Discrete-Time Low-Gain Control of Uncertain Infinite-Dimensional Systems

Hartmut Logemann and Stuart Townley

***Abstract*—Using a frequency-domain analysis, it is shown that the application of a feedback controller of the form $k/(z-1)$ or $kz/(z-1)$, where $k \in \mathbb{R}$, to a power-stable infinite-dimensional discrete-time system with square transfer-function matrix $\mathbf{G}(z)$ will result in a power-stable closed-loop system which achieves asymptotic tracking of arbitrary constant reference signals, provided that i) all the eigenvalues of $\mathbf{G}(1)$ have positive real parts, and ii) the gain parameter k is positive and sufficiently small. Moreover, if $\mathbf{G}(1)$ is positive definite, we show how the gain parameter gain k can be tuned adaptively. The resulting adaptive tracking controllers are universal in the sense that they apply to any power-stable system with $\mathbf{G}(1) \succ 0$; in particular, they are not based on system identification or plant parameter estimation algorithms, nor is the injection of probing signals required. Finally, we apply these discrete-time results to obtain adaptive sampled-data low-gain controllers for the class of regular systems, a rather general class of infinite-dimensional continuous-time systems for which convenient representations are known to exist, both in state space and in frequency domain. We emphasize that our results guarantee not only asymptotic tracking at the sampling instants but also in the sampling interval.**

***Index Terms*— Adaptive tracking, discrete-time systems, frequency-domain methods, infinite-dimensional systems, sampled-data control, state-space methods.**

I. Introduction

THE synthesis of low-gain I and PI-controllers for uncertain stable continuous-time plants has received considerable attention in the last 20 years. Let $\mathbf{G}_c$ be a stable proper rational continuous-time transfer function matrix. The main existence result on robust low-gain I-control states that if

$$\text{spectrum}\,[\mathbf{G}_c(0)] \subset \{s \in \mathbb{C} \,|\, \text{Re}\; s > 0\}$$

then there exists $k^* > 0$ such that for all $k \in (0, k^*)$ the controller kI/s stabilizes $\mathbf{G}_c$, and the resulting closed-loop system asymptotically tracks arbitrary constant reference signals. This result has been proved by Davison [5] and Lunze [20] using state-space methods and by Grosdidier *et al.* [6] and Morari [26] using frequency-domain methods (see also the book by Lunze [22, ch. 10] and the textbook by Morari and Zafiriou [27, p. 362]). This controller design approach, called "tuning regulator theory" [5], has been successfully applied to industrial control problems; see Coppus *et al.* [3] and Lunze [21].

The above tuning regulator result has been extended by Pohjolainen [28], [29], Pohjolainen and Lätti [30], Logemann *et al.* [14], Logemann and Owens [18], and Logemann and Townley [19] to various classes of (abstract) infinite-dimensional continuous-time systems and by Koivo and Pohjolainen [12] and Jussila and Koivo [9] to differential-delay systems.

If the plant uncertainty is large, the parameter k needs to be tuned adaptively. For continuous-time plants low-gain universal adaptive controllers which achieve asymptotic tracking of constant reference signals have been presented by Cook [2] and by Miller and Davison [24], [25] in the finite-dimensional case and by Logemann and Townley [19] in the infinite-dimensional case.[1] By "universal" we mean that the controllers are not based on system identification or plant parameter estimation algorithms.

In this paper, we consider the problem of low-gain I-control for the class of discrete-time power-stable infinite-dimensional systems. We apply our results to the sampled-data control of continuous-time regular systems, a large class of infinite-dimensional systems introduced and studied by Weiss [40], [41]. Regular systems encompass a large class of partial differential equations with boundary control and observation and functional differential equations with delays in the state, input, and output variables. The low-gain control problem for discrete-time systems appears to have received less attention than its continuous-time counterpart. Kobayashi [10], [11] has obtained nonadaptive sampled-data versions of the existence result above for various classes of infinite-dimensional systems. However, to the best of our knowledge there are no results on discrete-time adaptive low-gain control available in the literature.

In Section II, we develop a frequency-domain approach to nonadaptive low-gain discrete-time control which is more general than the state-space approach in [10] and [11]. We show that if $\mathbf{G}$ is any given input–output stable discrete-time transfer function matrix such that all eigenvalues of the steady-state gain $\mathbf{G}(1)$ have positive real parts, then the integrators $k/(z-1)$ and $kz/(z-1)$ achieve closed-loop stability and asymptotic tracking of constant reference signals, provided the gain parameter k is positive and small enough.

[1] Surprisingly, the low-gain adaptive tracking problem has received less attention than its high-gain counterpart; see Ilchmann [7], Logemann and Ilchmann [15], Ryan [34], and the references therein.

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It is natural to tune the scalar gain k adaptively, and therefore Section III is devoted to universal adaptive low-gain control of discrete-time infinite-dimensional systems. Whilst universal adaptive continuous-time control of infinite-dimensional systems has been developed quite extensively (see, e.g., [15], [16], [19], and [37]), surprisingly, to the best of our knowledge the only result on universal adaptive discrete-time control of infinite-dimensional systems is contained in the note by Logemann and Mårtensson [17] which extends an earlier finite-dimensional result by Mårtensson [23]. One possibility for adaptive tuning of k is to use discrete-time versions of Mårtensson's switching controllers, see [23] and [17], to perform a "dense" search for the gain in an interval of the form $(0, \delta)$, $\delta > 0$. However, we do not pursue this approach because it is not clear how this type of algorithm would exploit the low-gain features of the problem. Instead we restrict our attention to the case when $\mathbf{G}(1)$ is positive or negative definite, where Proposition 2.8 will prove to be extremely useful in constructing adaptive controllers which do, indeed, exploit the low-gain features of the problem. A basic idea in Section III is to set the integrator gain k equal to γ_n^{-p}, where $0 < p < 1/2$, and to adjust γ_n by a suitable adaptation law.

In Section IV, we develop an approach to adaptive low-gain sampled-data control for regular systems. We emphasize that our results guarantee not only asymptotic tracking at the sampling instants but also in the sampling interval. In Section V, we illustrate our results by some examples and simulations. Finally, we draw some conclusions in Section VI.

Notation: For $\alpha > 0$ and $\omega \in \mathbb{R}$ define $\mathbb{E}_\alpha := \{z \in \mathbb{C} \,|\, |z| > \alpha\}$ and $\mathbb{C}_\omega := \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > \omega\}$. Moreover, set

$$\mathcal{M}(\mathbb{E}_\alpha, \mathbb{C}^{m\times m}) := \{f : \mathbb{E}_\alpha \to \mathbb{C}^{m\times m} | f \text{ is meromorphic}\},$$
$$H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{m\times m}) := \{f : \mathbb{E}_\alpha \to \mathbb{C}^{m\times m} | f \text{ is holomorphic and bounded}\},$$
$$H^\infty_<(\mathbb{E}_\alpha, \mathbb{C}^{m\times m}) := \bigcup_{0<\beta<\alpha} H^\infty(\mathbb{E}_\beta, \mathbb{C}^{m\times m}).$$

If $m = 1$, then we write $\mathcal{M}(\mathbb{E}_\alpha) := \mathcal{M}(\mathbb{E}_\alpha, \mathbb{C}^{m\times m})$, $H^\infty(\mathbb{E}_\alpha) := H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{m\times m})$, and $H^\infty_<(\mathbb{E}_\alpha) := H^\infty_<(\mathbb{E}_\alpha, \mathbb{C}^{m\times m})$.

Let V and W be Hilbert spaces. The set of all linear bounded operators from V to W is denoted by $\mathcal{B}(V, W)$. Moreover, if S is a linear operator from V into V, we set

$$D(S) := \text{ domain of } S$$
$$\sigma(S) := \text{ spectrum of } S$$
$$\varrho(S) := \text{ resolvent set of } S.$$

Finally, for $p > 0$ we define

$$l^p_+(V) := \{(v_0, v_1, v_2, \cdots) \,|\, v_i \in V, \sum_{i=0}^\infty \|v_i\|^p < \infty\}.$$

If $V = \mathbb{C}$ or $V = \mathbb{R}$, then we write $l^p_+ := l^p_+(V)$. As usual, $L^p_{loc}(0, \infty; V)$ denotes the space of all locally p-integrable functions with values in V.

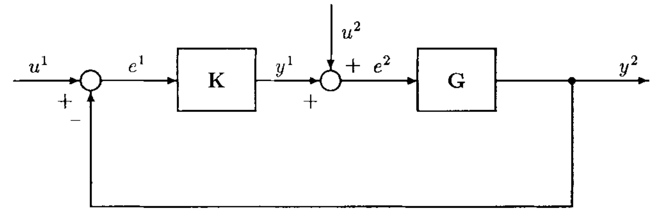

Fig. 1. Closed-loop system $\mathcal{F}(\mathbf{G}, \mathbf{K})$.

II. Nonadaptive Low-Gain Control of Discrete-Time Systems

A function $\mathbf{G}$ is called a (discrete-time) *transfer-function* or a (discrete-time) *transfer-function matrix* if $\mathbf{G} \in \mathcal{M}(\mathbb{E}_\alpha, \mathbb{C}^{m\times m})$ for some $\alpha > 0$. For any $m \times m$ transfer-function matrices $\mathbf{G}$ and $\mathbf{K}$, the feedback system shown in Fig. 1 will be denoted by $\mathcal{F}(\mathbf{G}, \mathbf{K})$. We shall call the feedback system $\mathcal{F}(\mathbf{G}, \mathbf{K})$ input–output stable if every transfer function $u^i \mapsto y^j$ that occurs around the loop has all its entries in $H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times m})$. More precisely, we give the following definition.

Definition 2.1: Let $\mathbf{G} \in \mathcal{M}(\mathbb{E}_\alpha, \mathbb{C}^{m\times m})$ and $\mathbf{K} \in \mathcal{M}(\mathbb{E}_\alpha, \mathbb{C}^{m\times m})$ for some $\alpha > 0$. The feedback system $\mathcal{F}(\mathbf{G}, \mathbf{K})$ is called *input–output stable* if $\det[I + \mathbf{G}(z)\,\mathbf{K}(z)] \not\equiv 0$ and

$$F(\mathbf{G}, \mathbf{K}) := \begin{bmatrix} \mathbf{K}(I+\mathbf{GK})^{-1} & -\mathbf{KG}(I+\mathbf{KG})^{-1} \\ \mathbf{GK}(I+\mathbf{GK})^{-1} & \mathbf{G}(I+\mathbf{KG})^{-1} \end{bmatrix} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{2m\times 2m}).$$

We say that $\mathbf{K}$ *stabilizes* $\mathbf{G}$ if $\mathcal{F}(\mathbf{G}, \mathbf{K})$ is input–output stable.

Notice that the above concept of input–output stability is stronger than l^2-stability, which is equivalent to $F(\mathbf{G}, \mathbf{K}) \in H^\infty(\mathbb{E}_1, \mathbb{C}^{2m\times 2m})$. However, Definition 2.1 has the advantage that it guarantees the analyticity of the closed-loop transfer function on $\mathbb{E}_\alpha$ for some $\alpha \in (0, 1)$, a property which will be needed in the following.

Remark 2.2:

i) It is trivial that $\mathbf{K}$ stabilizes $\mathbf{G}$ if and only if $\mathbf{G}$ stabilizes $\mathbf{K}$.
ii) Let $\mathcal{Q}$ denote the quotient field of $H^\infty_<(\mathbb{E}_1)$, i.e., $\mathcal{Q} = \{n/d \,|\, n, d \in H^\infty_<(\mathbb{E}_1), d(z) \not\equiv 0\}$. If $\mathcal{F}(\mathbf{G}, \mathbf{K})$ is input–output stable, then $\mathbf{G} \in \mathcal{Q}^{m\times m}$ and $\mathbf{K} \in \mathcal{Q}^{m\times m}$.
iii) $\mathbf{G} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times m})$, then $\mathcal{F}(\mathbf{G}, \mathbf{K})$ is input–output stable if and only if $\det[I + \mathbf{G}(z)\mathbf{K}(z)] \not\equiv 0$ and $\mathbf{K}(I + \mathbf{GK})^{-1}$ is in $H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times m})$.
iv) A *left coprime factorization* of $\mathbf{G}$ over $H^\infty_<(\mathbb{E}_1)$ is a pair $(\mathbf{D}, \mathbf{N}) \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times m}) \times H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times m})$ such that $\det \mathbf{D} \not\equiv 0$, $\mathbf{G} = \mathbf{D}^{-1}\mathbf{N}$, and there exist $\mathbf{X}, \mathbf{Y} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times m})$ satisfying $\mathbf{DX} + \mathbf{NY} = I$. *Right coprime factorizations* over $H^\infty_<(\mathbb{E}_1)$ are defined in an analogous way. It follows from Smith [36] that $\mathbf{G}$ and $\mathbf{K}$ admit left and right coprime factorizations over $H^\infty_<(\mathbb{E}_1)$ if $\mathcal{F}(\mathbf{G}, \mathbf{K})$ is input–output stable.

An application of a standard result in fractional representation theory (c.f., Vidyasagar *et al.* [38]) yields the following necessary and sufficient algebraic condition for closed-loop stability in the terms of coprime factors.

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Proposition 2.3: Let $\mathbf{G} \in \mathcal{M}(\mathbb{E}_\alpha, \mathbb{C}^{m\times m})$ and $\mathbf{K} \in \mathcal{M}(\mathbb{E}_\alpha, \mathbb{C}^{m\times m})$ for some $\alpha > 0$. Suppose that there exist a right-coprime factorization $(\mathbf{N_G}, \mathbf{D_G})$ of $\mathbf{G}$ and a left-coprime factorization $(\mathbf{D_K}, \mathbf{N_K})$ of $\mathbf{K}$ [both over $H^\infty_<(\mathbb{E}_1)$]. Then the feedback system $\mathcal{F}(\mathbf{G}, \mathbf{K})$ is input–output stable if and only if the matrix $\mathbf{N_K N_G} + \mathbf{D_K D_G}$ is unimodular over $H^\infty_<(\mathbb{E}_1)$, i.e., if and only if

$$\inf_{z\in\mathbb{E}_1} |\det[\mathbf{N_K}(z)\mathbf{N_G}(z) + \mathbf{D_K}(z)\mathbf{D_G}(z)]| > 0.$$

In the following, let $\mathcal{Z}$ denote the z-transform and let $\star$ denote the convolution of two sequences. Moreover, we set

$$\theta_n := \begin{cases} 1 & \text{if } n = 0, 1, 2, \cdots \\ 0 & \text{if } n = -1, -2, \cdots. \end{cases}$$

For completeness we state and prove the final-value theorem for transfer functions in $H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times m})$.

Lemma 2.4: If $\mathbf{G} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times m})$ and $v \in \mathbb{C}^m$, then $\lim_{j\to\infty}[\mathcal{Z}^{-1}(\mathbf{G}) \star \{v\theta_n\}]_j = \mathbf{G}(1)v$.

Proof: Since there exists $\alpha \in (0, 1)$ such that $\mathbf{G}$ is holomorphic and bounded on $\mathbb{E}_\alpha$, it follows that $\mathbf{G}$ can be written as

$$\mathbf{G}(z) = \sum_{i=0}^{\infty} G_i z^{-i}, \quad \text{for all } z \in \mathbb{E}_\alpha$$

where $\{G_i\} \in l^1_+(\mathbb{C}^{m\times m})$. Now

$$\left[\mathcal{Z}^{-1}(\mathbf{G}) \star \{v\theta_n\}\right]_j = \sum_{i=0}^{j} G_{j-i} v\theta_i = \sum_{i=0}^{j} G_i v$$

and therefore

$$\lim_{j\to\infty} \left[\mathcal{Z}^{-1}(\mathbf{G}) \star \{v\theta_n\}\right]_j = \sum_{i=0}^{\infty} G_i v = \mathbf{G}(1)v. \qquad \square$$

We are now in the position to formulate the main result of this section. It forms the discrete-time counterpart of the continuous-time results due to Pohjolainen [28], [29], Logemann and Owens [18], and Logemann and Townley [19], and it shows that low-gain integrators achieve stability and asymptotic tracking of constant reference signals for large classes of input–output stable plants.

Theorem 2.5: Suppose that $\mathbf{G} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times m})$ and set $\mathbf{K}_k(z) := kI/(z-1)$, where $k \in \mathbb{R}$. If

$$\sigma[\mathbf{G}(1)] \subset \mathbb{C}_0 \tag{1}$$

then there exists $k^* > 0$ such that for all $k \in (0, k^*)$ the feedback system $\mathcal{F}(\mathbf{G}, \mathbf{K}_k)$ is input–output stable and moreover

$$[\mathbf{G}\mathbf{K}_k(I + \mathbf{G}\mathbf{K}_k)^{-1}](1) = I. \tag{2}$$

Lemma 2.4 and (2) imply that the closed-loop system asymptotically tracks reference signals of the form $\{r\theta_n\}$, where $r \in \mathbb{R}^m$.

Proof of Theorem 2.5: Setting $\mathbf{N}_k := kI/z$ and $\mathbf{D}(z) := [(z-1)/z]I$, it is clear that $(\mathbf{D}, \mathbf{N}_k)$ is a left-coprime factorization of $\mathbf{K}_k$. By Proposition 2.3, we only need to show that there exists $k^* > 0$ such that

$$\inf_{z\in\mathbb{E}_1} \left|\det\left[\frac{k}{z}\mathbf{G}(z) + \frac{z-1}{z}I\right]\right| > 0 \quad \text{for all } k \in (0, k^*). \tag{3}$$

Seeking a contradiction, suppose that there is no $k^* > 0$ such that (3) is true. Then there exists a sequence $k_n \downarrow 0$ such that for all $n \in \mathbb{N}$

$$\inf_{z\in\mathbb{E}_1} \left|\det\left[\frac{k_n}{z}\mathbf{G}(z) + \frac{z-1}{z}I\right]\right| = 0. \tag{4}$$

Since $\lim_{n\to\infty} k_n = 0$ and $\mathbf{G}$ is bounded on $\mathbb{E}_1$, it follows from (4) that there exist numbers $z_n \in \mathbb{C}$ with $|z_n| \geq 1$ and such that for all $n \in \mathbb{N}$

$$\det[k_n\mathbf{G}(z_n) + (z_n - 1)I] = 0. \tag{5}$$

Since $\lim_{n\to\infty} k_n\mathbf{G}(z_n) = 0$ we may conclude that

$$\lim_{n\to\infty} z_n = 1. \tag{6}$$

Moreover, we obtain from (5) that

$$\frac{1-z_n}{k_n} \in \sigma[\mathbf{G}(z_n)]. \tag{7}$$

As an immediate consequence of (7) we have

$$\left|\frac{1-z_n}{k_n}\right| \leq \|\mathbf{G}\|_\infty, \qquad \text{for all } n \in \mathbb{N}. \tag{8}$$

By (1) it is clear that there exists $\beta > 0$ such that $\sigma[\mathbf{G}(1)] \subset \mathbb{C}_\beta$. Hence, it follows from (6) that there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\sigma[\mathbf{G}(z_n)] \subset \mathbb{C}_\beta$. Combining this with (7) shows that for all $n \geq N$

$$\frac{1-z_n}{k_n} \in \mathbb{C}_\beta. \tag{9}$$

In particular, we have for all $n \geq N$

$$\operatorname{Re} z_n < 1. \tag{10}$$

Setting $z_n' := 1 + \imath \operatorname{Im} z_n$, and making use of (10) and the fact that $|z_n| \geq 1$, we obtain for all $n \geq N$

$$\frac{|z_n' - z_n|}{|1 - z_n|} = \frac{1 - \operatorname{Re} z_n}{\sqrt{2(1 - \operatorname{Re} z_n) + |z_n|^2 - 1}} \leq \frac{1 - \operatorname{Re} z_n}{\sqrt{2(1 - \operatorname{Re} z_n)}} = \frac{1}{\sqrt{2}}\sqrt{1 - \operatorname{Re} z_n}.$$

Since, by (6), $\lim_{n\to\infty} \operatorname{Re} z_n = 1$, we conclude that

$$\lim_{n\to\infty} \frac{|z_n' - z_n|}{|1 - z_n|} = 0. \tag{11}$$

Now

$$\frac{1 - z_n'}{k_n} = \frac{1 - z_n}{k_n} + \frac{z_n - z_n'}{1 - z_n}\frac{1 - z_n}{k_n}$$

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and therefore by (8), (9), and (11)

$$\liminf_{n\to\infty} \operatorname{Re} \frac{1-z_n'}{k_n} \geq \beta > 0.$$

This yields a contradiction, since by construction

$$\operatorname{Re} \frac{1-z_n'}{k_n} = 0, \qquad \text{for all } n \in \mathbb{N}.$$

Hence there exists $k^* > 0$ such that $\mathcal{F}(\mathbf{G}, \mathbf{K}_k)$ is input–output stable for all $k \in (0, k^*)$. In particular, this means that $\mathbf{GK}_k(I + \mathbf{GK}_k)^{-1} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times m})$ for all $k \in (0, k^*)$. Therefore we obtain, using the invertibility of $\mathbf{G}(1)$

$$\begin{aligned}[\mathbf{GK}_k(I+\mathbf{GK}_k)^{-1}](1) &= \lim_{z\to 1} k\mathbf{G}(z)[(z-1)I + k\mathbf{G}(z)]^{-1} \\ &= I\end{aligned}$$

which establishes (2). □

In the following we apply Theorem 2.5 to state-space systems. To this end consider a discrete-time system

$$x_{n+1} = Ax_n + Bu_n \tag{12a}$$
$$y_n = Cx_n + Du_n \tag{12b}$$

evolving on a real Hilbert space X. Here $A \in \mathcal{B}(X, X)$, $B \in \mathcal{B}(\mathbb{R}^m, X)$, $C \in \mathcal{B}(X, \mathbb{R}^m)$, and $D \in \mathcal{B}(\mathbb{R}^m, \mathbb{R}^m)$. A system of the form (12) is called *power stable* if A is power stable, i.e., there exist $M > 0$ and $\rho \in (0, 1)$ such that

$$\|A^n\| \leq M\rho^n, \qquad \text{for all } n \in \mathbb{N}.$$

The transfer function $\mathbf{G}$ of (12) is given by

$$\mathbf{G}(z) = C(zI - A)^{-1}B + D.$$

Clearly, if (12) is power stable, then $\mathbf{G} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times m})$.

Let $r \in \mathbb{R}^m$ and consider the control law given by

$$u_{n+1} = u_n + k(r\theta_n - y_n), \qquad \text{where } k \in \mathbb{R}. \tag{13}$$

Setting

$$\tilde{A} := \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad \tilde{C} := (C, D) \tag{14}$$

the closed-loop system can be written as

$$(x_{n+1}, u_{n+1}) = \tilde{A}_k(x_n, u_n) + k\tilde{B}r\theta_n$$

where

$$\tilde{A}_k := \tilde{A} - k\tilde{B}\tilde{C}. \tag{15}$$

Corollary 2.6: Suppose that (12) is power stable and $\sigma[\mathbf{G}(1)] \subset \mathbb{C}_0$. Then there exists $k^* > 0$ such that for all $k \in (0, k^*)$ the closed-loop operator $\tilde{A}_k$ is power stable and moreover $\lim_{n\to\infty} y_n = r$ for all $(x_0, u_0) \in X \times \mathbb{R}^m$.

The proof follows from a combination of Theorem 2.5 and a result by Logemann [13] on the equivalence of input–output and power stability (see [13, Th. 2]).

Remark 2.7: Condition (1) is crucial both in Theorem 2.5 and Corollary 2.6. Notice that in principle it can be checked by performing step-response experiments on the plant. If (1) is replaced by the weaker assumption $\det \mathbf{G}(1) \neq 0$, and if a matrix Γ is known such that $\sigma[\mathbf{G}(1)\Gamma] \subset \mathbb{C}_0$, then Theorem 2.5 and Corollary 2.6 remain true, provided the integrator gain k is replaced by $k\Gamma$.

For a matrix $M \in \mathbb{C}^{m\times m}$, we write in the following $M \succ 0$ if M is positive definite, i.e., $\langle \xi, M\xi \rangle > 0$ for all $\xi \in \mathbb{C}^m$, $\xi \neq 0$. We write $M \succeq 0$ if M is positive semidefinite, i.e., $\langle \xi, M\xi \rangle \geq 0$ for all $\xi \in \mathbb{C}^m$. Moreover, we write $M \prec 0$ if M is negative definite, and $M \preceq 0$ if M is negative semidefinite. For a complex matrix M let M^H denote the conjugate transpose of M. Recall that if M is positive or negative semidefinite, then $M = M^H$.

The next result will be an important tool in Section III, although it is interesting in its own right. For $\mathbf{G} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times m})$ we define $\tilde{\mathbf{G}}(z) := [1/(z-1)]\mathbf{G}(z)$ and

$$\tilde{\mathbf{G}}_k(z) := \tilde{\mathbf{G}}(z)\,[I + k\tilde{\mathbf{G}}(z)]^{-1}. \tag{16}$$

Proposition 2.8: Let $\mathbf{G} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times m})$ and suppose that $\det \mathbf{G}(1) \neq 0$. Then there exists $k^* > 0$ such that for all $k \in (0, k^*)$

$$\|\tilde{\mathbf{G}}_k\|_\infty = \frac{1}{k} \tag{17}$$

if and only if $\mathbf{G}(1) \succ 0$.

The H^∞-norm in (17) is defined to be the supremum over $\mathbb{E}_1$ of $\sigma_{\max}[\tilde{\mathbf{G}}_k(z)]$, the largest singular value of $\tilde{\mathbf{G}}_k(z)$.

Proposition 2.8 is an immediate consequence of the following Lemma.

Lemma 2.9: If $\mathbf{G} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times m})$, then the following statements hold.

i) Suppose that $\det \mathbf{G}(1) \neq 0$ and $k > 0$. Then (17) is true if and only if $I + k\tilde{\mathbf{G}}(z) + k\tilde{\mathbf{G}}^H(z) \succeq 0$ for all $z \in \mathbb{E}_1$.
ii) There exists $k^* > 0$ such that $I + k\tilde{\mathbf{G}}(z) + k\tilde{\mathbf{G}}^H(z) \succeq 0$ for all $z \in \mathbb{E}_1$ and for all $k \in (0, k^*)$ if and only if $\mathbf{G}(1) \succeq 0$.

Note that if $\mathbf{G}(z) \in \mathbb{R}^{m\times m}$ for real z, then $I + k\tilde{\mathbf{G}}(z) + k\tilde{\mathbf{G}}^H(z) \succeq 0$ for all $z \in \mathbb{E}_1$ if and only if $(1/2)I + k\tilde{\mathbf{G}}(z)$ is positive real.

Proof of Lemma 2.9:

i) The continuous-time argument given in [19] (see the proof of Lemma 3.10) applies to the discrete-time case as well and will not be repeated here.
ii) Since $\mathbf{G}$ is holomorphic at $z = 1$, we can write

$$\mathbf{G}(z) = \mathbf{G}(1) + \sum_{i=1}^{\infty} G_i(z-1)^i \tag{18}$$

where $G_i \in \mathbb{C}^{m\times m}$ and the power series in (18) converges and is bounded in some disc Δ_ε centered at one and with radius $\varepsilon > 0$. Consequently

$$\begin{aligned} I &+ k\tilde{\mathbf{G}}(z) + k\tilde{\mathbf{G}}^H(z) \\ &= I + \frac{k}{z-1}\mathbf{G}(1) + \frac{k}{\bar{z}-1}\mathbf{G}^H(1) \\ &\quad + k\mathbf{H}(z), \quad \text{for all } z \in \Delta_\varepsilon \end{aligned} \tag{19}$$

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where

$$\mathbf{H}(z) := \sum_{i=1}^{\infty} G_i(z-1)^{i-1} + \sum_{i=1}^{\infty} G_i^H(\bar{z}-1)^{i-1}.$$

Moreover, since $\tilde{\mathbf{G}}(z)$ is bounded on $\mathbb{E}_1 \setminus \Delta_\varepsilon$, there exists $k_1 > 0$ such that

$$I + k\tilde{\mathbf{G}}(z) + k\tilde{\mathbf{G}}^H(z) \succeq 0, \quad \text{for all } z \in \mathbb{E}_1 \setminus \Delta_\varepsilon,\ k \in (0,\, k_1). \tag{20}$$

Suppose first that $\mathbf{G}(1) \succeq 0$. Then, using (19) and (20) it remains to show that

$$I + \frac{2k(\mathrm{Re}\, z - 1)}{|z-1|^2}\,\mathbf{G}(1) + k\mathbf{H}(z) \succeq 0, \quad \text{for all } z \in \mathbb{E}_1 \cap \Delta_\varepsilon,\ k \in (0,\, k_2) \tag{21}$$

for some $k_2 > 0$.

If $z \in \mathbb{E}_1 \cap \Delta_\varepsilon$ has $\mathrm{Re}\, z \geq 1$, then, since $\mathbf{H}(z)$ is bounded on Δ_ε, it follows that the left-hand side of (21) is positive semidefinite for all sufficiently small k. If $z \in \mathbb{E}_1 \cap \Delta_\varepsilon$ has $\mathrm{Re}\, z < 1$, then

$$\begin{aligned}\frac{\mathrm{Re}\, z - 1}{|z-1|^2} &= \frac{\mathrm{Re}\, z - 1}{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2 + 1 - 2\mathrm{Re}\, z}\\ &\geq \frac{\mathrm{Re}\, z - 1}{2(1 - \mathrm{Re}\, z)}\\ &= -\tfrac{1}{2}.\end{aligned}$$

Thus we may conclude that the left-hand side of (21) is positive semidefinite for all sufficiently small k. Hence we have shown that there exists a k_2 such that (21) holds true. Choosing $k^* = \min(k_1, k_2)$ we see that $I + k\tilde{\mathbf{G}}(z) + k\tilde{\mathbf{G}}^H(z) \succeq 0$ for all $z \in \mathbb{E}_1$ and all $k \in (0,\, k^*)$.

Conversely, suppose that there exists a $k^* > 0$ such that $I + k\tilde{\mathbf{G}}(z) + k\tilde{\mathbf{G}}^H(z) \succeq 0$ for all $z \in \mathbb{E}_1$ and all $k \in (0,\, k^*)$. Then, by (19), we obtain for any $\xi \in \mathbb{C}^m$ that

$$2\mathrm{Re}\left\langle \xi,\, \frac{k}{z-1}\,\mathbf{G}(1)\xi \right\rangle + \|\xi\|^2 + k\langle \xi,\, \mathbf{H}(z)\xi\rangle \geq 0, \quad \text{for all } z \in \mathbb{E}_1 \cap \Delta_\varepsilon,\ k \in (0,\, k^*).$$

Hence, it follows that for all $z \in \mathbb{E}_1 \cap \Delta_\varepsilon$, all $\xi \in \mathbb{C}^m$, and all $k \in (0,\, k^*)$

$$\begin{aligned}&\frac{2k}{|z-1|^2}[(\mathrm{Re}\, z - 1)\mathrm{Re}\,\langle \xi,\, \mathbf{G}(1)\xi\rangle\\ &\quad - \mathrm{Im}\, z\, \mathrm{Im}\,\langle \xi,\, \mathbf{G}(1)\xi\rangle] + \|\xi\|^2\\ &\quad + k\langle \xi,\, \mathbf{H}(z)\xi\rangle \geq 0.\end{aligned} \tag{22}$$

It follows immediately from (22), by considering $z \in \mathbb{E}_1 \cap \Delta_\varepsilon$ with $\mathrm{Im}\, z = 0$, that for any $\xi \in \mathbb{C}^m$

$$a := \mathrm{Re}\,\langle \xi,\, \mathbf{G}(1)\xi\rangle \geq 0.$$

To prove that $\mathbf{G}(1) \succeq 0$, it only remains to show

$$b := \mathrm{Im}\,\langle \xi,\, \mathbf{G}(1)\xi\rangle = 0, \quad \text{for all } \xi \in \mathbb{C}^m. \tag{23}$$

Seeking a contradiction, suppose that (23) is not true. Without loss of generality we may assume that $b > 0$. For $z_n = x_n + \imath y_n \in \mathbb{E}_1 \cap \Delta_\varepsilon$ with

$$x_n = 1 + \frac{1}{n}, \quad y_n = \frac{\beta}{n}$$

where

$$\beta > \frac{a}{b}$$

we obtain

$$\begin{aligned}\frac{\mathrm{Re}\, z_n - 1}{|z_n - 1|^2} &= \frac{n}{1+\beta^2}\\ \frac{\mathrm{Im}\, z_n}{|z_n - 1|^2} &= \frac{\beta n}{1+\beta^2}.\end{aligned}$$

Denoting the left-hand side of (22) by $L(z,\, \xi)$, it then follows that

$$\begin{aligned}L(z_n,\, \xi) &= \frac{2kn}{1+\beta^2}(a - \beta b) + \|\xi\|^2 + k\langle \xi,\, \mathbf{H}(z_n)\xi\rangle\\ &\to -\infty \quad \text{as } n \to \infty\end{aligned}$$

which contradicts (22). □

Remark 2.10: It is not difficult to show that Theorem 2.5, Corollary 2.6, and Proposition 2.8 remain true if the integrator $z/(z-1)$ is applied instead of the (strictly proper) integrator $1/(z-1)$.

III. Adaptive Low-Gain Control of Discrete-Time Systems

Throughout this section we assume that the quadruple $(A,\, B,\, C,\, D)$ defines a power-stable m-input/m-output discrete-time system given by (12), with transfer function $\mathbf{G}(z) = C(zI - A)^{-1}B + D$. Let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ denote the state-space operators for the state-space realization of the series connection of a discrete-time integrator followed by $\mathbf{G}(z)$. The transfer function of the series connection is denoted by $\tilde{\mathbf{G}}(z)$. We consider the integrators $1/(z-1)$ and $z/(z-1)$. In the first case $\tilde{\mathbf{G}}(z) = [1/(z-1)]\mathbf{G}(z)$, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are given by (14) and $\tilde{D} = 0$; in the second case $\tilde{\mathbf{G}}(z) = [z/(z-1)]\mathbf{G}(z)$, $\tilde{A}$, and $\tilde{C}$ remain the same, whilst

$$\tilde{D} = D \quad \text{and} \quad \tilde{B} = \begin{pmatrix} B \\ I \end{pmatrix}.$$

From the results of Section II we know that if $\sigma[\mathbf{G}(1)] \subset \mathbb{C}_0$, then proportional negative output feedback applied to $\tilde{\mathbf{G}}$ will result in a stable closed-loop system, provided the feedback gain k is positive and sufficiently small. It is natural to tune the scalar gain k, adaptively, and this section is devoted to this problem. As already mentioned in the introduction, for most of the results in this section we need to assume that $\mathbf{G}(1) \succ 0$, which is of course stronger than $\sigma[\mathbf{G}(1)] \subset \mathbb{C}_0$ (c.f. Remark 3.3).

First, we record a simple result which shows that the reference signal can be realized internally. This internal realization of the reference signal is inspired by the internal model principle and converts the tracking problem ($r \neq 0$) into the stabilization problem $r = 0$).

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Lemma 3.1: Suppose $\det \mathbf{G}(1) \neq 0$. For each $r \in \mathbb{R}^m$, if $u^r = \mathbf{G}^{-1}(1)r$ and $x^r = (I - A)^{-1}Bu^r$, then

$$\tilde{A}(x^r, u^r) = (x^r, u^r) \tag{24}$$

and hence

$$\begin{aligned}\tilde{C}\tilde{A}^n(x^r, u^r) &= \tilde{C}(x^r, u^r)\\ &= r, \qquad \text{for all } n \in \mathbb{N}.\end{aligned} \tag{25}$$

The easy proof of the above lemma is left to the reader.

Theorem 3.2: Let (12) describe an m-input/m-output, power-stable system. Suppose that the transfer function of the plant satisfies $\mathbf{G}(1) \succ 0$. Let $r\theta_n$, $r \in \mathbb{R}^m$ be an arbitrary constant reference signal and consider the control law

$$u_{n+1} = u_n + \gamma_n^{-p} e_n \tag{26a}$$

$$\gamma_{n+1} = \gamma_n + \|e_n\|^2 \tag{26b}$$

where $e_n = r - y_n$ and $0 < p < \frac{1}{2}$. If $(x_0, u_0) \in X \times \mathbb{R}^m$ and $\gamma_0 > 0$, then:

i) $\lim_{n\to\infty} \gamma_n = \gamma_\infty < \infty$;
ii) $\lim_{n\to\infty} u_n = u^r = \mathbf{G}^{-1}(1)r$ and $\{u_n - u^r\} \in l_+^2(\mathbb{R}^m)$;
iii) $\lim_{n\to\infty} x_n = x^r = (I-A)^{-1}Bu^r \in X$ and $\{x_n - x^r\} \in l_+^2(X)$;
iv) $\lim_{n\to\infty} y_n = r$.

Proof: First note that by Lemma 3.1, $r = \tilde{C}\tilde{A}^n(x^r, u^r)$. Hence, we can rewrite the error, $\{e_n\}$, as

$$e_n = \tilde{C}\tilde{x}_n \tag{27}$$

where

$$\tilde{x}_n = \tilde{A}^n(x^r - x_0, u^r - u_0) - \sum_{j=0}^{n-1} \tilde{A}^{n-j-1}\tilde{B}\gamma_j^{-p} e_j. \tag{28}$$

The next step is to show that $\{\gamma_n\}$ is bounded. If $\{\gamma_n\}$ is unbounded, then by Proposition 2.8 there exists $n_1 \geq 0$ such that, with $k_1 = \gamma_{n_1}^{-p}$, $k_1/(z-1)$ is a stabilizing controller, in the sense that $\tilde{\mathbf{G}}_{k_1}$ given by (16) is in $H_<^\infty(\mathbb{E}_1, \mathbb{C}^{m\times m})$. Moreover, $\|\tilde{\mathbf{G}}_{k_1}\|_\infty = 1/k_1$. It follows from Corollary 2.6 that the operator $\tilde{A}_{k_1}$ given by (15) is power stable. For all $n \geq n_1$ we can rewrite (27) and (28) as

$$e_n = \tilde{C}\,\tilde{A}_{k_1}^{n-n_1}\tilde{x}_{n_1} - \sum_{j=n_1}^{n-1} \tilde{C}\,\tilde{A}_{k_1}^{n-j-1}\tilde{B}\,(\gamma_j^{-p} - k_1)e_j. \tag{29}$$

Let $\{w_n\}$ be a sequence in $\mathbb{R}^m$. Then using Schwartz's inequality and Parseval's theorem

$$\begin{aligned}&\left(\sum_{j=n_1}^{n}\left\|\sum_{i=n_1}^{j-1}\tilde{C}\tilde{A}_{k_1}^{j-i-1}\tilde{B}w_i\right\|^2\right)^{1/2}\\ &\leq \|\tilde{\mathbf{G}}_{k_1}\|_\infty\left(\sum_{j=n_1}^{n}\|w_j\|^2\right)^{1/2}\\ &= \frac{1}{k_1}\left(\sum_{j=n_1}^{n}\|w_j\|^2\right)^{1/2}.\end{aligned}$$

Hence

$$\begin{aligned}&\left(\sum_{j=n_1}^{n}\|e_j\|^2\right)^{1/2}\\ &\leq \left(\sum_{j=n_1}^{n}\|\tilde{C}\tilde{A}_{k_1}^{j-n_1}\tilde{x}_{n_1}\|^2\right)^{1/2}\\ &\quad + \left(\sum_{j=n_1}^{n}\left\|\sum_{i=n_1}^{j-1}\tilde{C}\tilde{A}_{k_1}^{j-i-1}\tilde{B}(\gamma_i^{-p} - k_1)e_i\right\|^2\right)^{1/2}\\ &\leq c_0 + \frac{1}{k_1}(k_1 - \gamma_n^{-p})\left(\sum_{j=n_1}^{n}\|e_j\|^2\right)^{1/2}\end{aligned} \tag{30}$$

where the existence of a positive constant c_0 is guaranteed by the power-stability of $\tilde{A}_{k_1}$. Hence

$$\begin{aligned}\sqrt{\gamma_{n+1} - \gamma_{n_1}} &= \left(\sum_{j=n_1}^{n}\|e_j\|^2\right)^{1/2}\\ &\leq c_0 k_1 \gamma_n^p \leq c_0 k_1 \gamma_{n+1}^p, \quad \text{for all } n \geq n_1.\end{aligned}$$

This inequality clearly contradicts the unboundedness of $\{\gamma_n\}$ and the assumption that $0 < p < \frac{1}{2}$. Therefore $\{\gamma_n\}$ is bounded. Hence, $\lim_{n\to\infty} \gamma_n = \gamma_\infty$ exists, proving i). Using (26b), we have that $\{e_n\} \in l_+^2(\mathbb{R}^m)$. To prove ii), iii), and iv), simply note that by (27) and (28)

$$\tilde{x}_{n+1} = \tilde{A}_{k_1}\tilde{x}_n + \tilde{B}(k_1 - \gamma_n^{-p})e_n$$

so that $\{\tilde{x}_n\}$ is the solution of a power-stable system driven by an l_+^2-input. Hence $\{\tilde{x}_n\} \in l_+^2(X)$ and $\lim_{n\to\infty} \tilde{x}_n = 0$, from which ii), iii), and iv) follow readily. □

Remark 3.3: Suppose that $\det \mathbf{G}(1) \neq 0$ and that A is power stable. Then, using a well-known result on positive definite operators (see Rudin [33, p. 313]), it follows that the condition $\mathbf{G}(1) \succ 0$ is satisfied, provided that

$$A = A^*, \quad B = C^*, \quad \text{and} \quad D = D^* \succeq 0.$$

Theorem 3.2 is the exact discrete-time analog of the continuous-time adaptive low-gain result given by [19, Proposition 4.6]. In the high-gain stabilization of continuous-time, finite-dimensional systems, exponential decay to zero of the state can be guaranteed by using piecewise-linear gain adaptation; see, for example, Ilchmann and Owens [8]. In the particular situation here we can exploit certain spectrum-decomposition properties of stabilizable discrete-time systems (see, e.g., Logemann [13]) to obtain similar results in the present setting.

Lemma 3.4: Let the assumptions of Theorem 3.2 be satisfied and let $k \in \mathbb{R}$. If $\tilde{A}_k^n\tilde{x}_0 \to 0$ as $n \to \infty$, then there exists $M > 0$ and $\rho \in (0, 1)$ such that

$$\|\tilde{A}_k^n\tilde{x}_0\| \leq M\rho^n, \quad \text{for all } n \in \mathbb{N}.$$

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Proof: By Corollary 2.6, there exists $k^* > 0$ and $\alpha \in (0, 1)$ such that

$$\sigma(\tilde{A}_{k^*}) \subset \{\lambda \in \mathbb{C} |\, |\lambda| < \alpha\}.$$

Thus the pair $(\tilde{A}_k, \tilde{B})$ is stabilizable. Now $\tilde{B}$ is compact, and hence it follows from [13, Th. 4] that $X \times \mathbb{R}^m$ admits a decomposition

$$X \times \mathbb{R}^m = \tilde{X}_U \oplus \tilde{X}_S$$

so that $\tilde{X}_U$ is finite-dimensional, $\tilde{A}_k(\tilde{X}_U) \subset \tilde{X}_U$, $\tilde{A}_k(\tilde{X}_S) \subset \tilde{X}_S$, $\tilde{A}_k : \tilde{X}_S \to \tilde{X}_S$ is power stable, and the eigenvalues of $\tilde{A}_k : \tilde{X}_U \to \tilde{X}_U$ all have modulus greater than or equal to one. If we decompose $\tilde{x}_0$ with respect to $\tilde{X}_S$ and $\tilde{X}_U$ as $\tilde{x}_0 = \tilde{x}_S + \tilde{x}_U$, then clearly $\tilde{x}_U = 0$, and therefore, using power-stability of $\tilde{A}_k$ restricted to $\tilde{X}_S$, we have

$$\| \tilde{A}_k^n \tilde{x}_0 \| = \| \tilde{A}_k^n x_S \| \le M\rho^n$$

for some $M > 0$ and $\rho \in (0, 1)$. □

Theorem 3.5: Suppose the conditions of Theorem 3.2 hold. Let $\{\kappa_j\}$ be any strictly decreasing sequence of positive real numbers with $\lim_{j\to\infty} \kappa_j = 0$. With the control law

$$\gamma_{n+1} = \gamma_n + \|e_n\|^2, \qquad \gamma_0 \ge \kappa_0^{1/p} \tag{31a}$$

$$u_{n+1} = u_n + \kappa_j e_n, \quad \text{if } \gamma_n^{-p} \in (\kappa_{j+1}, \kappa_j] \tag{31b}$$

where $0 < p < 1/2$, the conclusions of Theorem 3.2 hold true. Moreover, all convergences are exponential.

Proof: Arguing as in the proof of Theorem 3.2 we can readily obtain

$$\sqrt{\gamma_{n+1} - \gamma_{n_1}} \le \frac{c_0 k_1}{\kappa_j} \le c_0 k_1 \gamma_n^p \le c_0 k_1 \gamma_{n+1}^p$$

so that $\{\gamma_n\}$ is bounded and therefore i)–iv) of Theorem 3.2 hold. In particular, $\tilde{x}_n \to 0$ as $n \to \infty$. Since $\lim_{n\to\infty} \gamma_n^{-p} = \gamma_\infty^{-p}$, there exist nonnegative integers n_2 and i so that $\gamma_\infty^{-p} \in (\kappa_{i+1}, \kappa_i]$ and

$$\tilde{x}_n = \tilde{A}_{\kappa_i}^{n-n_2} \tilde{x}_{n_2}, \quad \text{for all } n \ge n_2.$$

It follows from Lemma 3.4 that $\|\tilde{x}_n\| \le M\rho^n$ for some $M > 0$ and $\rho \in (0, 1)$. □

So far we have assumed that the discrete-time integrator is $1/(z-1)$. If $D = 0$, then we can also use the integrator $z/(z-1)$. Due to the proportional part contained in $z/(z-1)$, this integrator will usually produce faster responses than the strictly proper integrator $1/(z-1)$. Under the conditions of Theorem 3.2, the conclusions of Theorems 3.2 and 3.5 remain true if $D = 0$ and if (26) is replaced by

$$u_{n+1} = u_n + v_{n+1}, \quad v_n = \gamma_n^{-p} e_n \tag{32a}$$

$$\gamma_{n+1} = \gamma_n + \|e_n\|^2 \tag{32b}$$

respectively, (31) is replaced by

$$u_{n+1} = u_n + v_{n+1} \tag{33a}$$

where

$$v_n = \kappa_j e_n \quad \text{if } \gamma_n^{-p} \in (\kappa_{j+1}, \kappa_j] \tag{33b}$$

$$\gamma_{n+1} = \gamma_n + \|e_n\|^2. \tag{33c}$$

However, if $D \ne 0$ (as will be the case in Section IV), then we can use (32) or (33) only if we assume that $r(D) \le d$ for some known $d > 0$, where $r(D)$ denotes the spectral radius of D.

Theorem 3.6: Suppose the conditions of Theorem 3.2 hold. If $r(D) \le d$ for some known $d > 0$ and $\{u_n\}$ and $\{\gamma_n\}$ are given by (32), respectively (33), with $\gamma_0^{-p} < 1/d$, then the conclusions of Theorem 3.2, respectively, Theorem 3.5, hold.

Proof: Suppose the controller is given by (32), then the error $\{e_n\}$ satisfies

$$e_n = \tilde{C}\tilde{x}_n - D\gamma_n^{-p} e_n \tag{34}$$

where $\tilde{x}_n$ is given by (28). Equation (34) is in general not solvable for e_n if $-\gamma_n^p \in \sigma(D)$. This possibility is ruled out by assuming that $\gamma_0^{-p} < 1/d$. Therefore

$$e_n = \tilde{C}_{k_1} \tilde{A}_{k_1}^{n-n_1} \tilde{x}_{n_1} - \sum_{j=n_1}^{n-1} \tilde{C}_{k_1} \tilde{A}_{k_1}^{n-j-1} \tilde{B}_{k_1} (\gamma_j^{-p} - k_1) e_j - \tilde{D}_{k_1} (\gamma_n^{-p} - k_1) e_n \tag{35}$$

where

$$\tilde{A}_{k_1} := \begin{pmatrix} A & B \\ 0 & I \end{pmatrix} - (I + k_1 D)^{-1} k_1 \begin{pmatrix} B \\ I \end{pmatrix} (CD)$$

$$\tilde{B}_{k_1} := (I + k_1 D)^{-1} \begin{pmatrix} B \\ I \end{pmatrix}$$

$$\tilde{C}_{k_1} := (I + k_1 D)^{-1} C$$

$$\tilde{D}_{k_1} := (I + k_1 D)^{-1} D.$$

In (35) we choose the gain $k_1 > 0$ sufficiently small to ensure that $\tilde{A}_{k_1}$ is power stable (see Corollary 2.6) and that $\det(I + k_1 D) \ne 0$. Using (35), the remainder of the proof follows closely that of Theorem 3.2 and Theorem 3.5 and is therefore omitted. The case when the controller is given by (33) follows similarly. □

So far we have assumed that $\mathbf{G}(1) \succ 0$. Trivially, we can also deal with the case $\mathbf{G}(1) \prec 0$, simply by changing the sign of the integrator gain. For completeness we now consider the situation when the steady-state gain $\mathbf{G}(1)$ is *sign-definite*, that is, where either $\mathbf{G}(1) \succ 0$ or $\mathbf{G}(1) \prec 0$. This situation arises most naturally in the single-input/single-output case where we only need to assume that the steady-state gain is nonzero, i.e., $\mathbf{G}(1) \ne 0$, but the sign of $\mathbf{G}(1)$ is unknown. Unlike the known sign case, where the current gain is determined purely as a function of the adaptation parameter γ_n, we now use a controller involving an additional sign-switching component.

Theorem 3.7: Suppose that (12) describes an m-input/m-output, power-stable system and that $\mathbf{G}(1)$ is sign-definite. Let $r\theta_n$, $r \in \mathbb{R}^m$ be an arbitrary constant reference signal, and consider the control law defined recursively by

$$\gamma_{n+1} = \gamma_n + \|e_n\|^2$$
$$u_{n+1} = u_n + S_n \log^{-p}(\gamma_n) e_n \tag{36}$$

where

$$S_{n+1} = \begin{cases} S_n & \text{if } \log^{-p} \gamma_{n+1} > \frac{1}{2} \log^{-p} \eta_n \\ -S_n & \text{else} \end{cases} \tag{37}$$

$$\eta_{n+1} = \begin{cases} \eta_n & \text{if } \log^{-p} \gamma_{n+1} > \frac{1}{2} \log^{-p} \eta_n \\ \gamma_{n+1} & \text{else} \end{cases} \tag{38}$$

$$S_0 = 1, \quad \eta_0 = \gamma_0$$

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and $0 < p < 1/2$. If $(x_0, u_0) \in X \times \mathbb{R}^m$ and $\gamma_0 > 1$, then the conclusions of Theorem 3.2 hold.

In the switching strategy given by (37) and (38) the current sign of the gain $S_n \log^{-p} \gamma_n$ is held in S_n, the modulus of the gain is determined by $\log^{-p} \gamma_n$, and subsequent halvings of the gain are monitored by η_n.

Proof: Note that it is sufficient to consider the stabilization problem ($r = 0$) and that the proof is complete if we can show that $\{\gamma_n\}$ is bounded. If $\{\gamma_n\}$ is unbounded, then we can choose a sequence $n_0 \le n_1 \le n_2 \le \cdots$ with $\lim_{j\to\infty} n_j = \infty$ so that:

i) $S_i \mathbf{G}(1) \succ 0$, for all $i = n_{2j}, \cdots, n_{2j+1}$, $j \in \mathbb{N}$;
ii) $\log^{-p} \gamma_i > \frac{1}{2} \log^{-p} \gamma_{n_{2j}}$, for all $i = n_{2j}, \cdots, n_{2j+1}$, $j \in \mathbb{N}$;
iii) $\log^{-p} \gamma_{n_{2j+1}+1} \le \frac{1}{2} \log^{-p} \gamma_{n_{2j}}$, for all $j \in \mathbb{N}$;
iv) $\|\tilde{\mathbf{G}}_{k_{n_{2j}}}\|_\infty = 1/|k_{n_{2j}}|$, for all $j \in \mathbb{N}$, where $k_{n_{2j}} := S_{n_{2j}} \log^{-p} \gamma_{n_{2j}}$.

Using arguments similar to those in the proof of Theorem 3.2, we obtain

$$\begin{aligned}\left(\sum_{i=n_{2j}}^{n_{2j+1}} \|e_i\|^2\right)^{1/2} &\le \left(\sum_{i=n_{2j}}^{n_{2j+1}} \left\|\tilde{C}\tilde{A}_{k_{n_{2j}}}^{i-n_{2j}} \tilde{x}_{n_{2j}}\right\|^2\right)^{1/2} \\ &\quad + \left(\sum_{i=n_{2j}}^{n_{2j+1}} \left\|\sum_{l=n_{2j}}^{i-1} \tilde{C}\tilde{A}_{k_{n_{2j}}}^{i-l-1}\tilde{B}(\log^{-p} \gamma_l - k_{n_{2j}})e_l\right\|^2\right)^{1/2} \\ &\le \frac{|k_{n_0}|}{|k_{n_{2j}}|}\left(\sum_{i=n_{2j}}^{n_{2j+1}} \left\|\tilde{C}\tilde{A}_{k_{n_0}}^{i-n_{2j}} \tilde{x}_{n_{2j}}\right\|^2\right)^{1/2} \\ &\quad + \frac{1}{|k_{n_{2j}}|}|k_{n_{2j}} - k_{n_{2j+1}}|\left(\sum_{i=n_{2j}}^{n_{2j+1}} \|e_i\|^2\right)^{1/2} \qquad (39)\end{aligned}$$

where $\tilde{A}_{k_{n_{2j}}}$ is given by (15). In (39) we used the fact that for $\tilde{x} \in X \times \mathbb{R}^m$ and $k \in \mathbb{R}$

$$\begin{aligned}(\tilde{A} - k\tilde{B}\tilde{C})^n\tilde{x} = (\tilde{A} - k_{n_0}\tilde{B}\tilde{C})^n\tilde{x} - \sum_{i=0}^{n-1}(\tilde{A} - k_{n_0}\tilde{B}\tilde{C})^{n-1-i} \\ \cdot (k - k_{n_0})\tilde{B}\tilde{C}(\tilde{A} - k\tilde{B}\tilde{C})^i\tilde{x}\end{aligned}$$

and $\|\tilde{\mathbf{G}}_{k_{n_0}}\| = 1/k_{n_0}$. Therefore

$$\begin{aligned}\sqrt{\gamma_{n_{2j+1}+1} - \gamma_{n_{2j}}} &= \left(\sum_{i=n_{2j}}^{n_{2j+1}} \|e_i\|^2\right)^{1/2} \\ &\le 2\frac{|k_{n_0}|}{|k_{n_{2j}}|}\left(\sum_{i=n_{2j}}^{n_{2j+1}} \left\|\tilde{C}\tilde{A}_{k_{n_0}}^{i-n_{2j}} \tilde{x}_{n_{2j}}\right\|^2\right)^{1/2}.\end{aligned}$$

Now

$$\tilde{x}_{n+1} = \tilde{A}_{k_{n_0}}\tilde{x}_n - \tilde{B}(\log^{-p} \gamma_n - k_{n_0})e_n$$

and using the fact that k_{n_0} is a stabilizing gain, we obtain

$$\|\tilde{x}_{n_{2j}}\| \le c_0 + c_1\left(\sum_{i=0}^{n_{2j}-1} \|e_i\|^2\right)^{1/2}$$

for some positive constants c_0 and c_1. Hence

$$\sqrt{\gamma_{n_{2j+1}+1} - \gamma_{n_{2j}}} \le 2|k_{n_0}| \log^p \gamma_{n_{2j}}\left(\tilde{c}_0 + \tilde{c}_1\sqrt{\gamma_{n_{2j}} - \gamma_0}\right)$$

where $\tilde{c}_0$ and $\tilde{c}_1$ are positive constants. If $\theta_{n_{2j}}$ is defined by $\log^{-p} \theta_{n_{2j}} = \frac{1}{2} \log^{-p} \gamma_{n_{2j}}$, then

$$\sqrt{\theta_{n_{2j}} - \gamma_{n_{2j}}} \le 2|k_{n_0}| \log^p \gamma_{n_{2j}}\left(\tilde{c}_0 + \tilde{c}_1\sqrt{\gamma_{n_{2j}} - \gamma_0}\right). \qquad (40)$$

Applying the mean-value theorem to $\log^{-p} \gamma$ on $[\gamma_{n_{2j}}, \theta_{n_{2j}}]$ gives

$$\frac{1}{(\log^{-p} \xi_{n_{2j}})'} = -\frac{2(\theta_{n_{2j}} - \gamma_{n_{2j}})}{\log^{-p} \gamma_{n_{2j}}}$$

where $\xi_{n_{2j}} \in [\gamma_{n_{2j}}, \theta_{n_{2j}}]$. Hence, it follows from (40) that

$$\sqrt{\frac{\xi_{n_{2j}} \log^{-p} \gamma_{n_{2j}}}{2p \log^{-p-1} \xi_{n_{2j}}}} \le 2|k_{n_0}| \log^p \gamma_{n_{2j}}\left(\tilde{c}_0 + \tilde{c}_1\sqrt{\gamma_{n_{2j}} - \gamma_0}\right).$$

Using the monotonicity of logarithm we obtain

$$\sqrt{\xi_{n_{2j}} \log^{1-2p} \xi_{n_{2j}}} \le 2\sqrt{2p}|k_{n_0}|\left(\tilde{c}_0 + \tilde{c}_1\sqrt{\xi_{n_{2j}} - \gamma_0}\right). \qquad (41)$$

But $0 < p < 1/2$ so that (41) contradicts the unboundedness of $\{\gamma_n\}$. It follows that $\{\gamma_n\}$ is bounded. The rest of the proof is exactly the same as that of Theorem 3.2. □

IV. Sampled-Data Low-Gain Control of Regular Systems

In Sections II and III we obtained results on low-gain control for discrete-time systems. We now apply these discrete-time results to the sampled-data control of regular systems, an important class of continuous-time infinite-dimensional systems introduced and studied by Weiss [40], [41]. Let the continuous-time system to be controlled have a transfer function $\mathbf{G}_c(s)$. In [19], it is shown that for continuous-time low-gain control it is necessary that $\det \mathbf{G}_c(0) \neq 0$ and natural to assume that the plant is exponentially stable. In this section we show that these two properties are mapped, under appropriate and naturally defined sampling, into the conditions needed in discrete-time low-gain control, which in turn permits application of the results obtained in Sections II and III. In the case of a regular system with a bounded observation operator, the system is sampled using sample/hold. In the case of a regular system with unbounded observation we have to "smooth" the output by averaging prior to sample/hold.

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A. Regular Systems

A general class of m-input/m-output continuous-time infinite-dimensional systems would be the *well-posed* systems as introduced by Salamon in [35]. The class of well-posed systems captures the systems theoretic properties of linearity, time-invariance, and causality together with natural continuity properties. Moreover, every well-posed system has a well-defined transfer function $\mathbf{G}_c(s)$. A *regular* system is a well-posed system satisfying the extra requirement that

$$\lim_{s\to\infty,\, s\in\mathbb{R}} \mathbf{G}_c(s) = D_c$$

exists. Let X be a real Hilbert space. Given an input function $u(\cdot) \in L^2_{loc}(0, \infty; \mathbb{R}^m)$, the state of a regular linear system with state-space X is described by

$$\dot{x}(t) = A_c x(t) + B_c u(t), \quad x(0) = x_0 \in X. \tag{42}$$

Here:

- A_c is the generator of a C_0-semigroup $T(t)$ on X;
- $B_c \in \mathcal{B}(\mathbb{R}^m, X_{-1})$, and X_{-1} is the completion of X with respect to $\|x\|_{X_{-1}} := \|(\beta I - A_c)^{-1}x\|_X$, where $\beta \in \varrho(A_c)$.

It is well known that $T(t)$ extends to a C_0-semigroup on X_{-1}. The generator of this semigroup is a bounded operator from X to X_{-1} which extends A_c. The extended semigroup and its generator will be denoted by the same symbols $T(t)$ and A_c, respectively. Equality in (42) holds in X_{-1}.

Continuity of the input-to-state map is expressed by

$$\left\| \int_0^t T(t-\nu) B_c u(\nu)\, d\nu \right\|_X \le b_t \|u(\cdot)\|_{L^2(0,\, t;\, \mathbb{R}^m)} \tag{43}$$

where $b_t \ge 0$. If $x_0 \in X$, then the mild solution, given by

$$x(t) = T(t)x_0 + \int_0^t T(t-\nu) B_c u(\nu)\, d\nu \tag{44}$$

evolves continuously in X. Moreover, (43) implies that $(sI - A_c)^{-1}B_c \in \mathcal{B}(\mathbb{R}^m, X)$ for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ greater than the exponential growth bound of $T(t)$.

To introduce an observation for (44), let X_1 denote the domain of A_c, as an operator defined on X, endowed with the graph norm. The semigroup $T(t)$ restricts to a C_0-semigroup $T(t)$ on X_1. The exponential growth bounds of $T(t)$ are the same on all three spaces X_1, X, and X_{-1}. If $u(\cdot) = 0$ and $x_0 \in X_1$, then the output of a regular (or well-posed system) is given by

$$y(t) = C_c T(t) x_0$$

where the observation operator C_c is in $\mathcal{B}(X_1, \mathbb{R}^m)$. Continuity of the state-to-output map guarantees that

$$\|C_c T(\cdot)x\|_{L^2(0,\, t;\, \mathbb{R}^m)} \le c_t \|x\|_X, \quad \text{for all } x \in X_1 \tag{45}$$

for some $c_t \ge 0$.

The continuity of the input-to-output map for a regular system, combined with continuity of the state-to-output map, guarantees that for each $x_0 \in X$ and $u(\cdot) \in L^2(0, T; \mathbb{R}^m)$ the output $y(\cdot)$ is well-defined in $L^2(0, T; \mathbb{R}^m)$ and satisfies

$$y(t) = C_L x(t) + D_c u(t) \qquad \text{for a.e. } t \ge 0. \tag{46}$$

Here C_L is the Lebesgue extension of C_c; see Weiss [39]. In particular, we have $X_1 \subset D(C_L)$. The following properties of C_L are consequences of regularity:

- for each $x \in X$, $T(t)x \in D(C_L)$ for a.e. $t \ge 0$;
- $\operatorname{im}[(sI - A_c)^{-1}B_c] \subset D(C_L)$ for all $s \in \varrho(A_c)$.

Moreover, the transfer function $\mathbf{G}_c(s)$ of a regular system can be written as

$$\mathbf{G}_c(s) = C_L(sI - A_c)^{-1}B_c + D_c.$$

Detailed definitions of regular and well-posed systems can be found in [35], [40], and [41] and are summarized in the context of continuous-time low-gain control in [19].

Let $\{u_n\} \subset \mathbb{R}^m$ be an arbitrary sequence. If

$$u(n\tau + t) = u_n \in \mathbb{R}^m, \qquad \text{for each } n \in \mathbb{N},\ t \in [0, \tau) \tag{47}$$

and if $0 \in \varrho(A_c)$, then the state $x(n\tau + t)$ will satisfy

$$x(n\tau + t) = T(t)x(n\tau) + [T(t) - I]\, A_c^{-1} B_c u_n. \tag{48}$$

We see that $x(n\tau + t) \in X$ for all $t \in [0, \tau)$ and $n \in \mathbb{N}$, and accordingly we define $x_n \in X$ by

$$x_n = x(n\tau). \tag{49}$$

Moreover, $T(\tau) \in \mathcal{B}(X)$ and $[T(\tau) - I]\, A_c^{-1}B_c \in \mathcal{B}(\mathbb{R}^m, X)$ define appropriate state-space operators for the state evolution of the discretization. However, in general, regularity only guarantees that $y(\cdot) \in L^2(0, \infty; \mathbb{R}^m)$ so that even with piecewise-constant input functions, standard sampling at the output is not defined. Moreover, even if the output function is continuous, so that standard sampling is defined, in general the resulting discrete-time system will not have a bounded observation operator. We therefore consider two cases.

i) The observation operator is bounded. In this case C_c extends to a bounded operator from X to $\mathbb{R}^m$ and $C_c = C_L \in \mathcal{B}(X, \mathbb{R}^m)$. This case includes, in particular, the well-known class of Pritchard–Salamon systems; see for example Pritchard and Salamon [32] and Curtain *et al.* [4].

ii) The observation operator is unbounded in the sense that C_c cannot be extended to an operator in $\mathcal{B}(X, \mathbb{R}^m)$.

In case i), if $x_0 \in X$, then the output is continuous and standard sampling is defined, while in case ii) we first average the output over one sampling interval.

B. Bounded Observation

Suppose that $C_c \in \mathcal{B}(X, \mathbb{R}^m)$. If $x_0 \in X$ and $u(\cdot)$ is given by (47), then the output $y(\cdot)$ given by (46) is piecewise-continuous, the discontinuities being at $n\tau$. It is clear that $y(\cdot)$ is right-continuous at $n\tau$. We define

$$y_n := y(n\tau) \tag{50}$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{bmatrix} T(\tau) & [T(\tau) - I]\, A_c^{-1} B_c \\ C_c & D_c \end{bmatrix}. \tag{51}$$

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Proposition 4.1: Suppose that (44) and (46) describe an exponentially stable, regular system with bounded observation operator C_c. Fix $\tau > 0$ as the sampling interval and let $\{u_n\} \subset \mathbb{R}^m$. If $u(\cdot)$ given by (47) is applied to (44) and (46), then x_n and y_n given by (49) and (50) satisfy

$$x_{n+1} = Ax_n + Bu_n \tag{52a}$$
$$y_n = Cx_n + Du_n \tag{52b}$$

where (A, B, C, D) is given by (51). Moreover, A is power stable and

$$\mathbf{G}(1) = C(I - A)^{-1}B + D = \mathbf{G}_c(0). \tag{53}$$

Here $\mathbf{G}(z)$ denotes the transfer function of (52).

Proof: It is clear that $\{x_n\}$ and $\{y_n\}$ satisfy (52). Moreover, A is power stable because $T(t)$ is exponentially stable on X. All that remains is to verify (53). Using (51) we have

$$\begin{aligned} C(I - A)^{-1}B + D &= C_c[I - T(\tau)]^{-1}[T(\tau) - I]\,A_c^{-1}B_c \\ &\quad + D_c \\ &= -C_cA_c^{-1}B_c + D_c \\ &= \mathbf{G}_c(0). \end{aligned}$$ □

If (44) and (46) describe an exponentially stable, regular system with a bounded observation operator and $\det\,\mathbf{G}_c(0) \neq 0$, then (52) describes a power-stable, discrete-time system with $\det\,\mathbf{G}(1) \neq 0$. We can therefore apply the discrete-time low-gain results of Section II, and in the case when the steady-state gain $\mathbf{G}_c(0)$ is sign-definite, the adaptive, low-gain discrete-time results of Section III, to guarantee that

$$\lim_{n\to\infty} u_n = u^r = \mathbf{G}_c^{-1}(0)r, \quad \lim_{n\to\infty} y_n = r. \tag{54}$$

We are also interested in the response of the continuous-time system to these digitally computed controls.

We will concentrate on applying the adaptive results of Section III to sampled-data control of square plants with sign-definite, steady-state gains. Applications of the nonadaptive results of Section II to general multivariable square systems can be obtained in the same manner.

Theorem 4.2: Suppose that the system given by (44) and (46) is an exponentially stable regular system with bounded observation operator C_c. Suppose that $\mathbf{G}_c(0) \succ 0$. Fix $r \in \mathbb{R}^m$ and set $e_n = r - y_n$, where y_n is given by (50).

a) If $\{u_n\}$ and $\{\gamma_n\}$ are obtained by applying (26) to (52), and $u(\cdot)$, defined by (47), is applied to (44) and (46), then for all $(x_0, u_0) \in X \times \mathbb{R}^m$:
   i) $\lim_{t\to\infty} x(t) = x^r := -A_c^{-1}B_cu^r \in X$;
   ii) $\lim_{t\to\infty} y(t) = r$.

b) If $\{u_n\}$ and $\{\gamma_n\}$ are obtained by applying (31) to (52), and $u(\cdot)$, defined by (47), is applied to (44) and (46), then for all $(x_0, u_0) \in X \times \mathbb{R}^m$ there exists $M, \omega > 0$ such that:
   i) $\|x(t) - x^r\|_X \le Me^{-\omega t}$;
   ii) $\|r - y(t)\| \le Me^{-\omega t}$.

Proof:

(a) Using Theorem 3.2 and Proposition 4.1, we see that $\lim_{n\to\infty} u_n = u^r = \mathbf{G}_c(0)^{-1}r$, and $\{u_n - u^r\} \in l_+^2(\mathbb{R}^m)$. Applying the corresponding $u(\cdot)$ obtained from $\{u_n\}$ via (44), we have that for each $n \in \mathbb{N}$ and all $t \in [n\tau, (n+1)\tau)$

$$\begin{aligned} x(t) &= T(t)x_0 + T(t - n\tau)[T(\tau) - I] \\ &\quad \cdot \sum_{j=0}^{n-1} T\left[(n - j - 1)\tau\right] A_c^{-1}B_c(u_j - u^r) \\ &\quad + \left[T(t - n\tau) - I\right] A_c^{-1}B_c(u_n - u^r) \\ &\quad + \left[T(t) - I\right] A_c^{-1}B_cu^r. \end{aligned} \tag{55}$$

Therefore i) follows from the exponential stability of $T(t)$ and the fact that $\{u_n - u^r\} \in l_+^2(\mathbb{R}^m)$. To prove ii), we use (46) so that

$$\lim_{t\to\infty} y(t) = -C_cA_c^{-1}B_cu^r + D_cu^r = r.$$

(b) In this case $u_n \to u^r$ exponentially, by Theorem 3.5. It follows that the right-hand side of (55) tends to $x^r = -A_c^{-1}B_cu^r$ exponentially. Statement ii) follows similarly. □

Unbounded Observation: In this case, we cannot define a sampled output via (50). Instead we first average the output over one sampling interval and define

$$y_n := \frac{1}{\tau}\int_0^\tau y\,(n\tau + t)\,dt. \tag{56}$$

Proposition 4.3: Suppose that the system given by (44) and (46) is exponentially stable and regular. Fix $\tau > 0$ as the sampling interval, and let $\{u_n\} \subset \mathbb{R}^m$. If $u(\cdot)$ is given by (47), then $\{x_n\}$ and $\{y_n\}$, given by (49) and (56), respectively, satisfy the discrete-time equations given by (52), where we have (57), as shown at the bottom of the page. Moreover, A is power stable and

$$\mathbf{G}(1) = C(I - A)^{-1}B + D = \mathbf{G}_c(0). \tag{58}$$

Proof: We already know from Proposition 4.1 that (52a) is true. Combining (48) and (46) gives

$$y(n\tau + t) = C_LT\,(t)\,x_n + C_LT\,(t)\,A_c^{-1}B_cu_n + \mathbf{G}_c\,(0)\,u_n \quad \text{for a.e. } t \in [0, \tau). \tag{59}$$

Let $\varepsilon > 0$, and choose $x_n^\varepsilon \in X_1$ and $b_n^\varepsilon \in X$ such that

$$\lim_{\varepsilon\to 0} \|x_n - x_n^\varepsilon\|_X = 0$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{Bmatrix} T(\tau) & [T(\tau) - I]\,A_c^{-1}B_c \\ \tau^{-1}C_L[T(\tau) - I]\,A_c^{-1} & \tau^{-1}C_L(T(\tau) - I)A_c^{-2}B_c + \mathbf{G}_c(0) \end{Bmatrix} \tag{57}$$

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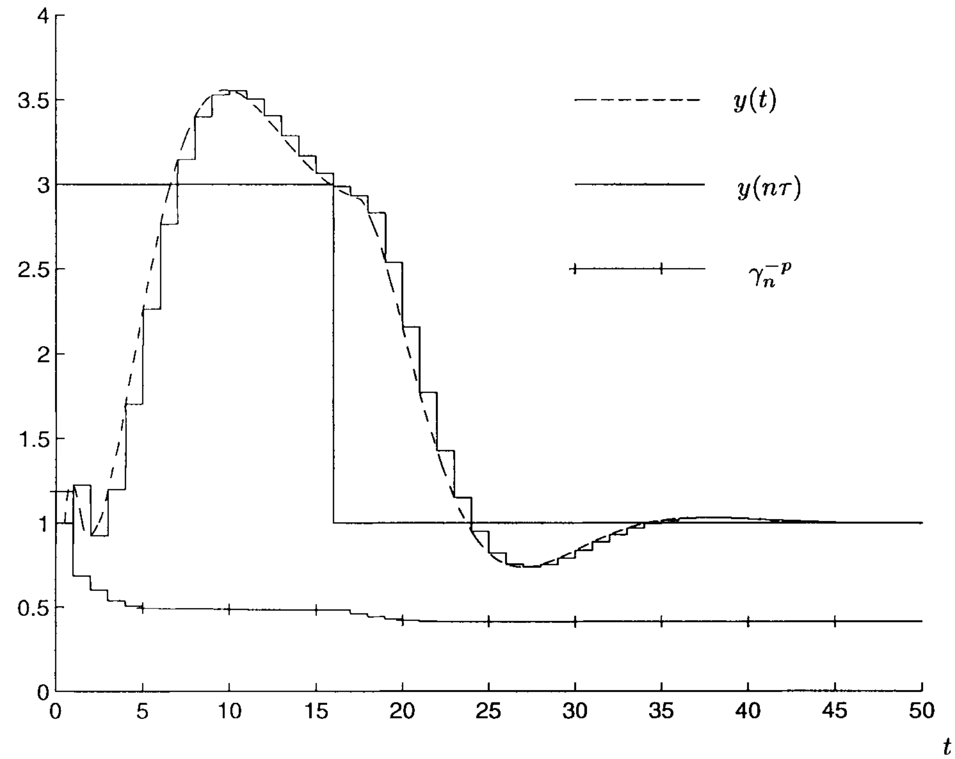

Fig. 2. Closed-loop plots with controller (26).

and

$$\lim_{\varepsilon\to 0} \|B_c u_n - b_n^\varepsilon\|_{X_{-1}} = 0.$$

Define

$$\begin{aligned} y^\varepsilon(n\tau + t) &:= C_L T(t) x_n^\varepsilon + C_L T(t) A_c^{-1} b_n^\varepsilon + \mathbf{G}_c(0) u_n \\ y_n^\varepsilon &:= \frac{1}{\tau}\int_0^\tau y^\varepsilon(n\tau + t)\, dt. \end{aligned} \tag{60}$$

Using the continuity of the state-to-output map [c.f. (45)] we see that

$$\lim_{\varepsilon\to 0} y_n^\varepsilon = y_n. \tag{61}$$

On the other hand, since $C_L \in \mathcal{B}(X_1, \mathbb{R}^m)$, we may take C_L out of the integral in (60) and hence obtain the following formula:

$$\begin{aligned} y_n^\varepsilon =& \tau^{-1} C_L [T(\tau) - I] A_c^{-1} x_n^\varepsilon \\ &+ \tau^{-1} C_L [T(\tau) - I] A_c^{-2} b_n^\varepsilon + \mathbf{G}_c(0) u_n. \end{aligned}$$

Now letting $\varepsilon \to 0$ and using (61) yields

$$\begin{aligned} y_n =& \tau^{-1} C_L [T(\tau) - I] A_c^{-1} x_n \\ &+ \left\{\tau^{-1} C_L [T(\tau) - I] A_c^{-2} B_c + \mathbf{G}_c(0)\right\} u_n \end{aligned} \tag{62}$$

which is (52b) with C and D given by (57). Since $T(t)$ is exponentially stable, A is power stable. Finally, to show (58), we use (57) to obtain

$$\begin{aligned} \mathbf{G}(1) =& \tau^{-1} C_L [T(\tau) - I] A_c^{-1} [I - T(\tau)]^{-1} [T(\tau) - I] A_c^{-1} B_c \\ &+ \left[\tau^{-1} C_L [T(\tau) - I] A_c^{-2} B_c + \mathbf{G}_c(0)\right] \\ =& \mathbf{G}_c(0). \end{aligned}$$

□

In the following let $\mathcal{L}$ denote the Laplace transform and let $\mathcal{A}$ denote the set of all distributions f of the form

$$g = g_a + \sum_{i=0}^{\infty} g_i \delta_{t_i}$$

where $g_a \in L^1(0, \infty)$, $t_i \geq 0$, δ_{t_i} denotes the unit point mass at t_i and $\{g_i\} \in l_+^1$. The set of all $m \times m$ matrices with entries in $\mathcal{A}$ is denoted by $\mathcal{A}^{m\times m}$.

We now combine Theorems 3.2 and 3.5 and Proposition 4.3 to prove the following result on adaptive low-gain sampled-data control for regular systems with unbounded observation.

Theorem 4.4: Let (44) and (46) describe a regular, exponentially stable system. Suppose that $\mathbf{G}_c(0) \succ 0$. Fix $r \in \mathbb{R}^m$ and set $e_n = r - y_n$, where y_n is given by (56).

a) If $\{u_n\}$ and $\{\gamma_n\}$ are obtained by applying (26) to (52), and $u(\cdot)$, defined by (47), is applied to (44) and (46), then for all $(x_0, u_0) \in X \times \mathbb{R}^m$:
   i) $\lim_{t\to\infty} x(t) = x^r := -A_c^{-1} B_c u^r \in X$;
   ii) $[r - y(\cdot)] \in L^2(0, \infty; \mathbb{R}^m)$.

   Under the extra assumptions that

   $$x_0 \in X_1 \quad \text{and} \quad \mathcal{L}^{-1}(\mathbf{G}_c) \in \mathcal{A}^{m\times m} \tag{63}$$

   we have $\lim_{t\to\infty}[r - y(t)] = 0$.

b) If $\{u_n\}$ and $\{\gamma_n\}$ are obtained by applying (31) to (52), and $u(\cdot)$, defined by (47), is applied to (44) and (46), then for $(x_0, u_0) \in X \times \mathbb{R}^m$:
   i) $\|x(t) - x^r\|_X \leq M e^{-\omega t}$ for some $M, \omega > 0$ and all $t \geq 0$;
   ii) $\int_0^\infty \|r - y(t)\|^2 e^{2\alpha t}\, dt < \infty$ for some $\alpha > 0$.

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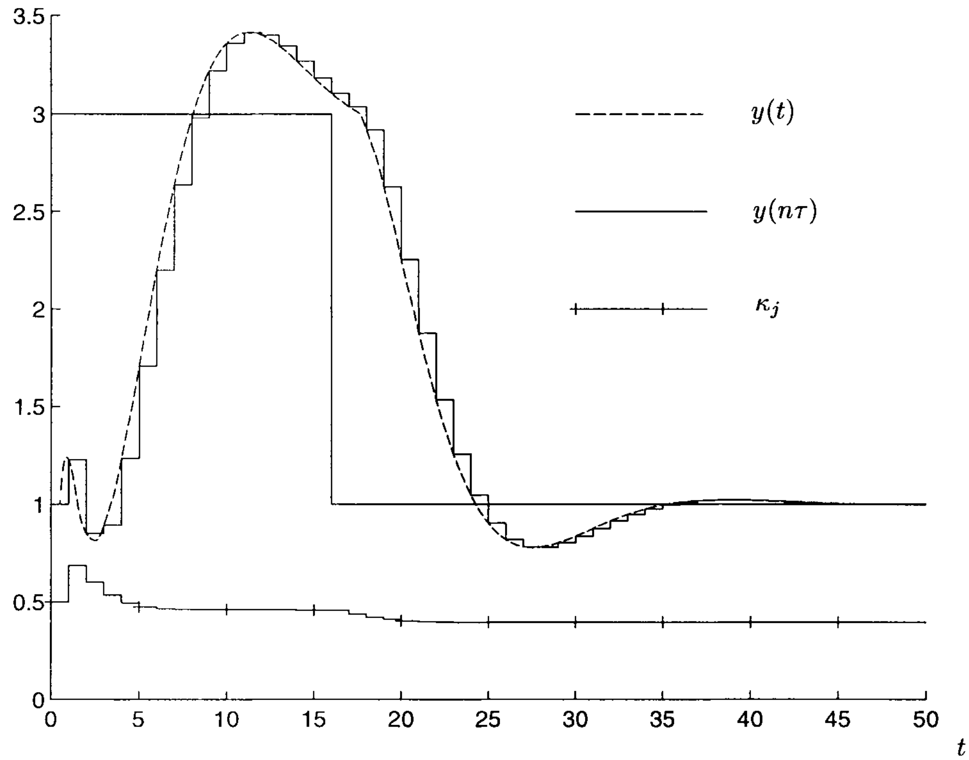

Fig. 3. Closed-loop plots with controller (31).

If the extra assumptions

$$x_0 \in X_1$$

and

$$\exp(\alpha\,\cdot)\mathcal{L}^{-1}(\mathbf{G}_c)(\cdot) \in \mathcal{A}^{m\times m} \quad \text{for some } \alpha > 0 \qquad (64)$$

are satisfied, then there exist $M,\,\omega > 0$ such that $\|r - y(t)\| \leq Me^{-\omega t}$ for all $t \geq 0$.

The assumption that $\mathcal{L}^{-1}(\mathbf{G}_c) \in \mathcal{A}^{m\times m}$ is not very restrictive and seems to be satisfied in all practical examples of systems with H^∞-transfer-function matrices.

Proof of Theorem 4.4:

a) Using Theorem 3.2 and Proposition 4.3 we know that $\lim_{n\to\infty} u_n = u^r = \mathbf{G}_c(0)^{-1}r$. Part i) is the same as that in Theorem 4.2. For part ii) note that for almost all $t \in [n\tau,\,(n+1)\tau)$

$$\begin{aligned} y(t) = &\, C_L T(t)x_0 + C_L T(t-n\tau)\,[T(\tau) - I] \\ &\cdot \sum_{j=0}^{n-1} T\,[(n-j-1)\tau]\,A_c^{-1}B_c(u_j - u^r) \\ &+ C_L\,[T(t-n\tau) - I]\,A_c^{-1}B_c(u_n - u^r) \\ &+ D_c(u_n - u^r) \\ &+ C_L T(t)A_c^{-1}B_c u^r + \mathbf{G}_c(0)u^r. \end{aligned}$$

Since $\mathbf{G}_c(0)u^r = r$, the result follows by using the fact that $\{u_n - u^r\} \in l^2_+(\mathbb{R}^m)$ and observing that $C_L T(\cdot)x_0 \in L^2(0,\,\infty;\mathbb{R}^m)$ for all $x_0 \in X$. If (63) is satisfied, then, by the exponential stability of $T(t)$ on X_1, we have that $\lim_{t\to\infty} C_L T(t)x_0 = 0$, and hence

$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{t\to\infty} \{C_c T(t)x_0 \\ &\qquad + [\mathcal{L}^{-1}(\mathbf{G}_c) \star u](t)\} \\ &= \lim_{t\to\infty} [\mathcal{L}^{-1}(\mathbf{G}_c) \star u](t) \end{aligned}$$

where $\star$ denotes continuous-time convolution applied component-wise. Since $\lim_{t\to\infty} u(t) = u^r$ and $\mathcal{L}^{-1}(\mathbf{G}_c) \in \mathcal{A}^{m\times m}$, it is easy to show that

$$\lim_{t\to\infty} [\mathcal{L}^{-1}(\mathbf{G}_c) \star u](t) = \mathbf{G}_c(0)u^r = r$$

and hence $\lim_{t\to\infty} y(t) = r$.

b) In this case $u_n \to u^r$ exponentially (by Theorem 3.5). It follows that the right-hand side of (55) tends to $x^r = -A_c^{-1}B_c u^r$ exponentially. Statement ii) follows similarly. Finally, suppose the extra assumptions (64) hold. Using the fact that $\mathcal{L}^{-1}(\mathbf{G}_c) = G\exp(-\alpha\,\cdot)$ for some $G \in \mathcal{A}^{m\times m}$ and some $\alpha > 0$, it is not difficult to show that there exist $M,\,\omega > 0$ such that

$$\begin{aligned} &\|r - [\mathcal{L}^{-1}(\mathbf{G}_c) \star u](t)\| \\ &\quad = \|\mathbf{G}_c(0)u^r - [\mathcal{L}^{-1}(\mathbf{G}_c) \star u](t)\| \\ &\quad \leq Me^{-\omega t} \end{aligned}$$

c.f. Callier and Winkin [1]. The claim now follows, since $C_L T(t)x_0$ converges to zero exponentially (by the assumption that $x_0 \in X_1$). □

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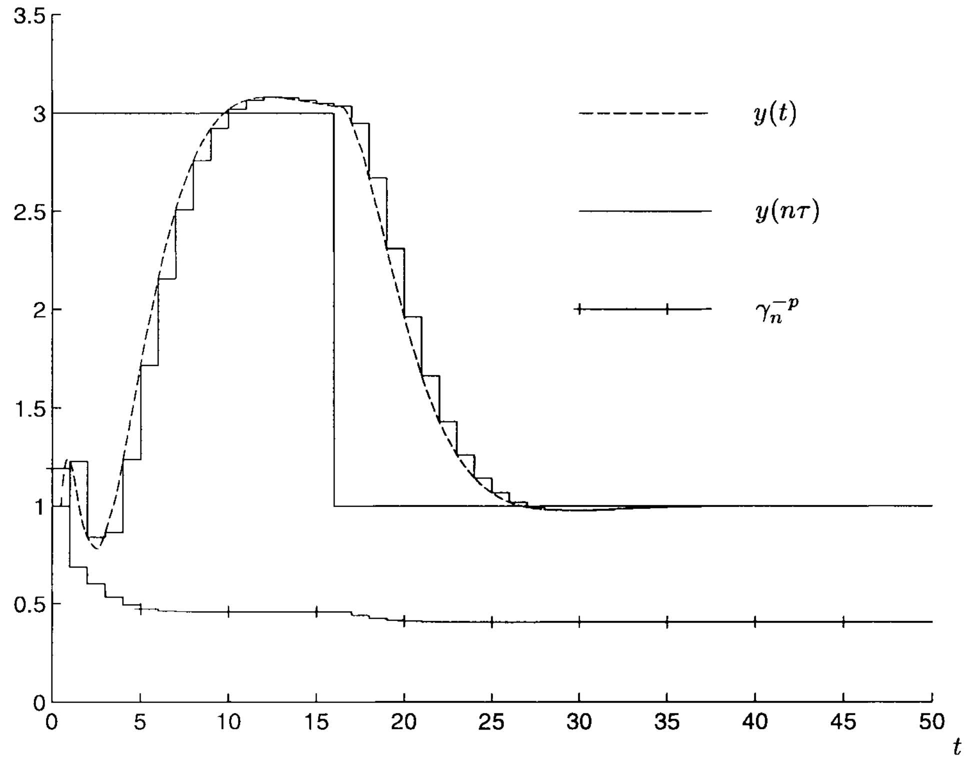

Fig. 4. Closed-loop plots with controller (32).

Remark 4.5: Theorems 4.2 and 4.4 have been obtained by applying Theorems 3.2 and 3.5 to the sampled system (52). In the same manner we may derive sampled-data versions of Theorems 3.6 and 3.7. □

V. Examples and Simulations

The results of Sections II and III apply to the general class of infinite-dimensional, discrete-time systems, whilst the results of Section IV apply to the general class of continuous-time regular linear systems. For the purpose of illustration we consider a simple example of an uncertain finite-dimensional system with output delay to which we apply the adaptive low-gain, sampled-data controllers of Section IV. In the simulations we used Matlab.

Example 5.1: We consider a class

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_c u(t) \\ y(t) &= C_c x(t-h) \end{aligned} \tag{65}$$

of systems with output delay. Here $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$, $C_c \in \mathbb{R}^{m\times n}$, and $h > 0$. The transfer function of (65) is given by $\mathbf{G}_c(s) = e^{-hs}C_c(sI - A_c)^{-1}B_c$. The system (65) can be represented as a Pritchard–Salamon system with state space $\mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n)$; see, e.g., Pritchard and Salamon [31], [32]. As remarked in Section IV, Pritchard–Salamon systems can be represented as regular systems with bounded observation operators. It follows from the results of Section IV that (65) can be sampled using Proposition 4.1 to obtain a discrete-time system. The results of Sections II–IV, and, in particular, the discrete-time, adaptive low-gain results, can be applied to the discrete-time system obtained by sampling, provided that $\sigma(A_c) \cap \mathbb{C}_0^{cl} = \emptyset$ and $\det(C_c A_c^{-1} B_c) \neq 0$. To use the results of Sections III and IV we must assume that $C_c A_c^{-1} B_c$ is sign-definite.

To be specific, let $m = 1$, $n = 2$, and

$$A_c = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad B_c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_c = (1, 0).$$

In this case $C_c A_c^{-1} B_c < 0$ so that $\mathbf{G}(1) = \mathbf{G}_c(0) = -C_c A_c^{-1} B_c > 0$, where $\mathbf{G}$ denotes the transfer function of the corresponding discretization.

In the simulations we set

$$\begin{aligned} &h = 0.5, \ x(t) = (1, 1.5)^T \text{ for all } t \in [-h, 0], \\ &\tau = 1, u_0 = 0.3, \gamma_0 = 0.5, p = 0.25 \end{aligned}$$

and

$$r(t) = \begin{cases} 3 & \text{if } t < 15 \\ 1 & \text{if } t \geq 15. \end{cases}$$

In Fig. 2 we show plots of $\{y(n\tau)\}$, $y(\cdot)$, and $\{\gamma_n^{-p}\}$ for the closed-loop system obtained by using (26), applied to the sampled version of (65), with $y(\cdot)$ the response of (65) to the control input computed using the corresponding $\{u_n\}$, and $u(\cdot)$ given by (47); see Theorem 4.2.

In Fig. 3 we show the same closed-loop plots but with (26) replaced by (31) and κ_j plotted instead of γ_n^{-p}. Recall that this controller guarantees that the error $e(t)$ converges to zero exponentially. Both of these controllers use the discrete-time integrator $1/(z-1)$.

In Figs. 4 and 5 we show the same closed-loop plots but with (26) and (31) replaced by (32) and (33), respectively,

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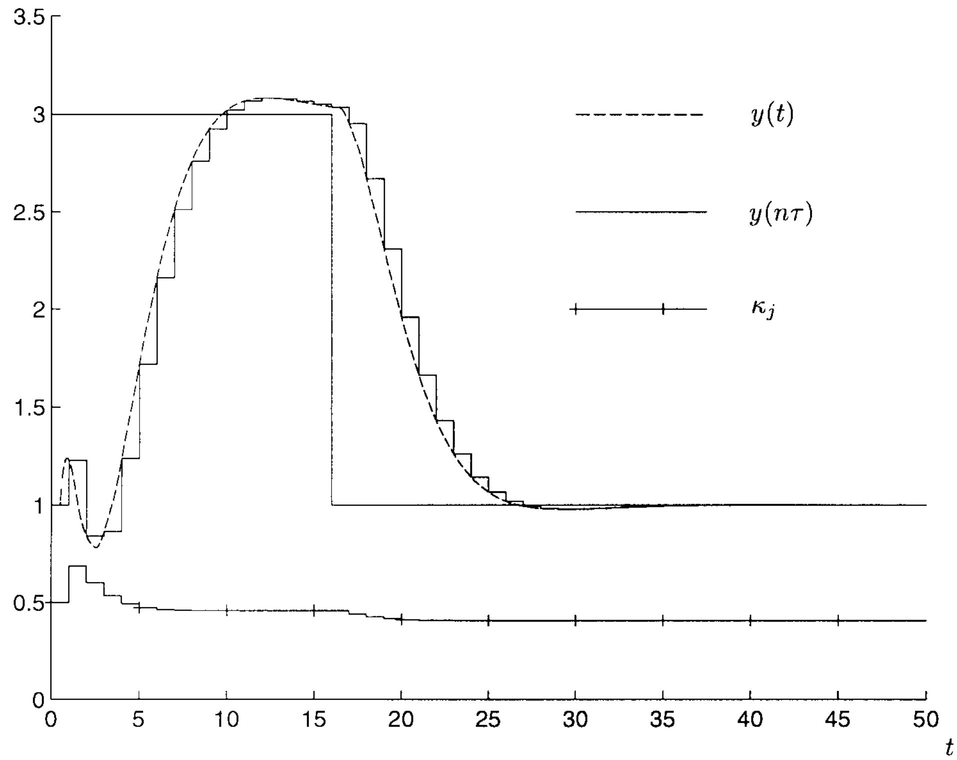

Fig. 5. Closed-loop plots with controller (33).

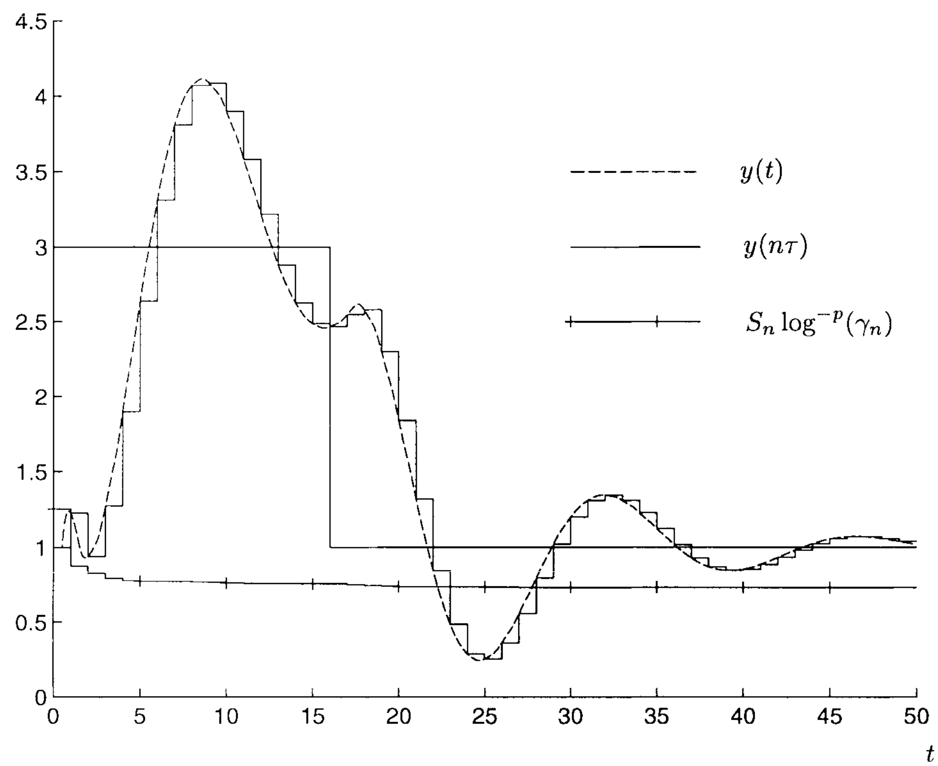

Fig. 6. Closed-loop plots with controller (36)–(38) and $S_0 = +1$.

that is, when we use the discrete-time integrator $z/(z-1)$. As expected, due to the proportional part in $z/(z-1)$, the closed-loop responses shown in Figs. 4 and 5 have smaller overshoot than those in Figs. 2 and 3.

In Figs. 6 and 7 we show the closed-loop plots for the controller given by (36)–(38) with $\gamma_0 = 1.5$, where $S_0 = +1$ and $S_0 = -1$, respectively. This controller is appropriate if we do not know the sign of $\mathbf{G}_c(0)$. Notice that the closed-loop

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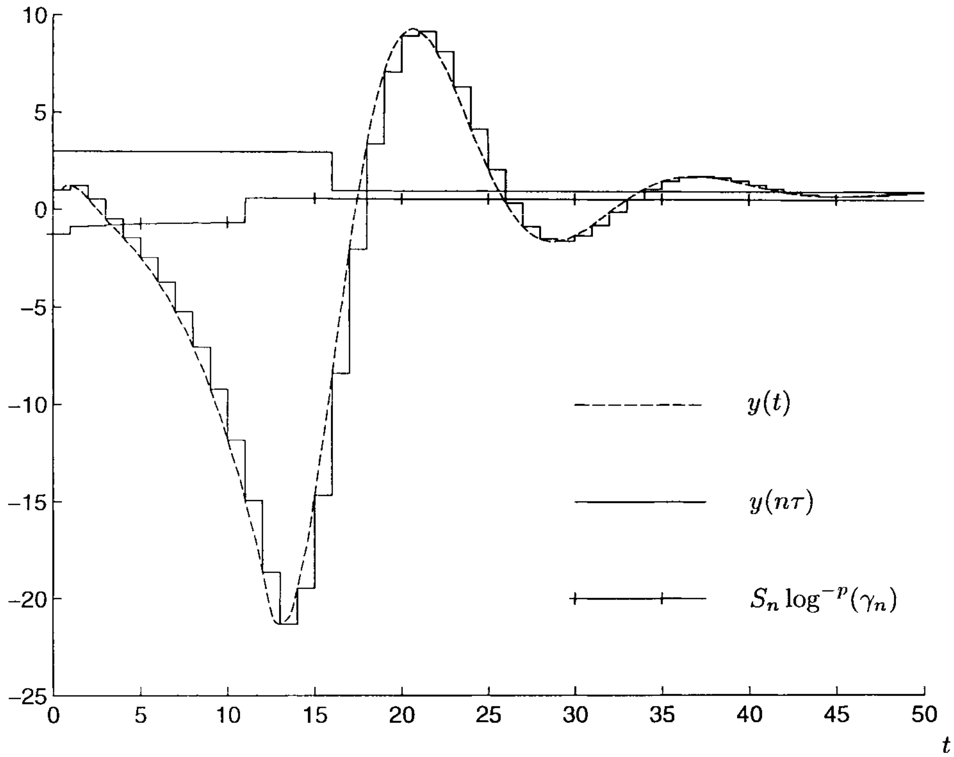

Fig. 7. Closed-loop plots with controller (36)–(38) and $S_0 = -1$.

response in the case when $S_0 = -1$ is poor. This corresponds to the case when the controller started with the incorrect sign, producing an unstable closed-loop system until the controller switches to the correct sign at approximately 13 s, after which the closed-loop system is stable and the error $e(t)$ converges to zero. □

VI. Concluding Remarks

We have considered nonadaptive and adaptive low-gain control for linear infinite-dimensional discrete-time systems. In particular, for power-stable systems with sign-definite, steady-state gain we have derived explicit techniques for tuning the integral gain in the low-gain controller. We have applied our results to the adaptive, sampled-data, low-gain control of regular linear (infinite-dimensional continuous-time) systems. Our results are illustrated by simulations for a finite-dimensional unknown (but stable) system with fixed but unknown output delay.

Several problems still require further work. For example, can our approach tolerate input–output nonlinearities such as saturation, deadzone, and hysteresis. Preliminary works suggest that this is the case, at least for nonadaptive low-gain control.

A more challenging problem would be to show that the sufficient condition for nonadaptive low-gain control, namely $\sigma[\mathbf{G}(1)] \subset \mathbb{C}_0$ (the steady-state gain matrix has unmixed spectrum), is also sufficient for adaptive-low gain control via techniques which make explicit use of the low-gain nature of the problem. We note that adaptive controllers based on the dense searching techniques of Logemann and Mårtensson [17] could be used in this context.

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