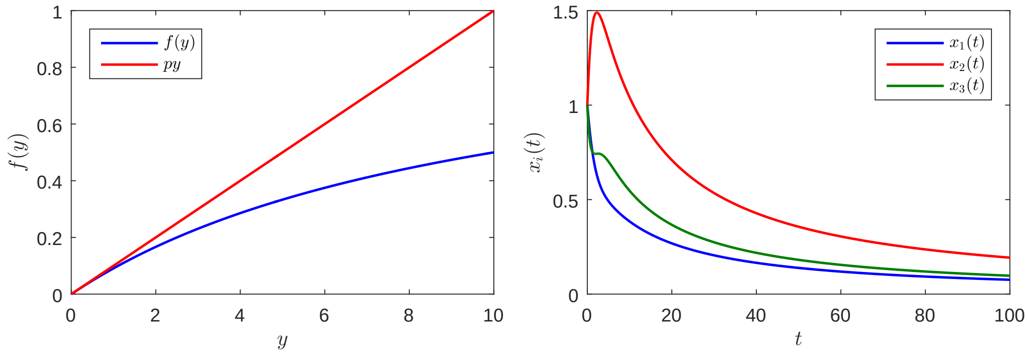
□

See Section 2.4.1 for a comparison of the different conditions on f appearing in Theorem 3.4.5.

The following example illustrates parts (2) and (3) of Theorem 3.4.5. This also demonstrates how much of a difference having exponential stability can make over just asymptotic stability.

Example 3.4.6. *Consider the Lur'e system* (3.4) *with linear part given by* (3.24). *First consider the nonlinearity $f_1(y) = y/(10 + y)$. We begin by noting that* **(A3.4)** *holds and that $f_1(0) = 0$. It can be shown that $f_1(y) < py$ for all $y > 0$, however $\sup_{y>0} f_1(y)/y < p$ is not true as $f_1'(0) = 0.1 = p$, therefore the lines $f_1(y)$ and py initially have the same gradient. This can be seen in Figure 3.6(a), where $f(y)$ is plotted in blue and py is plotted in red. Application of*

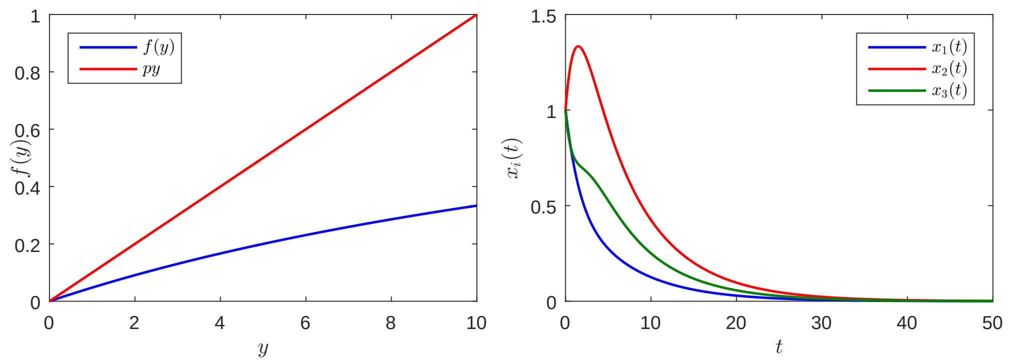
Theorem 3.4.5 tells us that for all $x^0 \in \mathbb{R}^n_+$, $x(t) \to 0$ as $t \to \infty$. This is illustrated in Figure 3.6(b) with the initial condition $x^0 = (1,1,1)^T$.



(a) (b)

Figure 3.6: Simulation for Example 3.4.6. (a) Plot of $f_1(y) = y/(10+y)$ in blue and the line py in red. It shows that $f_1(0) = 0$ and that $f_1(y) < py$ for all $y > 0$ and that both lines have the same gradient at $y = 0$. (b) Is a time history plot of the three components of $x(t)$ which can be seen slowly converging to 0.

Now consider the nonlinearity $f_2(y) = y/(20+y)$. Like with $f_1(y)$, $f_2(0) = 0$ and **(A3.4)** *is satisfied. However, we now have that $\sup_{y>0} f_2(y)/y < p$ as the initial gradient of $f_2 < p$. This can be seen in Figure 3.7(a) where again, $f(y)$ is plotted in blue and py is plotted in red. Application of Theorem 3.4.5 now tells us that there exists $\gamma > 0$ and $g \geq 1$ such that, for every $x^0 \in \mathbb{R}^n_+$, $\|x(t;x^0)\| \leq ge^{-\gamma t}\|x^0\|$ for all $t \geq 0$. In other words, 0 is globally exponentially stable. Figure 3.7(b) demonstrates this with initial condition $x^0 = (1,1,1)^T$.*



(a) (b)

Figure 3.7: Simulation for Example 3.4.6. (a) Plot of $f_2(y) = y/(20+y)$ in blue and the line py in red. It shows that $f_2(0) = 0$ and that $\sup_{y>0} f_2(y)/y < p$. (b) Is a time history plot of the three components of $x(t)$ which can be seen rapidly converging to 0.

A comparison of Figures 3.6(b) and 3.7(b) demonstrates the huge difference which can occur between asymptotic convergence and exponential convergence.

We introduce additional assumptions before we start looking at a system with a nonzero equilibrium. For these assumptions to make sense we need to assume **(A3.1)**-**(A3.4)** hold, which imply that $p > 0$.

(A3.5) There exists $y^* > 0$ such that $f(y^*) = py^*$.

(A3.6) f satisfies

$$\left| \frac{f(y) - f(y^*)}{y - y^*} \right| \leq p, \qquad \forall \quad y \in \mathbb{R}_+ \backslash \{y^*\}.$$

(A3.7) f satisfies

$$\left| \frac{f(y) - f(y^*)}{y - y^*} \right| < p \qquad \forall \quad y \in \mathbb{R}_+ \backslash \{0, y^*\}.$$

(A3.8) f satisfies

$$\limsup_{y \to y^*} \left| \frac{f(y) - f(y^*)}{y - y^*} \right| < p.$$

(A3.9) f satisfies

$$\limsup_{y \to \infty} \left| \frac{f(y) - f(y^*)}{y - y^*} \right| < p.$$

(A3.10) For all $y_0 > 0$

$$\sup_{y \geq y_0,\, y \neq y^*} \left| \frac{f(y) - f(y^*)}{y - y^*} \right| < p.$$

Assumptions **(A3.6)** and **(A3.7)** are sector conditions in the sense that they are equivalent to the graph of f being sandwiched between the straight lines py and $2py^* - py$ as illustrated in Figure 3.1. If f is differentiable at y^*, then **(A3.8)** is equivalent to the condition $|f'(y^*)| < p$. Assumption **(A3.9)** says that, for all sufficiently large y, the points $(y, f(y))$ lie in a sector defined by two straight lines of slope r and $-r$, where $0 < r < p$. Note that neither **(A3.8)** or **(A3.9)** is implied by **(A3.7)**. Collectively **(A3.7)**-**(A3.9)** are equivalent to **(A3.10)**, and can been seen as a uniform version of **(A3.7)**.

These assumptions are also explained in Section 2.4.2.

Lemma 3.4.7. *Assume that* **(A3.1)**-**(A3.5)** *are satisfied and* $f(0) \in (0, 2py^*)$. *Then* $x^* := -A^{-1}bpy^* > 0$ *is an equilibrium of* (3.4). *If in addition* **(A3.7)** *is satisfied, then there are no other equilibria in* $\mathbb{R}^n_+$.

Proof. Assume **(A3.1)**-**(A3.5)** are satisfied. We begin by verifying that $x^* > 0$. By Lemma 3.3.3, $\|\boldsymbol{G}\|_{H^\infty} = \boldsymbol{G}(0) > 0$ and noting $y^* > 0$ we have $f(y^*) > 0$.

From Corollary 2.3.6, **(A3.1)** and **(A3.2)** if follows that $-A^{-1} \geq 0$. We also have that $b \geq 0$ by **(A3.1)**. Combining the above we have that $x^* = -A^{-1}bpy^* \geq 0$. Assume that $x^* = 0$. This implies $bpy^* = 0$, therefore $b = 0$ which invalidates **(A3.1)**, therefore $x^* > 0$.

We now demonstrate that x^* is an equilibrium of (3.4). Noting that $c^T x^* = y^*$, we conclude

$$Ax^* + bf(c^T x^*) = Ax^* + bf(y^*) = 0,$$

thus $x^* > 0$ is an equilibrium of (3.4).

Finally assume that **(A3.7)** also holds and that $x^\dagger \in \mathbb{R}^n_+$ is an equilibrium of (3.4), that is $Ax^\dagger + bf(c^T x^\dagger) = 0$. We demonstrate that $x^\dagger = x^*$. Since A is Hurwitz by **(A3.2)**, A must be invertible, therefore,

$$x^\dagger = -A^{-1}bf(c^T x^\dagger). \tag{3.29}$$

Firstly assume that $c^T x^\dagger = 0$. Noting $f(0) > 0$,

$$0 = c^T x^\dagger = -c^T A^{-1} bf(0) = \boldsymbol{G}(0)f(0) > 0,$$

which does not hold, we conclude that $c^T x^\dagger > 0$. Since

$$c^T x^\dagger = -c^T A^{-1} bf(c^T x^\dagger) = \boldsymbol{G}(0)f(c^T x^\dagger) = \frac{1}{p} f(c^T x^\dagger),$$

it follows that

$$f(c^T x^\dagger) - f(y^*) = p(c^T x^\dagger - y^*).$$

Invoking **(A3.7)**, we conclude that $c^T x^\dagger = y^*$, which, together with (3.29) implies that $x^\dagger = x^*$. □

Theorem 3.4.8. *Consider the system* (3.4) *and assume that* **(A3.1)**-**(A3.5)** *hold.*

(1) If the additional assumption **(A3.6)** *is satisfied, there exist $g \geq 1$ such that $x^* = -A^{-1}bpy^*$ is stable in the large in the sense that, for every $x^0 \in \mathbb{R}^n_+$,*

$$\|x(t; x^0) - x^*\| \leq g\|x^0 - x^*\| \qquad \forall \quad t \geq 0.$$

(2) If the additional assumption **(A3.7)** *is satisfied and $f(0) \in (0, 2py^*)$, the equilibrium $x^* = -A^{-1}bpy^*$ is globally asymptotically stable in the sense that it is stable in the large and, for every $x^0 \in \mathbb{R}^n_+$, $x(t; x^0) \to x^*$ as $t \to \infty$.*

(3) If the additional assumptions **(A3.7)**-**(A3.9)** *or* **(A3.10)** *are satisfied and $f(0) \in (0, 2py^*)$, then the equilibrium $x^* = -A^{-1}bpy^*$ is globally*

exponentially stable in the sense that, for every $x^0 \in \mathbb{R}^n_+$*, there exists constants* $\gamma > 0$ *and* $g \geq 1$ *such that*

$$\|x(t;x^0) - x^*\| \leq g e^{-\gamma t}\|x^0 - x^*\| \qquad \forall \quad t \geq 0.$$

Proof. Note that, by the linear boundedness of f and assumptions **(A3.1)** and **(A3.4)**, we have that, for every $x^0 \in \mathbb{R}^n_+$, $x(\cdot, x^0)$ is defined on $\mathbb{R}_+$ and $x(t;x^0) \in \mathbb{R}^n_+$ for all $t \geq 0$.

Define

$$\tilde{f}(y) = \begin{cases} f(y+y^*) - f(y^*) & \text{for } y \geq -y^* \\ -f(y^*) & \text{for } y < -y^*. \end{cases} \tag{3.30}$$

See Figure 3.8 for a comparison of $f(y)$ and $\tilde{f}(y)$.

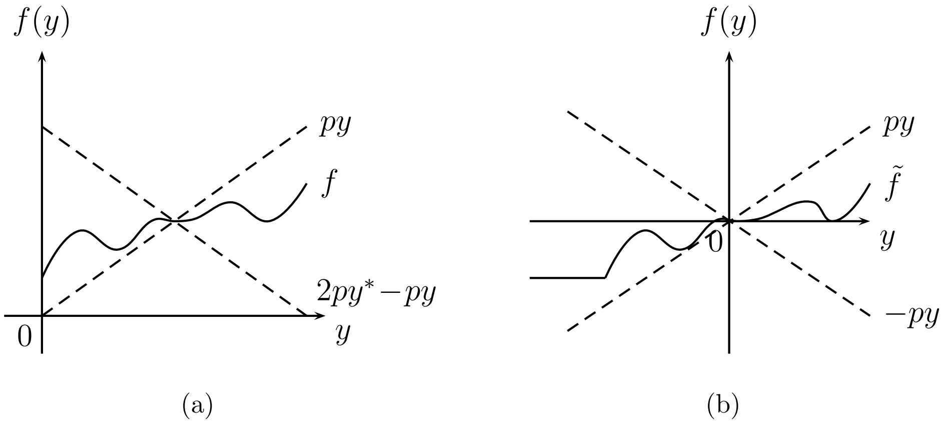


Figure 3.8: (a) A graph of the original nonlinearity f satisfying the sector condition given by the lines $l_1(y) = py$ and $l_2(y) = 2py^* - py$. (b) A graph of the shifted and extended nonlinearity $\tilde{f}$ given by (3.30) bounded by the lines $\tilde{l_1}(y) = py$ and $\tilde{l_2}(y) = -py$.

Furthermore, let $x^0 \in \mathbb{R}^n_+$ and set $\tilde{x}(t) := x(t;x^0) - x^*$. It follows that

$$\dot{\tilde{x}} = A\tilde{x} + b\tilde{f}(c^T\tilde{x}), \qquad \tilde{x}(0) = x^0 - x^*. \tag{3.31}$$

It follows from above that, for every $x^0 \in \mathbb{R}^n_+$, $\tilde{x}(\cdot)$ is the maximally defined forward solution of (3.31) and thus $x(\,\cdot\,;x^0) = \tilde{x}(\,\cdot\,) + x^*$ is the maximally defined forward solution of (3.4).

It follows as a consequence of elementary stability radius theory, (see [59]) that

$$\mathbb{D}(0,p) \subseteq \mathbb{S}(A,b,c^T). \tag{3.32}$$

To prove statement (1), note that

$$\left|\frac{\tilde{f}(y)}{y}\right| \leq p \qquad \forall \quad y \in \mathbb{R}\backslash\{0\}. \tag{3.33}$$

Combining (3.31)-(3.33) with statement (1) of Theorem 3.2.3 yields the existence of a constant $g \geq 1$ such that for every $x^0 \in \mathbb{R}^n_+$,

$$\|x(t;x^0) - x^*\| \leq g\|x^0 - x^*\|, \qquad \forall \quad t \geq 0.$$

Proceeding to prove statement (2), we observe that

$$\left|\frac{\tilde{f}(y)}{y}\right| < p \qquad \forall \quad y \in \mathbb{R}\backslash\{0\}. \tag{3.34}$$

Combining (3.31), (3.32) and (3.34) with statement (2) of Theorem 3.2.3 yields that for every $x^0 \in \mathbb{R}^n_+$, $\tilde{x}(t) \to 0$ as $t \to \infty$ or equivalently, $x(t;x^0) \to x^*$ as $t \to \infty$.

Finally we proceed to prove statement (3). From the assumptions we have that

$$\left|\frac{\tilde{f}(y)}{y}\right| < r \qquad \forall \quad y \in \mathbb{R}\backslash\{0\}, \tag{3.35}$$

where $0 < r < p$. Combining (3.31), (3.32) and (3.35) with statement (3) of Theorem 3.2.3 yields that for every $x^0 \in \mathbb{R}^n_+$, there exists $\gamma > 0$ and $g \geq 1$ such that

$$x(t;x^0)\| \leq ge^{-\gamma t}\|x^0\| \qquad \forall \quad t \geq 0.$$

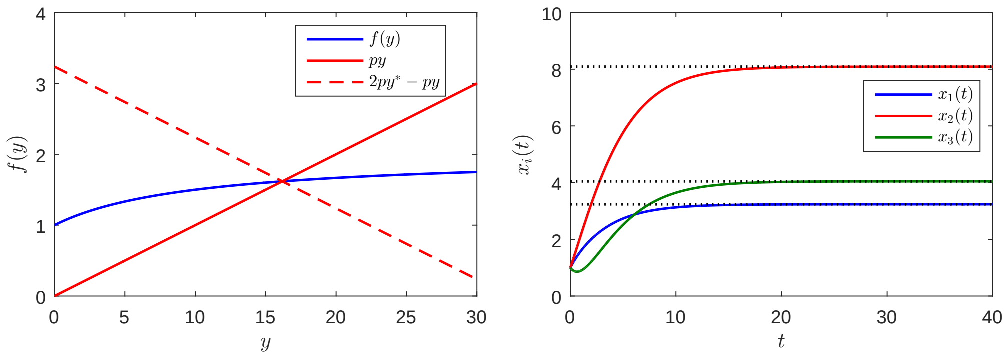
□

Example 3.4.9. *Consider the Lur'e system* (3.4) *with linear part given by* (3.24)*. Let $f(y) = 1 + y/(10 + y)$ which satisfies* **(A3.4)***. It is easily verified that for $y^* = 5(1+\sqrt{5}) \approx 16.1803$, $f(y^*) = py^*$, thus satisfying* **(A3.5)***. Using this value yields that $y(0) \in (0, 2py^*)$. Finally we note that* **(A3.8)** *holds for this particular $f(y)$ which can be seen in Figure 3.9(a) with $f(y)$ shown in blue, $l_1(y) = py$ in red and $l_2(y) = 2py^* - py$ as a red dashed line. Application of Theorem 3.4.8, part (3) tells us that for all $x^0 \in \mathbb{R}^n_+$, that $x^* = -A^{-1}bpy^* \approx (3.2361, 8.0902, 4.0451)^T$ is globally exponentially stable. This is illustrated in Figure 3.9(b) with an arbitrary initial condition.*

3.4.3 Systems With Two Equilibria

We introduce an additional assumption which acts as an addition to **(A3.4)**.

(A3.11) $f(0) = 0$.



(a) (b)

Figure 3.9: Simulation for Example 3.4.9. (a) Plot of $f_2(y) = 1 + y/(10 + y)$ in blue, py in red and $2py^* - py$ as a dashed red line. It shows that $f(y)$ lies in the sector for all $t \geq 0$ and that $f(y) = py$ at a unique point, y^*. (b) A time history plot of the three components of $x(t)$ which can be seen converging to $x^* \approx (3.2361, 8.0902, 4.0451)^T$ denoted by the dotted lines.

We begin by reformulating Lemma 3.4.7 for systems (3.4) when **(A3.11)** is satisfied.

Lemma 3.4.10. *Assume* **(A3.1)**-**(A3.5)** *and* **(A3.11)** *are satisfied. Then* 0 *and* $x^* = -A^{-1}bpy^* > 0$ *are equilibria of the system* (3.4)*. If additionally* **(A3.7)** *holds, then there are no other equilibria in* $\mathbb{R}^n_+$*.*

Proof. Clearly, since $f(0) = 0$ by **(A3.11)**, 0 is an equilibrium of (3.4). By Lemma 3.4.7, we also have that x^* is an equilibrium of (3.4) satisfying $x^* > 0$.

Now assume that **(A3.7)** holds and that $x^\dagger \in \mathbb{R}^n_+$ is an equilibrium of (3.4), that is, $Ax_0 + bf(c^T x_0) = 0$. We have to show that $x^\dagger = 0$ or $x^\dagger = x^*$. Since A is Hurwitz, A is invertible, and so,

$$x^\dagger = -A^{-1}bf(c^T x^\dagger). \tag{3.36}$$

If $c^T x^\dagger = 0$, then $x^\dagger = 0$. Assume that $c^T x^\dagger > 0$. Since

$$c^T x^\dagger = -c^T A^{-1} f(c^T x^\dagger) = \boldsymbol{G}(0) f(c^T x^\dagger) = \frac{1}{p} f(c^T x^\dagger),$$

it follows that

$$f(c^T x^\dagger) - f(y^*) = p(c^T x^\dagger - y^*).$$

Invoking **(A3.7)**, we conclude that $c^T x^\dagger = y^*$, which, together with (3.36) implies that $x^\dagger = x^*$. □

The following result shows in particular that, under suitable assumptions, the equilibrium $x^* = -A^{-1}bpy^*$ is stable in the large and attracts every nonzero initial vector $x^0 \in \mathbb{R}^n_+$. We ofter refer to this as "global" asymptotically stable as we must remove a singularity from the whole domain to get the domain of attraction, due to this point being an equilibrium.

Theorem 3.4.11. *Consider the system* (3.4). *Assume that* **(A3.1)**-**(A3.5)** *and* **(A3.11)** *hold.*

(1) Under the additional assumption that **(A3.6)** *is satisfied, there exists* $g \geq 1$ *such that* $x^* = -A^{-1}bpy^*$ *is stable in the large in the sense that, for every* $x^0 \in \mathbb{R}^n_+$,

$$\|x(t;x^0) - x^*\| \leq g\|x^0 - x^*\| \qquad \forall \quad t \geq 0.$$

(2) Under the additional assumption that **(A3.7)** *is satisfied, the equilibrium* $x^* = -A^{-1}bpy^*$ *is "globally" asymptotically stable in the sense that it is stable in the large and, for every* $x^0 \in \mathbb{R}^n_+$ *with* $x^0 \neq 0$, $x(t;x^0) \to x^*$ *as* $t \to \infty$.

Proof. Statement (1) is a specific case covered in statement (1) of Theorem 3.4.8 with $f(0) = 0$ so there is nothing to show. We therefore move straight on to the proof of statement (2).

Note that, by the linear boundedness of f and assumptions **(A3.1)** and **(A3.4)**, we have that, for every $x^0 \in \mathbb{R}^n_+$ with $x^0 \neq 0$, $x(\,\cdot\,, x^0)$ is defined on $\mathbb{R}_+$ and $x(t;x^0) \in \mathbb{R}^n_+$ for all $t \geq 0$.

Assuming **(A3.1)-(A3.5)**, **(A3.7)** and **(A3.11)** are true, by statement (2) of Theorem 3.2.5 we have that

$$\lim_{t\to\infty} x(t;x^0) = x^* \quad \text{or} \quad \lim_{t\to\infty} x(t;x^0) = 0.$$

Fix $x^0 \in \mathbb{R}^n_+$ with $x^0 \neq 0$ and write $x(t) := x(t;x^0)$ for all $t \geq 0$. Seeking a contradiction, suppose that $x(t) \to 0$ as $t \to \infty$. Then there exists $\tau \geq 0$ such that $c^T x(t) \leq y^*$ for all $t \geq \tau$. Thus, since

$$\begin{aligned} x(t+\tau) = {} & e^{(A+pbc^T)t}x(\tau) \\ & + \int_\tau^{t+\tau} e^{(A+pbc^T)(t+\tau-s)} \left[b(f(c^T x(s)) - pc^T x(s))\right] ds \qquad \forall \quad t \geq 0, \end{aligned}$$

and

$$f(y) - py \geq 0 \qquad \forall \quad y \in [0, y^*],$$

we have

$$x(t+\tau) \geq e^{(A+pbc^T)t}x(\tau) \qquad \forall \quad t \geq 0. \tag{3.37}$$

By Lemma 3.3.5, $\alpha(A + pbc^T) = 0$ and so, it follows from Theorem 2.3.8 that there exists $v \gg 0$ such that $v^T(A + pbc^T) = 0$. Consequently,

$$v^T e^{(A+pbc^T)t} = v^T \qquad \forall \quad t \geq 0.$$

By (3.37),

$$v^T x(t+\tau) \geq v^T x(\tau) \qquad \forall \quad t \geq 0.$$

Since $v \gg 0$ and $x(\tau) \in \mathbb{R}^n_+$, $x(\tau) \neq 0$, it is clear that $v^T x(\tau) > 0$ and so,

$$v^T x(t) \geq v^T x(\tau) > 0 \qquad \forall \quad t \geq \tau,$$

contradicting the supposition that $\lim_{t\to\infty} x(t) = 0$. Thus $x(t) \to x^* = -A^{-1}bpy^*$ as $t \to \infty$, which completes the proof of statement (2). $\square$

Example 3.4.12. *Consider the Lur'e system* (3.4) *with linear part given by* (3.24)*. Let* $f(y) = 2y/(5+y)$*. Clearly* **(A3.4)** *and* **(A3.11)** *are satisfied. Simple calculation yields that* $y^* = 15$*. Figure 3.10(a), which is a plot of* $f(y)$ *in blue and the lines* $l_1 = py$ *and* $l_2 = 2py^* - py$ *in red, you can see that clearly* **(A3.5)** *and* **(A3.7)** *are satisfied. Therefore, for any* $x^0 \in \mathbb{R}^n_+$ *with* $x^0 \neq 0$*, Theorem 3.4.11 tells us that* $x(t; x^0) \to x^* = -A^{-1}bpy^* = (3, 7.5, 3.75)^T$ *as* $t \to \infty$*. A simulation for an arbitrary initial condition can be found Figure 3.10(b).*

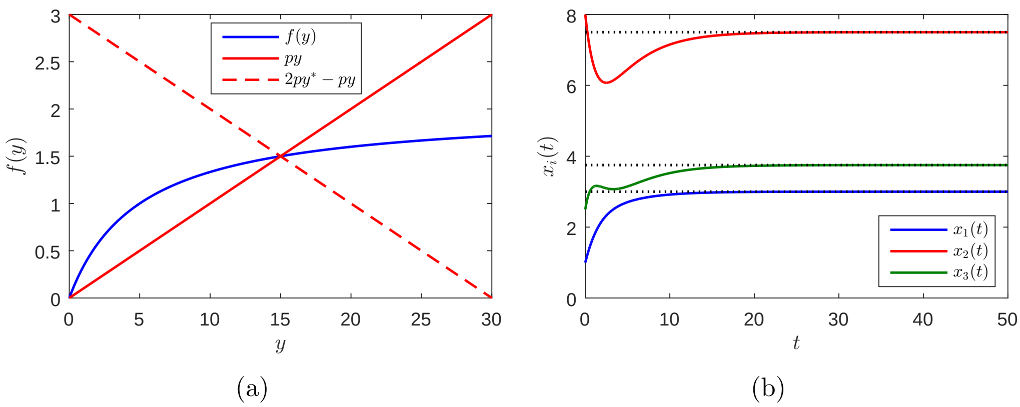


Figure 3.10: Simulation for Example 3.4.12. (a) Plot of $f(y) = 2y/(5+y)$ in blue and the line py in red. It shows that $f(y)$ lies in the sector for all $t \geq 0$ and that $f(y) = py$ at a unique point, y^*. (b) A time history plot of the three components of $x(t)$ which can be seen converging to $x^* = (3, 7.5, 3.75)^T$ which is denoted by the dotted lines.

The issue of exponential stability is far more difficult to establish than stability in the large or asymptotic stability for a system with multiple equilibria. We recap some of the major points covered thus far in this part of the section.

Assumptions **(A3.1)-(A3.5)** and **(A3.11)** imply that the Lur'e system (3.4) has at least two equilibria which include 0 and x^*. If in addition **(A3.6)** holds, then x^* is stable in the large and if **(A3.7)** holds then 0 and x^* are the only equilibria and x^* is asymptotically stable with a domain of attraction equal to $\mathbb{R}^n_+\backslash\{0\}$.

To have "global" exponential stability of x^* we would require there to exist $g \geq 1$ and $\gamma > 0$ such that, for every $x^0 \in \mathbb{R}^n_+$, $x^0 \neq 0$,

$$\|x(t;x^0) - x^*\| \leq ge^{-\gamma t}\|x^0 - x^*\| \qquad \forall\, t \geq 0, \tag{3.38}$$

which is not possible. This is a straightforward consequence of the continuity of the flow map $(t;x^0) \mapsto x(t;x^0)$ together with the facts that 0 is an equilibrium of (3.4) and $x^* \neq 0$. Indeed, $x(t;0) = 0$ for each $t \geq 0$, and so, if (t_n) is a sequence in $\mathbb{R}_+$ with $t_n \to \infty$, then there exists a sequence (x^0_n) in $\mathbb{R}^n_+\backslash\{0\}$ such that

$$t_n \to \infty, \quad x^0_n \to 0, \quad x(t_n;x^0_n) \to 0 \qquad \text{as} \quad n \to \infty,$$

and thus,

$$\|x(t_n;x^0_n) - x^*\| \to \|x^*\| > 0 \qquad \text{as} \quad n \to \infty.$$

Hence, there do not exist constants g and γ such that (3.38) holds for all $x^0 \in \mathbb{R}^n_+$ with $x^0 \neq 0$.

In the following we develop a series of results which play an important role in the development of an exponential stability result for the case where we have two equilibria. They will allows us to bound $c^T x(t;x^0)$ away from 0 after a certain amount of time has passed, however we are required to limit our choice of initial condition x^0 and thus instead of reaching a "global" result, we establish a "quasi-global" result.

It is convenient to introduce some new notation. Assuming **(A3.1)**-**(A3.3)** hold, let $q \geq p$, set $a_q := \alpha(A + qbc^T)$ and let $v_q, w_q \in \mathbb{R}^n_+$ denote the unique positive vectors such that

$$v_q^T(A + qbc^T) = a_q v_q^T = a_q v_q^T, \qquad (A + qbc^T)w_q = a_q w_q, \qquad \|v_q\|_1 = \|w_q\|_1 = 1,$$

the existence of which are ensured by statement (4) of Theorem 2.3.8 applied to $A + qbc^T$. By Lemma 3.3.5, $0 = a_p < a_q$ for all $q > p$.

Invoking statement (5) of Theorem 2.3.8, there exists $\tau_q > 0$ such that

$$e^{-a_q t}e^{(A+qbc^T)t} \geq \frac{1}{2v_q^T w_q} w_q v_q^T =: L_q \gg 0 \qquad \forall\, t \geq \tau_q. \tag{3.39}$$

We define the constants

$$\mu := \delta(A + pbc^T) \geq 0, \qquad \lambda_q := \text{smallest component of } c^T L_q. \tag{3.40}$$

Note that $\lambda_q > 0$ by the positivity of L_q. Furthermore, for every $l > 0$, we set

$$\omega(l) := \inf\{\|z\|_1 \ : \ c^T z \geq l\} > 0. \tag{3.41}$$

Lemma 3.4.13. *Consider the system* (3.4). *Assume that* **(A3.1)**-**(A3.5)**, **(A3.7)** *and* **(A3.11)** *hold and fix* $y^\dagger \in (0, y^*)$.

(1) If, for $x^0 \in \mathbb{R}^n_+$ *with* $x^0 \neq 0$*, there exists* $t^\dagger \geq 0$ *such that* $c^T x(t^\dagger; x^0) = y^\dagger$*, then*

$$c^T x(t; x^0) \geq \min\{\lambda_p \omega(y^\dagger), e^{-\mu\tau_p} y^\dagger\} > 0 \qquad \forall\ t \geq t^\dagger.$$

(2) For each $\varepsilon > 0$*, there exists* $t_\varepsilon \in (0, \infty)$ *such that, for all* $x^0 \in \mathbb{R}^n_+$ *with* $\|x^0\| \geq \varepsilon$ *and* $c^T x^0 < y^\dagger$*, there exists* $t^\dagger \in (0, t_\varepsilon]$ *such that* $c^T x(t^\dagger; x^0) = y^\dagger$.

Proof. Let $x^0 \in \mathbb{R}^n_+$ with $x^0 \neq 0$ and set $x(t) := x(t; x^0)$ for all $t \geq 0$.

To prove statement (1), let $t^\dagger \geq 0$ be such that $c^T x(t^\dagger) = y^\dagger$. If $c^T x(t) \geq y^\dagger$ for all $t \geq t^\dagger$, then there is nothing to show as $y^\dagger \geq \min\{\lambda_p \omega(y^\dagger), e^{-\mu\tau_p} y^\dagger\}$. Therefore, let us assume that there exists $t_1 > t^\dagger$ such that $c^T x(t_1) < y^\dagger$. It is sufficient to show that

$$c^T x(t_1) \geq \min\{\lambda_p \omega(y^\dagger), e^{-\mu\tau_p} y^\dagger\}.$$

To this end, as $t \mapsto c^T x(t)$ is continuous, note that there exists $t_0 \in [t^\dagger, t_1)$ such that $c^T x(t_0) = y^\dagger$ and

$$c^T x(t) \leq y^\dagger \qquad \forall\ t \in [t_0, t_1].$$

Invoking the sector condition **(A3.7)**, we obtain

$$f(c^T x(t)) \geq p c^T x(t) \qquad \forall\ t \in [t_0, t_1]. \tag{3.42}$$

Now, for $t \geq 0$, it follows from the variation-of-parameters formula

$$\begin{aligned} x(t + t_0) &= e^{(A+pbc^T)t} x(t_0) \\ &+ \int_{t_0}^{t+t_0} e^{(A+pbc^T)(t+t_0-s)} \left[b \left(f(c^T x(s)) - p c^T x(s) \right) \right] ds. \end{aligned} \tag{3.43}$$

By the hypotheses and (3.42), the integrand on the right hand side of (3.43) is nonnegative for all $s \in [t_0, t_1]$, and so

$$x(t+t_0) \geq e^{(A+pbc^T)t} x(t_0) \qquad \forall\, t \in [0, t_1 - t_0]. \tag{3.44}$$

By definition of μ in (3.40), it follows that $\mu I + A + pbc^T$ is nonnegative, and thus

$$e^{(A+pbc^T)t} \geq e^{-\mu t} I \geq e^{-\mu \tau_p} I \qquad \forall\, t \in [0, \tau_p]. \tag{3.45}$$

with τ_p defined by (3.39). Combining (3.44) and (3.45), we see that

$$c^T x(t+t_0) \geq e^{-\mu\tau_p} c^T x(t_0) = e^{-\mu\tau_p} I,$$

for all t such that $0 \leq t \leq \min\{\tau_p, t_1 - t_0\}$. Hence, if $t_1 - t_0 \leq \tau_p$, then

$$c^T x(t_1) = c^T x(t_1 - t_0 + t_0) \geq e^{-\mu\tau_p} y^\dagger. \tag{3.46}$$

Furthermore, if $t_1 - t_0 > \tau_p$, then, by (3.39), (3.40) and (3.44)

$$c^T x(t+t_0) \geq c^T L_p x(t_0) \geq \lambda_p \|x(t_0)\|_1 \qquad \forall \quad t \in (\tau_p, t_1 - t_0].$$

Since $c^T x(t_0) = y^\dagger$, we have $\|x(t_0)\|_1 \geq \omega(y^\dagger)$ and so,

$$c^T x(t_1) = c^T x(t_1 - t_0 + t_0) \geq \lambda_p \omega(y^\dagger). \tag{3.47}$$

Combining (3.46) and (3.47) yields that

$$c^T x(t_1) \geq \min\{\lambda_p \omega(y^\dagger), e^{-\mu\tau_p y^\dagger}\},$$

which completes the proof of statement (1).

We proceed to prove statement (2). To this end, given $\varepsilon > 0$, let $x^0 \in \mathbb{R}^n_+$ be such that $c^T x^0 < y^\dagger$ and $\|x^0\|_1 \geq \varepsilon$. We consider two cases.

Case 1. There does not exist $t \in (0, \tau_p]$ such that $c^T x(t) = y^\dagger$, in which case

$$c^T x(t) < y^\dagger \qquad \forall \quad t \in [0, \tau_p]. \tag{3.48}$$

Set

$$\mathcal{T} := \{t \geq 0 \ : \ c^T x(s) \geq y^\dagger \ \forall\ s \in [\tau_p, \tau_p + t]\}$$

and

$$r := \inf\{f(y)/y \ : \ \lambda_p \varepsilon \leq y \leq y^\dagger\} > p.$$

Note that **(A3.7)** guarantees that $r > p$. It is clear that $\mathcal{T} \neq \emptyset$ and $t^* := \sup \mathcal{T}$ satisfies $0 < t^* \leq \infty$ by (3.48) and the definition of t^*.

Noting that $c^T x(t) \leq y^\dagger$ for all $t \in [0, \tau_p + t^*)$, we can argue as in the derivation of statement (1) to obtain

$$c^T x(t) \geq c^T e^{(A+pbc^T)t} x^0 \qquad \forall \quad t \in [0, \tau_p + t^*).$$

Now

$$c^T e^{(A+pbc^T)t} x^0 \geq c^T L_p x^0 \geq \lambda_p \varepsilon \qquad \forall \quad t \geq \tau_p,$$

and so

$$c^T x(t) \geq \lambda_p \varepsilon \qquad \forall\, t \in [\tau_p, \tau_p + t^*). \tag{3.49}$$

Consequently,

$$\lambda_p \varepsilon \leq c^T x(t) \leq y^\dagger \qquad \forall \quad t \in [\tau_p, \tau_t^*), \tag{3.50}$$

and thus, by definition of r,

$$f(c^T x(t)) \geq r c^T x(t) \qquad \forall \quad t \in [\tau_p, \tau_p + t^*).$$

The variation-of-parameters formula then yields

$$\begin{aligned} x(t) &= e^{(A+rbc^T)(t-\tau_p)} x(\tau_p) + \int_{\tau_p}^{t} e^{(A+rbc^T)(t-s)} \left[b \left(f(c^T x(s)) - r c^T x(s) \right) \right] ds \\ &\geq e^{(A+rbc^T)(t-\tau_p)} x(\tau_p) \qquad \forall \quad t \in [\tau_p, \tau_p + t^*). \end{aligned} \tag{3.51}$$

Since

$$e^{(A+rbc^T)(t-\tau_p)} x(\tau_p) \geq e^{a_r(t-\tau_p)} L_r x(\tau_p) \qquad \forall \quad t \geq \tau_p + \tau_r, \tag{3.52}$$

we use the positivity of $a_r > 0$ to conclude from (3.50) and (3.51) that $t^* < \infty$. Setting $t^\dagger := \tau_p + t^*$, it is clear that $c^T x(t^\dagger) = y^\dagger$. If $t^\dagger > \tau_p + \tau_r$, which is equivalent to $t^* > \tau_r$, then, by (3.51) and (3.52),

$$y^\dagger = c^T x(t^\dagger) \geq c^T e^{(A+rbc^T)(t^\dagger - \tau_p)} x(\tau_p) \geq e^{a_r(t^\dagger - \tau_p)} c^T L_r x(\tau_p),$$

and so, invoking (3.49),

$$y^\dagger \geq e^{a_r(t^\dagger - \tau_p)} \lambda_r \omega(\lambda_p \varepsilon),$$

which in turn leads to

$$t^\dagger \leq \tau_p + \frac{1}{a_r} \ln \frac{y^\dagger}{\lambda_r \omega(\lambda_p \varepsilon)} =: s_\varepsilon.$$

Consequently, we have that

$$t^\dagger \leq \max\{s_\varepsilon, \tau_p + \tau_r\} =: t_\varepsilon. \tag{3.53}$$

Case 2. There exists $t \in (0, \tau_p]$ such that $c^T x(t) = y^\dagger$. In which case, setting $t^\dagger := t$, (3.53) is trivially satisfied. $\square$

Informally, the following proposition says that, under certain assumptions, the output $c^T x(t; x^0)$ is uniformly, ultimately bounded away from 0.

Proposition 3.4.14. *Assume that* **(A3.1)**-**(A3.5)**, **(A3.7)** *and* **(A3.11)** *hold and let* $\varepsilon > 0$. *Then there exists* $\eta > 0$ *and* $\theta \geq 0$ *such that, for all* $x^0 \in \mathbb{R}^n_+$ *with* $\|x^0\| \geq \varepsilon$, *the solution* $x(\cdot\,; x^0)$ *of* (3.4) *satisfies*

$$c^T x(t; x^0) \geq \eta \qquad \forall \quad t \geq \theta. \tag{3.54}$$

Proof. Fix $y^\dagger \in (0, y^*)$ and $\varepsilon > 0$. For $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$, set $x(t) := x(t; x^0)$. Furthermore, define

$$\eta := \min\{\lambda_p \omega(y^\dagger), e^{-\mu\tau_p} y^\dagger\}$$

and $\theta := t_\varepsilon$, where t_ε is the number guaranteed to exist by statement (2) of Lemma 3.4.13. We demonstrate that

$$c^T x(t) \geq \eta \qquad \forall \quad t \geq \theta \tag{3.55}$$

by considering three exhaustive cases.

Case 1. If $c^T x(0) = c^T x^0 < y^\dagger$, appealing to statement (2) of Lemma 3.4.13, we see that there exists $t^\dagger \in (0, \theta]$ such that $c^T x(t^\dagger) = y^\dagger$. Application of statement (1) of Lemma 3.4.13 yields that $c^T x(t) \geq \eta$ for all $t \geq t^\dagger$, thus (3.55) is satisfied.

Case 2. If $c^T x(0) = c^T x^0 = y^\dagger$, by statement (1) of Lemma 3.4.13, $c^T x(t) \geq \eta$ for all $t \geq 0$, and hence, (3.55) hold.

Case 3. If $c^T x(0) = c^T x^0 > y^\dagger$ we consider two scenarios. Firstly if $c^T x(t) > y^\dagger$ for all $t \geq 0$ then (3.55) is satisfied since $y^\dagger \geq \eta$. Alternatively, there exists $t^\dagger > 0$ such that

$$c^T x(t^\dagger) = y^\dagger \qquad \text{and} \qquad c^T x(t) > y^\dagger \qquad \forall \quad t \in [0, t^\dagger),$$

By statement (1) of Lemma 3.4.13, $c^T x(t) \geq \eta$ for all $t \geq t^\dagger$. It now follows that (3.55) holds, since $c^T x(t) > y^\dagger \geq \eta$ for all $t \in [0, t^\dagger)$. $\square$

Now that we have reached our goal of establishing a bound (3.55) we can move on to the "quasi-global" exponential stability result for (3.4).

Theorem 3.4.15. *Assume that* **(A3.1)**-**(A3.5)**, **(A3.7)**-**(A3.9)** *and* **(A3.11)** *hold. The equilibrium* $x^* = -A^{-1}bpy^*$ *of the Lur'e system* (3.4) *is quasi-globally exponentially stable in the sense that, for every* $\varepsilon > 0$, *there exists* *constants* $\gamma > 0$ *and* $g \geq 1$ *such that* (3.38) *holds for every* $x^0 \in \mathbb{R}^n_+$ *with* $\|x^0\| \geq \varepsilon$.

Proof. Let $\varepsilon > 0$. By Proposition 3.4.14, there exists $\eta > 0$ and $\theta \geq 0$ such that $c^T x(t; x^0) \geq \eta$ for all $t \geq \theta$ and all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$. Invoking assumptions **(A3.7)-(A3.9)**, it follows that

$$r := \sup \left\{ \frac{|f(y + y^*) - f(y^*)|}{|y|} \ : \ -y^* + \eta \leq y < \infty, \ y \neq 0 \right\} < p. \qquad (3.56)$$

Consider a fixed, yet arbitrary, $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$ and write $\tilde{x}(t) := x(t; x^0) - x^*$ for $t \geq 0$. Choose a locally Lipschitz function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} \tilde{f}(y) = f(y + y^*) - f(y^*) \qquad & \forall\ y \in [-y^* + \eta, \infty) \\ \text{and} \qquad \left| \tilde{f}(y)/y \right| \leq r \qquad & \forall \quad y \in \mathbb{R} \backslash \{0\}. \end{aligned} \qquad (3.57)$$

Since $c^T \tilde{x}(t) \geq -y^* + \eta$ for all $t \geq \theta$ and using (3.57), it is straightforward to show

$$\dot{\tilde{x}}(t) = A\tilde{x} + b\tilde{f}(c^T \tilde{x}) \qquad \forall \quad t \geq \theta.$$

By (3.56), $r < p$ and thus it follows from statement (3) of Theorem 3.2.3 that there exists $\gamma > 0$ and $h \geq 1$, which do not depend on x^0, such that

$$\|\tilde{x}(t)\| \leq he^{-\gamma(t-\theta)} \|\tilde{x}(\theta)\| \qquad \forall \quad t \geq \theta.$$

Combining the stability in the large of x^* established in statement (1) of Theorem 3.4.11, this shows that there exists $g > h$, also not dependent on x^0, such that

$$\|\tilde{x}(t)\| \leq ge^{-\gamma t} \|x^0 - x^*\| \qquad t \geq 0.$$

Rewriting this in terms of $x(t; x^0)$, gives (3.38) thus completing the proof. $\square$

When **(A3.10)** was introduced, it was noted that it was equivalent to **(A3.7)-(A3.9)** collectively. Therefore, Theorem 3.4.15 can also be formulated assuming **(A3.10)** holds instead of **(A3.7)-(A3.9)**.

Note that assumption **(A3.9)** defines the sector condition for all sufficiently large y. This assumption is essential for quasi-global exponential stability as it allows us to have initial conditions, x^0, which yield large initial values for y. There is however, a weaker concept called "semi-global" exponential stability which does not require **(A3.9)** to hold. This is presented in the following theorem.

Theorem 3.4.16. *Assume that* **(A3.1)**-**(A3.5)**, **(A3.7)**, **(A3.8)** *and* **(A3.11)** *hold. The equilibrium* $x^* = -A^{-1}bpy^*$ *of the Lur'e system* (3.4) *is semi-globally exponentially stable in the sense that, for every compact set* $\Gamma \subseteq \mathbb{R}^n_+$ *with* $0 \notin \Gamma$, *there exists constants* $\gamma > 0$ *and* $g \geq 1$ *such that* (3.38) *holds for every* $x^0 \in \Gamma$.

Proof. Let $\Gamma \subseteq \mathbb{R}^n_+$ be compact with $0 \notin \Gamma$. Then there exists $\varepsilon > 0$ such that $\|x^0\|_1 \geq \varepsilon$ for all $x^0 \in \Gamma$. Consequently, invoking Proposition 3.4.14, there exists $\eta > 0$ and $\theta \geq 0$ such that

$$c^T x(t; x^0) \geq \eta \qquad \forall \quad t \geq \theta, \quad \forall \quad x^0 \in \Gamma.$$

Furthermore, by statement (1) of Theorem 3.4.11, the equilibrium x^* is stable in the large and thus, there exists a constant $h > 0$ such that

$$c^T x(t; x^0) \leq h \qquad \forall \quad t \geq 0, \quad \forall \quad x^0 \in \Gamma.$$

Replacing the definition of r in (3.56) by

$$r := \sup \left\{ \frac{|f(y + y^*) - f(y^*)|}{|y|} \;:\; -y^* + \eta \leq y \leq h,\ y \neq 0 \right\}$$

and $\tilde{f} : \mathbb{R} \to \mathbb{R}$ in (3.57) by

$$\begin{aligned} \tilde{f}(y) = f(y + y^*) - f(y^*) \qquad & \forall \quad y \in [-y^* + \eta, \infty) \\ \text{and} \qquad \left| \tilde{f}(y)/y \right| \leq r \qquad & \forall \quad y \in \mathbb{R} \backslash \{0\}, \end{aligned}$$

and noting that $r < p$ by **(A3.7)** and **(A3.8)**, we can argue as in the proof of Theorem 3.4.15 to establish the claim. □

There is an alternative method of proving the semi-global exponential stability property guaranteed by Theorem 3.4.16 that does not make use of Proposition 3.4.14. This proof rests on a combination of local exponential stability of x^* which is not difficult to establish, statement (2) of Theorem 3.4.11 and a well known uniformity property enjoyed by compact subsets of the region of attraction of an asymptotically stable equilibrium, see [90, Proposition 5.20]. We emphasize that this approach cannot be used to establish the quasi-global exponential stability property, Theorem 3.4.15, which pertains to initial vectors of arbitrary large norm.

Note that no examples are given for any form of exponential stability explicitly. This is because the system considered in Example 3.4.12 satisfies the assumptions required for some sort of exponential stability.

3.5 Input-to-State Stability of Nonnegative Lur'e Systems in Continuous Time

In this section we consider forced nonnegative Lur'e systems of the form (3.18). As in the previous section, nonnegative means that the state $x(t)$ of (3.18) remains nonnegative for all $t \in \mathbb{R}_+$. We therefore require **(A3.1)** and **(A3.4)** to hold.

The unique maximally defined forward solution of (3.18) is denoted by $x(\,\cdot\,; x^0, d)$. If f is affine-linearly bounded and the disturbance $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$ is nonnegative, then the maximally defined forward solution exists for all times $t \geq 0$ (that is, there is no finite escape time from the nonnegative orthant). Obviously, if $d \in L^\infty_{\text{loc}}$ is not nonnegative, then the interval of existence of maximally defined forward solutions may be bounded (finite escape time from the nonnegative orthant).

3.5.1 Disturbed Systems Without Stable Equilibria

The results in the section extend the results in Section 3.4.1 to disturbed Lur'e systems. The proofs in this section will be omitted as the proofs in Section 3.4.1, mutatis mutandis, carry over to Lur'e systems with disturbance.

Theorem 3.5.1. *Consider the system* (3.18)*. Assume that* **(A3.1)**-**(A3.4)**, **(A3.11)** *and*

$$\inf_{y>0} \frac{f(y)}{y} > p.$$

If $x^0 \in \mathbb{R}^n_+$ with $x^0 \neq 0$ and $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$ are such that the solution $x(t; x^0, d)$ exists for $t \geq 0$, then

$$\lim_{t\to\infty} x_i(t; x^0, d) = \infty, \qquad \forall \quad i \in \{1, \ldots, n\},$$

where $x_i(t; x^0, d)$ denotes the i-th component of $x(t; x^0, d)$.

Theorem 3.5.2. *Consider the system* (3.18)*. Assume that* **(A3.1)**-**(A3.4)** *hold, $f(0) > 0$ and*

$$\inf_{y\geq 0} \frac{f(y)}{y} > p.$$

If $x^0 \in \mathbb{R}^n_+$ and $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$ are such that the solution $x(t; x^0, d)$ exists for $t \geq 0$, then

$$\lim_{t\to\infty} x_i(t; x^0, d) = \infty, \qquad \forall\ i \in \{1, \ldots, n\},$$

where $x_i(t; x^0, d)$ denotes the i-th component of $x(t; x^0, d)$.

3.5.2 ISS of Systems with A Unique Stable Equilibrium

This section provides counterpart results to those covered in Section 3.4.2 for systems with a disturbance. It is convenient at this stage to introduce a final assumption which requires **(A3.1)-(A3.3)** to hold.

> **(A3.12)** $py - f(y) \to \infty$ as $y \to \infty$.

Theorem 3.5.3. *Consider the system* (3.18) *and assume* **(A3.1)-(A3.4)**, **(A3.11)** *and* **(A3.12)** *hold. Further assume*

$$\frac{f(y)}{y} < p, \qquad \forall \quad y > 0.$$

Then, 0 *is ISS in the sense that there exists* $\psi \in \mathcal{KL}$ *and* $\varphi \in \mathcal{K}$ *such that, for all* $x^0 \in \mathbb{R}_+$ *and all nonnegative disturbances* $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$,

$$\|x(t; x^0, d)\| \leq \psi(\|x^0\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \qquad \forall \quad t \geq 0. \tag{3.58}$$

Proof. By Lemma 3.3.3, $p = 1/\|\boldsymbol{G}\|_{H^\infty} > 0$ and therefore,

$$\mathbb{D}(0, p) \subseteq \mathbb{S}(A, b, c^T).$$

Aiming to apply Theorem 3.2.7 with $r = p$ and $k = 0$ consider the function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ given by (3.27) which extends f to the whole real line. Furthermore, by hypothesis on f, we have that

$$p|y| - |\tilde{f}(y)| > 0 \quad \forall \quad y \neq 0$$

and

$$p|y| - |\tilde{f}(y)| \to \infty \quad \text{as} \quad |y| \to \infty.$$

Hence there exists $\beta \in \mathcal{K}_\infty$ such that

$$|\tilde{f}(y)| \leq p|y| - \beta(|y|) \qquad \forall \quad y \in \mathbb{R}.$$

Note that by linear boundedness of f and assumption **(A3.1)** and **(A3.4)**, we have that, for every $x^0 \in \mathbb{R}^n_+$ and every $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^n_+)$, $x(\,\cdot\,; x^0, d)$ is defined on $\mathbb{R}_+$ and $x(t; x^0, d) \in \mathbb{R}^n_+$ for all $t \geq 0$. Therefore, for every $x^0 \in \mathbb{R}^n_+$ and every $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^n_+)$, $x(\,\cdot\,; x^0, d)$ is also the uniquely defined forward solution of

$$\dot{x} = Ax + b\tilde{f}(c^T x) + d, \qquad x(0) = x^0. \tag{3.59}$$

An application of Theorem 3.2.7 to (3.59) shows that there exists $\psi \in \mathcal{KL}$ and

$\varphi \in \mathcal{K}$ such that, for all $x^0 \in \mathbb{R}^n_+$ and all $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^n_+)$,

$$\|x(t; x^0, d)\| \leq \psi(\|x^0\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \qquad \forall \quad t \geq 0,$$

which completes the proof. □

Example 3.5.4. *Consider the system* (3.18) *with linear part given by* (3.24)*, nonlinearity* $f(y) = y/(20 + y)$ *and disturbance*

$$d(t) = \begin{pmatrix} 0.25(1 + \sin(t/5)) \\ 0.5(1 + \sin(t/10)) \\ 1 + \sin(t/2) \end{pmatrix}.$$

Theorem 3.5.3 tells us that 0 *is ISS. We demonstrate this in Figure 3.11 which is a simulation of this system with an arbitrary initial condition.*

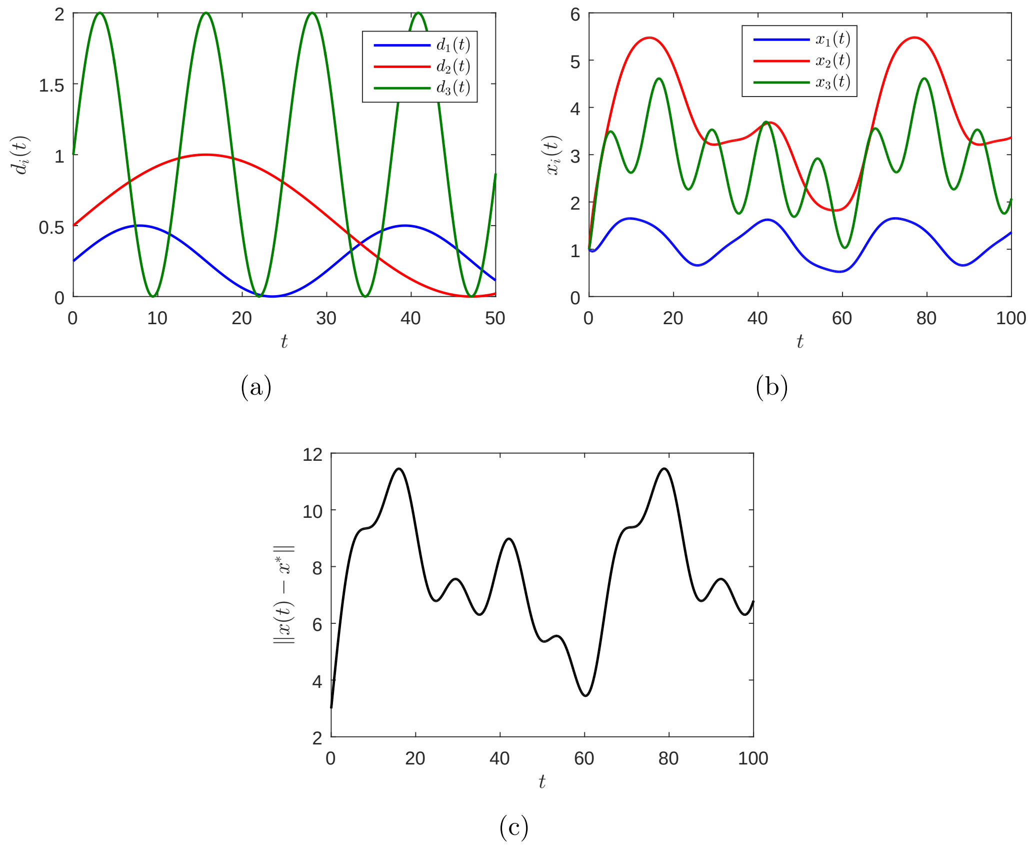


Figure 3.11: Simulation for Example 3.5.4. (a) Plot of three components $d(t)$. (b) A time history plot of the three components of $x(t)$. (c) Plot of the error $\|x(t) - x^*\|$.

Theorem 3.5.5. *Consider the system* (3.18) *and assume* **(A3.1)-(A3.5)**, **(A3.7)** *and* **(A3.12)** *hold and that* $f(0) \in (0, 2py^*)$. *Then* $x^* = -A^{-1}bpy^*$ *is ISS in the sense that, there exists* $\psi \in \mathcal{KL}$ *and* $\varphi \in \mathcal{K}$ *such that, for all*

$x^0 \in \mathbb{R}^n_+$ *and all nonnegative disturbances* $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$,

$$\|x(t; x^0, d) - x^*\| \le \psi(\|x^0 - x^*\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \qquad \forall \quad t \ge 0.$$

Proof. By Lemma 3.3.3, $p = 1/\|\boldsymbol{G}\|_{H^\infty} > 0$ and therefore, $\mathbb{D}(0, p) \subseteq \mathbb{S}(A, b, c^T)$. Define $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by (3.30). Then

$$p|y| - |\tilde{f}(y)| > 0 \quad \forall \quad y \ne 0$$

and

$$p|y| - |\tilde{f}(y)| \to \infty \quad \text{as} \quad |y| \to \infty.$$

Hence there exists $\beta \in \mathcal{K}_\infty$ such that

$$|\tilde{f}(y)| \le p|y| - \beta(|y|) \qquad \forall \quad y \in \mathbb{R}.$$

Let $x^0 \in \mathbb{R}^n_+$ and $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$ and set $\tilde{x}(t) = x(t; x^0, d) - x^*$. It follows that

$$\dot{\tilde{x}} = A\tilde{x} + b\tilde{f}(c^T\tilde{x}) + d, \qquad \tilde{x}(0) = x^0 - x^* := \tilde{x^0}. \tag{3.60}$$

Theorem 3.2.7 yields that (3.60) is ISS in the sense that there exists $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that, for every $x^0 \in \mathbb{R}^n$ and every $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$,

$$\|\tilde{x}(t)\| \le \psi(\|\tilde{x^0}\|, t) + \varphi(\|d\|_{L^\infty(0,t)}), \qquad \forall \quad t \ge 0,$$

where $\tilde{x}(\,\cdot\,; \tilde{x^0}, d)$ denotes the unique forward solution of (3.60). This is equivalent to

$$\|x(t; x^0, d) - x^*\|\psi(\|x^0 - x^*\|) + \varphi(\|d\|_{L^\infty(0,t)}) \qquad \forall \quad t \ge 0,$$

completing the proof. □

Example 3.5.6. *Consider the system* (3.18) *with linear part given by* (3.24)*, nonlinearity* $f(y) = 1 + y/(10 + y)$ *and disturbance*

$$d(t) = \begin{pmatrix} 0.2(1 + \sin(t/4)) \\ 1 + \sin(t/2) \\ 0.5(1 + \sin(t)) \end{pmatrix}.$$

Theorem 3.5.5 tells us that $x^* \approx (3.2361, 8.0902, 4.0451)^T$ *is ISS. We demonstrate this in Figure 3.12(b) which is a simulation of this system with a very small, arbitrary initial condition.*

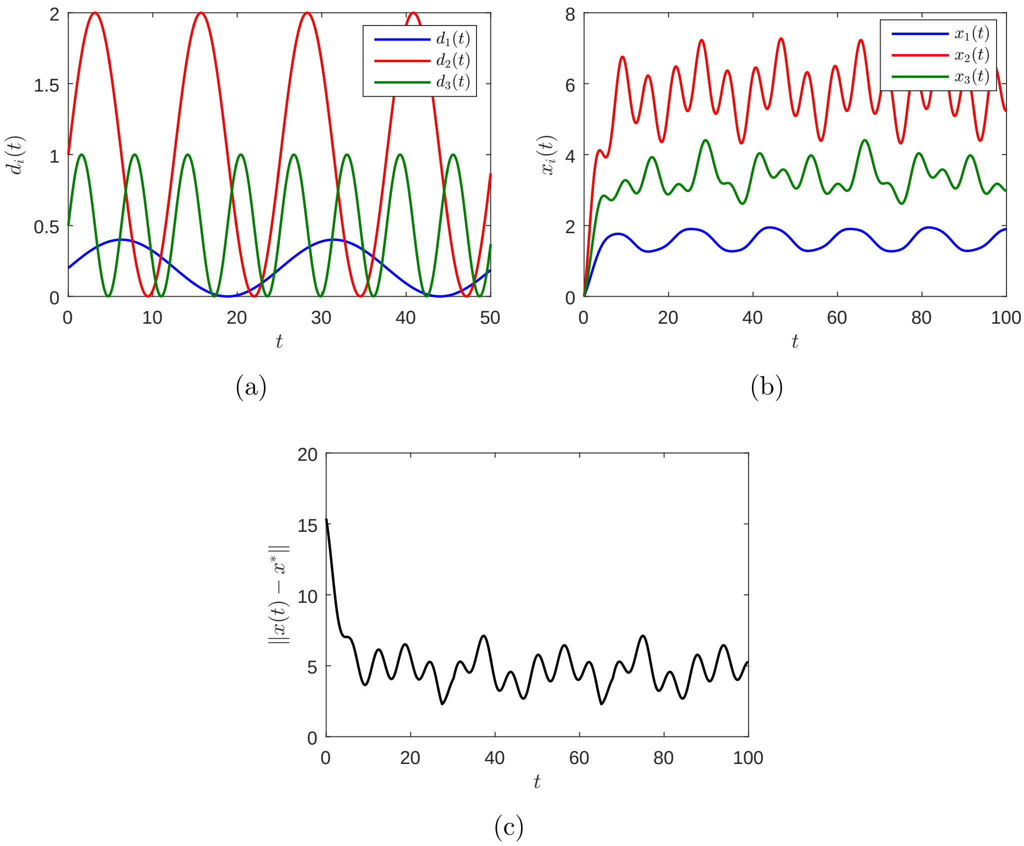


Figure 3.12: Simulation for Example 3.5.4. (a) Plot of three components $d(t)$. (b) A time history plot of the three components of $x(t)$. (c) Plot of the error $\|x(t) - x^*\|$.

3.5.3 ISS of Systems With Two Equilibria

We reformulate Lemma 3.4.13 and Proposition 3.4.14 in terms of the forced Lur'e system (3.18).

Lemma 3.5.7. *Consider the system* (3.18)*. Assume* **(A3.1)**-**(A3.5)**, **(A3.7)** *and* **(A3.11)** *hold, fix* $y^\dagger \in (0, y^*)$ *and let* $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$*.*

(1) If, for $x^0 \in \mathbb{R}^n_+$ *with* $x^0 \neq 0$*, there exists* $t^\dagger \geq 0$ *such that* $c^T x(t^\dagger; x^0, d) = y^\dagger$*, then*

$$c^T x(t; x^0, d) \geq \min\{\lambda_p \omega(y^\dagger), e^{-\mu\tau_p} y^\dagger\} > 0 \qquad \forall \quad t \geq t^\dagger,$$

where τ_p *is defined by* (3.39)*.*

(2) For each $\varepsilon > 0$*, there exists* $t_\varepsilon \in (0, \infty)$ *such that for all* $x^0 \in \mathbb{R}^n_+$ *with* $\|x^0\|_1 \geq \varepsilon$ *and* $c^T x^0 < y^\dagger$*, there exists* $t^\dagger \in (0; t_\varepsilon]$ *such that* $c^T x(t^\dagger; x^0, d) = y^\dagger$*.*

The proof of Lemma 3.5.7 is omitted as the proof of Lemma 3.4.13, mutatis mutandis, carries over to disturbed Lur'e systems.

Proposition 3.5.8. *Consider the system* (3.18) *and assume* **(A3.1)**-**(A3.5)**, **(A3.7)** *and* **(A3.11)** *hold and let* $\varepsilon > 0$. *Then there exists* $\eta > 0$ *and* $\theta \geq 0$ *such that, for all* $x^0 \in \mathbb{R}^n_+$ *with* $\|x^0\| \geq \varepsilon$ *and all nonnegative disturbances* $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$, *the solution* $x(\,\cdot\,; x^0, d)$ *of* (3.18) *satisfies*

$$c^T x(t; x^0, d) \geq \eta \qquad \forall \quad t \geq \theta. \tag{3.61}$$

The proof of Proposition 3.5.8 is omitted as the proof of Proposition 3.4.14, mutatis mutandis, caries over to disturbed Lur'e systems.

We now state and prove the main result of this section. It can be viewed as a counterpart of statement (2) of Theorem 3.4.11 for Lur'e systems with disturbances.

Theorem 3.5.9. *Consider the system* (3.18) *and assume* **(A3.1)**-**(A3.5)**, **(A3.7)**, **(A3.11)** *and* **(A3.12)** *hold. Then* $x^* = -A^{-1}bpy^*$ *is "quasi-globally" ISS in the sense that, for all* $\varepsilon > 0$, *there exists* $\psi \in \mathcal{KL}$ *and* $\varphi \in \mathcal{K}$ *such that, for all* $x^0 \in \mathbb{R}^n_+$ *with* $\|x^0\| \geq \varepsilon$ *and all nonnegative disturbances* $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$,

$$\|x(t; x^0, d) - x^*\| \leq \psi(\|x^0 - x^*\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \qquad \forall \quad t \geq 0. \tag{3.62}$$

Proof. Let $\varepsilon > 0$. By Proposition 3.5.8, there exists $\eta > 0$ and $\theta \geq 0$ such that for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$ and all $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$,

$$c^T x(t; x^0, d) \geq \eta \qquad \forall \quad t \geq \theta. \tag{3.63}$$

Define $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(y) = \begin{cases} f(y + y^*) - f(y^*) & \text{for } y \geq -y^* + \eta \\ f(\eta) - f(y^*) & \text{for } y < -y^* + \eta. \end{cases} \tag{3.64}$$

Then

$$p|y| - |\tilde{f}(y)| > 0 \quad \forall \quad y \neq 0$$

and

$$p|y| - |\tilde{f}(y)| \to \infty \quad \text{as} \quad |y| \to \infty.$$

Hence, there exists $\beta \in \mathcal{K}_\infty$ such that

$$|\tilde{f}(y)| \leq p|y| - \beta(|y|) \qquad \forall \quad y \in \mathbb{R}.$$

Combining this with the fact that $\mathbb{D}(0, p) \subseteq \mathbb{S}(A, b, c^T)$, it follows by Theorem

3.2.7 that the system

$$\dot{z} = Az + b\tilde{f}(c^T z) + \tilde{d}, \qquad z(0) = z^0. \tag{3.65}$$

is ISS in the sense that there exists $\tilde{\psi} \in \mathcal{KL}$ and $\tilde{\varphi} \in \mathcal{K}$ such that, for every $z^0 \in \mathbb{R}^n$ and every $\tilde{d} \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$,

$$\|z(t; z^0, \tilde{d})\| \le \tilde{\psi}(\|z^0\|, t) + \tilde{\varphi}(\|\tilde{d}\|_{L^\infty(0,t)}), \qquad \forall \quad t \ge 0, \tag{3.66}$$

where $z(\,\cdot\,; z^0, \tilde{d})$ denotes the unique forward solution of (3.65).

Let $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \ge \varepsilon$ and let $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$ be a nonnegative disturbance. Define $\tilde{x}(t) := x(t; x^0, d) - x^*$ for all $t \ge 0$ and set

$$\tilde{x}_\theta(t) := \tilde{x}(t + \theta) \quad \text{and} \quad d_\theta(t) := d(t + \theta) \qquad \forall \quad t \ge 0.$$

By (3.63),

$$c^T \tilde{x}_\theta(t) \ge -y^* + \eta \qquad \forall \quad t \ge 0,$$

and it is easy to see that $\tilde{x}_\theta$ solves (3.65) with $z^0 = \tilde{x}_\theta(0) = x(\theta; x^0, d) - x^*$ and $\tilde{d} = d_\theta$. Hence, by (3.66), we have that

$$\|\tilde{x}_\theta(t)\| \le \tilde{\psi}(\|\tilde{x}_\theta(0)\|, t) + \tilde{\varphi}(\|d_\theta\|_{L^\infty(0,t)}) \qquad \forall \quad t \ge 0. \tag{3.67}$$

Moreover, on the interval $[0, \theta]$, $\tilde{x}$ satisfies

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + b\hat{f}(c^T \tilde{x}(t)) + d(t) \qquad \forall \quad t \in [0, \theta],$$

where the function $\hat{f} : [-y^*, \infty) \to [-py^*, \infty)$ is defined by

$$\hat{f}(y) = f(y + y^*) - f(y^*) = f(y + y^*) - py^* \qquad \forall \quad y \ge -y^*.$$

It is clear that $|\hat{f}(y)| \le p|y|$ for all $y \ge -y^*$ and, using the variation-of-parameters formula, it follows that there exists constants $k_1 > 0$ and $k_2 > 0$ (not depending on x^0 and d) such that

$$\|\tilde{x}(t)\| \le k_1(\|x^0 - x^*\| + \|d\|_{L^\infty(0,\theta)}) + k_2 \int_0^t \|\tilde{x}(s)\| ds \qquad \forall \quad t \in [0, \theta]. \tag{3.68}$$

Applying Gronwall's Lemma to the estimate (3.68) yields

$$\begin{aligned} \|\tilde{x}(t)\| &\le k_1 e^{k_2\theta}(\|x^0 - x^*\| + \|d\|_{L^\infty(0,\theta)}) \\ &= k(\|x^0 - x^*\| + \|d\|_{L^\infty(0,\theta)}) \qquad \forall \quad t \in [0, \theta], \end{aligned} \tag{3.69}$$

where $k := k_1 e^{k_2\theta}$. Defining $\psi_1 \in \mathcal{KL}$ and $\varphi_1 \in \mathcal{K}$ by

$$\psi_1(s,t) := k e^{\theta} e^{-t} s \qquad \forall \quad s,t \geq 0$$

and

$$\varphi_1(s) := ks \qquad \forall \quad s \geq 0,$$

respectively, and noting that $ks \leq k e^{\theta} e^{-t} s = \psi_1(t,s)$ for all $t \in [0,\theta]$ and $s \geq 0$, it follows from (3.69) that

$$\|\tilde{x}(t)\| \leq \psi_1(\|x^0 - x^*\|, t) + \varphi_1(\|d\|_{L^\infty(0,t)}) \qquad \forall \quad t \in [0,\theta]. \tag{3.70}$$

Note that here we have made use of the causality of the underlying Lur'e system (on the right hand side of (3.70) the L^∞-norm is taken over $[0,t]$ and not over $[0,\theta]$ as in (3.69)). Furthermore, evaluating (3.69) at $t = \theta$ we see that

$$\|\tilde{x}_\theta(0)\| = \|\tilde{x}(\theta)\| \leq k(\|x^0 - x^*\| + \|d\|_{L^\infty(0,\theta)}). \tag{3.71}$$

Inserting (3.71) into (3.67) and invoking the inequality

$$\begin{aligned}\tilde{\psi}(s_1 + s_2, t) &\leq \tilde{\psi}(2s_1, t) + \tilde{\psi}(2s_2, t)\\ &\leq \tilde{\psi}(2s_1, 0) + \tilde{\psi}(2s_2, 0) \qquad \forall \quad s_1, s_2, t \geq 0,\end{aligned}$$

we obtain

$$\begin{aligned}|\tilde{x}(t+\theta)\| \leq \tilde{\psi}(2k\|x^0 - x^*\|, t) &+ \tilde{\psi}(2k\|d\|_{L^\infty(0,\theta)}, 0)\\ &+ \tilde{\varphi}(\|d\|_{L^\infty(0,t+\theta)}) \qquad \forall \quad t \geq 0.\end{aligned} \tag{3.72}$$

Defining $\psi_2 \in \mathcal{KL}$ and $\varphi_2 \in \mathcal{K}$ by

$$\psi_2(s,t) := \begin{cases} \tilde{\psi}(2ks, 0), & (s,t) \in \mathbb{R}_+ \times [0,\theta] \\ \tilde{\psi}(2ks, t-\theta), & (s,t) \in \mathbb{R}_+ \times (\theta, \infty) \end{cases}$$

and

$$\varphi_2(s) := \tilde{\varphi}(s) + \tilde{\psi}(2ks, 0) \qquad \forall \quad s \geq 0$$

respectively, the estimate (3.72) can be written as

$$\|\tilde{x}(t+\theta)\| \leq \psi_2(\|x^0 - x^*\|, t+\theta) + \varphi_2(\|d\|_{L^\infty(0,t+\theta)}) \qquad \forall\, t \geq 0.$$

Finally, setting

$$\psi := \max(\psi_1, \psi_2) \in \mathcal{KL},$$

and

$$\varphi := \max(\varphi_1, \varphi_2) \in \mathcal{K},$$

it is clear that ψ and φ do not depend on x^0 and d. Invoking (3.70), we obtain

$$\|\tilde{x}(t)\| \leq \psi(\|x^0 - x^*\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \qquad \forall \quad t \geq 0,$$

and hence (3.62), completing the proof. □

Example 3.5.10. *Consider the disturbed Lur'e system* (3.18) *with* (A, b, c) *given by* (3.24)*. If the nonlinearity is given by*

$$f(y) = \frac{2y}{5+y},$$

for $y \geq 0$ *and the disturbance is given by*

$$d(t) = \begin{pmatrix} 0.1(1+\sin(t/2)) \\ 0.5(1+\sin(t/5)) \\ 0.25(1+\sin(t/10)) \end{pmatrix},$$

then Theorem 3.5.9 tells us that

$$x^* = -A^{-1}bpy^* = \begin{pmatrix} 3 \\ 7.5 \\ 3.75 \end{pmatrix}$$

is "quasi-globally" ISS in the sense that for all $\varepsilon > 0$*, there exists* $\psi \in \mathcal{KL}$ *and* $\varphi \in \mathcal{K}$ *such that for all* $x^0 \in \mathbb{R}^n_+$ *with* $\|x^0\| > \varepsilon$*,*

$$\|x(t; x^0, d) - x^*\| \leq \psi(\|x^0 - x^*\|, t) + \varphi(\|d\|_{L^\infty(0,t)}), \qquad \forall \quad t \geq 0.$$

This is illustrated in Figure 3.13(b) which simulates this system far an arbitrary initial condition.

It is clear from these plots that the error is bounded, or at least for t ≤ 100.

To conclude this section we comment on forced Lur'e systems with arbitrary, not necessarily nonnegative, disturbances. It is clear if the disturbance d is not nonnegative, then the solution may not exist on the whole interval $\mathbb{R}_+$, that is, the solution may approach the boundary of the nonnegative orthant in finite time.

The following result shows that if $x^* \gg 0$, then, under the assumptions of Theorem 3.5.9, the forward solution exists on $\mathbb{R}_+$ in the interior of the nonnegative orthant for all initial conditions $x^0 \in \mathbb{R}^n_+$ and all, not necessarily

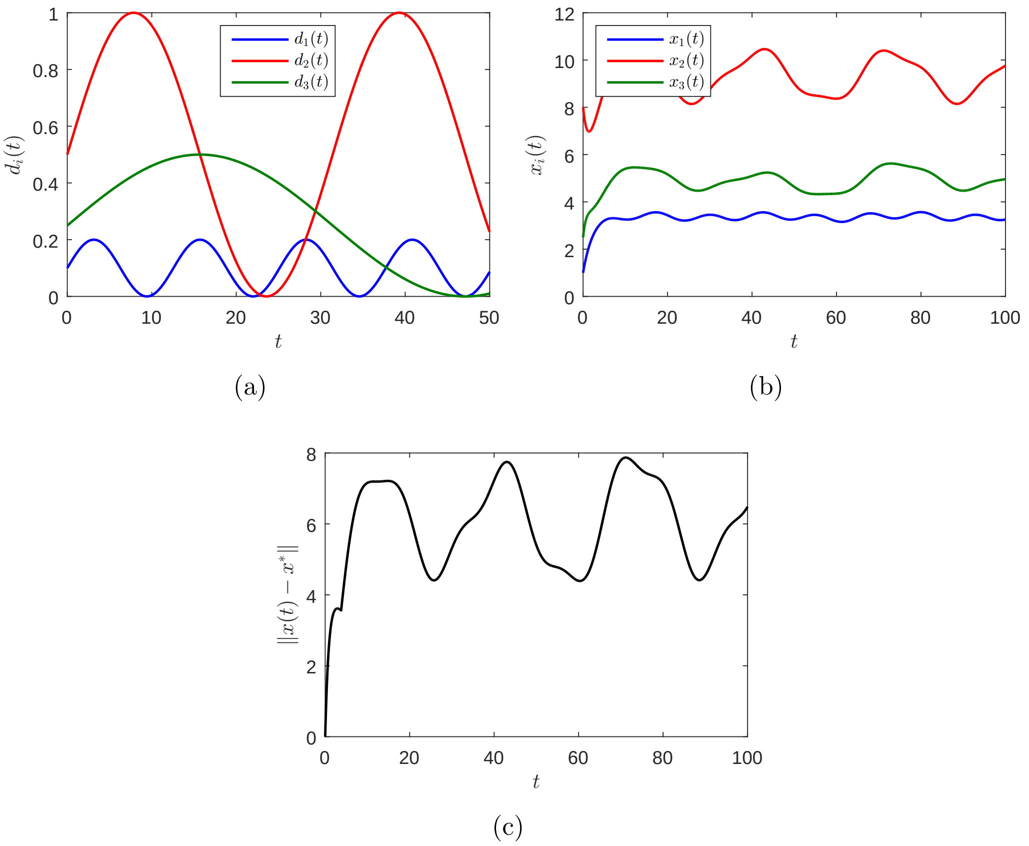


Figure 3.13: Simulation for Example 3.5.10. (a) Plot of three components $d(t)$. (b) A time history plot of the three components of $x(t)$. (c) Plot of the error $\|x(t) - x^*\|$.

nonnegative disturbances $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ with $\|x^0 - x^*\| + \|d\|_{L^\infty(\mathbb{R}_+)}$ sufficiently small.

Proposition 3.5.11. *Consider the system* (3.18) *and assume* **(A3.1)**-**(A3.5)**, **(A3.7)**, **(A3.11)** *and* **(A3.12)** *hold and* $x^* = -A^{-1}bpy^* \gg 0$. *Then there exists* $\varepsilon > 0$, *such that, for all* $x^0 \in \mathbb{R}^n_+$ *and all disturbances* $d \in L(\mathbb{R}_+, \mathbb{R}^n)$ *with* $\|x^0 - x^*\| + \|d\|_{L^\infty(\mathbb{R}_+)} < \varepsilon$, *the maximally defined solution* $x(\,\cdot\,; x^0, d)$ *exists on* $\mathbb{R}_+$ *with values in the interior of* $\mathbb{R}^n_+$.

Proof. Since $x^* \gg 0$ and $c^T x^* = y^*$, there exists $\varepsilon_0 > 0$ such that

$$\mathbb{B}(x^*, \varepsilon_0) \subseteq \operatorname{int} \mathbb{R}^n_+, \tag{3.73}$$

and

$$c^T z \geq \frac{y^*}{2} \qquad \forall \quad z \in \mathbb{B}(x^*, \varepsilon_0). \tag{3.74}$$

Defining the nonlinearity $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by (3.64) with $\eta = y^*/2$, there exists $\beta \in \mathcal{K}_\infty$ such that $|\tilde{f}(y)| \leq p|y| - \beta(|y|)$ for all $y \in \mathbb{R}$, and thus, in the context of the system

$$\dot{w} = Aw + b\tilde{f}(c^T w) + d, \qquad w(0) = w^0, \tag{3.75}$$

the origin is ISS, as follows from Theorem 3.2.7. Consequently, there exists $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that,

$$\|w(t; w^0, d)\| \leq \psi(\|w^0\|, t) + \varphi(\|d\|_{L^\infty}) \tag{3.76}$$

for all $t \geq 0$ and all $(w^0, d) \in \mathbb{R}^n_+ \times L^\infty(\mathbb{R}_+, \mathbb{R}^n)$, where $w(\,\cdot\,; w^0, d)$ denotes the unique solution of (3.75). Obviously, $w(t; x^0, d)$ is defined for all $t \geq 0$. Note that in (3.76) all disturbances $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$, which are not necessarily nonnegative are considered. Now choose $\varepsilon > 0$ such that

$$\psi(\|w^0\|, 0) + \varphi(\|d\|_{L^\infty}) < \varepsilon_0$$

for all $(w^0, d) \in \mathbb{R}^n \times L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ with $\|w^0\| + \|d\|_{L^\infty} < \varepsilon$. With this choice of ε, it follows from (3.76) that

$$\|w(t; w^0, d)\| < \varepsilon_0 \tag{3.77}$$

for all $t \geq 0$ and all $(w^0, d) \in \mathbb{R}^n \times L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ such that $\|w^0\| + \|d\|_{L^\infty} < \varepsilon$.

Finally, let $(x^0, d) \in \mathbb{R}^n \times L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ such that $\|x^0 - x^*\| + \|d\|_{L^\infty} < \varepsilon$, and set $z(t) := w(t; x^0 - x^*, d) + x^*$. By (3.73), (3.74) and (3.77),

$$z(t) \in \mathbb{B}(x^*, \varepsilon_0) \subseteq \operatorname{int} \mathbb{R}^n_+ \qquad \forall \quad t \geq 0, \tag{3.78}$$

and

$$c^T z(t) \geq \frac{y^*}{2} \qquad \forall\, t \geq 0.$$

Consequently, by the latter,

$$c^T w(t; x^0 - x^*, d) = c^T z(t) - y^* \geq -\frac{y^*}{2} \qquad \forall \quad t \geq 0,$$

from which it follows that

$$\begin{aligned} \tilde{f}(c^T w(t; x^0 - x^*, d)) &= f(c^T w(t; x^0 - x^*, d) + y^*) - f(y^*) \\ &= f(c^T z(t)) - p y^* \qquad \forall \quad t \geq 0. \end{aligned}$$

An immediate consequence of this identity is that $\dot{z} = Az + bf(c^T z) + d$. Now $z(0) = x^0$, and thus, by uniqueness of solutions, $z(t) = x(t; x^0, d)$ for all $t \geq 0$. The claim now follows from (3.78). □

3.6 Applications to Biology

In this section we apply the theory developed in this chapter to two applications. The first of which is an application to population modeling and the second applies it to enzyme synthesis.

3.6.1 Population Dynamics

In this first application we return to the continuous time population model introduced in Section 2.3.1. For this we assume that $d_i = 0$ for all $1 \leq i \leq n$ and instead that the birth rate is a density dependent function which depends on the final age-class. We also include an external disturbance to each age-class which will represent migration.

We have n coupled differential equations given by

$$\begin{aligned} \dot{x}_1 &= -a_1 x_1 + f(x_n) + d_1, & x_1(0) &= {x^0}_1, & & \\ \dot{x}_k &= a_{2(k-1)} x_{k-1} - a_{2k-1} x_k + d_k, & x_k(0) &= {x^0}_k, & \text{for} \quad & k \in \{2, \ldots, n\}, \end{aligned} \tag{3.79}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$, $a_i > 0$ for all $i \in \{1, \ldots, 2n-1\}$, $d_i \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ is nonnegative for all $i \in \{1, \ldots, n\}$ and ${x^0}_i \geq 0$ for all $i \in \{1, \ldots, n\}$. Note that the a_i notation has been introduced for convenience to limit the number of parameters.

Introducing $x := (x_1, \ldots, x_n)^T$ and $d := (d_1, \ldots, d_n)^T$, the system (3.79) may be rewritten in the form (3.18) with

$$A := \begin{pmatrix} -a_1 & 0 & \cdots & 0 \\ a_2 & -a_3 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & a_{2n-2} & -a_{2n-1} \end{pmatrix}, \qquad b := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad c := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{3.80}$$

Obviously, **(A3.1)** holds. Since

$$\sigma(A) = \{-a_1, -a_3, \ldots, -a_{2n-1}\},$$

and $a_i > 0$ for all $i \in \{1, \ldots, 2n-1\}$, assumption **(A3.2)** is also satisfied. Moreover, it is readily verified that **(A3.3)** holds.

A straightforward calculation yields that:

$$p = \frac{1}{\boldsymbol{G}(0)} = -\frac{1}{c^T A^{-1} b} = \frac{\prod_{i=1}^{n} a_{2i-1}}{\prod_{i=1}^{n-1} a_{2i}}. \tag{3.81}$$

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be given by

$$f(y) = \frac{my}{k+y} \tag{3.82}$$

where $m, k > 0$ are positive constants. See Section 2.4.3 for additional details on this function.

Consider the case where $d = 0$. It can easily be seen that $f(y)$ satisfies **(A3.4)** and **(A3.11)**. For $p > 0$ we can fall into one of three cases.

- If $m/k < p$ then the conditions for statement (3) of Theorem 3.4.5 are satisfied so 0 will be globally exponentially stable.
- If $m/k = p$ then the conditions for statement (2) of Theorem 3.4.5 are satisfied so for all $x^0 \in \mathbb{R}^n_+$, $x(t; x^0) \to 0$ as $t \to \infty$.
- If $m/k > p$ then **(A3.5)** and **(A3.10)** and satisfied, thus the assumptions of Theorem 3.4.15 are satisfied and there exists a "quasi-globally" exponentially stable nonzero equilibrium.

In a similar fashion, we can say the same about the forced system with $d \neq 0$.

- If $m/k \leq p$ then the conditions of Theorem 3.5.3 are satisfies and 0 will be ISS.
- If $m/k > p$ then the conditions of Theorem 3.5.9 are satisfied and there exists an $x^* \neq 0$ which is "quasi-globally" ISS.

Consider the system (3.79) with f given by (3.82) and constants given by

$$n = 3,\ a_1 = 1,\ a_2 = 0.8,\ a_3 = 0.9,\ a_4 = 0.6,\ a_5 = 0.8,\ m = 3,\ k = 1. \tag{3.83}$$

For this choice of constants it follows from (3.81) that $p = 3/2$. A simple calculation shows that $y^* = 1$ is the unique positive value of y^* such that $py^* = f(y^*)$. Furthermore,

$$x^* := -A^{-1}bpy^* = \begin{pmatrix} 3/2 \\ 4/3 \\ 1 \end{pmatrix}$$

is the unique nonzero, quasi-globally, exponentially stable equilibrium of the system where $d = 0$. If $d \neq 0$ then x^* is quasi-globally ISS.

We illustrate these two properties in the following simulations for three arbitrary initial conditions in $\mathbb{R}^n_+$. Figure 3.14 contains a time history plot of $x(t)$ with $d = 0$ to illustrate that x^* is quasi-globally, exponentially stable.

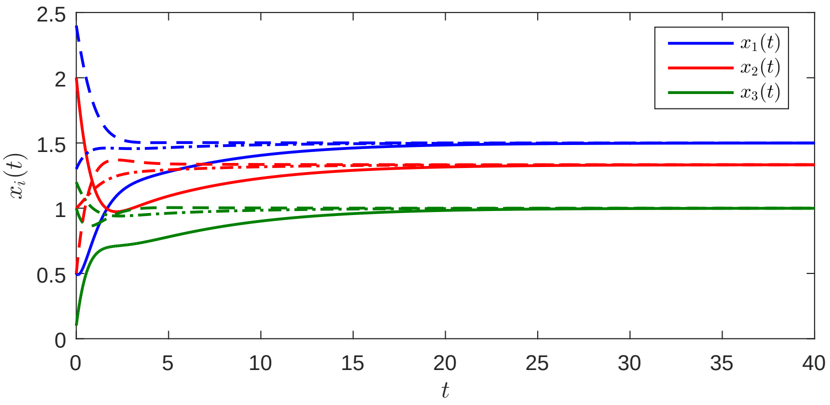


Figure 3.14: Simulations of the system given by (3.80) with nonlinearity given by (3.82), constants given by (3.83), $d = 0$ with three arbitrary initial conditions where the solid lines, dashed lines and dashed/dotted lines represent a different initial condition.

Now, consider two disturbances,

$$d_1(t) = \begin{pmatrix} 0.1(1+\sin(t/5)) \\ 0.2(1+\sin(3t/5)) \\ 0.15(1+\sin(t/5)) \end{pmatrix} \quad \text{and} \quad d_2(t) = \begin{pmatrix} 0.9(1+\sin(2t/3)) \\ 0.75(1+\sin(t/6)) \\ 0.1(1+\sin(t/3)) \end{pmatrix}. \tag{3.84}$$

Figure 3.15 contains a time history plots of $x(t)$ and error plots of $\|x(t;x^0)-x^*\|$ with disturbances given by (3.84) for three arbitrary nonnegative and nonzero initial conditions to illustrate that x^* is quasi-globally ISS. In this figure the solid, dashed and dashed/dotted lines correspond to the three arbitrary initial conditions.

3.6.2 Enzymatic Control Processes

The following example is based on [103, Section 7.2], which in turn was based on [49]. This example was chosen as we can bring a new idea to this old example, namely ISS.

Certain metabolites repress the enzymes which are essential for their own synthesis. This is achieved by inhibiting the transcription of the molecule DNA to messenger RNA or mRNA (M). This mRNA is the template which produces the enzyme (E). The enzyme will combine with a substrate and form a product (P). It is this product which inhibits the production of mRNA. A simple model of this is given in Figure 3.16.

The DNA is readily available so does not need to be modeled. The production of mRNA is inhibited by the product and degrades according to first order kinetics. Both the enzyme and the product are produced and degraded by first order kinetics.

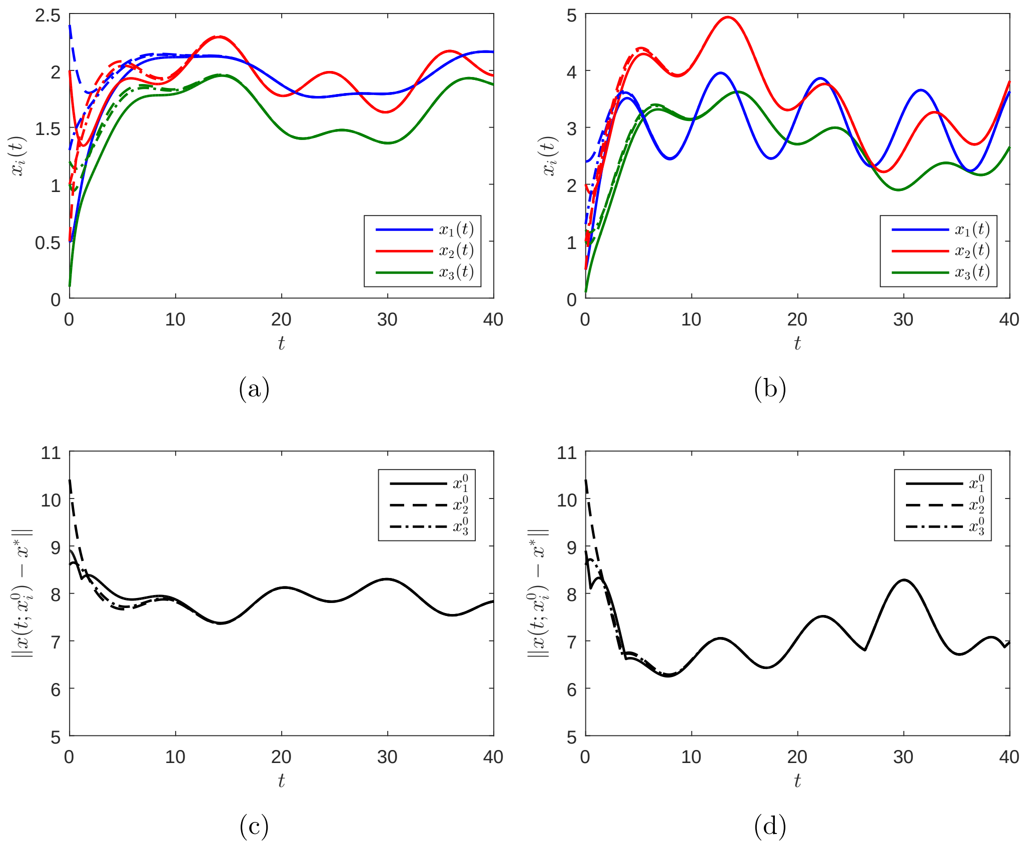


Figure 3.15: Simulations of the system given by (3.80) with nonlinearity given by (3.82) and constants given by (3.83), $d(t)$. Three arbitrary initial conditions are used, where solid lines, dashed lines and dashed/dotted lines correspond to an initial condition. (a) Time history of $x(t)$ with disturbance $d_1(t)$ given by (3.84). (b) Time history of $x(t)$ with disturbance $d_2(t)$ given by (3.84). (c) Error $\|x(t;x_i^0)-x^*\|$ where $d(t)=d_1(t)$. (d) Error $\|x(t;x_i^0)-x^*\|$ where $d(t)=d_2(t)$.

A model for this system therefore is:

$$\frac{dM}{dt} = \frac{v}{k+P} - a_1 M,$$

$$\frac{dE}{dt} = a_2 M - a_3 E, \tag{3.85}$$

$$\frac{dP}{dt} = a_4 E - a_5 P, \tag{3.86}$$

where M represents the concentration of mRNA, E is the concentration of the enzyme and P is the concentration of the product being produced from the action between the enzyme and substrate. v, k and a_i for $i = 1, \ldots, 5$ are positive constants.

It is perhaps more biologically realistic if the third equation is replaced by

$$\frac{dP}{dt} = a_4 E - \frac{v_2 P}{k_2 + P},$$

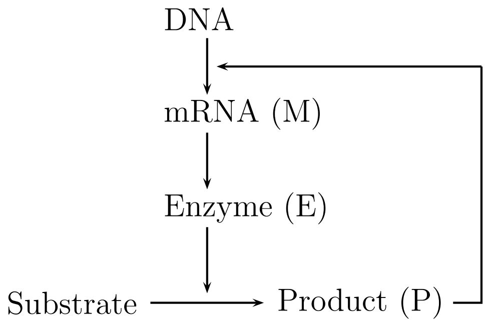


Figure 3.16: A schematic for the production of a self repressive enzyme.

where $v_2, k_2 > 0$ (see [103, Section 7.2]). This is a more realistic model for the degradation of the product as it saturates for large values of P. For the purpose of this example however we will not be using this equation for P.

Set $x = (M, E, P)^T$, then rewriting (3.85) in vector form

$$\dot{x} = \begin{pmatrix} -a_1 & 0 & 0 \\ a_2 & -a_3 & 0 \\ 0 & a_4 & -a_5 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} f\left(\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x\right), \tag{3.87}$$

where

$$f(y) = \frac{v}{k + y}. \tag{3.88}$$

It is easy to verify that for all $a_i > 0$, $i = 1, \ldots, 5$ that **(A3.1)**-**(A3.3)** are satisfied. It is also true that **(A3.4)** hold for all $v, k > 0$. Clearly we have that $f(0) > 0$ therefore systems with a nonlinearity such as this are a candidate to fit the framework of Theorem 3.4.8. The following lemma demonstrates why this is the case.

Lemma 3.6.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be twice continuously differentiable and $p > 0$. If f is nonincreasing ($f' \leq 0$), convex ($f'' \geq 0$) and if $f'(0) > -p$, then there exists a unique $y^* \geq 0$ such that $f(y^*) = py^*$ and*

$$\sup_{y \geq 0,\, y \neq y^*} \left| \frac{f(y) - f(y^*)}{y - y^*} \right| < p. \tag{3.89}$$

Proof. Noting that f is nonincreasing, it is clear that there exists a unique $y^* \geq 0$ such that $f(y^*) = py^*$.

First consider the case when $y^* = 0$. Noting that f is nonincreasing implies that $f(y) = 0$ for all $y \geq 0$, from which (3.89) holds trivially.

Now suppose that $y^* > 0$. Combining nonnegativity, convexity and $f'(0) > -p$ yields

$$-p < f'(0) \leq f'(y) \leq 0 \qquad \forall \quad y \geq 0. \tag{3.90}$$

From this it follows that

$$py^* \le f(0) < 2py^*. \tag{3.91}$$

We can now conclude on the interval $[0, y^*)$ that $f(y) > py$, which follows from f being nonincreasing and that $f(y) < 2py^* - py$, which follows from (3.90) and (3.91). On the interval (y^*, ∞) using the fact f is nonincreasing it follows that $f(y) < py$, and using (3.90), $f(y) > 2py^* - py$.

Combining the above yields (3.89), completing the proof. □

We demonstrate that a nonlinearity of the form (3.88) satisfies the requirements of Lemma 3.6.1. Begin by noting that

$$f'(y) = -\frac{v}{(k+y)^2}, \qquad f''(y) = \frac{v}{(k+y)^3}.$$

Clearly $f'(y) \ge 0$ for all $y \ge 0$, thus f is nonincreasing. Additionally $f''(y) \ge 0$ for all $y \ge 0$, therefore f is convex. The final assumption that $f'(0) > -p$ depends on v, k and p. If $v/k^2 < p$ then all of the assumptions of Lemma 3.6.1 are satisfied.

We consider the specific example described in [49], which provides values for the constants appearing in (3.87) and (3.88). These give the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 0 & 0 \\ 1 & -0.6 & 0 \\ 0 & 1 & -0.8 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} f\left(\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x \right), \\ x(0) &= x^0, \\ f(y) &= \frac{360}{43+y}. \end{aligned} \tag{3.92}$$

We first calculate p for this system and show that $v/k^2 < p$.

$$p = \frac{-1}{c^T A^{-1} b} = 0.48 > 0.19 = \frac{v}{k^2}.$$

We can therefore apply Lemma 3.6.1 to this system and yield the existence of a unique $y^* > 0$ such that $py^* = f(y^*)$ and

$$\sup_{y \ge 0,\, y \ne y^*} \left| \frac{f(y) - f(y^*)}{y - y^*} \right| < p.$$

This value can be computed to be $y^* = 13.3174$.

This system is an ideal candidate for application of Theorem 3.4.8. As previously stated, assumptions **(A3.1)**-**(A3.4)** are satisfied and for the parameters we have, Lemma 3.6.1 tells us that **(A3.5)** and **(A3.10)** are satisfied.

Therefore an application of Theorem 3.4.8(3) tells us that the equilibrium

$$x^* = -A^{-1}bpy^* = \begin{pmatrix} 6.3923 \\ 10.6539 \\ 13.3174 \end{pmatrix}$$

is is globally exponentially stable in the sense that, for all $x^0 \in \mathbb{R}^n_+$, there exists constants $\gamma > 0$ and $g \geq 1$ such that

$$\|x(t;x^0) - x^*\| \leq ge^{-\gamma t}\|x^0 - x^*\| \qquad \forall\ t \geq 0.$$

Illustrated in Figure 3.17 is a simulation for this system starting at 3 different initial points,

$$x^1(0) = \begin{pmatrix} 2 \\ 5.5 \\ 11 \end{pmatrix}, \qquad x^2(0) = \begin{pmatrix} 19 \\ 19 \\ 3 \end{pmatrix}, \qquad x^3(0) = \begin{pmatrix} 15 \\ 0.5 \\ 10 \end{pmatrix}. \qquad (3.93)$$

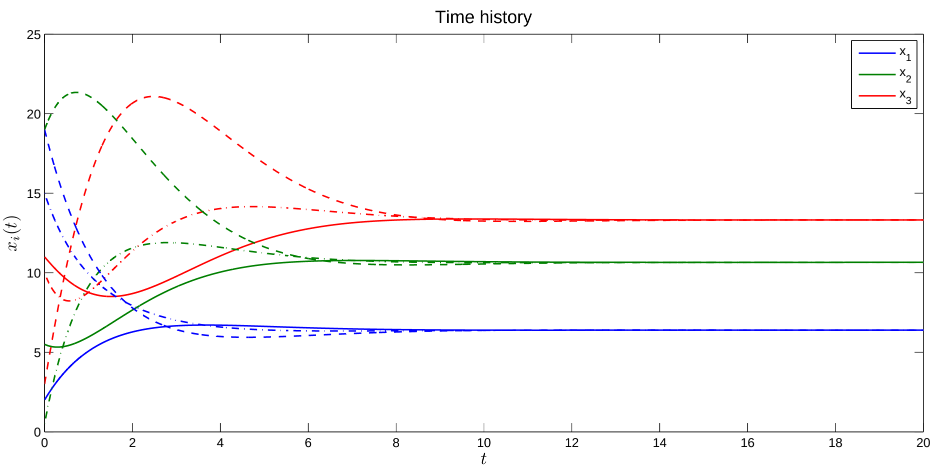


Figure 3.17: A simulation of the system (3.92) with initial conditions given by (3.93). The solid lines are for initial condition $x^1(0)$, the dashed for $x^2(0)$ and the dashed/dotted line for $x^3(t)$.

It can clearly be seen that for all three initial conditions that $x(t) \to x^*$ as $t \to \infty$ as expected.

We now demonstrate that in the presence of an additional input to this system, we have that the system is ISS. We now consider the system

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -1 & 0 & 0 \\ 1 & -0.6 & 0 \\ 0 & 1 & -0.8 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} f\left(\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x(t)\right) + d(t), \\ x(0) &= x^0, \qquad f(y) = \frac{360}{43+y}, \qquad d(t) = \begin{pmatrix} 2+2\sin(2t) \\ 0.75+0.75\sin(t/2) \\ 2+2\sin(4t) \end{pmatrix}. \end{aligned} \tag{3.94}$$

A disturbance in an enzymatic control process could represent a number of different things such as such as underestimated parameters in the linear system which are supplemented by an additive disturbance or a time varying linear system. The implications of ISS are that even if we supplement an estimated linear system with an additive disturbance, the state remains close to the equilibrium of the unforced system.

Due to the nature of the nonlinearity, it is clear that

$$py - f(y) \to \infty \qquad \text{as} \qquad y \to \infty,$$

and that $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$. We can therefore apply Theorem 3.5.5 to reach the conclusion that $x^* = -A^{-1}bpy^*$ is ISS in the sense that, there exists $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that, for all $x^0 \in \mathbb{R}^n_+$,

$$\|x(t;x^0,d) - x^*\| \leq \psi(\|x^0 - x^*\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \qquad \forall\, t \geq 0.$$

Illustrated in Figure 3.18 is a simulation of (3.94) for the three initial conditions given in (3.93).

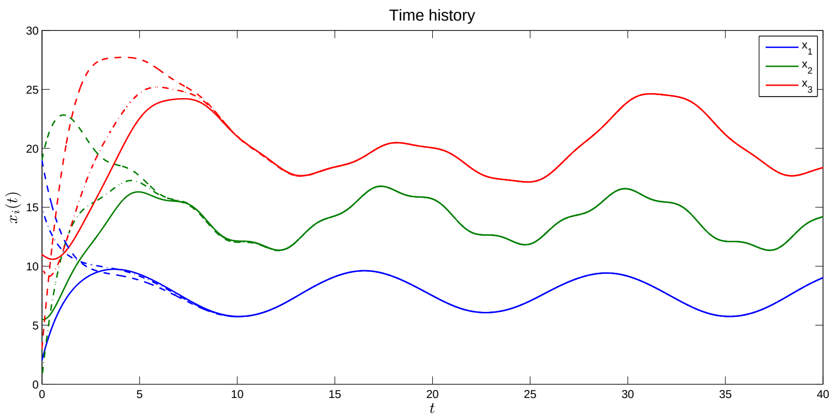


Figure 3.18: A simulation of the system (3.94) with initial conditions given by (3.93). The solid lines are for initial condition $x^1(0)$, the dashed for $x^2(0)$ and the dashed/dotted line for $x^3(t)$.

These simulations verify that our theory is correct for this example as clearly $\|x(t; x^0, d) - x^*\|$ is bounded for the values of $t \in [0, 40]$.

Chapter 4

Converging-Input Converging-State Property of Continuous Time Lur'e Systems

This chapter is mainly based on [11].

4.1 Introduction

We consider forced Lur'e systems in continuous-time of the form

$$\dot{x} = Ax + Bf(Cx) + v, \qquad x(0) = x^0 \in \mathbb{R}^n, \tag{4.1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$ are matrices, $f : \mathbb{R}^p \to \mathbb{R}^m$ is a (nonlinear) function, x denotes the state and v is a control function (also interpreted as and named a disturbance, forcing term or input). It is often useful to think of (4.1) as a closed-loop system obtained by static output feedback applied to the linear system specified by (A, B, C), namely,

$$\dot{x} = Ax + Bu + v, \qquad y = Cx, \qquad u = f(y),$$

where u and y denote the input and output variables, respectively, see also Figure 4.1. Lur'e systems are a common and important class of nonlinear systems and are at the center of the classical subject of absolute stability theory which includes the well known circle and Popov criteria, see [54, 55, 72, 79, 86, 146, 153]. An absolute stability criterion for (4.1) is a sufficient condition for stability, usually formulated in terms of frequency-domain properties of the linear system given by (A, B, C) and sector or boundedness conditions for f, guaranteeing stability for all nonlinearities f satisfying these conditions. Traditionally, Lyapunov approaches to the stability theory of systems of the

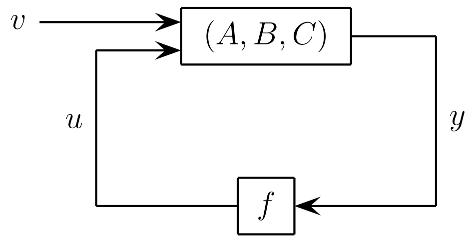


Figure 4.1: Block diagram of the controlled Lur'e system (4.1).

form (4.1) consider uncontrolled ($v = 0$) Lur'e systems with forcing (usually acting through B, that is, v is of the form $v = Bw$) and have been studied using the input-output framework initiated by Sandberg and Zames in the 1960s, see, for example, [31, 146]. More recently, forced Lur'e systems have been analyzed in the context of input-to-state stability (ISS) theory, see [2, 71, 72, 124].

In this chapter, we investigate the following problem (and variations thereof):

> *Given $v^\infty \in \mathbb{R}^n$, find conditions (necessary or sufficient) for the existence of $x^\infty \in \mathbb{R}^n$ such that, for every x^0 and every v with $v(t) \to v^\infty$ as $t \to \infty$, the solution x of* (4.1) *converges to x^∞.*

In particular, we consider the so-called *converging-input converging-state* (often written as CICS) property: (4.1) is said to have the CICS property if, for every $v^\infty \in \mathbb{R}^n$, there exists $x^\infty \in \mathbb{R}^n$ such that $\lim_{t\to\infty} x(t) = x^\infty$ for all x^0 and all inputs v converging to v^∞.

One of the main contributions of this chapter is the establishment of sufficient conditions for the CICS property which are reminiscent of the complex Aizerman conjecture [60, 61, 72, 124], the circle criterion for ISS [71, 72, 124] and the "nonlinear" ISS small-gain condition for Lur'e systems [124] and involve the transfer function matrix of the linear system (A, B, C) and an incremental condition (in terms of norm or sector inequalities) on the nonlinearity f.

By way of further background and motivation, we comment that if (4.1) is linear and asymptotically stable, that is, $f(z) = Kz$ and $A + BKC$ is Hurwitz for some matrix K, then (4.1) has the CICS and *converging-input converging-output* properties. Indeed, for given v^∞ and v converging to v^∞, it is well-known that for every x^0 the state x and output y have respective limits

$$x^\infty := -(A + BKC)^{-1} v^\infty$$

and

$$y^\infty := Cx^\infty = -C(A+BKC)^{-1}v^\infty.$$

The matrices $-(A+BKC)^{-1}$ and $-C(A+BKC)^{-1}$ are sometimes referred to as steady-state gains. When a Lur'e system has the CICS property, it is possible to define (nonlinear) steady-state gains, that is, nonlinear mappings $v^\infty \mapsto x^\infty$ or $v^\infty \mapsto y^\infty$, respectively, which generalize the above linear relationship. We mention that Lur'e systems which are globally asymptotically stable when controlled ($v = 0$), need not have the CICS property. Indeed, there may exist inputs converging to 0 such that, for some initial states, the corresponding state trajectory is asymptotically divergent (see Example 4.3.4(b)).

This chapter is organized as follows. In Section 4.2, we discuss a number of preliminaries, present some auxiliary results and prove necessary conditions for CICS. Section 4.3 is devoted to sufficient conditions for the CICS property, the main result begin Theorem 4.3.3, from which several CICS criteria are derived as corollaries. These criteria have the flavor of well-known absolute stability results (complex Aizerman conjecture, circle criterion and small gain). In Section 4.4, we consider Lur'e systems of the form

$$\dot{x} = Ax + Bf(Cx - v), \qquad x(0) = x^0 \in \mathbb{R}^n. \tag{4.2}$$

Note that (4.2) can be thought of as a closed-loop system obtained by linear feedback applied to the linear system (A, B, C) subject to an input nonlinearity f:

$$\dot{x} = Ax + Bf(w), \qquad y = Cx, \qquad w = y - v,$$

see Figure 4.2. We derive a CICS criterion for Lur'e systems of the form (4.2)

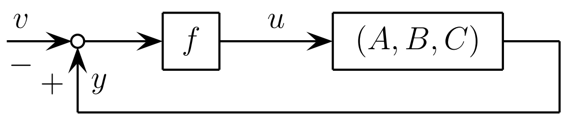


Figure 4.2: Block diagram of the controlled Lur'e system (4.2).

and use it to generalize the well-known result on integral control for linear systems to this class of nonlinear systems. Section 4.5 is devoted to nonnegative Lur'e systems. These arise naturally in a variety of applied contexts: a common key feature is that the state variables x, which may represent population abundances, chemical concentrations or economic quantities (such as prices) are, necessarily, nonnegative. In a population model, the function f may describe density-dependence (typically a sublinear function) owing to increased competition for resources at higher population abundances. In a chemical reaction model, the function f may describe a nonlinear reaction rate between certain components. Unforced biological, ecological and chemical models often admit (at least) two equilibria: the zero equilibrium and some nonzero equilibrium, the latter corresponding to the co-existence of populations or chemical compounds. The control v in (4.1) may model immigration or emigration in a population model or the addition of a new reagent in a chemical reaction model. The main result in Section 4.5 is a sufficient condition for a "quasi CICS" property for Lur'e systems which, for zero control $v = 0$, have two equilibria (see Theorem 4.5.6). In this context, we shall make contact with Chapter 3 on stability properties on nonnegative Lur'e systems: a certain "repelling property" established in Chapter 3 will play a pivotal role in the proof of Theorem 4.5.6.

For general nonlinear systems, the CICS property has been studied in [137, 120]. Concepts related to or reminiscent of the CICS property have been introduced in [1, 136]. Whilst results in [1, 120, 136] have little overlap with the material presented in this chapter, [137] plays an important role in the proof of statement (1) of Theorem 4.3.3, one of the main results in this chapter. With the exception of [123], there does not seem to be any previous work on the CICS property for Lur'e systems. We will make detailed comments on the relationship of our results to those in [123] after the proof of Corollary 4.3.15.

4.2 Preliminary Results and A Necessary Condition for CICS

Consider the forced Lur'e system

$$\dot{x} = Ax + Bf(Cx) + v, \qquad x(0) = x^0 \in \mathbb{R}^n, \qquad y = Cx, \tag{4.3}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $f : \mathbb{R}^p \to \mathbb{R}^m$ is locally Lipschitz and $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ is an input function, otherwise known as a forcing or control function. If $v = 0$, then we will refer to (4.3) as an uncontrolled system. Frequently, the input v will be of the form $v = Ew$, where $E \in \mathbb{R}^{n\times q}$ and $w \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^q)$. If $q = m$ and $E = B$, then (4.3) can be written in the form

$$\dot{x} = Ax + Bu, \qquad x(0) = x^0 \in \mathbb{R}^n, \qquad y = Cx, \qquad u = v + f(y).$$

Let $x(\,\cdot\,; x^0, v)$ denote the unique maximally defined forward solution of the initial-value problem (4.3). We say that $(x^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^n$ is an *equilibrium pair* of (4.3) if $Ax^* + Bf(Cx^*) + v^* = 0$, that is, if x^* is an equilibrium of the autonomous differential equation

$$\dot{x} = Ax + Bf(Cx) + v^*. \tag{4.4}$$

In the following, let $\theta : \mathbb{R}_+ \to \mathbb{R}$ denote the constant function given by $\theta(t) = 1$ for all $t \geq 0$. It is clear that if, for some $v^\infty \in \mathbb{R}^n$ and $x^0 \in \mathbb{R}^n$, $x(t; x^0, v^\infty\theta)$ converges to x^∞ as $t \to \infty$, then (x^∞, v^∞) is an equilibrium pair of (4.4). An equilibrium pair (x^*, v^*) is said to be *globally asymptotically stable* (GAS), if x^* is a globally asymptotically stable equilibrium of (4.4).

Obviously, if $(0, 0)$ is an equilibrium pair of (4.3), then $(0, 0)$ is GAS if, and only if, the equilibrium 0 of the uncontrolled system (4.3) is GAS.

We say an equilibrium pair (x^*, v^*) of (4.3) is *input-to-state stable* (ISS) if there exists $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that, for every $x^0 \in \mathbb{R}^n$ and every $v \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$,

$$\|x(t; x^0, v) - x^*\| \leq \psi(\|x^0 - x^*\|, t) + \varphi(\|v - v^*\theta\|_{L^\infty(0,t)}) \qquad \forall\, t \geq 0. \tag{4.5}$$

The concept of ISS was first formulated in [133] and for surveys of ISS we refer the reader to [25, 138].

Let $\mathbb{S}_\mathbb{C}(A, B, C)$ denote the set of *complex* stabilizing output feedback gains for the linear system (A, B, C), that is,

$$\mathbb{S}_\mathbb{C}(A, B, C) := \{K \in \mathbb{C}^{m \times p} \ : \ A + BKC \text{ is Hurwitz}\}.$$

Moreover, we define

$$\mathbb{S}_\mathbb{R}(A, B, C) := \mathbb{S}_\mathbb{C}(A, B, C) \cap \mathbb{R}^{m \times p},$$

to be the set of *real* stabilizing output feedback gains for (A, B, C).

In the following, let $\boldsymbol{G}$ be the transfer function of the linear system

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{4.6}$$

that is, $\boldsymbol{G}(s) = C(sI - A)^{-1}B$. Note that if A is Hurwitz, then all poles of $\boldsymbol{G}$ have negative real parts. Applying output feedback of the form $u = Ky + w$ to (4.6), where $K \in \mathbb{R}^{m \times p}$ and w is an input signal, leads to the closed-loop system

$$\dot{x} = (A + BKC)x + Bw, \qquad y = Cx. \tag{4.7}$$

For notational convenience, we set

$$A_K := A + BKC. \tag{4.8}$$

The transfer function of the system (4.7) will be denoted by $\boldsymbol{G}_K$, and is given by

$$\begin{aligned}\boldsymbol{G}_K(s) &= C(sI - A_K)^{-1}B = C(sI - A - BKC)^{-1}B \\ &= \boldsymbol{G}(s)(I - K\boldsymbol{G}(s))^{-1}.\end{aligned}$$

The next result is a key tool which we make use of throughout this chapter.

Theorem 4.2.1. *Let $K \in \mathbb{S}_{\mathbb{R}}(A, B, C)$ and set $\gamma := 1/\|\boldsymbol{G}_K\|_{H^\infty}$, where $\gamma := \infty$ if $\|\boldsymbol{G}_K\|_{H^\infty} = 0$. The following statements hold.*

(1) If $\gamma < \infty$ and

$$\|f(z) - Kz\| < \gamma\|z\| \qquad \forall\ z \in \mathbb{R}^p,\ z \neq 0, \tag{4.9}$$

then the equilibrium 0 of the uncontrolled system (4.3) is GAS.

(2) If $\gamma < \infty$ and there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\|f(z) - Kz\| \leq \gamma\|z\| - \alpha(\|z\|) \qquad \forall\ z \in \mathbb{R}^p, \tag{4.10}$$

then the equilibrium pair $(0, 0)$ of (4.3) is ISS.

(3) If $\gamma = \infty$, then the conclusions of statements (1) and (2) hold for every locally Lipschitz $f : \mathbb{R}^p \to \mathbb{R}^m$ such that $f(0) = 0$.

Proof. Since $K \in \mathbb{S}_{\mathbb{R}}(A, B, C)$, the matrix $A_K = A + BKC$ is Hurwitz. The structured complex stability radius of A_K with respect to the weights B and C is defined by

$$r_{\mathbb{C}}(A_K, B, C) := \inf\{\|P\| \ :\ P \in \mathbb{C}^{m\times p} \text{ such that } A_K + BPC \text{ is not Hurwitz}\}.$$

It is well known, see [59, 61], that

$$r_{\mathbb{C}(A_K,B,C)} = \frac{1}{\|\boldsymbol{G}_K\|_{H^\infty}} = \gamma. \tag{4.11}$$

To prove statements (1) and (2), let $x^0 \in \mathbb{R}^n$ and write $x(t) := x(t; x^0, 0)$. Obviously, x satisfies $\dot{x} = A_K x + B f_K(Cx)$, where $f_K : \mathbb{R}^p \to \mathbb{R}^m$ is defined by

$$f_K(z) = f(z) - Kz \qquad \forall\ z \in \mathbb{R}^p. \tag{4.12}$$

By hypothesis, $\|f_K(z)\| < \gamma\|z\|$ for all nonzero $z \in \mathbb{R}^p$ and thus the claim follows from (4.11) and [61, Theorem 4.5.22] or [60, Corollary 3.15]. Moreover, by (4.11), $\mathbb{B}_{\mathbb{C}}(K,\gamma) \subseteq \mathbb{S}_{\mathbb{C}}(A,B,C)$, and thus, statement (2) is a consequence of [124, Theorem 3.2].

We proceed to prove statement (3). To this end, let $f : \mathbb{R}^p \to \mathbb{R}^m$ be locally Lipschitz and such that $f(0) = 0$. We show that the equilibrium pair $(0,0)$ of (4.3) is ISS. Let $x^0 \in \mathbb{R}^n$ and $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^n)$ be arbitrary and set $x(t) := x(t;x^0,v)$. Then $\dot{x} = A_K x + B f_K(Cx) + v$, where f_K is defined by (4.12). Thus, by the variation-of-parameters formula,

$$x(t) = e^{A_K t}x^0 + \int_0^t e^{A_K(t-s)}\left(Bf_K(Cx(s)) + v(s)\right) ds \qquad \forall\, t \in [0,\omega), \quad (4.13)$$

where $0 < \omega \leq \infty$ and $[0,\omega)$ is the maximal interval of existence of the forward solution x. Note that, since f is not necessarily affine linearly bounded, finite escape time cannot be ruled out at this stage. Now $Ce^{A_K t}B$ is the inverse Laplace transform of $\boldsymbol{G}_K$ and hence $Ce^{A_K t}B = 0$ for all $t \in \mathbb{R}$. Consequently, it follows from (4.13),

$$Cx(t) = Ce^{A_K t}x^0 + \int_0^t Ce^{A_K(t-s)}v(s)ds \qquad \forall\, t \in [0,\omega). \qquad (4.14)$$

Since A_K is Hurwitz, it follows that Cx is bounded on any bounded subinterval of $[0,\omega)$ and thus, by (4.13), x is also bounded on any bounded subinterval of $[0,\omega)$. We may therefore conclude that $\omega = \infty$.

By the Hurwitz property of A_K, there exists $M \geq 1$ and $\mu > 0$ such that

$$\|e^{A_K t}\| \leq Me^{-\mu t} \qquad \forall\, t \geq 0.$$

Combining this with (4.14) shows that there exists positive constants M_1 and M_2 such that, for all $x^0 \in \mathbb{R}^n$ and $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^n)$,

$$\|Cx(t)\| \leq M_1 e^{-\mu t}\|x^0\| + M_2\|v\|_{L^\infty(0,t)} \qquad \forall\, t \geq 0. \qquad (4.15)$$

Moreover, let $\eta \in \mathcal{K}$ be such that

$$\|B\|\|f_K(z)\| \leq \eta(\|z\|) \qquad \forall\, z \in \mathbb{R}^p. \qquad (4.16)$$

The existence of such a function η follows from the continuity of f_K and the fact that $f_K(0) = 0$. Invoking (4.15) and (4.16), we obtain

$$\|B\|\|f_K(Cx(t))\| \leq \eta_1(e^{-\mu t}\|x^0\|) + \eta_2(\|v\|_{L^\infty(0,t)}) \qquad \forall\, t \geq 0, \qquad (4.17)$$

where the $\mathcal{K}$-functions η_1 and η_2 are defined by $\eta_1(s) = \eta(2M_1 s)$ and $\eta_2(s) = \eta(2M_2 s)$.

Next we estimate the term

$$I(t) := \int_0^t e^{A_K(t-s)} B f_K(Cx(s))ds.$$

To this end, writing

$$I(t) = \int_0^{t/2} e^{A_K(t-s)} B f_K(Cx(s))ds + \int_{t/2}^t e^{A_K(t-s)} B f_K(Cx(s))ds,$$

we note that, by (4.17),

$$\begin{aligned} \|I(t)\| \le Me^{-(\mu/2)t} \big(\eta_1(\|x^0\|) + \eta_2(\|v\|_{L^\infty(0,t)})\big) \int_0^{t/2} e^{-\mu(t/2-s)} ds \\ + \frac{M}{\mu} \big(\eta_1(e^{-(\mu/2)t}\|x^0\|) + \eta_2(\|v\|_{L^\infty(0,t)})\big) \qquad \forall\ t \ge 0. \end{aligned}$$

Consequently,

$$\begin{aligned} \|I(t)\| \le \frac{M}{\mu} \big(e^{-(\mu/2)t}\eta_1(\|x^0\|) + \eta_1(e^{-(\mu/2)t}\|x^0\|)\big) \\ + \frac{2M}{\mu}\eta_2(\|v\|_{L^\infty(0,t)}) \qquad \forall\ t \ge 0. \end{aligned}$$

Furthermore,

$$\left\| \int_0^t e^{A_K(t-s)} v(s) \right\| \le \frac{M}{\mu} \|v\|_{L^\infty(0,t)} \qquad \forall\ t \ge 0,$$

and therefore, by (4.13),

$$\begin{aligned} \|x(t)\| \le Me^{-\mu t}\|x^0\| + \frac{M}{\mu} \big(e^{-(\mu/2)t}\eta_1(\|x^0\|) + \eta_1(e^{-(\mu/2)t}\|x^0\|)\big) \\ + \frac{M}{\mu} \big(2\eta_2(\|v\|_{L^\infty(0,t)}) + \|v\|_{L^\infty(0,t)}\big) \qquad \forall\ t \ge 0. \end{aligned}$$

Hence, defining $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ by

$$\psi(s,t) := Me^{-\mu t}s + \frac{M}{\mu} \big(e^{-(\mu/2)t}\eta_1(s) + \eta_1(e^{-(\mu/2)t}s)\big)$$

and

$$\varphi(s) := \frac{M}{\mu}(2\eta_2(s) + s),$$

respectively, we conclude that, for every $x^0 \in \mathbb{R}^n$ and every $v \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$,

$$\|x(t; x^0, v)\| = \|x(t)\| \le \psi(\|x^0\|, t) + \varphi(\|v\|_{L^\infty(0,t)}) \qquad \forall\ t \ge 0,$$

showing that the equilibrium pair $(0,0)$ of (4.3) is ISS. $\square$

The scenario which is considered in statement (3) of Theorem 4.2.1, wherein $\|\boldsymbol{G}_K\|_{H^\infty} = 0$ (or, equivalently, $\boldsymbol{G}_K = 0$), is not very interesting, but is included for mathematical completeness. Note that $\|\boldsymbol{G}_K\|_{H^\infty} = 0$ if, and only if, $\|\boldsymbol{G}\|_{H^\infty} = 0$. Consequently, if (A, B) is controllable ((C, A) is observable) and $C \neq 0$ ($B \neq 0$), then $\|\boldsymbol{G}_K\|_{H^\infty} \neq 0$.

As an element in $L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, strictly speaking, v is not a function, but an equivalence class of functions. In the material which follows we often say that

$$\lim_{t\to\infty} v(t) = v^\infty, \tag{4.18}$$

however, we should clarify what we mean by this. We say (4.18) holds if

$$\|v - v^\infty \theta\|_{L^\infty(t,\infty)} \to 0 \qquad \text{as} \qquad t \to \infty,$$

or equivalently, if there exists a representative w in the equivalence class v such that $w(t) \to v^\infty$ as $t \to \infty$.

The following proposition is a special case of a well-known result from ISS theory.

Proposition 4.2.2. *Assume that* $(0, 0)$ *is an ISS equilibrium pair of* (4.3). *Then* (4.3) *has the* 0*-converging-input converging-state property: for every* $x^0 \in \mathbb{R}^n$ *and for every* $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ *such that* $v(t) \to 0$ *as* $t \to \infty$*, we have that* $x(t; x^0, v) \to 0$ *as* $t \to \infty$.

We emphasize that ISS is not a necessary condition for (4.3) to have the 0-converging-input converging-state property (0-CICS), see Example 4.26. We now introduce a concept which strengthens the notion of the 0-CICS property and is the primary focus of the chapter.

Definition 4.2.3. *We say that* (4.3) *has the* converging-input converging-state property *(CICS property) if, for every* $v^\infty \in \mathbb{R}^n$*, there exists* $x^\infty \in \mathbb{R}^n$ *such that, for all* $x^0 \in \mathbb{R}^n$ *and all* $v \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ *with* $\lim_{t\to\infty} v(t) = v^\infty$,

$$\lim_{t\to\infty} x(t; x^0, v) = x^\infty.$$

If (4.3) has the CICS property and if $f(0) = 0$ (that is, the origin is an equilibrium of the uncontrolled Lur'e system (4.3)), then (4.3) has the 0-CICS property.

The CICS property enables us to define steady-state gains for the Lur'e system (4.3). Assuming that (4.3) has the CICS property, the map

$$\Gamma_{\text{is}} : \mathbb{R}^n \to \mathbb{R}^n, \qquad v^\infty \mapsto x^\infty$$

is well defined and is called the *input-to-state steady-state gain* (ISSS gain). The map

$$\Gamma_{\mathrm{io}} : \mathbb{R}^n \to \mathbb{R}^p, \qquad v^\infty \mapsto C\Gamma_{\mathrm{is}}(v^\infty) = Cx^\infty$$

is called the *input-to-output steady-state gain* (IOSS gain). If (4.3) has the CICS property, then, for every v^∞, the point

$$x^\infty := \Gamma_{\mathrm{is}}(v^\infty)$$

is a globally attractive equilibrium of the system $\dot{x} = Ax + Bf(Cx) + v^\infty\theta$.

In the following, the map

$$F_K : \mathbb{R}^p \to \mathbb{R}^p, \qquad z \mapsto z - \boldsymbol{G}_K(0)(f(z) - Kz), \tag{4.19}$$

where $K \in \mathbb{S}_{\mathbb{R}}(A, B, C)$, will play a key role. For a set $W \subseteq \mathbb{R}^p$, we shall denote the preimage of W under F_K by $F_K^{-1}(W)$. For $w \in \mathbb{R}^p$, it is convenient to set $F_K^{-1}(w) := F_K^{-1}(\{w\})$. We note two simple, but important properties of F_K:

$$F_K(\operatorname{im} C) \subseteq \operatorname{im} C, \qquad F_K^{-1}(\operatorname{im} C) \subseteq \operatorname{im} C. \tag{4.20}$$

Definition 4.2.4. *For a set S, the symbol $\#S$ denotes the cardinality of S. If S is infinite then we write $\#S = \infty$.*

The next proposition describes properties of the map F_K and shows how F_K relates to equilibrium pairs (x^∞, v^∞) of (4.3).

Proposition 4.2.5. *Assume that $K \in \mathbb{S}_{\mathbb{R}}(A, B, C)$.*

(1) Let $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ have a limit $v^\infty := \lim_{t\to\infty} v(t)$ and assume that, for some $x^0 \in \mathbb{R}^n$, the limit $x^\infty := \lim_{t\to\infty} x(t; x^0, v)$ exists. Then (x^∞, v^∞) is an equilibrium pair of (4.3),

$$x^\infty = -A_K^{-1}(B(f(Cx^\infty) - KCx^\infty) + v^\infty), \tag{4.21}$$

where A_K is given by (4.8)*, and $F_K(Cx^\infty) = -CA_K^{-1}v^\infty$.*

(2) Let $v^\infty \in \mathbb{R}^n$ and assume that there exists $x^\infty \in \mathbb{R}^n$ such that, for all $x^0 \in \mathbb{R}^n$, $x(t; x^0, v^\infty\theta) \to x^\infty$ as $t \to \infty$. Then $\#F_K^{-1}(-CA_K^{-1}v^\infty) = 1$.

(3) Let $v^\infty \in \mathbb{R}^n$, $y^\infty \in F_K^{-1}(-CA_K^{-1}v^\infty)$ and set

$$x^\infty := -A_K^{-1}(B(f(y^\infty) - Ky^\infty) + v^\infty).$$

Then $Cx^\infty = y^\infty$ and (x^∞, v^∞) is an equilibrium pair of (4.3).

Proof. To prove statement (1), set $x(t) := x(t; x^0, v)$ and note that x satisfies

$$\dot{x} = A_K x + B(f(Cx) - KCx) + v.$$

Since A_K is Hurwitz, it follows immediately that (4.21) holds. As an immediate consequence of (4.21), we have

$$0 = A_K x^\infty + B(f(Cx^\infty) - KCx^\infty) + v^\infty = Ax^\infty + Bf(Cx^\infty) + v^\infty,$$

showing that (x^∞, v^∞) is an equilibrium pair of (4.3). Furthermore, applying C to both sides of (4.21) and rearranging shows that $F_K(Cx^\infty) = -CA_K^{-1}v^\infty$.

We proceed to prove statement (2). By statement (1), x^∞ satisfies (4.21), and $Cx^\infty \in F_K^{-1}(-CA_K^{-1}v^\infty)$, showing that $F_K^{-1}(-CA_K^{-1}v^\infty) \neq \emptyset$. Let $y_1, y_2 \in F_K^{-1}(-CA_K^{-1}v^\infty)$. It remains to show that $y_1 = y_2$. To this end, set

$$\xi_i := -A_K^{-1}(B(f(y_i) - Ky_i) + v^\infty), \qquad i \in \{1, 2\}. \tag{4.22}$$

Then

$$F_K(y_i) = y_i - \boldsymbol{G}_K(0)(f(y_i) - Ky_i) = y_i - C\xi_i - CA_K^{-1}v^\infty, \qquad i \in \{1, 2\}.$$

But $F_K(y_i) = -CA_K^{-1}v^\infty$ for $i \in \{1, 2\}$ and so, $y_i = C\xi_i$ for $i \in \{1, 2\}$. Consequently, by (4.22),

$$A_K\xi_i + B(f(C\xi_i) - KC\xi_i) + v^\infty = 0, \qquad i \in \{1, 2\},$$

and so

$$A\xi_i + Bf(C\xi_i) + v^\infty = 0, \qquad i \in \{1, 2\},$$

showing that (ξ_1, v^∞) and (ξ_2, v^∞) are equilibrium pairs of (4.3). Hence for $i \in \{1, 2\}$, $x(t; \xi_i, v^\infty\theta) = \xi_i$ for all $t \geq 0$ and it follows from hypothesis that $\xi_1 = x^\infty = \xi_2$. Thus, $y_1 = C\xi_1 = C\xi_2 = y_2$, completing the proof.

Finally we proceed to prove statement (3). Note that

$$\begin{aligned} Cx^\infty &= \boldsymbol{G}_K(0)(f(y^\infty) - Ky^\infty) - CA_K^{-1}v^\infty \\ &= y^\infty - F_K(y^\infty) - CA_K^{-1}v^\infty = y^\infty. \end{aligned}$$

Therefore,

$$Ax^\infty + Bf(Cx^\infty) + v^\infty = A_K x^\infty + B(f(y^\infty) - Ky^\infty) + v^\infty = 0,$$

showing that (x^∞, v^∞) is an equilibrium pair of (4.3).

□

The following is a corollary of Proposition 4.2.5 and provides, in terms of F_K, a necessary condition for the CICS property to hold.

Corollary 4.2.6. *Let $K \in \mathbb{S}_{\mathbb{R}}(A, B, C)$. If the Lur'e system (4.3) has the CICS property, then $\#F_K^{-1} = 1$ for all $z \in \operatorname{im} C$.*

Proof. Let $z \in \operatorname{im} C$. Then there exists $v^\infty \in \mathbb{R}^n$ such that $z = -CA_K^{-1}v^\infty$. By the CICS property, it is clear that there exists $x^\infty \in \mathbb{R}^n$, such that for all $x^0 \in \mathbb{R}^n$, $x(t; x^0, v^\infty\theta) \to x^\infty$ as $t \to \infty$. Hence, by statement (2) of Proposition 4.2.5, $\#F_K^{-1}(z) = \#F_K^{-1}(-CA_K^{-1}v^\infty) = 1$. □

It follows from (4.20) and Corollary 4.2.6 that, if (4.3) has the CICS property, then the restriction of F_K to $\operatorname{im} C$ provides a bijection from the subspace $\operatorname{im} C$ into itself.

4.3 Sufficient conditions for CICS

In this section, we provide conditions which ensure that the Lur'e system (4.3) has the CICS property. The main result is Theorem 4.3.3, which, in turn, yields a host of sufficient conditions for the CICS property, formulated as Corollaries 4.3.7, 4.3.9, 4.3.10, 4.3.13 and 4.3.15.

We begin this section by stating and proving two technical results, Lemma 4.3.1 and Proposition 4.3.2, the later of which provides conditions which guarantee certain surjectivity and injectivity properties of the map F_K. We denote the restriction of F_K to $\operatorname{im} C$ by $\hat{F}_K$. It follows from (4.20) that $\hat{F}_K$ maps into $\operatorname{im} C$ and we define the co-domain of $\hat{F}_K$ to be equal to $\operatorname{im} C$.

Lemma 4.3.1. *Let $g : \mathbb{R}^p \to \mathbb{R}^m$ be an arbitrary function and let $r > 0$.*

(1) If there exists $\zeta \in \mathbb{R}^p$ such that

$$r\|z\| - \|g(z + \zeta) - g(\zeta)\| \to \infty \qquad \text{as} \quad \|z\| \to \infty, \tag{4.23}$$

then, for every $\xi \in \mathbb{R}^p$,

$$r\|z\| - \|g(z + \xi) - g(\xi)\| \to \infty \qquad \text{as} \quad \|z\| \to \infty.$$

(2) If g is continuous, $\|g(z)\| < r\|z\|$ for all nonzero $z \in \mathbb{R}^p$ and $r\|z\| - \|g(z)\| \to \infty$ as $\|z\| \to \infty$, then there exists $\alpha \in \mathcal{K}_\infty$ such that $\|g(z)\| \leq r\|z\| - \alpha(\|z\|)$ for all $z \in \mathbb{R}^p$.

Proof. To prove statement (1), let $\xi \in \mathbb{R}^p$, set $w := z + \xi - \zeta$ and note that

$$r\|z\| - \|g(z+\xi) - g(\xi)\| = r\|w + \zeta - \xi\| - \|g(w+\zeta) - g(\zeta) + g(\zeta) - g(\xi)\|.$$

Consequently,

$$r\|z\| - \|g(z+\xi) - g(\xi)\| \geq r\|w\| - \|g(w+\zeta) - g(\zeta)\| - r\|\zeta - \xi\| - \|g(\zeta) - g(\xi)\|,$$

and since $\|w\| \to \infty$ as $\|z\| \to \infty$, the claim follows from (4.23).

To prove statement (2), define $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\beta(s) := \inf_{\|z\| \geq s} (r\|z\| - \|g(z)\|), \qquad s \geq 0.$$

Then β is continuous by the continuity of g, $\beta(0) = 0$, $\beta(s) > 0$ for $s > 0$, β is nondecreasing, $\beta(s) \to \infty$ as $s \to \infty$ and $\|g(z)\| \leq r\|z\| - \beta(\|z\|)$ for all $z \in \mathbb{R}^p$. Therefore, setting $\alpha(s) := (1 - e^{-s})\beta(s)$, it is clear that $\alpha \in \mathcal{K}_\infty$ and $\|g(z)\| \leq r\|z\| - \alpha(\|z\|)$ for all $z \in \mathbb{R}^p$, completing the proof. □

Proposition 4.3.2. *Let $Y \subseteq \operatorname{im} C$ be nonempty, $K \in \mathbb{S}_\mathbb{R}(A, B, C)$, set $\gamma := 1/\|\boldsymbol{G}_K\|_{H^\infty}$, where $\gamma := \infty$ if $\|\boldsymbol{G}_K\|_{H^\infty} = 0$, and assume that f satisfies the condition:*

(A4.1) *For all, $\xi \in Y$, and all $z \in \mathbb{R}^p$ with $z \neq 0$*

$$\|f(z+\xi) - f(\xi) - Kz\| < \gamma\|z\|.$$

The following statements hold.

(1) $\#F_K^{-1}(z) = 1$ for every $z \in \operatorname{im} C$ such that $F_K^{-1}(z) \cap Y \neq \emptyset$.

(2) If

$$\|\boldsymbol{G}_K(0)\| < \|\boldsymbol{G}_K\|_{H^\infty}, \tag{4.24}$$

then F_K is surjective.

(3) If there exists $\zeta \in \mathbb{R}^p$ such that

$$\gamma\|z\| - \|f(z+\zeta) - f(\zeta) - Kz\| \to \infty \qquad \text{as} \quad \|z\| \to \infty, \tag{4.25}$$

then F_K is surjective.

(4) If $Y = \operatorname{im} C$ and (4.24) or (4.25) hold, then $\hat{F}_K$ is bijective.

Proof. If $\boldsymbol{G}_K(0) = 0$, then $F_K(z) = z$ for all $z \in \mathbb{R}^p$. Consequently, the maps F_K and $\hat{F}_K$ are bijective and there is nothing to prove. Let us now assume that $\boldsymbol{G}_K(0) \neq 0$. Then, $\|\boldsymbol{G}_K\|_{H^\infty} \neq 0$, and so, $0 < \gamma < \infty$.

To prove statement (1), let $z \in \operatorname{im} C$ and assume that $F_K^{-1}(z) \cap Y \neq \emptyset$. Let $\xi_1 \in F_K^{-1}(z) \cap Y$ and $\xi_2 \in F_K^{-1}(z)$. To establish that $\#F_K^{-1}(z) = 1$, it suffices to show that $\xi_1 = \xi_2$. Since $F_K(\xi_1) = F_K(\xi_2)$, it follows that

$$\|\xi_2 - \xi_1\| = \|\boldsymbol{G}_K(0)(f(\xi_2) - f(\xi_1) - K(\xi_2 - \xi_1))\|.$$

If $\xi_1 \neq \xi_2$, then, by condition **(A4.1)**,

$$\|\xi_2 - \xi_1\| < \|\boldsymbol{G}_K(0)\|\gamma\|\xi_2 - \xi_1\| \leq \|\xi_2 - \xi_1\|,$$

which is impossible, hence $\xi_1 = \xi_2$.

We proceed to prove statement (2). To show that F_K is surjective, note that, by [116, Theorem 9.36], it is sufficient to prove that F_K is coercive, that is,

$$\frac{1}{\|z\|}\langle F_K(z), z\rangle \to \infty \qquad \text{as} \quad \|z\| \to \infty. \tag{4.26}$$

To establish (4.26), we note that, for all $z \in \mathbb{R}^p$,

$$\begin{aligned}\frac{1}{\|z\|}\langle F_K(z), z\rangle &= \|z\| + \frac{1}{\|z\|}\langle \boldsymbol{G}_K(0)(f(z) - Kz), z\rangle \\ &\geq \|z\| - \|\boldsymbol{G}_K(0)\|\|f(z) - Kz\|,\end{aligned}$$

and hence

$$\frac{1}{\|z\|}\langle F_K(z), z\rangle \geq \|z\| - \|\boldsymbol{G}_K(0)\|(\|f(z) - f(\xi) - Kz\| + \|f(\xi)\|) \qquad \forall\, z \in \mathbb{R}^p, \tag{4.27}$$

where $\xi \in Y$. By condition **(A4.1)**,

$$\|f(z + \xi) - f(\xi) - Kz\| \leq \gamma\|z\| \qquad \forall\, z \in \mathbb{R}^p.$$

Consequently, for all $z \in \mathbb{R}^p$,

$$\begin{aligned}\|f(z) - f(\xi) - Kz\| &\leq \|f(z - \xi + \xi) - f(\xi) - K(z - \xi)\| + \|K\xi\| \\ &\leq \gamma\|z - \xi\| + \|K\xi\|,\end{aligned}$$

and thus,

$$\|f(z) - f(\xi) - Kz\| \leq \gamma\|z\| + (\|K\| + \gamma)\|\xi\| \qquad \forall\, z \in \mathbb{R}^p. \tag{4.28}$$

Setting

$$\kappa := \|\boldsymbol{G}_K(0)\|(\|f(\xi)\| + (\|K\| + \gamma)\|\xi\|), \tag{4.29}$$

and invoking (4.27) and (4.28), we conclude that

$$\frac{1}{\|z\|}\langle F_K(z), z\rangle \geq (1-\gamma\|\boldsymbol{G}_K(0)\|)\|z\| - \kappa \qquad \forall\, z \in \mathbb{R}^p.$$

Now, by hypothesis, $\|\boldsymbol{G}_K(0)\| < \|\boldsymbol{G}_K\|_{H^\infty}$, or, equivalently, $1-\gamma\|\boldsymbol{G}_K(0)\| > 0$, implying that (4.27) holds, and so surjectivity of F_K follows.

To prove statement (3), let $\xi \in Y$. By hypothesis and statement (1) of Lemma 4.3.1 applied to $g(z) = f(z) - Kx$,

$$\gamma\|z\| - \|f(z+\xi) - f(\xi) - Kz\| \to \infty \qquad \text{as} \quad \|z\| \to \infty.$$

Together with assumption **(A4.1)** and an application of statement (3) of Lemma 4.3.1 this shows that there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\|f(z+\xi) - f(\xi) - Kz\| \leq \gamma\|z\| - \alpha(\|z\|) \qquad \forall\, z \in \mathbb{R}^p.$$

An argument very similar to that leading to (4.28) yields

$$\|f(z) - f(\xi) - KZ\| \leq \gamma\|z\| - \alpha(\|z-\xi\|) + (\|K\| + \gamma)\|\xi\| \qquad \forall\, z \in \mathbb{R}^p.$$

Together with (4.27) this implies

$$\frac{1}{\|z\|}\langle F_K(z), z\rangle \geq (1-\gamma\|\boldsymbol{G}(0)\|)\|z\| + \|\boldsymbol{G}_K(0)\|\alpha(\|z-\xi\|) - \kappa \qquad \forall\, z \in \mathbb{R}^p,$$

with κ being defined by (4.29). Now $1-\gamma\|\boldsymbol{G}_K(0)\| \geq 1 - \gamma\|\boldsymbol{G}_K\|_{H^\infty} = 0$ and (4.26) follows, showing that F_K is coercive and hence surjective.

Finally, to prove statement (4), assume that $Y = \operatorname{im} C$ and that (4.24) or (4.25) are satisfied. Then the map F_K is surjective, which follows from statement (2) if (4.24) holds and from statement (3) if (4.25) holds. Surjectivity of F_K, (4.20), and statement (1) guarantee that $\# F_K^{-1}(z) = 1$ for all $z \in \operatorname{im} C$. Writing $F_K^{-1}(z) = \{y_z\}$ for every $z \in \operatorname{im} C$ and, once again, invoking (4.20), we conclude that $y_z \in \operatorname{im} C$ and bijectivity of $\hat{F}_K$ follows. □

For $\tau \geq 0$, we define the *left-shift operator* Λ_τ by $(\Lambda_\tau v)(t) = v(t+\tau)$ for all $t \geq 0$, where v is an arbitrary function $\mathbb{R}_+ \to \mathbb{R}^n$. A subset $\mathcal{V} \subseteq L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ is said to be *equi-convergent* to $v^\infty \in \mathbb{R}^n$ if, for every $\varepsilon > 0$, there exists $\tau \geq 0$ such that

$$\|\Lambda_\tau v - v^\infty\theta\|_{L^\infty} \leq \varepsilon \qquad \forall\, v \in \mathcal{V},$$

or, equivalently,

$$\|v(t) - v^\infty\| \leq \varepsilon \qquad \forall\, t \geq \tau \quad \forall\, v \in \mathcal{V}.$$

The following theorem is the main result in this section.

Theorem 4.3.3. *Let $Y \subseteq \operatorname{im} C$ be nonempty, $K \in \mathbb{S}_{\mathbb{R}}(A,B,C)$, $v^\infty \in \mathbb{R}^n$ and set $\gamma := 1/\|\boldsymbol{G}_K\|_{H^\infty}$, where $\gamma := \infty$ if $\|\boldsymbol{G}\|_{H^\infty} = 0$. Assume that condition* **(A4.1)** *holds and that $F_K^{-1}(-CA_K^{-1}v^\infty) \cap Y \neq \emptyset$. Then $\#F_K^{-1}(-CA_K^{-1}v^\infty) = 1$ and, writing $\{y^\infty\} = F_K^{-1}(-CA_K^{-1}v^\infty)$, the pair (x^∞, v^∞), where*

$$x^\infty := -A_K^{-1}(B(f(y^\infty) - Ky^\infty) + v^\infty), \tag{4.30}$$

is an equilibrium pair of the system (4.3). Furthermore, $Cx^\infty = y^\infty$ and the following statements hold.

(1) The equilibrium pair (x^∞, v^∞) is GAS, and, for every $x^0 \in \mathbb{R}^n$ and every $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ such that $\lim_{t\to\infty} v(t) = v^\infty$, we have that $x(t; x^0, v) \to x^\infty$.

(2) Under the additional assumption that, for some $\zeta \in \mathbb{R}^p$,

$$\gamma\|z\| - \|f(z+\zeta) - f(\zeta) - Kz\| \to \infty \quad \text{as} \quad \|z\| \to \infty, \tag{4.31}$$

(x^∞, v^∞), with x^∞ given by (4.30), is an ISS equilibrium pair of (4.3) and there exists $\psi_1, \psi_2 \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that, for all $(x^0, v) \in \mathbb{R}^n \times L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ and all $t \geq 0$,

$$\begin{aligned}\|x(t; x^0, v) - x^\infty\| \leq \psi_1(\|x^0 - x^\infty\|, t) + \psi_2(\|v - v^\infty\theta\|_{L^\infty}, t) \\ + \varphi(\|\Lambda_{t/2}(v - v^\infty\theta)\|_{L^\infty}).\end{aligned} \tag{4.32}$$

In particular, for every $x^0 \in \mathbb{R}^n$ and every $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ such that $\lim_{t\to\infty} v(t) = v^\infty$,

$$\lim_{t\to\infty} x(t; x^0, v) = x^\infty,$$

and the convergence is uniform in the following sense: given a set of inputs $\mathcal{V} \subseteq L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ which is equi-convergent to v^∞ and $\kappa > 0$, the set of solutions

$$\{x(\,\cdot\,; x^0, v) \,:\, (x^0, v) \in \mathbb{R}^n \times \mathcal{V} \text{ such that } \|x^0\| + \|v\|_{L^\infty} \leq \kappa\}$$

is equi-convergent to x^∞.

Proof. By hypothesis, assumption **(A4.1)** holds and $F_K^{-1}(-CA_K^{-1}v^\infty) \cap Y \neq \emptyset$, and thus, statement (1) of Proposition 4.3.2 yields that $\#F_K^{-1}(-CA_K^{-1}v^\infty) = 1$. For x^∞ given by (4.30), it follows from statement (3) of Proposition 4.2.5 that (x^∞, v^∞) is an equilibrium pair of (4.3) and $Cx^\infty = y^\infty$.

Define $\tilde{f} : \mathbb{R}^p \to \mathbb{R}^m$ by

$$\tilde{f}(z) := f(z + y^\infty) - f(y^\infty) \qquad \forall\, z \in \mathbb{R}^p.$$

We manipulate (4.30):

$$\begin{aligned} x^\infty &= -A_K^{-1}(B(f(y^\infty) - Ky^\infty) + v^\infty) \\ (A + BKC)x^\infty &= -B(f(y^\infty) - Ky^\infty) - v^\infty \\ Ax^\infty + BKCx^\infty &= -Bf(y^\infty) + BKCx^\infty - v^\infty \\ Ax^\infty + Bf(y^\infty) + v^\infty &= 0. \end{aligned}$$

A simple calculation shows that

$$A(\chi + x^\infty) + Bf(C(\chi + x^\infty)) + v^\infty = A\chi + B\tilde{f}(C\chi) \qquad \forall\, \chi \in \mathbb{R}^n. \quad (4.33)$$

Moreover, since $y^\infty \in Y$, it follows from assumption **(A4.1)** that

$$\|\tilde{f}(z) - Kz\| < \gamma\|z\| \qquad \forall\, z \in \mathbb{R}^p,\ z \neq 0. \quad (4.34)$$

To prove statement (1), note that, by (4.33), a function x satisfies

$$\dot{x} = Ax + Bf(Cx) + v^\infty\theta \quad (4.35)$$

if, and only if, $\tilde{x} := x - x^\infty\theta$ satisfies

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{f}(C\tilde{x}). \quad (4.36)$$

Consequently, the equilibrium x^∞ of (4.35) is GAS if, and only if, the equilibrium 0 of (4.36) is GAS. Invoking (4.34) in conjunction with statements (1) and (3) of Theorem 4.2.1 shows that the equilibrium 0 of (4.36) is GAS and hence, x^∞ is a GAS equilibrium of system (4.35). An application of [137, Theorem 1] or [89, Theorem 4.3] allows us to conclude that, for $x^0 \in \mathbb{R}^n$ and $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ with $v(t) \to v^\infty$ as $t \to \infty$, we have that $x(t; x^0, d) \to x^\infty$ as $t \to \infty$, completing the proof of statement (1).

We proceed to prove statement (2). To this end, let $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ and set $\tilde{v} = v - v^\infty\theta$. Invoking (4.33) shows that a function x solves (4.3) if, and only if, $\tilde{x} := x - x^\infty\theta$ solves

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{f}(C\tilde{x}) + \tilde{v}. \quad (4.37)$$

Consequently, the equilibrium pair (x^∞, v^∞) of (4.3) is ISS if, and only if, the pair $(0,0)$ of (4.37) is ISS. By (4.31) and statement (1) of Lemma 4.3.1,

$$\gamma\|z\| - \|\tilde{f}(z) - Kz\| \to \infty \qquad \text{as} \quad \|z\| \to \infty.$$

This, together with (4.33) and statement (2) of Lemma 4.3.1, shows that there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\|\tilde{f}(z) - Kz\| \leq \gamma\|z\| - \alpha(\|z\|) \qquad \forall\, z \in \mathbb{R}^p.$$

Statements (2) and (3) of Theorem 4.2.1 now show that $(0,0)$ is an ISS equilibrium pair of the system (4.37) and thus, the equilibrium pair (x^∞, v^∞) of (4.3) is ISS. Consequently, there exists $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that for all $x^0 \in \mathbb{R}^n$, and all $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$

$$\|x(t;x^0,v) - x^\infty\| \leq \psi(\|x^0 - x^\infty\|, t) + \varphi(\|v - v^\infty\theta\|_{L^\infty(0,t)}), \qquad \forall\, t \geq 0. \quad (4.38)$$

It remains to show that (4.32) holds. To this end, let $x^0 \in \mathbb{R}^n$ and $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$, and note that, by the state transition property of system (4.3),

$$x(t;x^0,v) = x\left(t/2, x\left(t/2;x^0,v\right), \Lambda_{t/2}v\right) \qquad \forall\, t \geq 0.$$

Hence, by (4.38),

$$\begin{aligned}\|x(t;x^0,v) - x^\infty\| \leq &\psi\left(\left\|x\left(t/2;x^0,v\right) - x^\infty\right\|, t/2\right) \\ &+ \varphi(\|\Lambda_{t/2}v - v^\infty\theta\|_{L^\infty}) \quad \forall\, t \geq 0.\end{aligned}$$

Another application of (4.38) yields

$$\begin{aligned}\|x(t;x^0,v) - x^\infty\| \leq &\psi\left(\psi\left(\|x^0 - x^\infty\|, t/2\right) + \varphi(\|v - v^\infty\theta\|_{L^\infty}), t/2\right) \\ &+ \varphi(\|\Lambda_{t/2}(v - v^\infty\theta)\|_{L^\infty}) \qquad \forall\, t \geq 0.\end{aligned}$$

Consequently, defining $\psi_1, \psi_2 \in \mathcal{KL}$ by

$$\psi_1(s,t) := \psi(2\psi(s,t/2), t/2)$$

and

$$\psi_2(s,t) := \psi(2\varphi(s), t/2) \qquad \forall\, s,t \geq 0,$$

we obtain, for $t \geq 0$,

$$\begin{aligned}\|x(t;x^0,v) - x^\infty\| \leq &\psi_1(\|x^0 - x^\infty\|, t) + \psi_2(\|v - v^\infty\theta\|_{L^\infty}, t) \\ &+ \varphi(\|\Lambda_{t/2}(v - v^\infty\theta)\|_{L^\infty}),\end{aligned}$$

which is (4.32).

□

We illustrate the conclusions of Theorem 4.3.3 with some simple examples.

Example 4.3.4. *Consider the one-dimensional Lur'e system*

$$\dot{x} = -x + f(x) + v. \tag{4.39}$$

Note that here $n = 1$, $A = -1$ and $B = C = 1$. We choose $K = 0$ and so

$$\boldsymbol{G}_K(s) = \boldsymbol{G}_0(s) = \boldsymbol{G}(s) = \frac{1}{s+1}.$$

Since $\|\boldsymbol{G}\|_{H^\infty} = \boldsymbol{G}(0) = 1$, we have $\gamma = 1$.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(z) = z - \operatorname{sign}(z)(1 - e^{-|z|}) \qquad \forall \quad z \in \mathbb{R}. \tag{4.40}$$

Since

$$f'(z) = 1 - e^{-|z|} \qquad \forall \quad z \in \mathbb{R},$$

the Mean-Value Theorem guarantees that

$$|f(z+\xi) - f(\xi)| < |z| \qquad \forall \quad \xi, z \in \mathbb{R},\, z \neq 0.$$

Furthermore,

$$F_0(z) = z - f(z) = \operatorname{sign}(z)(1 - e^{-|z|}) \qquad \forall \quad z \in \mathbb{R},$$

and so, $F_0(\mathbb{R}) = (-1, 1)$. Setting $Y := \operatorname{im} C = \mathbb{R}$, we see that, for every $v^\infty \in (-1, 1)$, the assumptions of statement (1) of Theorem 4.3.3 are satisfied. Therefore, if $v^\infty \in (-1, 1)$, then, for all $x^0 \in \mathbb{R}$ and all $v \in L^\infty(\mathbb{R}_+, \mathbb{R})$ such that $\lim_{t\to\infty} v(t) = v^\infty$, we have that either $x(t; x^0, v) \to x^\infty = F_0^{-1}(v^\infty)$ or $|x(t; x^0, v)| \to \infty$ as $t \to \infty$. We show that divergence is not possible. Seeking a contradiction, suppose that there exists $v^\infty \in (-1, 1)$, $v \in L^\infty(\mathbb{R}_+, \mathbb{R})$ with $\lim_{t\to\infty} v(t) = v^\infty$ and $x^0 \in \mathbb{R}$ such that $|x(t; x^0, v)| \to \infty$ as $t \to \infty$. Setting $x(t) := x(t; x^0, v)$, we have that either $x(t) \to \infty$ or $x(t) \to -\infty$. If $x(t) \to \infty$ as $t \to \infty$, then there exists $\tau \geq 0$ such that

$$\dot{x}(t) = -1 + e^{-x(t)} + v(t) \leq (v^\infty - 1)/2 < 0 \qquad \forall \quad t \geq \tau.$$

But this implies that $x(t) \to -\infty$ as $t \to \infty$, providing the desired con-

tradiction. Similarly, if $x(t) \to -\infty$ as $t \to \infty$, then there exists $\tau \geq 0$ such that

$$\dot{x}(t) = 1 - e^{x(t)} + v(t) \geq (v^\infty + 1)/2 > 0 \qquad \forall \quad t \geq \tau,$$

showing that $x(t) \to \infty$ as $t \to \infty$, which is impossible.

The above analysis shows that in particular that the system (4.39) has the 0-CICS property. Note that the equilibrium pair $(0, 0)$ of (4.39) is not ISS (since the input $v(t) = 1 + \varepsilon$, $\varepsilon > 0$, produces an unbounded solution).

(b) *Consider again the system (4.39), but now with $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(z) = z - \mathrm{sat}(z)e^{-|z|}, \qquad \forall \quad z \in \mathbb{R},$$

where $\mathrm{sat}(z) := z$ for $|z| \leq 1$ and $\mathrm{sat}(z) := \mathrm{sign}(z)$ for $|z| > 1$. Set $Y := \{0\}$ and let $v^\infty = 0$. Since,

$$|f(z)| < |z| \qquad \forall \quad z \neq 0,$$

the assumptions of statement (1) of Theorem 4.3.3 are satisfied and it follows that $y^\infty = x^\infty = 0$, the equilibrium 0 of the uncontrolled system (4.39) is GAS, and, for every $x^0 \in \mathbb{R}$ and every $v \in L^\infty(\mathbb{R}_+, \mathbb{R})$ with $\lim_{t\to\infty} v(t) = 0$, either $x(t; x^0, v) \to 0$ or $|x(t; x^0, v)| \to \infty$ as $t \to \infty$. Divergence is possible, indeed, with input v given by $v(t) = 2/(t + e)$, it is straightforward to verify that $x(t; 1, v) = \ln(t + e)$.

Example 4.3.5. *Consider the two-dimensional Lur'e system*

$$\left.\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - f(2x_1 + x_2) + v_1 \\ \dot{x}_2 &= -x_1 - 3x_2 + 3f(2x_1 + x_2) + v_2, \end{aligned}\right\} \tag{4.41}$$

with nonlinearity $f \in \mathcal{F}$, where $\mathcal{F}$ is the set of all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} &f(0) = 0, \qquad f'(z) \geq 0 \;\; \forall\, z \in \mathbb{R}, \qquad \max_{y\in[3,4]} f'(z) = 2 \\ &\qquad\text{and} \qquad f'(z) \leq 1/2 \;\; \forall\, z \in \mathbb{R}\backslash(3, 4). \end{aligned} \tag{4.42}$$

Setting

$$A := \begin{pmatrix} -1 & 1 \\ -1 & -3 \end{pmatrix}, \qquad B := \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \qquad C := \begin{pmatrix} 2 & 1 \end{pmatrix},$$

it is clear that system (4.41) is of the form (4.3). The matrix A is Hurwitz (as -2 is an eigenvalue of algebraic multiplicity two) and the transfer function of the linear system (A, B, C) is $\boldsymbol{G}(s) = (s+4)/(s+2)^2$. Choosing $K = 0$, we have,

$$\|\boldsymbol{G}_K\|_{H^\infty} = \|\boldsymbol{G}_0\|_{H^\infty} = \|\boldsymbol{G}\|_{H^\infty} = \boldsymbol{G}(0) = 1,$$

and thus, $\gamma = 1$. It follows from (4.42) that $|z| - |f(z)| \to \infty$ as $|z| \to \infty$ and so, (4.31) holds with $\zeta = 0$. Using elementary calculus, it is not difficult to show that, for every $\xi \in \mathbb{R}\backslash(1,6)$, there exists $a_\xi \in (0,1)$ such that

$$|f(z+\xi) - f(\xi)| \leq a_\xi |z| \qquad \forall \quad z \in \mathbb{R}.$$

Hence, condition **(A4.1)** *holds with $Y := \mathbb{R}\backslash(1,6)$. Furthermore, $F_0(z) = F_K(z) = z - f(z)$, and so, using (4.42),*

$$F_0(Y) = (-\infty, 1 - f(1)] \cup [6 - f(6), \infty) \supseteq (-\infty, 1/2] \cup [6, \infty).$$

According to statement (2) of Theorem 4.3.3, for every $v^\infty = (v_1^\infty, v_2^\infty)^T \in \mathbb{R}^2$ such that

$$\frac{5v_1^\infty + 3v_2^\infty}{4} = -CA^{-1}v^\infty \in F_0(Y),$$

there exists $x^\infty \in \mathbb{R}^2$ such that, for all $x^0 \in \mathbb{R}^2$ and all $v \in L^\infty(\mathbb{R}_+, \mathbb{R})$ with $\lim_{t\to\infty} v(t) = v^\infty$, the solution $x(t; x^0, v)$ of (4.41) converges to x^∞ as $t \to \infty$.

Let $\xi_0 \in (1,6)$. Then it is not difficult to show that there exists $f \in \mathcal{F}$ such that

$$\sup_{z\neq 0} \frac{|f(z+\xi_0) - f(\xi_0)|}{|z|} = \sup_{z\neq 0} \frac{f(z+\xi_0) - f(\xi_0)}{z} > 1, \qquad (4.43)$$

and it is clear that condition **(A4.1)** *does not hold for $\xi = \xi_0$. We claim that, for $v^\infty = (v_1^\infty, v_2^\infty)^T \in \mathbb{R}^2$ such that*

$$\frac{5v_1^\infty + 3v_2^\infty}{4} = -CA^{-1}v^\infty = F_0(\xi_0), \qquad (4.44)$$

there does not exist $x^\infty \in \mathbb{R}^2$ such that $\lim_{t\to\infty} x(t; x^0, v) = x^\infty$ for all $x^0 \in \mathbb{R}^2$ and all $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^2)$ with $\lim_{t\to\infty} v(t) = v^\infty$. To this end note that, by (4.43), there exists $z_0 \neq 0$ such that

$$\frac{f(z_0+\xi_0) - f(\xi_0)}{z_0} > 1,$$

and thus

$$z_0(F_0(z_0+\xi_0) - F_0(\xi_0)) < 0.$$

Now (4.42) *guarantees that*

$$F_0(z) \to \pm\infty \qquad \text{as} \qquad z \to \pm\infty,$$

and we conclude that there exists $\xi_1 \neq \xi_0$ *such that* $F_0(\xi_0) = F_0(\xi_1)$. *As a consequence,* $\#F_0^{-1}(-CA^{-1}v^\infty) > 1$, *and so, by statement (2) of Proposition 4.2.5, it follows that there does not exist* $x^\infty \in \mathbb{R}^2$ *such that* $\lim_{t\to\infty} x(t; x^0, v) = x^\infty$ *for all* $x^0 \in \mathbb{R}^2$ *and all* $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^2)$ *with* $\lim_{t\to\infty} v(t) = v^\infty$.

To illustrate the last point, we consider a specific example. Fix $\xi_0 = 7/2 \in (1, 6)$ *and let* $f \in \mathcal{F}$ *be given by*

$$f(z) := \begin{cases} z/2 & z \in (-\infty, 3) \\ q(z) & z \in [3, 4] \\ \dfrac{z-4}{2} + 3 & z \in (4, \infty), \end{cases} \tag{4.45}$$

where $q(z) := -2z^3 + 21z^2 - 143z/2 + 81$. *See Figure 4.3 for the graph of* f.

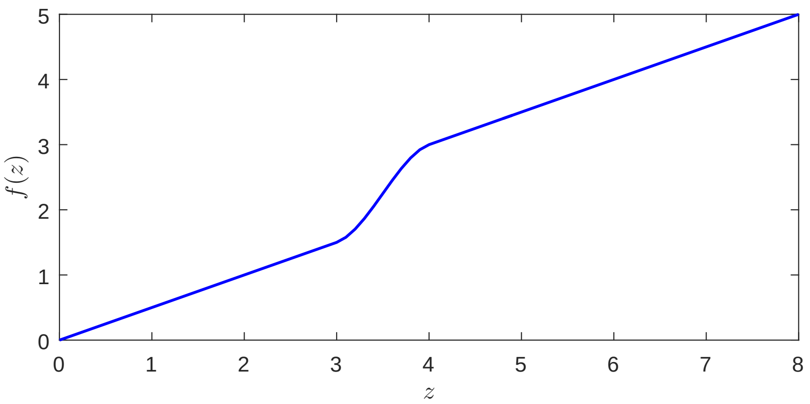


Figure 4.3: Graph of the function f from (4.45).

It is straightforward to verify that the function f *belongs in* $\mathcal{F}$, *in particular:*

$$f(3) = \frac{3}{2}, \qquad f(4) = 3, \qquad f'(3) = \frac{1}{2} = f'(4)$$

$$\text{and} \qquad \max_{z\in[3,4]} f'(z) = f'(\xi_0) = 2.$$

The last identity shows that condition (4.43) *holds. Moreover,* $F_0(\xi_0) = \xi_0 - f(\xi_0) = 5/4$, *and thus,* $v^\infty := (2, -5/3)^T$ *satisfies* (4.44). *A straightforward argument shows that* $F_0^{-1}(5/4) = \{5/2, 7/2, 9/2\}$. *Calculating* $x^\infty = -A^{-1}(Bf(y^\infty) + v^\infty)$ *for* $y^\infty \in \{5/2, 7/2, 9/2\}$, *we see that* (x^∞, v^∞) *is an* *equilibrium pair for every x^∞ of the form*

$$x^\infty = \begin{pmatrix} 13/12 \\ x_2^\infty \end{pmatrix}, \qquad \text{where} \quad x_2^\infty \in \left\{ \frac{1}{3}, \frac{4}{3}, \frac{7}{3} \right\}.$$

In particular, there does not exist $x^\infty \in \mathbb{R}^2$ such that $\lim_{t\to\infty} x(t; x^0, v) = x^\infty$ for all $x^0 \in \mathbb{R}^2$ and all $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^2)$ with $\lim_{t\to\infty} v(t) = v^\infty$.

The following corollary is a consequence of statement (1) of Theorem 4.3.3.

Corollary 4.3.6. *Let $K \in \mathbb{S}_\mathbb{R}(A, B, C)$, set $\gamma := 1/\|\boldsymbol{G}_K\|_{H^\infty}$, where $\gamma := \infty$ if $\|\boldsymbol{G}_K\|_{H^\infty} = 0$, and assume that* **(A4.1)** *holds with $Y := F_K^{-1}(\operatorname{im} C) \subseteq \operatorname{im} C$ and set $V := \{w \in \mathbb{R}^n : -CA_K^{-1}w \in F_K(Y)\}$. Furthermore, assume that, for every $x^0 \in \mathbb{R}^n$ and every $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ such that $\lim_{t\to\infty} v(t) = v^\infty \in V$, the function $Cx(\,\cdot\,; x^0, v)$ is bounded. Then, for every $v^\infty \in V$, $\#F_K^{-1}(-CA_K^{-1}v^\infty) = 1$ and, for every $x^0 \in \mathbb{R}^n$ and every $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ such that $\lim_{t\to\infty} v(t) = v^\infty \in V$, we have that $x(t; x^0, v) \to x^\infty$ as $t \to \infty$, where $x^\infty := -A_K^{-1}(B(f(t^\infty) - Ky^\infty) + v^\infty)$ with y^∞ given by $\{y^\infty\} = F_K^{-1}(-CA_K^{-1}v^\infty)$.*

Proof. Let $v^\infty \in V$ and set $z := -CA_K^{-1}v^\infty$. Obviously, $z \in \operatorname{im} C$ and it follows from the definitions of the sets Y and V that $F_K^{-1}(z) \cap Y \neq \emptyset$. Consequently, by Proposition 4.3.2, $\#F_K^{-1}(-CA_K^{-1}v^\infty) = \#F_K^{-1}(z) = 1$.

To prove the convergence property, let $x^0 \in \mathbb{R}^n$, $v^\infty \in V$ and let $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ be such that $v(t) \to v^\infty$ as $t \to \infty$ and write $x(t) := x(t; x^0, v)$. By hypothesis, Cx is bounded, and so, since x satisfies

$$\dot{x} = A_K x + B(f(Cx) - KCx) + v,$$

the Hurwitz property of A_K guarantees that x is bounded. An application of statement (1) of Theorem 4.3.3 shows that $x(t) \to x^\infty$ as $t \to \infty$, completing the proof. □

We note that Corollary 4.3.6 is particularly useful as usually $\operatorname{rk} C = p$, in which case $\operatorname{im} C = \mathbb{R}^p$, $Y = \mathbb{R}^p$ and $F_K(Y) = F_K(\mathbb{R}^p) = \operatorname{im} F_K$.

The next result, a corollary of statement (2) of Theorem 4.3.3, provides a sufficient condition for the CICS property.

Corollary 4.3.7. *Let $K \in \mathbb{S}_\mathbb{R}(A, B, C)$ and set $\gamma := 1/\|\boldsymbol{G}_K\|_{H^\infty}$, where $\gamma := \infty$ if $\|\boldsymbol{G}\_K\|_{H^\infty} = 0$. If there exists $\zeta \in \mathbb{R}^p$ such that* (4.25) *holds and f satisfies*

(A4.2) *For all $\xi \in \operatorname{im} C$ and all $z \in \mathbb{R}^p$ with $z \neq 0$,*

$$\|f(z + \xi) - f(\xi) - Kz\| < \gamma \|z\|,$$

then (4.3) *has the CICS property.*

Proof. The map F_K is surjective, as follows from hypothesis **(A4.2)**, (4.25) and statement (3) of Proposition 4.3.2. Hence, by (4.20),

$$F_K^{-1}(-CA_K^{-1}v^\infty) \cap \operatorname{im} C \neq \emptyset \qquad \forall\, v^\infty \in \mathbb{R}^n.$$

Invoking statement (2) of Theorem 4.3.3 with $Y = \operatorname{im} C$ shows that the Lur'e system (4.3) has the CICS property. □

As an illustration of Corollary 4.3.7, consider the system (4.39) with f given by (4.40) and $K = 0$, see part (a) of Example 4.3.4. In this case, $\gamma = 1$, $Y = \mathbb{R}$ and $V = F_0(\mathbb{R}) = (-1, 1)$. As has been shown in part (a) of Example 4.3.4, assumption **(A4.1)** holds with $Y = \mathbb{R}$ and $Cx(\,\cdot\,; x^0, v) = x(\,\cdot\,; x^0, v)$ is bounded for all $x^0 \in \mathbb{R}$ and all convergent $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ with limit in $(-1, 1)$. Consequently, all assumptions of Corollary 4.3.6 are satisfied and so, for all $x^0 \in \mathbb{R}$ and all $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ such that $\lim_{t\to\infty} v(t) = v^\infty$, we have that $\lim_{t\to\infty} x(t; x^0, v) = x^\infty$, where x^∞ is given by $\{x^\infty\} = F_0^{-1}(v^\infty)$. Note that the system does not have the CICS property, since the input $v(t) \equiv 1+\varepsilon$, $\varepsilon > 0$, generates a divergent state trajectory. Moreover, note that Corollary 4.3.7 does not apply: whilst assumption **(A4.2)** is satisfied, there does not exist $\zeta \in \mathbb{R}$ such that (4.25) holds.

We give a sufficient condition for **(A4.2)** to hold.

Lemma 4.3.8. *Assume that $f : \mathbb{R}^p \to \mathbb{R}^m$ is continuously differentiable, with derivative denoted by Df. Let $\Delta \subseteq \mathbb{R}^p$ be a set which does not have any accumulation points. If*

$$\|(Df)(z) - K\| < \gamma \qquad \forall\, z \in \mathbb{R}^p \backslash \Delta,$$

then condition **(A4.2)** *holds.*

In the following we derive a number of further corollaries which provide "interpretations" of Corollary 4.3.7 in terms of the complex Aizerman conjecture, small-gain theorems and circle criteria respectively.

The first result is reminiscent of the complexified Aizerman conjecture [60, 61, 124]

Corollary 4.3.9. *Let $K \in \mathbb{R}^{m\times p}$, $r > 0$ and assume that*

$$\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(A, B, C).$$

If

$$\|f(z+\xi) - f(\xi) - Kz\| < r\|z\| \qquad \forall\, \xi \in \operatorname{im} C,\ \forall\, z \in \mathbb{R}^p,\ z \neq 0 \qquad (4.46)$$

and there exists $\zeta \in \mathbb{R}^p$ *such that*

$$r\|z\| - \|f(z+\zeta) - f(\zeta) - Kz\| \to \infty \qquad \text{as} \quad \|z\| \to \infty, \tag{4.47}$$

then (4.3) *has the CICS property.*

Proof. By hypothesis $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(A,B,C)$ and so, $A_K = A + BKC$ is Hurwitz and $\mathbb{B}_{\mathbb{C}}(0,r) \subseteq \mathbb{S}_{\mathbb{C}}(A_K,B,C)$. Thus, appealing to elementary stability radius theory [59, 61], we have that $r \leq 1/\|\boldsymbol{G}\|_{H^\infty}$. The claim now follows from Corollary 4.3.7. □

Corollary 4.3.9 says, roughly speaking, that linear stability, namely

$$\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(A,B,C),$$

implies CICS for all nonlinearities f satisfying the incremental ball condition (4.46) and the divergence property (4.47).

Consider the following incremental small-gain condition:

(A4.3) For every $\xi \in \operatorname{im} C$ there exists $\alpha_\xi \in \mathcal{K}_\infty$ such that

$$\|\boldsymbol{G}_K\|_{H^\infty} \frac{\|f(z+\xi) - f(\xi) - Kz\|}{\|z\|} \leq 1 - \frac{\alpha_\xi(\|z\|)}{\|z\|} \tag{4.48}$$

for all $z \in \mathbb{R}^p$ with $z \neq 0$.

We are now in the position to state a "nonlinear" small-gain criterion for the CICS property.

Corollary 4.3.10. *Let* $K \in \mathbb{S}_{\mathbb{R}}(A,B,C)$*. If* f *satisfies* **(A4.3)***, then* (4.3) *has the CICS property.*

Proof. It is clear that if **(A4.3)** is satisfied, then **(A4.2)** and (4.25) hold. Thus, the claim follows from Corollary 4.3.7. □

Note that **(A4.3)** is not a small-gain condition in the sense of classical input-output theory of feedback systems as presented in [31, 54, 55, 79, 146]. In the classical sense, for every fixed $\xi \in \operatorname{im} C$, the RHS of (4.48) is smaller that 1 for all $z \neq 0$, it is in general not uniformly bounded away from 1. Indeed, it is possible that, for fixed ξ, the RHS of (4.48) is converging to 1 as $\|z\| \to 0$ or $\|z\| \to \infty$. Therefore, rather than comparing Corollary 4.3.10 with classical small-gain theorems, it is more appropriate to view it in the context of "modern" nonlinear ISS small-gain results, see for example [26, 74, 124, 139].

If $\operatorname{im} C = \mathbb{R}^p$, then condition **(A4.2)** implies that $f_K : \mathbb{R}^p \to \mathbb{R}^m$, $z \mapsto f(z) - Kz$ is globally Lipschitz and γ is a Lipschitz constant for f_K. If the map f_K is globally Lipschitz and has a Lipschitz constant $\lambda < \gamma$, then

$$\|\boldsymbol{G}_K\|_{H^\infty} \frac{\|f_K(z+\xi) - f_K(\xi)\|}{\|z\|} \le \frac{\lambda}{\gamma} < 1 \qquad \forall\, z, \xi \in \mathbb{R}^p,\ z \neq 0. \tag{4.49}$$

This inequality is an incremental small-gain condition in the sense of classical input-output theory and is sufficient for **(A4.3)** to hold. Consequently, (4.49) is a sufficient condition for the CICS property.

In the following example we present a simple nonlinearity f such that f satisfies **(A4.2)**, f_K has minimal Lipschitz constant equal to γ and (4.31) holds.

Example 4.3.11. *Let*

$$A := \begin{pmatrix} -2 & -1 & 0 \\ 1 & -1 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \qquad B := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad C := \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial of A is $\det(sI - A) = (s+1)^3$. *Hence, A is Hurwitz and so we may choose $K = 0$, leading to*

$$\boldsymbol{G}_K(s) = \boldsymbol{G}_0(s) = \boldsymbol{G}(s) = \frac{1}{(s+1)^3}.$$

A routine argument shows that

$$\|\boldsymbol{G}\|_{H^\infty} = \boldsymbol{G}(0) = 1,$$

and thus $\gamma = 1/\|\boldsymbol{G}\|_{H^\infty} = 1$. In the following, we consider the Lur'e system

$$\left.\begin{aligned} \dot{x} &= Ax + Bf(Cx) + v \\ f(z) &= \operatorname{sign}(z)\ln(1+|z|). \end{aligned}\right\} \tag{4.50}$$

The function f is continuously differentiable and

$$f'(0) = 1 \qquad \text{and} \qquad 0 < f'(z) < 1 \quad \forall\, z \neq 0. \tag{4.51}$$

It follows from Lemma 4.3.8 that condition **(A4.2)** *is satisfied. Moreover, trivially, $|z| - |f(z)| \to \infty$ as $|z| \to \infty$, and so, Corollary 4.3.7 guarantees that* (4.50) *has the CICS property.*

If the assumptions of Corollary 4.3.7 hold, then, by Proposition 4.3.2, the map $\hat{F}_K : \operatorname{im} C \to \operatorname{im} C$ restricting F_K to $\operatorname{im} C$ is bijective and the ISSS gain of (4.3) can be written as

$$\Gamma_{\text{is}}(z) = -A_K^{-1}(B(f_K \circ \hat{F}_K^{-1})(-CA_K^{-1}z) + z) \qquad \forall\, z \in \mathbb{R}^n,$$

where $f_K(z) := f(z) - Kz$. Similarly, the IOSS gain of (4.3) can be expressed as

$$\Gamma_{\text{io}}(z) = \hat{F}_K^{-1}(-CA_K^{-1}z) \qquad \forall\, z \in \mathbb{R}^n.$$

Note that if A is Hurwitz, $f = 0$ and $K = 0$, then (4.3) can be rewritten as the linear system

$$\dot{x} = Ax + v, \qquad y = Cx,$$

which has transfer function $\boldsymbol{H}(s) = C(sI - A)^{-1}$. In this case $F_K(z) = F_0(z) = z$ for all $z \in \mathbb{R}^n$ and $\Gamma_{\text{io}}(z) = -CA^{-1}z = \boldsymbol{H}(0)z$, that is, the familiar linear steady-state gain is recovered.

Definition 4.3.12. *A square rational matrix-valued function $s \mapsto \boldsymbol{H}(s)$ of a complex variable s is said to be* positive real *if for every $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$, which is not a pole of $\boldsymbol{H}$, the matrix $\boldsymbol{H}^*(s) + \boldsymbol{H}(s)$ is positive-semi-definite.*

Next we present, in a form of two corollaries, sufficient conditions for the CICS property which are reminiscent of the well-known circle criterion, see, for example, [55, 79, 124, 146].

Corollary 4.3.13. *Let $K_1, K_2 \in \mathbb{R}^{m\times p}$. Assume that (A, B, C) is stabilizable and detectable, $(I - K_2\boldsymbol{G})(I - K_1\boldsymbol{G})^{-1}$ is positive real, for all $\xi \in \operatorname{im} C$ and all $z \in \mathbb{R}^p$ with $z \neq 0$,*

$$\langle f(z+\xi) - f(\xi) - K_1 z, f(z+\xi) - f(\xi) - K_2 z\rangle < 0 \tag{4.52}$$

and there exists $\zeta \in \mathbb{R}^p$ and $\alpha \in \mathcal{K}_\infty$ such that for all $z \in \mathbb{R}^p$,

$$\langle f(z+\zeta) - f(\zeta) - K_1 z, f(z+\zeta) - f(\zeta) - K_2 z\rangle \leq -\alpha(\|z\|)\|z\|. \tag{4.53}$$

Then the Lur'e system (4.3) has the CICS property.

Proof. We shall rewrite the Lur'e system in a form which will allow the application of Corollary 4.3.7. For $\xi \in \mathbb{R}^p$, define $f_\xi : \mathbb{R}^p \to \mathbb{R}^m$ by

$$f_\xi(z) = f(z+\xi) - f(\xi) \qquad \forall\, z \in \mathbb{R}^p. \tag{4.54}$$

Setting

$$L := \frac{1}{2}(K_1 - K_2) \qquad \text{and} \qquad M := \frac{1}{2}(K_1 + K_2),$$

we have that

$$\begin{aligned}\langle f_\xi(z) - K_1 z, f_\xi(z) - K_2 z\rangle &= \langle f_\xi(z) - (M-L)z, f_\xi(z) - (M-L)z\rangle \\ &= \|f_\xi(z) - Mz\|^2 - \|Lz\|^2 \qquad \forall\, z \in \mathbb{R}^p.\end{aligned}$$

Note that in conjunction with (4.52) this implies $\ker L = \{0\}$. Thus L^*L is invertible and $L^\# := (L^*L)^{-1}L^* \in \mathbb{R}^{p\times m}$ is a left inverse of L. Define the nonlinearity $g : \mathbb{R}^m \to \mathbb{R}^m$ by $g(z) := f(L^\# z) - K_1 L^\# z$ for all $z \in \mathbb{R}^m$ and consider the Lur'e system

$$\dot{x} = A_{K_1} x + Bg(LCx) + v, \tag{4.55}$$

where $A_{K_1} := A + BK_1C$. The linear state space system (A_{K_1}, B, LC) has transfer function

$$\boldsymbol{H}(s) = LC(sI - A_{K_1})^{-1}B = L\boldsymbol{G}_{K_1}(s), \qquad \text{where} \quad \boldsymbol{G}_{K_1} = \boldsymbol{G}(I - K_1\boldsymbol{G})^{-1}.$$

It is obvious that x solves the original Lur'e system $\dot{x} = Ax + Bf(Cx) + v$ if, and only if, x solves (4.55). Therefore, it is sufficient to show that (4.55) has the CICS property. To this end, set $K := -LL^\#$. Using, mutatis mutandis, arguments from [124, proof of Corollary 3.10], it follows that $K \in \mathbb{S}_\mathbb{R}(A_{K_1}, B, LC)$,

$$\gamma := \frac{1}{\|\boldsymbol{H}_K\|_{H^\infty}} \geq 1, \qquad \text{where} \quad \boldsymbol{H}_K := \boldsymbol{H}(I - K\boldsymbol{H})^{-1},$$

there exists $\beta \in \mathcal{K}_\infty$ such that

$$\|g(z + L\zeta) - g(L\zeta) - Kz\| \leq \|z\| - \beta(\|z\|) \leq \gamma\|z\| - \beta(\|z\|) \qquad \forall\, z \in \mathbb{R}^m,$$

and

$$\|g(z + \eta) - g(\eta) - Kz\| < \|z\| \leq \gamma\|z\| \qquad \forall\, \eta \in \operatorname{im}(LC),\ \forall\, z \in \mathbb{R}^m,\ z \neq 0.$$

Consequently, the assumptions of Corollary 4.3.7 are satisfied in the context of the Lur'e system (4.55) and therefore, (4.55) has the CICS property, completing the proof. □

Definition 4.3.14. *A rational square matrix* $\boldsymbol{H}$ *is said to be* strictly positive real *if there exists* $\varepsilon > 0$ *such that the rational matrix function* $s \mapsto \boldsymbol{H}(s - \varepsilon)$ *is positive real.*

Corollary 4.3.15. *Let* $K_1, K_2 \in \mathbb{R}^{m\times p}$. *Assume that* $\ker(K_1 - K_2) = \{0\}$, (A, B, C) *is stabilizable and detectable,* $(I - K_2\boldsymbol{G})(I - K_1\boldsymbol{G})^{-1}$ *is strictly pos-*

itive real and for all $\xi \in \operatorname{im} C$ *and all* $z \in \mathbb{R}^p$*,*

$$\langle f(z+\xi) - f(\xi) - K_1 z, f(z+\xi) - f(\xi) - K_2 z \rangle \leq 0 \tag{4.56}$$

Then the Lur'e system (4.3) *has the CICS property.*

Proof. Set $M := K_2 - K_1$, let $\xi \in \operatorname{im} C$ and define $f_\xi : \mathbb{R}^p \to \mathbb{R}^m$ by (4.54). Then, mutatis mutandis, arguments from [124, proof of Corollary 3.13] can be invoked to show that there exists $k > 0$ and $\mu > 0$ such that, for all $\kappa \in (0, k)$, the rational matrix function

$$(I - (K_2 + \kappa M)\boldsymbol{G})(I - (K_1 - \kappa M)\boldsymbol{G})^{-1}$$

is positive real and

$$\langle f_\xi(z) - (K_1 - \kappa M)z, f_\xi(z) - (K_2 + \kappa M)z \rangle \leq -\mu\kappa(\kappa+1)\|z\|^2 \qquad \forall\, z \in \mathbb{R}^p.$$

It follows that the conditions of Corollary 4.3.13 hold with $\alpha(s) = \mu\kappa(\kappa+1)s$ and K_1 and K_2 replaced by $K_1 - \kappa M$ and $K_2 + \kappa M$ respectively. Hence (4.3) has the CICS property. □

Note that the assumptions in Corollary 4.3.15 are essentially identical to those in the "classical" circle criterion which guarantees global asymptotic stability (see [54, Theorem 5.1], [55, Corollary 5.8] and [79, Theorem 7.1]), the only difference being that (4.56) is the incremental version of the standard sector condition in the circle criterion.

We further note that Corollary 4.3.15 is reminiscent of the main result in [123] which provides a description of the steady-state error of single-input single-output Lur'e systems of the form (4.2) in response to a class of polynomial inputs under the assumption that the conditions of the SISO circle criterion are met. Whilst the CICS property is not mentioned in [123], part (1) of [123, Theorem 1] can be interpreted in CICS terms. We must emphasize that Corollary 4.3.13 and Corollary 4.3.15 are not equivalent. We illustrate this in the following example.

Example 4.3.16. *Consider the one-dimensional Lur'e system*

$$\dot{x} = f(x) + v, \tag{4.57}$$

with $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(z) = -\operatorname{sign}(z)\ln(1 + |z|)$*. The function* f *is continuously differentiable,* $f'(0) = -1$ *and* $-1 < f'(z) < 0$ *for all* $z \neq 0$*. Obviously,* (4.57) *is of the form* (4.3) *with* $(A, B, C) = (0, 1, 1)$*, and so* $\boldsymbol{G}(s) =$

1/s. Let $K_1 < K_2$ *and note that*

$$\frac{1 - K_2\boldsymbol{G}(s)}{1 - K_1\boldsymbol{G}(s)} = \frac{s - K_2}{s - K_1} \tag{4.58}$$

is positive real (strictly positive real) if, and only if, $K_2 \le 0$ *(*$K_2 < 0$*).*

Now if $K_1 < K_2 < 0$*, then, for every* $\xi \in \mathbb{R}$*,*

$$(f(z+\xi) - f(\xi) - K_1 z)(f(z+\xi) - f(\xi) - K_2 z) > 0,$$

for $|z|$ *sufficiently large, and we conclude that Corollary 4.3.15 does not apply.*

However, choosing $K_1 < -1$ *and* $K_2 = 0$*, it is not difficult to show that the conditions of Corollary 4.3.13 are satisfied. Indeed, for* $K_1 < -1$ *and* $K_2 = 0$*, the rational function in* (4.58) *is positive real and, by the mean-value theorem for differentiation,*

$$(f(z+\xi) - f(\xi) - K_1 z)(f(z+\xi) - f(\xi)) < 0 \qquad \forall\ \zeta, z \in \mathbb{R},\ z \neq 0. \tag{4.59}$$

Furthermore, it is clear that

$$\operatorname{sign}(z)\left(\frac{f(z)}{z} - K_1\right) f(z) \to -\infty \qquad \text{as} \quad |z| \to \infty$$

which together with (4.59)*, shows that there exists* $\alpha \in \mathcal{K}_\infty$ *such that*

$$(f(z) - K_1 z) f(z) \le -\alpha(|z|)|z| \qquad \forall\ z \in \mathbb{R}.$$

We have now established that the assumptions of Corollary 4.3.13 hold with $K_1 < -1$*,* $K_2 = 0$ *and* $\eta = 0$*, and consequently, system* (4.57) *has the CICS property.*

4.4 The CICS Property for Another Class of Lur'e Systems

In this section, we shall briefly consider forced Lur'e systems of the form

$$\dot{x} = Ax + Bf(Cx - v), \qquad x(0) = x^0 \in \mathbb{R}^n, \qquad y = Cx, \tag{4.60}$$

where, as in Sections 4.2 and 4.3, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $f : \mathbb{R}^p \to \mathbb{R}^m$, y denotes the output and $v \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^p)$ is the control (forcing, input) function. In the uncontrolled case ($v = 0$), the Lur'e systems (4.3) and (4.60) are identical. The Lur'e system (4.60) can be thought of as a closed-loop system obtained by applying the linear feedback $w = y - v$ to the system $\dot{x} = Ax + Bf(w)$.

Let $\hat{x}(\,\cdot\,; x^0, v)$ denote the unique maximally defined forward solution of the initial-value problem (4.60). The Lur'e system (4.60) is said to have the CICS property if, for every $v^\infty \in \mathbb{R}^p$, there exists $x^\infty \in \mathbb{R}^n$ such that $\lim_{t\to\infty} \hat{x}(t; x^0, v) = x^\infty$ for all $x^0 \in \mathbb{R}^n$ and all $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ with $\lim_{t\to\infty} v^\infty$.

The following proposition provides a sufficient condition for (4.60) to have the CICS property.

Proposition 4.4.1. *Let $K \in \mathbb{S}_{\mathbb{R}}(A, B, C)$ and set $\gamma := 1/\|\boldsymbol{G}_K\|_{H^\infty}$, where $\gamma := \infty$ if $\|\boldsymbol{G}_K\| = 0$. Furthermore, assume that there exists $\eta \in \mathbb{R}^p$ such that* (4.25) *holds and f satisfies*

(A4.4) *For all $\xi, z \in \mathbb{R}^p$ with $z \neq 0$,*

$$\|f(z + \xi) - f(\xi) - Kz\| < \gamma\|z\|.$$

Then the map F_K is bijective and, for all $v^\infty \in \mathbb{R}^p$, all $x^0 \in \mathbb{R}^n$ and all $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ with $\lim_{t\to\infty} v(t) = v^\infty$,

$$\lim_{t\to\infty} \hat{x}(t; x^0, v) = x^\infty := -A_K^{-1} B(f(y^\infty - v^\infty) - Ky^\infty),$$

where $y^\infty \in \mathbb{R}^p$ is given by

$$y^\infty := F_K^{-1}(-(I + \boldsymbol{G}_K(0)K)v^\infty) + v^\infty$$

and satisfies $y^\infty = Cx^\infty$. In particular, the Lur'e system (4.60) *has the CICS property.*

Proof. It follows from statement (3) of Proposition 4.3.2 that F_K is surjective. Injectivity of F_K can be shown by an argument similar to that used in the proof of statement (1) of Proposition 4.3.2.

To prove the convergence property, let $x^0 \in \mathbb{R}^n$, $v^\infty \in \mathbb{R}^p$ and $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ such that $v(t) \to v^\infty$ as $t \to \infty$. Setting $\tilde{x}(t) := \hat{x}(t; x^0, v) - x^\infty$, $\tilde{v}(t) := v(t) - v^\infty$ and $\tilde{f}(z) := f(z + y^\infty - v^\infty) - f(y^\infty - v^\infty)$, a routine calculation shows that $\tilde{x}$ satisfies

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{f}(C\tilde{x} - \tilde{v}).$$

Consequently, writing $w := B[f(C\tilde{x} - \tilde{v}) - f(C\tilde{x})]$, it follows that

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{f}(C\tilde{x}) + w, \tag{4.61}$$

and we note that (4.61) is a forced Lur'e system of the form (4.3). Note that the hypotheses of f combined with Lemma 4.3.1 guarantee that there exists

$\alpha \in \mathcal{K}_\infty$ such that

$$\|\tilde{f}(z) - Kz\| \leq \gamma\|z\| - \alpha(\|z\|) \qquad \forall\ z \in \mathbb{R}^p.$$

Consequently, by Theorem 4.2.1, the equilibrium pair $(0, 0)$ of (4.61) is ISS. Moreover, hypothesis **(A4.4)** implies that

$$\|w(t)\| \leq \|B\|(\gamma + \|K\|)\|\tilde{v}(t)\| \qquad \forall\ t \geq 0,$$

showing that $w(t) \to 0$ as $t \to \infty$, noting that $\gamma < \infty$ by hypothesis. An application of Proposition 4.2.2 now shows that $\tilde{x}(t) \to 0$ as $t \to \infty$ and thus, $\hat{x}(t; x^0, v) \to x^\infty$ as $t \to \infty$.

It remains to show that $y^\infty = Cx^\infty$. To see this, note that

$$Cx^\infty = \boldsymbol{G}_K(0)(f(y^\infty - v^\infty) - Ky^\infty).$$

Hence

$$\begin{aligned} y^\infty - Cx^\infty = y^\infty - v^\infty - \boldsymbol{G}_K(0)(f(y^\infty - v^\infty) - K(y^\infty - v^\infty)) \\ + (I + \boldsymbol{G}_K(0)K)v^\infty, \end{aligned}$$

and so,

$$y^\infty - Cx^\infty = F_K(y^\infty - v^\infty) + (I + \boldsymbol{G}_K(0)K)v^\infty.$$

But

$$F_K(y^\infty - v^\infty) = -(I - \boldsymbol{G}_K(0)K)v^\infty,$$

implying that $y^\infty = Cx^\infty$. □

Note that under the assumptions of Proposition 4.4.1, it is natural to define the IOSS gain of (4.60) to be the map

$$v^\infty \mapsto F_K^{-1}(-(I + \boldsymbol{G}_K(0)K)v^\infty) + v^\infty.$$

Proposition 4.4.1 allows us to extend a classical result on integral control to Lur'e systems of the form (4.60). To this end, assume that the assumptions of Proposition 4.4.1 are satisfied, $f(0) = 0$ and the linear system (A, B, C) contains an integrator, that is, $\boldsymbol{G}$ has a Laurent expansion of the form

$$\boldsymbol{G}(s) = \sum_{j=-1}^{\infty} G_j s^j,$$

for all sufficiently small $|s|$, $s \neq 0$, where $G_j \in \mathbb{R}^{p \times m}$ and $G_{-1} \neq 0$. If $G_{-1}K$ is invertible, then $\boldsymbol{G}_K(0)K = -I$ and so, $y^\infty = F_K^{-1}(0) + v^\infty = v^\infty$, where we have used $f(0) = 0$, showing that every input v with limit v^∞ produces an output y converging also to v^∞, or equivalently, the IOSS gain of (4.60) is equal to the identity.

4.5 CICS Property for Nonnegative Lur'e Systems

In this section we study nonnegative Lur'e systems which arise naturally in a variety of applied contexts, such as population dynamics and chemical reaction models. We restrict attention to models with scalar feedback f ($m = p = 1$ and f is a scalar function), that is, Lur'e systems of the form

$$\dot{x} = Ax + bf(c^T x) + v, \qquad x(0) = x^0 \in \mathbb{R}^n_+, \qquad y = c^T x, \tag{4.62}$$

so that, in particular, the linear system (A, b, c^T) is a single-input, single-output (SISO) system. We assume that the following positivity conditions hold:

(A4.5) $A \in \mathbb{R}^{n\times n}$ is Metzler and $b, c \in \mathbb{R}^n_+$, $b, c > 0$,

(A4.6) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz.

Furthermore, we only consider nonnegative control functions, $v \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$.

As before, we denote the unique maximally defined forward solution of the initial-value problem (4.62) by $x(\,\cdot\,; x^0, v)$. It is well-known that if **(A4.5)** and **(A4.6)** hold, then for all nonnegative initial states $x^0 \in \mathbb{R}^n_+$ and $v \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, the solution $x(t; x^0, v)$ remains in the nonnegative orthant $\mathbb{R}^n_+$ for all $t \in [0, \omega)$, where $[0, \omega)$, $0 < \omega \le \infty$, denotes the maximal interval of existence. If $\omega < \infty$, then $\|x(t; x^0, v)\| \to \infty$ as $t \to \infty$. If **(A4.5)** and **(A4.6)** hold, then we will refer to (4.62) as a *nonnegative* Lur'e system.

For later purposes, we introduce a further "positivity" assumption on the linear system (A, b, c^T).

(A4.7) The matrix $A + bc^T$ is irreducible.

Note that $A + bc^T$ is irreducible if, and only if, $A + kbc^T$ is irreducible for every $k > 0$.

Let $s \mapsto \boldsymbol{G}(s) = c^T(sI - A)^{-1}b$ denote the transfer function of the linear SISO system (A, b, c^T).

For completeness we restate Lemma 3.3.3.

Lemma 4.5.1. *Assume that* **(A4.5)** *is satisfied, then*

$$\|\boldsymbol{G}\|_{H^\infty} = |\boldsymbol{G}(0)| = \boldsymbol{G}(0) \ge 0.$$

If additionally **(A4.7)** *is satisfied, then* $\boldsymbol{G}(0) > 0$.

It follows from Lemma 4.5.1 that if A is Hurwitz, **(A4.5)** holds and (A, b) is controllable or (c^T, A) is observable, then $\boldsymbol{G}(0) > 0$.

Theorem 4.5.2. *Let* $Y \subseteq \mathbb{R}_+$ *be nonempty and assume that* **(A4.5)** *and* **(A4.6)** *hold and* A *is Hurwitz. Set* $\gamma := 1/\boldsymbol{G}(0)$*, where* $\gamma := \infty$ *if* $\boldsymbol{G}(0) = 0$*, and assume further that*

$$\left|\frac{f(z) - f(\xi)}{z - \xi}\right| < \gamma \qquad \forall\ (\xi, z) \in Y \times \mathbb{R}_+ \quad \text{such that} \quad z \neq \xi, \tag{4.63}$$

and

$$\gamma z - f(z) \to \infty \quad \text{as} \quad z \to \infty. \tag{4.64}$$

Then the following statements hold.

(1) The map

$$F : \mathbb{R}_+ \to \mathbb{R},\ z \mapsto z - \boldsymbol{G}(0) f(z) \tag{4.65}$$

has the following properties: $\mathbb{R}_+ \subseteq F(\mathbb{R}_+)$ *and* $\#F^{-1}(z) = 1$ *for every* $z \in \mathbb{R}$ *such that* $F^{-1}(z) \cap Y \neq \emptyset$.

(2) Let $v^\infty \in \mathbb{R}_+$ *and assume that* $F^{-1}(-c^T A^{-1} v^\infty) \cap Y \neq \emptyset$. *Then,*

$$\#F^{-1}(-c^T A^{-1} v^\infty) = 1$$

and, for all $x^0 \in \mathbb{R}^n$ *and all* $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n_+)$ *such that* $\lim_{t\to\infty} v(t) = v^\infty$,

$$\lim_{t\to\infty} x(t; x^0, v) = -A^{-1}(b f(y^\infty) + v^\infty) =: x^\infty \in \mathbb{R}^n_+,$$

where $\{y^\infty\} = F^{-1}(-c^T A^{-1} v^\infty)$ *and* x^∞ *satisfies* $c^T x^\infty = y^\infty \geq 0$.

Proof. We begin by extended f to the whole real line, $\mathbb{R}$ by defining

$$\tilde{f} : \mathbb{R} \to \mathbb{R}, \qquad z \mapsto \begin{cases} f(z), & \text{for } z \geq 0 \\ f(0), & \text{for } z < 0. \end{cases} \tag{4.66}$$

Using (4.63), it is straightforward to show that

$$\left|\frac{\tilde{f}(z + \xi) - \tilde{f}(\xi)}{z}\right| < \gamma \qquad \forall\ \xi \in Y, \forall\ z \in \mathbb{R},\ z \neq 0.$$

Consequently,

$$|\tilde{f}(z + \xi) - \tilde{f}(\xi)| < \gamma |z| \qquad \forall\ \xi \in Y, \forall\ z \in \mathbb{R},\ z \neq 0. \tag{4.67}$$

Furthermore, by (4.64),

$$\gamma|z| - |\tilde{f}(z+\xi) - \tilde{f}(\xi)| \to \infty \quad \text{as} \quad |z| \to \infty. \tag{4.68}$$

Defining $\tilde{F} : \mathbb{R} \to \mathbb{R}$ by $\tilde{F}(z) = z - \boldsymbol{G}(0)\tilde{f}(z)$, an application of Proposition 4.3.2 shows that $\tilde{F}$ is surjective and

$$\#\tilde{F}^{-1}(z) = 1 \qquad \text{for every } z \in \mathbb{R} \text{ such that } \tilde{F}^{-1}(z) \cap Y \neq \emptyset. \tag{4.69}$$

Now $\tilde{F}(z) < -\boldsymbol{G}(0)f(0) \leq 0$ for all $z < 0$ and so surjectivity of $\tilde{F}$ implies that $\mathbb{R}_+ \subseteq \tilde{F}(\mathbb{R}_+) = F(\mathbb{R}_+)$. Moreover, let $z \in \mathbb{R}$ be such that $F^{-1}(z) \cap Y \neq \emptyset$. If $w \in F^{-1}(z) \subseteq \mathbb{R}_+$ then $z = F(w) = \tilde{F}(w)$, and so $F^{-1}(z) \subseteq \tilde{F}^{-1}(z)$. Consequently, $\tilde{F}^{-1}(z) \cap Y \neq \emptyset$, whence, by (4.69), $\#F^{-1}(z) = \#\tilde{F}^{-1}(z) = 1$, completing the proof of statement (1).

To prove statement (2), let $v^\infty \in \mathbb{R}_+$ be such that $F^{-1}(-c^T A^{-1} v^\infty) \cap Y \neq \emptyset$. It follows from the proof of statement (1) that

$$F^{-1}(-c^T A^{-1} v^\infty) \cap Y \subseteq \tilde{F}^{-1}(-c^T A^{-1} v^\infty) \cap Y \neq \emptyset. \tag{4.70}$$

Let $x^0 \in \mathbb{R}^n_+$ and let $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n_+)$ be such that $\lim_{t\to\infty} v(t) = v^\infty$. Setting $x(t) := x(t; x^0, v)$, it is clear that $x(t) \in \mathbb{R}^n_+$ for $t \geq 0$, implying that $c^T x(t) \geq 0$ for all $t \geq 0$. Consequently, x is a solution of

$$\dot{\chi} = A\chi + b\tilde{f}(c^T\chi) + v. \tag{4.71}$$

Appealing to (4.67), (4.68) and (4.70), an application of statement (2) of Theorem 4.3.3 to the Lur'e system (4.70) then shows that

$$\#\tilde{F}^{-1}(-c^T A^{-1} v^\infty) = 1, \tag{4.72}$$

and

$$\lim_{t\to\infty} x(t) = -A^{-1}(b\tilde{f}(y^\infty) + v^\infty), \tag{4.73}$$

where $\{y^\infty\} = \tilde{F}^{-1}(-c^T A^{-1} v^\infty)$. By hypothesis, $F^{-1}(-c^T A^{-1} v^\infty) \cap Y \neq \emptyset$ and thus, invoking (4.70) and (4.72), we obtain that

$$\#F^{-1}(-c^T A^{-1} v^\infty) = 1.$$

Finally, since $-c^T A^{-1} v^\infty \geq 0$, we have $y^\infty \geq 0$, implying that $f(y^\infty) = \tilde{f}(y^\infty)$ and $\{y^\infty\} = F^{-1}(-c^T A^{-1} v^\infty)$. In particular, the RHS of (4.73) is equal to $-A^{-1}(bf(t^\infty) + v^\infty)$ and the proof is complete. $\square$

As an immediate consequence of Theorem 4.5.2 we obtain the following result.

Corollary 4.5.3. *Assume that* **(A4.5)** *and* **(A4.6)** *hold and A is Hurwitz. Set $\gamma := 1/\boldsymbol{G}(0)$, where $\gamma := \infty$ if $\boldsymbol{G}(0) = 0$, and assume further that*

$$\left|\frac{f(z) - f(\xi)}{z - \xi}\right| < \gamma \qquad \forall\ (\xi, z) \in \mathbb{R}_+ \times \mathbb{R}^n_+,\ \text{such that } z \neq \xi, \tag{4.74}$$

and (4.64) *is satisfied. Then, for every $v^\infty \in \mathbb{R}^n_+$, $\#F^{-1}(-c^T A^{-1} v^\infty) = 1$, with F given by* (4.65)*, the nonnegative Lur'e system* (4.62) *has the CICS property: for all $x^0 \in \mathbb{R}^n_+$ and all $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n_+)$ with $\lim_{t\to\infty} v(t) = v^\infty$,*

$$\lim_{t\to\infty} x(t; x^0, v) = -A^{-1}(bf(y^\infty) + v^\infty) =: x^\infty \in \mathbb{R}^n_+,$$

where $\{y^\infty\} = F^{-1}(-c^T A^{-1} v^\infty)$.

The following lemma, which is an immediate consequence of the mean-value theorem for differentiation, provides a sufficient condition for (4.74) to hold.

Lemma 4.5.4. *Assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable and let $\Delta \subseteq \mathbb{R}_+$ be a subset which does not have any accumulation points. If*

$$|f'(z)| < \gamma \qquad \forall\ z \in \mathbb{R}_+ \backslash \Delta,$$

then (4.74) *holds for all $(\xi, z) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $z \neq \xi$.*

Example 4.5.5. *Nonnegative Lur'e systems of the form* (4.62) *with*

$$A := \begin{pmatrix} -a_1 & 0 & \cdots & 0 \\ a_2 & -a_3 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & a_{2n-2} & -a_{2n-1} \end{pmatrix}, \quad b := \begin{pmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad c := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

where $a_i > 0$ for all $i \in \{1, \ldots, 2n-1\}$ and $b_1 > 0$, arise in both population modeling [51] and reaction kinetics, see, for example [103, Section 7.2]. Obviously, A is Metzler and Hurwitz. The matrix A can represent a continuous time population matrix as introduces in Section 2.3.1. In a population dynamics context, the a_{2k-1} represents mortality rates and growth rates progressing to the next age class, the a_{2k} represents growth rates from the previous stage class and f models nonlinear recruitment. The function v could model, for example, immigration effects.

Here we consider the following specific example of the above structure.

$$A := \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1/2 & 0 \\ 0 & 1 & -2 \end{pmatrix}, \qquad b := \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \qquad c := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.75}$$

Then,

$$\boldsymbol{G}(s) = c^T (sI - A)^{-1} b = \frac{2}{(s+1/2)(s+1)(s+2)},$$

and a routine argument shows that

$$\|\boldsymbol{G}\|_{H^\infty} = \boldsymbol{G}(0) = 2,$$

whence $\gamma = 1/\|\boldsymbol{G}\|_{H^\infty} = 1/2$. *We consider the nonnegative Lur'e system*

$$\dot{x} = Ax + bf(c^T x) + v \tag{4.76}$$

for three different nonlinearities $f : \mathbb{R}_+ \to \mathbb{R}_+$.

(a) *Let* $f(z) = z/(2+z)$ *for* $z \geq 0$. *Then,* $f'(z) = 2(z+2)^{-2}$ *and so,*

$$f'(0) = \frac{1}{2} \qquad \text{and} \qquad f'(z) < \frac{1}{2} \quad \forall\ z > 0.$$

By Lemma 4.5.4, condition (4.74) *holds. Furthermore,* (4.64) *is trivially satisfied. Consequently, Corollary 4.5.3 guarantees that* (4.76) *has the CICS property.*

(b) *Let* $f(z) = 1/(2+z)$ *for* $z \geq 0$. *Then,* $f'(z) = -(z+2)^{-2}$, *and, arguing as in part (a), we see that* (4.76) *has the CICS property.*

Figure 4.4(a) displays numerical simulations of the state trajectories generated by the input signals v^1 *and* v^2 *given by*

$$v^j(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} w_j(t), \qquad \text{with} \qquad \begin{aligned} w_1(t) &:= \frac{1}{1 + 1 + e^{-0.8(t-10)}}, \\ w_2(t) &:= 1 + (-1)^{S(t)} (0.65)^{\lfloor t/10 \rfloor}, \end{aligned} \tag{4.77}$$

where $\lfloor z \rfloor \in \mathbb{N}_0$ *denotes the largest integer less or equal to* $z \in \mathbb{R}_+$ *and the "switching function"* $S : \mathbb{R}_+ \to \{0, 1\}$ *is defined by*

$$S(t) := \begin{cases} 0, & \lfloor t/10 \rfloor \text{ even}, \\ 1, & \lfloor t/10 \rfloor \text{ odd}. \end{cases}$$

The functions w_1 *and* w_2 *are plotted in Figure 4.4(b). Obviously,* $w_1(t) \to$

1 and $w_2(t) \to 1$ as $t \to \infty$, and so

$$\lim_{t\to\infty} v^1(t) = \lim_{t\to\infty} v^2(t) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T =: v^\infty.$$

By the CICS property, the limit

$$\lim_{t\to\infty} x(t; x^0, v^j) =: x^\infty$$

exists, is independent of $j \in \{1, 2\}$ and the initial condition x^0, and is given by $x^\infty = -A^{-1}(bf(y^\infty)+v^\infty)$, where $\{y^\infty\} = F^{-1}(-c^T A^{-1} v^\infty)$ (see Corollary 4.5.3). The condition for y^∞ can be expressed in the form

$$y^\infty - \boldsymbol{G}(0) f(y^\infty) + c^T A^{-1} v^\infty = 0,$$

which is a quadratic equation in y^∞ and has nonnegative solution $y^\infty =$ 1.5616. Now x^∞ can be computed and we obtain

$$x^\infty := \begin{pmatrix} 0.5615 \\ 3.1231 \\ 1.5615 \end{pmatrix}.$$

See Figure 4.4(a) for an illustration.

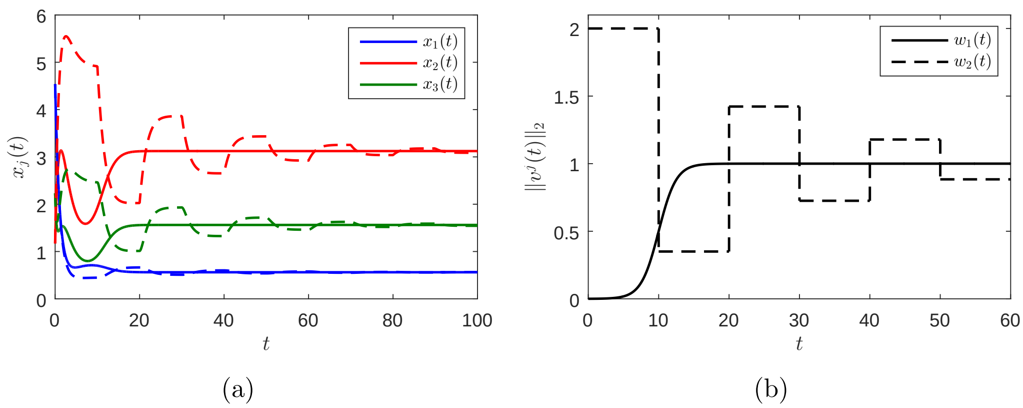


Figure 4.4: (a) State components generated by input signal shown in (b) and given by (4.77). The nonzero initial states x^0 have been chosen randomly.

(c) *Let $f(z) = 2z/(z+1)$ for $z \geq 0$, in which case*

$$f(z) - f(\xi) = \frac{2(z-\xi)}{(z+1)(\xi+1)} \qquad \forall \quad (\xi, z) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

Note that, for any $\xi \in [0, 3]$, there exists $z \geq 0$, $z \neq \xi$, such that

$$\left| \frac{f(z) - f(\xi)}{z - \xi} \right| \geq \frac{1}{2} = \gamma.$$

In particular, for $\xi = 3$*:*

$$\frac{f(3)}{3} = \frac{f(0) - f(3)}{0 - 3} = \frac{1}{2} = \gamma.$$

On the other hand, for every $\xi > 3$*:*

$$\left| \frac{f(z) - f(\xi)}{z - \xi} \right| < \frac{1}{2} = \gamma \qquad \forall \quad z \geq 0.$$

It is obvious that $z/2 - f(z) \to \infty$ *as* $z \to \infty$*, and so, Theorem 4.5.2, with* $Y := (3, \infty)$*, can be applied to (4.76). To this end, note that the function* $F : \mathbb{R}_+ \to \mathbb{R}_+$ *is given by*

$$F(z) = z - \boldsymbol{G}(0) f(z) = z - \frac{4z}{z+1},$$

and so, $F(y) = (0, \infty)$*. Now,*

$$A^{-1} := \begin{pmatrix} -1 & 0 & 0 \\ -2 & -2 & 0 \\ -1 & -1 & -1/2 \end{pmatrix},$$

and thus, $-c^T A^{-1} = (1, 1, 1/2)$*, showing that*

$$-c^T A^{-1} v^\infty > 0 \qquad \forall \quad v^\infty \in \mathbb{R}^3_+ \backslash \{0\}.$$

Consequently,

$$F^{-1}(-c^T A^{-1} v^\infty) \cap Y \neq 0 \qquad \forall \quad v^\infty \in \mathbb{R}^3_+ \backslash \{0\}.$$

Theorem 4.5.2 guarantees that, for every $v^\infty \in \mathbb{R}^3_+ \backslash \{0\}$*, there exists* $x^\infty \in \mathbb{R}^3_+$ *such that* $\lim_{t \to \infty} x(t; x^0, v) = x^\infty$ *for all* $x^\infty \in \mathbb{R}^3_+$ *and all* $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^3_+)$ *with* $\lim_{t \to \infty} v(t) = v^\infty$*.*

To consider a specific numerical example, let

$$v^\infty = \begin{pmatrix} 1/4 \\ 1/4 \\ 1 \end{pmatrix},$$

in which case, $-c^T A^{-1} v^\infty = 1$*. Now* $F^{-1}(1) = \{2 + \sqrt{5}\}$*, and so* $y^\infty =$

$2+\sqrt{5}$, $f(y^\infty) = (1+\sqrt{5})/2$, *and*

$$x^\infty = -A^{-1}(bf(y^\infty) + v^\infty) = (1+\sqrt{5})\begin{pmatrix}1\\2\\1\end{pmatrix} + \begin{pmatrix}1/4\\1\\1\end{pmatrix}.$$

Finally, we comment on input functions v *which converge to* 0. *There does not exist* x^∞ *such that* $\lim_{t\to\infty} x(t;x^0,v) = x^\infty$ *for all* $x^0 \in \mathbb{R}^3_+$ *and all* $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^3_+)$ *with* $\lim_{t\to\infty} v(t) = 0$. *Indeed, this follows from the fact that, for* $v = 0$, *the system* (4.76) *has two equilibria in* $\mathbb{R}^3_+$, *namely* $(0,0,0)^T$ *and* $(3,6,3)^T$ *. Also note that* $F^{-1}(0) = \{0,3\}$ *and thus* $\#F^{-1}(0) > 1$ *(cf. Proposition 4.2.5).*

In the context of the Lur'e system discussed in part (c) of Example 4.5.5, it is interesting to note that the nonzero equilibrium $x^* = (3,6,3)^T$ of the uncontrolled system is asymptotically stable with region of attraction equal to $\mathbb{R}^3_+\backslash\{0\}$. This gives rise to the following question: does $x(t;x^0,v)$ converge to x^* for all *nonzero* initial-conditions $x^0 \in \mathbb{R}^3_+$ and all $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^3_+)$ with $\lim_{t\to\infty} v(t) = 0$? We shall now state and prove a CICS result which implies that the answer to the question is "yes".

Theorem 4.5.6. *Assume that* **(A4.5)**-**(A4.7)** *hold and* A *is Hurwitz. Set* $\gamma := 1/\boldsymbol{G}(0)$ *and assume further that* $f(0) = 0$, *there exists* $y^* > 0$ *such that* $f(y^*) = \gamma y^*$, (4.64) *is satisfied,*

$$\liminf_{z\to 0} \frac{f(z)}{z} > \gamma, \tag{4.78}$$

and

$$\left|\frac{f(z) - f(\xi)}{z - \xi}\right| < \gamma \qquad \forall\ (\xi, z) \in [y^*, \infty) \times (0,\infty),\ z \neq \xi. \tag{4.79}$$

Then the following statements hold.

(1) The points 0 *and* $x^* := -\gamma y^* A^{-1} b$ *are equilibria of the uncontrolled system* $\dot{x} = Ax + bf(c^T x)$.

(2) The map

$$F^* : [y^*, \infty) \to \mathbb{R}_+,\ z \mapsto z - \boldsymbol{G}(0) f(z)$$

is a bijection.

(3) The nonnegative Lur'e system (4.62) *has the following "quasi-CICS" property: for all* $x^0 \in \mathbb{R}^n_+$, *all* $v^\infty \in \mathbb{R}^n_+$ *and all* $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n_+)$ *such*

that $\|x^0\| + \|v\|_{L^\infty} > 0$ *and* $\lim_{t\to\infty} v(t) = v^\infty$,

$$\lim_{t\to\infty} x(t; x^0, v) = -A^{-1}(bf(y^\infty) + v^\infty) =: x^\infty \in \mathbb{R}^n_+,$$

where $y^\infty = F^{*-1}(-c^T A^{-1} v^\infty)$. *In particular, if* $v^\infty = 0$, *then* $y^\infty = y^*$ *and* $x^\infty = x^* = -\gamma y^* A^{-1} b$.

Proof. Since $f(0) = 0$, it is obvious that 0 is an equilibrium of $\dot{x} = Ax + bf(c^T x)$. Invoking the hypothesis that $f(y^*) = \gamma y^*$, a straightforward calculation shows the x^* is also an equilibrium of $\dot{x} = Ax + bf(c^T x)$, completing the proof of statement (1).

To prove statements (2) and (3), let $x^0, v^\infty \in \mathbb{R}^n_+$ and $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n_+)$ be such that $\|x^0\| + \|v\|_{L^\infty} > 0$ and $\lim_{t\to\infty} v(t) = v^\infty$. We consider two cases: $x^0 \neq 0$ and $x^0 = 0$.

Case 1: $x^0 \neq 0$.

Invoking **(A4.5)-(A4.7)** and conditions (4.78) and (4.79), it follows from Proposition 3.4.14 that there exists $\varepsilon \in (0, y^*)$ and $\tau \geq 0$ such that

$$c^T x(t; x^0, v) \geq \varepsilon \qquad \forall\, t \geq \tau. \tag{4.80}$$

Consider

$$\tilde{f} : \mathbb{R} \to \mathbb{R}, \quad z \mapsto \begin{cases} f(z + y^*) - f(y^*), & \text{for } z \geq -y^* + \varepsilon \\ f(\varepsilon) - f(y^*), & \text{for } z < -y^* + \varepsilon \end{cases} \tag{4.81}$$

and

$$\tilde{F} : \mathbb{R} \to \mathbb{R}, \quad z \mapsto z - \boldsymbol{G}(0)\tilde{f}(z),$$

and note that, since $f(y^*) = \gamma y^*$,

$$\tilde{F}(z) = z + y^* - \boldsymbol{G}(0) f(z + y^*) = F^*(z + y^*) \qquad \forall\, z \geq 0. \tag{4.82}$$

In particular, $\tilde{F}(0) = 0$ and, by (4.64) and (4.79),

$$\tilde{F}(z) > 0, \qquad \forall\, z > 0$$

and

$$\tilde{F}(z) \to \infty \quad \text{as} \quad z \to \infty,$$

implying that $\tilde{F}(\mathbb{R}_+) = \mathbb{R}_+$ and so

$$\tilde{F}^{-1}(z) \cap \mathbb{R}_+ \neq \emptyset \qquad \forall\, z \in \mathbb{R}_+. \tag{4.83}$$

Next, we prove that

$$\left|\frac{\tilde{f}(z+\xi)-\tilde{f}(\xi)}{z}\right| < \gamma \qquad \forall\, \xi \in \mathbb{R}_+, \quad \forall\, z \in \mathbb{R},\, z \neq 0. \tag{4.84}$$

To see this, let $\xi \geq 0$. Then, invoking (4.79), we obtain that, for nonzero $z \geq -(\xi + y^*) + \varepsilon$,

$$\left|\frac{\tilde{f}(z+\xi)-\tilde{f}(\xi)}{z}\right| = \left|\frac{f(z+\xi+y^*)-f(\xi+y^*)}{z+\xi+y^*-(\xi+y^*)}\right| < \gamma.$$

Furthermore, for $z < -(\xi + y^*) + \varepsilon$, we have $|z| = -z > \xi + y^* - \varepsilon > 0$ and so

$$\left|\frac{\tilde{f}(z+\xi)-\tilde{f}(\xi)}{z}\right| = \left|\frac{f(\varepsilon)-f(\xi+y^*)}{z}\right| < \left|\frac{f(\varepsilon)-f(\xi+y^*)}{\varepsilon-(\xi+y^*)}\right| < \gamma,$$

where the final inequality follows from (4.79). Therefore, (4.84) holds. Consequently,

$$|\tilde{f}(z+\xi)-\tilde{f}(\xi)| < \gamma|z| \qquad \forall\, \xi \in \mathbb{R}_+, \quad \forall\, z \in \mathbb{R},\, z \neq 0. \tag{4.85}$$

Moreover, by (4.64),

$$\gamma|z| - |\tilde{f}(z+\xi)-\tilde{f}(\xi)| \to \infty \quad \text{as} \quad |z| \to \infty. \tag{4.86}$$

Setting $x^* := -A^{-1}bf(y^*) = -\gamma y^* A^{-1}b$ and

$$\tilde{x}(t) := x(t+\tau; x^0, v) - x^* \qquad \forall\, t \geq 0,$$

we have, by (4.80),

$$c^T\tilde{x}(t) = c^T x(t+\tau; x^0, v) - y^* \geq -y^* + \varepsilon \qquad \forall\, t \geq 0,$$

where we have used that $c^T x^* = y^*$. Consequently,

$$\tilde{f}(c^T\tilde{x}(t)) = f(c^T x(t+\tau; x^0, v)) - f(y^*) \qquad \forall\, t \geq 0,$$

and so, $\tilde{x}$ satisfies

$$\dot{\tilde{x}} = A\tilde{x} + b\tilde{f}(c^T\tilde{x}) + \Lambda_\tau v, \tag{4.87}$$

where, as before, Λ_τ denotes the left-shift by τ. Appealing to (4.83), (4.85) and (4.86), we may apply Theorem 4.3.3 with $K = 0$ and $Y = \mathbb{R}_+$, in the context of the controlled Lur'e system (4.87) and obtain that

$$\#\tilde{F}^{-1}(z) = 1 \qquad \forall\, z \in \mathbb{R}_+ \tag{4.88}$$

and

$$\lim_{t\to\infty} \tilde{x}(t) = -A^{-1}(b\tilde{f}(\tilde{y}^\infty) + v^\infty) =: \tilde{x}^\infty, \tag{4.89}$$

where $\{\tilde{y}^\infty\} = \tilde{F}^{-1}(-c^T A^{-1} v^\infty)$. Equations (4.82), (4.83) and (4.88) shows that F^* is a bijection. Finally, setting $y^\infty := \tilde{y}^\infty + y^*$, we obtain from (4.82), that $y^\infty = F^{*-1}(-c^T A^{-1} v^\infty)$, and, by (4.89),

$$\begin{aligned} x(t;x^0,v) \to \tilde{x}^\infty + x^* &= -A^{-1}(b(f(y^\infty) - \gamma y^*) + v^\infty) - \gamma y^* A^{-1} b \\ &= -A^{-1}(bf(y^\infty) + v^\infty), \qquad \text{as} \quad t \to \infty. \end{aligned}$$

Case 2: $x^0 = 0$.

Then, by hypothesis, $\|v\|_{L^\infty} > 0$ and thus, there exists $t_0 \geq 0$ such that

$$x(t_0;0,v) = \int_0^{t_0} e^{A(t_0-s)} \left(bf(c^T x(s;0,v)) + v(s)\right) ds \geq \int_0^{t_0} e^{A(t_0-s)} v(s) ds > 0.$$

The function χ defined by $\chi(t) := x(t+t_0;0,x)$ satisfies

$$\dot{\chi} = A\chi + bf(c^T\chi) + \Lambda_{t_0} v, \qquad \chi(0) = x(t_0;0,v) > 0,$$

and so, by case 1,

$$\lim_{t\to\infty} x(t;0,v) = \lim_{t\to\infty} \chi(t) = -A^{-1}(bf(y^\infty) + v^\infty) = x^\infty,$$

completing the proof. □

Example 4.5.7. *Here we re-visit part (c) of Example 4.5.5: A, b and c are given by* (4.75) *and* $f(z) = 2z/(z+1)$ *for all* $z \geq 0$. *It is readily verified that* $A+bc^T$ *is irreducible, that is* **(A4.7)** *is satisfied. We recall that the uncontrolled Lur'e system* (4.76) *has two equilibria, namely* 0 *and* $x^* = (3,6,3)^T$ *(the latter being asymptotically stable with domain of attraction* $\mathbb{R}^3_+\backslash\{0\}$*, as follows from Theorem 3.4.11) and that* $\gamma = 1/\|\boldsymbol{G}\|_{H^\infty} = 1/2$. *We note that* $f(3) = 3/2 = 3\gamma$, $\liminf_{z\to 0} f(z)/z = f'(0) = 2$, *and*

$$\left|\frac{f(z)-f(\xi)}{z-\xi}\right| = \left|\frac{2}{(z+1)(\xi+1)}\right| < \frac{1}{2} \qquad \forall \quad (\xi,z) \in [3,\infty)\times(0,\infty),\ z \neq \xi.$$

We may now apply Theorem 4.5.6 with $y^* = 3$ *and obtain that the Lur'e system under consideration has the quasi-CICS property (in the sense of Theorem 4.5.6).*

Consider the input signals v^1 and v^2 given by $v^j(t) = w_j(t)(0, 1, 0)^T$, where

$$w_1(t) = \frac{1}{1 + e^{-0.8(t-10)}}$$

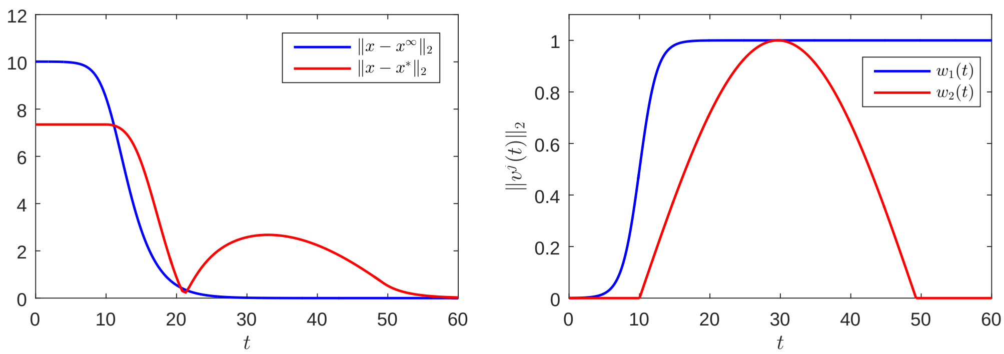
and

$$w_2(t) = \begin{cases} 0, & 0 \le t \le 10, \\ \sin\left(\dfrac{2(t-10)}{25}\right), & 10 < t \le 10 + \dfrac{25\pi}{2}, \\ 0, & 10 + \dfrac{25\pi}{2} < t. \end{cases}$$

See Figure 4.5(b) for an illustration. Note that $v^1(t) \to (0, 1, 0)^T$ and $v^2(t) \to (0, 0, 0)^T$ as $t \to \infty$. By Theorem 4.5.6, for all $x^0 \in \mathbb{R}^3_+$, we have $x(t; x^0, v^1) \to x^\infty$ and $x(t; x^0, v^2) \to x^*$ as $t \to \infty$, where

$$x^\infty = -A^{-1}\left[bf(y^\infty) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 3.2361 \\ 8.4721 \\ 4.2361 \end{pmatrix}.$$

Figure 4.5(a) shows that plots of $\|x(t; x^0, v^1) - x^\infty\|_2$ and $\|x(t; x^0, v^2) - x^*\|_2$ for $x^0 = 0$. In particular, we see that the state trajectory $x(t; 0, v^2)$ is at the zero equilibrium for $0 \le t \le 10$ since the input v^2 is zero in this time interval. On the interval $(10, 10 + 25\pi/2)$, v^2 is positive and correspondingly, $x(t; 0, v^2)$ moves away from the origin and eventually converges to x^*.



(a) (b)

Figure 4.5: Numerical simulations for Example 4.5.7. (a) shows the norm of state errors for $x^0 = 0$ corresponding to the input signals shown in panel (b).

The following lemma provides a sufficient condition for (4.79) to hold.

Lemma 4.5.8. *Assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable, $f(0) = 0$, $f'(z) \ge 0$ for all $z \ge 0$, $f'(0) > \gamma$, f' is nonincreasing and*

$\lim_{z\to\infty} f'(z) < \gamma$. *Then there exists* $y^* > 0$ *such that* $f(y^*) = \gamma y^*$ *and*

$$\left|\frac{f(z)-f(\xi)}{z-\xi}\right| = \frac{f(z)-f(\xi)}{z-\xi} < \gamma \qquad \forall\ (\xi, z) \in [y^*, \infty) \times (0, \infty),\ z \neq \xi.$$

Proof. It follows immediately from the hypothesis that there exists $y^* > y^\dagger > 0$ such that $f(y^*) = \gamma y^*$, $f'(y^\dagger) = \gamma$, $f'(z) > \gamma$ if $z \in [0, y^\dagger)$ and $f'(z) < \gamma$ if $z > y^\dagger$. We consider two cases.

Case 1: $\xi \geq y^*$ and $z > y^\dagger$, $z \neq \xi$.

In this case,

$$|f(z) - f(\xi)| = \left|\int_\xi^z f'(s)ds\right| < \gamma|z-\xi|.$$

Case 2: $\xi \geq y^*$ and $z \in (0, y^\dagger]$.

Note that, by case 1, $|f(y^*) - f(\xi)| \leq \gamma|y^* - \xi|$ and thus,

$$|f(z) - f(\xi)| \leq |f(z) - f(y^*)| + \gamma|y^* - \xi| = \gamma|y^* - \xi| + \gamma y^* - f(z).$$

Now

$$f(z) = \int_0^z f'(s)ds > \gamma z,$$

and we conclude that

$$|f(z) - f(\xi)| < \gamma|\xi - y^*| + \gamma(y^* - z) = \gamma|z - \xi|.$$

In both cases we have

$$|f(z) - f(\xi)| < \gamma|z - \xi|,$$

completing the proof. □

Chapter 5

Stability of Nonnegative Lur'e Systems in Discrete Time

This chapter acts as a discrete time counterpart to Chapter 3.

5.1 Introduction

Lur'e systems are common nonlinear feedback systems in mathematical control theory and comprises of a linear system with state x, input u and output y, given by

$$x(t+1) = Ax(t) + bu(t), \qquad x(0) = x^0 \in \mathbb{R}^n_+, \qquad y(t) = c^T x(t) \tag{5.1}$$

and a nonlinear feedback $u = f(y)$. This system can be written in closed loop form as

$$x(t+1) = Ax(t) + bf(c^T x(t)), \qquad x(0) = x^0 \in \mathbb{R}^n_+. \tag{5.2}$$

We restrict our attention to nonnegative systems, that is, systems in which the state x remains nonnegative for all time t.

Lur'e systems arise in various contexts, in particular in population dynamics such as [143]. In this application $x(t)$ describes the population structure at time t, A models linear transition rates such as survival or growth, and $bf(c^T)$ is a density dependent birth rate.

The main inspiration for this chapter is [143] in which a trichotomy of stability/instability is derived for nonnegative Lur'e systems. We develop this trichotomy more by implementing *absolute stability* theory. A common assumption that the nonlinearity f satisfies $f(0) = 0$ yields that 0 is an equilibrium of (5.2). The study of stability properties of the zero equilibrium of Lur'e systems is termed absolute stability and generally refers to the situation where the linear system (5.1) is known and the nonlinearity f is unknown, but usually sector bounded. Common nonlinearities used in population dynamic models are the Beverton-Holt nonlinearity [8] and the Ricker nonlinearity [117], which both satisfy $f(0) = 0$. The sector boundedness of these nonlinearities is considered in Section 2.4.2.

Although we gain a lot by implementing absolute stability theory such as exponential asymptotic stability and the ability to apply the results to a larger class of nonlinear systems, we do however loose the strict trichotomy from [143]. We organize the results in this chapter in a similar manner by considering three separate cases, instability, stability of the 0 equilibrium and stability of a nonzero equilibrium.

If the Lur'e system (5.2) is subject to an external additive time-dependence d, otherwise known as a forcing term, the system (5.2) can be replaced by

$$x(t+1) = Ax(t) + bf(c^T x(t)) + d(t), \qquad x(0) = x^0 \in \mathbb{R}^n_+, \tag{5.3}$$

where $d : \mathbb{N}_0 \to \mathbb{R}^n_+$. We adapt recent research [125] to show that under certain conditions this system is input-to-state stable (ISS). What this means is that the mapping $(x^0, d) \mapsto x(t)$ has nice boundedness and asymptotic properties. The study of these nonnegative, forced Lur'e systems provides an extension to the material in [143] which has biological interpretations such as migration and can account for model error.

We provide a descriptive example of how the theory developed in this chapter can be applied to population modeling and what the nonlinearity and disturbance could represent.

This chapter is organized as follows. Section 5.2 collects material on absolute stability and input-to-state material which are essential to the results in the remainder of this chapter. Section 5.3 builds on the concept of nonnegativity established in this introduction and provides assumption which guarantee that the system remains nonnegative, including some basic results as well as an example system which will be used throughout this chapter. Section 5.4 contains results in which we apply absolute stability results to nonnegative Lur'e systems. Section 5.5 contains results in which we apply input-to-state stability results to forced nonnegative Lur'e systems. Finally Section 5.6 contains an overview of one particular way we can apply the results from this chapter to a model the population of a specific species.

5.2 Stability of Nonnegative Lur'e Systems in Discrete Time

Consider the discrete time Lur'e system

$$x(t+1) = Ax(t) + bf(c^T x(t)), \qquad x(0) = x^0 \in \mathbb{R}^n, \tag{5.4}$$

where $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous with $f(0) = 0$. Let $x(\,\cdot\,, x^0)$ denote the solution of the system (5.4).

Let $\mathbb{S}(A, b, c^T)$ denote the set of complex stabilizing gains of the linear system (A, b, c^T), that is

$$\mathbb{S}(A, b, c^T) := \{\kappa \in \mathbb{C} \ : \ \rho(A + \kappa bc^T) < 1\}.$$

Define $\boldsymbol{G}$ to be the transfer function of the linear system (A, b, c^T), that is

$$\boldsymbol{G}(z) := c^T(zI - A)^{-1}b.$$

Let $k \in \mathbb{S}(A, b, c^T)$ and define $\boldsymbol{G}_k \in H^\infty$ by

$$\boldsymbol{G}_k(z) := c^T(zI - A - kbc^T)^{-1}b,$$

or equivalently,

$$\boldsymbol{G}_k(z) := \frac{\boldsymbol{G}(z)}{1 - k\boldsymbol{G}(z)}.$$

Define $r = 1/\|\boldsymbol{G}_k\|_{H^\infty}$, then, by stability radius theory,

$$\mathbb{D}(k, r) \subseteq \mathbb{S}(A, b, c^T),$$

where $\mathbb{D}(k, r)$ denotes a disc centered at k with radius r, that is,

$$\mathbb{D}(k, r) = \{\kappa \in \mathbb{R} \ : \ |k - \kappa| < r\}.$$

Let $\overline{\mathbb{D}}(k, r)$ denote the closed ball centered at k with radius r, that is,

$$\overline{\mathbb{D}}(k, r) = \{\kappa \in \mathbb{R} \ : \ |k - \kappa| \le r\}.$$

We now give precise definitions to three types of stability which we will be considering in this chapter.

Definition 5.2.1. *Consider the system* (5.4).

1. *The equilibrium* 0 *is said to be* stable in the large *in the sense that there*

exists exists $g \geq 1$ such that, for every $x_0 \in \mathbb{R}$,

$$\|x(t;x^0)\| \leq g\|x^0\| \qquad \forall \quad t \in \mathbb{N}_0.$$

2. *The equilibrium* 0 *is said to be* globally asymptotically stable *if* 0 *is stable in the large and for every $x^0 \in \mathbb{R}^n$, $x(t;x^0) \to 0$ as $t \to \infty$.*

3. *The equilibrium* 0 *is said to be* globally exponentially stable *if there exists $\gamma > 0$ and $g \geq 1$ such that, for every $x^0 \in \mathbb{R}^n$,*

$$\|x(t;x^0)\| \leq ge^{-\gamma t}\|x^0\| \qquad \forall \quad t \in \mathbb{N}_0.$$

The following is a key result for this chapter and is an Aizerman version of the circle criterion developed in [125].

Theorem 5.2.2. *Let $A \in \mathbb{R}^{n\times n}$ and $b, c \in \mathbb{R}^n$. Moreover, let $f : \mathbb{R} \to \mathbb{R}$ be continuous with $f(0) = 0$. Assume that $k \in \mathbb{S}(A, b, c^T)$, where $k \in \mathbb{R}$, and set*

$$r := \frac{1}{\|\boldsymbol{G}_k\|_{H^\infty}}.$$

Further assume that at least one of the following assumptions holds true:

- *There exists z_0 with $|z_0| = 1$ such that*

$$r|\boldsymbol{G}_k(z_0)| < 1.$$

- *The linear triple (A, b, c^T) is controllable and observable.*

Then the following statements hold.

(1) If

$$\frac{f(y)}{y} \subseteq \overline{\mathbb{D}}(k,r), \qquad \forall \quad y \in \mathbb{R}\backslash\{0\},$$

then there exists $g \geq 1$ such that

$$\|x(t;x^0)\| \leq g\|x^0\|, \qquad \forall \quad t \in \mathbb{N}_0, \quad \forall \quad x^0 \in \mathbb{R}^n.$$

In particular, the equilibrium 0 *of* (5.4) *is said to be stable in the large.*

(2) If

$$\frac{f(y)}{y} \subseteq \mathbb{D}(k,r), \qquad \forall \quad y \in \mathbb{R}\backslash\{0\},$$

then the equilibrium 0 *of* (5.4) *is globally asymptotically stable in the sense that* 0 *is stable in the large and, for all $x^0 \in \mathbb{R}^n$, $x(t;x^0) \to 0$ as $t \to \infty$.*

(3) If there exists $r_1 \in (0, r)$ such that

$$\frac{f(y)}{y} \subset \mathbb{D}(k, r_1), \qquad \forall \quad y \in \mathbb{R}\backslash\{0\},$$

then the equilibrium 0 of (5.4) is globally exponentially stable, that is, there exists $\gamma > 0$ and $g \geq 1$ such that

$$\|x(t; x^0)\| \leq g e^{-\gamma t} \|x^0\|, \qquad \forall \quad t \in \mathbb{N}_0, \quad \forall \quad x^0 \in \mathbb{R}^n.$$

Now consider the forced discrete time Lur'e system of the form

$$x(t+1) = Ax(t) + bf(c^T x(t)) + d(t), \qquad x(0) = x^0 \in \mathbb{R}^n, \qquad (5.5)$$

where $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, $f : \mathbb{R} \to \mathbb{R}$ is continuous with $f(0) = 0$ and $d : \mathbb{N}_0 \to \mathbb{R}^n$. The forcing term d is also known as a disturbance or input. Let $x(\,\cdot\,; x^0, d)$ denote the solution of the system (5.5).

Definition 5.2.3. *Let $d : \mathbb{N}_0 \to \mathbb{R}^n$. Define*

$$\|d\|_t := \max\{\|d(\tau)\|_1 \ : \ \tau \in \{0, 1, \ldots, t\}\}.$$

The following theorem is the second key result which we will be using throughout this chapter and also comes from [125].

Theorem 5.2.4. *Let $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$ and let $f : \mathbb{R} \to \mathbb{R}$ be continuous with $f(0) = 0$. Let $d : \mathbb{N}_0 \to \mathbb{R}^n$. Assume that $k \in \mathbb{S}(A, b, c^T)$, where $k \in \mathbb{R}$, and set*

$$r := \frac{1}{\|\boldsymbol{G}_k\|_{H^\infty}},$$

where $\boldsymbol{G}_k(z) = c^T(zI - A - kbc^T)^{-1}b$. Therefore,

$$\mathbb{D}(k, r) \subseteq \mathbb{S}(A, b, c^T).$$

Assume that

$$r|y| - |f(y)| \to \infty \qquad \text{as} \qquad |y| \to \infty.$$

Further assume that at least one of the following assumptions holds true:

- *There exists z_0 with $|z_0| = 1$ such that*

$$r|\boldsymbol{G}_k(z_0)| < 1.$$

- *The linear triple (A, b, c^T) is controllable and observable.*

Then there exists $\psi \in \mathcal{KL}_D$ *and* $\varphi \in \mathcal{K}$ *such that, for all* $x^0 \in \mathbb{R}^n$ *and all* $d : \mathbb{N}_0 \to \mathbb{R}^n$,

$$\|x(t; x^0, d)\| \leq \psi(\|x^0\|, t) + \varphi(\|d\|_t) \qquad \forall \quad t \in \mathbb{N}_0. \tag{5.6}$$

Obviously, if $d = 0$, then 0 is an equilibrium of (5.4). If there exists $\psi \in \mathcal{KL}_D$ and $\varphi \in \mathcal{K}$ such that (5.6) holds for all $x^0 \in \mathbb{R}^n$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n$, then the equilibrium 0 of the system (5.5) is said to be *input-to-state stable* (ISS).

5.3 Nonnegative Lur'e Systems in Discrete Time

In this section we introduce assumptions which ensure that the state $x(t)$ of a Lur'e system given by (5.4) remains nonnegative for all $t \in \mathbb{R}_+$. We then make a series of remarks and lemmas about these nonnegative Lur'e systems. To conclude this section we introduce a nonnegative Lur'e system which will be used as an example throughout this chapter and show that it satisfies the assumptions which have been introduced.

We first make a trivial remark.

Remark 5.3.1. *Consider the system* (5.4)*. If* $f(0) = 0$*, then* 0 *is an equilibrium of the system.*

We proceed to introduce assumptions which will be used throughout this chapter.

(A5.1) The matrix A is nonnegative and the vectors b and c are nonnegative and nonzero.

(A5.2) The matrix A is stable, that is $\rho(A) < 1$.

(A5.3) The matrix $A + bc^T$ is primitive.

(A5.4) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous.

We note that these assumptions are very similar to **(A3.1)**-**(A3.4)** appearing in Chapter 3. **(A5.1)** is a nonnegativity result on the linear system where A is now nonnegative as opposed to Metzler. **(A5.2)** is a stability condition on the matrix A which is now discrete time stability instead of Hurwitz. **(A5.3)** is a primitivity assumption which replaces the irreducibility assumption **(A3.3)** as in Chapter 3, the matrix $A + bc^T$ was not nonnegative. Finally **(A5.4)** is a nonnegativity assumption on the nonlinearity and includes a constraint on the smoothness of f.

A series of remarks and lemmas based on a system (5.4) satisfying these assumptions is made. The first remark is about the system (5.4) being non-negative.

Remark 5.3.2. *If* **(A5.1)** *and* **(A5.4)** *hold, then, for every* $x^0 \in \mathbb{R}^n_+$*, the solution* $x(\,\cdot\,, x^0)$ *of* (5.4) *satisfies* $x(t; x^0) \in \mathbb{R}^n_+$ *for all* $t \in \mathbb{N}_0$*.*

The next remark is about primitivity.

Remark 5.3.3. *If* **(A5.3)** *is satisfied, then* $A + \kappa bc^T$ *is primitive for all* $\kappa > 0$*.*

The following remark plays an important role in this chapter. It demonstrates the nonnegativity of the steady-state gain of the linear system (A, b, c^T) and relates it to the H^∞-norm, under certain assumptions.

Lemma 5.3.4. *Assume* **(A5.1)**-**(A5.3)** *hold. Then*

$$\|\boldsymbol{G}\|_{H^\infty} = \boldsymbol{G}(1) > 0.$$

Proof. We begin by demonstrating that $\boldsymbol{G}(1) > 0$. By **(A5.1)** we have that $c^T A^j b \geq 0$ for all j. Noting that

$$\boldsymbol{G}(1) = c^T (I - A)^{-1} b = \sum_{j=0}^{\infty} c^T A^j b, \tag{5.7}$$

by **(A5.2)**, it is sufficient to show that $c^T A^j b > 0$ for some j.

To this end, note that by **(A5.3)**, $A + bc^T$ is primitive, therefore $(A + bc^T)^k \gg 0$ for some $k \in \mathbb{N}$. Combining this with **(A5.1)**, we obtain $c^T (A + bc^T)^k b > 0$. Now,

$$0 < c^T (A + bc^T)^k b = \sum_{i=1}^{2^k} s_i,$$

where $s_i = c^T A^{j_i} b \sigma_i$, for suitable $\sigma_i \geq 0$ and $0 \leq j_i \leq k$. Consequently, $c^T A^{j_i} b > 0$ for some i between 1 and 2^k. Therefore, invoking (5.7), $\boldsymbol{G}(1) > 0$.

Now we proceed to show that $\|\boldsymbol{G}\|_{H^\infty} = \boldsymbol{G}(1)$. By definition,

$$\|\boldsymbol{G}\|_{H^\infty} = \sup_{|z|>1} |\boldsymbol{G}(z)| = \sup_{|z|=1} |\boldsymbol{G}(z)|.$$

For $|z| \geq 1$,

$$\boldsymbol{G}(z) = c^T (zI - A)^{-1} b = \sum_{j=0}^{\infty} \frac{1}{z} c^T \left(\frac{1}{z^j} A^j \right) b,$$

and so, for $|z| = 1$,

$$|\boldsymbol{G}(z)| \le \sum_{j=0}^{\infty} |c^T A^j b| = \sum_{j=0}^{\infty} c^T A^j b = \boldsymbol{G}(1),$$

which completes the proof. $\square$

We now define a quantity which will be key to all of the results which we shall be considering in this chapter.

Definition 5.3.5. *Define $p \in \mathbb{R}$ to be the inverse of the steady-state gain of (A, b, c^T), that is*

$$p := \frac{1}{\boldsymbol{G}(1)} = \frac{1}{c^T(I - A)^{-1}b}.$$

Lemma 5.3.6. *Assume that* **(A5.1)-(A5.3)** *hold and let $q > p$. Then*

$$1 = \rho(A + pbc^T) < \rho(A + qbc^T). \tag{5.8}$$

Proof. We begin by showing $\rho(A + pbc^T) < \rho(A + qbc^T)$ for $p < q$. Noting that $A + pbc^T < A + qbc^T$ and the fact that primitivity implies irreducibility by Definition 2.1.11, it follows from Corollary 2.1.24 that $\rho(A+pbc^T) < \rho(A+qbc^T)$.

Now we shall show that $1 = \rho(A + pbc^T)$. We have that $p = 1/\|\boldsymbol{G}\|_{H^\infty}$ by Lemma 5.3.4, and, by a stability radius result for nonnegative systems (see [63, Theorem 3.4]), p is a destabilizing perturbation of minimal modulus, implying that $\rho(A + pbc^T) = 1$. $\square$

Throughout this chapter we will illustrate main results by simulation. For simplicity the linear system will remain unchanged with just the nonlinearity varying to fit the assumptions of the theorem which we are illustrating. We introduce an extra assumption, which not necessarily linked to nonnegative Lur'e systems, will be required for a lot of the results.

(A5.5) At least one of the following statements hold.

- There exists z_0 with $|z_0| = 1$ such that $p|\boldsymbol{G}(z_0)| < 1$.
- (A, b, c^T) is controllable and observable.

We now introduce our example system and verify that **(A5.1)-(A5.3)** and **(A5.5)** are satisfied.

Example 5.3.7. *Consider the Lur'e system* (5.4) *with the following choice of linear system,*

$$A = \begin{pmatrix} 0.75 & 0 & 0 \\ 0.1 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \tag{5.9}$$

We begin by noting that A, b, c are all nonnegative and nonzero, therefore **(A5.1)** *hold. By simple calculation it can be shown that $\rho(A) = 0.75$, therefore* **(A5.2)** *is satisfied. Now*

$$A + bc^T = \begin{pmatrix} 0.75 & 1 & 1 \\ 0.1 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{pmatrix},$$

therefore, by the results in Section 2.1.2, $A+bc^T$ is primitive, therefore **(A5.3)** *holds.*

To demonstrate that **(A5.5)** *holds we demonstrate that (A, b, c^T) is controllable and observable. Beginning with controllable, note that the controllability matrix of the system is*

$$\mathcal{C} = \begin{pmatrix} b & Ab & A^2 b \end{pmatrix} = \begin{pmatrix} 1 & 0.75 & 0.5625 \\ 0 & 0.1 & 0.125 \\ 0 & 0 & 0.05 \end{pmatrix}.$$

This is of full rank, therefore (A, b) is controllable. To show that (A, c^T) is observable, we consider the observability matrix of the system is

$$\mathcal{O} = \begin{pmatrix} c^T \\ c^T A \\ c^T A^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0.1 & 1 & 0.5 \\ 0.175 & 0.75 & 0.25 \end{pmatrix}.$$

Clearly this is of full rank, therefore (A, c^T) is observable. Combing the above, (A, b, c^T) is controllable and observable, therefore, **(A5.5)** *holds.*

We end this example noting that $p = 0.625$.

5.4 Absolute Stability of Nonnegative Lur'e Systems in Discrete Time

We divide this section into three parts. The first considers the case where we do not have stability and all solutions diverge to $+\infty$. The second considers systems with a unique equilibrium at 0 which is stable. The final part considers systems with two equilibria, 0 and $x^* \neq 0$, where we demonstrate that under certain assumptions, x^* is stable.

5.4.1 Systems Without Stable Equilibria

In this section we consider systems of the form (5.4) which lack a stable equilibria. This occurs when $\inf_{y>0} f(y)/y > p$, as illustrated in Figure 5.1. In this case we will see that all nonzero $x^0 \in \mathbb{R}_+^n$ leads to $x(t; x^0)$ diverging to $+\infty$ in all components as $t \to \infty$.

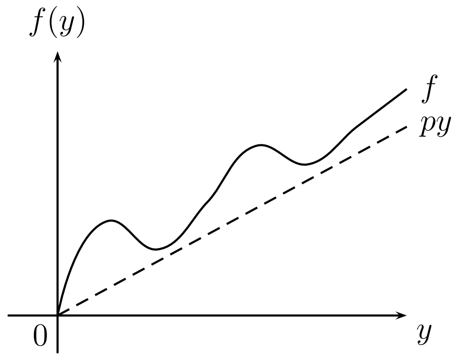


Figure 5.1: A graph of a function f satisfying **(A5.4)** and $\inf_{y>0} f(y)/y > p$.

Theorem 5.4.1. *Consider the system* (5.4) *and assume* **(A5.1)-(A5.4)** *hold. If*

$$\inf_{y>0} \frac{f(y)}{y} > p,$$

then for all $x^0 \in \mathbb{R}_+$ *with* $x^0 \neq 0$*,*

$$\lim_{t\to\infty} x_i(t; x^0) = \infty, \qquad \forall \quad i \in \{1, \ldots, n\},$$

where $x_i(t; x^0)$ *denotes the* i*-th component of* $x(t; x^0)$*.*

Proof. Let $x^0 \in \mathbb{R}_+^n$ with $x^0 \neq 0$. By hypothesis on f, there exists $q > p$ such that

$$f(y) \geq qy \qquad \forall \quad y \in \mathbb{R}_+.$$

Therefore,

$$\begin{aligned} x(t+1; x^0) &= Ax(t; x^0) + bf(c^T x(t; x^0)) \geq Ax(t; x^0) + bqc^T x(t; x^0) \\ &= (A + qbc^T)x(t; x^0), \qquad \forall \quad t \in \mathbb{N}_0, \end{aligned}$$

and so, $x(t;x^0) \geq (A+qbc^T)^t x^0$. By Lemma 5.3.6, $r := \rho(A+qbc^T) > 1$. Now

$$r^{-t}(A+qbc^T)^t \to \frac{vw^T}{w^Tv} \gg 0 \qquad \text{as} \quad t \to \infty,$$

where v and w are the left and right Perron vectors of $A+qbc^T$, respectively (see Theorem 2.1.23). Thus

$$\liminf_{t\to\infty} r^{-t} x_i(t;x^0) \geq \xi_i,$$

for $i = 1, \ldots, n$, where ξ_i is the i-th component of

$$\xi := \frac{vw^T}{w^Tv} x^0 \gg 0.$$

Consequently

$$\lim_{t\to\infty} x_i(t;x^0) = \infty,$$

for every $i = 1, \ldots, n$. □

Example 5.4.2. *Recall the system from Example 5.3.7 and let* $f(y) = 0.75y + \sin(y/3)$. *Note that* **(A5.4)** *holds for this nonlinearity. Clearly,*

$$\inf_{y>0} \frac{f(y)}{y} > p,$$

which can be seen in Figure 5.2(a), therefore all of the assumptions of Theorem 5.4.1 apply, therefore, for all initial condition $x^0 \in \mathbb{R}^n_+$ *with* $x^0 \neq 0$, $\lim_{t\to\infty} x_i(t,x^0) = \infty$ *for* $i = 1, 2, 3$, *where* $x_i(t;x^0)$ *is the* i*-th component of* $x(t;x^0)$. *This is illustrated in Figure 5.2(b), where* x^0 *was randomly chosen. Although the divergence is slow, it is the case that* $x(t;x^0)$ *is diverging to* $+\infty$ *in all three components.*

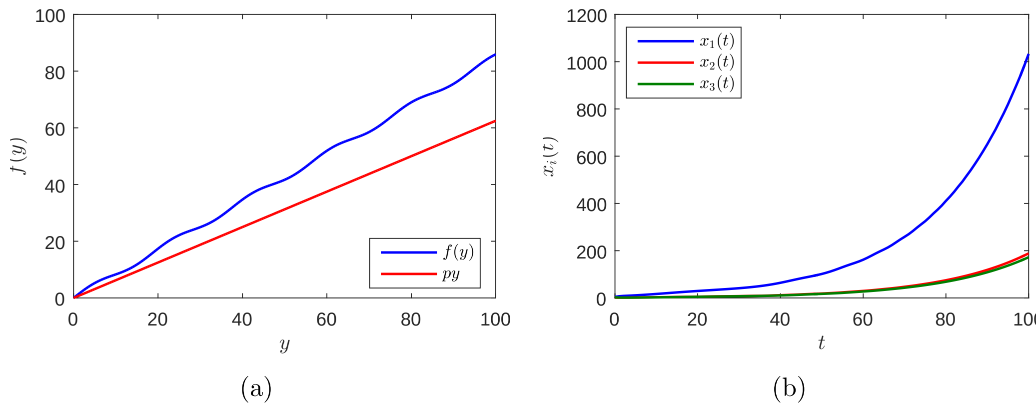


Figure 5.2: Numerical simulations for Example 5.4.2. (a) A plot of the nonlinearity and the line py. (b) Time history of $x(t)$.

5.4.2 Systems With A Unique Stable Equilibrium

In this section we consider systems which have a single equilibrium, which exhibits somewhat nice stability properties. This unique equilibrium is $x = 0$ and occurs when inequalities of the form $f(y)/y \leq p$ hold for all $y > 0$.

Theorem 5.4.3. *Consider the system* (5.4) *and assume* **(A5.1)**-**(A5.5)** *hold.*

(1) If $f(y)/y \leq p$ for all $y > 0$, then the equilibrium 0 is stable in the large in the sense that there exists $g \geq 1$ such that, for every $x^0 \in \mathbb{R}^n_+$,

$$\|x(t; x^0)\| \leq g\|x^0\| \qquad \forall \quad t \in \mathbb{N}_0.$$

(2) If $f(y)/y < p$ for all $y > 0$, then the equilibrium 0 is globally asymptotically stable in the sense that 0 is stable in the large and that for every $x^0 \in \mathbb{R}^n_+$, $x(t; x^0) \to 0$ as $t \to \infty$.

(3) If $\sup_{y>0} f(y)/y < p$, then the equilibrium 0 is globally exponentially stable in the sense that there exists $\gamma > 0$ and $g \geq 1$ such that, for every $x^0 \in \mathbb{R}^n_+$,

$$\|x(t; x^0)\| \leq g e^{-\gamma t}\|x^0\| \qquad \forall \quad t \in \mathbb{N}_0.$$

Proof. By Lemma 5.3.4, $p = 1/\|\boldsymbol{G}\|_{H^\infty}$ and therefore $\mathbb{D}(0, p) \subseteq \mathbb{S}(A, b, c^T)$. Aiming to apply Theorem 5.2.2, define an extension $\tilde{f} : \mathbb{R} \to \mathbb{R}$ of f by

$$\tilde{f}(y) = \begin{cases} f(y) & \text{for } y > 0 \\ 0 & \text{for } y \leq 0. \end{cases} \tag{5.10}$$

By Remark 5.3.2 we have that for every $x^0 \in \mathbb{R}^n_+$, $x(t; x^0) \in \mathbb{R}^n_+$ for all $t \in \mathbb{N}_0$. Therefore, for every $x^0 \in \mathbb{R}^n_+$, $x(t; x^0)$ is also the solution of

$$x(t + 1) = Ax(t) + b\tilde{f}(c^T x(t)), \qquad x(0) = x^0. \tag{5.11}$$

To prove statement (1), assume $f(y)/y \leq p$ for all $y > 0$. Trivially,

$$\frac{\tilde{f}(y)}{y} \subseteq \overline{\mathbb{D}}(0, p) \qquad \forall \quad y \in \mathbb{R}, \ y \neq 0.$$

Application of statement (1) of Theorem 5.2.2 yields the existence of $g \geq 1$ such that, for all $x^0 \in \mathbb{R}^n_+$,

$$\|x(t; x^0)\| \leq g\|x^0\| \qquad \forall \quad t \in \mathbb{N}_0.$$

To prove statement (2), assume $f(y)/y < p$ for all $y > 0$. Now

$$\frac{\tilde{f}(y)}{y} \subseteq \mathbb{D}(0, p) \qquad \forall \quad y \in \mathbb{R},\ y \neq 0.$$

Application of statement (2) of Theorem 5.2.2 to system (5.11) implies that $x(t; x^0) \to 0$ as $t \to \infty$ for all $x^0 \in \mathbb{R}^n_+$.

To prove statement (3), assume $\sup_{y>0} f(y)/y < p$. Therefore

$$\frac{\tilde{f}(y)}{y} \subseteq \mathbb{D}(0, p_1) \qquad \forall \quad y \in \mathbb{R},\ y \neq 0,$$

where $p_1 = \sup_{y>0} f(y)/y$. Application of statement (3) of Theorem 5.2.2 to the system (5.11) yields the existence of $\gamma > 0$ and $g \geq 1$ such that, for all $x^0 \in \mathbb{R}^n_+$,

$$\|x(t; x^0)\| \leq g e^{-\gamma t} \|x^0\| \qquad \forall \quad t \in \mathbb{N}_0.$$

This completes the proof. □

See Section 2.4.1 for a comparison of the different conditions on f appearing in Theorem 5.4.3.

Example 5.4.4. *We return to the Lur'e system given in Example 5.3.7. We consider two different nonlinearities, the first of which is*

$$f_1(y) = \frac{5y}{8 + y}, \qquad y \geq 0.$$

We begin by noting that **(A5.4)** *holds. Now note that as $y \to 0$, $f(y) \to 5/8 = 0.625 = p$, and for all $y > 0$, $f(y)/y < p$, as seen in Figure 5.3(a), therefore the conditions of statement (2) of Theorem 5.4.3 hold, therefore 0 is stable in the large and, for every $x^0 \in \mathbb{R}_+$, $x(t; x^0) \to 0$ as $t \to \infty$. This can be seen in Figure 5.3(b) where x^0 is randomly chosen.*

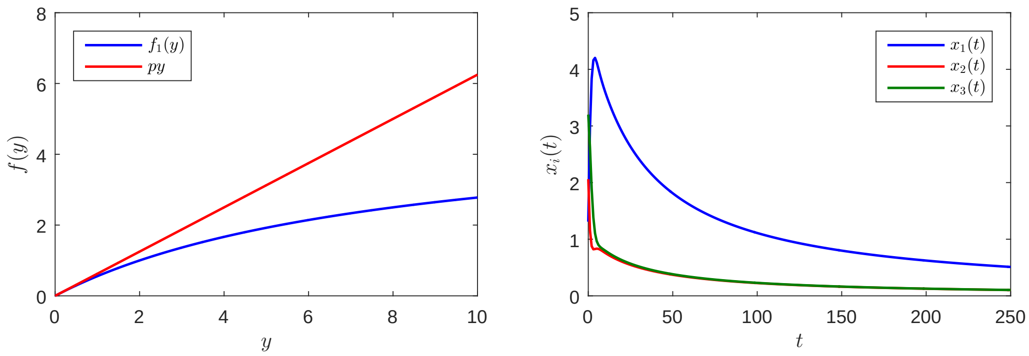
Now consider the same system, now with nonlinearity given by

$$f_2(y) = \frac{5y}{16 + y}, \qquad y \geq 0.$$

Again this nonlinearity satisfies **(A5.4)**. *However we now have that*

$$\sup_{y>0} \frac{f(y)}{y} < p,$$

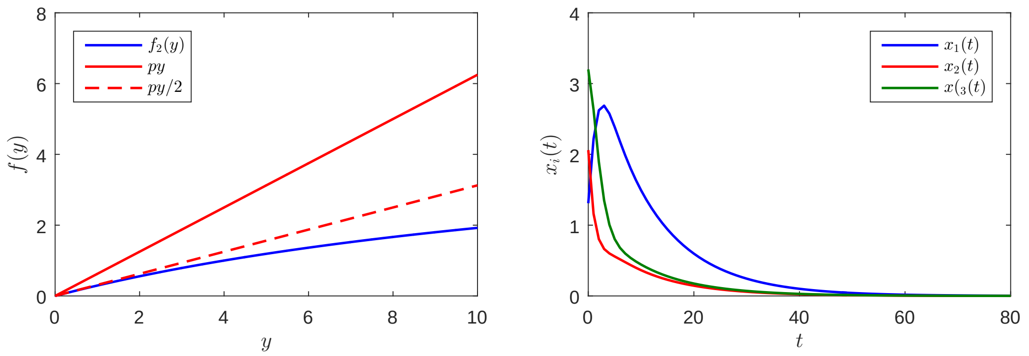
in fact we have $f(y)/y < 2p$ for all $y > 0$. This can be seen in Figure 5.4(a) We can therefore apply statement (3) of Theorem 5.4.3 to this system which yields that 0 is globally exponentially stable, in the sense that there exists $\gamma > 0$



(a) (b)

Figure 5.3: Numerical simulations for Example 5.4.4 with nonlinearity $f_1(y) = 5y/(8+y)$. (a) A plot of the nonlinearity and the line py. (b) Time history of $x(t)$.

and $g \geq 1$ *such that, for every* $x^0 \in \mathbb{R}^n_+$, $\|x(t; x^0)\| \leq g e^{-\gamma t} \|x^0\|$, *for all* $t \in \mathbb{N}_0$. *This exponential stability is illustrated in Figure 5.4(b) where* x^0 *is randomly chosen.*



(a) (b)

Figure 5.4: Numerical simulations for Example 5.4.4 with nonlinearity $f_2(y) = 5y/(16+y)$. (a) A plot of the nonlinearity and the lines py and $py/2$. (b) Time history of $x(t)$.

The difference between asymptotic stability and exponential stability becomes very clear when considering Figures 5.3(b) and 5.4(b). In the first figure we has asymptotic stability and the three components can be seen converging to 0*. In the second we have exponential stability, in which case the three components converge to* 0 *at a much faster rate. In fact it takes just* 100 *time steps to converge to* 0 *within a small tolerance, where the asymptotic stability case still has not converged in* 250 *time steps with the same tolerance.*

5.4.3 Systems With Two Equilibria

In this section we consider systems with two equilibria, namely 0 and $x^* \neq 0$. We begin this section by introducing some more assumptions.

(A5.6) There exists $y^* > 0$ such that $f(y^*) = py^*$.

(A5.7) f satisfies

$$\left|\frac{f(y) - f(y^*)}{y - y^*}\right| \leq p \qquad \forall \quad y \geq 0,\ y \neq y^*.$$

(A5.8) f satisfies

$$\left|\frac{f(y) - f(y^*)}{y - y^*}\right| < p, \qquad \forall \quad y > 0,\ y \neq y^*.$$

(A5.9) f satisfies

$$\limsup_{y \to y^*} \left|\frac{f(y) - f(y^*)}{y - y^*}\right| < p.$$

We firslty draw the readers attention to how these assumptions are the same as **(A3.5)**-**(A3.8)** in Chapter 3. Assumptions **(A5.7)** and **(A5.8)** are sector conditions in the sense that they are equivalent to the graph of f begin sandwiched between the straight lines $l_1(y) = py$ and $l_2(y) = 2py^* - py$. Assumption **(A5.9)** means that the graph of $f(y)$ does not cross the lines py and $2py^* - py$ tangentially.

Further details of these sector conditions are given in Section 2.4.2.

As mentioned previously, we are dealing with systems with two equilibria. A more precise meaning of this is given in the following lemma.

Lemma 5.4.5. *Assume that* **(A5.1)**-**(A5.4)** *and* **(A5.6)** *hold. Then* 0 *and* $x^* = (I - A)^{-1}bpy^* > 0$ *are equilibria of the system* (5.4). *If in addition* **(A5.8)** *holds, then there are no other equilibria in* $\mathbb{R}^n_+$.

Proof. By Remark 5.3.1, 0 is an equilibrium of (5.4). By **(A5.2)**,

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k \geq 0.$$

Now $bpy^* \geq 0$ and so $x^* \geq 0$. Obviously, $c^T x^* = \boldsymbol{G}(1)py^* = y^* > 0$ and so $x^* \neq 0$, therefore, $x^* > 0$.

Since

$$x^* = Ax^* + bpy^* = Ax^* + bf(y^*) = Ax^* + bf(c^T x^*),$$

we have that x^* is an equilibrium.

Now assume that **(A5.8)** holds and $x_e \in \mathbb{R}^n_+$ is an equilibrium of (5.4), that is $x_e = Ax_e + bf(c^T x_e)$. We show that either $x_e = 0$ or $x_e = x^*$. Firstly note that

$$x_e = (I - A)^{-1} bf(c^T x_e). \tag{5.12}$$

If $c^T x_e = 0$, then $x_e = 0$. Assume that $c^T x_e > 0$. Noting that

$$c^T x_e = c^T (I - A)^{-1} bf(c^T x_e) = \boldsymbol{G}(1) f(c^T x_e) = \frac{1}{p} f(c^T x_e),$$

it follows that

$$f(c^T x_e) - f(y^*) = p(c^T x_e - y^*).$$

Since $c^T x_e \neq 0$, we can invoke assumption **(A5.8)** to conclude that $c^T x_e = y^*$. It now follows from (5.12) that $x_e = x^*$. □

Theorem 5.4.6. *Consider the system* (5.4) *and assume that* **(A5.1)**-**(A5.7)** *hold. Then the equilibrium* $x^* = (I - A)^{-1} bpy^*$ *is stable in the large in the sense that there exists* $g \geq 1$ *such that, for every* $x^0 \in \mathbb{R}^n_+$,

$$\|x(t; x^0) - x^*\| \leq g \|x^0 - x^*\| \qquad \forall \quad t \in \mathbb{N}_0.$$

Proof. Let $\tilde{x}(t; x^0) := x(t; x^0) - x^*$ and

$$\tilde{f}(y) = \begin{cases} f(y + y^*) - f(y^*) & \text{for } y \geq -y^* \\ -f(y^*) & \text{for } y < -y^*. \end{cases} \tag{5.13}$$

See Figure 5.5 for a comparison of f and $\tilde{f}$.

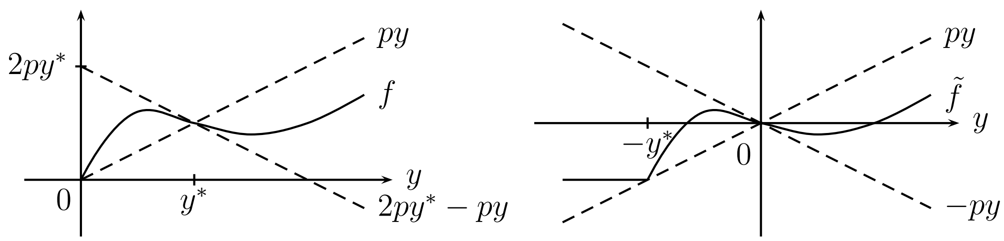


Figure 5.5: A comparison of a function f satisfying **(A5.4)**, **(A5.6)** and **(A5.7)** and the modified $\tilde{f}$ given by (5.13)

.

By **(A5.7)**

$$\left|\tilde{f}(y)\right| \leq p|y| \qquad \forall \quad y \in \mathbb{R}, \tag{5.14}$$

and $\tilde{f}(-y^*) = -f(y^*) = -py^*$. Moreover,

$$\begin{aligned}\tilde{x}(t+1;x^0) &= Ax(t;x^0) + bf(c^T x(t;x^0)) - x^* \\ &= A\tilde{x}(t;x^0) + (A-I)x^* + bf(c^T x(t;x^0)) \\ &= A\tilde{x}(t;x^0) - bf(y^*) + bf(c^T\tilde{x}(t;x^0) + y^*), \qquad (5.15)\end{aligned}$$

where (5.15) follows from

$$(A-I)x^* = (A-I)(I-A)^{-1}bf(y^*) = -bf(y^*),$$

and

$$bf(c^T x(t;x^0)) = bf(c^T(\tilde{x}(t;x^0) + x^*)) = bf(c^T\tilde{x}(t;x^0) + y^*).$$

Consequently, since $c^T\tilde{x}(t;x^0) \geq -y^*$,

$$\tilde{x}(t+1;x^0) = A\tilde{x}(t;x^0) + b\tilde{f}(c^T\tilde{x}(t;x^0)), \qquad \tilde{x}(0,x^0) = x^0 - x^* =: \tilde{x^0}. \quad (5.16)$$

Since $\mathbb{D}(0,p) \subseteq \mathbb{S}(A,b,c^T)$, and (5.14) holds, we can apply statement (1) of Theorem 5.2.2 to the system (5.16). Hence there exists a constant $g \geq 1$ such that for every $x^0 \in \mathbb{R}^n_+$,

$$\|\tilde{x}(t;x^0)\| \leq g\|\tilde{x^0}\|, \qquad \forall \quad t \in \mathbb{N}_0.$$

Reverting back to the original system we have

$$\|x(t;x^0) - x^*\| \leq g\|x^0 - x^*\|, \qquad \forall \quad t \in \mathbb{N}_0,$$

completing the proof. □

In Section 3.4.2, when considering the continuous time counterparts of the results in this section, to establish exponential stability we required a result which allowed us to bound $c^T x(t;x^0)$ away from zero for sufficiently large t, namely Proposition 3.4.14. The following counterexamples demonstrate that a discrete time counterpart of Proposition 3.4.14 does not hold true.

Example 5.4.7. *Consider the system* (5.4) *where*

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Firstly, note that **(A5.1)** *is clearly satisfied. The spectral radius of A is* 0*,* *therefore,* **(A5.2)** *is satisfied. Noting*

$$(A + bc^T)^5 = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \gg 0,$$

$A + bc^T$ *is primitive therefore* **(A5.3)** *is satisfied. Now*

$$\begin{aligned} \boldsymbol{G}(1) = c^T(I - A)^{-1}b &= \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = 2, \end{aligned}$$

and so $p = 1/2$.

The controllability matrix of (A, b, c^T) *is*

$$\mathcal{C} = \begin{pmatrix} b & Ab & A^2b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I,$$

which is of full rank thus (A, b) *is controllable. The observability matrix is*

$$\mathcal{O} = \begin{pmatrix} c^T \\ c^T A \\ c^T A^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which is also of full rank so (A, c^T) *is observable. Therefore* **(A5.5)** *is satisfied.*

Choose $f : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* f *is continuous,* $f(0) = 0$ *(thus* **(A5.4)** *is satisfied),* $f(1) = p = 1/2$,

$$\left| \frac{f(y) - f(1)}{y - 1} \right| < \frac{1}{2} \qquad \forall \quad y > 0, \ y \neq 1,$$

and $f(3) = 0$. *Note that* **(A5.5)** *holds with* $y^* = 1$.

Let $x^0 = (0, 0, 3)^T$. *Then*

$$x(1; x^0) = Ax^0 + bf(c^T x^0) = 0 + bf(3) = 0.$$

Hence,

$$x(2; x^0) = Ax(1; x^0) + bf(c^T x(1; x^0)) = 0,$$

and so on. We see that $x(t;x^0) \to 0$*, therefore,* $c^T x(t;x^0)$ *is not bounded away from zero for sufficiently large* t*.*

Now let f *satisfy the earlier conditions with the exception of* $f(3) = 0$*. We now assume that* $f(y) > 0$ *for all* $y > 0$ *and* $f(y) \to 0$ *as* $y \to \infty$*. Consider the system* (5.4) *with*

$$x^n = \begin{pmatrix} 0 \\ 0 \\ n \end{pmatrix}, \qquad n \in \mathbb{N}.$$

Since

$$x(1;x^n) = Ax^n + bf(c^T x^n) = bf(n),$$

we see that since

$$x(1;x^n) \to 0 \quad \text{as} \quad n \to \infty.$$

For given $\tau \in \mathbb{N}$*, we have*

$$x(\tau+1;x^n) = x(\tau;x(1;x^n)) \to 0 \quad \text{as} \quad n \to \infty$$

therefore,

$$c^T x(\tau+1;x^n) \to 0 \quad \text{as} \quad n \to \infty.$$

We have seen two examples of systems with two equilibria which satisfy **(A5.6)** and the sector condition **(A5.8)** and have $x(t;x^0) \to 0$ as $t \to \infty$. It is therefore clear that $c^T x(t;x^0)$ is not bounded away from zero for sufficiently large t, which in the continuous time case, would be assured. We therefore have to take a different approach to the one from Section 3.4.2.

In the following three lemmas we demonstrate that, under additional assumptions, we can bound $c^T x(t;x^0)$ away from 0 for sufficiently large t.

Lemma 5.4.8. *Consider the system* (5.4)*. Assume that* **(A5.1)**-**(A5.4)**, **(A5.6)** *and* **(A5.8)** *hold. Also assume that* $c^T A^\kappa \gg 0$ *for some* $\kappa \in \mathbb{N}_0$*. For every* $\varepsilon > 0$ *there exist* $\eta > 0$ *and* $\theta \in \mathbb{N}_0$ *such that for all* $x^0 \in \mathbb{R}^n_+$ *with* $\|x^0\| \geq \varepsilon$*,*

$$c^T x(t;x^0) \geq \eta \qquad \forall \quad t \in \mathbb{N}_0,\ t \geq \theta.$$

Proof. Let $\varepsilon > 0$. We begin by demonstrating that for some $\delta > 0$, $\|x(t;x^0)\| \geq \delta$ for all $t \in \mathbb{N}_0$, for all $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$.

As **(A5.1)**-**(A5.3)** hold, $p > 0$ by Lemma 5.3.4. Note that if $0 \leq y \leq y^*$, then $f(y) \geq py^*$ by **(A5.8)**. By **(A5.6)** and **(A5.8)** there exists $y^{**} > y^*$ such that $f(y) > 0$ for all $y \in [y^*, y^{**}]$. Set

$$\lambda := \inf_{y^* \leq y \leq y^{**}} \frac{f(y)}{y} > 0. \tag{5.17}$$

Define

$$\varepsilon_0 := \inf\{\|z\|_1 \ : \ c^T z \geq y^{**}\} > 0.$$

Set $y(t) := c^T x(t; x^0)$, where $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$.

If $y(t) < y^*$, then $f(y(t)) \geq py(t)$, and therefore,

$$v^T x(t+1; x^0) \geq v^T A(t; x^0) + v^T bpy(t) = v^T (A + pbc^T) x(t; x^0) = v^T x(t; x^0),$$

where v^T is the left Perron vector of $A + pbc^T$ associated with the dominant eigenvalue 1 (see Lemma 5.3.6).

If $y(t) \in [y^*, y^{**}]$, then $f(y(t)) \geq \lambda y(t)$, where λ is given by (5.17). Hence

$$v^T x(t+1; x^0) \geq v^T A(t; x^0) + v^T \lambda b y(t) \geq v^T \lambda b y^*.$$

If $y(t) > y^{**}$, then $\|x(t; x^0)\|_1 \geq \varepsilon_0$. Hence,

$$v^T x(t+1; x^0) = v^T A x(t; x^0) + v^T b f(y(t)) \geq v^T A x(t; x^0) \geq \min(v^T A)_i \varepsilon_0,$$

where $(v^T A)_i$ is the i-th component of the row vector $v^T A$. Noting that $v^T \gg 0$ and A has no zero columns, as $c^T A^\kappa \gg 0$ for some $\kappa \in \mathbb{N}$, it follows that $v^T A \gg 0$, thus $\min(v^T A)_i > 0$.

For all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$

$$v^T x(t+1; x^0) \geq \min(v^T x(t; x^0), v^T b \lambda y^*, \min(v^T A)_i \varepsilon_0), \qquad \forall \quad t \in \mathbb{N}_0.$$

Thus, for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$

$$v^T x(t; x^0) \geq \min(v^T x^0, v^T b \lambda y^*, \min(v^T A)_i \varepsilon_0) > 0 \qquad \forall \quad t \in \mathbb{N}_0.$$

Noting that $v^T \gg 0$, there exists $\delta > 0$ such that for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$

$$\|x(t; x^0)\|_1 \geq \delta, \qquad \forall \quad t \in \mathbb{N}_0.$$

Now, note that for all $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$

$$x(t+1; x^0) = A x(t; x^0) + b(f(c^T x(t; x^0))) \geq A x(t; x^0),$$

by **(A5.1)** and **(A5.4)**. For $k \in \mathbb{N}_0$ and all $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$,

$$x(t; x^0) \geq A^k x(t-k; x^0), \qquad \forall \quad t = k, k+1, \ldots.$$

Note that $c^T A^\kappa \gg 0$ for some $\kappa \in \mathbb{N}_0$. Let $k = \kappa$ and writing $w^T = (w_1, \ldots, w_n) = c^T A^k$, $w_i > 0$ for all $i = 1, \ldots, n$ and furthermore, for all

$x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$,

$$c^T x(t; x^0) \geq \min_{1 \leq i \leq n} (w_i)\delta > 0 \qquad \forall \quad t = k, k+1, \ldots.$$

This completes the proof with $\theta = k$ and $\eta = \min_{1 \leq i \leq n}(w_i)\delta$. □

There are many different structures of c^T and A such that $c^T A^\kappa \gg 0$ holds for some $\kappa \in \mathbb{N}$. All of these require that A has no zero columns. One of the simplest structures is if c has no zero entries, however this is a very strong assumption. Another way of ensuring that $c^T A^\kappa \gg 0$ for some $\kappa \in \mathbb{N}$ is if A is a primitive matrix. Again this is a strong assumption. There are examples such that A is not primitive and c has some zero components. One particular example is the linear system (5.9).

We proceed to consider the next lemma for bounding $x(t; x^0)$ away from 0 for sufficiently large t, this time by introducing a stronger assumption on the nonlinearity.

Lemma 5.4.9. *Consider the system* (5.4). *Assume that* **(A5.1)**-**(A5.4)**, **(A5.6)** *and* **(A5.8)** *hold. Also assume for some* $\alpha > 0$ *that* $f(y) \geq \alpha y$ *for all* $y \geq 0$. *For* $\varepsilon > 0$ *there exist* $\eta > 0$ *and* $\theta \in \mathbb{N}_0$ *such that for all* $x^0 \in \mathbb{R}^n_+$ *with* $\|x^0\| \geq \varepsilon$,

$$c^T x(t; x^0) \geq \eta \qquad \forall \quad t \in \mathbb{N}_0,\ t \geq \theta.$$

Proof. Let $\varepsilon > 0$. We begin by demonstrating that for some $\delta > 0$, $\|x(t; x^0)\| \geq \delta$ for all $t \in \mathbb{N}_0$, for all $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$.

As **(A5.1)**-**(A5.3)** hold, $p > 0$ by Lemma 5.3.4. Note that if $0 \leq y \leq y^*$, then $f(y) \geq py^*$ by **(A5.8)**. We also know that as $f(y) \geq \alpha y$ for all $y \geq 0$ that

$$\frac{f(y)}{y} \geq \alpha > 0 \qquad \forall \quad y \geq y^*.$$

Set $y(t) := c^T x(t; x^0)$, where $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$.

If $y(t) < y^*$, then for all $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$,

$$v^T x(t+1; x^0) \geq v^T A x(t; x^0) + v^T b p y(t) = v^T (A + pbc^T) x(t; x^0) = v^T x(t; x^0),$$

where v^T is the left Perron vector of $A + pbc^T$ associated with the dominant eigenvalue 1 (see Lemma 5.3.6).

If $y(t) \geq y^*$ then for all $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$,

$$v^T (t+1; x^0) \geq v^T A x(t; x^0) + v^T b \alpha y(t) \geq v^T \alpha b y^*.$$

Thus, for all $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$

$$v^T x(t+1; x^0) \geq \min(v^T x(t; x^0), v^T b\alpha y^*), \qquad \forall \quad t \in \mathbb{N}_0.$$

Therefore, for all $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$

$$v^T x(t; x^0) \geq \min(v^T x^0, v^T b\alpha y^*) > 0, \qquad \forall \quad t \in \mathbb{N}_0.$$

Noting that $v^T \gg 0$, there exists $\delta > 0$ such that for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$

$$\|x(t; x^0)\| \geq \delta > 0, \qquad \forall \quad t \in \mathbb{N}_0.$$

As $f(y) \geq \alpha y$ for all $y \geq 0$ where $\alpha > 0$, for all $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$

$$x(t+1; x^0) = Ax(t; x^0) + bf(c^T x(t; x^0)) \geq (A + \alpha bc^T)x(t; x^0).$$

Noting that $A + \alpha bc^T$ is a primitive matrix by **(A5.3)**, there exists $k \in \mathbb{N}$ such that $(A + \alpha bc^T)^k \gg 0$. Therefore, for all $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$

$$x(t; x^0) \geq (A + \alpha bc^T)^k x(t-k; x^0) \qquad \forall \quad t = k, k+1, \ldots,$$

thus,

$$c^T x(t; x^0) \geq c^T (A + \alpha bc^T)^k x(t-k; x^0) \qquad \forall \quad t = k, k+1, \ldots.$$

By **(A5.1)**, $c^T(A + \alpha bc^T)^k \gg 0$ for some $k \in \mathbb{N}$. Writing $w^T = (w_1, \ldots, w_n) = c^T(A + \alpha bc^T)^k$, $w_i > 0$ for all $i = 1, \ldots, n$ and furthermore, for all $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$,

$$c^T x(t; x^0) \geq \min_{1 \leq i \leq n}(w_i)\delta > 0, \qquad \forall \quad t = k, k+1, \ldots.$$

This completes the proof with $\theta = k$ and $\eta = \min_{1 \leq i \leq n}(w_i)\delta$. □

We have seen two lemmas which allow us to bound $c^T x(t; x^0)$ away from 0 for sufficiently large t, one adding an extra assumption to the linear system and the other adding one to the nonlinearity. Both of these are fairly strong assumptions to make, however there is a third case where we need only make a weaker additional assumption about the nonlinearity and impose restrictions on the size of the initial condition x^0.

Lemma 5.4.10. *Consider the system* (5.4)*. Assume that* **(A5.1)**-**(A5.6)** *and* **(A5.8)** *hold and assume $f(y) > 0$ for all $y > 0$. For every compact set $\Gamma \subset \mathbb{R}^n_+$*

with $0 \notin \Gamma$ *there exist* $\eta > 0$ *and* $\theta \in \mathbb{N}_0$ *such that for all* $x^0 \in \Gamma$*,*

$$c^T x(t; x^0) \geq \eta \qquad \forall \quad t \in \mathbb{N}_0,\ t \geq \theta.$$

Proof. Let $\Gamma \subseteq \mathbb{R}^n_+$ be a compact set with $0 \notin \Gamma$. We note that there exists $y^\# > 0$ such that

$$c^T x(t; x^0) \leq y^\# \qquad \forall \quad t \in \mathbb{N}_0, \quad \forall \quad x^0 \in \Gamma,$$

by Theorem 5.4.6. Let

$$\inf_{y^* \leq y \leq y^\#} \frac{f(y)}{y} = \lambda > 0,$$

noting that $f(y) > 0$ for all $y > 0$.

If $0 \leq y \leq y^*$ we have $f(y) \geq py$ by **(A5.8)**, thus for all $x^0 \in \Gamma$

$$v^T x(t+1; x^0) \geq v^T A(t; x^0) + v^T bpy(t) = v^T (A + pbc^T) x(t; x^0),$$

where v^T is the left Perron vector of $A + pbc^T$ associated with the dominant eigenvalue 1 (see Lemma 5.3.6).

If $y(t) \geq y^*$ then for all $x^0 \in \Gamma$

$$v^T x(t+1; x^0) \geq v^T A x(t; x^0) + v^T b\lambda y(t) \geq v^T b \lambda y^*.$$

Combining the above, for all $x^0 \in \Gamma$

$$v^T x(t+1; x^0) \geq \min(v^T x(t; x^0), v^T b \lambda y^*), \qquad \forall \quad t \in \mathbb{N}_0.$$

Therefore, for all $x^0 \in \Gamma$

$$v^T x(t; x^0) \geq \min(v^T x^0, v^T b \lambda y^*) > 0, \qquad \forall \quad t \in \mathbb{N}_0.$$

Noting that $v^T \gg 0$, there exists $\delta > 0$ such that for all $x^0 \in \Gamma$

$$\|x(t; x^0)\| \geq \delta > 0, \qquad \forall \quad t \in \mathbb{N}_0.$$

Setting $\mu = \min(\lambda, p) > 0$, we have for all $x^0 \in \Gamma$

$$x(t+1; x^0) \geq A x(t; x^0) + \mu b c^T x(t; x^0) = (A + \mu b c^T) x(t; x^0),$$

and for some $k \in \mathbb{N}_0$,

$$x(t; x^0) \geq (A + \mu b c^T)^k x(t-k; x^0), \qquad \forall \quad t = k, k+1, \ldots,$$

therefore,

$$c^T x(t; x^0) \geq c^T (A + \mu b c^T)^k x(t - k; x^0), \qquad \forall \quad t = k, k+1, \ldots.$$

Finally, noting $c^T(A + \mu bc^T)^k \gg 0$ for some $k \in \mathbb{N}_0$ by **(A5.1)** and **(A5.3)**, we can write $w^T = (w_1, \ldots, w_n) = c^T(A + \mu bc^T)^k$ we have $w_i > 0$ for all $i = 1, \ldots, n$ and furthermore, for all $x^0 \in \Gamma$

$$c^T x(t; x^0) \geq \min_{1 \leq i \leq n} (w_i) \delta > 0.$$

This completes the proof with $\theta = k$ and $\eta = \min_{1 \leq i \leq n}(w_i)$. □

We note that the assumption on the nonlinearity in Lemma 5.4.10 is weaker than that in Lemma 5.4.9, therefore when we proceed with the next result, Lemma 5.4.9 does not play a part.

Theorem 5.4.11. *Consider the system* (5.4). *Assume that* **(A5.1)**-**(A5.6)** *and* **(A5.8)** *hold. Further assume that one of the following also holds:*

- $c^T A^k \gg 0$ *for some* $k \in \mathbb{N}_0$.
- $f(y) > 0$ *for all* $y > 0$.

Then the equilibrium $x^* = (I - A)^{-1} bpy^*$ *is "globally" asymptotically stable in the sense that it is stable in the large, and for all* $x^0 \in \mathbb{R}^n_+$ *with* $x^0 \neq 0$, $x(t; x^0) \to x^*$ *as* $t \to \infty$.

Proof. Firstly note that the conditions of Theorem 5.4.6 are satisfied, therefore x^* is stable in the large. Let $x^0 \in \mathbb{R}^n_+$ with $x^0 \neq 0$.

Assume that $c^T A^k \gg 0$ for some $k \in \mathbb{N}_0$. By Lemma 5.4.8, there exists $\eta_1 > 0$ and $\theta_1 \in \mathbb{N}_0$ such that

$$c^T x(t; x^0) \geq \eta_1 \qquad \forall \quad t \in \mathbb{N}_0, \ t \geq \theta_1.$$

Now, alternatively assume that $f(y) > 0$ for all $y > 0$. Then by Lemma 5.4.10 with $\Gamma = \{x^0\}$, there exists $\eta_2 > 0$ and $\theta_2 \in \mathbb{N}_0$ such that

$$c^T x(t; x^0) \geq \eta_2 \qquad \forall \quad t \in \mathbb{N}_0, \ t \geq \theta_2.$$

Therefore for all $x^0 \in \mathbb{R}^0_+$ with $x^0 \neq 0$,

$$c^T x(t; x^0) \geq \eta \qquad \forall \quad t \in \mathbb{N}_0, \ t \geq \theta,$$

where $\eta = \min(\eta_1, \eta_2)$ and $\theta = \max(\theta_1, \theta_2)$.

Define

$$\tilde{x}(t) := x(t-\theta; x^0) - x^*,$$

and

$$\tilde{f}(y) := \begin{cases} f(y+y^*) - f(y^*) & \text{for } y \geq \eta - y^* \\ f(\eta) - f(y^*) & \text{for } y < \eta - y^*. \end{cases} \tag{5.18}$$

Now

$$\begin{aligned} \tilde{x}(t+1) = x(t+1; x^0) - x^* &= Ax(t; x^0) + bf(c^T x(t; x^0)) - x^* \\ &= A\tilde{x}(t) + (A - I)x^* + bf(c^T x(t; x^0)) \\ &= A\tilde{x}(t) - bf(y^*) + bf(c^T \tilde{x} + y^*) \\ &= A\tilde{x}(t) + b\tilde{f}(c^T \tilde{x}(t)), \qquad \tilde{x}(0) = x^0 - x^* \end{aligned} \tag{5.19}$$

where (5.19) follows from $c^T \tilde{x}(t) \geq \eta - y^*$ for all $t \in \mathbb{N}_0$. From **(A5.8)** it follows that

$$\left| \frac{\tilde{f}(y)}{y} \right| < p, \qquad \forall \quad y \in \mathbb{R} \backslash \{0\}.$$

We have that $\mathbb{D}(0, p) \subseteq \mathbb{S}(A, b, c^T)$. Applying statement (2) of Theorem 5.2.2 to (5.19) yields that $\lim_{t\to\infty} \tilde{x}(t) = 0$. Therefore, for an arbitrary $x^0 \in \mathbb{R}^n_+$ with $x^0 \neq 0$,

$$\lim_{t\to\infty} x(t; x^0) = x^*.$$

□

We illustrate Theorem 5.4.11 in the following two examples. The first uses the Lur'e system given in Example 5.3.7 for which $c^T A \gg 0$, and the second uses a different linear system such that $c^T A^k \not\gg 0$ for all $k \in \mathbb{N}_0$.

Example 5.4.12. *We return to the Lur'e system given in Example 5.3.7. Let* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *be given by*

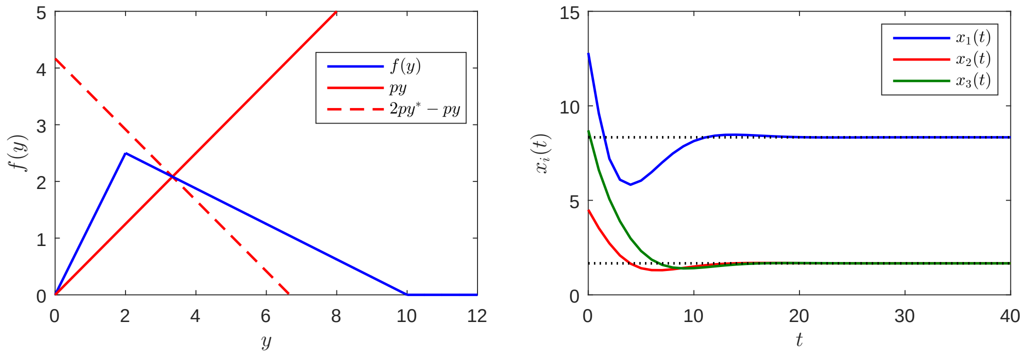
$$f(y) = \begin{cases} 2py & 0 \leq y \leq 2 \\ 0.5p(10 - y) & 2 \leq y \leq 10 \\ 0 & y > 10 \end{cases}$$

Note that **(A5.4)** *holds. A trivial calculation shows that* **(A5.6)** *holds for* $y^* = 10/3$ *and it can easily be seen that* **(A5.8)** *holds, which can be seen in Figure 5.6(b). Application of Theorem 5.4.11 yields that for all* $x^0 \in \mathbb{R}^n_+$ *with*

$x_0 \neq 0$,

$$x(t; x^0) \to x^* = (I - A)^{-1} bpy^* = \begin{pmatrix} 25/3 \\ 5/3 \\ 5/3 \end{pmatrix} \quad \text{as} \quad t \to \infty.$$

This is illustrated in Figure 5.6(b) for an arbitrary initial condition.



(a) (b)

Figure 5.6: Numerical simulations for Example 5.4.12. (a) A plot of the nonlinearity and the lines py and $2py^* - py$. (b) Time history of $x(t)$.

Example 5.4.13. *Consider the system*

$$x(t+1) = \begin{pmatrix} 0 & 0.5 \\ 0 & 0.1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} f\left(\begin{pmatrix} 0 & 1 \end{pmatrix} x(t)\right), \qquad x(0) = x^0$$
$$f(y) = \frac{9y}{2+y}.$$

This system is of the form of (5.4) *with*

$$A = \begin{pmatrix} 0 & 0.5 \\ 0 & 0.1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad c^T = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

We verify which assumptions this system satisfies. We begin by noting that A, b, c are all nonnegative and nonzero so **(A5.1)** *is satisfied. The eigenvalues of A are* 0 *and* 0.1*, therefore, A is stable and* **(A5.2)** *is satisfied. Noting*

$$A + bc^T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \gg 0,$$

we have that $A + bc^T$ is primitive, thus **(A5.3)** *holds. The nonlinearity $f(y)$ is continuous, and so* **(A5.4)** *holds. The controllability matrix of the system* *is*

$$\mathcal{C} = \begin{pmatrix} b & Ab \end{pmatrix} = \begin{pmatrix} 1 & 0.5 \\ 1 & 0.1 \end{pmatrix},$$

which is of full rank and the observability matrix is

$$\mathcal{O} = \begin{pmatrix} c^T \\ c^T A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix},$$

which is also of full rank, therefore (A, b, c^T) is controllable and observable, meaning **(A5.5)** *is satisfied.*

A simple calculation yields that $p = 0.9$ and $y^* = 8$ is the unique, nonzero solution of $py^* = f(y^*)$, thus **(A5.6)** *holds. Figure 5.7(a) contains a plot of $f(y)$ and the lines py and $2py^* - py$ and shows that* **(A5.8)** *holds. From this plot it is also obvious that $f(y) > 0$ for all $y > 0$.*

We finally note that $c^T A^k \not\gg 0$ for any $k \in \mathbb{N}_0$. To do this we simple calculate the first few vectors and observe the trend. Now

$$\begin{aligned} c^T &= (0, 1) \\ c^T A &= (0, 0.5) \\ c^T A^2 &= (0, 0.05) \\ c^T A^3 &= (0, 0.005). \end{aligned}$$

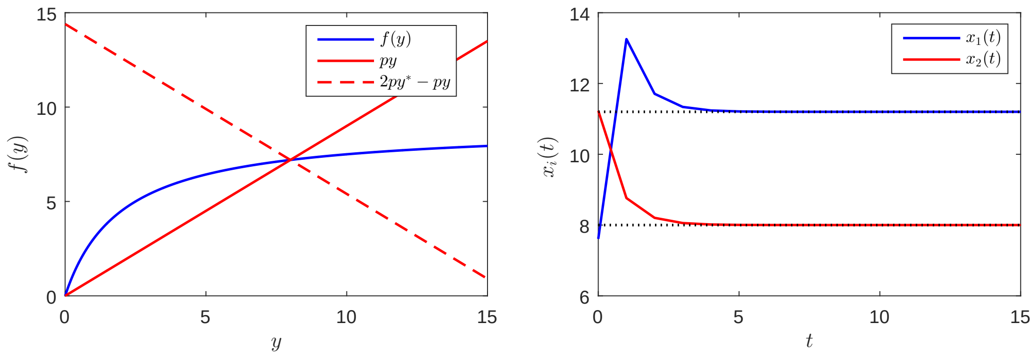
Clearly $c^T A^k = (0, 0.5 \times 0.1^{k-1})$, therefore, all of these vectors have a zero component in the first entry, and will continue to do so for all time.

We have seen that all of the assumptions required for Theorem 5.4.11 holds, however unlike the previous example, we now have that $f(y) > 0$ for all $y > 0$ instead of $c^T A^k \gg 0$ for some $k \in \mathbb{N}_0$. We therefore have that, for all $x^0 \in \mathbb{R}^2_+$, with $x^0 \neq 0$,

$$x(t; x^0) \to x^* = (I - A)^{-1} bpy^* = \begin{pmatrix} 11.2 \\ 8 \end{pmatrix} \quad \text{as} \quad t \to \infty.$$

This is illustrated in Figure 5.7(b) for an arbitrary nonzero initial condition.

We proceed to study exponential stability properties of $x^* \neq 0$. We do this in the following two theorems. The first of which involves the bound on $c^T x(t; x^0)$ from Lemmas 5.4.8 and 5.4.9, that is the additional assumption on the linear system is required or the strong condition on the nonlinearity. Using these lemmas allow us to formulate a quasi-global result. It is deemed a quasi-global result as we restrict $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon > 0$.



(a) (b)

Figure 5.7: Numerical simulations for Example 5.4.13. (a) A plot of the non-linearity and the lines py and $2py^* - py$. (b) Time history of $x(t)$.

Theorem 5.4.14. *Consider the system* (5.4)*. Assume that* **(A5.1)**-**(A5.6)**, **(A5.8)** *and* **(A5.9)** *hold. Also assume that one of the following holds:*

- $c^T A^k \gg 0$ *for some* $k \in \mathbb{N}_0$.
- *For some* $\alpha > 0$, $f(y) \geq \alpha y$ *for all* $y > 0$.

Then the equilibrium $x^* = (I - A)^{-1} b p y^*$ *is quasi-globally exponentially stable in the sense that, for every* $\varepsilon > 0$ *there exists constants* $\gamma > 0$ *and* $g \geq 1$ *such that, for all* $x^0 \in \mathbb{R}^n_+$ *with* $\|x^0\| \geq \varepsilon$,

$$\|x(t; x^0) - x^*\| \leq g e^{-\gamma t} \|x^0 - x^*\| \qquad \forall \quad t \in \mathbb{N}_0.$$

Proof. Let $\varepsilon > 0$. Assume that $c^T A^k \gg 0$ for some $k \in \mathbb{N}_0$. By Lemma 5.4.8 there exists constants $\eta_1 > 0$ and $\theta_1 \in \mathbb{N}_0$ such that, for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$,

$$c^T x(t; x^0) \geq \eta_1 \qquad \forall \quad t \in \mathbb{N}_0, \ t \geq \theta_1.$$

Now alternatively assume that for some $\alpha > 0$, $f(y) > \alpha y$ for all $y > 0$. Then by Lemma 5.4.9, there exists constants $\eta_2 > 0$ and $\theta_2 \in \mathbb{N}_0$ such that, for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$,

$$c^T x(t; x^0) \geq \eta_2 \qquad \forall \quad t \in \mathbb{N}_0, \ t \geq \theta_2.$$

Combining the above, for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$,

$$c^T x(t; x^0) \geq \eta \qquad \forall \quad t \in \mathbb{N}_0, \ t \geq \theta,$$

where $\eta = \min\{\eta_1, \eta_2\}$ and $\theta = \max\{\theta_1, \theta_2\}$.

Define

$$\tilde{x}(t) := x(t - \theta; x^0) - x^*,$$

and

$$\tilde{f}(y) : \begin{cases} f(y+y^*) - f(y^*) & \text{for} \quad y \geq \eta - y^* \\ f(\eta) - f(y^*) & \text{for} \quad y < \eta - y^*. \end{cases}$$

Now

$$\tilde{x}(t+1) = A\tilde{x} + b\tilde{f}(c^T\tilde{x}(t)), \qquad \tilde{x}(0) = x^0 - x^*$$

which follows as in the proof of Theorem 5.4.11. From **(A5.8)** and **(A5.9)** it follows that

$$\sup_{y \in \mathbb{R}\setminus\{0\}} \left| \frac{\tilde{f}(y)}{y} \right| < p. \tag{5.20}$$

We have that $\mathbb{D}(0,p) \subseteq \mathbb{S}(A,b,c^T)$. Applying statement (3) of Theorem 5.2.2 to (5.20) yields the existence of $\gamma > 0$ and $g \geq 1$ such that

$$\|\tilde{x}(t)\| \leq ge^{-\gamma t}\|\tilde{x}(0)\| \qquad \forall \quad t \in \mathbb{N}_0.$$

Therefore, returning to our original system we have

$$\|x(t;x^0) - x^*\| \leq fge^{-\gamma t}\|x^0 - x^*\|, \qquad \forall \quad t \in \mathbb{N}_0.$$

□

The system from Example 5.4.12 satisfies the assumptions of Theorem 5.4.14, therefore the equilibrium x^* is quasi-globally exponentially stable. The same is not true for the system in Example 5.4.13. We revisit this example.

Example 5.4.15. *Consider the system*

$$\begin{aligned} x(t+1) &= \begin{pmatrix} 0 & 0.5 \\ 0 & 0.1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} f\left(\begin{pmatrix} 0 & 1 \end{pmatrix} x(t)\right), \qquad x(0) = x^0 \\ f(y) &= \frac{9y}{2+y} + 0.4y. \end{aligned}$$

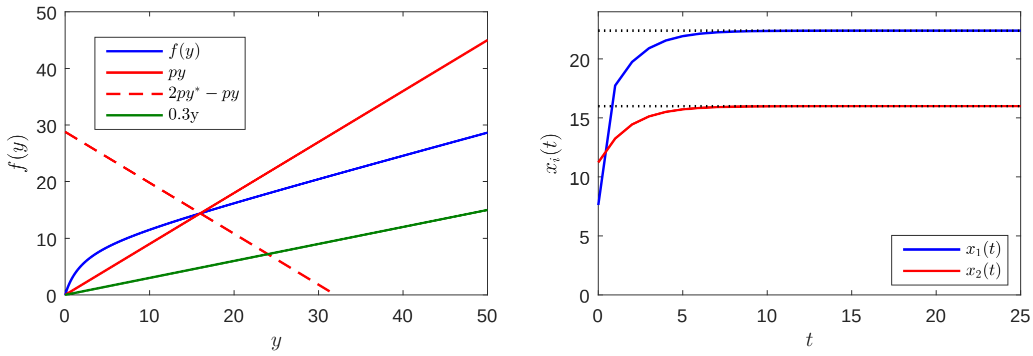
This is a similar system to that in Example 5.4.13 with a slightly modified nonlinearity. We therefore know **(A5.1)**-**(A5.3)** *and* **(A5.5)** *hold and that* $p = 0.9$.

Trivially, **(A5.4)** *holds and* $y^* = 16$ *is the unique nonzero solution of* $py^* = f(y^*)$*, this* **(A5.6)** *holds. Figure 5.8(a) contains a plot of* $f(y)$*, the lines* py *and* $2py^* - py$ *and the line* $0.3y$*. From this it is seem that* **(A5.8)** *and* **(A5.9)** *holds and that* $f(y) \geq 0.3y$ *for all* $y > 0$.

The assumptions for Theorem 5.4.14 all hold, therefore, there exists $\gamma > 0$ *and* $g \geq 1$ *such that for all* $x^0 \in \mathbb{R}^n_+$ *with* $\|x^0\| \geq \varepsilon > 0$,

$$\|x(t;x^0) - x^*\| \leq ge^{-\gamma t}\|x^0 - x^*\| \qquad \forall \quad t \in \mathbb{N}_0.$$

This is illustrated in Figure 5.8(b) for an arbitrary initial condition.



(a) (b)

Figure 5.8: Numerical simulations for Example 5.4.12. (a) A plot of the nonlinearity and the lines py, $2py^* - py$ and $0.075y$. (b) Time history of $x(t)$.

The second exponential stability result uses the bound on $c^T x(t; x^0)$ from Lemma 5.4.10, and as such requires the weaker assumption on $f(y)$, namely, $f(y) > 0$ for all $y > 0$. This exponential stability result is a semi-global result as we require that x^0 is in a compact set of $\mathbb{R}^n_+$ which does not contain 0.

Theorem 5.4.16. *Consider the system* (5.4). *Assume that* **(A5.1)-(A5.6)**, **(A5.8)** *and* **(A5.9)** *hold. Also assume that* $f(y) > 0$ *for all* $y > 0$. *Then the equilibrium* $x^* = (I - A)^{-1} bpy^*$ *is semi-globally exponentially stable in the sense that, for every compact set* $\Gamma \subset \mathbb{R}^n_+$ *with* $0 \notin \Gamma$, *there exists constants* $\gamma > 0$ *and* $g \geq 1$ *such that, for every* $x^0 \in \Gamma$,

$$\|x(t; x^0) - x^*\| \leq g e^{-\gamma t} \|x^0 - x^*\| \qquad \forall \quad t \in \mathbb{N}_0.$$

Proof. Let $\Gamma \in \mathbb{R}^n_+$ be a compact set with $0 \notin \Gamma$. From Theorem 5.4.6 we know that x^* is stable in the large, that is there exists $\hat{g} \geq 1$ such that for all $x^0 \in \Gamma$

$$\|x(t; x^0) - x^*\| \leq \hat{g} e^{-\gamma t} \|x^0 - x^*\| \qquad \forall \quad t \in \mathbb{N}_0. \tag{5.21}$$

Noting that Γ is a compact set, it follows that it is bounded and therefore for all $x^0 \in \Gamma$

$$0 \leq c^T x(t; x^0) \leq \lambda \qquad \forall \quad t \in \mathbb{N}_0, \tag{5.22}$$

where $\lambda > 0$ is a suitable constant. Since Γ is closed and $0 \notin \Gamma$, there exists $\varepsilon > 0$ such that

$$\|x^0\|_1 \geq \varepsilon \qquad \forall \quad \varepsilon \in \Gamma. \tag{5.23}$$

From Lemma 5.4.10, we have that there exists constants $\eta > 0$ and $\theta \in \mathbb{N}_0$

such that for all $x^0 \in \Gamma$,

$$0 < \eta \le c^T x(t; x^0) \qquad \forall \quad t \in \mathbb{N}_0. \tag{5.24}$$

Set

$$\tilde{x}(t; x^0) := x(t; x^0) - x^* \qquad \forall \quad t \in \mathbb{N}_0.$$

It follows from (5.22)-(5.24) that for all $x^0 \in \Gamma$

$$-y^* + \eta \le c^T \tilde{x}(t; x^0) \le -y^* + \lambda \qquad \forall \quad t \in \mathbb{N}_0, \ t \ge \theta. \tag{5.25}$$

By **(A5.8)** and **(A5.9)**

$$r := \sup \left\{ \frac{|f(y + y^*) - f(y^*)|}{|y|} \ : \ -y^* + \eta \le y \le -y^* + \lambda \right\} < p.$$

Now choose a continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{f}(y) = f(y + y^*) - f(y^*) \qquad \forall \quad y \in [-y^* + \eta, -y^* + \lambda],$$

and

$$\left| \frac{\tilde{f}(y)}{y} \right| \le r < p \qquad \forall \quad y \in \mathbb{R}. \tag{5.26}$$

Now note that for all $x^0 \in \Gamma$

$$\dot{\tilde{x}}(t + 1; x^0) = A\tilde{x}(t; x^0) + \tilde{f}(c^T \tilde{x}(t; x^0)) \qquad \forall \quad t \in \mathbb{N}_0, \ t \ge \theta, \tag{5.27}$$

which follows from (5.25). We now have that $\mathbb{D}(0, p) \subseteq \mathbb{S}(A, b, c^T)$. By statement (5.26) we can apply statement (3) of Theorem 5.2.2 and conclude that there exists $\gamma > 0$ and $\tilde{g} \ge 1$ such that for all $x^0 \in \Gamma$

$$\|\tilde{x}(t + \theta; x^0)\| \le \tilde{g} e^{-\gamma t} \|\tilde{x}(\theta, x^0)\| \qquad \forall \quad t \in \mathbb{N}_0.$$

Using (5.21) and noting $\hat{g} \ge 1$ and $\tilde{g} \ge 1$, we obtain that for all $x^0 \in \Gamma$

$$\|\tilde{x}(t; x^0)\| \le \hat{g}\tilde{g} e^{\gamma\theta} e^{-\gamma t} \|x^0 - x^*\| \qquad \forall \quad t = 0, 1, \ldots, \theta.$$

The result now follows by reverting back to the original system and setting $g := \hat{g}\tilde{g}e^{\gamma\theta}$. □

Example 5.4.13 illustrates this theorem as it can easily be seen that **(A5.9)** holds and that is the only additional assumption required for Theorem 5.4.16 over Theorem 5.4.11.

5.5 Input-to-State Stability of Nonnegative Discrete Time Lur'e Systems

In this section we consider forced nonnegative Lur'e systems of the form (5.5). As with the previous section nonnegative in this context means that the state $x(t)$ of (5.5) remains nonnegative for all $t \in \mathbb{N}_0$. Therefore we shall always be assuming that **(A.5.1)** and **(A.5.4)** hold. We denote the solution of (5.5) by $x(\,\cdot\,; t, d)$.

5.5.1 Disturbed Systems Without Stable Equilibria

The result given in this section is an extension of Theorem 5.4.1 to disturbed Lur'e systems of the form (5.5).

Theorem 5.5.1. *Consider the system* (5.5). *Assume that* **(A5.1)**-**(A5.4)** *hold and that*

$$\inf_{y>0} \frac{f(y)}{y} > p.$$

If $x^0 \in \mathbb{R}^n_+$, $x^0 \neq 0$ *and* $d : \mathbb{N}_0 \to \mathbb{R}^n_+$, *then*

$$\lim_{t\to\infty} x_i(t; x^0, d) = \infty \qquad \forall \quad i \in \{1, \dots, n\},$$

where $x_i(t; x^0, d)$ *denoted the* i*-th component of* $x(t; x^0, d)$.

Proof. Let $x^0 \in \mathbb{R}^n_+$ with $x^0 \neq 0$. By hypothesis on f there exists $q > p$ such that

$$f(y) \geq qy \qquad \forall \quad y \in \mathbb{R}_+.$$

Therefore,

$$\begin{aligned} x(t+1; x^0, d) &= Ax(t; x^0, d) + bf(c^T x(t; x^0, d)) + d(t) \\ &\geq Ax(t; x^0, d) + bqc^T x(t; x^0, d) \\ &= (A + qbc^T)x(t; x^0, d), \qquad \forall \quad t \in \mathbb{N}_0, \end{aligned}$$

and so, $x(t; x^0, d) \geq (A + qbc^T)^t x^0$. By Lemma 5.3.6, $r := \rho(A + qbc^T) > 1$. Now

$$r^{-t}(A + qbc^T)^t \to \frac{vw^T}{w^T v} \gg 0 \qquad \text{as} \quad t \to \infty,$$

where v and w are the left and right Perron vectors of $A + qbc^T$, respectively, as given in Theorem 2.1.22. Thus

$$\liminf_{t\to\infty} r^{-t} x_i(t; x^0, d) \geq \xi_i,$$

for $i = 1, \dots, n$ where ξ_i is the i-th component of

$$\xi := \frac{vw^T}{w^T v} x^0 \gg 0.$$

Consequently $\lim_{t\to\infty} x_i(t; x^0) = \infty$, for every $i = 1, \dots, n$. □

Example 5.5.2. *We recall Example 5.4.2, where (A, b, c^T) is given by (5.9) and $f(y) = 0.75y + \sin(y/3)$. We apply these to a system of the form (5.5), where $d(t) = (d_1(t), d_2(t), d_3(t))^T$, and $d_i(t)$ is a random number in the interval $[0, 1]$ for each $t \in \mathbb{N}_0$ and $i = 1, 2, 3$.*

Hypotheses for Theorem 5.5.1 apply, therefore for all $x^0 \in \mathbb{R}^n_+$ with $x^0 \neq 0$ we have

$$\lim_{t\to\infty} x_i(t; x^0, d) = \infty,$$

for every $i = 1, 2, 3$. This is illustrated for a random x^0 in Figure 5.9(b) and can be compared to the same system with $d(t) = 0$ for all $t \in \mathbb{N}_0$ illustrated in Figure 5.9(a).

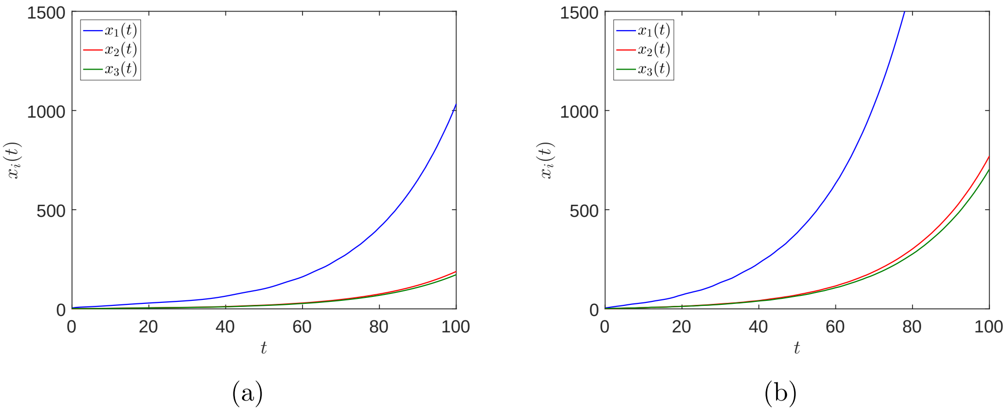


Figure 5.9: Numerical simulations for Example 5.5.2. (a) Time history of $x(t)$ for $d(t) = 0$ for all $t \in \mathbb{N}_+$. (b) Time history of $x(t)$ where $d(t)$ is a vector and all three components are random numbers in $[0, 1]$.

5.5.2 ISS of Systems With A Unique Stable Equilibrium

The result in this section is an extension of Theorem 5.4.3 for systems with a disturbance. Before stating this result we first introduce a new assumption.

(A5.10) $py - f(y) \to \infty$ as $y \to \infty$.

Theorem 5.5.3. *Consider the system* (5.5)*. Assume that* **(A5.1)**-**(A5.5)** *and* **(A5.10)** *hold. Also assume that $f(y)/y < p$ for all $y > 0$. Then 0 is ISS in the sense that there exists $\psi \in \mathcal{KL}_D$ and $\varphi \in \mathcal{K}$ such that, for all $x^0 \in \mathbb{R}^n_+$ and* *all* $d : \mathbb{N}_0 \to \mathbb{R}^n_+$,

$$\|x(t; x^0, d)\| \leq \psi(\|x^0\|, t) + \varphi(\|d\|_t) \qquad \forall \quad t \in \mathbb{N}_0.$$

Proof. By Lemma 5.3.4, $p = 1/\|\boldsymbol{G}\|_{H^\infty}$ and therefore $\mathbb{D}(0,p) \subseteq \mathbb{S}(A, b, c^T)$. To apply Theorem 5.2.4, consider the function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ given by (5.10) which extends f to the whole real line. Furthermore, by the hypothesis made on f,

$$p|y| - |\tilde{f}(y)| > 0 \qquad \forall \quad y \neq 0,$$

and

$$p|y| - |\tilde{f}(y)| \to \infty \qquad \text{as} \quad |y| \to \infty.$$

Hence, there exists $\beta \in \mathcal{K}_\infty$ such that

$$|\tilde{f}(y)| \leq p|y| - \beta(|y|) \qquad \forall \quad y \in \mathbb{R}.$$

Note that by assumptions **(A5.1)** and **(A5.4)** we have that, for every $x^0 \in \mathbb{R}^n_+$ and every $d : \mathbb{N}_0 \to \mathbb{R}^n_+$, $x(t; x^0, d) \in \mathbb{R}^n_+$ for all $t \in \mathbb{N}_0$. Therefore, for every $x^0 \in \mathbb{R}^n_+$ and every $d : \mathbb{N}_0 \to \mathbb{R}^n_+$, $x(\,\cdot\,; x^0, d)$ is also the solution of

$$x(t+1; x^0, d) = Ax(t; x^0, d) + b\tilde{f}(c^T x(t; x^0, d)) + d(t), \qquad x(0) = x^0. \quad (5.28)$$

Applying Theorem 5.2.4 to (5.28) shows that there exists $\psi \in \mathcal{KL}_D$ and $\varphi \in \mathcal{K}$ such that, for all $x^0 \in \mathbb{R}^n_+$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$,

$$\|x(t; x^0, d)\| \leq \psi(\|x^0\|, t) + \varphi(\|d\|_t) \qquad \forall \quad t \in \mathbb{N}_0,$$

completing the proof. □

Example 5.5.4. *Consider the system* (5.5) *and recall Example 5.4.4, where* (A, b, c^T) *is given by* (5.9) *and*

$$f(y) = \frac{5y}{16 + y}, \qquad y \geq 0.$$

Let $d(t) = (d_1(t), d_2(t), d_3(t))$, *and* $d_i(t)$ *is a random number in the interval* $[0, 1]$ *for each* $t \in \mathbb{N}_0$ *and* $i = 1, 2, 3$.

The hypotheses for Theorem 5.5.3 apply, therefore 0 *is ISS in the sense that there exists* $\psi \in \mathcal{KL}_D$ *and* $\varphi \in \mathcal{K}$ *such that, for all* $x^0 \in \mathbb{R}^n_+$ *and all* $d : \mathbb{N}_0 \to \mathbb{R}^n_+$,

$$\|x(t; x^0, d)\| \leq \psi(\|x^0\|, t) + \varphi(\|d\|_t) \qquad \forall \quad t \in \mathbb{N}_0.$$

.

Figure 5.10(a) provides a time history plot of the three components of $x(t)$ and Figure 5.10(b) provides a plot of $\|x(t)\|_1$.

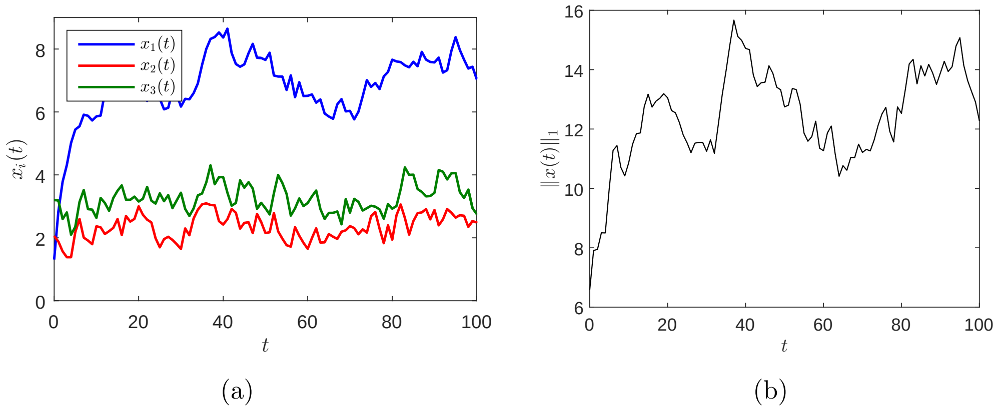


Figure 5.10: Numerical simulations for Example 5.5.4. (a) Time history of $x(t)$. (b) Plot of $\|x(t)\|_1$.

5.5.3 ISS of Systems With Two Equilibria

Before stating the main results in this section we must first reformulate Lemmas 5.4.8-5.4.10 for the disturbed system (5.5).

Lemma 5.5.5. *Consider the system* (5.5)*. Assume* **(A5.1)**-**(A5.4)**, **(A5.6)** *and* **(A5.8)** *hold. Also assume that $c^T A^\kappa \gg 0$ for some $\kappa \in \mathbb{N}_0$. For every $\varepsilon > 0$ there exists $\eta > 0$ and $\theta \in \mathbb{N}_0$ such that, for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$,*

$$c^T x(t; x^0, d) \geq \eta \qquad \forall \quad t \in \mathbb{N}_0, \ t \geq \theta.$$

The proof of this lemma follows from the proof of Lemma 5.4.8, mutatis mutandis, carries over to disturbed Lur'e systems.

Lemma 5.5.6. *Consider the system* (5.5)*. Assume* **(A5.1)**-**(A5.4)**, **(A5.6)** *and* **(A5.8)** *hold. Also assume for some $\alpha > 0$ that $f(y) \geq \alpha y$ for all $y \geq 0$. For every $\varepsilon > 0$ there exist $\eta > 0$ and $\theta \in \mathbb{N}_0$ such that, for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$,*

$$c^T x(t; x^0, d) \geq \eta \qquad \forall \quad t \in \mathbb{N}_0, \ t \geq \theta.$$

The proof of this lemma is omitted as the proof of Lemma 5.4.9, mutatis mutandis, carries over to disturbed Lur'e systems.

Lemma 5.5.7. *Consider the system* (5.5)*. Assume* **(A5.1)**-**(A5.6)**, **(A5.8)** *and* **(A5.10)** *hold. For every compact set $\Gamma \subseteq \mathbb{R}^n_+$ with $0 \notin \Gamma$ and all $\Delta > 0$,* *there exist $\eta > 0$ and $\theta \in \mathbb{N}_0$ such that for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty \leq \Delta$,*

$$c^T x(t; x^0, d) \geq \eta \qquad \forall \quad t \in \mathbb{N}_0,\ t \geq \theta.$$

Proof. Let $\Gamma \in \mathbb{R}^n_+$ be a compact set with $0 \notin \Gamma$ and $\Delta > 0$. We begin by demonstrating that there exists $y^\# > 0$ such that, for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty < \Delta$,

$$c^T x(t; c^0, d) \leq y^\# \qquad \forall \quad t \in \mathbb{N}_0.$$

Let $y^{**} > y^*$. Then $f(y^{**}) < py^{**}$ and thus

$$q := \frac{f(y^{**})}{y^{**}} < p.$$

Define $g : \mathbb{R}_+ \to \mathbb{R}_+$ to be the continuous function given by

$$g(y) = \begin{cases} qy & \text{for} \quad 0 \leq y \leq y^{**}, \\ f(y) & \text{for} \quad y > y^{**}. \end{cases}$$

Note that $0 < g(y) < py$ for all $y > 0$ and

$$py - g(y) \to \infty \quad \text{as} \quad y \to \infty$$

by **(A5.10)**. Now

$$x(t+1; x^0, d) = Ax(t; x^0, d) + bg(c^T x(t; x^0, d)) + d(t) + e(t; x^0, d) \quad \forall \quad t \in \mathbb{N}_0, \tag{5.29}$$

where

$$e(t; x^0, d) := b(f(c^T x(t; x^0, d)) - g(c^T x(t; x^0, d)).$$

Moreover,

$$\|e(t; x^0, d)\| \leq \|b\| \sup_{0 \leq y \leq y^{**}} |f(y) - g(y)| =: \kappa < \infty.$$

Applying Theorem 5.5.3 to (5.29) (with the nonlinearity given by g and the disturbance given by $d + e$) we conclude that there exists a constant $x^\# > 0$ such that, for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty \leq \Delta$,

$$\|x(t; x^0, d)\| \leq x^\# \qquad \forall \quad t \in \mathbb{N}_0.$$

Therefore, setting $y(t) := c^T x(t; x^0, d)$, we conclude that $y(t) \leq y^\#$ for all

$t \in \mathbb{N}_0$. Let
$$\inf_{y^* \le y \le y^\#} \frac{f(y)}{y} = \lambda > 0,$$
noting that $f(y) > 0$ for all $y > 0$.

If $0 \le y \le y^*$ we have $f(y) \ge py$ by **(A5.8)**, thus for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty \le \Delta$

$$\begin{aligned} v^T x(t+1; x^0, d) &= v^T A x(t; x^0, d) + v^T b f(c^T x(t; x^0, d)) + v^T d(t) \\ &\ge v^T A(t; x^0, d) + v^T b p y(t) = v^T (A + pbc^T) x(t; x^0, d), \end{aligned}$$

where v^T is the left Perron vector of $A + pbc^T$ associated with the dominant eigenvalue 1 (see Lemma 5.3.6).

If $y(t) \ge y^*$ then, for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}$ with $\|d(t)\|_\infty \le \Delta$,

$$\begin{aligned} v^T x(t+1; x^0, d) &= v^T A x(t; x^0, d) + v^T b f(c^T x(t; x^0, d)) + v^T d(t) \\ &\ge v^T A x(t; x^0, d) + v^T \lambda b y(t) \ge v^T \lambda b y^*. \end{aligned}$$

Combining the above, for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty \le \Delta$

$$v^T x(t+1; x^0, d) \ge \min(v^T x(t; x^0, d), v^T b \lambda y^*), \qquad \forall \quad t \in \mathbb{N}_0.$$

Therefore, for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty \le \Delta$

$$v^T x(t; x^0, d) \ge \min(v^T x^0, v^T b \lambda y^*) > 0, \qquad \forall \quad t \in \mathbb{N}_0.$$

Noting that $v^T \gg 0$, there exists $\delta > 0$ such that, for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty \le \Delta$,

$$\|x(t; x^0,)\| \ge \delta > 0, \qquad \forall \quad t \in \mathbb{N}_0.$$

Setting $\mu = \min(\lambda, p) > 0$, we have for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty \le \Delta$

$$x(t+1; x^0, d) \ge A x(t; x^0, d) + \mu b c^T x(t; x^0, d) = (A + \mu bc^T) x(t; x^0, d),$$

and for some $k \in \mathbb{N}_0$,

$$x(t; x^0, d) \ge (A + \mu bc^T)^k x(t-k; x^0, d), \qquad \forall \quad t = k, k+1, \ldots,$$

therefore,

$$c^T x(t; x^0, d) \ge c^T (A + \mu bc^T)^k x(t-k; x^0, d), \qquad \forall \quad t = k, k+1, \ldots.$$

Finally, noting $c^T(A+\mu bc^T)^k \gg 0$ for some $k \in \mathbb{N}_0$ by **(A5.1)** and **(A5.3)**, we can write $w^T = (w_1, \ldots, w_n) = c^T(A+\mu bc^T)^k$ we have $w_i > 0$ for all $i = 1, \ldots, n$ and furthermore, for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty < \Delta$,

$$c^T x(t; x^0, d) \geq \min_{1\leq i\leq n}(w_i)\delta > 0.$$

This completes the proof with $\theta = k$ and $\eta = \min_{1\leq i\leq n}(w_i)$. $\square$

The following lemma is a discrete time version of Gronwall's Lemma which we will be making use of in the proof of later results.

Lemma 5.5.8. *Let $\alpha \geq 0$, $w > 0$ and $u : \mathbb{N}_0 \to \mathbb{R}_+$ be such that $u(0) \leq \alpha$ and*

$$u(t) \leq \alpha + w\sum_{k=0}^{t-1} u(k) \qquad \forall \quad t \in \mathbb{N}.$$

Then

$$u(t) \leq \alpha e^{tw}, \qquad \forall \quad t \in \mathbb{N}_0.$$

Proof. We begin by demonstrating that

$$u(t) \leq \alpha(1+w)^t, \qquad \forall \quad t \in \mathbb{N}_0. \tag{5.30}$$

Firstly we note that (5.30) holds for $t = 0$ by $u(0) \leq \alpha$. We demonstrate that (5.30) holds for all $t \in \mathbb{N}$ by strong induction. Assume (5.30) holds for $t = \tau$, that is

$$u(\tau) \leq \alpha(1+w)^\tau.$$

Now

$$\begin{aligned} u(\tau+1) &\leq \alpha + w\sum_{k=0}^{\tau} u(k) \leq \alpha + \alpha w\sum_{k=0}^{\tau}(1+w)^k \\ &= \alpha\left(1 + w\sum_{k=0}^{\tau}(1+w)^k\right). \end{aligned}$$

Noting that

$$\sum_{k=0}^{\tau}(1+w)^k = \frac{(1+w)^{\tau+1}-1}{(1+w)-1} = \frac{(1+w)^{\tau+1}-1}{w},$$

it follows that

$$u(\tau+1) \leq \alpha(1+w)^{\tau+1}.$$

Therefore (5.30) holds for $t = \tau+1$, therefore by strong induction, (5.30) holds for all $t \in \mathbb{N}_0$.

Finally, noting $w > 0$ we have that $1 + w \le e^w$, therefore,

$$u(t) \le \alpha e^{tw}, \qquad \forall \quad t \in \mathbb{N}_0.$$

□

The first ISS result which we consider is a quasi-ISS result. We name it this because we must restrict our initial condition.

Theorem 5.5.9. *Consider the system* (5.5). *Assume* **(A5.1)**-**(A5.6)**, **(A5.8)** *and* **(A5.10)** *hold. Also assume that one of the following holds:*

- $c^T A^k \gg 0$ *for some* $k \in \mathbb{N}_0$.
- *For some* $\alpha > 0$, $f(y) \ge \alpha y$ *for all* $y \ge 0$

The equilibrium $x^* = -(I - A)^{-1} b p y^*$ *is quasi-ISS in the sense that, for every* $\varepsilon > 0$, *there exist* $\psi \in \mathcal{KL}_D$ *and* $\varphi \in \mathcal{K}$ *such that for all* $x^0 \in \mathbb{R}^n_+$ *with* $\|x^0\| \ge \varepsilon$ *and all* $d : \mathbb{N}_0 \to \mathbb{R}^n_+$,

$$\|x(t; x^0, d) - x^*\| \le \psi(\|x^0 - x^*\|, t) + \varphi(\|d\|_t) \qquad \forall \quad t \in \mathbb{N}_0.$$

Proof. Let $\varepsilon > 0$. Assume that $c^T A^k \gg 0$ for some $k \in \mathbb{N}_0$. By Lemma 5.5.5, there exists $\eta_1 > 0$ and $\theta_1 \in \mathbb{N}_0$ such that for all $x^0 \in \mathbb{R}^n_+$ with $\|x\| \ge \varepsilon$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$,

$$c^T x(t; x^0, d) \ge \eta_1 \qquad \forall \quad t \in \mathbb{N}_0,\ t \ge \theta_1.$$

Now alternatively assume that for $\alpha > 0$, $f(y) \ge \alpha$ for all $y \ge 0$. Then by Lemma 5.5.6, there exists $\eta_2 > 0$ and $\theta_2 \in \mathbb{N}_0$ such that for all $x^0 \in \mathbb{R}^n_+$ with $\|x\| \ge \varepsilon$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$,

$$c^T x(t; x^0, d) \ge \eta_2 \qquad \forall \quad t \in \mathbb{N}_0,\ t \ge \theta_2.$$

Therefore for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \ge \varepsilon$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$

$$c^T x(t; x^0, d) \ge \eta \qquad \forall \quad t \in \mathbb{N}_0,\ t \ge \theta, \tag{5.31}$$

where $\eta = \min(\eta_1, \eta_2)$ and $\theta = \max(\theta_1, \theta_2)$.

Define $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(y) = \begin{cases} f(y + y^*) - f(y^*) & \text{for } y \ge -y^* + \eta \\ f(\eta) - f(y^*) & \text{for } y < -y^* + \eta. \end{cases}$$

Then, by **(A5.8)**

$$p|y| - |\tilde{f}(y)| > 0 \qquad \forall \quad y \ne 0$$

and, by **(A5.10)**,

$$p|y| - |\tilde{f}(y)| \to \infty \qquad \text{as} \quad |y| \to \infty.$$

Combining this with the fact that $\mathbb{D}(0,p) \subseteq \mathbb{S}(A,b,c^T)$, it follows from Theorem 5.2.4 that the system

$$z(t+1) = Az(t) + b\tilde{f}(c^T z(t)) + \tilde{d}, \qquad z(0) = z^0, \tag{5.32}$$

is ISS in the sense that there exists $\tilde{\psi} \in \mathcal{KL}_D$ and $\tilde{\varphi} \in \mathcal{K}$ such that, for every $z^0 \in \mathbb{R}$, and every $\tilde{d} : \mathbb{N}_0 \to \mathbb{R}^n$,

$$\|z(t; z^0, \tilde{d})\| \le \tilde{\psi}(\|z^0, t) + \tilde{\varphi}(\|\tilde{d}_t), \qquad \forall \quad t \in \mathbb{N}_0, \tag{5.33}$$

where $z(t; z^0, \tilde{d})$ denoted the unique solution of (5.32).

Let $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \ge \varepsilon$ and let $d : \mathbb{N}_0 \to \mathbb{R}^n_+$. Define $\tilde{x}(t) := x(t; x^0, d) - x^*$ for all $t \in \mathbb{N}_0$ and set

$$\tilde{x}_\theta(t) := \tilde{x}(t+\theta) \quad \text{and} \quad d_\theta(t) := d(t+\theta) \qquad \forall \quad t \in \mathbb{N}_0.$$

By (5.31),

$$c^T \tilde{x}_\theta(t) \ge -y^* + \eta \qquad \forall \quad t \in \mathbb{N}_0,$$

and it is easy to see that $\tilde{x}_\theta$ solves (5.32) with $z^0 = \tilde{x}_\theta(0) = x(\theta; x^0, d) - x^*$ and $\tilde{d} = d_\theta$. Hence, by (5.33), we have that

$$\|\tilde{x}_\theta(t)\| \le \tilde{\psi}(\|\tilde{x}_\theta(0)\|, t) + \tilde{\varphi}(\|d_\theta\|_t) \qquad \forall \quad t \in \mathbb{N}_0. \tag{5.34}$$

Moreover, for $t = 0, 1, \ldots, \theta$, $\tilde{x}$ satisfies

$$\tilde{x}(t+1) = A\tilde{x}(t) + b\hat{f}(c^T \tilde{x}(t)) + d(t) \qquad \forall \quad t = 0, 1, \ldots, \theta,$$

where the function $\hat{f} : [-y^*, \infty) \to [-py^*, \infty)$ is defined by

$$\hat{f}(y) = f(y + y^*) - f(y^*) = f(y + y^*) - py^* \qquad \forall \quad y \ge -y^*.$$

It is clear that $|\hat{f}(y)| \le p|y|$ for all $y \ge -y^*$ and, using the variation-of-parameters formula, it follows that there exists constants $k_1 \ge 1$ and $k_2 > 0$

such that, for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$,

$$\begin{aligned}\|\tilde{x}(t; x^0, d)\| &\leq k_1(\|x^0 - x^*\| + \|d\|_{t-1}) + k_2 \sum_{s=0}^{t-1} \|\tilde{x}(s)\| \qquad \forall \quad t = 1, \ldots, \theta \\ &\leq k_1(\|x^0 - x^*\| + \|d\|_\theta) + k_2 \sum_{s=0}^{t-1} \|\tilde{x}(s)\| \qquad \forall \quad t = 1, \ldots, \theta.\end{aligned}$$

Hence, by Lemma 5.5.8,

$$\|\tilde{x}(t; x^0, d)\| \leq k_1 e^{k_2\theta}(\|x^0 - x^*\| + \|d\|_\theta) \qquad \forall \quad t = 0, 1, \ldots, \theta,$$

holds for all $x^0 \in \mathbb{R}^n_+$ such that $\|x^0\| \geq \varepsilon$, and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$.

Setting $k := k_1 e^{k_2\theta}$ and defining $\psi_1 \in \mathcal{KL}_D$ and $\varphi_1 \in \mathcal{K}$ by

$$\psi_1(s, t) := k e^{\theta - t} s \qquad \forall \quad s \in \mathbb{R}_+, \quad \forall \quad t \in \mathbb{N}_0,$$

and

$$\varphi_1(s) := ks \qquad \forall \quad s \in \mathbb{R}_+,$$

it can be seen that for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$,

$$\begin{aligned}\|\tilde{x}(t; x^0, d)\| &\leq k(\|x^0 - x^*\| + \|d\|_\theta) \\ &\leq \psi_1(\|x^0 - x^*\|, t) + \varphi_1(\|d\|_\theta) \qquad \forall \quad t = 0, 1, \ldots, \theta.\end{aligned} \tag{5.35}$$

In particular, for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$,

$$\|\tilde{x}_\theta(0; x^0, d)\| = \|\tilde{x}(\theta; x^0, d)\| \leq k(\|x^0 - x^*\| + \|d\|_\theta),$$

Combining this with (5.34), yields that, for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \geq \varepsilon$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$,

$$\|\tilde{x}(t+\theta; x^0, d)\| \leq \tilde{\psi}(2k\|x^0 - x^*\|, t) + \tilde{\psi}(2k\|d\|_\theta, 0) + \tilde{\varphi}(\|d\|_{t+\theta}) \qquad \forall \quad t \in \mathbb{N}_0. \tag{5.36}$$

Defining $\psi_2 \in \mathcal{KL}_D$ by

$$\psi_2(s, t) = \begin{cases} \tilde{\psi}(2ks, 0), & (s, t) \in \mathbb{R}_+ \times \{0, 1, \ldots, \theta\} \\ \tilde{\psi}(2ks, t - \theta), & (s, t) \in \mathbb{R}_+ \times \{\theta + 1, \theta + 2, \ldots\} \end{cases}$$

and $\varphi_2 \in \mathcal{K}$ by

$$\varphi_2(s) := \tilde{\varphi}(s) + \tilde{\psi}(2ks, 0) \qquad \forall \quad s \in \mathbb{R}_+,$$

(5.36) can be written as

$$\|\tilde{x}(t+\theta;x^0,d)\| \le \psi_2(\|x^0-x^*\|,t+\theta)+\varphi_2(\|d\|_{t+\theta}) \qquad \forall \quad t\in\mathbb{N}_0,$$

which holds for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \ge \varepsilon$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$.

Finally, setting

$$\psi := \max(\psi_1,\psi_2) \in \mathcal{KL}_D$$

and

$$\varphi := \max(\varphi_1,\varphi_2) \in \mathcal{K},$$

and invoking (5.35), we obtain

$$\|\tilde{x}(t;x^0,d)\| \le \psi(\|x^0-x^*\|,t)+\varphi(\|d\|_t) \qquad \forall \quad t\in\mathbb{N}_0,$$

from which it follows

$$\|x(t;x^0,d)-x^*\| \le \psi(\|x^0-x^*\|,t)+\varphi(\|d\|_t) \qquad \forall \quad t\in\mathbb{N}_0,$$

which holds for all $x^0 \in \mathbb{R}^n_+$ with $\|x^0\| \ge \varepsilon$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$. □

The next ISS result we present is a semi-ISS result. This terminology is motivated by the fact that we restrict our attention to a bounded set of initial conditions and a bounded set of disturbances.

Theorem 5.5.10. *Consider the system* (5.5)*. Assume* **(A5.1)**-**(A5.6)**, **(A5.8)** *and* **(A5.10)** *hold. Also assume that $f(y) > 0$ for all $y > 0$. The equilibrium $x^* = (I-A)^{-1}bpy^*$ is semi-ISS in the sense that, for every compact set $\Gamma \subset \mathbb{R}^n_+$ with $0 \notin \Gamma$ and $\Delta > 0$, there exists $\psi \in \mathcal{KL}_D$ and $\varphi \in \mathcal{K}$ such that for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ such that $\|d(t)\|_\infty \le \Delta$,*

$$\|x(t;x^0,d)-x^*\| \le \psi(\|x^0-x^*\|,t)+\varphi(\|d\|_t) \qquad \forall \quad t\in\mathbb{N}_0.$$

Proof. Let $\Gamma \in \mathbb{R}^n_+$ be a compact set with $0 \notin \Gamma$ and $\Delta > 0$. By Lemma 5.5.7 there exists $\eta > 0$ and $\theta \in \mathbb{N}_0$ such that for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty \le \Delta$,

$$c^T x(t;x^0,d) \ge \eta \qquad \forall \quad t\in\mathbb{N}_0,\ t \ge \theta. \tag{5.37}$$

Define $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(y) = \begin{cases} f(y+y^*)-f(y^*) & \text{for } y \ge -y^*+\eta \\ f(\eta)-f(y^*) & \text{for } y < -y^*+\eta. \end{cases}$$

Then, by **(A5.8)**,

$$p|y| - |\tilde{f}(y)| > 0 \qquad \forall \quad y \neq 0,$$

and, by **(A5.10)**,

$$p|y| - |\tilde{f}(y)| \to \infty \qquad \text{as} \quad |y| \to \infty.$$

Combining this with the fact that $\mathbb{D}(0,p) \subseteq \mathbb{S}(A,b,c^T)$, it follows from Theorem 5.2.4 that the system

$$z(t+1) = Az(t) + b\tilde{f}(c^T z(t)) + \tilde{d}(t), \qquad z(0) = z^0, \tag{5.38}$$

is ISS in the sense that there exists $\tilde{\psi} \in \mathcal{KL}_D$ and $\tilde{\varphi} \in \mathcal{K}$ such that, for every $z^0 \in \mathbb{R}^n$ and every $\tilde{d} : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|\tilde{d}\|_\infty \leq \Delta$,

$$\|z(t; z^0, \tilde{d})\| \leq \tilde{\psi}(\|x^0\|, t) + \tilde{\varphi}(\|\tilde{d}\|_t), \qquad \forall \quad t \in \mathbb{N}_0, \tag{5.39}$$

where $z(t; z^0, \tilde{d})$ denotes the solution of (5.38).

Let $x^0 \in \Gamma$ and $d : \mathbb{N}_0$ with $\|d\|_\infty \leq \Delta$. Define

$$\tilde{x}(t; x^0, d) = x(t; x^0, d) - x^* \qquad \forall \quad t \in \mathbb{N}_0,$$

and set

$$\tilde{x}_\theta(t; x^0, d) := \tilde{x}(t+\theta; x^0, d) \quad \text{and} \quad d_\theta(t) := d(t+\theta) \qquad \forall \quad t \in \mathbb{N}_0.$$

By (5.37),

$$c^T \tilde{x}_d(t; x^0, d) \geq -y^* + \eta \qquad \forall \quad t \in \mathbb{N}_0,$$

and it can easily be seen that $\tilde{x}_\theta(t; x^0, d)$ solves (5.38) with $z^0 = \tilde{x}_\theta(0, x^0, d)$ and $\tilde{d} = d_\theta$. Therefore, by (5.39), for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty \leq \Delta$,

$$\|\tilde{x}_\theta(t; x^0, d)\| \leq \tilde{\psi}(\|\tilde{x}_\theta(0; x^0, d)\|, t) + \tilde{\varphi}(\|d_\theta\|_t) \qquad \forall \quad t \in \mathbb{N}_0. \tag{5.40}$$

Moreover, for $t = 0, 1, \ldots, \theta$, $\tilde{x}(t; x^0, d)$ satisfies

$$\tilde{x}(t+1; x^0, d) = A\tilde{x}(t; x^0, d) + b\hat{f}(c^T \tilde{x}(t; x^0, d)) + d(t) \qquad \forall \quad t = 0, 1, \ldots, \theta,$$

where $\hat{f} : [-y^*, \infty) \to [-py^*, \infty)$ is defined by

$$\hat{f}(y) = f(y + y^*) - f(y^*) = f(y + y^*) - py^* \qquad \forall \quad y \geq -y^*.$$

It is clear that $|\hat{f}(y)| \leq p|y|$ for all $y \geq -y^*$ and using the variation-of-parameters formula, it follows that there exist constants $k_1 \geq 1$ and $k_2 > 0$ such that, for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\| \leq \Delta$,

$$\begin{aligned}\|\tilde{x}(t; x^0, d)\| &\leq k_1(\|x^0 - x^*\| + \|d\|_{t-1}) + k_2 \sum_{s=0}^{t-1} \|\tilde{x}(s)\| \qquad \forall \quad t = 1, \ldots, \theta \\ &\leq k_1(\|x^0 - x^*\| + \|d\|_\theta) + k_2 \sum_{s=0}^{t-1} \|\tilde{x}(s)\| \qquad \forall \quad t = 1, \ldots, \theta.\end{aligned}$$

Hence, by Lemma 5.5.8,

$$\|\tilde{x}(t; x^0, d)\| \leq k_1 e^{k_2\theta}(\|x^0 - x^*\| + \|d\|_\theta) \qquad \forall \quad t = 0, 1, \ldots, \theta,$$

holds for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty \leq \Delta$.

Setting $k := k_1 e^{k_2\theta}$ and defining $\psi_1 \in \mathcal{KL}_D$ and $\varphi_1 \in \mathcal{K}$ by

$$\psi_1(s, t) := k e^{\theta - t} s \qquad \forall \quad s \in \mathbb{R}_+, \quad \forall \quad t \in \mathbb{N}_0,$$

and

$$\varphi_1(s) := ks \qquad \forall \quad s \in \mathbb{R}_+,$$

it can be seen that for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty \leq \Delta$,

$$\begin{aligned}\|\tilde{x}(t; x^0, d)\| &\leq k(\|x^0 - x^*\| + \|d\|_\theta) \\ &\leq \psi_1(\|x^0 - x^*\|, t) + \varphi_1(\|d\|_\theta) \qquad \forall \quad t = 0, 1, \ldots, \theta.\end{aligned} \tag{5.41}$$

In particular, for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\theta \leq \Delta$,

$$\|\tilde{x}_\theta(0; x^0, d)\| = \|\tilde{x}(\theta; x^0, d)\| \leq k(\|x^0 - x^*\| + \|d\|_\theta),$$

Combining this with (5.40), yields that, for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty \leq \Delta$,

$$\|\tilde{x}(t + \theta; x^0, d)\| \leq \tilde{\psi}(2k\|x^0 - x^*\|, t) + \tilde{\psi}(2k\|d\|_\theta, 0) + \tilde{\varphi}(\|d\|_{t+\theta}) \qquad \forall \quad t \in \mathbb{N}_0. \tag{5.42}$$

Defining $\psi_2 \in \mathcal{KL}_D$ by

$$\psi_2(s, t) = \begin{cases} \tilde{\psi}(2ks, 0), & (s, t) \in \mathbb{R}_+ \times \{0, 1, \ldots, \theta\} \\ \tilde{\psi}(2ks, t - \theta), & (s, t) \in \mathbb{R}_+ \times \{\theta + 1, \theta + 2, \ldots\} \end{cases}$$

and $\varphi_2 \in \mathcal{K}$ by

$$\varphi_2(s) := \tilde{\varphi}(s) + \tilde{\psi}(2ks, 0) \qquad \forall \quad s \in \mathbb{R}_+,$$

(5.42) can be written as

$$\|\tilde{x}(t+\theta; x^0, d)\| \leq \psi_2(\|x^0 - x^*\|, t+\theta) + \varphi_2(\|d\|_{t+\theta}) \qquad \forall \quad t \in \mathbb{N}_0,$$

which holds for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty \leq \Delta$.

Finally, setting

$$\psi := \max(\psi_1, \psi_2) \in \mathcal{KL}_D$$

and

$$\varphi := \max(\varphi_1, \varphi_2) \in \mathcal{K},$$

and invoking (5.41), we obtain

$$\|\tilde{x}(t; x^0, d)\| \leq \psi(\|x^0 - x^*\|, t) + \varphi(\|d\|_t) \qquad \forall \quad t \in \mathbb{N}_0,$$

from which it follows

$$\|x(t; x^0, d) - x^*\| \leq \psi(\|x^0 - x^*\|, t) + \varphi(\|d\|_t) \qquad \forall \quad t \in \mathbb{N}_0,$$

which holds for all $x^0 \in \Gamma$ and all $d : \mathbb{N}_0 \to \mathbb{R}^n_+$ with $\|d\|_\infty \leq \Delta$. □

Example 5.5.11. *Consider the system considered in Example 5.5.4, however the nonlinearity is now given by*

$$f(y) = \frac{6y}{5+y}, \qquad y \geq 0.$$

Assume that the disturbance is the same as that considered in Example 5.5.4. Noting that $p = 0.625$*, there exists a unique* $y^* > 0$ *such that* $f(y^*) = py^*$ *meaning that* **(A5.6)** *holds, where* $y^* = 4.6$*. By the results in Section 2.4.3,* **(A5.8)** *and* **(A5.9)** *hold. It is also clear that* **(A5.10)** *holds.*

Now we can apply Theorem 5.5.10 to this system, noting that $\|d\|_\infty \leq 3 < \infty$*. This means for every compact set* $\Gamma \in \mathbb{R}^n_+$ *with* $0 \notin \Gamma$*, there exist* $\psi \in \mathcal{KL}$ *and* $\varphi \in \mathcal{K}$ *such that, for every* $x^0 \in \Gamma$*,*

$$\|x(t; x^0, d) - x^*\| \leq \psi(\|x^0 - x^*\|, t) + \varphi(\|d\|_t) \qquad \forall \quad t \in \mathbb{N}_0,$$

where

$$x^* = \begin{pmatrix} 11.5 & 2.3 & 2.3 \end{pmatrix}^T.$$

We illustrate this bound in Figure 5.11 in which we simulate this example for three random disturbance vectors and plot the error $\|x(t; x^0, d) - x^*\|$*, for an arbitrary* $x^0 \neq 0$ *with* $\|x^0\| < 50$*.*

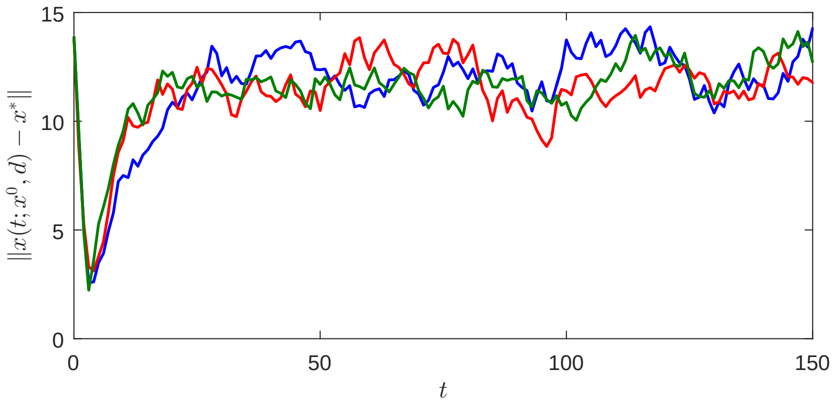


Figure 5.11: Numerical simulations for Example 5.5.11. Error $\|x(t;x^0,d)-x^*\|$ for three different $d:\mathbb{N}_0\to[0,1]^3$.

5.6 Application to Population Ecology

In this section we will demonstrate how the results developed in this chapter can be used in an ecological context. We will be using the Leslie-Plus matrix of a species of wallabies (*Onychogalea fraenata*) from Section 2.2.2, given by

$$L_+ = \begin{pmatrix} 0 & 0 & 0 & 3.1 \\ 0.93 & 0 & 0 & 0 \\ 0 & 0.82 & 0 & 0 \\ 0 & 0 & 0.47 & 0.8 \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.93 & 0 & 0 & 0 \\ 0 & 0.82 & 0 & 0 \\ 0 & 0 & 0.47 & 0.8 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3.1 \end{pmatrix}. \tag{5.43}$$

We begin by noting that $L_+ = A + bc^T$, therefore, using what we established in Section 2.2.2, $A + bc^T$ is a primitive matrix and so **(A5.3)** holds. Also note that trivially **(A5.1)** holds. It is easily shown that $\rho(A) = 0.8$, therefore A is stable and **(A5.2)** is satisfied.

If we choose $f(y) = y$, then the systems

$$x(t+1) = Ax(t) + bf(c^T x(t)), \qquad x(0) = x^0 \in \mathbb{R}^n_+ \tag{5.44}$$

and

$$x(t+1) = L_+ x(t), \qquad x(0) = x^0 \in \mathbb{R}^n_+$$

are identical. Noting that $p = 0.18$, if follows that

$$\inf_{y>0} \frac{f(y)}{y} = 1 > 0.18 = p.$$

By Theorem 5.4.1, for any initial condition $x^0 \in \mathbb{R}^n_+$ with $x^0 \neq 0$,

$$\lim_{t\to\infty} x_i(t) = \infty, \tag{5.45}$$

where $x_i(t)$ is the i-th component of $x(t)$.

Figure 5.12 is a time history of $x(t)$ for an arbitrary nonnegative and nonzero initial condition which clearly illustrates the divergence property (5.45).

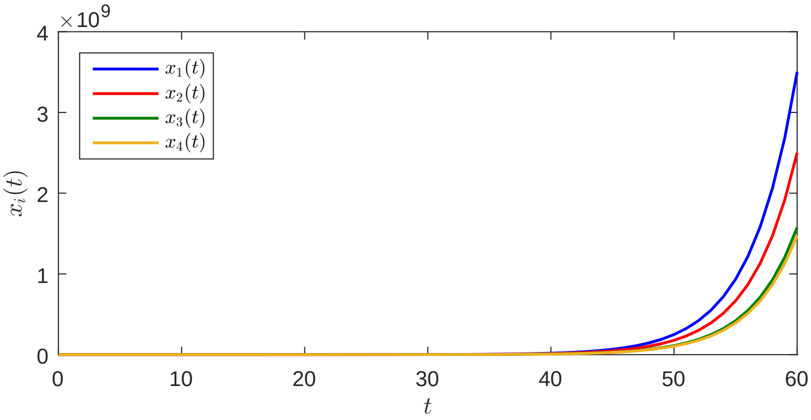


Figure 5.12: Time history of $x(t)$ for the system (5.44).

What this means is using this linear model for wallabies, the total population, and indeed the population in each stage-class, will diverge to infinity. This clearly is not a biologically realistic model for asymptotic behavior. One way of dealing with this population divergence is to introduce an upper bound on the number of adults (members of the final stage-class) reproducing each year. This can by achieved by replacing f by a Beverton-Holt type nonlinearity. This nonlinearity was introduced in Section 2.4.3, and is given by

$$f(y) = \frac{my}{k+y}.$$

It is easily seen that as $y \to \infty$, $f(y) \to m$ and as $y \to 0$, $f(y)/y \to m/k$. If we wish to introduce an upper bound of 500 adults reproducing each per time step, we therefore require that $m = 500$. It is easily seen that $f(y)$ is an increasing function with decreasing derivative, therefore $f(y) \leq m$ for all $y \geq 0$. If we also want the system to behave in a similar manner to the linear system for small values of y we require $m/k = 1$, therefore, $k = 500$.

We now consider the system (5.4) with nonlinearity given by

$$f(y) = \frac{500y}{500+y}, \tag{5.46}$$

and linear part given by (5.43). This will model population of wallabies in a similar way to (5.44), however there will be at most 500 adults reproducing per time step. Noting that $p = 0.18$, there exists a unique $y^* > 0$ such that $f(y^*) = py^*$, meaning that **(A5.6)** holds, where $y^* = 20500/9$. By the results in Section 2.4.3, **(A5.8)** and **(A5.9)** hold.

It remains to show that **(A5.5)** is satisfied, then we will be able to apply Theorem 5.4.16 to establish the existence of a semi-globally exponentially stable, nonzero equilibrium of the system. This is easily done noting that the controllability and observability matrices for (5.43) are

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.93 & 0 & 0 \\ 0 & 0 & 0.7626 & 0 \\ 0 & 0 & 0 & 0.3584 \end{pmatrix}$$

and

$$\mathcal{O} = \begin{pmatrix} 0 & 0 & 0 & 3.1 \\ 0 & 0 & 1.457 & 2.48 \\ 0 & 1.1947 & 1.1656 & 1.9840 \\ 1.1111 & 0.9558 & 0.9325 & 1.5872 \end{pmatrix}$$

respectively, which both are of full rank.

Now by Theorem 5.4.16, there exist constants $\gamma > 0$ and $g \geq 1$ such that, for every x^0 which lies in a compact set, not containing zero,

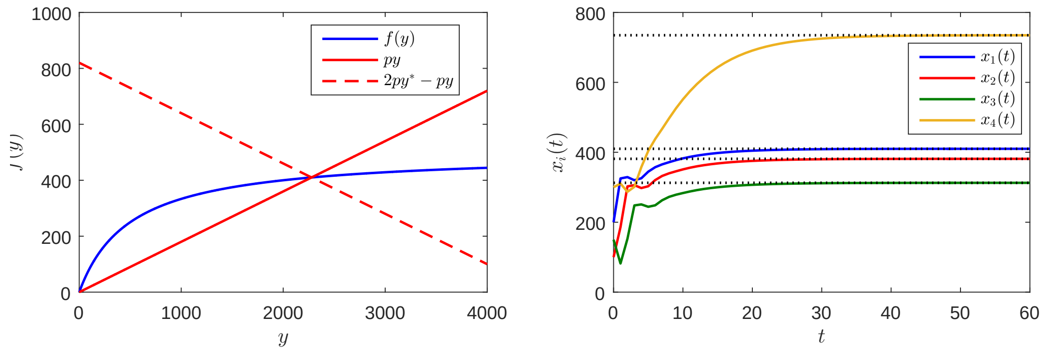
$$\|x(t;x^0) - x^*\| \leq ge^{-\gamma t}\|x^0 - x^*\| \qquad \forall \quad t \in \mathbb{N}_0,$$

where

$$x^* = (I-A)^{-1}bpy^* = \begin{pmatrix} 410.001 \\ 381.301 \\ 312.667 \\ 734.767 \end{pmatrix}. \tag{5.47}$$

Figure 5.13(a) plots the nonlinearity and shows that it satisfies the strict sector condition and Figure 5.13(b) contains the time history of $x(t)$ with the initial condition $x^0 = (200, 100, 150, 300)^T$ and shows it converges to x^*.

We now address the issue of a hard limit we have imposed on the maximum number of parents reproducing each year. In some years, the number of newborns could be more than 500, which we do not currently allow. To deal



(a) (b)

Figure 5.13: Example of a population model for wallaby where we include an upper limit on the number of adults reproducing each year. (a) The nonlinearity $f(y) = 500y/(500 + y)$ satisfying a sector condition. (b) Time history plot of $x(t)$ in the colored lines and the steady state x^* as a dotted black line.

with this we can think of $f(y)$ being the lower bound for the number of adults reproducing each year. The actual number of adults reproducing each year will now be $500y/(500 + y) + d_1$, where $d_1 : \mathbb{N}_0 \to [0, 50]$. d_1 is a disturbance term which represents additional adults reproducing each year and takes a random value between 0 and 50. The system now takes the form (5.5) where (A, b, c^T) is given by (5.43), $f(y)$ is given by (5.46) and

$$d(t) = \begin{pmatrix} d_1(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \text{where} \qquad d_1 \ : \ \mathbb{N}_0 \to [0, 50]. \tag{5.48}$$

Clearly **(A5.10)** holds. With $\Delta = 50$, $\|d\|_\infty < \Delta$, therefore we are in a position to apply Theorem 5.5.10. This means that for every compact set $\Gamma \subseteq \mathbb{R}^n_+$, with $0 \notin \Gamma$, there exist $\psi \in \mathcal{KL}_D$ and $\varphi \in \mathcal{K}$ such that for all $x^0 \in \Gamma$ and $d(t)$ given by, (5.48)

$$\|x(t; x^0, d) - x^*\| \leq \psi(\|x^0 - x^*\|, t) + \varphi(\|d\|_t) \qquad \forall \quad t \in \mathbb{N}_0. \tag{5.49}$$

What this means in a biological context is that even if we do not know the exact number of adults which reproduce each time step, just that it lies in the interval $[500y/(500 + y), 500y/(500 + y) + 50]$, where y is the number of individuals in the final stage-class, we do know that the difference between the total population and the population x^* given by (5.47) is bounded by (5.49). This is illustrated in Figure 5.14, where the error, $\|x(t; x^0, d) - x^*\|$ is plotted

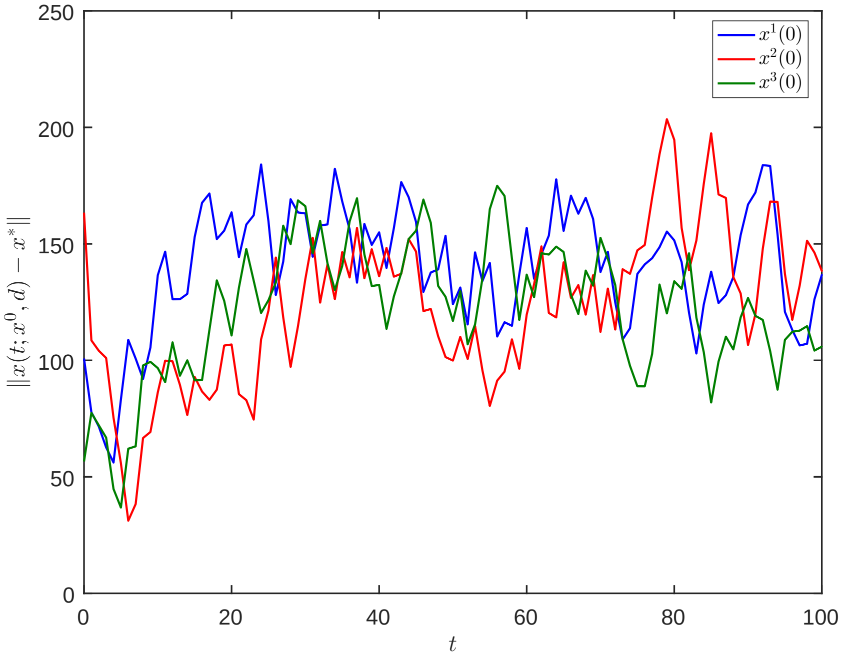


Figure 5.14: Example of a population model for a wallaby where the number of adults reproducing lies in the interval $[500y/(500+y), 500y(500+y)+50]$ with three initial conditions given by (5.50).

for the three initial conditions

$$x^1(0) = \begin{pmatrix} 400 \\ 400 \\ 350 \\ 700 \end{pmatrix}, \qquad x^2(0) = \begin{pmatrix} 350 \\ 350 \\ 350 \\ 700 \end{pmatrix}, \qquad x^3(0) = \begin{pmatrix} 400 \\ 400 \\ 300 \\ 750 \end{pmatrix}, \tag{5.50}$$

and three different, yet similarly defined disturbances, $d(t)$.

Chapter 6

Integral Control of Discrete Time Nonnegative Lur'e Systems for Population Management

This chapter is mainly based on [53].

6.1 Introduction

Regulation or management to a constant set-point is fundamental across the natural and man-made world. Examples include the regulation of blood sugar by insulin [127]; bacterial chemotaxis in living cells [154]; calcium homeostasis [36]; the regulation of temperature in a central heating system [57]; or the navigation of a supertanker across stormy seas [76, 4]. Such examples span a huge range of time and length scales. In conservation management or pest control, population managers would aim to regulate the population to a desired density. A key feature in all of these applications is that set-point regulation must be robust to parametric uncertainty and observation errors. So how is such robust set-point regulation achieved? [154] argue that the robustness of many homeostatic mechanisms must use integral control. Integral control is a simple yet powerful technique developed by control engineers, and is one component of a family of so-called PID (P for proportional, I for integral and D for derivative) controllers. PID controllers are used widely in industrial processes [5] and have been described as one of the "Success Stories in Control" [122, p. 103]. One striking feature of integral controllers, and PID controllers in general, is that they can be implemented on the basis of both minimal knowledge of the system to be managed or regulated, and in the presence of considerable system uncertainty. It is these two features that makes them appealing for population management/conservation.

Conservation is crucial to maintaining biodiversity and species viability in environments facing a range of pressures, such as those from habitat destruction, climate change, invasion and changing land use. Likewise, pest control or management is key to controlling unwanted or invasive populations which possibly have uncertain or unmodeled vital rates. However, that said, the first two sentences of the abstract of [147] read "Too much of wildlife management is today still more of an art than a science. Turning the art into a much needed predictive science requires including research in the management process". In response to Walker's claim there have been many theoretical approaches to population management in the ecological and conservation literature (see Section 6.2.1 for references). As far as we can tell, integral control (and PID control more generally) *has not* been considered as a technique for regulating a population by restocking or removing members. Here we present such an approach to conservation; describing how integral control arises naturally and is suitable for the task. In doing so we draw on a large body of existing theoretical work on integral control, which we adapt to a context of population management. Our focus in this chapter is conservation and so we concentrate on supplementing populations. We comment, however, that managing a (possibly growing ambient) unwanted population to lower population densities can also be achieved using a *combined* proportional and integral (PI) control strategy.

This chapter is organized as follows. Section 6.2 contains a nontechnical overview of integral control and describes the key concepts. Integral control, indeed PID control in general, is an extensively studied subject and it is clearly not possible, or indeed our purpose, to include a complete treatment here. Similarly, there are many other theoretical approaches to population management in the ecological and conservation literature, and in Section 6.2.1 we compare and contrast the methods proposed here to some existing techniques, such as partially observable Markov decision processes. Section 6.3 describes the mathematics of integral control and progressively adds additional features to the model necessitated by the specific demands of population modeling. These additional features are described on page 199 and addressed in Sections 6.3.1-6.3.5. Throughout this chapter we will illustrate theoretical concepts with ecological examples. We seek to give a workable overview of integral control, the suitable modifications geared towards conservation using ecological models and cite relevant sources for further reading. We summaries our results and their applications in the discussion in Section 6.5.

6.2 Integral Control for Population Conservation

Our objective is to present a method to restock a managed, but declining, population. We assume that the population is modeled by an age- or stage-structured population projection model, see Section 2.2. These are discrete-time models where the time-steps are assumed fixed: a week, month, or breeding cycle for instance. First, we need to have access to an *observation* of the population. In a typical application, we do not know, and cannot measure, the entire population distribution at any given time; in fact, in practice there are stage-classes about which we have no knowledge. For instance, we might be able to measure population density of only the reproductive adults, and so in this case it is that stage only which is the observation. It is this part of the system which we seek to regulate. An important specification in the problem statement, therefore, is that only information of the measured stage-class (or classes) is available.

Second, we need to be able to replenish a stage (or combination of stages), that is, add new (or remove existing) population members. In a context of conservation, say of an endangered plant, such an action might be restocking by planting seedlings grown in a greenhouse. We describe a method for choosing management actions that result in the densities of the measured stages reaching a chosen reference value. Figure 6.1 contains a diagram of the setup described thus far.

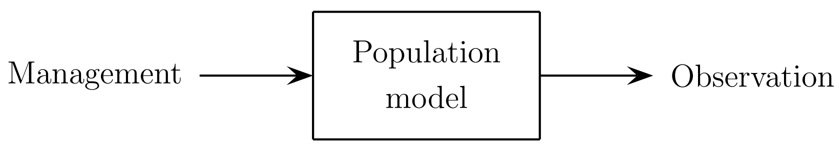
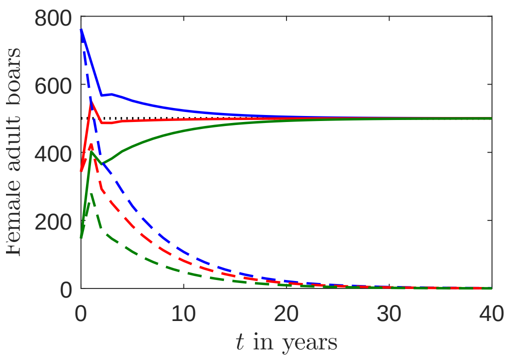


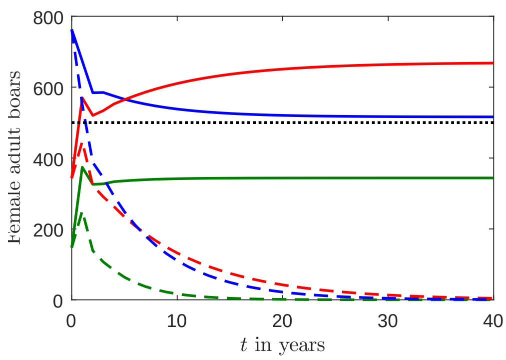
Figure 6.1: Diagram of the restocking scheme: management acts by adding or removing members of the population of certain stage-classes and a portion of the population is observed. The goal is to choose a management strategy so that the observed observations reach a chosen reference value.

The above problem fits naturally into a "classical" control theory setting, and we draw on techniques developed in that field to present a solution. A precomputed or open-loop control is a choice of management strategy that is determined entirely by the model parameters and the chosen reference value. It is called open-loop because the corresponding block diagram, Figure 6.1, is an open-loop as there is no feedback loop. It is straightforward to show that under mild assumptions on the model, such as stability of the unforced system, a suitably chosen constant management strategy, that is, a fixed number of new members of the population being added at each time-step results in the observations converging to the reference value.

As an illustrative example, a matrix population projection model for females of the declining population of wild boar (*Sus scrofa*), in poor environmental conditions, is given in [9]. The matrix has three stage-classes, structured according to age. Suppose that at each time-step the density of the third stage-class, here denoting adult female boar, is measured. Similarly, assume that we have access to the same stage-class, so that we can release female adults into the population. The model is described in detail in Example 6.3.3. From each of three random initial population distributions our goal is to raise the female adult density to 500 (and to maintain that density). Here the chosen reference abundance is arbitrary but typical of wild boar density from [73, p. 447-449]. Figure 6.2(a) contains the results of applying a precomputed control; the observed abundances of female adult boar of the unmanaged population are declining with time and the observed abundances of female adult boar of the managed population are converging to the target reference of 500.



(a)



(b)

Figure 6.2: Precomputed control applied to the declining wild boar matrix PPM considered in Section 6.2. In both plots the solid lines and dashed lines denote observations with and without precomputed control respectively. The black dotted lines denote the target reference abundance $r = 500$. The three different colours represent three initial conditions. (a) Observed female adult boar population. (b) Observed female adult boar with randomly perturbed model parameters.

Precomputed control provides a simple method for raising population density via restocking. It does suffer from a major flaw, however. Precomputed control is not updated according to observations taken and requires exact knowledge of the model parameters, here denoting modeled vital rates, in order to be implemented as intended. Applying precomputed control when these model parameters are uncertain can often result in the management objective failing. For example, Figure 6.2(b) contains projections of the wild boar projection model considered above, but with randomly perturbed model parameters. The precomputed control is based on the nominal estimate of these parameters; those given in Example 6.3.3. It is evident that here the precomputed control does not achieve the desired outcome of 500 female adult boar. Although there are perturbations where the precomputed control does give rise to eventual observations *larger* than the reference r, there are also cases where the observations are *smaller* than r. Furthermore, in general it is not possible to predict the effect of arbitrary model uncertainty on the resulting observations of a precomputed control strategy, greatly limiting the appeal of precomputed control in this situation.

The above example shows that precomputed control is in general *not robust* to parameter uncertainty, which is a particular instance of model uncertainty; a term we make precise in the present context in Section 6.3.1. The lack of robustness of precomputed control is problematic because ecological models are inherently noisy, often parametrized statistically from limited time-series data (see [104] or [17, Chapter 6]) and subject also to many other forms of uncertainty (see [149, 114]). Naturally, based on the above remarks we desire a method for raising population density via restocking that is robust to these sources of uncertainty.

The problem statement, therefore, is:

> *Design a method to restock a managed, but declining, population. The method should be implemented with only access to specified observations of that population and in a manner that is both independent of the initial population distribution and is robust to model uncertainty.*

Similar problem statements arise in many engineering contexts (as discussed earlier). It is well-known to engineers that the solution is to base the management strategy on a feedback law. In words, the management action to be taken at each time-step is based on observations of the population. Such a scheme is represented in Figure 6.3. Feedback control is often called *closed-loop* control because the loop in Figure 6.3 is closed.

Without yet going into the mathematical details; the choice of feedback control used depends on both the model to be controlled and the desired goal. The choice of feedback control is guided by the internal model principle [43] which states that the controller, in this case the management strategy, must be able to reproduce the dynamics of the reference signal. Hence, if we wish to use a feedback control to regulate the population to a constant value, it will need to include an integrator and hence will be an integral controller. Furthermore, there is inherent robustness in this type of control, as we explain in Section

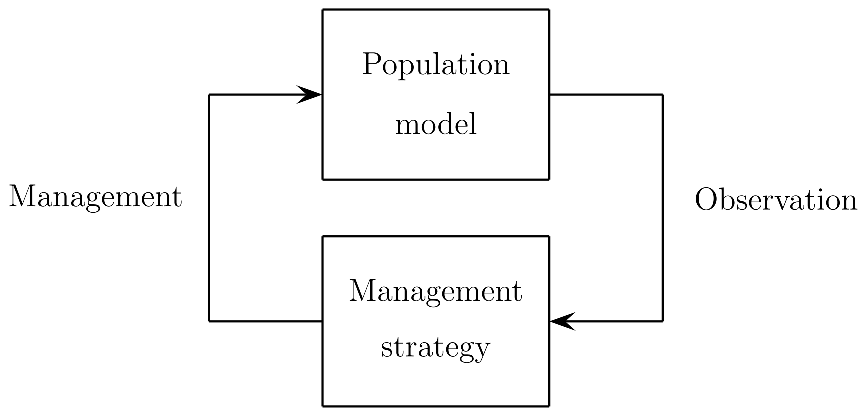


Figure 6.3: Feedback control for population management: the management strategy is determined by the observations of that population. The goal is to design a management strategy so that the observed observations reach a chosen reference value.

6.3.1.

In the remainder of this chapter we demonstrate that integral control is a suitable feedback strategy for population management via restocking. We proceed in Section 6.3 to give a mathematical presentation of integral control. Figure 6.4 shows projections of the uncertain wild boar model subject to an integral control management strategy. We see that the desired outcome of 500 female adult boar is achieved.

Integral control, as presented in this chapter, dates back to the 1970s and early contributions include [27, 95, 102, 52], while the later results we present draw on contemporary material, which we cite in the text. We conclude this section with a brief overview of other modeling approaches to population management prevalent in the literature to which we compare and contrast integral control.

6.2.1 Comparison with existing approaches to population management

There are both deterministic and stochastic modeling approaches to population management in the literature. For populations modeled by matrix PPMs one approach is to investigate the effects of changing life history parameters on the dominant eigenvalue, which characterizes the asymptotic growth rate of the population. A dominant eigenvalue greater than one gives rise to an asymptotically increasing population under a few technical, but reasonable, mathematical conditions such as primitivity (see Theorem 2.1.23). This can be achieved by sufficient increase in the entries of the matrix specifying the

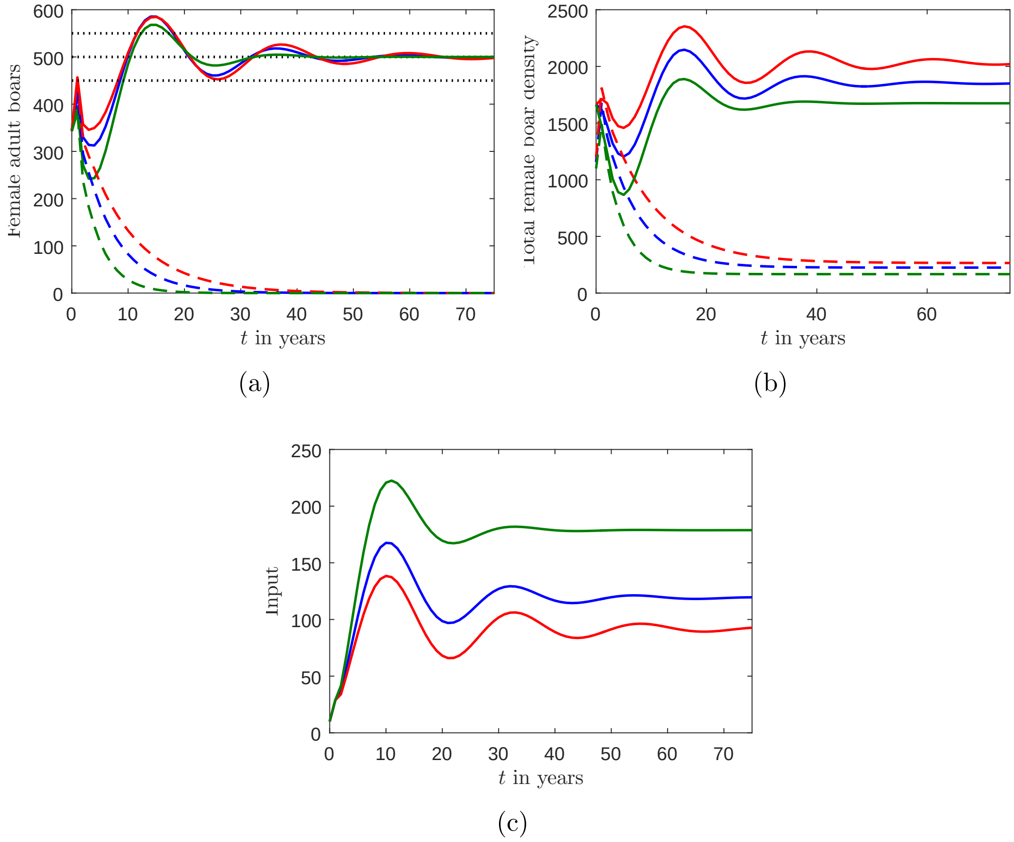


Figure 6.4: Integral control (6.9) applied to the declining wild boar matrix PPM of Example 6.3.3 with randomly perturbed model parameters. In each plot the solid lines are corresponding simulations subject to the integral control system (6.9). The dashed lines are projections from the uncontrolled model (6.1). (a) Observations of female adult boar. The dotted lines are the reference $r = 500$ and $r \pm 10\%$. (b) Total female population density. (c) The number of new individuals added at each time-step, determined by the integral control management strategy (6.8).

PPM [7, p. 27]. A sensitivity [30] or elasticity [29] analysis can be used to quantify how small changes in particular vital rates affect the dominant eigenvalue; often guiding or even directing conservation efforts. Examples include, but are by no means restricted to, [24, 67, 152, 93, 141].

Biologically, the above procedure corresponds to improving the vital rates for a population, for example by improving the quality or access to food, or by removing or limiting predation or poaching. Mathematically, the above procedure is a form of perturbation analysis and over recent years new tools have been added by [65, 66, 32, 92] to analytically describe the dependence of the dominant eigenvalue on the perturbation. These methods largely draw on the stability radius for robust control developed by [59, 58]. The above framework is not directly comparable to integral control because (a) it is not a restocking or reintroduction scheme and (b) perturbations to vital rates are generally not modeled dynamically; they are considered as a static (that is, instant) intervention.

Stochastic models for population management are also prevalent in the literature. Markov decision processes (MDP) (see, for example, [110]) are, roughly speaking, Markov chains where at each time-step the state transition function depends on an action chosen by the modeler. Associated with each action and state are rewards (and/or costs), which are combined to form a so-called value function. As with feedback control, MDPs have been extended to the situation where, at each time-step, the entire state is not available to the modeler and instead only an observation (which is a stochastic or deterministic function of the state) is available. In this situation a partially observable MDP (POMDP) is used instead. Since their inception POMDPs have been used in a wide variety of fields and we refer the reader to the survey [101] or the tutorial paper [87] for examples and a history of their development.Worked examples in the conservation literature include [18, 19] and POMDPs have also been applied for detecting and managing an ecological invasion, for example, in [56] and [115] and the references therein.

Although POMDPs are used in the literature with the same population management objective as that here (in some sense); we note that POMDPs are used in a slightly different fashion and as a result have different advantages and disadvantages. In the examples given above, the aim is to choose actions *optimally*, that is, to maximize the expected rewards obtained (and/or minimize the expected costs incurred) through the value function. Integral control is an example of feedback control – it is not an optimal control technique, and thus is a complimentary method. Two advantages of integral control are, first, that the models are very straightforward to use. This is especially pertinent because finding optimal policies for POMDPs is, in general, computationally very intensive [16], especially as the size of the state-space grows. The same is also true for models for population management that use stochastic dynamic programming (SDP), such as [129, 97, 148, 142, 99]. Second, integral control is demonstrably robust to model uncertainty, a key consideration in ecological models. Optimal controls (including those obtained from classical results such as the Pontryagin Maximum Principle) are not always robust to model uncertainty [33, 121]; an increase in performance is traded-off against a loss of robustness. Robustness to model uncertainty for POMDP models has been discussed in [41] and appeals to the theory of (active) adaptive management [150, 151].

We conclude this section by remarking on active adaptive management [147, 130, 151]. Precomputed control is an example of management that is not adaptive – the same number of individuals are released every time-step and no monitoring of the resulting population takes place. Conversely integral control, and feedback control more generally, is an example of active adaptive management. After every management event (that is, at each time-step) observations are collected and used to update the management action at the next step; this is the fundamental ingredient of feedback control, as depicted in Figure 6.3.

6.3 Mathematical Formulation of Integral Control

This section contains a mathematical presentation of integral control for population management. We first consider matrix population projection models (PPMs) and the reader is also referred to Section 2.2 for further details. Suppose that the population can be described by n distinct age- or stage-classes. If the population density in each stage-class is x_j, for $j = 1, \ldots, n$, then we let $x(t) = (x_1(t), \ldots, x_n(t))^T$ denote the population vector which has dynamics described by the PPM

$$x(t+1) = Ax(t), \qquad x(0) = x^0, \qquad \forall \quad t \in \mathbb{N}_0, \tag{6.1}$$

where x^0 denotes the initial population distribution. Throughout this chapter we assume that the unmanaged population modeled by (6.1) is in asymptotic decline for every initial population distribution x^0, which means that the *spectral radius* of A is less that one. Recall that the spectral radius of a matrix M can be defined as

$$\rho(M) = \lim_{t \to \infty} \|M^t\|^{\frac{1}{t}},$$

where $\| \cdot \|$ denotes any matrix norm, which captures the asymptotic growth rate of the norm of M^t.

Since we shall always consider matrices A that are nonnegative it follows from [7, p. 26] that the spectral radius of A equals the dominant eigenvalue. For such matrices this is often referred to as the *asymptotic growth rate.*

Throughout this chapter we introduce a set of assumptions, the first of which is given below.

(A6.1) $A \in \mathbb{R}_+^{n \times n}$ and $\rho(A) < 1$.

It is often the case that we do not know the entire population distribution $x(t)$ in (6.1) precisely because there are stage-classes about which we have no information. It is probable, for instance, that the full initial population x^0 is unavailable. However, we assume that we do have access to a measured variable, or observation, $y(t)$ described by

$$y(t) = c^T x(t), \qquad \forall \quad t \in \mathbb{N}_0. \tag{6.2}$$

The variable $y(t)$ represents the total knowledge about $x(t)$ available for management decisions, and might take the form of the results of a census or survey. Here c in (6.2) is a column vector, so that c^T is a row vector, called the observation vector. By way of an example, suppose that we are considering a population with five stage-classes. If the abundance of the penultimate stage is measured at each time-step, then

$$c^T = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \text{with} \qquad y(t) = c^T x(t) = x_4(t).$$

The second facet of the model is to allow the population to be supplemented or depleted by the arrival or removal of new members respectively. To describe this we include a control term $bu(t)$ in (6.1), to obtain the controlled population model

$$x(t+1) = Ax(t) + bu(t), \qquad x(0) = x^0, \qquad \forall \quad t \in \mathbb{N}_0. \tag{6.3}$$

The term $bu(t)$ describes the addition (if $bu(t) \geq 0$) or removal (if $bu(t) < 0$) of population members distributed across population stages through the column vector b. The vector b is the choice of the modeler, although probably subject to implementation constraints. The population model (6.3) together with the observation (6.2) is combined to give

$$\left.\begin{aligned} x(t+1) &= Ax(t) + bu(t)\,, \quad x(0) = x^0, \\ y(t) &= c^T x(t), \end{aligned}\right\} \qquad \forall \quad t \in \mathbb{N}_0. \tag{6.4}$$

The time-dependent variable $u(t)$ is the management strategy and $y(t)$ is the observation, both at time step t.

Recalling that we do not know the population $x(t)$ exactly, we are interested in what effect $u(t)$ has on $y(t)$. Under the assumption **(A6.1)**, the linearity of (6.4) means that it is straightforward to demonstrate that if

$$\lim_{t\to\infty} u(t) = \tilde{u}, \qquad \text{then} \qquad \lim_{t\to\infty} y(t) =: \tilde{y} = c^T (I-A)^{-1} b\tilde{u}. \tag{6.5}$$

The constant $c^T(I-A)^{-1}b$ is called the *steady-state gain* as it is the multiplier (or gain) that when applied to a constant input signal gives the resulting eventual observation. Using the fact

$$c^T (I-A)^{-1} b = \sum_{k=0}^{\infty} c^T A^k b = c^T (I + A + A^2 + \ldots) b, \tag{6.6}$$

another interpretation of the steady-state gain is that it is the measure cumulative contribution to the observation over all time from a constant influx of $\tilde{u} = 1$ population members structured by b. When b and c are nonnegative vectors, then from (6.6) it follows that $c^T(I-A)^{-1}b \geq 0$ as well. If $c^T(I-A)^{-1}b > 1$ then $\tilde{u}$ is amplified after a long period of time and conversely if $c^T(I-A)^{-1}b < 1$ then $\tilde{u}$ it is attenuated.

Assuming that $c^T(I-A)^{-1}b > 0$, we see from (6.5) that in order for the observations to eventually reach a chosen value r, so that $\tilde{y} = r$, then

$$u(t) = \tilde{u} := \frac{r}{c^T(I-A)^{-1}b}, \qquad \forall \quad t \in \mathbb{N}_0, \tag{6.7}$$

and this precomputed control achieves $y(t)$ tending to r for any initial population distribution x^0.

We introduce a second assumption which rules out the degenerate case that the steady-state gain of (A, b, c^T) is zero.

> **(A6.2)** The matrix A and vectors b and c^T are such that $c^T(I-A)^{-1}b > 0$.

Remark 6.3.1. **(A6.2)** *is always satisfied if A satisfies* **(A6.1)**, *A is irreducible and b and c^T are nonnegative and nonzero.*

Irreducibility is a natural assumption for ecologically meaningful PPMs (see [140]) and hence **(A6.2)** is not overly restrictive.

The integral control feedback scheme is the dynamic, time-dependent strategy

$$u(0) = u^0, \qquad u(t) = u^0 + g\sum_{j=0}^{t-1}(r - y(j)), \qquad \forall \quad t \in \mathbb{N}, \tag{6.8}$$

where r is the chosen reference value, $g > 0$ is a design parameter (often called a gain parameter) and the value of u^0 is arbitrary. The strategy (6.8) is a discrete time integrator because at time-step t the control signal $u(t)$ is determined by summing the previous deviations of the observation $y(t)$ from the reference r. This is equivalent to integrating in discrete time. The combination of (6.4) and (6.8) leads to the feedback system

$$\left.\begin{aligned} &x(t+1) = Ax(t) + bu(t)\,, \quad x(0) = x^0, \\ &u(t+1) = u(t) + g(r - c^T x(t))\,, \quad u(0) = u^0, \end{aligned}\right\} \qquad \forall \quad t \in \mathbb{N}_0. \tag{6.9}$$

Before stating the first result we need some more notation. The transfer function $\boldsymbol{G}$ of the linear system (6.4) is defined by

$$z \mapsto \boldsymbol{G}(z) := c^T(zI - A)^{-1}b, \qquad z \in \mathbb{C}, \tag{6.10}$$

which is certainly defined for every complex z that is not an eigenvalue of A. The transfer function is a ubiquitous concept in control engineering with many uses, and has also been used in ecological modeling, see [65]. For our present purposes it is sufficient to note that under assumption **(A6.1)** the steady-state gain is equal to $\boldsymbol{G}(1)$, the transfer function evaluated at one.

Theorem 6.3.2. *Assume that the linear system* (6.4) *satisfies assumptions* **(A6.1)** *and* **(A6.2)**. *Then there exists $g^* > 0$ such that for all $g \in (0, g^*)$, every $r > 0$ and all initial conditions $(x^0, u^0) \in \mathbb{R}_+^n \times \mathbb{R}_+$, the solution (x, u) of* (6.9) *has the following properties:*

(1) $\lim_{t\to\infty} u(t) = \dfrac{r}{G(1)}$,

(2) $\lim_{t\to\infty} x(t) = (I - A)^{-1} b \dfrac{r}{G(1)}$,

(3) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} c^T x(t) = r$.

The proof of this result can be found in [91]. We do however provide an illustration of both how integral control works and the role of g. First, note that if (x^*, u^*) is an equilibrium of the feedback system (6.9), then by definition

$$\begin{aligned} x^* &= Ax^* + bu^* &\quad \Rightarrow \quad & x^* = (I - A)^{-1} bu^*, \\ u^* &= u^* + g(r - c^T x^*) &\quad \Rightarrow \quad & c^T x^* = r, \end{aligned} \tag{6.11}$$

where for the second implication we have used that $g > 0$. The final equality in (6.11) shows that the x^* component of any equilibrium (x^*, u^*) of (6.9) gives rise to an output $c^T x^*$ equal to the reference r.

Using (6.11), the feedback system (6.9) can be written as

$$\begin{pmatrix} x(t+1) - x^* \\ u(t+1) - u^* \end{pmatrix} = \underbrace{\begin{pmatrix} A & b \\ -gc^T & 1 \end{pmatrix}}_{=:A_g} \begin{pmatrix} x(t) - x^* \\ u(t) - u^* \end{pmatrix}, \qquad \forall \quad t \in \mathbb{N}_0. \tag{6.12}$$

By inspection of (6.12) we see that Theorem 6.3.2 holds precisely for $g > 0$ such that $\rho(A_g) < 1$, where $\rho(A_g)$ is the spectral radius of A_g. Under assumption **(A6.1)**, when $g = 0$ the eigenvalues of A_0 are those of A and 1, thus $\rho(A_0) = 1$. However, for small but positive g it can be shown that $\rho(A_g) < 1$. If g is too large then $\rho(A_g) \geq 1$ and the theorem fails. As such, Theorem 6.3.2 is a so-called "low-gain" result since it guarantees that, if the gain parameter g is small enough, then the control objective is achieved. Consequently, in these circumstances, integral control provides a solution to our original problem of restoring population levels via restocking, in a manner that only requires knowledge of the available observations $y(t)$ and for any initial population distribution x^0.

The conclusions (1)-(3) of Theorem 6.3.2 demonstrate that the integral controller (6.9) solves the replenishment problem. The model (6.9) is reasonable general and is suited to a wide range of scientific and engineering applications. In the context of population management, the following potential problems need to be addressed:

(P1) *What types of uncertainty can integral control tolerate?* Ecological systems are inherently noisy, with many forms of uncertainty that the model (6.9) does not yet address.

(P2) *Can integral control be extended to incorporate additional feasibility constraints on the input $u(t)$?* The feedback strategy (6.8) can generate either very large or negative values of $u(t)$. Large input signals might be too large for practical implementation given limited resources. Negative $u(t)$ requires managers to remove members from the population, which seems illogical when our ultimate goal is to boost or at least conserve population density. Negative control signals may even result in the integral control system (6.9) predicting negative populations, which is clearly absurd.

(P3) *How small does the gain g in the feedback strategy* (6.8) *need to be?* Theorem 6.3.2 requires that the parameter g is small enough and although it is always possible to choose such a g, the theorem gives no indication of what this is or how to find it.

(P4) *Can the rate of convergence of the observations to the reference be improved?* Theorem 6.3.2 guarantees that the observations converge to the reference, but the integral control model (6.9) does not yet include additional features that can alter the rate of convergence.

(P5) *Can integral control be applied to other population models?* Matrix PPMs model a single population in discrete stage-classes and, for example, have no explicit spatial components.

Sections 6.3.1-6.3.5 sequentially address the above problems. Each subsection begins with a verbal outline of the solution that proceeds the mathematical details. Section 6.4.1 describes how the solutions of these problems combine.

6.3.1 What types of uncertainty can integral control tolerate?

Here we describe types of uncertainty likely to be present in integral control and qualify the extent to which integral control can tolerate these uncertainties **(P1)**.

Several authors have proposed frameworks for describing and reducing uncertainty in ecological modeling, and we appeal to the terminology of [149, 150, 114]. Since we are describing the modeling aspects of integral control, we are focusing on *epistemic uncertainty*, in the language of [114], as opposed to *linguistic uncertainty*. Mathematically, we argue that there are three types of uncertainty present that integral control needs to be able to cope with: Several authors have proposed frameworks for describing and reducing uncertainty in ecological modeling, and we appeal to the terminology of [149, 150, 114]. Since we are describing the modeling aspects of integral control, we are focusing on *epistemic uncertainty*, in the language of [114], as opposed to *linguistic uncertainty*. 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Since we are describing the modeling aspects of integral control, we are focusing on *epistemic uncertainty*, in the language of [114], as opposed to *linguistic uncertainty.* Mathematically, we argue that there are three types of uncertainty present that integral control needs to be able to cope with:

(i) model uncertainty;

(ii) measurement errors;

(iii) activation errors.

The connections between these descriptions and those already established in the literature are described in Table 6.1.

	Williams (2001) [149]	Regan et al. (2002) [114]
(i)	Environmental variation Structural uncertainty	Natural variation Inherent randomness Model uncertainty
(ii)	Partial observability	Measurement error Systematic error
(iii)	Partial controllability	

Table 6.1: Connecting types of uncertainty to which integral control is subject with existing descriptions of uncertainty in the ecology literature.

Robust control is an important and well-studied topic in control engineering with many textbooks dedicated to the subject, for example [34, 47, 156, 155]. Quoting [34, p. 8], "Generally speaking, the notion of robustness means that some characteristic of the feedback system holds for every plant in the set P". The term *plant* in control engineering denotes the model to be studied or controlled and comes historically from power and chemical plants. We need to identify the *set of plants* and the desired *characteristics*. In our context the set of plants P is all integral control models of the form (6.9) with the collection of uncertainties *(i)*-*(iii)*. The desired characteristics to hold are the conclusions of Theorem 6.3.2. Quoting [47, p. xi], "Systems that can tolerate plant variability and uncertainty are called robust - $\cdots$".

We now discuss the types of uncertainty in more detail.

(i) *Model uncertainty* amounts to not knowing the model parameters A, b and c^T in (6.1). Uncertainty in A can arise quite naturally. Parameter values in A may be only estimates or statistical means of some "true" value. Or, the structure of A may be uncertain. For instance, A could be age-structured or stage-structured, which can model the same underlying process but have different mathematical realizations. in some cases the input vector b will be known, for example, when b represents restocking into a well-defined developmental stage-class in the model. However, b could be uncertain; say, when restocking seedlings which recruit into an unknown distribution of size classes. Often the observation vector c^T is known, for the same reason as b - when c^T captures counting abundance of a well-defined development stage, such as female nesting adult turtles. However, c^T could be uncertain; in a size based model, not all of the stage-classes need to be specified in order to count the

abundances of a given size. Such a situation leaves c^T unknown. Finally, the dimension n of the model itself could be uncertain. Integral control is robust to all of these model uncertainties for the following reasons.

The two crucial assumptions placed on the model parameters A, b and c^T for integral control are **(A6.1)** and **(A6.2)**. Assumption **(A6.1)** does not require knowledge of A and holds for any population model of the form (6.1) in asymptotic decline. Similarly, assumption **(A6.2)** does not require knowledge of A, b and c^T , or indeed the exact value of $\boldsymbol{G}(1) = c^T(I - A)^{-1}b$, only that it is positive, which is true when A is nonnegative and irreducible and b and c^T are nonnegative and nonzero. As we have commented earlier, irreducibility is a natural assumption for matrix PPMs [140]. Knowledge of A, b or c^T is not needed for the implementation of integral control. In fact, assumptions **(A6.1)** and **(A6.2)** are necessary for low-gain integral control and so we cannot allow greater uncertainty.

Example 6.3.3. *The wild boar matrix PPM considered in Section 6.2 had matrix A, control vector b and observation vector c^T given by*

$$A = \begin{pmatrix} 0.13 & 0.56 & 1.64 \\ 0.25 & 0 & 0 \\ 0 & 0.31 & 0.58 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad c^T = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}. \tag{6.13}$$

For the simulations in Figure 6.4 each of the nonzero entries of A is randomly perturbed by up to 20%. The same gain parameter $g = 0.12$ is used for each simulation. We see that each simulated observation converges to the reference $r = 500$. However, the total female population densities and the number of new individuals added per time-step in Figure 6.4(b) and (c) respectively are converging to different limits. This is because by Theorem 6.3.2 (1) and (2), the respective limits

$$\lim_{t\to\infty} \|x(t)\|_1 = \lim(x_1(t) + x_2(t) + x_3(t)) = \left\| (I - A)^{-1} b \frac{r}{\boldsymbol{G}(1)} \right\|_1$$

and

$$\lim_{t\to\infty} u(t) = \frac{r}{\boldsymbol{G}(1)},$$

both depend on A (noting that $\boldsymbol{G}(1)$ also depends on A), which is being perturbed in this example.

(ii) Observation errors. The integral control model (6.9) assumes that the observations $y(t)$ taken at each time-step are correct. In practice there are bound to be errors incurred in the counting or measuring process. This is conceivably a problem because the integrator (6.8) feeds back the observation

$y(t)$ into the control signal.

Here we describe how integral control responds in the presence of measurement errors. In what follows $y(t)$ denotes the measured observation, whilst the actual observation is $c^T x(t)$. As always we are assuming that A, b and c^T in (6.4) satisfy **(A6.1)** and **(A6.2)** and further that $g > 0$ in (6.8) is chosen sufficiently small so that Theorem 6.3.2 holds for the integral control system (6.9). A general additive observation error $d(t)$ can be incorporated into (6.9) as

$$\left.\begin{aligned} x(t+1) &= Ax(t) + bu(t), & x(0) &= x^0, \\ y(t) &= c^T x(t) + d(t), & & \\ u(t+1) &= u(t) + g(r - y(t)), & u(0) &= u^0, \end{aligned}\right\} \quad t \in \mathbb{N}_0. \tag{6.14}$$

If $d(t)$ equals a constant $\tilde{d}$ for each t (that is, a constant systematic observation error is made), or $d(t)$ converges to $\tilde{d}$, then it is elementary to demonstrate that the measured variable $y(t)$ converges to $r - \tilde{d}$. In words, there is offset in the tracking.

If $d(t)$ is periodic (the observation error is seasonal for example), say

$$d(t) = \tilde{d}\cos(\theta t)$$

for some $\tilde{d} \in \mathbb{R}$ and $\theta > 0$, then again it is elementary to demonstrate that the measured variable $y(t)$ settles to the periodic signal

$$r - \tilde{d}A_\theta \cos(\theta t + \varphi_\theta),$$

which oscillates around r with magnitude $\tilde{d}A_\theta$ and phase shift φ_θ, where

$$A_\theta = \left| \frac{g\boldsymbol{G}(e^{i\theta})}{e^{i\theta} - 1} \left(1 + \frac{g\boldsymbol{G}(e^{i\theta})}{e^{i\theta} - 1}\right)^{-1} \right|$$

and

$$\varphi_\theta = \arg\left(\frac{g\boldsymbol{G}(e^{i\theta})}{e^{i\theta} - 1} \left(1 + \frac{g\boldsymbol{G}(e^{i\theta})}{e^{i\theta} - 1}\right)^{-1} \right).$$

For complex z, the notation $\arg(z)$ denotes the argument of z. For arbitrary additive observation error $d(t)$ one can show that

$$\limsup_{t\to\infty} |y(t) - r| \leq \mu_g \limsup_{t\to\infty} |d(t)|, \tag{6.15}$$

where the constant μ_g can be computed and is given by

$$\mu_g := \sum_{j=0}^{\infty} \left| \begin{pmatrix} c^T & 0 \end{pmatrix} \begin{pmatrix} A & b \\ -gc^T & 1 \end{pmatrix}^j \begin{pmatrix} 0 \\ g \end{pmatrix} \right|,$$

and is finite since by assumption $g > 0$ is such that

$$A_g = \begin{pmatrix} A & b \\ -gc^T & 1 \end{pmatrix}$$

has $\rho(A_g) < 1$. The significance of the bound in (6.15) is that for large t the error between the measured observation and the reference is linear in the magnitude of $d(t)$.

It is important to note that assumptions **(A6.1)** and **(A6.2)** and the size of the gain parameter g are all independent of measurement errors when these errors occur additively, as in (6.14).

Example 6.3.4. *Simulations of the integral control system with additive output error* (6.14) *applied to the wild boar model of Example 6.3.3 are plotted in Figure 6.5. For the same A, b, c^T, x^0, u^0, r and g as in that example, Figure 6.5(a) contains three projected observations subject to the additive observation errors plotted in Figure 6.5(b). The specific $d(t)$ considered are constant with value -50 (blue), converging to 125 (red) and periodic (green). The resulting observations are convergent to $r - d = 500 - (-50) = 550$, $500 - 125 = 375$ and periodic respectively.*

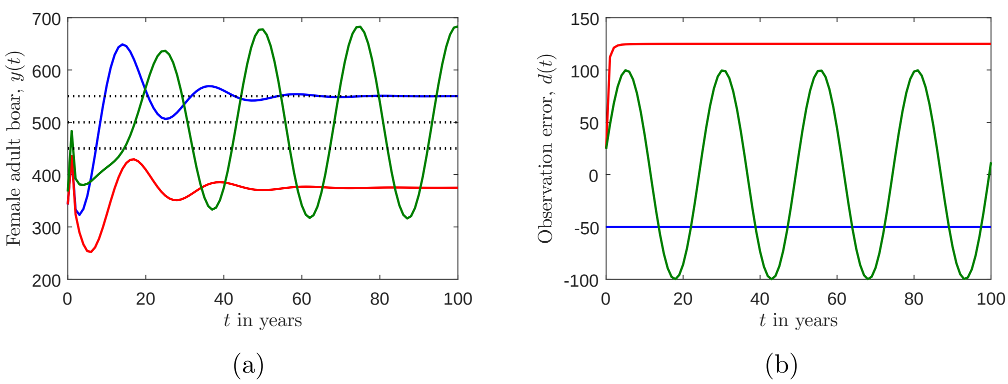


Figure 6.5: Integral control with additive observation errors (6.14) applied to the wild boar matrix PPM of Example 6.3.3. See Example 6.3.4. (a) Observations. The dotted lines are the reference $r = 500$ and $r \pm 10\%$. (b) Observation errors

A potentially more plausible description of observation error is that it is proportional to the observation taken, which is described by

$$y(t) = (1 + \varepsilon(t))c^T x(t), \qquad \forall \quad t \in \mathbb{N}_0. \tag{6.16}$$

The term $\varepsilon(t)$ is the error which is unknown and assumed to be close to zero. For example, $\varepsilon(t)$ taking value -0.1, -0.12 and 0.05 in three consecutive time-steps corresponds to measuring 90, 88 and 105% of the actual population respectively. The case $\varepsilon(t) = 0$ corresponds to the measured and actual observations coinciding so that (6.2) is recovered. We assume that $\varepsilon(t) > -1$ for every t, so that a positive observation is always taken. For applications, what is often important is knowing the "worst case scenario", which amounts to knowing the largest possible observation errors.

If we assume that the observation errors are random, that is, each $\varepsilon(t)$ is a random variable, then each observation $y(t)$ is also a random variable. The main result of this section is Theorem 6.3.5 below that states that if the errors are assumed independent and identically distributed (IID) then the expectation of the observations $y(t)$ converge to the reference r . If additionally the variance of the errors is not too large then the variance of the observation $y(t)$ converges to a finite computable quantity.

Let $\otimes$ denote that Kronecker product and $0_{m\times p}$ denote the $m \times p$ zero matrix.

Theorem 6.3.5. *Assume that the linear system* (6.4) *satisfies assumptions* **(A6.1)** *and* **(A6.2)**, *and that $g > 0$ is such that*

$$\rho(A_g) < 1, \qquad \text{where} \qquad A_g = \begin{pmatrix} A & b \\ -gc^T & 1 \end{pmatrix}.$$

Assume that $(\varepsilon(t))_{t=0}^{\infty}$ is a sequence of IID random variables with zero mean and variance σ^2 and let $y(t)$ denote the measured observations of the integral control system (6.9) *with observation error* (6.16). *It follows that*

(1) $y(t)$ converges in expectation to r, that is

$$\lim_{t\to\infty} \mathbb{E}(y(t)) = r.$$

(2) If

$$\sigma^2 < \frac{1}{g^2 \max_{|z|=1} \left| \tilde{E}(zI - A_g \otimes A_g)^{-1} \tilde{D} \right|}, \tag{6.17}$$

where $\tilde{D} = (0_{1\times(n^2+2n)}, 1)^T$ and $\tilde{E} = \big((c^T, 0) \otimes (c^T, 0)\big)$, then

$$\lim_{t\to\infty} \operatorname{var} y(t) = \begin{pmatrix} c^T & 0 \end{pmatrix} C_\infty \begin{pmatrix} c \\ 0 \end{pmatrix} < \infty.$$

Here the matrix $C_\infty = C_\infty^T$ solves the symmetric linear matrix equation [111]

$$C_\infty - A_g C_\infty A_g^T - g^2\sigma^2 (DE) C_\infty (DE)^T - g^2 r^2 \sigma^2 DD^T = 0,$$

where

$$D = \begin{pmatrix} 0_{1\times n} \\ 1 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} c^T & 0 \end{pmatrix}.$$

Proof. Let (x, u) denote the solution of

$$\left.\begin{aligned} x(t+1) &= Ax(t) + bu(t), & x(0) &= x^0, \\ u(t+1) &= u(t) + g(r - (1+\varepsilon(t))c^T x(t)), & u(0) &= u^0, \end{aligned}\right\} \quad t \in \mathbb{N}_0, \tag{6.18}$$

the integral control system (6.9) with proportional observation errors $\varepsilon(t)$. When ε is a sequence of random variables then so are x and u. We let

$$x^* = (I - A)^{-1} b \frac{r}{\boldsymbol{G}(1)}, \qquad u^* = \frac{r}{\boldsymbol{G}(1)},$$

which are equilibria of (6.9) as in (6.11). For notational convenience we define the random variable

$$z(t) := \begin{pmatrix} x(t) - x^* \\ u(t) - u^* \end{pmatrix}, \qquad \forall \quad t \in \mathbb{N}_0, \tag{6.19}$$

a vector with $n+1$ components. A short calculation using (6.11) and (6.18) demonstrates that $z(t)$ has dynamics given by

$$\begin{aligned} z(t+1) = &\left(\begin{pmatrix} A & b \\ -gc^T & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} g\varepsilon(t) \begin{pmatrix} c^T & 0 \end{pmatrix} \right) z(t) \\ &- \begin{pmatrix} 0 \\ 1 \end{pmatrix} gr\varepsilon(t), \qquad \forall \quad t \in \mathbb{N}. \end{aligned} \tag{6.20}$$

We introduce the notation

$$A_g := \begin{pmatrix} A & b \\ -gc^T & 1 \end{pmatrix}, \quad D := \begin{pmatrix} 0_{n\times 1} \\ 1 \end{pmatrix}, \quad E := \begin{pmatrix} c^T & 0 \end{pmatrix},$$

where recall that $0_{n\times 1}$ is a column vector of n zeros. With this notation (6.20) can be more concisely expressed as

$$z(t+1) = (A_g - g\varepsilon(t)DE)z(t) - Dgr\varepsilon(t), \qquad \forall \quad t \in \mathbb{N}_0. \tag{6.21}$$

Letting $\overline{z(t)} = \mathbb{E}(z(t))$ denote the expectation of $z(t)$, we take expectations in (6.21) to yield that

$$\overline{z(t+1)} = A_g \overline{z(t)}, \qquad \forall \quad t \in \mathbb{N}_0, \tag{6.22}$$

where we have used the facts that expectation is linear, $\overline{\varepsilon(t)} = 0$ and that $\varepsilon(t)$ and $z(t)$ are independent. We are assuming that the gain parameter $g > 0$ is such that $\rho(A_g) < 1$, and hence from (6.22) we conclude that

$$\lim_{t\to\infty} \overline{z(t)} = 0,$$

and thus

$$\lim_{t\to\infty} \overline{y(t)} = \lim_{t\to\infty} \begin{pmatrix} c^T & 0 \end{pmatrix} \overline{z(t)} + r = r,$$

establishing claim (1).

We now consider the covariance

$$\begin{aligned} \operatorname{cov}(z(t), z(t)) &= \mathbb{E}\left(\left(z(t) - \overline{z(t)}\right)\left(z(t) - \overline{z(t)}\right)^T\right) \\ &= \mathbb{E}\left(z(t)z^T(t)\right) - \overline{z(t)} \cdot \overline{z^T(t)} \\ &=: C(t) - \overline{z(t)} \cdot \overline{z^T(t)}, \qquad \forall \quad t \in \mathbb{N}_0 \end{aligned} \tag{6.23}$$

where $z^T(t) = (z(t))^T$. We focus on the quantity $C(t)$, which, appealing to (6.21), has dynamics

$$\begin{aligned} C(t+1) &= \mathbb{E}\left(z(t+1)z^T(t+1)\right) \\ &= \mathbb{E}([(A_g - g\varepsilon(t)DE)z(t) - Dgr\varepsilon(t)] \\ &\qquad\qquad [(A_g - g\varepsilon(t)DE)z(t) - Dgr\varepsilon(t)]^T) \\ &= \mathbb{E}\left(A_g z(t)(A_g z(t))^T\right) \\ &\quad + \mathbb{E}\left(A_g z(t)z^T(t)(-g\varepsilon(t))(DE)^T\right) \\ &\quad + \mathbb{E}\left(\left(A_g z(t)z^T(t)(-g\varepsilon(t))(DE)^T\right)^T\right) \\ &\quad + \mathbb{E}\left(A_g z(t)\varepsilon(t)D^T gr\right) + \mathbb{E}\left(\left(A_g z(t)\varepsilon(t)D^T gr\right)^T\right) \\ &\quad + g^2\sigma^2\mathbb{E}\left(DEz(t)z^T(t)(DE)^T\right) \\ &\quad + g^2r^2\sigma^2 DD^T + \mathbb{E}\left(D(-g\varepsilon(t))Ez(t)\varepsilon(t)(-D^T gr)\right) \\ &\quad + \mathbb{E}\left(\left(D(-g\varepsilon(t))Ez(t)\varepsilon(t)(-D^T gr)\right)^T\right), \end{aligned} \tag{6.24}$$

for all $t \in \mathbb{N}_0$. Equation (6.24) simplifies to

$$\begin{aligned} C(t+1) = A_g C(t) A_g^T &+ g^2\sigma^2(DE)C(t)(DE)^T + g^2\sigma^2r^2DD^T \\ &+ rg^2\sigma^2 DE\overline{z(t)}D^T + rg^2\sigma^2(DE)^T D\overline{z^T(t)}, \qquad \forall \quad t \in \mathbb{N}_0. \end{aligned} \tag{6.25}$$

Defining $A_1 := A_g$, $A_2 := g\sigma DE$ and writing

$$\begin{aligned} c(t) &: = \text{vec } C(t), \\ p(t) &: = \text{vec } \left(g^2\sigma^2 r^2 DD^T + rg^2\sigma^2 DE\overline{z(t)}D^t + rg^2\sigma^2 (DE)^T D\overline{z^T(t)} \right), \end{aligned} \tag{6.26}$$

where if x_i are the columns of the $n \times n$ matrix $X = (x_1, \ldots, x_n)$ then

$$\text{vec } X := \begin{pmatrix} x_1^T & x_2^T & \cdots & x_n^T \end{pmatrix}^T \in \mathbb{R}^{n^2}.$$

Arguing as in [111], the matrix difference equation (6.25) can be written as the $(n+1)^2 \times (n+1)^2$ linear system

$$c(t+1) = \left(\sum_{i=1}^{2} A_i \otimes A_i \right) c(t) + p(t), \qquad \forall \quad t \in \mathbb{N}, \tag{6.27}$$

where $\otimes$ denotes the Kronecker product. Using the fact $\overline{z(t)} \to 0$ as $t \to \infty$ it follows from (6.26) that

$$\lim_{t \to \infty} p(t) = \text{vec } (g^2\sigma^2 r^2 DD^T) =: p_\infty.$$

Consequently, if

$$\rho \left(\sum_{i=1}^{2} A_i \otimes A_i \right) = \rho(A_1 \otimes A_1 + A_2 \otimes A_2) < 1, \tag{6.28}$$

then for any initial condition $c(0)$ the solution of c of (6.27) converges to a finite limit c_∞ satisfying

$$c_\infty = \left(\sum_{i=1}^{2} A_i \otimes A_i \right) c_\infty + p_\infty. \tag{6.29}$$

Assuming that (6.28) holds, defining C_∞ as the matrix such that

$$c_\infty = \text{vec } C_\infty,$$

we have from (6.29) that C_∞ must satisfy

$$C_\infty = A_g C_\infty A_g^T + g^2\sigma^2 (DE) C_\infty (DE)^T + g^2\sigma^2 r^2 DD^T.$$

Furthermore, as $C(t)$ converges to C_∞ the iterative scheme (6.25) provides a method for approximating C_∞.

It remains to find a characterization of the stability condition (6.28). Recalling that for square matrices X,Y

$$\sigma(X \otimes Y) = \{\lambda\mu \ : \ \lambda \in \sigma(X), \mu \in \sigma(Y)\},$$

we have

$$\rho(A_1 \otimes A_1) = \rho(A_g \otimes A_g) = \rho(A_g)^2 < 1,$$

and thus we can view $A_1 \otimes A_1 + A_2 \otimes A_2$ as a structured perturbation of $A_1 \otimes A_1$. Therefore we can characterize the condition (6.28) by appealing to stability radius arguments [58, 59]. A calculation shows that $A_2 \otimes A_2$ is a rank one perturbation, namely

$$\begin{aligned} A_2 \otimes A_2 &= g^2\sigma^2 \begin{pmatrix} 0 & 0 \\ c^T & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ c^T & 0 \end{pmatrix} \\ &= g^2\sigma^2 \begin{pmatrix} 0_{(n^2+2n)\times 1} \\ 1 \end{pmatrix} \left(\begin{pmatrix} c^T & 0 \end{pmatrix} \otimes \begin{pmatrix} c^T & 0 \end{pmatrix} \right) \\ &=: g^2\sigma^2 \tilde{D}\tilde{E}. \end{aligned}$$

Hence, condition (6.28) is satisfied if, and only if,

$$\sigma^2 g^2 < \frac{1}{\max_{|z|=1} \left| \tilde{E}(zI - A_g \otimes A_g)^{-1}\tilde{D} \right|},$$

which is equivalent to the condition (6.17). We can now take the limit as $t \to \infty$ in (6.23) and use that $\overline{z(t)}$ converges to zero to deduce that

$$\lim_{t\to\infty} \operatorname{cov}(z(t), z(t)) = \lim_{t\to\infty} C(t) = C_\infty. \tag{6.30}$$

The variance of the output satisfies

$$\begin{aligned} \operatorname{var} y(t) &= \operatorname{var}(y(t) - r) = \operatorname{var}\left(\begin{pmatrix} c^T & 0 \end{pmatrix} z(t) \right) \\ &= \operatorname{cov}\left(\begin{pmatrix} c^T & 0 \end{pmatrix} z(t), \begin{pmatrix} c^T & 0 \end{pmatrix} z(t) \right) \\ &= \begin{pmatrix} c^T & 0 \end{pmatrix} \operatorname{cov}(z(t), z(t)) \begin{pmatrix} c \\ 0 \end{pmatrix}, \qquad \forall \quad t \in \mathbb{N}_0. \end{aligned} \tag{6.31}$$

Therefore taking limits in (6.31) and invoking (6.30) we have that

$$\lim_{t\to\infty} \operatorname{var} y(t) = \begin{pmatrix} c^T & 0 \end{pmatrix} C_\infty \begin{pmatrix} c \\ 0 \end{pmatrix} < \infty,$$

proving claim (2). □

The quantity of the right hand side of (6.17) can be readily computed numerically, and provides an estimate for the largest permitted variance in observation error so that the resulting observation has finite variance.

Example 6.3.6. *Theorem 6.3.5 is applied to the wild boar model of Example 6.3.3. For the same* A*,* b*,* c^T*,* x^0*,* u^0*,* r *and* g *as in that example the integral control system* (6.9) *with proportional observation error* $\varepsilon(t)$ *as in* (6.16) *is simulated. The errors* $\varepsilon(t)$ *are normally distributed with zero mean and constant variance* $\sigma^2 = 0.09$*. Figure 6.6(a) plots three observation simulations* $y(t)$ *as well as the expected observation* $\mathbb{E}(y(t))$*. Figure 6.6(b) contains the corresponding three sequences of input signals* $u(t)$*, as well as the expected input sequence. In this example the variance of* $y(t)$ *converges to* ~ 530*, so that the standard deviation of* $y(t)$ *is* ~ 72*, and the constant in* (6.17) *equals* 3.04*. Hence, in this example the variance of* $y(t)$ *will converge for any observation error with* $\sigma^2 < 3.04$*.*

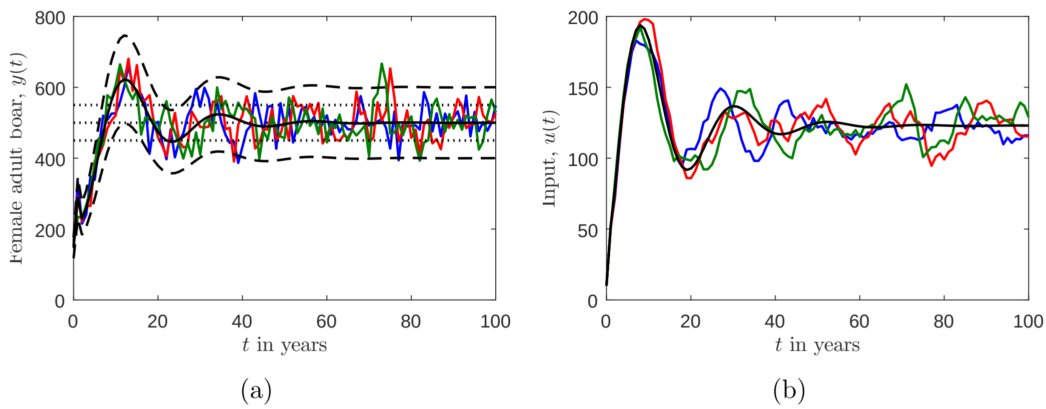


Figure 6.6: Integral control with proportional observation errors (6.9), (6.16) applied to the wild boar matrix PPM of Example 6.3.3. See Example 6.3.6. (a) Observations plotted blue, red and green solid lines. The solid black line and the dashed black lines are the expected observation $\mathbb{E}(y(t))$ and $\mathbb{E}(y(t)) \pm \sqrt{\text{var}\, y(t)}$. The dotted black lines are the reference $r = 500$ and $r \pm 10\%$. (b) Inputs plotted in the matching line style as the corresponding observations in (a)

(iii) Activation errors. The integral control model (6.9) assumes that the input signals are exact, that is, the number of individuals specified by the integral control strategy (6.8) is equal to the number of individuals released (or planted etc.) at each time-step. In the context of restocking schemes we expect that activation errors are generally less prevalent than measurement errors, and so only give a brief treatment. Accommodating an additive activation error

$d(t)$, (6.9) becomes

$$\left.\begin{aligned} x(t+1) &= Ax(t) + b(u(t)+d(t)), & x(0) &= x^0, \\ y(t) &= c^T x(t), & & \\ u(t+1) &= u(t) + g(r - y(t)), & u(0) &= u^0, \end{aligned}\right\} \quad \forall \quad t \in \mathbb{N}_0, \quad (6.32)$$

where g is small enough so that the conclusions of Theorem 6.3.2 apply to the integral control system (6.9). One advantage of integral control is it rejects constant, or convergent activation errors. Specifically, if $d(t)$ equals a constant $\tilde{d}$ for each t (that is, a constant systematic activation error is made), or $d(t)$ converges to $\tilde{d}$, then the observations $y(t)$ still converge to r.

The effect of periodic or general additive activation errors on the observations mirror those in the observation error case. Specifically, if $d(t) = \hat{d}\cos(\omega t)$ for some $\hat{d} \in \mathbb{R}$ and $\omega > 0$, then again it is elementary to demonstrate that the measured variable $y(t)$ settles to the periodic signal

$$r + \hat{d}M_\omega \cos(\omega t + \psi_\omega),$$

which oscillates around r with magnitude $\hat{d}M_\omega$ and phase shift ψ_ω, where

$$M_\omega = \left| \frac{(e^{i\omega} - 1)\boldsymbol{G}(e^{i\omega})}{e^{i\omega} - 1 + g\boldsymbol{G}(e^{i\omega})} \right|$$

and

$$\psi_\omega = \arg \left(\frac{(e^{i\omega} - 1)\boldsymbol{G}(e^{-\omega})}{e^{i\omega} - 1 + g\boldsymbol{G}(e^{i\omega})} \right).$$

For arbitrary additive activation errors $d(t)$ one can show that

$$\limsup_{t\to\infty} |y(t) - r| \le \nu_g \limsup_{t\to\infty} |d(t)|, \tag{6.33}$$

where

$$\nu_g := \sum_{j=0}^{\infty} \left| \begin{pmatrix} c^T & 0 \end{pmatrix} \begin{pmatrix} A & b \\ -gc^T & 1 \end{pmatrix}^j \begin{pmatrix} b \\ 0 \end{pmatrix} \right|.$$

As with the estimate (6.15), the bound (6.33) depends linearly on the magnitude of the activation error. As with observation errors, we note that assumptions **(A6.1)** and **(A6.2)** and the size of the gain parameter g are all independent of the activation errors considered in (6.32).

6.3.2 Can integral control be extended to incorporate additional feasibility constraints on the input $u(t)$?

If we require that the input $u(t)$ satisfied $0 \leq u(t) \leq U$, where U is a chosen per time-step upper bound, and if the reference r is such that $0 < r < \boldsymbol{G}(1)U$ then a modified integral control model still achieves the desired control objective **(P2)**. Furthermore, if $r \geq \boldsymbol{G}(1)U$ then the control objective *cannot* be solved by replenishment alone. The main result of this section which establishes the above claims is Theorem 6.3.7, and is a special case of [23, Theorem 3.2].

We bound the input in the integral control system (6.9) by applying a filter to the input. To that end we introduce the saturation nonlinearity φ, which replaces negative control signals by zero and includes the upper bound U chosen by the modeler for the maximum control signal:

$$\varphi \,:\, \mathbb{R} \to \mathbb{R}, \quad \varphi(v) := \begin{cases} 0, & \text{for} \quad 0 < v, \\ v, & \text{for} \quad 0 \leq v \leq U, \\ U, & \text{for} \quad U < v. \end{cases} \tag{6.34}$$

The feedback system (6.9) is replaced by

$$\left.\begin{aligned} x(t+1) &= Ax(t) + bu(t), & x(0) &= x^0, \\ w(t+1) &= w(t) + g(r - c^T x(t)), & w(0) &= w^0, \\ u(t) &= \varphi(w(t)), \end{aligned}\right\} \qquad \forall \quad t \in \mathbb{N}_0. \tag{6.35}$$

The inclusion of φ in (6.35) ensures that a nonnegative population is always predicted. The scalar $w(t)$ is the integrator state, and is generated by the integrator (6.8). The control $u(t)$ is the *filtered* integrator state $\varphi(w(t))$. Figure 6.7 contains a diagram of this arrangement.

In addition to tackling **(P2)**, Theorem 6.3.7 also provides the upper bound $1/|\gamma|$ for the integrator gain g, where the constant γ is given by

$$\gamma := \sup_{q \geq 0} \left\{ \inf_{\theta \in [0, 2\pi)} \operatorname{Re} \left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta} - 1} \right) \boldsymbol{G}(e^{i\theta}) \right] \right\} \in \mathbb{R}, \tag{6.36}$$

and $\boldsymbol{G}$ is the transfer function from (6.10).

Theorem 6.3.7. *Assume that the linear system* (6.4) *satisfies assumption* **(A6.1)** *and* **(A6.2)** *and let γ be as in* (6.36)*. Then, for every $U > 0$, every $r \in (0, \boldsymbol{G}(1)U)$, every $g \in (0, 1/|\gamma|)$ and all initial conditions $(x^0, w^0) \in \mathbb{R}^n_+ \times \mathbb{R}_+$, the solution (x, u) of* (6.35) *has properties (1)-(3) of Theorem 6.3.2 and furthermore the integrator state $w(t)$ converges to $r/\boldsymbol{G}(1)$ as $t \to \infty$.*

Theorem 6.3.7 is proven in [23, Theorem 3.2]. We do however provide some

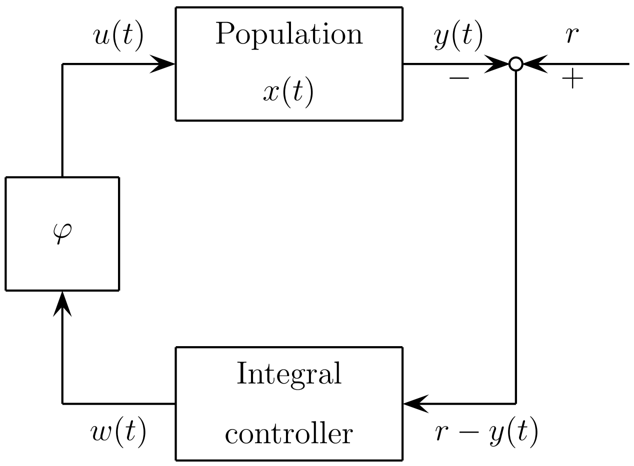


Figure 6.7: Block diagram of the feedback system (6.35). The control signal $u(t)$ applied to the population equals the filtered integrator state $\varphi(w(t))$, where $w(t)$ is generated by the integrator (6.8).

remarks.

Remark 6.3.8. *Although the conclusions (1)-(3) of Theorem 6.3.7 are the same as those in Theorem 6.3.2, there is a crucial difference in the hypotheses of these theorems. Specifically, in Theorem 6.3.7 the desired reference value r is not completely free: it is constrained by the steady-state gain $\boldsymbol{G}(1)$ and input bound U by the requirement that $r < \boldsymbol{G}(1)U$. This is not unreasonable; in the presence of no control, population density is declining. If the upper limit on the number of new arrivals U is too low, or alternatively, the chosen reference r is too high, then the observations of the population cannot reach r by restocking alone. We comment further that, mathematically, this limitation is not unique to integral control. A consequence of the model under consideration (in particular* (6.5)*) is that if $u(t)$ is bounded from above by U then* any *restocking scheme cannot lead to the eventual observations ever being larger than $\boldsymbol{G}(1)U$. If $r > \boldsymbol{G}(1)U$ then the observations* cannot *asymptotically reach r by restocking alone.*

Remark 6.3.9. *As with Theorem 6.3.2, Theorem 6.3.7 is a low-gain result and provides the upper bound $1/|\gamma|$ for the gain g that will ensure convergence. It is shown in [22] that*

$$-\infty < \gamma \leq \frac{-\boldsymbol{G}(1)}{2}.$$

The parameter γ can be estimated numerically from its definition (6.36) *although this may not always be straightforward. If* **(A6.1)** *and* **(A6.2)** *hold and if b and c^T are nonnegative then*

$$\kappa := \frac{2}{\boldsymbol{G}(1) + 2|\boldsymbol{G}'(1)|} \leq \frac{1}{|\gamma|}, \tag{6.37}$$

where $\boldsymbol{G}'$ denotes the derivative of $\boldsymbol{G}$. The constant κ is much easier to compute than γ. The derivation of (6.37) *follows after this remark. Consequently, under the assumptions* **(A6.1)** *and* **(A6.2)***, every gain $g \in (0, \kappa)$ is a "regulating gain" in the sense that conclusions (1)-(3) of Theorem 6.3.2 hold for* (6.35)*.*

We prove the inequality in (6.37). For a sequence v we use the notation $\hat{v}$ to denote the Z-transform of v given by

$$\hat{v}(z) = \sum_{j=0}^{\infty} v(j) z^{-j},$$

defined for all complex z where the summation converges absolutely. The step response of the linear system (6.4) is the output of (6.4) subject to zero initial state ($x^0 = 0$) and constant input $\tilde{u} = 1$ and is given by

$$s(0) = 0, \qquad s(t) = \sum_{j=0}^{t-1} c^T A^j b, \qquad \forall \quad t \in \mathbb{N}.$$

Assumption **(A6.1)** ensures that $s(t) \to \boldsymbol{G}(1)$ as $t \to \infty$. Furthermore, a calculation shows that s has Z-transform

$$\hat{s}(z) = \frac{z\boldsymbol{G}(z)}{z-1}, \qquad z \in \mathbb{C}, \ |z| > 1.$$

Under the assumptions that $A, b, c^T \geq 0$ and **(A6.2)** it follows that $s(t) \geq 0$ and is nondecreasing. We define the step response error

$$e(t) = s(t) - \boldsymbol{G}(1), \qquad \forall \quad t \in \mathbb{N}_0,$$

which is consequently nonpositive, nondecreasing and converges to 0. Furthermore, the Z-transform of e satisfies

$$\frac{\hat{e}(z)}{z} = \frac{\boldsymbol{G}(z) - \boldsymbol{G}(1)}{z-1}, \tag{6.38}$$

for every complex z with modulus greater than one. Since $\boldsymbol{G}$ is differentiable at $z = 1$ we note that

$$\lim_{z\to 1} \hat{e}(z) = \lim_{z\to 1} \frac{\hat{e}(z)}{z} = \lim_{z\to 1} \frac{\boldsymbol{G}(z) - \boldsymbol{G}(1)}{z-1} = \boldsymbol{G}'(1). \tag{6.39}$$

As $z \mapsto \hat{e}(z)/z$ is continuous outside of the unit disc the above shows that we can extend $z \mapsto \hat{e}(z)/z$ continuously to $z = 1$ with

$$\hat{e}(1) = \frac{\hat{e}(1)}{1} = \boldsymbol{G}'(1). \tag{6.40}$$

We now use (6.40) and the property that $e(t) \le 0$ for every t to show that for any complex z with modulus one,

$$\begin{aligned} -\boldsymbol{G}'(1) = -\hat{e}(1) &= -\sum_{k=0}^{\infty} e(k) = \sum_{k=0}^{\infty} |e(k)| \cdot |e^{-(k+1)}| \ge \left| \frac{\hat{e}(z)}{z} \right| \\ &\ge \text{Re}\left(\frac{-\hat{e}(z)}{z} \right) = \text{Re}\left(\frac{\boldsymbol{G}(1) - \boldsymbol{G}(z)}{z-1} \right) \\ &= -\frac{\boldsymbol{G}(1)}{2} - \text{Re}\left(\frac{\boldsymbol{G}(z)}{z-1} \right). \end{aligned} \tag{6.41}$$

Rearranging (6.41) gives

$$\text{Re}\left(\frac{\boldsymbol{G}(z)}{z-1} \right) \ge \boldsymbol{G}'(1) - \frac{\boldsymbol{G}(1)}{2}, \qquad \forall \quad z \in \mathbb{C},\ |z| = 1,$$

which implies that

$$\inf_{|z|=1} \text{Re}\left(\frac{\boldsymbol{G}(z)}{z-1} \right) = \inf_{\theta\in[0,2\pi)} \text{Re}\left(\frac{\boldsymbol{G}(e^{i\theta})}{e^{i\theta}-1} \right) \ge \boldsymbol{G}'(1) - \frac{\boldsymbol{G}(1)}{2}. \tag{6.42}$$

From [23] we have that γ satisfies

$$-\infty < \gamma \le \frac{-\boldsymbol{G}(1)}{2} < 0$$

as $\boldsymbol{G}(1) > 0$, and so

$$0 > \gamma \ge \inf_{\theta\in[0,2\pi)} \text{Re}\left(\frac{\boldsymbol{G}(e^{i\theta})}{e^{i\theta}-1} \right) \ge \boldsymbol{G}'(1) - \frac{\boldsymbol{G}(1)}{2}, \tag{6.43}$$

where we have used the estimate (6.42). It is clear from $\gamma \le -\boldsymbol{G}(1)/2$ and (6.43) that $\boldsymbol{G}'(1) < 0$ and consequently inequality (6.43) is equivalent to

$$0 < -\gamma = |\gamma| \le -\boldsymbol{G}'(1) + \frac{\boldsymbol{G}(1)}{2} = |\boldsymbol{G}'(1)| + \frac{\boldsymbol{G}(1)}{2} =: \frac{1}{\kappa},$$

which implies (6.37).

Example 6.3.10. *Consider a planting program to raise levels of the savannah grass Setaria incrassata in the presence of heavy grazing. [105] contains matrix PPMs of Setaria incrassata where the population is partitioned into five stage-classes according to tuft circumference in cm. The specific divisions are given in [105, Table 2]. The matrix we use is the average over* 4 *years [105, Table 3, first row]. We control the second stage-class, plants of tuft diameter* $11-20$ *cm, and observe the total density of all plants with tuft diameter greater than* 11 *cm, that is, stages two to five. The matrix* A*, control vector* b *and observation vector* c^T *are thus given by*

$$\begin{gathered} A = \begin{pmatrix} 0.5925 & 0.5900 & 0.5825 & 0.8100 & 4.5650 \\ 0.2075 & 0.3775 & 0.2475 & 0.4675 & 0.1675 \\ 0.0050 & 0.1250 & 0.4225 & 0.1850 & 0.2625 \\ 0 & 0 & 0.0850 & 0.2750 & 0.1225 \\ 0 & 0 & 0 & 0.0325 & 0.6600 \end{pmatrix}, \\ b = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad c^T = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \end{pmatrix}. \end{gathered} \tag{6.44}$$

Figure 6.8(a) demonstrates the results of the filtered integral control system (6.35) *for different* U *and also the original integral control system* (6.9)*. Here* U *denotes the* maximum *number of individuals that can be planted each year. From a random initial population distribution with total density* 200 *the goal is to raise the total measured population density to* 800*. In this example,* $\boldsymbol{G}(1) = 8.1081$ *and so for fixed* $r = 800$ *the condition* $r < \boldsymbol{G}(1)U$ *necessitates that* U *satisfies*

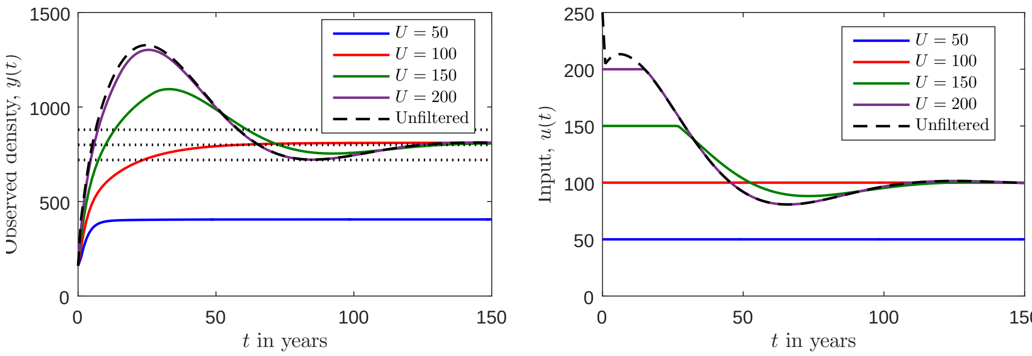
$$\frac{r}{\boldsymbol{G}(1)} = 98.6673 < U,$$

for the conclusions of Theorem 6.3.7 to hold. As expected, therefore, for $U = 50$ *the observation does not reach the reference. As* A, b *and* c^T *are nonnegative we can use the constant* κ *in* (6.37) *as an upper bound for a regulating gain* g *which gives*

$$\kappa = \frac{2}{\boldsymbol{G}(1) + 2|\boldsymbol{G}'(1)|} = \frac{2}{8.10 + 2 \times 126.42} = 0.0077.$$

We thus take $g = 0.0076 < \kappa$*. Figure 6.8(b) contains the resulting filtered input signals* $\varphi(w(t))$ *for each* U *and the unfiltered signal* $u(t)$ *given by* (6.8)*. We see that the linear feedback system* (6.9) *exhibits a large transient amplification, but also that the tracking takes longer and there is larger subsequent undershoot.*

Observe that here each filtered signal is truncated at U and that as U gets larger both the input and the observed population density behave more like the linear case as the filter effect is reduced.



(a) (b)

Figure 6.8: Integral control with filtered input (6.35) applied to the savannah grass matrix PPM of Example 6.3.10 with different U values. (a) Observations as colored solid lines, labeled with the corresponding U value. The black dashed line is the observation subject to the unfiltered integral control system (6.9). The dotted lines are the reference $r = 800$ and $r \pm 10\%$. (b) Filtered input signal $u(t) = \varphi(w(t))$ in colored lines labeled with corresponding U value. The black dashed line is the unfiltered input generated by (6.8).

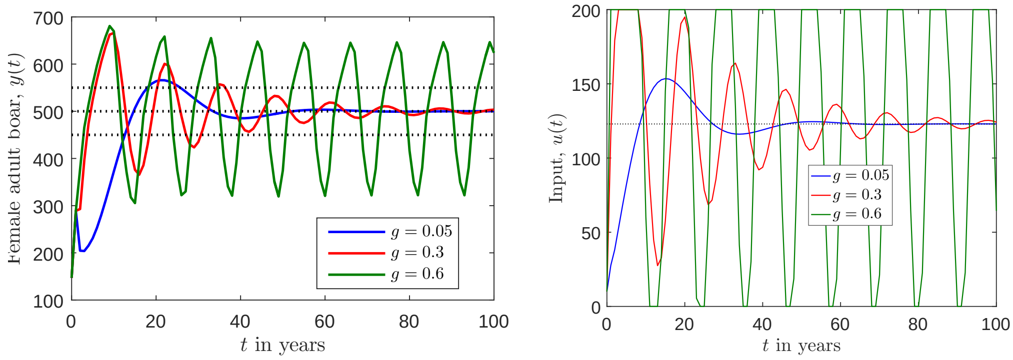
Remark 6.3.11. *We comment that Theorem 6.3.7 can be extended to the feedback system* (6.35) *with the nonlinear filter φ replaced by other nonlinearities. For example, if φ is replaced by a function that grows sublinearly then there are increasingly diminishing returns from larger input signals. In the context of a plant population, if the control term $bu(t)$ denotes sowing seeds, then at high densities the proportion of seeds that become plantlings may not depend linearly on the number of seeds sown owing to density-dependence effects. Such an effect can be modeled by a suitable choice of φ in* (6.35).

6.3.3 How small does the gain g in the feedback strategy (6.8) need to be?

Here we discuss the design parameter g in more detail. We seek to explain its role and how suitable g can be chosen or estimated. Finally, we include another feature in the integral control model which computes g adaptively, circumventing the need to choose it altogether **(P3)**.

The choice of g affects the performance of integral control. As a tuning parameter; a larger g usually corresponds to a faster response, which is sometimes desirable. As the next precautionary example demonstrates, however, choosing g too large may result in failure of the control objective.

Example 6.3.12. *We revisit the wild boar PPM considered in Example 6.3.3. For fixed A, b, c^T, x^0, u^0 and r as in that example, we project the filtered integral control system* (6.35) *with* $U = 200$ *for increasing gains* $g = 0.05, 0.3, 0.6$ *and have plotted the results in Figure 6.9 . We see that the observations oscillate around r with greater magnitude as g increases, and fails to converge to the reference for* $g = 0.6$*. Note that the filtered input $u(t)$ is truncated at both* 0 *and U.*



(a) (b)

Figure 6.9: Integral control with filtered input (6.35) applied to the wild boar matrix PPM of Example 6.3.3 with different gain parameters. See Example 6.3.12. The three colored lines represent the three values of g, see the legend. (a) Observations. The dotted black lines are the reference $r = 500$ and $r \pm 10\%$. (b) Filtered input signals with $U = 200$. The dotted line is $r/\boldsymbol{G}(1)$

Recall the characterization from (6.12) of which gains g result in convergence of the observations - those such that $\rho(A_g) < 1$. Describing the dependence of $\rho(A_g)$ on g analytically is, in general, intractable. It is of course true that for each candidate $g > 0$, $\rho(A_g)$ can be computed numerically, but this does not provide a systematic method of finding how large g can be, or the qualitative behavior of the resulting dynamics. Notwithstanding the above, the *root locus* method developed in [39, 40] is a graphical method of describing how the eigenvalues of A_g in this instance (more precisely, the poles of the closed-loop system (6.12)) change with the parameter g. This powerful technique can be used to choose g in such a manner that both of the conclusions of Theorem 6.3.2 apply and qualitative and quantitative properties of the resulting dynamics are specified. Many textbooks provide a modern treatment of the root locus method and we refer the reader to [44, Chapter 4] for more information.

Regarding model uncertainty, we comment that the choice of g is robust to model uncertainty in the following sense. If $g^* > 0$ is such that $\rho(A_{g^*}) < 1$ then there exists $\varepsilon > 0$ such that $\rho(\tilde{A}_{g^*}) < 1$ for all $\tilde{A}_{g^*}$ with $\|A_{g^*} - \tilde{A}_{g^*}\| < \varepsilon$.

In words, if g^* is a regulating gain for a given A_g then g^* is a regulating gain for all $\tilde{A}_{g^*}$ "close-enough" to A_{g^*}. Recalling that A_g depends on A, b, c^T and g, this amounts to model uncertainty in A, b and c^T that is "small enough". The terms "close enough" and "small enough" can be precisely quantified by appealing to stability radius arguments [58, 59].

The presence of the nonlinear filter φ in the integral control system (6.35) prevents the root locus method from being applied here and the proof of Theorem 6.3.7 is more subtle. Here it is very difficult in general to find an exact expression for the largest gain that results in convergence, and so in order to apply Theorem 6.3.7 a positive lower bound for $1/|\gamma|$ is required. The constant κ in (6.37) is such a bound in the (usual) case where A, b and c^T are nonnegative. However, the same problem arises as with the precomputed control (6.7) because the formula for κ depends on the model data A, b and c^T. Although γ and κ are robust to model uncertainty in a similar sense to g as described above (that is, "small" perturbations to A, b and c^T can be tolerated), in the presence of severe uncertainty in A, b and c^T, using (6.37) may not give a correct lower bound for the "true" $1/|\gamma|$.

A different approach, therefore, may be desirable for choosing g. The next method we present is an example of *adaptive control* (see [84, 3]), where in this instance the parameter g is determined via a suitable adaptation rule. That is, we allow the gain parameter g also to change with time, determined by a dynamical system included in the feedback loop. Specifically, we set

$$g(t) = \frac{1}{h(t)}, \qquad h(t+1) = h(t) + |r - y(t)|, \qquad \forall \quad t \in \mathbb{N}_0,$$

which yields the adaptive integral control system

$$\left.\begin{aligned} x(t+1) &= Ax(t) + bu(t), & x(0) &= x^0, \\ w(t+1) &= w(t) + (h(t))^{-1}(r - c^T x(t)), & w(0) &= w^0, \\ h(t+1) &= h(t) + |r - y(t)|, & h(0) &= h^0, \\ u(t) &= \varphi(w(t)). \end{aligned}\right\} \quad \forall \quad t \in \mathbb{N}_0. \tag{6.45}$$

Figure 6.10 contains a diagram of the arrangement in (6.45). The main result of this section is Theorem 6.3.13 below, which is a special case of a result in [88], and is an adaptive version of Theorem 6.3.7 which obviates the need to choose a gain parameter g.

Theorem 6.3.13. *Assume that the linear system* (6.4) *satisfies assumptions* **(A6.1)** *and* **(A6.2)**. *Then, for every $U > 0$, every $r \in (0, \boldsymbol{G}(1)U)$ and all initial conditions $(x^0, w^0, h^0) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times (0, \infty)$, the solution (x, u, h) of* (6.45) *has properties (1)-(3) of Theorem 6.3.2, the integrator state $w(t)$ converges to*

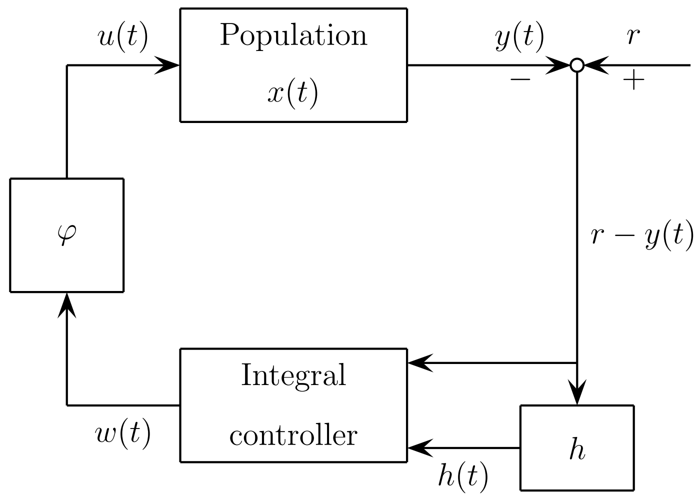


Figure 6.10: Block diagram of the adaptive feedback system (6.45). The constant gain parameter g is replaced by a dynamic signal $h(t)$ which itself is determined by the difference $r - y(t)$.

$r/\boldsymbol{G}(1)$ *as* $t \to \infty$ *and additionally*

(4) the nonincreasing gain $g(t) = 1/h(t)$ *converges to a positive limit depending on* (x^0, w^0, h^0) *as* $t \to \infty$.

Remark 6.3.14. *Theorem 6.3.13 is remarkable because it ensures that the integral control system* (6.45) *achieves the desired objective in the presence of very little information. The reference* r*, observations* $y(t)$ *and assurance that* $r < \boldsymbol{G}(1)U$ *are required, but knowledge of* A*,* b*,* c^T*,* x^0 *and crucially a suitable gain* $g > 0$ *is not.*

As with Theorem 6.3.7, the version of Theorem 6.3.13 presented is a special case of a more general result, where the filter φ can be replaced by other function. We provide more details of these.

Theorem 6.3.7 applies when φ in (6.34) is replaced by any function $\varphi : \mathbb{R} \to \mathbb{R}$ that satisfies a so-called Lipschitz condition, namely:

(A6.3) There exists $L > 0$ such that $0 \leq \varphi(v) - \varphi(w) \leq L(v - w)$ for all $v \geq w$.

The constant L in assumption **(A6.3)** is called the Lipschitz constant of φ and, for example, the function φ in (6.34) satisfies **(A6.3)** with $L = 1$.

For a function $\varphi : \mathbb{R} \to \mathbb{R}$ and a set $X \subseteq \mathbb{R}$ we let $\operatorname{im}\varphi$ and $\varphi^{-1}(X)$ denote the image of φ and the preimage of X under the function φ respectively.

In this more general setting, Theorem 6.3.7 can be restated as: Assume that (6.35) satisfy **(A6.1)-(A6.3)**. Then, for every $r \in \mathbb{R}$ such that $r/\boldsymbol{G}(1) \in \operatorname{im} \varphi$, every $g \in (0, 1/|\gamma L|)$ and all initial conditions $(x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}$, statements (1)-(3) of Theorem 6.3.2 hold. Moreover, if additionally $\varphi^{-1}(r/\boldsymbol{G}(1))$ is a singleton then (x^r, u^r) is a globally asymptotically stable equilibrium of (6.35).

The adaptive integral control result, Theorem 6.3.13, can be restated as: Assume that (6.45) satisfies assumptions **(A6.1)-(A6.3)**. Then, for every $r \in \mathbb{R}$ such that $r/\boldsymbol{G}(1) \in \operatorname{im} \varphi$, and all initial conditions $(x^0, u^0, h^0) \in \mathbb{R}^n \times \mathbb{R} \times (0, \infty)$,

(1)
$$\lim_{t\to\infty} u(t) = \frac{r}{\boldsymbol{G}(1)},$$

(2)
$$\lim_{t\to\infty} x(t) = x^r := (I - A)^{-1} b \frac{r}{\boldsymbol{G}(1)},$$

(3)
$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} c^T x(t) = r.$$

Moreover, if $\varphi^{-1}(r/\boldsymbol{G}(1))$ is a singleton, then

(4) the nonincreasing gain $k(t) = 1/h(t)$ converges to a positive limit as $t \to \infty$,

(5)
$$\lim_{t\to\infty} w(t) = w^r,$$
where $\varphi(w^r) = r/\boldsymbol{G}(1)$.

Example 6.3.15. *Theorem 6.3.13 is applied to the wild boar model of Example 6.3.3. For the same A, b, c^T as in that example, but with $r = 200$, the adaptive integral control system* (6.35) *for gains g determined adaptively via* (6.45) *is projected for three different (x^0, w^0, h^0) triples. The results are plotted in Figure 6.11. Here the convergence of the observations to the reference ensured by Theorem 6.3.13 is slow, note the $\log x$-axes in the figure. This is because in the adaptive control scheme* (6.45) *the gain $g(t) = 1/h(t)$ always decreases and can become small very quickly resulting in sluggish performance. Recall, however, that the control scheme has no knowledge of A, b or c^T, only that $\rho(A) < 1$, $\boldsymbol{G}(1) > 0$ and that $r < \boldsymbol{G}(1)U$.*

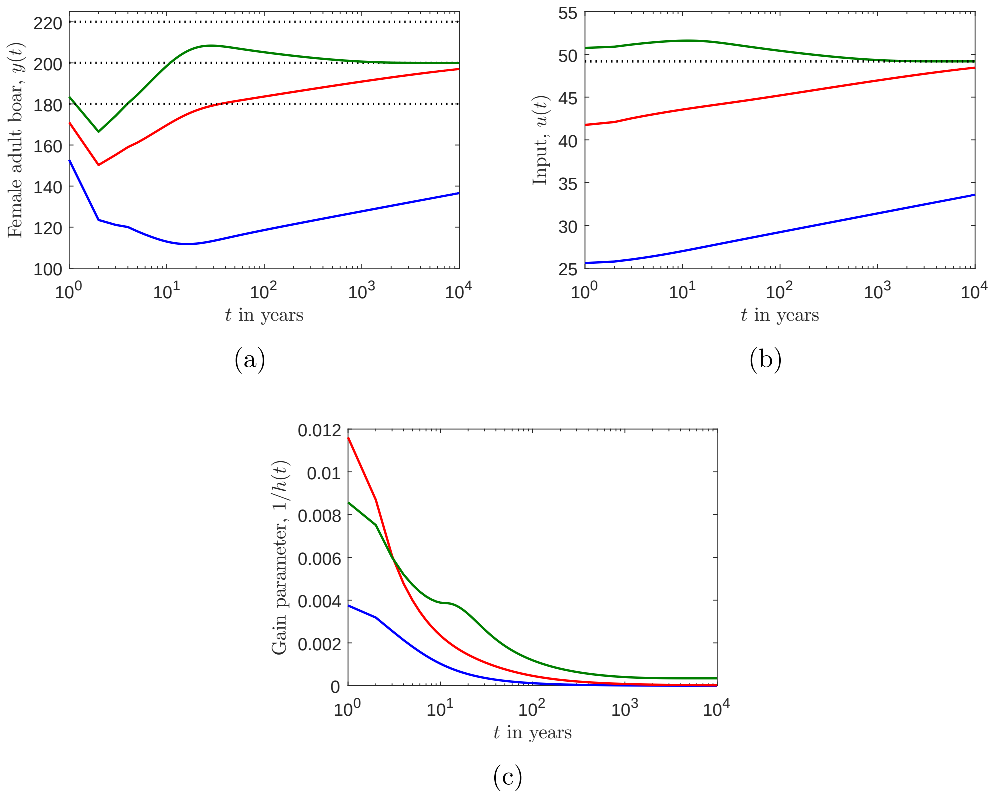


Figure 6.11: Adaptive integral control (6.45) applied to the wild boar matrix PPM of Example 6.3.3 with different initial triples (x^0, w^0, h^0). See Example 6.3.15. In each plot the blue, red and green lines are corresponding simulations. (a) Observations. The dotted lines are the reference $r = 200$ and $r \pm 10\%$. (b) Filtered input signals and limiting input $r/\boldsymbol{G}(1)$ in dotted line. (c) Adaptive gain parameters $g(t) = 1/h(t)$.

6.3.4 Can the rate of convergence of the observations to the reference be improved?

By adding a proportional part to the integral (PI) control feedback strategy (6.8) the resulting rate of convergence of the observations to the reference can be increased **(P4)**.

So far, we have been using integral control to move the equilibrium of a declining model to a chosen nonzero equilibrium as mentioned in the introduction of this chapter. Integral control is just one part of PID-control. Loosely speaking, the observations resulting from a PI control strategy converge faster to the reference.

We proceed to give the details. In the first instance we replace the integral control system (6.9) by

$$\left.\begin{aligned} x(t+1) &= Ax(t) + bu(t), & x(0) &= x^0, \\ w(t+1) &= w(t) + g(r - c^T x(t)), & w(0) &= w^0, \\ u(t) &= w(t) + g(r - c^T x(t)), \end{aligned}\right\} \qquad \forall \quad t \in \mathbb{N}_0. \tag{6.46}$$

In (6.46), w is the integrator state and u is the control, now given by

$$u(t) = w^0 + g \sum_{j=0}^{t} (r - y(j)), \qquad \forall \quad t \in \mathbb{N}_0. \tag{6.47}$$

Recall (6.8) which is

$$u(0) = u^0, \qquad u(t) = u^0 + g \sum_{j=0}^{t-1} (r - y(j)), \qquad \forall \quad t \in \mathbb{N}.$$

The difference between (6.8) and (6.47) is that in the later, at each time-step t, $u(t)$ depends of the current observation error $r - y(t)$ and not just the *previous* errors. In our original system (6.9) we had $u(t) = w(t)$, that is, the control was simply an integrator - I control. We now compute u by adding to w the current error $r - y(t)$. The motivation for using such a control strategy is that the current error $r - y(t)$ acts as a proportional (P) part which increases the rate of convergence.

As we are considering population models, where $x(t)$ needs to be nonnegative, for the model (6.46) to be meaningful we require the constraint that $A - gbc^T$ is component-wise nonnegative, which we note may not always be satisfied. However, whenever this is the case, the conclusions of Theorem 6.3.2 and Theorem 6.3.13 hold for the integral control system (6.46) with small enough gain g and suitably modified adaptive case respectively. The conclusions of Theorem 6.3.7 also hold (see [22]), but with γ in (6.36) replaced by

$$\gamma_0 := \sup_{q \geq 0} \left\{ \inf_{\theta \in [0, 2\pi)} \operatorname{Re} \left[\left(q + \frac{e^{i\theta}}{e^{i\theta} - 1} \right) \boldsymbol{G}(e^{i\theta}) \right] \right\}.$$

Under the assumptions **(A6.1)** and **(A6.2)** and if b and c^T are nonnegative, we demonstrate that

$$\kappa_0 := \frac{2}{\boldsymbol{G}(1) + 2|\boldsymbol{G}(1) + \boldsymbol{G}'(1)|} \leq \frac{1}{|\gamma_0|}, \tag{6.48}$$

so that the conclusions of the theorem hold for the system (6.46) for every gain g such that $g \in (0, \kappa_0)$, Furthermore, we show that $\kappa < \kappa_0$, so that certainly the range of regulating gains for (6.46) is not smaller than that for (6.9).

For complex z with modulus greater than or equal to one the transfer function $\boldsymbol{G}$ given by (6.10) of the linear system (6.4) can be written as

$$G(z)\sum_{j=0}^{\infty} g_j z^{-j}, \qquad \text{where} \qquad g_j = \begin{cases} 0, & j=0, \\ c^T A^{j-1} b, & j \geq 1\,. \end{cases}$$

We define $z \mapsto \tilde{\boldsymbol{G}}(z) := z\boldsymbol{G}(z)$ and introduce the constant

$$\tilde{\gamma} := \sup_{q\geq 0}\left\{ \inf_{\theta\in[0,2\pi)} \operatorname{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta}-1}\right)\tilde{\boldsymbol{G}}(e^{i\theta})\right]\right\}.$$

We know that $-\infty < \tilde{\gamma} \leq -\tilde{\boldsymbol{G}}(1)/2 = -\boldsymbol{G}(1)/2$. By inspection of the definition of $\tilde{\boldsymbol{G}}$, the constant $\tilde{\gamma}$, and γ_0 in (6.48) we see that

$$\tilde{\gamma} = \gamma_0, \tag{6.49}$$

$$\tilde{\boldsymbol{G}}'(z) = \boldsymbol{G}(z) + z\boldsymbol{G}'(z) \quad \text{and so} \quad \tilde{\boldsymbol{G}}'(1) = \boldsymbol{G}(1) + \boldsymbol{G}'(1). \tag{6.50}$$

We note that from (6.50) it follows that

$$\begin{aligned} \tilde{\boldsymbol{G}}'(1) = \boldsymbol{G}(1) + \boldsymbol{G}'(1) &= \sum_{j=1}^{\infty} g_j z^{-j} - \sum_{j=1}^{\infty} j g_j z^{-j} \\ &= \sum_{j=1}^{\infty} (1-j) g_j z^{-j} \leq 0, \end{aligned} \tag{6.51}$$

and consequently we can apply the estimate (6.37) to $\tilde{\boldsymbol{G}}$ to yield that

$$\frac{2}{\tilde{\boldsymbol{G}}(1) + 2|\tilde{\boldsymbol{G}}'(1)|} \leq \frac{1}{|\tilde{\gamma}|}. \tag{6.52}$$

In light of (6.49), (6.50) and the following definition of κ_0, (6.52) implies that

$$\kappa_0 := \frac{2}{\boldsymbol{G}(1) + 2|\boldsymbol{G}(1) + \boldsymbol{G}'(1)|} = \frac{2}{\tilde{\boldsymbol{G}}(1) + 2|\tilde{\boldsymbol{G}}'(1)|} \leq \frac{1}{|\tilde{\gamma}|} = \frac{1}{|\gamma_0|}.$$

Finally, as $\boldsymbol{G}(1) > 0$, it is clear from (6.51) that $\boldsymbol{G}'(1) \leq \tilde{\boldsymbol{G}}'(1) \leq 0$ and thus

$$|\boldsymbol{G}(1) + \boldsymbol{G}'(1)| = |\tilde{\boldsymbol{G}}'(1)| < |\boldsymbol{G}'(1)|. \tag{6.53}$$

From inequality (6.53) we deduce that

$$\kappa = \frac{2}{\boldsymbol{G}(1) + 2|\boldsymbol{G}'(1)|} < \frac{2}{\boldsymbol{G}(1) + 2|\boldsymbol{G}(1) + \boldsymbol{G}'(1)|} = \kappa_0,$$

as required.

The rate of convergence of the observation to the reference can be tuned even further in the linear integral control case by making the following alteration. We consider now the feedback scheme

$$\left.\begin{aligned} x(t+1) &= Ax(t) + bu(t), & x(0) &= x^0, \\ w(t+1) &= w(t) + g(r - c^T x(t)), & w(0) &= w^0, \\ u(t) &= w(t) + k(r - c^T x(t)), \end{aligned}\right\} \quad \forall \quad t \in \mathbb{N}_0. \tag{6.54}$$

The term $k(r - c^T x(t))$ is a proportional feedback and the parameter $k > 0$ is called the proportional feedback gain. We note that the integral control system (6.46) is a special case of (6.54) where $k = g$, but in general they need not be the same. Although the parameter k introduces another choice that has to be made by the modeler, its inclusion often results in faster convergence of the observations to the reference. Our main result for PI control is Theorem 6.3.16 below.

Theorem 6.3.16. *Assume that the linear system* (6.4) *satisfies assumptions* **(A6.1)** *and* **(A6.2)** *and assume that* $k > 0$ *is such that* $A - kbc^T$ *is nonnegative with* b *and* c^T *also assumed nonnegative. Then there exists* $g^* > 0$*, which depends on* k*, such that for all* $g \in (0, g^*)$*, every* $r > 0$ *and all initial conditions* $(x^0, w^0) \in \mathbb{R}^n_+ \times \mathbb{R}_+$*, the solution* (x, u) *of* (6.54) *satisfies properties (1)-(3) of Theorem 6.3.2 and additionally the integrator state* $w(t)$ *converges to* $r/\boldsymbol{G}(1)$ *as* $t \to \infty$*.*

Proof. By assumption $k > 0$ is chosen so that $A - kbc^T$ is component-wise nonnegative. Since A, b and c^T are also component-wise nonnegative we clearly have that $A \geq A - kbc^T$ and so Corollary 2.1.24 implies that

$$0 \leq \rho(A - kbc^T) \leq \rho(A) < 1.$$

We deduce that assumption **(A6.1)** holds for $A - kbc^T$. Moreover, one can show that the transfer function of $(A - kbc^T, b, c^T)$ is

$$z \mapsto \boldsymbol{G}_k(z) = \frac{\boldsymbol{G}(z)}{1 + k\boldsymbol{G}(z)},$$

so that

$$\boldsymbol{G}_k(1) = \frac{\boldsymbol{G}(1)}{1 + k\boldsymbol{G}(1)} > 0,$$

implying that assumption **(A6.2)** applies to $(A - kbc^T, b, c^T)$. Therefore, Theorem 6.3.2 now applies to the feedback system (6.54), that is the original integral control system (6.9) with A replaced by $A - kbc^T$. It is straightforward to demonstrate that the equilibria (x^*, u^*) of (6.54) are the same as those of (6.9). □

Example 6.3.17. *We compare the rates of convergence of the observations to the reference of the integral control systems* (6.9), (6.46) *and* (6.54) *when applied to the wild boar model of Example 6.3.3. The results are plotted in Figure 6.12. The systems* (6.9) *and* (6.46) *both have the same gain parameter* $g = 0.12$*, as in Example 6.3.3. We see that the observations of* (6.46) *converges faster and in a less oscillatory manner than those of* (6.9). *For the PI system* (6.54) *we take increasing proportional gain parameter* $k = 0.2, 0.3$*, and* $k = 0.4$ *and note the progressively faster convergence of the observations to the reference.*

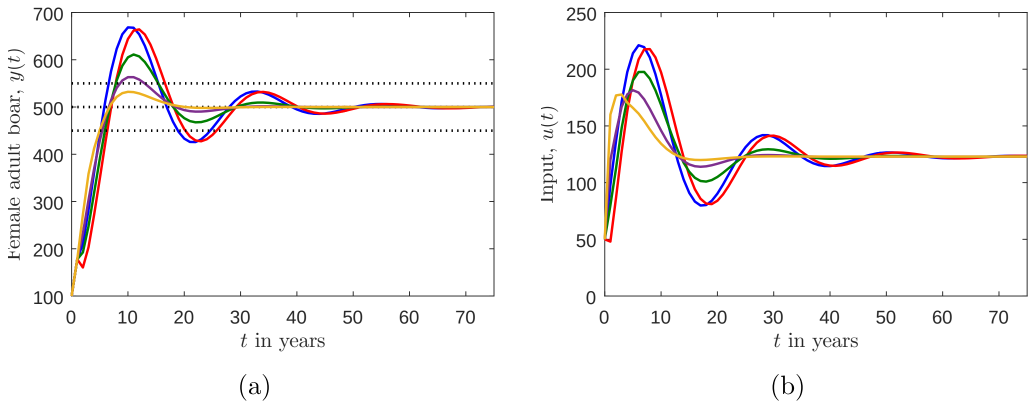


Figure 6.12: Integral control (6.9), integral control with proportional feedback (6.46) and PI system (6.54) applied to the wild boar matrix PPM of Example 6.3.3. See Example 6.3.17. (a) Observations. The dotted black lines are the reference $r = 500$ and $r \pm 10\%$. (b) Inputs. In both (a) and (b): the blue line is the original system (6.9) with $g = 0.12$, the red line is the system (6.46) with $g = 0.12$ and the green, purple and gold lines are the PI system (6.54) with increasing $k = 0.2$, 0.3 and 0.4 respectively.

6.3.5 Can integral control be applied to other population models?

Here we demonstrate that integral control can be applied to integral projection models (IPMs) and that the results on integral control for PPMs from Sections 6.3.2 and 6.3.3 extend to IPMs. We also comment on how certain spatially structured models fit into an integral control framework **(P5)**.

IPMs are a relatively recent approach to population modeling, introduced in [35]. Since their inception several models have been published in, for example [37, 21, 113, 107]. We refer the reader to [35], or the tutorial paper [14], for full details and only give a brief overview here. Typically an IPM takes the form

$$n(\xi, t+1) = \int_{s\in\Omega} k(s,\xi)n(s,t)ds, \qquad n(\xi,0) = n_0(\xi), \qquad \xi \in \Omega, \qquad t \in \mathbb{N}_0. \tag{6.55}$$

Here $n(\xi, t)$ denotes the population at stage $\xi \in \Omega$ and time-step t, where Ω is the range of size or stage-classes and is usually an interval of real numbers, although general sets are permitted (see [37]). For each fixed t, $n(\xi, t)$ is a function of ξ. The function k is called a projection kernel and describes the life history parameters of survival, growth and fecundity of the population.

The model (6.55) can be written in the form (6.1), where A now denotes the operator

$$A : L^1(\Omega) \to L^1(\Omega), \qquad (Av)(\xi) = \int_{s\in\Omega} k(s,\xi)v(s)ds, \tag{6.56}$$

where $L^1(\Omega)$ is the space of Lebesgue measurable functions (see, for example [38, p. 647]) with finite L^1 norm

$$L^1(\Omega) = \left\{ f : \Omega \to \mathbb{R} \ : \ f \text{ Lebesgue measurable and } \int_{x\in\Omega} |f(x)|dx < \infty \right\}.$$

In order to convert the IPM (6.1) (where A is now given by (6.56)) into a controlled and observed system (6.4) we need to introduce appropriate control vector b and observation vector c (the superscript ^{T} is omitted as we are no longer considering matrix transposition).

Example 6.3.18. *Suppose that for an IPM, $\Omega = [\alpha, \beta]$, the interval from the minimal size α to the maximal size β. In such a framework the control action is a mapping $\mathbb{R} \to L^1(\Omega)$ and a suitable choice for b is a function in $L^1(\Omega)$ so that the control term $bu(t)$ in (6.4) is b multiplied by the scalar $u(t)$. To model the distribution of new individuals arriving uniformly between stage-classes ξ_1 and ξ_2 with $\alpha \le \xi_1 < \xi_2 \le \beta$ we define b by*

$$b(s) = \begin{cases} \dfrac{1}{\xi_2 - \xi_1}, & s \in [\xi_1, \xi_2], \\ 0, & \text{otherwise.} \end{cases} \tag{6.57}$$

The function b distributes new arrivals uniformly between ξ_1 and ξ_2. In some applications, it may be more realistic that the distribution of new arrivals is not uniform, and perhaps centered around some point between ξ_1 and ξ_2. Such a control vector represents a 'smoother' version of b in (6.57). There are *many such functions with this property. The quartic function*

$$b'(s) = \begin{cases} \frac{30}{(\xi_2 - \xi_1)^5}(s-\xi_1)^2(s-\xi_2)^2, & s \in [\xi_1, \xi_2], \\ 0, & \text{otherwise}, \end{cases} \tag{6.58}$$

is one example. The scaling of b' is chosen so that b' integrates to one.

For matrix PPMs the observation vectors we consider are row vectors. The equivalent of a row vector in the IPM context is a linear mapping $L^1(\Omega) \to \mathbb{R}$. For example, the mapping

$$v \mapsto cv := \int_{\xi_1}^{\xi_2} v(s)ds, \tag{6.59}$$

models the measurement of the population density of v between stage-classes ξ_1 and ξ_2. When $\Omega = [\alpha, \beta]$ and $\xi_1 = \alpha$, $\xi_2 = \beta$ then c in (6.59) measures the entire population density.

Mathematically, PPMs and IPMs are very similar, although the latter involves some extra technicalities. Theorem 6.3.19 is the main result of this section and demonstrates that our main results for matrix PPMs carry over to IPMs. Theorem 6.3.19 is a combination of special cases of results originally proven in [23] and [88].

The two key assumptions **(A6.1)** and **(A6.2)** in the matrix PPM case captured the properties that the uncontrolled population is in asymptotic decline and that the control, model and observation are chosen so that the steady-state gain is nonzero respectively. The same assumptions are required for IPMs although the formulation is slightly more technical: specifically, let X denote a Banach space,

(A6.4) the bounded linear operator $A : X \to X$ has $\rho(A) < 1$,

(A6.5) the operators $A : X \to X$, $b : \mathbb{R} \to X$ and $c : X \to \mathbb{R}$ are all bounded and such that $c(I - A)^{-1}b > 0$.

We comment that assumption **(A6.4)** can be checked numerically and assumption **(A6.5)** generally holds for the IPMs presented here. In more detail, for $\Omega = [\alpha, \beta]$ the space $L^1(\Omega)$ is a Banach space and for "reasonable" kernels k, (for instance, if k is square integrable) the operator A in (6.56) is compact. Compact operators can be uniformly approximated by finite dimensional operators, so the spectral radius A can be estimated by computing the spectral radii of a sequence of finite dimensional approximations of A. More precisely, if $(A_n)_{n=1}^{\infty}$ is a matrix sequence that approximates A uniformly, then by, for example, [28, Theorem 2.1] the spectral radii $\rho(A_n)$ converges to $\rho(A)$.

Assumption **(A6.5)** means that a constant positive input signal eventually gives rise to a positive observation. Alternatively, suppose that the controlled and observed IPM is given by A (for reasonable kernels k), input b and observation c as in (6.56), (6.57) and (6.59) respectively. If A, b and c are uniformly approximated by A_n, b_n and c_n then

$$\boldsymbol{G}_n(1) := c_n(I - A_n)^{-1}b_n \to c(I - A)^{-1}b = \boldsymbol{G}(1) \qquad \text{as} \quad n \to \infty,$$

and so the computable steady-state gain of A_n, b_n and c_n converge to that of A, b and c.

Theorem 6.3.19. *Given the controlled and observed projection system* (6.9) *in the IPM case, then under assumptions* **(A6.4)** *and* **(A6.5)** *the conclusions of Theorem 6.3.2 hold. If additionally the input bound* $U > 0$ *and reference* $r > 0$ *are such that* $r \in (0, \boldsymbol{G}(1)U)$ *then the conclusions of Theorems 6.3.7 and 6.3.13 apply to the IPM versions of* (6.35) *and* (6.45) *respectively.*

For a proof of the above result we refer the reader to [23, 22] for the first two claims and [88] for the third.

Example 6.3.20. *We consider an IPM for platte thistle (Cirsium canescens) based on that from [118], discussed also in [14]. Here the stages are structured according to stem diameter; a continuous variable assumed to take values between* ~ 0.6 *and* ~ 33 *mm. The distribution of plants of stage* ξ *at time* t *is denoted by* $n(\xi, t)$*. We have altered some of the parameters in the model from those in [118] so that the ambient population is in asymptotic decline.*

To supplement this population we suppose that individuals of stem diameter centered around 2.5 *mm are planted at each step, distributed by* b' *from* (6.58) *with* $\xi_1 = 2.5 - e^{0.5}$ *mm and* $\xi_2 = 2.5 + e^{0.5}$ *mm. The distribution of new plants is plotted in Figure 6.13.*

The observation $y(t)$ *of the population at each time-step is the total density of all plants with diameter between* 22 *and* 30*mm, described by*

$$y(t) = (cn)(t) = \int_{22}^{30} n(s,t)ds.$$

From a random initial population of total density 10 *we seek to raise the total density of thistles with diameter in the range* $22-30$ *mm to* $r = 40$*. In order to simulate the model we discretize the IPM; which we do so via a finite element (FE) method, a standard technique in numerical analysis. Such a scheme produces a matrix equation that approximates the controlled and observed IPM, but is different from one obtained by parameterizing a PPM model.*

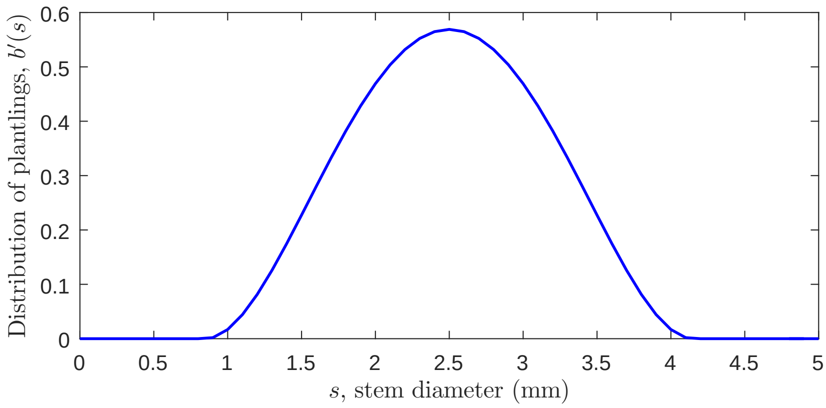


Figure 6.13: Graph of function b' describing distribution of new plants at each time-step of IPM Example 6.3.20. Here $\xi_1 = 2.5 - e^{0.5}$ and $\xi_2 = 2.5 + e^{0.5}$.

We provide details of the approximations. Following [14] we take $\Omega = [e^{-0.5}, e^{3.5}]$, *so that* $\alpha = e^{-0.5} \approx 0.6$ *and* $\beta = e^{3.5} \approx 33$. *The kernel* k *is divided into*

$$k(y, x) = p(y, x) + f(y, x),$$

where p *denotes the survival components and* f *denotes the reproductive components. These have respective decompositions*

$$p(y, x) = s(x)(1 - f_p(x))g(y, x)$$

and

$$f(y, x) = P_e J(y)s(x)f_p(x)S(x).$$

For our simulations we use the functions given in [14, Table 1] for f_p, g *and* J. *For the functions* s *and* S, *and the constant* P_e *we make modifications so that the population is declining and to demonstrate the results better. We use*

$$\begin{aligned} s(x) &= 0.7\frac{e^{0.85x-0.65}}{1 + e^{0.85x-0.65}} \\ S(x) &= e^{0.05x+0.04} \\ P_e &= 0.05. \end{aligned}$$

Finite element approximations are one method of reducing the infinite-dimensional IPM to a finite-dimensional difference equation by discretizing the spatial domain. That is, the function space $L^1(\Omega)$ *is approximated by an indexed sequence of finite-dimensional subspaces which get 'closer' to* $L^1(\Omega)$ *as the index* N *increases. In what follows we give a very brief description of how finite elements is used to derive an approximation of the IMP and refer the reader to the texts [75] or [13] for a thorough treatment.*

For an integer N, the interval $[\alpha, \beta]$ is partitioned into N subintervals with $N+1$ equally spaced endpoints s_i defined by

$$s_i = \alpha + \frac{(i-1)(\beta - \alpha)}{N}, \qquad 1 \le i \le N+1.$$

In particular $s_1 = \alpha$ and $s_{N+1} = \beta$. The $N+1$ 'hat' or 'tent' functions δ_i are defined by

$$\delta_i(s) = \begin{cases} \dfrac{s - s_{i-1}}{s_i - s_{i-1}} & s \in [s_{i-1}, s_i], \\ \dfrac{s_{i+1} - s}{s_{i+1} - s_i} & s \in [s_i, s_{i+1}], \\ 0 & \text{otherwise}, \end{cases} \qquad 1 \le i \le N+1, \tag{6.60}$$

where $s_0 = s_1 = \alpha$ and $s_{N+2} = s_{N+1} = \beta$. The hat functions are more readily understood visually, and some examples are plotted in Figure 6.14.

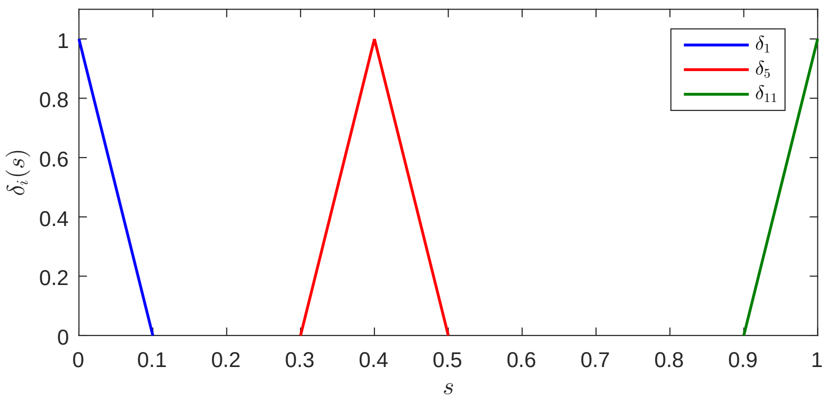


Figure 6.14: Three sample hat functions defined by (6.60) with $\alpha = 0$, $\beta = 1$ and $N = 10$. The functions δ_1, δ_5 and δ_{11} are plotted in blue, red and green lines respectively.

Loosely speaking, the finite element method assumes that functions in $L^1(\Omega)$ are well approximated by a linear combination of finitely many of the δ_i functions. And so, supposing that n is a solution of the IPM (6.4)*, using* (6.56)*,* (6.58) *and* (6.59)*, with input u and output y then for any continuous function v the following equation is satisfied*

$$\int_{\xi \in \Omega} v(\xi) \left[n(\xi, t+1) - (An)(\xi, t) - b(\xi) u(t) \right] d\xi = 0, \qquad \forall \quad t \in \mathbb{N}_0. \tag{6.61}$$

We assume that v and n can be written as a linear combination of the δ_i, that *is, as*

$$v(t,\xi) = \sum_{i=1}^{N+1} v_i(t)\delta_i(\xi), \qquad n(t,\xi) = \sum_{j=1}^{N+1} n_j(t)\delta_j(\xi), \tag{6.62}$$

for some coefficients v_i and n_j. Substituting (6.62) *into* (6.61) *and simplifying gives the following matrix equation*

$$M\boldsymbol{n}(t+1) = D\boldsymbol{n}(t) + Ju(t), \qquad \forall \quad t \in \mathbb{N}_0, \tag{6.63}$$

where $\boldsymbol{n}(t) = (n_1(t), \ldots, n_{N+1}(t))^T$ and the matrices M, D and the vector J have components given by

$$\left.\begin{aligned} M_{ij} &= \int_{\xi\in\Omega} \delta_i(\xi)\delta_j(\xi)d\xi, \\ D_{ij} &= \int_{\xi\in\Omega} \delta_i(\xi)\int_{s\in\Omega} k(\xi,s)\delta_j(s)dsd\xi, \\ J_i &= \begin{pmatrix} J_1 & \cdots & J_{N+1}\end{pmatrix}^T, \\ J_i &= \int_{\xi\in\Omega} \delta_i(\xi)b(\xi)d\xi, \end{aligned}\right\} 1 \le i,j \le N+1.$$

It is straightforward to see that the matrix M is invertible; if $q \in \mathbb{C}^{N+1}$ has i-th component q_i then we see that

$$\overline{q}^T Mq = \sum_{i,j=1}^{N+1} \overline{q_i} M_{ij} q_j = \int_{\xi\in\Omega} \left\| \sum_{i=1}^{N+1} q_i\delta_i(\xi) \right\|^2 d\xi \ge 0. \tag{6.64}$$

Furthermore, if $\overline{q}^T Mq = 0$ then as $\xi \mapsto \sum_{i=1}^{N+1} q_i\delta_i(\xi)$ is continuous if follows from (6.64) *that*

$$\sum_{i=1}^{N+1} q_i\delta_i(\xi) = 0, \qquad \forall \quad \xi \in \Omega \qquad \Rightarrow \qquad q_i = 0 \qquad \forall \quad i \in \{1,\ldots,N+1\},$$

and thus $q = 0$, proving that M is invertible.

When the output is of the form

$$y(t) = \int_{\xi_1}^{\xi_2} n(s,t)ds, \qquad \forall \quad t \in \mathbb{N}_0, \tag{6.65}$$

where $\xi_1 < \xi_2$ denotes the range of stage-classes observed, then substituting (6.62) *into* (6.65) *gives $y(t) = F\boldsymbol{n}(t)$, where the row vector*

$$F = \begin{pmatrix} F_1 & \cdots & F_{N+1}\end{pmatrix}$$

has components

$$F_i = \int_{\xi_1}^{\xi_2} \delta_i(s) ds, \qquad 1 \le i \le N+1.$$

Therefore, we have the following system with $N+1$ *states*

$$\left. \begin{aligned} \boldsymbol{n}(t+1) &= M^{-1}D\boldsymbol{n}(t) + M^{-1}Ju(t), \\ y(t) &= F\boldsymbol{n}(t), \end{aligned} \right\} \quad \forall \quad t \in \mathbb{N}_0, \tag{6.66}$$

which is an approximation of the IPM (6.4) *and can be readily implemented. The matrix* M *and vector* F *can be found analytically, whilst* D *and* J *generally need to be computed numerically. This can be achieved using quadrature, or for example the Matlab function* `integral` *and* `integral2`*. In principle, larger* N *gives rise to a closer approximation, but clearly adds complexity to simulations. We denote by* $\boldsymbol{G}_N$ *the transfer function of* (6.66) *so the steady-state gain of* (6.66) *is*

$$\boldsymbol{G}_N(1) = F(I - M^{-1}D)^{-1}M^{-1}J,$$

whenever $\rho(M^{-1}D) < 1$*. For our example we worked on the* log *of the interval* $[\alpha, \beta]$*, as this gave better results. As such the above goes through with* $s_1 = -0.5$*,* $s_{N+1} = 3.5$*. Figure 6.15 plots both the spectral radius of* $M^{-1}D$ *and the steady state gain* $\boldsymbol{G}_N(1)$ *for increasing* N*. The figure suggests that both converge for* $N \ge 10$ *and thus we choose* $N = 12$ *for the simulations in Figure 6.16. Furthermore, this suggests that the model in this example satisfies both assumptions* **(A6.4)** *and* **(A6.5)***.*

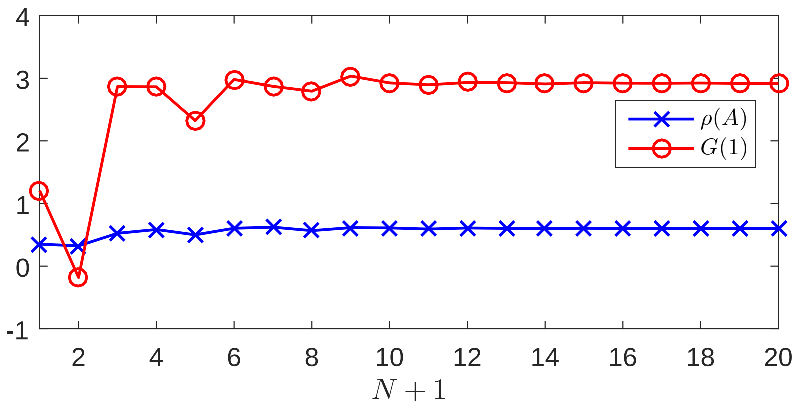


Figure 6.15: Spectral radius in blue and steady-state gain in red of the finite element approximations (6.66) of the IPM model of platte thistle of Example 6.3.20.

We assume that the input filter φ *is present, with input bound* $U = 17.5$*, and since* $\boldsymbol{G}(1) = 2.9324$*, in order for the results of Theorem 6.3.19 to apply* *we require*

$$r < \boldsymbol{G}(1)U = 51.32.$$

The results of the simulations are plotted in Figure 6.16. We see that, as expected, each control scheme achieves the desired control objective.

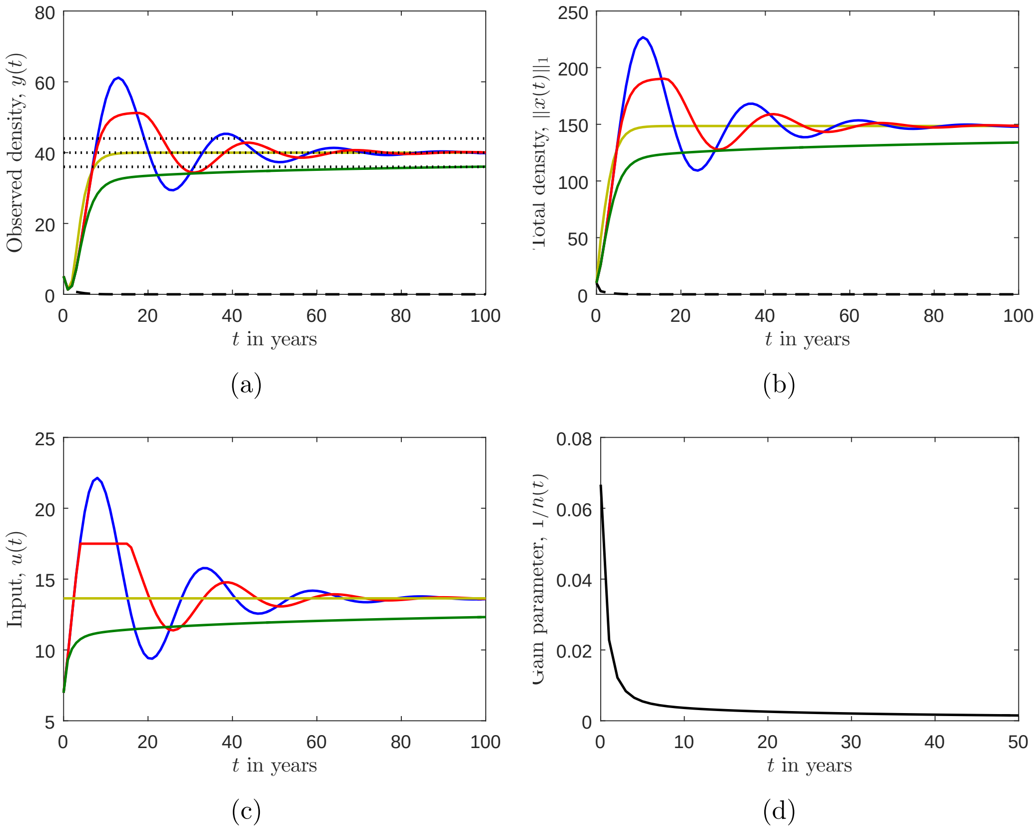


Figure 6.16: Integral control applied to the (discretization of the) platte thistle IPM of Example 6.3.20. (a) Observations. (b) Total population densities. (c) Inputs. (d) Adaptive gains. In (a)-(c) the blue lines denote the original integral control system (6.9), the red lines are the filtered integral control system (6.35) and the green lines are the adaptive integral control system (6.45). The gold line is the precomputed control and the dotted black lines denote the reference $r = 40$ and $r \pm 10\%$. The dashed black lines in (a)-(b) denote projections from the uncontrolled model. Each projection is from the same random initial population distribution. Here $r = 40$, $U = 17.5$ and $g = 0.075$

6.4 Further Development to Integral Control

In this section with two remarks on other directions in which integral control can be developed.

Remark 6.4.1. *Integral control can be developed for population models that contain a spatial component. The theoretical results we have drawn upon and* *derived here are predicated on the underlying population model being density-independent (that is, linear) and provided that linearity is preserved in the presence of spatial dynamics, then integral control is still applicable. It is beyond the scope of the present contribution to give comprehensive details for such situations but we do consider two examples. The first is a controlled and observed matrix metapopulation model (for example [109] or more recently [119]), so that a population changes over time and across N discrete patches, for integer N. The stage-structured population in the i-th patch at time-step t is denoted $x_i(t)$ and has dynamics described by*

$$x_i(t+1) = A_i x_i(t) + \sum_{j=1}^{N} D_{ij} x_j(t) + b_i u(t), \qquad x_i(0) = x_i^0, \qquad \forall \quad t \in \mathbb{N}_0, \tag{6.67}$$

for $i \in \{1, 2, ..., N\}$. Here A_i describes the survival and recruitment of the i-th patch, D_{ij} are dispersal matrices, describing the movements of individuals to patch i from patch j and b_i is the control vector of the i-th patch. Spatial inhomogeneity is incorporated when the vital rates and dispersal rates vary across patches. The model (6.67) *can be reformulated in the form* (6.4) *by concatenating the population vectors and patch matrices as*

$$x(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{pmatrix}, \qquad A := \begin{pmatrix} (A_1 + D_{11}) & D_{12} & \cdots & D_{1N} \\ D_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & D_{(N-1)N} \\ D_{N1} & \cdots & D_{N(N-1)} & (A_N + D_{NN}) \end{pmatrix},$$
$$b := \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} \tag{6.68}$$

and by defining an observation $y(t)$ as some linear combination of the states $y(t) = c^T x(t)$ as usual. It is important to note that the D_{ij} may not be component-wise nonnegative, as they describe both movement in to and out of a given patch and so therefore A in (6.68) *may have negative components. However, the nonnegativity assumed in* **(A6.1)** *is not required for integral control, only that $\rho(A) < 1$. Assumption* **(A6.2)** *is unchanged, and when these assumptions hold for the above A, b and c^T then integral control is applicable and the results we have presented carry over. As mentioned in Section 6.3.1, full knowledge of A_i , D_{ij} is not required for these assumptions to hold.*

The second example is a linear, integro-difference model (for examples in *ecology, see [80] or [81] and the references therein). A single stage-structured population over a (possibly inhomogeneous) spatial domain* Ω *at time-step* t *and position* $\xi \in \Omega$ *is denoted by* $n(\xi, t)$ *and has dynamics given by*

$$\left.\begin{aligned} n(\xi, t+1) &= \int_{s\in\Omega} k(\xi, s) R n(s,t) ds + b(\xi) u(t)\,, \\ n(\xi, 0) &= n_0(\xi), \\ y(t) &= \int_{\Omega_1} \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix} n(s,t) ds, \end{aligned}\right\} \quad \forall \quad t \in \mathbb{N}_0. \tag{6.69}$$

In (6.69)*,* R *is a matrix that models survival and recruitment of the population,* n_0 *denotes the initial population distribution,* $k = k(\xi, s)$ *is a dispersal kernel which is a probability distribution describing the probability that an individual from position* s *disperses to position* ξ *at each time-step and the function* $b = b(\xi)$ *describes the distribution of new individuals at position* ξ*. The observation* $y(t)$ *has been chosen as the number of individuals in the region* $\Omega_1 \subseteq \Omega$*, although of course other observations are permitted. Similarly to the IPM* (6.55)*,* (6.69) *can be reformulated as* (6.4)*, although we do not give the details here.*

Remark 6.4.2. *Further developments of integral control allow regulation of more than one observation and with access to more than one management action at each time-step. For example, suppose that we seek to regulate both the total population abundance and the abundance of a given single stage-class, and we can replenish more than one stage-class (or combination of stage-classes) independently. This leads to a framework called multi-input, multi-observation in control engineering and conceptually the extension from the single-input, single-observation case is usually straightforward, although mathematically there are often additional difficulties to overcome. That said, integral control feedback systems have been designed where at each time-step* t*,* m *control actions are made and* p *observations are recorded for positive integers* m *and* p*; for example [78]. The reference is now a vector of chosen values* $r \in \mathbb{R}^p$*. However, existing results do not address integral control where additionally component-wise nonnegativity has to be preserved; clearly a requirement for meaningful population models. Combining these two ideas is seemingly not straightforward. One immediate issue is that not every nonnegative reference vector can be a target for management. In our example considered above, obviously the former observation (total population abundance) shall always be larger than the latter (abundance of a single stage-class). Such a constraint must therefore also be present in the choice of reference. We comment that integral control that preserves nonnegativity in the multi-input,* *multi-observation case is the subject of ongoing research.*

6.4.1 How the solutions to (P1)-(P5) interact

The solutions proposed to problems **(P1)**-**(P5)** interact as follows. Robustness to model uncertainty **(P1)** (i) is encapsulated in assumptions **(A6.1)** and **(A6.2)**, which are necessary and sufficient conditions for low-gain integral control and are hence assumed throughout. The same is true of the infinite-dimensional versions of these assumptions **(A6.4)** and **(A6.5)**. Thus the solutions to **(P2)**-**(P5)** include this same robustness to model uncertainty. The material presented in addressing **(P2)**, **(P3)** and **(P5)** is cumulative, so our solution to **(P3)** (adaptive gain selection) incorporated the solution to **(P2)** (filtering the input signal). We addressed problem **(P4)**, namely that of increasing the rate of convergence of the observations to the reference, by including a proportional controller to augment the integral controller. For simplicity our main result of Section 6.3.4, Theorem 6.3.16, only considered the linear integral control system (6.9). However, Theorems 6.3.7, 6.3.13 and 6.3.19 can all be extended to the PI feedback system (6.46) (where the proportional k and integral gains g are equal). It is possible to extend versions of all the theorems presented to incorporate additive observation errors and additive activation errors (**(P1)** (ii) and (iii) respectively). The proportional observation errors (6.16) are trickier to incorporate into the solutions to **(P2)**-**(P5)**, and a treatment of such is beyond the scope of this contribution. However, appealing to techniques such as λ-tracking [68, 69] and funnel control [70] would provide insight in this direction.

6.5 Discussion

We have introduced integral control as a potential tool for population management. A brief overview of the method has been given, which seeks to motivate both the necessity of integral control for robust population management via restocking and indeed further how integral control is suitable for such a task. Sections 6.2 and 6.3 contain a verbal and mathematical "road-map" respectively of how integral control is applied. Although well-established in control engineering and, as mentioned in the introduction, now starting to appear in the biological literature; PI control has not been applied to population management, to which it seems well suited. It has been suggested elsewhere in the literature that there is ample scope for using control theory in ecology [51, 12] but often it seems that the focus is on optimal control [85]. As mentioned in Section 6.2.1, the trade-off between performance and robustness has produced an unfortunate discord between theory and practice, so much so that in [121] it says "By 1975, the much lamented gap between academic theory and engineering practice in the control field has grown to prodigious proportions".

Integral control is a particular instance of feedback control, which is known to control engineers to be incredibly robust to model uncertainty. Moreover, appealing in part to recent mathematical results, the basic integral control model can be extended to meet several challenges that arise in population ecology.

Furthermore, integral control is straightforward to implement (at least theoretically) once a PPM or IPM is available. It does not suffer the so-called "curse of dimensionality" present in SDP which necessitates low-dimensional models to be realized practically. Of course population management models that use POMDPs and SDP (such as those cited in Section 6.2.1) treat an issue that we have omitted; namely that of managing optimality. The reason for this omission is, in part, because it is not the aim of this chapter. We have sought to describe a robust approach to population management via restocking. With the material presented, however, and given costs of reintroduction and observation one could easily investigate by simulation which choices of b, c^T and g (reintroductions, measurements, and gain) give rise to lowest cost or fastest responses. Such costs could also be traded off against set rewards of having certain abundances of populations.

Another important consideration is that we, to use a medical analogy, have presented a treatment of symptoms rather than a cure of the underlying condition, as managing via integral control requires that populations are restocked indefinitely to secure persistence. Such a policy is clearly infeasible in practice, at least in many cases. Although conservation biologists often rely on captive rearing, translocations and species reintroductions [126]; methods that fit our mathematical framework, such conservation programs are expensive, laborious and risk the welfare of endangered species. A possibly more practical approach would be to combine integral control in the short term to raise population abundances with additional conservation efforts to ensuring future population persistence, for example by improving environmental conditions. The aim might be to restock to sufficient population densities that ensure population viability; that is, the population persists unaided.

Main Assumptions

Chapter 3

(A3.1) The matrix A is a Metzler matrix and $b, c \in \mathbb{R}_+^n$ are nonzero.

(A3.2) The matrix A is Hurwitz.

(A3.3) The matrix $A + bc^T$ is irreducible.

(A3.4) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz.

(A3.5) There exists $y^* > 0$ such that $f(y^*) = py^*$.

(A3.6) f satisfies

$$\left| \frac{f(y) - f(y^*)}{y - y^*} \right| \leq p, \qquad \forall \quad y \in \mathbb{R}_+ \backslash \{y^*\}.$$

(A3.7) f satisfies

$$\left| \frac{f(y) - f(y^*)}{y - y^*} \right| < p \qquad \forall \quad y \in \mathbb{R}_+ \backslash \{0, y^*\}.$$

(A3.8) f satisfies

$$\limsup_{y \to y^*} \left| \frac{f(y) - f(y^*)}{y - y^*} \right| < p.$$

(A3.9) f satisfies

$$\limsup_{y \to \infty} \left| \frac{f(y) - f(y^*)}{y - y^*} \right| < p.$$

(A3.10) For all $y_0 > 0$

$$\sup_{y \geq y_0,\, y \neq y^*} \left| \frac{f(y) - f(y^*)}{y - y^*} \right| < p.$$

(A3.11) $f(0) = 0$.

(A3.12) $py - f(y) \to \infty$ as $y \to \infty$.

Chapter 4

(A4.1) For all, $\xi \in Y$, and all $z \in \mathbb{R}^p$ with $z \neq 0$

$$\|f(z+\xi) - f(\xi) - Kz\| < \gamma \|z\|.$$

(A4.2) For all $\xi \in \operatorname{im} C$ and all $z \in \mathbb{R}^p$ with $z \neq 0$,

$$\|f(z+\xi) - f(\xi) - Kz\| < \gamma \|z\|.$$

(A4.3) For every $\xi \in \operatorname{im} C$ there exists $\alpha_\xi \in \mathcal{K}_\infty$ such that

$$\|\boldsymbol{G}_K\|_{H^\infty} \frac{\|f(z+\xi) - f(\xi) - Kz\|}{\|z\|} \leq 1 - \frac{\alpha_\xi(\|z\|)}{\|z\|}$$

for all $z \in \mathbb{R}^p$ with $z \neq 0$.

(A4.4) For all $\xi, z \in \mathbb{R}^p$ with $z \neq 0$,

$$\|f(z+\xi) - f(\xi) - Kz\| < \gamma \|z\|.$$

(A4.5) $A \in \mathbb{R}^{n \times n}$ is Metzler and $b, c \in \mathbb{R}^n_+$, $b, c > 0$,

(A4.6) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz.

(A4.7) The matrix $A + bc^T$ is irreducible.

Chapter 5

(A5.1) The matrix A is nonnegative and the vectors b and c are nonnegative and nonzero.

(A5.2) The matrix A is stable.

(A5.3) The matrix $A + bc^T$ is primitive.

(A5.4) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous.

(A5.5) At least one of the following statements hold.

- There exists z_0 with $|z_0| = 1$ such that $p|\boldsymbol{G}(z_0)| < 1$.
- (A, b, c^T) is controllable and observable.

(A5.6) There exists $y^* > 0$ such that $f(y^*) = py^*$.

(A5.7) f satisfies

$$\left|\frac{f(y)-f(y^*)}{y-y^*}\right| \le p \qquad \forall \quad y \ge 0,\ y \ne y^*.$$

(A5.8) f satisfies

$$\left|\frac{f(y)-f(y^*)}{y-y^*}\right| < p, \qquad \forall \quad y > 0,\ y \ne y^*.$$

(A5.9) f satisfies

$$\limsup_{y\to y^*}\left|\frac{f(y)-f(y^*)}{y-y^*}\right| < p.$$

(A5.10) $py - f(y) \to \infty$ as $y \to \infty$.

Chapter 6

(A6.1) $A \in \mathbb{R}_+^{n\times n}$ and $\rho(A) < 1$.

(A6.2) The matrix A and vectors b and c^T are such that $c^T(I-A)^{-1}b > 0$.

(A6.3) There exists $L > 0$ such that $0 \le \varphi(v) - \varphi(w) \le L(v-w)$ for all $v \ge w$.

(A6.4) the bounded linear operator $A : X \to X$ has $\rho(A) < 1$,

(A6.5) the operators $A : X \to X$, $b : \mathbb{R} \to X$ and $c : X \to \mathbb{R}$ are all bounded and such that $c(I-A)^{-1}b > 0$.

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