SIAM J. CONTROL AND OPTIMIZATION
Vol. 30, No. 3, pp. 573-593, May 1992

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ON ROBUST PI-CONTROL OF INFINITE-DIMENSIONAL SYSTEMS*

HARTMUT LOGEMANN† AND HANS ZWART‡

Abstract. A PI-controller is applied to a class of linear multivariable infinite-dimensional minimum-phase systems satisfying a generalized "relative-degree one" condition. It is shown that the closed-loop system is stable and tracks asymptotically constant reference signals in the presence of asymptotically constant disturbances, provided that the controller gains are sufficiently large. It turns out that the closed-loop system has nice robustness properties under high-gain conditions. In particular, robustness criteria for external and internal stability are given if the closed-loop system is subjected to perturbations induced by nonlinearities in the feedback loop. The analysis is based on frequency-domain as well as state-space methods.

Key words. infinite-dimensional systems, PI-controllers, high-gain feedback control, robust stability, robust tracking, measurement nonlinearities, input-output stability, internal stability

AMS(MOS) subject classifications. 93C25, 93C35, 93D15, 93D20, 93D25

1. Introduction. The concepts of classical control theory such as root locus plots, Nyquist diagrams, and PI- and PID-controllers are still very popular among control engineers because of their simplicity and their applicability to a great variety of practical problems. Designs based on classical frequency-domain methods lead to low-dimensional controllers, which are easy to implement. Although it was developed mainly for finite-dimensional systems, classical control theory has been applied by engineers to infinite-dimensional systems for many years, despite the fact that few precise theoretical results were available. Since the late 1970s, there has been a renewed theoretical interest in the use of methods from classical control theory for designing control laws for (multivariable) infinite-dimensional systems; see, e.g., Pohjolainen [33], Banks and Abbasi-Ghelmansarai [1], and Byrnes and Gilliam [3] for root-locus techniques; Boyd and Desoer [2] and Freudenberg and Looze [9] for a priori performance bounds on feedback systems such as Bode-type integral relationships; Desoer and Wang [7], Harris and Valenca [11], and Logemann [18], [19], [21] for Nyquist-type stability criteria; Pohjolainen [34], Jussila and Koivo [15], Kobayashi [17], and Logemann and Owens [24] for low-gain PI-control; and Logemann and Owens [22], [23] for high-gain PI-control.

In this paper we continue the work on high-gain PI-control of infinite-dimensional systems started by Logemann and Owens [22], [23]. We investigate stability, tracking, disturbance rejection, and robustness properties achieved by a high-gain PI-controller applied to an infinite-dimensional minimum-phase system satisfying a generalized "relative-degree one" condition. In particular, we study the robustness of closed-loop stability with respect to various classes of measurement nonlinearities. Our analysis is based on time-domain input-output methods, frequency-domain methods, as well as state-space methods. To relate frequency-domain results, on one hand, and state-space results, on the other hand, we express frequency-domain conditions in state-space terms, and vice versa (cf. § 4). Moreover, recent results on the relationship between input-output stability and internal stability of linear infinite-dimensional systems (see Jacobson [13], Jacobson and Nett [14], and Curtain [5]) will play an important role in § 5.

* Received by the editors May 14, 1990; accepted for publication (in revised form) December 18, 1990.

† University of Bremen, Institute for Dynamical Systems, Postfach 330 440, 2800 Bremen 33, Germany.

‡ University of Twente, Department of Applied Mathematics, Post Office Box 217, 7500 AE Enschede, the Netherlands.

The content of the paper is as follows. Section 2 contains some preliminaries and introduces the notation used in the sequel. In § 3 we consider systems described by (not necessarily rational) transfer matrices G of the form

$$G(s)=\left(I+\frac{1}{s}D^{-1}H(s)\right)^{-1}\frac{1}{s}D^{-1}, \tag{1.1}$$

where H is a "stable" transfer matrix (see § 3 for a precise definition) and D is a nonsingular constant matrix. We derive a necessary and sufficient condition for a transfer matrix to be of the form (1.1) in terms of its zeros and its behaviour as $|s|\to\infty$ in the right half-plane. A PI-controller is given that achieves input-output stability and tracking of asymptotically constant reference signals in the presence of asymptotically constant disturbances, provided that the controller gains are sufficiently large. It turns out that the transient performance of the closed-loop system improves as the controller gains increase and that it is perfect for infinitely large gains. Moreover, we investigate the robustness of closed-loop stability with respect to (possibly time-varying) finite-gain stable nonlinearities in the feedback loop. Static, as well as dynamic, nonlinearities are considered, and sufficient conditions for input-output stability of the nonlinear closed-loop system are given. We emphasize that wellposedness questions (i.e., the problem of existence and uniqueness of solutions) are carefully treated. Sections 4 and 5 are devoted to the special situation when the plant transfer matrix G can be realized by an abstract infinite-dimensional state-space system with bounded control and observation operators. In § 4 we prove that the zeros of an infinite-dimensional state-space system (as defined, e.g., in Zwart [40]) coincide with the zeros of its transfer matrix (as defined in § 2), provided that the system is exponentially stabilizable and exponentially detectable. Furthermore, we derive various sufficient conditions in state-space terms for (1.1) to be satisfied. In § 5 we deal with the problem of *internal* stability of the closed-loop system perturbed by nonlinearities in the feedback loop. Assuming that the realizations of the plant and the controller are both exponentially stabilizable and exponentially detectable, we show that the criterion for input-output stability given in § 3 also ensures global exponentially stability of the nonlinear feedback scheme if the nonlinearities in the loop are static. In the case of dynamical nonlinearities, we can prove that the origin of the closed-loop system is globally attractive. The proofs of some of the results in §§ 3–5 are relegated to Appendices 1 and 2. For the convenience of the reader, we have included some recent material on exponential stabilizability and exponential detectability of infinite-dimensional systems in Appendix 3.

2. Notation and preliminaries.

—$\mathbb{C}_\alpha := \{s\in\mathbb{C} \mid \mathrm{Re}(s)>\alpha\}$, $\quad \alpha\in\mathbb{R}$.

—Let $U\subset\mathbb{C}$ be open, then $\mathcal{H}(U)$ and $\mathcal{M}(U)$ denote the holomorphic and meromorphic functions on U, respectively.

—$H^\infty_\alpha := \{f:\mathbb{C}_\alpha\to\mathbb{C} \mid f$ bounded and holomorphic$\}$.

—$H^\infty_- := \bigcup_{\alpha<0} H^\infty_\alpha$.

—Consider distributions of the form

$$f=f_a+\sum_{j=0}^{\infty} f_j\delta_{t_j}, \tag{2.1}$$

where $f_a:\mathbb{R}_+\to\mathbb{C}$ is measurable, $f_j\in\mathbb{C}$, $t_0=0$, $t_j>0$ for $j\geqq 1$, and δ_{t_j} denotes the Dirac distribution with support in $\{t_j\}$. Let $\mathcal{A}$ be the set of all distributions f of the form (2.1) such that

$$\|f\|_{\mathcal{A}} := \int_0^\infty |f_a(t)|\,dt+\sum_{j=0}^{\infty}|f_j|<\infty. \tag{2.2}$$

$\mathscr{A}$ is a convolution algebra and, provided with the norm given by (2.2), it becomes a Banach algebra (cf. Hille and Phillips [12, p. 141]).

—$\mathscr{A}_- := \{f \in \mathscr{A} \mid$ there exists $\varepsilon > 0: fe^{\varepsilon\cdot} \in \mathscr{A}\}$.

—$\hat{\mathscr{A}}, \hat{\mathscr{A}}_-$ is the set consisting of the Laplace transformed elements of $\mathscr{A}, \mathscr{A}_-$, respectively. Realize that $\hat{\mathscr{A}} \subset H_0^\infty$ and $\hat{\mathscr{A}}_- \subset H_-^\infty$.

—$\check{f}$ denotes the inverse Laplace transform of f.

—θ denotes the unit step.

—Let $M = (m_{ij}) \in \mathbb{C}^{p\times p}$, then $\|M\| := \max_{1\leqq i\leqq p} \sum_{j=1}^p |m_{ij}|$ unless stated otherwise, $\bar{\sigma}(M) :=$ largest singular value of M, $W(M) :=$ numerical range of M (cf. Halmos [10]).

—Let X be a Banach space and $A: D(A) \subset X \to X$ a linear operator, then $\sigma(A) :=$ spectrum of A, $\sigma_p(A) :=$ point spectrum of A, and $\rho(A) :=$ resolvent set of A.

—For $F = (f_{ij}) \in (L^1(\mathbb{R}_+))^{p\times p}$ define $\|F\|_1 := \max_{1\leqq i\leqq p} \sum_{j=1}^p \|f_{ij}\|_1$.

—Let $F = (f_{ij}) \in \mathscr{A}^{p\times p}$, then $\|F\|_{\mathscr{A}} := \max_{1\leqq i\leqq p} \sum_{j=1}^p \|f_{ij}\|_{\mathscr{A}}$.

—If $f = (f_1, \cdots, f_p)^t \in (L^q(\mathbb{R}_+))^p$, then $\|F\|_q := \max_{1\leqq j\leqq p} \|f_j\|_q$.

—For $F \in (H_0^\infty)^{p\times p}$ define $\|F\|_\infty := \sup_{s\in\mathbb{C}_0} \bar{\sigma}(F(s))$.

—Let f be a function defined on an interval $[a, b)$, $a < b \leqq \infty$; then we define, for $t \geqq a$,

$$(\pi_t f)(\tau) := \begin{cases} f(\tau), & a \leqq \tau \leqq t, \\ 0, & \tau > t. \end{cases}$$

—Define the space $LL^q(\mathbb{R}_+)$ by $LL^q(\mathbb{R}_+) := \{f: \mathbb{R}_+ \to \mathbb{C} \mid f$ measurable and $\pi_t f \in L^q(\mathbb{R}_+)$ for all $t \geqq 0\}$, i.e., $f \in LL^q(\mathbb{R}_+)$, if and only if $|f|^q$ is locally integrable.

We need three additional concepts:

H_-^∞-stability. Let $G \in \mathscr{M}(\mathbb{C}_\alpha)^{p\times p}$ and $K \in \mathscr{M}(\mathbb{C}_\alpha)^{q\times p}$ for some $\alpha < 0$; then it is convenient to denote the feedback system shown in Fig. 1 by $\mathscr{F}[G, K]$, and we say that $\mathscr{F}[G, K]$ is H_-^∞-stable if

$$\begin{pmatrix} (I+KG)^{-1}K & -(I+KG)^{-1}KG \\ (I+GK)^{-1}GK & (I+GK)^{-1}G \end{pmatrix}$$

is in $(H_-^\infty)^{(p+q)\times(p+q)}$.

Zeros of a square meromorphic matrix. Let $M \in \mathscr{M}(U)^{p\times p}$. Since it is well known that $\mathscr{M}(U)$ is the quotient field of $\mathscr{H}(U)$ and $\mathscr{H}(U)$ is a Bezout domain[1] (see, e.g., Rudin [35]), it follows that M admits a right coprime factorization over $\mathscr{H}(U)$, i.e., there exist matrices $N, D, X, Y \in \mathscr{H}(U)^{p\times p}$ such that $\det(D) \not\equiv 0$, $M = ND^{-1}$, and $XD + YN \equiv I$. Right coprime factorizations are unique up to multiplication from the right by units of $\mathscr{H}(U)^{p\times p}$ (see, for example, Vidyasagar, Schneider, and Francis [37]). The zeros of M are, by definition, the zeros of $\det(N)$.

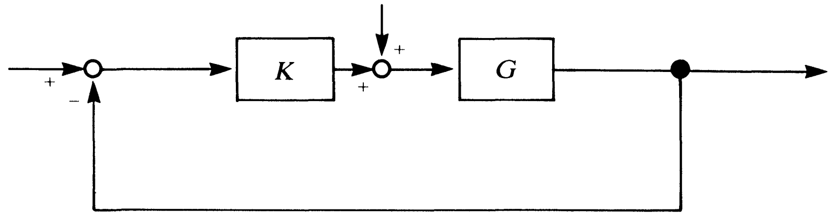


FIG. 1

[1] An integral domain is called Bezout domain if every finitely generated ideal is principal.

Asymptotically constant signals. A function $f: \mathbb{R}_+ \to \mathbb{C}^p$ is called asymptotically constant if there exists $c \in \mathbb{C}^p$ such that $\lim_{t\to\infty} f(t) = c$.

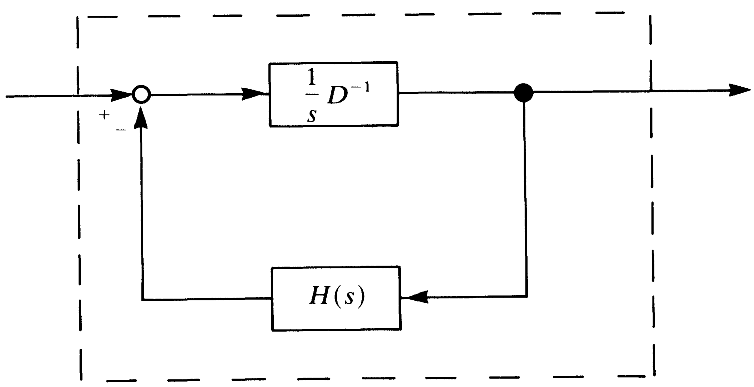
3. Robust PI-control of infinite-dimensional systems: Results on input-output stability. Let G be a meromorphic transfer function of size $p \times p$ such that on $\mathbb{C}_\alpha$ (for some $\alpha < 0$)

(3.1) $$G^{-1}(s) = sD + H(s), \quad \text{where } D \in \mathbb{C}^{p\times p}, \quad \det(D) \neq 0, \quad \text{and} \quad H \in (H_-^\infty)^{p\times p}.$$

Of course, (3.1) is equivalent to

(3.2) $$G(s) = \left(I + \frac{1}{s} D^{-1} H(s)\right)^{-1} \frac{1}{s} D^{-1}.$$

Hence (3.1) means that G can be decomposed, as shown in Fig. 2. The following proposition gives a necessary and sufficient condition for G to be of the form (3.2).



$G(s)$

FIG. 2

PROPOSITION 3.1. *Let G be a transfer matrix of size $p \times p$, which is meromorphic in $\mathbb{C}_\alpha$ for some $\alpha < 0$. Then G^{-1} is of the form* (3.1) *if and only if there exist a number β with $\alpha < \beta < 0$ and an invertible matrix $D \in \mathbb{C}^{p\times p}$ such that G has no zeros in $\bar{\mathbb{C}}_\beta$ and $sG(s) - D^{-1} = O(1/s)$ as $|s| \to \infty$ in $\mathbb{C}_\beta$.*

As a consequence, we have the following corollary.

COROLLARY 3.2. *Suppose that $A \in (H_-^\infty)^{n\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times n}$, and define*

$$G(s) := C(sI - A(s))^{-1}B \quad \text{and} \quad \chi(s) := \det\begin{pmatrix} sI - A(s) & -B \\ C & 0 \end{pmatrix}.$$

If $\det(CB) \neq 0$ *and if χ has no zeros in $\bar{\mathbb{C}}_0$, then G^{-1} is of the form* (3.1) *with $D = (CB)^{-1}$.*

The proof of the proposition and the corollary can be found in Appendix 1.

We give two classes of infinite-dimensional systems whose transfer matrices satisfy (3.1).

Example 3.3 (Retarded systems). Consider the retarded system

$$\dot{x}(t) = \int_{-h}^{0} dA(\tau)x(t+\tau) + Bu(t), \qquad y(t) = Cx(t),$$

where $h > 0$ is the length of the delay, the function $A: [-h, 0] \to \mathbb{R}^{n\times n}$ is of bounded variation, $B \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{p\times n}$. It is straightforward to show that $\hat{A}(s) := \int_{-h}^{0} dA(\tau)\, e^{s\tau}\, d\tau$ is holomorphic and bounded on $\mathbb{C}_\alpha$ for any $\alpha \in \mathbb{R}$. In particular, we have $\hat{A} \in (H^{\infty}_{-})^{n\times n}$. The transfer matrix of the above retarded system is given by $G(s) = C(sI - \hat{A}(s))^{-1}B$. It follows from Corollary 3.2 that G^{-1} is of the form (3.1), provided the conditions $\det(CB) \neq 0$ and

$$\det\begin{pmatrix} sI - \hat{A}(s) & -B \\ C & 0 \end{pmatrix} \neq 0 \quad \text{for all } s \in \bar{\mathbb{C}}_0$$

are satisfied.

Example 3.4 (Volterra integrodifferential systems). Consider the system

$$\dot{x}(t) = A_0 x(t) + \int_0^t A_1(t-\tau)x(\tau)\, d\tau + Bu(t), \qquad y(t) = Cx(t),$$

where $A_0 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times n}$, and $e^{\varepsilon\cdot}A_1(\cdot) \in L^1(\mathbb{R}_+)$ for some $\varepsilon > 0$. Noting that the Laplace transform $\hat{A}_1$ of A_1 is in $(H^{\infty}_{-})^{n\times n}$, it follows from Corollary 3.2 that the transfer matrix $G(s) = C(sI - A_0 - \hat{A}_1(s))^{-1}B$ will satisfy (3.1) if $\det(CB) \neq 0$ and

$$\det\begin{pmatrix} sI - A_0 - \hat{A}_1(s) & -B \\ C & 0 \end{pmatrix} \neq 0 \quad \text{for all } s \in \bar{\mathbb{C}}_0.$$

It is fairly obvious that Examples 3.3 and 3.4 can be extended to certain classes of retarded systems with infinite delay and Volterra–Stieltjes integrodifferential systems.

Consider the PI-controller

$$K_k(s) := \Gamma \operatorname*{diag}_{1\leq j\leq p}\left(k_j + c_j + \frac{k_j c_j}{s}\right), \tag{3.3}$$

where $\Gamma \in \mathbb{C}^{p\times p}$, $\det(\Gamma) \neq 0$, $k = (k_1, \cdots, k_p)'$, $k_j > 0$, $c_j > 0$ for all $j = 1, 2, \cdots, p$. Sometimes it will be useful to emphasize the dependence of the controller (3.3) on the "gain vector" k, since we will be interested in the high-gain situation, where $k_j \to \infty$, $j = 1, \cdots, p$. That is why we introduced the subscript k in (3.3).

The above controller was investigated in Owens and Chotai [31] when applied to finite-dimensional systems. The infinite-dimensional case is studied in Logemann and Owens [22], [23].

The following theorem gives sufficient conditions for robust stability when the controller (3.3) is applied to a system satisfying (3.1).

THEOREM 3.5. *Let G be a transfer matrix such that* (3.1) *is satisfied. Then* (i) *the feedback system $\mathcal{F}[G, K_k]$ is H^{∞}_{-}-stable for all sufficiently large k_j, $j = 1, \cdots, p$, if*

$$\|\Gamma^{-1}(\Gamma - D)\| < 1, \tag{3.4}$$

where $\|\cdot\|$ is any submultiplicative norm on $\mathbb{C}^{p\times p}$ with the additional property that $\|\operatorname{diag}(a_j)\| \leq \max_j |a_j|$ for arbitrary $a_1, \cdots, a_p \in \mathbb{C}$; and (ii) *under the additional assumption that $k_j = \gamma_j\kappa$ with $\gamma_j > 0$ fixed $(j = 1, \cdots, p)$ the feedback scheme $\mathcal{F}[G, K_k]$ is H^{∞}_{-}-stable for a sufficiently large κ if*

$$\sigma\left(\operatorname*{diag}_j(\gamma_j)\Gamma^{-1}D\right) \subset \mathbb{C}_0 \tag{3.5}$$

or

$$W(\Gamma^{-1}D) \subset \mathbb{C}_0. \tag{3.6}$$

Proof. See Logemann and Owens [22].

Remark 3.6. (i) Note that K_k does not depend on H. Trivially, (3.4)–(3.6) are satisfied if $\Gamma = D$. Obviously, $\Gamma = D$ would be a natural choice in (3.3). However, D might not be exactly known to the designer.

(ii) Further applications of the concept of numerical range to control problems can be found in Mees [25].

To study the tracking and output disturbance rejection properties of $\mathscr{F}[G, K_k]$ (cf. Fig. 3), define $L_k := (I+GK_k)^{-1}GK_k$ and $H_k := (I+GK_k)^{-1}$.

PROPOSITION 3.7. *Let G be a square meromorphic transfer matrix such that G^{-1} is of the form* (3.1). *If $\mathscr{F}[G, K_k]$ is H^∞_--stable, then the closed-loop system tracks asymptotically constant reference signals in the presence of asymptotically constant output disturbances, i.e., for $r: \mathbb{R}_+ \to \mathbb{C}^p$ and $d: \mathbb{R}_+ \to \mathbb{C}^p$ such that $\lim_{t\to\infty} r(t) = r_\infty$ and $\lim_{t\to\infty} d(t) = d_\infty$, we have*

$$\lim_{t\to\infty} (\check{L}_k * r)(t) = r_\infty \quad \text{and} \quad \lim_{t\to\infty} (\check{H}_k * d)(t) = 0.$$

Proof. It follows from the stability of $\mathscr{F}[G, K_k]$ that $L_k, H_k \in (H^\infty_-)^{p\times p}$. Now it is easy to see that $sL_k(s)$ and $s(H_k(s)-I)$ are bounded on $\mathbb{C}_\beta$ for some $\beta < 0$, and hence we obtain, using a result in Mossaheb [27] (cf. also Logemann [18]), that $e^{\alpha\cdot}\check{L}_k \in (L^1(\mathbb{R}_+))^{p\times p}$ and $e^{\alpha\cdot}\check{H}_k \in (\mathbb{R}\delta_0 + L^1(\mathbb{R}_1))^{p\times p}$ for all $\alpha \in (0, -\beta)$, and thus L_k and $H_k \in (\hat{\mathscr{A}}_-)^{p\times p}$. By the final value theorem for transfer functions in $\hat{\mathscr{A}}_-$ (cf. Callier and Winkin [4]), it is sufficient to show that $L_k(0) = I$ and $H_k(0) = 0$.

An elementary calculation gives

$$\begin{aligned} L_k(s) &= ((G(s)K_k(s))^{-1} + I)^{-1} \\ &= \left\{ \operatorname*{diag}_{1\le j\le p} \left(\frac{s}{s(k_j+e_j)+k_jc_j} \right) \Gamma^{-1}(sD + H(s)) + I \right\}^{-1}. \end{aligned}$$

Since k_j, $c_j > 0$, $j = 1, \cdots, p$, it follows that $L_k(0) = I$. Since $H_k = I - L_k$, we obtain $H_k(0) = 0$. □

To investigate the transient performance of $\mathscr{F}[G, K_k]$ and the robustness of closed-loop stability with respect to measurement nonlinearities, the following lemma is useful.

LEMMA 3.8. *Let G be a square meromorphic transfer matrix such that G^{-1} is of the form* (3.1), *define K_k as in* (3.3), *and set*

$$G^*(s) := \frac{1}{s}\Gamma^{-1} \quad \text{and} \quad L_k^*(s) := (I + G^*K_k)^{-1}G^*K_k.$$

Then the following hold: (i) *If $\Gamma = D$, we have*

$$\lim_{k\to\infty} \|\check{L}_k^* - \check{L}_k\|_1 = 0 \tag{3.7}$$

(by $k\to\infty$, we mean $\min_{1\le j\le p}(k_j) \to \infty$);

(ii) *If $\|\Gamma^{-1}(\Gamma - D)\| =: \varepsilon < \frac{1}{2}$, then*

$$\limsup_{k\to\infty} \|\check{L}_k^* - \check{L}_k\|_1 \le \frac{2\varepsilon}{1-2\varepsilon}. \tag{3.8}$$

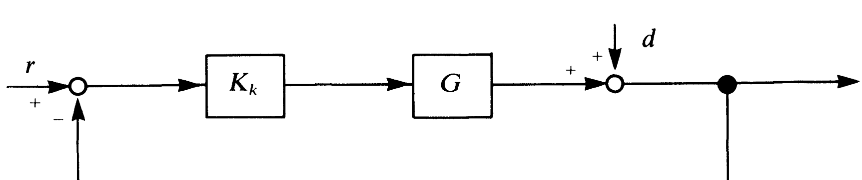


FIG. 3

The proof of Lemma 3.8 can be found in Appendix 2. Part (i) was proved in Logemann and Owens [22] under the extra assumption that H in (3.1) belongs to $\hat{\mathscr{A}}_-^{p\times p}$. It should be noted that it is, in general, considerably more difficult to check if a given transfer function is in $\hat{\mathscr{A}}_-$ than to verify that it belongs to H_-^∞.

Remark 3.9. (i) Define $\theta_j := e_j\theta$, where $e_j = (0, \cdots, 1, \cdots, 0)' \in \mathbb{R}^p$. We consider the transient performance of the feedback system $\mathscr{F}[G^*, K_k]$. It is easy to see that the transfer matrix L_k^* is given by

$$L_k^*(s) = \operatorname*{diag}_{1\leqq j\leqq p}\left(\frac{(k_j+c_j)s+k_jc_j}{(s+k_j)(s+c_j)}\right).$$

Since we are interested in high-gain feedback, we may assume without loss of generality that $k_j > c_j$, $j = 1, \cdots, p$. A routine calculation gives the following estimates for the overshoot O_j, the rise time T_j^r, and the settling time T_j^s in the jth loop (see, e.g., Franklin, Powell, and Emami-Naeini [8] for the notions of overshoot, rise time, and so forth).

We have $O_j \leqq c_j/(k_j - c_j)$, $T_j^r \leqq \frac{1}{2}\tau_j$, and $T_j^s \leqq \max(\frac{1}{2}\tau_j, -1/c_j \ln((k_j - c_j)/100\, c_j))$, where $\tau_j = (2(\ln(k_j) - \ln(c_j))/(k_j - c_j))$ is the time when the maximal overshoot occurs. The settling time T_j^s is defined here as the time required for the signal $(L_k^* * \theta_j)_j(t)$ to stay within the interval [0.99, 1.01].

The estimates show that the transient performance of the feedback system $\mathscr{F}[G^*, K_k]$ improves as the gains $k_j, j = 1, \cdots, p$, increase.

(ii) Suppose that $\Gamma = D$. In this case, part (i) of this remark and (3.7) show that the transient performance of the feedback system $\mathscr{F}[G, K_k]$ improves as the gains k_j, $j = 1, \cdots, p$ increase. If $\Gamma \neq D$, then (3.8) gives a bound on the performance degradation, provided that the condition of Lemma 3.8 (ii) is satisfied.

In the following, we investigate the effect of measurement nonlinearities on the stability of $\mathscr{F}[G, K_k]$. First, we will concentrate on memoryless nonlinearities. We consider functions $\varphi: \mathbb{R}_+ \times \mathbb{C}^p \to \mathbb{C}^p$, which satisfy the following conditions:

(N1) $\varphi(t, x)$ is continuous in t and locally Lipschitz continuous in x, uniformly in t on bounded intervals;

(N2) φ is unbiased, i.e., $\varphi(t, 0) = 0$, for all $t \geqq 0$;

(N3) $\varphi = \mathrm{id}_{\mathbb{C}^p} + \varphi_1 + \varphi_2$, where φ_1 and φ_2 satisfy

$$\text{and}\quad \left.\begin{array}{l}|\varphi_1(t,x)| \leqq \lambda_1|x|\\ |\varphi_2(t,x)| \leqq \lambda_2\end{array}\right\} \quad \begin{array}{l}\text{for all } t \geqq 0,\ x \in \mathbb{C}^p,\\ \text{and some constants } \lambda_1, \lambda_2 \geqq 0.\end{array}$$

Furthermore, for a function $\varphi: \mathbb{R}_+ \times \mathbb{C}_p \to \mathbb{C}^p$, let N_φ denote the operator induced by φ, i.e., $(N_\varphi u)(t) = \varphi(t, u(t))$ for any function $u: \mathbb{R}_+ \to \mathbb{C}^p$.

The following result gives a sufficient condition for the stability of the feedback system shown in Fig. 4.

THEOREM 3.10. *Let G be a square transfer matrix such that G^{-1} is of the form* (3.1). *Let the controller K_k be given by* (3.3), *suppose that $\varphi: \mathbb{R}_+ \times \mathbb{C}^p \to \mathbb{C}^p$ satisfies*

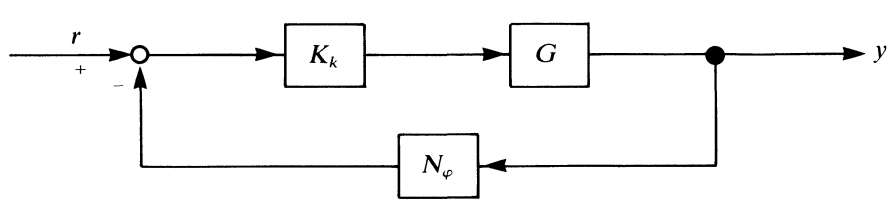


FIG. 4

(N1)-(N3), *and assume that the reference signal r is bounded on bounded intervals. Then, if* $\lambda_1 < 1$ *and*

$$\|\Gamma^{-1}(\Gamma - D)\| < \tfrac{1}{2}(1-\lambda_1), \tag{3.9}$$

there exists $k^* > 0$ *such that for all* $k_j > k^*$ $(j = 1, \cdots, p)$ *the nonlinear feedback system given by*

$$y = \check{G} * e, \qquad e = \check{K}_k * (r - N_\varphi(y)) \tag{3.10}$$

is well posed (i.e., *there exists a unique globally defined solution of* (3.10)) *and* L^∞*-stable in the sense that there exist nonnegative constants* m_1 *and* m_2 *such that*

$$\|y\|_\infty \leqq m_1 + m_2 \|r\|_\infty \tag{3.11}$$

for all $r \in (L^\infty(\mathbb{R}_+))^p$.

Proof. First, realize that for all sufficiently large k_j $(j = 1, \cdots, p)$

$$\|\check{L}_k\|_1 \lambda_1 < 1. \tag{3.12}$$

This follows from (3.9), Lemma 3.8(ii), and the fact that $\lim_{k\to\infty} \|\check{L}_k^*\|_1 = 1$ (cf. Logemann and Owens [22]), where L_k^* is defined as in Lemma 3.8. Equation (3.10) can be written in the form

$$y = \check{L}_k * r - \check{L}_k * (N_{\varphi_1}(y) + N_{\varphi_2}(y)), \tag{3.13}$$

which is a nonlinear Volterra integral equation in y. It follows from Theorem 1.2 and Corollary 2.7, in Miller [26] that (3.13) has a unique (continuous) solution that can be extended to the right as long as it remains bounded. Now pick $t > 0$ such that the solution of (3.13) exists on $[0, t]$. Application of the truncation operator π_t to (3.13) gives

$$\begin{aligned} \|\pi_t(y - y_l)\|_\infty &\leqq \|\pi_t \check{L}_k * (\pi_t(N_{\varphi_1}(y) + N_{\varphi_2}(y))\|_\infty \\ &\leqq \|\check{L}_k\|_1 (\|\pi_t(N_{\varphi_1}(y))\|_\infty + \lambda_2) \\ &\leqq \|\check{L}_k\|_1 (\lambda_1 \|\pi_t y\|_\infty + \lambda_2), \end{aligned} \tag{3.14}$$

where $y_l := \check{L}_k * r$ is the output of the linear feedback system $\mathcal{F}[G, K_k]$. Setting $\lambda := \|\check{L}_k\|_1$, we obtain

$$(1 - \lambda\lambda_1)\|\pi_t(y - y_l)\|_\infty \leqq \lambda(\lambda_1 \|\pi_t y_l\|_\infty + \lambda_2),$$

and hence ($\lambda\lambda_1 < 1$ by (3.12))

$$\|\pi_t(y - y_l)\|_\infty \leqq \frac{\lambda}{1 - \lambda\lambda_1}(\lambda_1 \|\pi_t y_l\|_\infty + \lambda_2). \tag{3.15}$$

Inequality (3.15) shows that the solution of (3.13) exists on $\mathbb{R}_+$, since $\|\pi_t y_l\|_\infty$ is finite for all $t \in \mathbb{R}_+$. Moreover, it follows that (3.11) holds with $m_1 = \lambda\lambda_2(1 - \lambda\lambda_1)^{-1}$ and $m_2 = \lambda(1 - \lambda\lambda_1)^{-1}$. □

Remark 3.11. (i) The proof shows that Theorem 3.10 remains true if we replace ∞ by $q = 1, 2, 3, \cdots$, provided that $\varphi_2 \equiv 0$.

(ii) Equation (3.15) yields an upper bound on the difference of the output signals of the linear and nonlinear feedback system corresponding to the same input signal r.

We now turn our attention to a certain class of dynamical measurement nonlinearities. We consider operators Φ from $(LL^q(\mathbb{R}_+))^p$ onto itself, which satisfy the following assumptions:

(N4) Φ is causal, i.e., $\pi_t\Phi = \pi_t\Phi\pi_t$ for all $t \geqq 0$ (cf. Willems [38]);

(N5) Φ is locally Lipschitz continuous, i.e., for all $t \geqq 0$ there exists $l_t \geqq 0$ such that $\|\pi_t(\Phi u - \Phi v)\|_q \leqq l_t \|\pi_t(u - v)\|_q$ for all $u, v \in (LL^q(\mathbb{R}_+))^p$ (cf. Willems [38]);

(N6) Φ is unbiased, i.e., $\Phi(0) = 0$;

(N7) $\Phi = \mathrm{id} + \Psi$, where Ψ satisfies $\|\Psi u\|_q \leqq \lambda_1 \|u\|_q + \lambda_2$ for all $u \in (L^q(\mathbb{R}_+))^p$ and some nonnegative constants λ_1 and λ_2.

Remark 3.12. Consider the nonlinearity N_φ induced by the function $\varphi: \mathbb{R}_+ \times \mathbb{C}^p \to \mathbb{C}^p$. Then, in general, N_φ will not fulfill (N5) unless φ satisfies a global Lipschitz condition.

THEOREM 3.13. *Let G be a square transfer matrix such that G^{-1} is of the form* (3.1), *let the controller K_k be given by* (3.3), *suppose that Φ: $(LL^q(\mathbb{R}_+))^p \to (LL^q(\mathbb{R}_+))^p$ satisfies* (N4)–(N7), *and assume that $r \in (LL^q(\mathbb{R}_+))^p$. Then we have* (i) *the nonlinear feedback system given by*

$$y = \check{G} * e, \qquad e = \check{K}_k * (r - \Phi(y)) \tag{3.16}$$

is well posed in the sense that there exists a unique (globally defined) solution $y \in (LL^q(\mathbb{R}_+))^p$; and (ii) *suppose that $\lambda_1 < 1$ and*

$$\|\Gamma^{-1}(\Gamma - D)\| < \tfrac{1}{2}(1 - \lambda_1); \tag{3.17}$$

then for all sufficiently large k_j $(j = 1, \cdots, p)$, the feedback system (3.16) *is L^q-stable in the sense that there exist nonnegative constants m_1 and m_2 such that $\|y\|_q \leqq m_1 + m_2\|r\|_q$, for all $r \in (L^q(\mathbb{R}_+))^p$.*

Proof. Equation (3.16) can be written in the form

$$y = \check{L}_k * r - \check{L}_k * \Psi(y). \tag{3.18}$$

It follows from Corollary 4.1.2 in Willems [38] that (3.18) admits a unique solution in $(LL^q(\mathbb{R}_+))^p$. The stability result can be shown, as in the proof of Theorem 3.10. □

4. Conditions in state-space terms for (3.1) to be satisfied. In this section we will give sufficient conditions for a system in state-space form to satisfy the decomposition (3.1). Our state-space system is given by

$$\dot{x} = Ax + Bu; \qquad x(0) = x_0, \tag{4.1a}$$

$$y = Cx, \tag{4.1b}$$

where (i) $A: D(A) \subset X \to X$, X is a Banach space, generates a strongly continuous semigroup, denoted by $T(t)$; (ii) $B: \mathbb{R}^p \mapsto X$; $(u_1, \cdots, u_p)^t \mapsto \sum_{i=1}^p b_i u_i$, where $b_i \in X$, $i = 1, \cdots, p$; and (iii) $C: X \mapsto \mathbb{R}^p$, $x \mapsto (\langle x, c_1\rangle, \cdots, \langle x, c_p\rangle)^t$, where $c_i \in X^*$; $i = 1, \cdots, p$.

In § 3 we have seen that the zeros of the system play an essential role in determining if it has a decomposition (3.1). The next definition gives an equivalent definition for zeros of a state-space system.

DEFINITION 4.1. Let $z \in \mathbb{C}$; then z is a zero of system (4.1) if the kernel of the operator $\left[\begin{smallmatrix} zI-A & B \\ C & 0 \end{smallmatrix}\right]: D(A) \oplus \mathbb{C}^p \mapsto X \oplus \mathbb{C}^p$ is nonzero.

LEMMA 4.2. *If system* (4.1) *is α-exponentially stabilizable and α-exponentially detectable, then the zeros in $\mathbb{C}_\alpha$ of* (4.1) *are the same as the zeros of $G(s) := C(sI - A)^{-1}B$ as defined in* § 2.

For the definitions of α-exponentially stabilizability and detectability, see Appendix 3.

Proof. Since the state-space system is α-exponentially stabilizable, there exists a bounded linear operator $F: X \to \mathbb{R}^p$ such that the semigroup generated by $A+BF$, $T_{BF}(t)$, satisfies $\|T_{BF}(t)\| \leqq M e^{\beta t}$, $\beta < \alpha$. With this feedback, we can construct the following right coprime factorization of $G(s)$ over $\mathscr{H}(\mathbb{C}_\alpha)$ (see Jacobson [13] and Nett, Jacobson, and Balas [30]):

$$G = NM^{-1}, \quad \text{where } N(s) = C(sI - A - BF)^{-1}B, \qquad M(s) = I + F(sI - A - BF)^{-1}B.$$

Let $u \in \mathbb{C}^p$, $u \neq 0$, satisfy $C(s_0 I - A - BF)^{-1}Bu = 0$ for some $s_0 \in \mathbb{C}_\alpha$. Then, by the coprimeness of N and M, $u + F(s_0 I - A - BF)^{-1}Bu \neq 0$, and

$$\begin{bmatrix} s_0 I - A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} (s_0 I - A - BF)^{-1}Bu \\ -u - F(s_0 I - A - BF)^{-1}Bu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.2}$$

On the other hand, if

$$\begin{bmatrix} s_0 I - A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.3}$$

for some $(x, u) \neq (0, 0) \in D(A) \oplus \mathbb{C}^p$, then

$$(s_0 I - A - BF)x + B(u + Fx) = 0. \tag{4.4}$$

Premultiplying (4.4) with $C(s_0 I - A - BF)^{-1}$ gives

$$0 = Cx + C(s_0 I - A - BF)^{-1}B(u + Fx) = 0 + C(s_0 I - A - BF)^{-1}B(u + Fx).$$

So, if $(u + Fx) \neq 0$, then s_0 is a zero of the transfer matrix G. If $u + Fx = 0$, then (4.4) with the invertibility of $(s_0 I - A - BF)$ would imply that $x = 0$. Hence also $u = 0$, since $u = u + Fx - Fx$. This is in contradiction with $(x, u) \neq (0, 0)$. □

For the system under consideration, we will prove that under certain conditions the transfer function can be decomposed in a similar way as in (3.1).

LEMMA 4.3. *Suppose that (A, B) is α-exponentially stabilizable. Assume further that $\det(CB) \neq 0$ and let $\gamma \in \mathbb{C}$. Then*

$$G^{-1}(s) = s[CB]^{-1} + (\gamma - s)[CB]^{-1}CA(sI - A - BF)^{-1}B[CB]^{-1} \tag{4.5}$$

holds on $\mathbb{C}_\alpha \cap \rho(A) \cap \rho(A + BF)$, where $G(s) = C(sI - A)^{-1}B$ and

$$F := [CB]^{-1}\{-CA + \gamma C\}. \tag{4.6}$$

Proof. The feedback F defined by (4.6) is an A-degenerate operator (Kato [16, Chap. IV, § 6]). Since (A, B) is α-exponentially stabilizable, the spectrum of A in $\mathbb{C}_\alpha$ is pure point spectrum with finite multiplicity (see Jacobson and Nett [14] or Appendix 3). Together with the fact that A generates a C_0-semigroup, this implies that $A + BF$ has pure point spectrum with finite multiplicity in $\mathbb{C}_\alpha$ (see Kato [16, Chap. IV, § 6]). So, with the exception of countably many points, we may calculate $(sI - A - BF)^{-1}$ for $s \in \mathbb{C}_2$. Let $s \in \rho(A + BF) \cap \rho(A)$; then

$$\begin{aligned} CA(sI - A - BF)^{-1}B &= CA(sI - A)^{-1}B + CA(sI - A - BF)^{-1}BF(sI - A)^{-1}B \\ &= CA(sI - A)^{-1}B \\ &\quad + CA(sI - A - BF)^{-1}B[CB]^{-1}(-1)CA(sI - A)^{-1}B \\ &\quad + CA(sI - A - BF)^{-1}B[CB]^{-1}\gamma C(sI - A)^{-1}B \\ &= \left\{ I - \left(1 - \frac{\gamma}{s}\right) CA(sI - A - BF)^{-1}B[CB]^{-1} \right\} CA(sI - A)^{-1}B \\ &\quad + \frac{\gamma}{s} CA(sI - A - BF)^{-1}B, \end{aligned}$$

where in the last equality we have used that $C(sI-A)^{-1}B = 1/s\{CB + CA(sI-A)^{-1}B\}$. So

$$\left(1-\frac{\gamma}{s}\right) CA(sI-A-BF)^{-1}B = \left\{I - \left(1-\frac{\gamma}{s}\right) CA(sI-A-BF)^{-1}B[CB]^{-1}\right\} \times CA(sI-A)^{-1}B$$

$$\Rightarrow CA(sI-A)^{-1}B[CB]^{-1} = \left\{I - \left(1-\frac{\gamma}{s}\right) CA(sI-A-BF)^{-1}B[CB]^{-1}\right\}^{-1} \times \left(1-\frac{\gamma}{s}\right) CA(sI-A-BF)^{-1}B[CB]^{-1}.$$

Using the equality $[I-Q(s)]^{-1}Q(s) = [I-Q(s)]^{-1} - I$ gives

$$CA(sI-A)^{-1}[CB]^{-1} = \left\{I - \left(1-\frac{\gamma}{s}\right) CA(sI-A-BF)^{-1}B[CB]^{-1}\right\}^{-1} - I.$$

So

$$sG(s) = sC(sI-A)^{-1}B = CB + CA(sI-A)^{-1}B$$
$$= \left\{I - \left(1-\frac{\gamma}{s}\right) CA(sI-A-BF)^{-1}B[CB]^{-1}\right\}^{-1} [CB].$$

Thus

$$G^{-1}(s) = s[CB]^{-1} + (\gamma - s)[CB]^{-1}CA(sI-A-BF)^{-1}B[CB]^{-1}. \qquad \square$$

So, the above lemma gives a decomposition similar to (3.1), but we do not know if $H(s) = (\gamma - s)[CB]^{-1}CA(sI-A-BF)^{-1}B[CB]^{-1} \in H_{-}^{\infty(p\times p)}$. This result will be given in Theorem 4.5. First, we will prove that, with the feedback F defined by (4.3), $(sI-A-BF)$ is invertible in $\mathbb{C}_\alpha$, provided that $G(\cdot)$ has no zeros there.

LEMMA 4.4. *Suppose that* (A, B) *is* α*-exponentially stabilizable and* (C, A) *is* α*-exponentially detectable. If system* (4.1) *has no zeros in* $\mathbb{C}_\alpha$*, and* $\gamma < \alpha$*, then* $(s-A-BF)$ *is invertible on* $\mathbb{C}_\alpha$*, where* F *is defined by* (4.6).

Proof. From the proof of Lemma 4.3 we know that $A+BF$ has only point spectrum on $\mathbb{C}_\alpha$. Let λ be an eigenvalue of $(A+BF)$ in $\mathbb{C}_\alpha$. Then there exists an $x \in X$, $x \neq 0$ such that

$$(\lambda I - A - BF)x = 0 \Leftrightarrow (\lambda I - A + B[CB]^{-1}CA)x - \gamma B[CB]^{-1}Cx = 0.$$

Premultiplying with C gives $\lambda Cx - CAx + CAx - \gamma Cx = 0$. Hence $\lambda = \gamma$ or $Cx = 0$. The first possibility is excluded by assumption, so suppose that $Cx = 0$. Then we have that

$$\begin{pmatrix} \lambda I - A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ [CB]^{-1}CAx \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus λ is a zero, but this is also excluded. So $\sigma(A+BF) \cap \mathbb{C}_\alpha = \varnothing$. $\square$

So, if the zeros of the system are in $\mathbb{C}\backslash\mathbb{C}_\alpha$, then $G^{-1}(s)$ exists everywhere on $\mathbb{C}_\alpha$. We may always write $G^{-1}(s)$ as $G^{-1}(s) = sD + H(s)$, where $H(s)$ has no poles in $\mathbb{C}_\alpha$. However, this is not enough to ensure that $H \in (H_{-}^{\infty})^{p\times p}$. The next theorem will give sufficient conditions for this to hold.

THEOREM 4.5. *Suppose that* (A, B) *is exponentially stabilizable,* (C, A) *is exponentially detectable, system* (4.1) *has no zeros in* $\mathbb{C}_\alpha$ *for some* $\alpha < 0$*, and* $\det(CB) \neq 0$*. Then* *the transfer matrix* $G(s)=C(sI-A)^{-1}B$ *has the property that* $G^{-1}(s)=s[CB]^{-1}+H(s)$ *with* $H\in(H_-^\infty)^{p\times p}$, *provided that either*

(i) $c_i\in D(A^{*2})$; $i=1,\cdots,p$, *or*

(ii) $b_i\in D(A)$, $c_i\in D(A^*)$; $i=1,\cdots,p$, *or*

(iii) $b_i\in D(A^2)$; $i=1,\cdots,p$, *or*

(iv) *A generates an analytic semigroup and* $b_i\in D(A)$, $i=1,\cdots,p$, *or*

(v) *A generates an analytic semigroup and* $c_i\in D(A^*)$, $i=1,\cdots,p$.

Furthermore,

$$H(s)=(\gamma-s)[CB]^{-1}CA(sI-A-BF)^{-1}B[CB]^{-1} \tag{4.7}$$

with

$$F=[CB]^{-1}\{-CA+\gamma C\}, \tag{4.8}$$

where $\gamma<0$.

Proof. Note that if (A, B, C) is exponentially stabilizable and detectable, then it is also β-exponentially stabilizable and detectable for some $\beta<0$. So, without loss of generality, we may assume that the system is α-exponentially stabilizable and detectable and that it has no zeros in $\mathbb{C}_\alpha$ for some negative α. Moreover, let $\gamma<\alpha$. These conditions ensure that on $\mathbb{C}_\alpha$ we have by Lemmas 4.3 and 4.4 that

$$G^{-1}(s)=s[CB]^{-1}+(\gamma-s)[CB]^{-1}CA(sI-A-BF)^{-1}B[CB]^{-1}.$$

So we must show that $H(s):=(\gamma-s)[CB]^{-1}CA(sI-A-BF)^{-1}B[CB]^{-1}$ is analytic and bounded on $\mathbb{C}_\alpha$.

Assume first that $c_i\in D(A^*)$; $i=1,\cdots,p$. So CA is a bounded operator from X to $\mathbb{R}^p$, and thus F is bounded. Since (A, B) is exponentially stabilizable and since $A+BF$ has no eigenvalues in $\mathbb{C}_\alpha$ (Lemma 4.4), we have that $T_{BF}(t)$ is exponentially stable (see Appendix 3, Theorem A.6), and hence $\gamma[CB]^{-1}CA(s-A-BF)^{-1}B[CB]^{-1}$ is in $(H_-^\infty)^{p\times p}$. This is, in general, not sufficient to ensure that $H\in(H_-^\infty)^{p\times p}$. We can rewrite the operator $sCA(sI-A-BF)^{-1}B$ as

$$\begin{aligned}
&CAs(sI-A-BF)^{-1}B\\
&\quad=CA(sI-A-BF)(sI-A-BF)^{-1}B+CA(A+BF)(sI-A-BF)^{-1}B\\
&\quad=CAB+CA(A+BF)(s-A-BF)^{-1}B \qquad (4.9)\\
&\quad=CAB+CA(sI-A-BF)^{-1}(A+BF)B. \qquad (4.10)
\end{aligned}$$

From (4.9) we see that $H\in(H_-^\infty)^{p\times p}$ if $c_i\in D(A^{*2})$; $i=1,\cdots,p$, and from (4.10) we see that $H\in(H_-^\infty)^{p\times p}$ if $c_i\in D(A^*)$ and $b_i\in D(A)$, $i=1,\cdots,p$. So we have proved that conditions (i) and (ii) imply the desired property. Now we will prove that condition (iii) does the same.

If $b_i\in D(A^2)$, $i=1,\cdots,p$, then let us consider a new realization (A_n, B_n, C_n) of $G(s)$, namely $A_n=A$, $B_n=AB$, and $C_n=CA^{-1}$, where we have assumed that $0\in\rho(A)$, but this is not essential. If (A, B, C) is exponentially stabilizable and exponentially detectable, then (A_n, B_n, C_n) is, also. This follows easily from Theorem A.5 in Appendix 3, the "dual" version of Theorem A.5, and the definitions of A_n, B_n, and C_n. Since $\operatorname{Im} B_n\subset D(A_n)$ and $\operatorname{Im} C_n^*\subset D(A_n^*)$, we have by part (ii) that

$$H_n(s)=(\gamma-s)[C_nB_n]^{-1}C_nA_n(sI-A_n-B_nF_n)^{-1}B_n[C_nB_n]^{-1} \tag{4.11}$$

is an element of $(H_-^\infty)^{p\times p}$, where $F_n:=[C_nB_n]^{-1}\{-C_nA_n+\gamma C_n\}$. It is clear that $F_n=FA^{-1}$, with F given by (4.8), and hence $H=H_n$. So it remains to show that condition (iv) or (v) is sufficient, also.

The feedback law F, as defined in (4.8), is an A-degenerate operator. From Zabczyk [39] it is known that if A generates an analytic semigroup, so does $A+BF$, for an A-degenerate feedback F. So, since any analytic semigroup satisfies the spectrum determined growth condition, and $\sigma(A+BF)\cap\mathbb{C}_\alpha=\emptyset$, there exists $M>0$ such that

$$\|(sI-A-BF)^{-1}\|\leqq\frac{M}{|s-\alpha|};\qquad s\in\mathbb{C}_\alpha.$$

Thus, if $c_i\in D(A^*)$; $i=1,\cdots,p$, then

$$\|H(s)\|\leqq\frac{|\gamma-s|}{|s-\alpha|}\|CA\|\,\|[CB]^{-1}\|^2\|B\|M;\qquad s\in\mathbb{C}_\alpha.$$

On the other hand, if $b_i\in D(A)$, then we can rewrite $H(s)$ as

$$H(s)=(\gamma-s)[CB]^{-1}C(sI-A-ABFA^{-1})^{-1}AB[CB]^{-1};\qquad s\in\mathbb{C}_\alpha,$$

and we have that

$$\|H(s)\|\leqq\frac{|\gamma-s|}{|s-\alpha|}\|C\|\,\|AB\|\,\|[CB]^{-1}\|^2\tilde{M};\qquad s\in\mathbb{C}_\alpha$$

for some suitable constant $\tilde{M}>0$. Thus in both cases $H\in(H_-^\infty)^{p\times p}$. □

Remark 4.6. Since $G(s)$ and $s[CB]^{-1}$ are independent of γ, we must have that $H(\cdot)$ is independent of γ. In fact, for $\gamma>\alpha$ the zero at γ introduced by the term $s-\gamma$ is cancelled by the pole at γ of $(s-A-BF)^{-1}$ (see the proof of Lemma 4.4).

Remark 4.7. Retarded systems do not satisfy any of the smoothness conditions (i)-(v) in Theorem 4.5. However, by Example 3.3, there is a whole class of retarded systems whose transfer matrices admit a decomposition of the form (3.1). This shows that the conditions of Theorem 4.5 are sufficient but not necessary for (3.1) to be satisfied.

5. Internal stability. The stability results in § 3 are formulated in input-output terms. Suppose that the transfer matrix G of a state-space system of the form (4.1) satisfies condition (3.1) (Theorem 4.5 gives conditions in state-space terms for this to be true). If we apply Theorem 3.10 or Theorem 3.13 to G, can we expect internal stability of the closed-loop system? In the *linear* case (i.e., N_φ in Theorem 3.10 and Φ in Theorem 3.10 are equal to the identity) the answer is yes, provided that the state-space realizations of the plant and the controller are both exponentially stabilizable and exponentially detectable. This follows from recent results on the equivalence of input-output and internal stability for infinite-dimensional systems (cf. Jacobson and Nett [14] and Curtain [5]). In this section we investigate the internal asymptotic behaviour of the *nonlinear* feedback systems considered in § 3.

LEMMA 5.1. *Let $T(t)$ be an exponentially stable, strongly continuous semigroup on the Banach space X, denote the generator of $T(t)$ by A, let $B:\mathbb{R}^p\to X$ and $C: X\to\mathbb{R}^p$ be bounded linear operators, and suppose that $f:\mathbb{R}_+\times\mathbb{R}^p$ satisfies* (N1) *and* $|f(t,x)|\leqq\lambda|x|$ *for all $t\geqq0$, $x\in\mathbb{R}^p$ for some $\lambda>0$. Moreover, set $R(t)=CT(t)B$ and for $t_0\geqq0$ and $x_0\in X$ let $x(t;t_0,x_0)$ denote the mild solution of*

$$(5.1)\qquad \begin{aligned}\dot{x}(t)&=Ax+Bf(t,Cx(t)),\qquad t\geqq t_0,\\ x(t_0)&=x_0.\end{aligned}$$

If $\|R\|_1\lambda<1$, then (5.1) *has a unique, globally defined mild solution and there exist positive constants M and ε such that $\|x(t;t_0,x_0)\|\leqq M e^{-\varepsilon(t-t_0)}\|x_0\|$ for all $x_0\in X$ and $t\geqq t_0\geqq0$.*

Remark 5.2. The assumptions made in Lemma 5.1 ensure that (5.1) admits a unique mild solution that can be continued to the right as long as it remains bounded (see Pazy [32]).

The proof of Lemma 5.1 is similar to the proof of Theorem 3.2 in Logemann [21] and is therefore omitted.

Now let us turn our attention to dynamical nonlinearities.

LEMMA 5.3. *Let X, $T(t)$, A, B, C, and $R(t)$ be as in Lemma 5.1. Suppose that $F: (LL^q(\mathbb{R}_+))^p \to (LL^q(\mathbb{R}_+))^p$ $(q = 1, 2, 3, \cdots;\ q \neq \infty)$ is causal, unbiased, and locally Lipschitz continuous. Then the following statements hold.* (i) *The equation*

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)BF(Cx(\cdot))(\tau)\, d\tau \tag{5.2}$$

admits for all $x_0 \in X$ a unique globally defined continuous solution $x(\cdot, x_0): [0, \infty) \to X$, which will be called the mild solution of

$$\dot{x}(t) = Ax(t) + BF(Cx(\cdot))(t), \qquad x(0) = x_0; \tag{5.3}$$

and (ii) *suppose that F additionally satisfies the condition*

$$\|F(u)\|_q \leqq \lambda_1 \|u\|_q + \lambda_2$$

for all $u \in (L^q(\mathbb{R}_+))^p$, where λ_1 and λ_2 are nonnegative constants; then the origin will be globally attractive (i.e., $\lim_{t\to\infty} x(t; x_0) = 0$ *for all* $x_0 \in X$) *if*

$$\|R\|_1 \lambda_1 < 1. \tag{5.4}$$

Proof. (i) It is clear that the mapping $u(\cdot) \mapsto F(Cu(\cdot))$ is causal, unbiased, and locally Lipschitz continuous. Hence it follows from Corollary 4.1.2 in Willems [38] that (5.2) has a unique solution x in $LL^q(\mathbb{R}_+, X)$. Since the right-hand side of (5.2) is continuous in t, we see that $x(t)$ is continuous as well.

(ii) Consider the equation

$$\begin{aligned} y(t) &= CT(t)x_0 + C\int_0^t T(t-\tau)BF(y(\cdot))(\tau)\, d\tau \\ &= CT(t)x_0 + \int_0^t R(t-\tau)F(y(\cdot))(\tau)\, d\tau. \end{aligned} \tag{5.5}$$

If $x(t) := x(t; x_0)$ is the solution of (5.2), then it is clear that $Cx(t)$ is a solution of (5.5). Using (5.4) and the fact that $CT(\cdot)x_0 \in (L^q(\mathbb{R}_+))^p$, it follows from the small-gain theorem that $Cx(\cdot) \in (L^q(\mathbb{R}_+))^p$ and hence $z(\cdot) := BF(Cx(\cdot)) \in L^q(\mathbb{R}_+, X)$. It remains to show that $w(t) := \int_0^t T(t-\tau)z(\tau)\, d\tau$ tends to zero as $t \to \infty$. By the exponential stability of $T(t)$, there exist positive constants N and γ such that $\|T(t)\| \leqq Ne^{-\gamma t}$ for all $t \geqq 0$.

Suppose for a moment that $q \neq 1$, and define q' by $1/q' + 1/q = 1$:

$$\begin{aligned} \|w(t)\| &\leqq N\left\{\int_0^{t/2} e^{-\gamma(t-\tau)}\|z(\tau)\|\, d\tau + \int_{t/2}^t e^{-\gamma(t-\tau)}\|z(\tau)\|\, d\tau\right\} \\ &= N\left\{\int_{t/2}^t e^{-\gamma\tau}\|z(t-\tau)\|\, d\tau + \int_{t/2}^t e^{-\gamma(t-\tau)}\|z(\tau)\|\, d\tau\right\} \\ &\leqq N\left\{\left(\int_{t/2}^t e^{-q'\gamma\tau}\, d\tau\right)^{1/q'}\left(\int_{t/2}^t \|z(t-\tau)\|^q\, d\tau\right)^{1/q}\right. \\ &\quad \left. + \int_{t/2}^t e^{-q'\gamma(t-\tau)}\, d\tau\right)^{1/q'}\left(\int_{t/2}^t \|z(\tau)\|^q\, d\tau\right)^{1/q}\right\}. \end{aligned}$$

We obtain

$$\|w(t)\| \leqq N\left\{\left(\int_{t/2}^{\infty} e^{-q'\gamma\tau}\,d\tau\right)^{1/q'}\|z\|_q + \|e^{-\gamma\cdot}\|_{q'}\left(\int_{t/2}^{\infty}\|z(\tau)\|^q\,d\tau\right)^{1/q}\right\}.$$

Now $\lim_{t\to\infty}\int_{t/2}^{\infty} e^{-q'\gamma\tau}\,d\tau=0$ and $\lim_{t\to\infty}\int_{t/2}^{\infty}\|z(\tau)\|^q\,d\tau=0$ and thus $\lim_{t\to\infty}\|w(t)\|=0$. A similar argument holds for the case where $q=1$. □

Remark 5.4. If in Lemma 5.3 (ii) $\lambda_2=0$, then it is not difficult to see that the origin of (5.2) is globally asymptotically stable; i.e., $\lim_{t\to\infty} x(t; x_0)=0$ for all $x_0\in X$, and for all $\varepsilon>0$ there exists $\delta>0$ such that $\|x_0\|\leqq\delta$ implies $\|x(t; x_0)\|\leqq\varepsilon$ for all $t\geqq 0$.

In the following, let the plant be given by

$$\dot{x}(t)=Ax(t)+Bu(t), \quad x(t_0)=x_0, \quad y(t)=Cx(t), \tag{5.6}$$

where the linear operator A generates a strongly continuous semigroup $T(t)$ on a Banach space X, B: $\mathbb{R}^p\to X$, C: $X\to\mathbb{R}^p$ are bounded linear operators, (A, B) is exponentially stabilizable, and (C, A) is exponentially detectable. A minimal realization of the controller K_k defined in (3.3) is given by

$$\dot{z}(t)=\operatorname*{diag}_{1\leqq j\leqq p}(k_jc_j)v(t), \quad z(t_0)=z_0, \quad w(t)=\Gamma z(t)+\Gamma \operatorname*{diag}_{1\leqq j\leqq p}(k_j+c_j)v(t). \tag{5.7}$$

Let Φ be an operator mapping $(LL^q(\mathbb{R}_+))^p$ into itself. We will interpret Φ as a measurement nonlinearity in the feedback interconnection of (5.6) and (5.7) as follows:

$$u=w, \qquad v=-\Phi(y). \tag{5.8}$$

THEOREM 5.5. *Suppose that* $G(s):=C(sI-A)^{-1}B$ *satisfies condition* (3.1) (*cf.* *Theorem* 4.5), *and define* $x_c(t):=(x(t), z(t))'$. *The following statements hold.* (i) *If* Φ *in* (5.8) *is given by* N_φ, *where* φ *satisfies* (N1)-(N3), $\lambda_1<1$, $\lambda_2=0$, *and* $\|\Gamma^{-1}(\Gamma-D)\|<\frac{1}{2}(1-\lambda_1)$ *then for all sufficiently large gains* k_j, $j=1,\cdots,p$, *there exist positive constants* M *and* ε (*dependent on* k) *such that* $\|x_c(t)\|\leqq M e^{-\varepsilon(t-t_0)}\|x_c(t_0)\|$ *for all* $x_c(t_0)\in X\times\mathbb{R}^p$, $t\geqq t_0\geqq 0$, *i.e., the nonlinear feedback system given by* (5.6)-(5.8) *is globally exponentially stable*; *and* (ii) *if* Φ *in* (5.8) *satisfies* (N4)-(N7), $q<\infty$, $\lambda_1<1$, *and* $\|\Gamma^{-1}(\Gamma-D)\|<\frac{1}{2}(1-\lambda_1)$, *then for all sufficiently large* k_j, $j=1,\cdots,p$, *the feedback system* (5.6)-(5.8) *is internally stable in the sense that the origin is globally attractive*; *i.e.,* $\lim_{t\to\infty} x_c(t)=0$ *for all* $x_c(0)\in X\times\mathbb{R}^p$.

Proof. (i) Set $B_k:=\operatorname{diag}_{1\leqq j\leqq p}(k_jc_j)$ and $D_k:=\Gamma\operatorname{diag}_{1\leqq j\leqq p}(k_j+c_j)$. A routine calculation then shows that

$$\dot{x}_c(t)=A_cx_c(t)-B_c\varphi_1(t, C_cx_c(t)), \tag{5.9}$$

where

$$A_c=A_c(k):=\begin{pmatrix} A-BD_kC & B\Gamma \\ -B_kC & 0\end{pmatrix}, \quad B_c=B_c(k):=\begin{pmatrix} BD_k \\ B_k\end{pmatrix}, \quad C_c:=(C\;\;0).$$

We know from Theorem 3.5(i) that $\mathscr{F}[G, K_k]$ is H_-^∞-stable for all sufficiently large k_j, $j=1,\cdots,p$, and thus, by a result of Jacobson and Nett [14] (cf. also Curtain [5]), $A_c(k)$ generates an exponentially stable semigroup $T_{c,k}(t)$ on $X\times\mathbb{R}^p$ for all large enough k_j, $j=1,\cdots,p$. It follows from the condition $\|\Gamma^{-1}(\Gamma-D)\|<\frac{1}{2}(1-\lambda_1)$ and Lemma 3.8(ii) that for all sufficiently large k_j $(j=1,\cdots,p)$ $\|\check{L}_k\|_1\lambda_1<1$. (Here we have used that $\lim_{k\to\infty}\|\check{L}_k^*\|_1=1$; cf. Logemann and Owens [22].) Finally, realize that $\check{L}_k(t)=C_cT_{c,k}(t)B_c(k)$, and apply Lemma 5.1 to (5.9).

(ii) Using the same arguments as in (i) and applying Lemma 5.3 instead of Lemma 5.1, we can prove the second claim. □

Remark 5.6. It is well known that the retarded system of Example 3.3 admits an abstract state-space realization of the form (5.6), where the state space X is given by $M^2(-h, 0; \mathbb{R}^n) := \mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n)$. Using the notation of Example 3.3, let us assume that

$$\det \begin{pmatrix} sI - \hat{A}(s) & -B \\ C & 0 \end{pmatrix} \neq 0 \quad \text{for all } s \in \bar{\mathbb{C}}_0. \tag{5.10}$$

In particular, it follows from (5.10) that

$$rk(sI \;\; -\hat{A}(s) \;\; B) = n \quad \text{for all } s \in \bar{\mathbb{C}}_0$$

and

$$rk\begin{pmatrix} sI - \hat{A}(s) \\ C \end{pmatrix} = n \quad \text{for all } s \in \bar{\mathbb{C}}_0.$$

Hence the abstract state-space realization of the retarded system is exponentially stabilizable and exponentially detectable (see, e.g., Salamon [36]), and using Example 3.3 we see that under the extra assumption $\det(CB) \neq 0$, Theorem 5.5 applies to retarded systems.

Appendix 1.

Proof of Proposition 3.1. **"Only if"** Since G^{-1} admits a decomposition of form (3.1) with $H \in (H_\gamma^\infty)^{p\times p}$ for some $\gamma < 0$ we obtain

$$\begin{aligned} s(sG(s) - D^{-1}) &= s\left(\left(D + \frac{1}{s}H(s)\right)^{-1} - D^{-1}\right) \\ &= \left(D + \frac{1}{s}H(s)\right)^{-1} s\left(I - \left(D + \frac{1}{s}H(s)\right)D^{-1}\right) \\ &= -\left(D + \frac{1}{s}H(s)\right)^{-1} H(s)D^{-1}, \end{aligned}$$

which shows that $sG(s) - D^{-1} = O(1/s)$ as $|s| \to \infty$ in $\bar{\mathbb{C}}_\beta$, where $\beta \in (\gamma, 0)$ is arbitrary. To show that G has no zeros in $\bar{\mathbb{C}}_\beta$, pick holomorphic matrices $N, D \in \mathcal{H}(\mathbb{C}_\gamma)^{p\times p}$ such that N and D are right coprime and $G = ND^{-1}$. Then, trivially, $G^{-1} = DN^{-1}$ and by the right coprimeness of D and N it follows from the analyticity of G^{-1} in $\mathbb{C}_\gamma$ that $\det(N)$ has no zeros in $\mathbb{C}_\gamma \supset \bar{\mathbb{C}}_\beta$.

"If" Setting $F(s) := (s+\gamma)G(s)$, $\gamma > |\beta|$, it follows from the assumption that

$$F(s) - D^{-1} = O(s^{-1}) \quad \text{as } |s| \to \infty \text{ in } \mathbb{C}_\beta \tag{A.1}$$

and, in particular,

$$\lim_{\substack{|s|\to\infty \\ s\in\mathbb{C}_\beta}} F(s) = D^{-1}. \tag{A.2}$$

Hence there exists $\rho > 0$ such that $F^{-1}(s)$ is bounded on $|s| > \rho$, $s \in \mathbb{C}_\beta$. Moreover, $F^{-1}(s) = (1/(s+\gamma))G^{-1}(s)$ and since G has no zeros in $\bar{\mathbb{C}}_\beta$, it follows that $F^{-1}(s)$ is bounded on $|s| \leqq \rho$, $s \in \mathbb{C}_\beta$. Therefore F^{-1} is a bounded holomorphic function on $\mathbb{C}_\beta$, i.e., $F^{-1} \in (H_\beta^\infty)^{p\times p}$. Now realize that

$$\tilde{H}(s) := (s+\gamma)(F^{-1}(s) - D) \tag{A.3}$$

$$= (s+\gamma)F^{-1}(s)(D^{-1} - F(s))D. \tag{A.4}$$

It follows from (A.3) that $\tilde{H}$ is holomorphic on $\mathbb{C}_\beta$ and, furthermore, we obtain from (A.1), (A.2), and (A.4), using the boundedness of F^{-1} on $\mathbb{C}_\beta$, that $\tilde{H}$ is bounded on $\mathbb{C}_\beta$; hence $\tilde{H} \in (H^\infty_\beta)^{p\times p}$. Finally, we obtain

$$G^{-1}(s) = (s+\gamma)F^{-1}(s) = (s+\gamma)D + \tilde{H}(s) = sD + H(s),$$

where $H(s) := \gamma D + \tilde{H}(s)$. □

Proof of Corollary 3.2. Since $A(s)$ is bounded on $\mathbb{C}_\gamma$ for some $\gamma < 0$ there exists $\rho > 0$ such that

$$G(s) = \frac{1}{s} C\left(\sum_{j=0}^{\infty}\left(\frac{1}{s}A(s)\right)^j\right)B = \frac{1}{s}CB + \frac{1}{s}C\left(\sum_{j=1}^{\infty}\left(\frac{1}{s}A(s)\right)^j\right)B$$

for all $s \in \mathbb{C}_\gamma$ such that $|s| \geqq \rho$. Hence

$$sG(s) - CB = O\left(\frac{1}{s}\right) \quad \text{as } |s| \to \infty \text{ in } \mathbb{C}_\gamma. \tag{A.5}$$

Moreover, since $\det(CB) \neq 0$, there exists an invertible matrix $Q \in \mathbb{C}^{n\times n}$ such that

$$Q^{-1}B = \begin{pmatrix} CB \\ 0 \end{pmatrix}, \qquad CQ = (I_p \;\; 0).$$

Partition the matrix $Q^{-1}A(\cdot)Q$ as follows:

$$Q^{-1}A(\cdot)Q = \begin{pmatrix} A_{11}(\cdot) & A_{12}(\cdot) \\ A_{21}(\cdot) & A_{22}(\cdot) \end{pmatrix},$$

where $A_{11}(\cdot)$, $A_{12}(\cdot)$, $A_{21}(\cdot)$, and $A_{22}(\cdot)$ are matrices with entries in H^∞_- of size $p\times p$, $p\times(n-p)$, $(n-p)\times p$, $(n-p)\times(n-p)$, respectively. As in Logemann [20], it follows that

$$\chi(s) = (-1)^p \det(CB)\det(sI - A_{22}(s)). \tag{A.6}$$

Now A_{22} is holomorphic and bounded on $\mathbb{C}_\alpha$ for some $\alpha < 0$, and therefore $\det(sI - A_{22}(s))$ has at most finitely many zeros in $\bar{\mathbb{C}}_\mu$ for any $\mu > \alpha$. Since, by assumption $\chi(s) \neq 0$ for all $s \in \bar{\mathbb{C}}_0$, we obtain, using (A.6),

$$\chi(s) \neq 0 \quad \text{for all } s \in \bar{\mathbb{C}}_\beta \tag{A.7}$$

for a negative β of sufficiently small modulus (without loss of generality, we may assume that $\gamma < \beta$). Let $G = ND^{-1}$ be a right coprime factorization over $\mathcal{H}(\mathbb{C}_\gamma)$ and use a well-known formula for the determinant of a four-block matrix to obtain

$$\begin{aligned}\chi(s) &= \det(sI - A(s))\det(G(s)) \\ &= \frac{\det(sI - A(s))}{\det(D(s))}\det(N(s)).\end{aligned} \tag{A.8}$$

It is known that $\det(D(s))$ divides $\det(sI - A(s))$ (in $\mathcal{H}(\mathbb{C}_\gamma)$) (cf. Logemann [18]), and hence, by (A.8), we have that $\det(N)$ divides χ (in $\mathcal{H}(\mathbb{C}_\gamma)$). Thus, by (A.7),

$$\det(N(s)) \neq 0 \qquad \text{for all } s \in \bar{\mathbb{C}}_\beta. \tag{A.9}$$

The claim now follows from (A.5), (A.9), and Proposition 3.1. □

Appendix 2.

Proof of Lemma 3.8. An elementary computation shows that

$$L_k - L_k^* = \{[I - J_k(I - \Gamma^{-1}D) + P_k]^{-1} - I\}L_k^*, \tag{A.10}$$

where

$$J_k(s) := \operatorname*{diag}_{1 \le j \le p} \left(\frac{s}{s+k_j} \right)$$

and

$$P_k(s) := \operatorname*{diag}_{1 \le j \le p} \left(\frac{s}{(s+k_j)(s+c_j)} \right) \left\{ \Gamma^{-1} H(s) + \operatorname*{diag}_{1 \le j \le p} (c_j)(I - \Gamma^{-1} D) \right\}.$$

Note that we can factorize P_k as $P_k = J_k Q$, where

$$Q(s) := \operatorname*{diag}_{j=1,\cdots,p} \left(\frac{1}{s+c_j} \right) \{ \Gamma^{-1} H(s) + \operatorname{diag}(c_j)(I - \Gamma^{-1} D) \}.$$

Using a result by Mossaheb [27] (cf. also Logemann [18]), we see that $\check{Q} e^{\varepsilon \cdot} \in (L^1(\mathbb{R}_+))^{p \times p}$ for all sufficiently small $\varepsilon > 0$. Moreover, we have

$$\text{(A.11)} \qquad \lim_{k \to \infty} \|\check{P}_k\|_1 = \lim_{k \to \infty} \|\check{J}_k * \check{Q}\|_1 = 0,$$

which can be derived using the equation $\check{J}_k = \delta_0 I - \operatorname{diag}_{1 \le j \le p} (k_j e^{-k_j \cdot})$ and Lemma A.1. In the case where $\Gamma = D$, we obtain from (A.10), by taking inverse Laplace transforms,

$$\text{(A.12)} \qquad \check{L}_k - \check{L}_k^* = \{ (\delta_0 I + \check{P}_k)^{-1} - \delta_0 I \} * \check{L}_k^*.$$

It follows from (A.11) that the inverse of $\delta_0 I + \check{P}_k$ exists (in the Banach algebra $\mathscr{A}^{p \times p}$) if $\min_{1 \le j \le p} (k_j)$ is sufficiently large. Hence (A.12) makes sense for large $k_j, j = 1, \cdots, p$. Part (i) of the lemma now follows from (A.12), (A.11), and the fact that

$$\text{(A.13)} \qquad \lim_{k \to \infty} \|\check{L}_k^*\|_1 = 1$$

(cf. Logemann and Owens [22] for (A.13)).

To prove part (ii), set $M_k := J_k(I - \Gamma^{-1} D)$ and realize that $\|\check{M}_k\|_{\mathscr{A}} \le \|\check{J}_k\|_{\mathscr{A}} \|I - \Gamma^{-1} D\| \le 2\varepsilon < 1$. Taking inverse Laplace transforms, using (A.11), and employing the fact that $\mathscr{A}^{p \times p}$ is a Banach algebra, it follows from (A.10) that

$$\check{L}_k - \check{L}_k^* = \{ [\delta_0 I - (\check{M}_k - \check{P}_k)]^{-1} - \delta_0 I) * \check{L}_k^* = \left(\sum_{n=1}^{\infty} (\check{M}_k - \check{P}_k)^n \right) * \check{L}_k^*$$

for all sufficiently large $k_j, j = 1, \cdots, p$. Moreover,

$$\|\check{L}_k - \check{L}_k^*\|_1 \le \left(\sum_{n=1}^{\infty} (2\varepsilon + \|\check{P}_k\|_1)^n \right) \|\check{L}_k^*\|_1 = \frac{2\varepsilon + \|\check{P}_k\|_1}{1 - (2\varepsilon + \|\check{P}_k\|_1)} \|\check{L}_k^*\|_1$$

for all sufficiently large $k_j, j = 1, \cdots, p$. We obtain, by using (A.13) and (A.11),

$$\limsup_{k \to \infty} \|\check{L}_k - \check{L}_k^*\|_1 \le \frac{2\varepsilon}{1 - 2\varepsilon},$$

which is (ii). □

LEMMA A.1. *Set* $e_k(t) := k e^{-kt} \theta(t)$, $t \ge 0$, $k \ge 0$. *Then* $\lim_{k \to \infty} \|e_k * f - f\|_1 = 0$ *for all* $f \in L^1(\mathbb{R}_+)$.

Remark A.2. Note that e_k is *not* a so-called approximate identity or Dirac sequence, because the support of e_k does not shrink to $\{0\}$ as $k \to \infty$.

Proof of Lemma A.1. In the following set $f(t) := 0$ for all $t < 0$

$$\|e_k * f - f\|_1 = \int_0^\infty \left| \int_0^t e_k(t-s)f(s)\,ds - f(t)\right| dt$$

$$(A.14) \qquad = \int_0^\infty \left| \int_0^\infty e_k(\tau)(f(t-\tau) - f(t))\,d\tau \right| dt$$

$$\leqq \int_0^\infty e_k(\tau)\left\{\int_0^\infty |f(t-\tau)-f(t)|\,dt\right\} d\tau.$$

It is well known that for a given $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_0^\infty |f(t-\tau)-f(t)|\,dt \leqq \varepsilon$ for all $\tau \in [0, \delta]$. Hence it follows from (A.14) that

$$\|e_k * f - f\|_1 \leqq \int_0^\delta e_k(\tau)\left\{\int_0^\infty |f(t-\tau)-f(t)|\,dt\right\} d\tau + 2\|f\|_1 \int_\delta^\infty e_k(\tau)\,d\tau$$

$$\leqq \varepsilon + 2\|f\|_1\, e^{-k\delta} \leqq 2\varepsilon$$

for all sufficiently large k.

Appendix 3. In this appendix we will present the most important results on stabilizability of infinite-dimensional systems. We use the same notation as in § 4.

DEFINITION A.3. System (A, B) is α-exponentially stabilizable if there exists a bounded linear operator $F \in L(X, \mathbb{R}^p)$ such that the semigroup $T_{BF}(t)$ generated by $A + BF$ satisfies $\|T_{BF}(t)\| \leqq M\,e^{\beta t}$ for some $M \geqq 1$ and $\beta < \alpha$. System (A, B) is exponentially stabilizable if it is 0-exponentially stabilizable. System (C, A) is α-exponentially detectable if there exists a bounded linear operator $K \in L(\mathbb{R}^p, X)$ such that the semigroup $T_{KC}(t)$ generated by $A + KC$ satisfies $\|T_{KC}(t)\| \leqq M\,e^{\beta t}$ for some $M \geqq 1$ and $\beta < \alpha$. System (C, A) is exponentially detectable if it is 0-exponentially detectable.

LEMMA A.4. *Suppose that the underlying space X is reflexive. Then system (A, B) is α-exponentially stabilizable if and only if (B^*, A^*) is α-exponentially detectable.*

We now have the following important theorem.

THEOREM A.5. *The following conditions are equivalent.* (i) *System (A, B) is α-exponentially stabilizable*; *and* (ii) *the state space can be decomposed in two semigroup-invariant subspaces $X = X_s \oplus X_u$, where X_s and X_u satisfy*

—$\|T(t)|_{X_s}\| \leqq M\,e^{\beta t}$; $M \geqq 1$, $\beta < \alpha$,

—$\dim(X_u) < \infty$,

—$\sigma(A|_{X_u}) = \sigma(A) \cap \bar{\mathbb{C}}_\alpha = \sigma_p(A) \cap \bar{\mathbb{C}}_\alpha$,

—*The finite-dimensional system $(A|_{X_u}, P_{X_u}B)$ is controllable, where P_{X_u} is the projection on X_u along X_s.*

For the proof, see Desch and Schappacher [6], Nefedov and Sholokhovich [29], Jacobson and Nett [14], or Curtain [5].

It is easy to show that X_u is the span of all unstable (generalized) eigenvectors of A.

The following theorem is used frequently in § 4.

THEOREM A.6. *Assume that (A, B) is α-exponentially stabilizable and let $Q \in L(X)$ be compact. Then $A + Q$ generates an α-exponentially stable semigroup if and only if $\sigma_p(A+Q) \cap \bar{\mathbb{C}}_\alpha = \varnothing$.*

Proof. It follows from Theorem A.5 that the essential exponential growth bound $\omega_e(T(\cdot))$ of $T(t)$ (cf., e.g., Nagel [28]) satisfies $\omega_e(T(\cdot)) < \alpha$. Since Q is compact, we have that $\omega_e(T_Q(\cdot)) = \omega_e(T(\cdot)) < \alpha$. Let $\omega(T_Q(\cdot))$ denote the exponential growth bound of the semigroup $T_Q(t)$. We must prove that $\omega(T_Q(\cdot)) < \alpha$. Let us assume the contrary. Then $\omega(T_Q(\cdot)) > \omega_e(T_Q(\cdot))$ and we can show, as in [28, p. 74], that there exists $\lambda \in \sigma_p(A+Q)$ satisfying $\operatorname{Re}(\lambda) = \omega(T_Q(\cdot)) \geqq \alpha$. This leads to a contradiction because $\sigma_p(A+Q) \cap \bar{\mathbb{C}}_\alpha = \varnothing$ by assumption.

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