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Some remarks on adaptive stabilization of infinite-dimensional systems

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Abstract: It is the purpose of this note to show that a first-order adaptive controller stabilizes a large class of infinite-dimensional systems described by strongly continuous semigroups. It is assumed that the plant is minimum-phase and has invertible high-frequency gain. Knowledge of the sign of the high-frequency gain is not required.

Keywords: Infinite-dimensional systems; adaptive stabilization; strongly continuous semigroups; semilinear evolution equations.

1. Introduction

Generalizing a result by Nussbaum [12] Willems and Byrnes [15] constructed a sign-switching high-gain adaptive controller which globally stabilizes any finite-dimensional single-input single-output minimum-phase system with invertible high-frequency gain. In recent years it was shown by several authors (see Dahleh [3], Dahleh and Hopkins [4], Kobayashi [7], Logemann [8] and Logemann and Owens [9]) that the adaptive algorithm presented in [15] stabilizes certain classes of *infinite-dimensional* systems as well. In [3], [4] and [8] the main result of [15] was extended to various classes of retarded systems. Generalizations to distributed parameter systems described by analytic semigroups were given in [7], while an input-output theory of high-gain adaptive stabilization of systems described by non-rational transfer functions was developed in [9].

In the following we shall consider systems of the form

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{1.1}$$

where A generates a strongly continuous semigroup $S(t)$ on a Banach space X and $B: \mathbb{R} \to X$ and $C: X \to \mathbb{R}$ are bounded linear operators. Suppose that the system (1.1) has no zeros in $\operatorname{Re}(s) \geq \alpha$ for some $\alpha < 0$ and $CB \neq 0$. Under these conditions it was shown by Kobayashi [7] that the adaptive control law given in [15] will globally stabilize the system (1.1) provided that

(i) X is a Hilbert space,

(ii) A is selfadjoint and has a complete orthonormal system of eigenvectors,

(iii) $S(t)$ is analytic,

(iv) im B and im C^* are contained in the domain of A.

In this paper we will answer the question posed in [7] whether the conditions (iii) and (iv) are really necessary for adaptive stabilization. We will show that (i)–(iv) can be relaxed considerably. In particular it will turn out that

- (i)–(iii) can be dropped,
- (iv) can be relaxed if (iii) holds.

The paper is organized as follows. Section 2 is devoted to preliminaries concerning the class of systems under consideration. Moreover it contains some technical lemmas which will be used in Section 3 in order to prove the main results of this paper. In the Appendix we prove the existence of a well-defined transfer function for a class of infinite-dimensional systems with unbounded observation operator. This result, which is needed in Section 3, might be of some independent interest.

Notation

For $\alpha \in \mathbb{R}$ define

$$\mathbb{C}_\alpha := \{ s \in \mathbb{C} \mid \operatorname{Re}(s) > \alpha \}.$$

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Let H_α^∞ denote the algebra of functions which are analytic and bounded on $\mathbb{C}_\alpha$.

Let X and Y be normed spaces. The vector space of all linear bounded operators from X to Y is denoted by $\mathscr{L}(X, Y)$.

Let A be a linear operator. Then we define $D(A) :=$ domain of A, $\sigma(A) :=$ spectrum of A and $\rho(A) :=$ resolvent set of A.

2. Preliminaries and system description

In the following we shall consider systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{2.1a}$$

$$y(t) = Cx(t), \quad t \geq 0, \tag{2.1b}$$

where A generates a strongly continuous semigroup $S(t)$ on a real Banach space X, $B \in \mathscr{L}(\mathbb{R}, X)$ and $C \in \mathscr{L}(X, \mathbb{R})$. Sometimes it will be necessary to consider the complexifications of X, A, B, and C. For simplicity these will be denoted by X, A, B, and C as well.

The notion of exponential stabilizability will play an important role in the sequel.

2.1. Definition. The system (2.1) (or the pair (A, B)) is called *exponentially stabilizable* if there exists $K \in \mathscr{L}(X, \mathbb{R})$ such that the strongly continuous semigroup generated by $A + BK$ is exponentially stable.

2.2. Lemma. *Suppose that the pair (A, B) is exponentially stabilizable and $\sigma(A) \subset \mathbb{C} \setminus \mathbb{C}_\alpha$ for some $\alpha < 0$. Then the strongly continuous semigroup $S(t)$ generated by A will be exponentially stable.*

The proof of the above lemma follows easily from Nefedov and Sholokhovich [11] or Jacobson and Nett [5] (cf. also Curtain [2]).

The following definition will make precise what we mean by a zero of the system (2.1).

2.3. Definition. A number $\lambda \in \mathbb{R}$ is called a *zero* of the system (2.1) if the kernel of the operator

$$\begin{pmatrix} \lambda I - A & B \\ C & 0 \end{pmatrix} : D(A) \oplus \mathbb{C} \to X \oplus \mathbb{C}$$

is non-trivial.

2.4. Remark. Let $\lambda \in \mathbb{C}$ be a zero of the system (2.1) and suppose that $\lambda \in \rho(A)$. Then it is easy to show that λ is a zero of the transfer function

$$G(s) = C(sI - A)^{-1}B$$

of (2.1).

Let us introduce the following assumptions

(A1) $CB \neq 0$.

(A2) The system (2.1) has no zeros in $\mathbb{C}_\alpha$ for some $\alpha < 0$.

(A3) The system (2.1) is exponentially stabilizable.

(A4) im $B \subset D(A)$.

(A5) im $C^* \subset D(A^*)$.

(A6) im $B \subset D(A^2)$.

(A7) im $C^* \subset D(A^{*2})$.

The next lemma establishes the existence of a feedback operator which shifts the spectrum of A into the left half plane.

2.5. Lemma. *Let* (A1)–(A3) *be satisfied and define*

$$F_\gamma := (CB)^{-1}(-CA + \gamma C), \tag{2.2}$$

where $\gamma < 0$. Then there exists $\alpha \in (\gamma, 0)$ such that

$$\sigma(A + BF_\gamma) \subset \mathbb{C} \setminus \mathbb{C}_\alpha.$$

Proof. By (A3) there exists $\beta < 0$ such that the spectrum of A in $\mathbb{C}_\beta$ consists of isolated eigenvalues with finite multiplicities (see Jacobson and Nett [5] or Curtain [2]). Moreover we have

$$\overline{\mathbb{C}}_0 \cap \rho(A + BF_\gamma) \neq \emptyset$$

by Appendix I. Since BF_γ is an A-degenerate operator it follows from Theorem 6.2 and Theorem 6.5 in Chapter IV of Kato's book [6] that the spectrum of $A + BF_\gamma$ in $\mathbb{C}_\beta$ consists of at most countably many eigenvalues with finite multiplicities. By (A2) there exists a number $\alpha < 0$ such that the system (2.1) has no zeros in $\mathbb{C}_\alpha$. W.l.o.g. we may assume $\max(\beta, \gamma) < \alpha$. Suppose that there exists λ in $\sigma(A + BF_\gamma) \cap \mathbb{C}_\alpha$. Then λ is an eigenvalue of $A + BF_\gamma$ and there exists $x \in X$, $x \neq 0$ such that

$$(\lambda I - A - BF_\gamma)x = 0.$$

Hence

$$(\lambda I - A + B(CB)^{-1}CA)x - \gamma B(CB)^{-1}Cx = 0. \tag{2.3}$$

Applying C to both sides of the above equation gives $(\lambda-\gamma)Cx=0$. Since $\gamma<\alpha<\mathrm{Re}(\lambda)$ it follows that $Cx=0$. We obtain using (2.3),

$$\begin{pmatrix}\lambda I-A & B\\ C & 0\end{pmatrix}\begin{pmatrix}x\\ (CB)^{-1}CAx\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Thus λ is a zero of (2.1) which is not possible by assumption. Hence we have shown

$$\sigma(A+BF_\gamma)\cap \mathbb{C}_\alpha=\emptyset.$$

2.6. Lemma. *Let* F_γ *be defined as in* (2.2). *Suppose that* (A1) *holds and that* $A+BF_\gamma$ *generates a strongly continuous semigroup* $S_\gamma(t)$. *Under these conditions we have*

$$S_\gamma(t)\ker C\subset\ker C\quad\forall t\geq 0.$$

Proof. Let $x\in\ker C$. For $s\in\rho(A+BF_\gamma)$ we have

$$\begin{aligned}x&=(sI-A-BF_\gamma)(sI-A-BF_\gamma)^{-1}x\\&=\left[sI-A-B(CB)^{-1}(-CA+\gamma C)\right]\\&\quad\cdot(sI-A-BF_\gamma)^{-1}x.\end{aligned}$$

Applying C to both sides of the above equation we obtain

$$\begin{aligned}0&=(sC-CA+CA-\gamma C)(sI-A-BF_\gamma)^{-1}x\\&=(s-\gamma)C(sI-A-BF_\gamma)^{-1}x.\end{aligned}$$

Hence we have shown for all $s\in\rho(A+BF_\gamma)$, $s\neq\gamma$, that

$$(sI-A-BF_\gamma)^{-1}\ker C\subset\ker C.$$

The claim now follows from Pazy [13], p. 121.

2.7. Remark. The feedback law F_γ was introduced by Curtain in [1], Section 8 in the context of disturbance decoupling for infinite-dimensional systems (cf. also Zwart [17]).

3. Main results

Let us recall the definition of a Nussbaum gain.

3.1. Definition. A measurable locally bounded function $N:\mathbb{R}\to\mathbb{R}$ is called a *Nussbaum gain* if for some $t_0\in\mathbb{R}$,

$$\sup_{t>t_0}\frac{1}{t-t_0}\int_{t_0}^{t}\tau N(\tau)\,\mathrm{d}\tau=+\infty$$

and

$$\inf_{t>t_0}\frac{1}{t-t_0}\int_{t_0}^{t}\tau N(\tau)\,\mathrm{d}\tau=-\infty.$$

3.2. Example. A continuously differentiable Nussbaum gain is given by

$$N(\tau)=\cos(\tfrac{1}{2}\pi\tau)\exp(\tau^2),$$

cf. Nussbaum [12] or Logemann and Owens [9].

In this section we shall apply the following control law to the system (2.1):

$$\begin{aligned}u(t)&=N(k(t))k(t)y(t),\\ \dot k(t)&=y^2(t),\quad k(0)=k_0\in\mathbb{R},\end{aligned}\tag{3.1}$$

where N is a Nussbaum gain. The control law (3.1) has been introduced by Willems and Byrnes [15] for finite-dimensional systems.

Defining

$$A_c:D(A)\times\mathbb{R}\to X\times\mathbb{R},$$
$$\begin{pmatrix}x\\k\end{pmatrix}\mapsto\begin{pmatrix}Ax\\0\end{pmatrix};$$
$$f:X\times\mathbb{R}\to X\times\mathbb{R},$$
$$\begin{pmatrix}x\\k\end{pmatrix}\mapsto\begin{pmatrix}N(k)kBCx\\(Cx)^2\end{pmatrix},$$

and

$$x_c(t):=\begin{pmatrix}x(t)\\k(t)\end{pmatrix},$$

we can write the closed-loop system as follows:

$$\dot x_c(t)=A_cx_c(t)+f(x_c(t)),\quad t\geq 0,\tag{3.2a}$$
$$x_c(0)=\begin{pmatrix}x_0\\k_0\end{pmatrix}\in X\times\mathbb{R}.\tag{3.2b}$$

A continuously differentiable $D(A_c)$-valued function which satisfies (3.2) is called a *classical solution* of (3.2). A *mild solution* of (3.2) is a continuous function satisfying

$$x_c(t)=S_c(t)x_c(0)+\int_0^t S_c(t-\tau)f(x_c(\tau))\,\mathrm{d}\tau,$$

where $S_c(t)$ denotes the strongly continuous semigroup generated by A_c.

The following lemma shows that (3.2) is well-posed.

3.3. Lemma. (i) *If N satisfies a local Lipschitz condition then for all $x_c(0) \in X \times \mathbb{R}$, (3.2) has a unique mild solution which can be continued to the right as long as it remains bounded.*

(ii) *If N is continuously differentiable then for all $x_c(0) \in D(A) \times \mathbb{R}$, (3.2) has a unique classical solution which can be continued to the right as long as it remains bounded.*

Proof. (i) It is easy to show that f satisfies a local Lipschitz condition, i.e. for any $l > 0$ there exists $L > 0$ such that

$$\| f(z) - f(z') \| \le L \| z - z' \|$$

for all $z, z' \in X \times \mathbb{R}$ satisfying $\|z\|, \|z'\| \le l$, where the norm $\|\cdot\|$ on $X \times \mathbb{R}$ is defined by $\|\cdot\| = \|\cdot\|_X + |\cdot|$. The claim follows now from Segal [14], Theorem 1 (cf. also Pazy [13], pp. 185).

(ii) It is routine to show that f is continuously Fréchet-differentiable. Moreover f satisfies a local Lipschitz condition (notice that this does not follow necessarily from the C^1-property in the infinite-dimensional case). Application of Theorem 1 and Lemma 3.1 in Segal [14] (cf. also Martin [10], pp. 347) proves the claim.

We are now in the position to state our main results.

3.4. Theorem. *Suppose that assumptions* (A1)–(A5) *are satisfied and that N is a continuously differentiable Nussbaum gain. The following statements hold true.*

(i) *For all $(x_0, k_0) \in D(A) \times \mathbb{R}$ the closed-loop system given by* (2.1) *and* (3.1) *has a unique globally defined classical solution $(x(t), k(t))$ with the following properties*:

$$\lim_{t\to\infty} k(t) \text{ exists and is finite}, \tag{3.3}$$

$$x(\cdot) \in L^2(0, \infty;\ X) \cap L^\infty(0, \infty;\ X), \tag{3.4}$$

$$\lim_{t\to\infty} x(t) = 0. \tag{3.5}$$

(ii) *For all $(x_0, k_0) \in X \times \mathbb{R}$ the closed-loop system given by* (2.1) *and* (3.1) *has a unique globally defined mild solution $(x(t), k(t))$ satisfying* (3.3)–(3.5).

Proof. (i) Define the linear bounded operator $P_1 : X \to X$ by

$$P_1 x = B(CB)^{-1}Cx.$$

Then P_1 is a projection and im P_1 = im B. Moreover set $P_2 := I - P_1$. It is obvious that im P_2 = ker C and X = im $B \oplus$ ker C. Let $(x(t), k(t))$ denote the classical solution of the feedback system given by (2.1) and (3.1) with initial value $(x_0, k_0) \in D(A) \times \mathbb{R}$ and let $[0, t_0)$ denote its maximal interval of existence. Realizing that

$$(A + BF_Y)(D(A) \cap \ker C) \subset \ker C,$$

$$\text{im } P_1 \subset D(A), \qquad P_2(D(A)) \subset D(A)$$

and

$$P_1 A P_2 x = -BF_Y P_2 x \quad \forall x \in D(A),$$

we obtain from (2.1),

$$\begin{aligned} P_1\dot{x}(t) &= P_1 A x(t) + Bu(t) \\ &= P_1 A P_1 x(t) - BF_Y P_2 x(t) + Bu(t) \end{aligned}$$

and

$$\begin{aligned} P_2\dot{x}(t) &= P_2(A + BF_Y)x(t) \\ &= (A + BF_Y)P_2 x(t) + P_2 A P_1 x(t). \end{aligned}$$

Noticing that $P_1 x(t) = B(CB)^{-1}y(t)$ and setting $z(t) := P_2 x(t)$ it follows

$$\begin{aligned} B(CB)^{-1}\dot{y}(t) &= B(CB)^{-1}CAB(CB)^{-1}y(t) \\ &\quad + B(u(t) - F_Y z(t)), \end{aligned}$$

$$\dot{z}(t) = (A + BF_Y)z(t) + P_2 AB(CB)^{-1}y(t).$$

We conclude that the initial value problem given by (2.1) and (3.1) can be written as

$$\dot{y}(t) = CBv_1(t), \quad y(0) = Cx_0, \tag{3.6}$$

$$\begin{aligned} \dot{z}(t) &= (A + BF_Y)z(t) \\ &\quad + P_2 AB(CB)^{-1}v_2(t), \quad z(0) = P_2 x_0, \end{aligned} \tag{3.7a}$$

$$w(t) = F_Y z(t) - (CB)^{-1}CAB(CB)^{-1}v_2(t), \tag{3.7b}$$

$$v_1(t) = u(t) - w(t), \quad v_2(t) = y(t), \tag{3.8}$$

$$\dot{k}(t) = y^2(t), \quad k(0) = k_0, \tag{3.9a}$$

$$u(t) = N(k(t))k(t)y(t). \tag{3.9b}$$

Hence we have shown that $(x(t), k(t))$ solves the initial value problem given by (2.1) and (3.1) (where $x_0 \in D(A)$) on $[0, t_0)$ if and only if

$$x(t) = z(t) + B(CB)^{-1}y(t), \tag{3.10}$$

where $(z(t),\ y(t),\ k(t))$ is a solution of the initial value problem defined by (3.6)–(3.9) on $[0,\ t_0)$.

We obtain from (A4) and (A5) that $P_2AB(CB)^{-1}$, F_γ and $(CB)^{-1}CAB(CB)^{-1})$ are bounded linear operators. Hence it follows in particular that $A+BF_\gamma$ generates a strongly continuous semigroup which will be denoted by $S_\gamma(t)$. Using Lemma 2.6 we obtain that $S_\gamma(t)$ is a strongly continuous semigroup on ker C. Therefore (3.7) is a well-defined semigroup system on ker C. Clearly, by (A3), the pair $(A+BF_\gamma,\ B)$ is exponentially stabilizable. Applying Lemma 2.2 and Lemma 2.5 we see that $S_\gamma(t)$ is exponentially stable. As a consequence the transfer function of (3.7) is in H_α^∞ for some $\alpha<0$. It now follows from Logemann and Owens [9] that the pair $(y(t),\ k(t))$ is bounded on $[0,\ t_0)$ which implies via (3.7) and (3.10) that $(x(t),\ k(t))$ is bounded on $[0,\ t_0)$. Using Lemma 3.3(ii) we obtain $t_0=\infty$, i.e. the closed-loop system given by (2.1) and (3.1) has a unique globally defined classical solution. Finally it follows again from Logemann and Owens [9] that (3.3)–(3.5) hold with x replaced by y, which proves the claim because of (3.10) and the exponential stability of (3.7).

(ii) It follows as in the proof of (i) that $(y(t),\ k(t))$ is bounded on $[0,\ t_0)$. Hence, by the exponential stability of (3.7) and Lemma 3.3(i) we have that the mild solution $(z(t),\ y(t),\ k(t))$ of the initial value problem (3.6)–(3.9) is globally defined. Moreover as in the proof of (i) we conclude that (3.3)–(3.5) hold true with x replaced by y. In order to prove the claim it is sufficient to show that

$$\left(z(t)+B(CB)^{-1}y(t),\ k(t)\right)$$

is the mild solution of the initial value problem given by (2.1) and (3.1). We have already shown in the proof of (i) that this is true if $x_0\in D(A)$. Therefore it remains true in the general case (i.e. $x_0\in X$), since $D(A)$ is dense in X and mild solutions depend continuously on their initial values (cf. Segal [14], Corollary 1.5).

3.5. Remark. (i) Notice that in the proof of Theorem 3.4 we have decomposed the original plant (2.1) into a feedback system consisting of an integrator in the forward loop and an (exponentially) stable system in the feedback loop (see (3.6)–(3.8)). Adaptive stabilization of systems admitting such a decomposition has been investigated by Logemann and Owens [9] using an input–output approach.

(ii) Kobayashi [7] proved a result similar to Theorem 3.4. However he had to assume that X is a Hilbert space and that A is a selfadjoint operator on X having complete orthonormal system of eigenvectors and generating an analytic semigroup. In particular Theorem 3.4 gives an affirmative answer to the question posed in [7] whether the assumption on the analyticity of the semigroup can be relaxed.

3.6. Corollary. *Suppose that assumptions* (A1)–(A3) *and* (A6) *are satisfied and that N is a continuously differentiable Nussbaum gain. Under these conditions statement* (i) *of Theorem* 3.4 *holds true.*

Proof. Let $\lambda\in\rho(A)$ and define a new state-space system $(\tilde{A},\ \tilde{B},\ \tilde{C})$ by $\tilde{A}:=A$, $\tilde{B}:=(\lambda I-A)B$ and $\tilde{C}:=C(\lambda I-A)^{-1}$. Notice that the transfer functions of $(A,\ B,\ C)$ and $(\tilde{A},\ \tilde{B},\ \tilde{C})$ are the same. It is clear that $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ satisfy (A1), (A2), (A4) and (A5). Moreover it follows from Jacobson and Nett [5] or Curtain [2] via (A3) that $(\tilde{A},\ \tilde{B})$ is exponentially stabilizable. Let $x_0\in D(A)$ and denote the mild solution of

$$\dot{x}(t)=\tilde{A}x(t)+\tilde{B}u(t),\quad x(0)=(\lambda I-A)x_0,$$
$$y(t)=\tilde{C}x(t),$$
$$u(t)=N(k(t))k(t)y(t),$$
$$\dot{k}(t)=y^2(t),\quad k(0)=k_0,$$

by $(\tilde{x}(t),\ \tilde{k}(t))$. It follows from Theorem 3.4(ii) that $(\tilde{x}(t),\ \tilde{k}(t))$ is globally defined and satisfies (3.3)–(3.5) with x and k replaced by $\tilde{x}$ and $\tilde{k}$. Finally notice that the pair $(x(t),\ k(t))$ defined by

$$x(t):=(\lambda I-A)^{-1}\tilde{x}(t)\quad\text{and}\quad k(t):=\tilde{k}(t)$$

is a classical solution of the initial value problem given by (2.1) and (3.1).

3.7. Corollary. *Suppose that the assumptions* (A1)–(A3) *and* (A7) *are satisfied and that N is a continuously differentiable Nussbaum gain. Under these conditions the statements* (i) *and* (ii) *of Theorem* 3.4 *hold true.*

Proof. Let $\lambda\in\rho(A)$ and define $\tilde{A}:=A$, $\tilde{B}:=(\lambda I-A)^{-1}B$ and $\tilde{C}:=C(\lambda I-A)$. As in the proof of

Corollary 3.6 we have that the system given by $(\tilde{A}, \tilde{B}, \tilde{C})$ satisfies (A1)–(A5). Let $x_0 \in X$ and denote the classical solution of

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t), \quad x(0) = (\lambda I - A)^{-1}x_0, \tag{3.11a}$$

$$y(t) = \tilde{C}x(t), \tag{3.11b}$$

$$u(t) = N(k(t))k(t)y(t), \tag{3.11c}$$

$$\dot{k}(t) = y^2(t), \quad k(0) = k_0, \tag{3.11d}$$

by $(\tilde{x}(t), \tilde{k}(t))$. By Theorem 3.4(i), $(\tilde{x}(t), \tilde{k}(t))$ is globally defined and satisfies (3.3)–(3.5) with x and k replaced by $\tilde{x}$ and $\tilde{k}$. Notice that the pair $(x(t), k(t))$ defined by

$$x(t) := (\lambda I - A)\tilde{x}(t) \quad \text{and} \quad k(t) := \tilde{k}(t)$$

is a mild solution of the initial value problem given by (2.1) and (3.1). It will be a solution in the classical sense if $x_0 \in D(A)$. We have

$$\tilde{x}(t) = S(t)\tilde{x}_0 + \int_0^t S(t-\tau)\tilde{B}\tilde{u}(\tau)\,\mathrm{d}\tau, \tag{3.12}$$

where $\tilde{x}_0 := (\lambda I - A)^{-1}x_0$ and

$$\tilde{u}(t) := N(\tilde{k}(t))\tilde{k}(t)C\tilde{x}(t).$$

Since the pair $(\tilde{A}, \tilde{B})$ is exponentially stabilizable, there exist closed subspaces X_s and X_u of X such that

- $X = X_s \oplus X_u$, X_u is finite-dimensional and $X_u \subset D(\tilde{A}) = D(A)$;
- the projections $P_s\colon X \to X_s$ and $P_u\colon X \to X_u$ commute with $S(t)$ and $\tilde{A} = A$;
- the strongly continuous semigroup $S_s(t) := S(t)|_{X_s}$ on X_s is exponentially stable.

(See Jacobson and Nett [5] or Curtain [2].)

Setting

$$z_s(t) := S_s(t)P_s x_0 + \int_0^t S_s(t-\tau)P_s B\tilde{u}(\tau)\,\mathrm{d}\tau$$

and

$$z_u(t) := S_u(t)P_u\tilde{x}_0 + \int_0^t S_u(t-\tau)P_u\tilde{B}\tilde{u}(\tau)\,\mathrm{d}\tau,$$

where $S_u(t) := S(t)|_{X_u}$, we obtain from (3.12),

$$\tilde{x}(t) = z_u(t) + (\lambda I - A)^{-1}z_s(t).$$

Since $\tilde{x}$, $z_s \in L^2(0, \infty;\ X) \cap L^\infty(0, \infty;\ X)$ and

$$\lim_{t\to\infty} \tilde{x}(t) = \lim_{t\to\infty} z_s(t) = 0,$$

the same is true for $z_u(t)$. Realizing that $(\lambda I - A)|_{X_u}$ is a bounded operator ($X_u \subset D(A)$ is finite-dimensional) it follows from

$$x(t) = (\lambda I - A)\tilde{x}(t) = z_s(t) + (\lambda I - A)|_{X_u} z_u(t)$$

that $x \in L^2(0, \infty;\ X) \cap L^\infty(0, \infty;\ X)$ and

$$\lim_{t\to\infty} x(t) = 0.$$

Hence the pair $(x(t), k(t))$ satisfies (3.3)–(3.5).

In Theorem 3.4 it was required that (A4) and (A5) hold. The next two results show that either (A4) or (A5) become superfluous provided that the semigroup $S(t)$ generated by A is analytic.

3.8. Theorem. *If* (A1)–(A4) *are satisfied, N is a continuously differentiable Nussbaum gain and the semigroup $S(t)$ generated by A is analytic, then statement* (i) *of Theorem* 3.4 *holds true.*

Proof. As in the proof of Theorem 3.4 we can show that the closed-loop system given by (2.1) and (3.1) is equivalent to the system (3.6)–(3.9). Since BF_γ is an A-degenerate operator it follows from Zabczyk [16] that $A + BF_\gamma$ generates an analytic semigroup $S_\gamma(t)$. Now analytic semigroups satisfy the spectrum determined growth assumption and hence $S_\gamma(t)$ is exponentially stable by Lemma 2.5. The stability result follows from [9] as in the proof of Theorem 3.4 provided that

(i) the transfer function H of (3.7) belongs to H_α^∞ for some $\alpha < 0$, and

(ii) the function $f(t) := F_\gamma S_\gamma(t)P_2x_0$ produced by the initial condition is in $L^2(0, \infty)$.

Notice that (i) and (ii) do not follow trivially because F_γ is unbounded. Define

$$R := A + BF_\gamma, \quad D := -(CB)^{-1}CAB(CB)^{-1}$$

and

$$E := P_2AB(CB)^{-1}.$$

It follows from Appendix II that the transfer function H of (3.7) is given by

$$H(s) = F_\gamma(sI - R)^{-1}E + D.$$

Using the fact that $0 \in \rho(R)$ we obtain

$$\begin{aligned} H(s) &= F_\gamma R^{-1}R(sI-R)^{-1}E + D \\ &= F_\gamma R^{-1}\big(s(sI-R)^{-1} - I\big)E + D \\ &= sF_\gamma R^{-1}(sI-R)^{-1}E - F_\gamma R^{-1}E + D. \end{aligned}$$

Now realizing that $F_\gamma R^{-1}$ is a bounded operator (by the closed-graph theorem) and using that R generates an exponentially stable analytic semigroup it follows that there exist $\beta < 0$ and $M > 0$ such that H is holomorphic on $\mathbb{C}_\beta$ and

$$\|(sI-R)^{-1}\| \le \frac{M}{|s-\beta|} \quad \text{for all } s \in \mathbb{C}_\beta.$$

Hence $H \in H_\alpha^\infty$ for all $\alpha > \beta$, which shows that (i) holds true.

In order to prove (ii), write

$$f(t) = F_\gamma R^{-1} R S_\gamma(t) P_2 x_0 = F_\gamma R^{-1} S_\gamma(t) R P_2 x_0$$

where we have used that $P_2 x_0 \in D(A)$ which is true because $x_0 \in D(A)$ and im $P_1 \subset D(A)$.

3.9. Corollary. *If* (A1)–(A3) *and* (A5) *are satisfied, N is a continuously differentiable Nussbaum gain and the semigroup generated by A is analytic then the statements* (i) *and* (ii) *of Theorem* 3.4 *hold.*

Proof. Define $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ as in the proof of Corollary 3.7 and verify that the system given by $(\tilde{A}, \tilde{B}, \tilde{C})$ fulfils (A1)–(A4). Application of Theorem 3.8 gives that for $x_0 \in X$ the solution $(\tilde{x}(t), \tilde{k}(t))$ of the initial value problem (3.11) satisfies (3.3)–(3.5) with x and k replaced by $\tilde{x}$ and $\tilde{k}$. Now proceed as in the proof of Corollary 3.7.

3.10. Remark. Notice that Theorem 3.8 and Corollary 3.9 improve the result by Kobayashi [7]. They give an affirmative answer to the question raised in [7] whether the assumption that both (A4) and (A5) are satisfied can be relaxed.

4. Appendices

Appendix I

In the proof of Lemma 2.5 we have made use of the following result:

4.1. Lemma. *If the operator F_γ is given by* (2.2) *then*

$$\bar{\mathbb{C}}_0 \cap \rho(A + BF_\gamma) \neq \emptyset.$$

Proof. Set $G_\gamma(s) := F_\gamma(sI-A)^{-1}B$. Since A generates a strongly continuous semigroup there exists $\alpha \in [0, \infty)$ such that $\mathbb{C}_\alpha \subset \rho(A)$. Hence $G_\gamma(s)$ is well defined for all $s \in \mathbb{C}_\alpha$.

Step 1: We claim that $s \in \rho(A + BF_\gamma)$ if $s \in \mathbb{C}_\alpha$ and $G_\gamma(s) \neq 1$. Notice that for $s \in \mathbb{C}_\alpha$,

$$I = (sI - A - BF_\gamma)(sI-A)^{-1} + BF_\gamma(sI-A)^{-1} \tag{4.1}$$

so that

$$B(1 - G_\gamma(s)) = (sI - A - BF_\gamma)(sI-A)^{-1}B. \tag{4.2}$$

For $s \in \mathbb{C}_\alpha$ satisfying $G_\gamma(s) \neq 1$ we obtain

$$B = (sI - A - BF_\gamma)(sI-A)^{-1}B(1 - G_\gamma(s))^{-1}. \tag{4.3}$$

Substituting (4.3) into (4.1) gives

$$I = (sI - A - BF_\gamma)(sI-A)^{-1} \cdot \left[I + B(1-G_\gamma(s))^{-1} F_\gamma(sI-A)^{-1}\right]. \tag{4.4}$$

We obtain from the definition of F_γ that the operator

$$H_\gamma(s) := (sI-A)^{-1} \cdot \left[I + B(1-G_\gamma(s))^{-1} F_\gamma(sI-A)^{-1}\right]$$

is bounded. Equation (4.4) shows that $H_\gamma(s)$ is a right inverse of $sI - A - BF_\gamma$. The claim now follows since it is not difficult to show that $H_\gamma(s)$ is a left inverse of $sI - A - BF_\gamma$ as well.

Step 2: It remains to show that there exists $\xi \in \mathbb{C}_\alpha$ satisfying $G_\gamma(\xi) \neq 1$. We will prove that

$$\lim_{\lambda \to \infty} G_\gamma(\lambda) = 0 \tag{4.5}$$

where λ is a real variable. Since A generates a strongly continuous semigroup there exist real numbers M and β such that

$$\|(\lambda I - A)^{-1}\| \le \frac{M}{\lambda - \beta} \quad \text{for all } \lambda > \beta. \tag{4.6}$$

In order to prove that (4.5) holds true it is sufficient to show

$$\lim_{\lambda \to \infty} \|A(\lambda I - A)^{-1}B\| = 0. \tag{4.7}$$

Notice that

$$\|A(\lambda I-A)^{-1}\| = \|\lambda(\lambda I-A)^{-1}-I\| \le 1+\frac{M\lambda}{\lambda-\beta} \quad \forall\lambda>\min(0,\beta).$$

Thus there exists $\lambda_0>\max(0,\beta)$ such that

$$\|A(\lambda I-A)^{-1}\|\le 1+2M \quad \forall\lambda>\lambda_0. \tag{4.8}$$

Let $\varepsilon>0$ be given. Set $x:=B(1)$ and choose $z\in D(A)$ satisfying

$$\|x-z\|\le\frac{\varepsilon}{2(1+2M)}. \tag{4.9}$$

Moreover let $\lambda_1\ge\lambda_0$ be such that

$$\frac{M}{\lambda-\beta}\|Az\|\le\tfrac{1}{2}\varepsilon \quad \forall\lambda>\lambda_1. \tag{4.10}$$

Then it follows from (4.6) and (4.8)–(4.10),

$$\begin{aligned}\|A(\lambda I-A)^{-1}B\| &= \|A(\lambda I-A)^{-1}x\| \\ &\le \|A(\lambda I-A)^{-1}\|\,\|x-z\| \\ &\quad+\|(\lambda I-A)^{-1}\|\,\|Az\| \\ &\le\varepsilon \quad \forall\lambda>\lambda_1,\end{aligned}$$

which proves (4.7).

Appendix II

Consider the system

$$\dot{x}(t)=Ax(t)+Bu(t),\quad x(0)=x_0, \tag{4.11a}$$

$$y(t)=Cx(t),\quad t\ge 0, \tag{4.11b}$$

where A generates a strongly continuous semigroup $S(t)$ on a Banach space X, $B\in\mathscr{L}(\mathbb{R},X)$ and $C: D(C)\to\mathbb{R}$ is an A-bounded linear operator. If $x_0\in D(A)$ and $u\in C^1(0,\infty;\mathbb{R})$ there exists a unique classical solution $x(t)\in D(A)$ ($\forall t\ge 0$) and hence the output y is well defined.

In the following let λ denote the exponential growth constant of $S(t)$. As usual the Laplace transformation is denoted by the superscript $\hat{}$.

4.2. Proposition. *Suppose* $x_0=0$ *and let* $u\in C^1(0,\infty;\mathbb{R})$ *be Laplace transformable such that* $\hat{u}(s)$ *exists on* $\mathbb{C}_\alpha$ *for some* $\alpha\in\mathbb{R}$. *Then the Laplace transform* $\hat{y}(s)$ *of the output of* (4.11) *exists for all* $s\in\mathbb{C}$ *satisfying* $\mathrm{Re}(s)>\max(\alpha,\lambda)$ *and is given by*

$$\hat{y}(s)=C(sI-A)^{-1}B\hat{u}(s).$$

Moreover the expression $C(sI-A)^{-1}B$ *is analytic in* $\mathbb{C}_\lambda$.

4.3. Remark. The above proposition says that there exists a transfer function for the system (4.11) and that it is given by $C(sI-A)^{-1}B$. This seems like a trivial fact. However, since C is unbounded, we have to *prove* that C can be taken out of the Laplace integral.

Proof of Proposition 4.2. W.l.o.g. we may assume that $\lambda<0$ and hence $A^{-1}\in\mathscr{L}(X,X)$. It is well known from semigroup theory that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(A^{-1}T(\tau)B\right)\big|_{\tau=t}=T(t)B. \tag{4.12}$$

Using (4.12), the variation-of-constants formula and partial integration we obtain

$$\begin{aligned}x(t)&=-\int_0^t \mathrm{d}_\tau\left(A^{-1}T(t-\tau)B\right)u(\tau)\,\mathrm{d}\tau \\ &=\int_0^t A^{-1}T(t-\tau)Bu'(\tau)\,\mathrm{d}\tau \\ &\quad -A^{-1}Bu(t)+A^{-1}T(t)Bu(0).\end{aligned}$$

Applying C to both sides of the equation, using the fact that CA^{-1} is bounded and taking Laplace transforms gives

$$\begin{aligned}\hat{y}(s)&=CA^{-1}(sI-A)^{-1}B(s\hat{u}(s)-u(0)) \\ &\quad -CA^{-1}B\hat{u}(s)+CA^{-1}(sI-A)^{-1}Bu(0) \\ &=CA^{-1}\left\{s(sI-A)^{-1}-I\right\}B\hat{u}(s) \\ &=CA^{-1}\left\{A(sI-A)^{-1}\right\}B\hat{u}(s) \\ &=C(sI-A)^{-1}B\hat{u}(s).\end{aligned}$$

It is clear that the above equations hold for all $s\in\mathbb{C}$ satisfying $\mathrm{Re}(s)>\max(\alpha,\lambda)$. Moreover it follows from the identity

$$C(sI-A)^{-1}B=CA^{-1}\left\{s(sI-A)^{-1}-I\right\}B$$

that $C(sI-A)^{-1}B$ is analytic in $\mathbb{C}_\lambda$.

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