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LOW-GAIN CONTROL OF UNCERTAIN REGULAR LINEAR SYSTEMS*

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Abstract. It is well known that closing the loop around an exponentially stable, finite-dimensional, linear, time-invariant plant with square transfer-function matrix $\mathbf{G}(s)$ compensated by a controller of the form $(k/s)\Gamma_0$, where $k \in \mathbb{R}$ and $\Gamma_0 \in \mathbb{R}^{m\times m}$, will result in an exponentially stable closed-loop system which achieves tracking of arbitrary constant reference signals, provided that (i) all the eigenvalues of $\mathbf{G}(0)\Gamma_0$ have positive real parts and (ii) the gain parameter k is positive and sufficiently small.

In this paper we consider a rather general class of infinite-dimensional linear systems, called regular systems, for which convenient representations are known to exist, both in time and in frequency domain. The purpose of the paper is twofold: (i) we extend the above result to the class of exponentially stable regular systems and (ii) we show how the parameters k and Γ_0 can be tuned adaptively. The resulting adaptive tracking controllers are not based on system identification or parameter estimation algorithms, nor is the injection of probing signals required.

Key words. regular infinite-dimensional systems, integral controllability, robust tracking, adaptive tracking, state-space methods, frequency-domain methods

AMS subject classifications. 93C20, 93C25, 93C40, 93D09, 93D15, 93D21, 93D25

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1. Introduction.

The synthesis of low-gain I and PI-controllers for uncertain stable plants has received considerable attention in the past 20 years. Let $\mathbf{G}$ be a stable proper rational transfer function matrix. The main existence result on robust low-gain I-control says that for any matrix Γ_0 satisfying

$$\text{spectrum}(\mathbf{G}(0)\Gamma_0) \subset \{s \in \mathbb{C} \,|\, \operatorname{Re} s > 0\}\,, \tag{1.1}$$

there exists $k^* > 0$ such that for all $k \in (0, k^*)$ the controller $(1/s)k\Gamma_0$ stabilizes $\mathbf{G}$ and the resulting closed-loop system asymptotically tracks arbitrary constant reference signals. This result has been proved by Davison [4][1] and Lunze [18] using state-space methods and by Grosdidier, Morari, and Holt [5] and Morari [25] using frequency-domain methods (see also the book by Lunze [20, Chapter 10], and the textbook by Morari and Zafiriou [26, p. 362]). There are consequently two parts to the design of low-gain tracking controllers: choosing Γ_0 and tuning k. Such a controller design approach, called "tuning regulator theory" [4], has been successfully applied to industrial control problems; see Coppus, Sha, and Wood [2] and Lunze [19].

In the case that $\mathbf{G}$ is square, $\mathbf{G}^{-1}(0)$ would be a natural choice for Γ_0, but in the presence of uncertainty, $\mathbf{G}(0)$ might not be known exactly. However, an estimate

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[1]In [4] the result is proven for the special choice $\Gamma_0 = \mathbf{G}^{-1}(0)$. However, an inspection of the Lyapunov argument in the proof of lemma 3 in [4] shows that it can be easily extended to the more general case when Γ_0 satisfies (1.1) (simply replace the identity I in equation (28) in [4] by N, where N is the positive definite solution of the Lyapunov equation $(\mathbf{G}(0)\Gamma_0)^T N + N(\mathbf{G}(0)\Gamma_0) = -I$).

G_0 of $\mathbf{G}(0)$ can be obtained, in principle, by performing step response experiments on the plant. In this case the matrix Γ_0 is then chosen such that (1.1) holds with $\mathbf{G}(0)$ replaced by G_0. Although Mustafa [28] has recently derived a formula for the maximal k^* in terms of a minimal realization (A, B, C, D) of $\mathbf{G}$, in the presence of uncertainty there are only crude methods available for determining a number $k^* > 0$ such that all gain parameters $k \in (0, k^*)$ will lead to a stable closed-loop system; see, e.g., Lunze [18] and Owens and Chotai [29]. Methods for tuning Γ_0 and k by means of experiments and simulation have been developed and discussed in many places; we mention only [4], [18], [20], [29], and the paper by Penttinen and Koivo [31].

The above-mentioned tuning regulator result has been extended by Pohjolainen [32], [33], Pohjolainen and Lätti [34], Logemann and Owens [15] and Logemann, Bontsema, and Owens [11] to various classes of (abstract) infinite-dimensional systems and by Koivo and Pohjolainen [9] and Jussila and Koivo [8] to differential delay systems.

If the plant uncertainty is large and/or if reliable plant step data are not available, then the parameters k and Γ_0 need to be tuned adaptively. It turns out that, once the tuning problem for k is solved, the tuning of Γ_0 can be achieved by applying the spectrum unmixing techniques used in multivariable high-gain adaptive stabilization, Mårtensson [21], [22]. Low-gain universal adaptive controllers which achieve asymptotic tracking of constant reference signals for finite-dimensional linear stable plants have been presented by Cook [1] and Miller and Davison [23], [24].[2] By "universal" we mean that the controllers are not based on system identification or parameter estimation algorithms. The controller given in [1] is smooth, while the control laws derived in [23], [24] are "piecewise constant." The controller given in [24] satisfies a control input constraint.

In this paper we consider the problem of low-gain I-control for the class of exponentially stable, linear, regular infinite-dimensional systems introduced and studied by Weiss; see [44], [45], [46], [47], [48], [49]. This class is rather general and includes all distributed parameter systems and all time-delay systems (retarded and neutral) which are of interest in applications. In particular, it includes the classes of infinite-dimensional systems considered in the references [8], [9], [15], [11], [32], [33], [34] mentioned earlier and the well-known class of Pritchard–Salamon systems; see Pritchard and Salamon [35], [36] and Curtain et al. [3]. Although there exist well-posed infinite-dimensional systems which are not regular, the authors believe that any physically motivated well-posed linear time-invariant control system is regular.

In section 2 we provide the necessary background on regular systems which will be needed in sections 3–5. With one exception, all the results in section 2 are due to Weiss [44], [45], [46], [47], [48], [49], the exception being a nonlinear existence result which is required for adaptive low-gain control. The proof of this result is relegated to an appendix.

Section 3 is devoted to nonadaptive low-gain control of regular systems. We first prove a frequency-domain result on the existence of low-gain tuning regulators of the form $(1/s)k\Gamma_0$ for all square transfer function matrices $\mathbf{G}$ which are holomorphic and bounded on some right-half plane $\operatorname{Re} s > \alpha$ for some $\alpha = \alpha(\mathbf{G}) < 0$ and satisfy $\det \mathbf{G}(0) \neq 0$. This result is then applied to regular state-space systems, and it is shown that for all sufficiently small k the closed-loop system will achieve asymptotic

[2] Surprisingly, the low-gain adaptive tracking problem has received less attention than its high-gain counterpart; see Ilchmann [7], Logemann and Ilchmann [12], Ryan [38], and the references therein.

tracking of constant reference signals, provided that the initial state of the open-loop system is sufficiently "smooth."

In sections 4 and 5 we consider the adaptive low-gain tracking problem for regular infinite-dimensional systems. While the problem of universal adaptive stabilization for infinite-dimensional systems has received some attention in recent years (see Logemann [10], Logemann and Mårtensson [13], Logemann and Owens [14], Logemann and Zwart [17], and Townley [41]), very little work has been done on adaptive tracking (see, however, the paper by Logemann and Ilchmann [12] on a high-gain adaptive servomechanism for a class of infinite-dimensional systems). In particular, it seems that so far no research has been carried out on the adaptive low-gain control problem in an infinite-dimensional setting. We mention that the main result in Cook [1] (at least as we understand it) relies on the Kalman–Yakubovich lemma. A straightforward extension of the approach in [1] to regular infinite-dimensional systems is not possible, since the existence of an appropriate infinite-dimensional version of the Kalman–Yakubovich lemma is a difficult open problem. The (discontinuous) piecewise constant controllers presented in Miller and Davison [23], [24] seem unnecessarily complicated and would not generalize to the infinite-dimensional case either. Section 4 is restricted to the case when the steady-state gain matrix $\mathbf{G}(0)$ is sign definite; i.e., $\mathbf{G}(0)$ is either positive or negative definite. We first give an alternative proof of the finite-dimensional result obtained by Cook [1]. Our proof illustrates certain special system theoretic properties of the low-gain problem, properties which can even be exploited in the infinite-dimensional case. The basic idea in [1] is to set the integrator gain k equal to $\mathcal{K}(\gamma)$, where $\mathcal{K}$ is a function, the so-called tuning function, and γ is a parameter which is adjusted by a suitable adaptation law. The class of tuning functions $\mathcal{K}$ given in [1] exploits the low-gain nature of the problem in the sense that $\mathcal{K}(\gamma) \to 0$ as $\gamma \to \infty$. We then prove the main result in section 4, a low-gain adaptive tuning regulator result for infinite-dimensional regular systems. The choice of tuning functions is more constrained than in the finite-dimensional case, although we can still work with functions $\mathcal{K}$ satisfying that $\mathcal{K}(\gamma) \to 0$ as $\gamma \to \infty$. In the sign-indefinite case, which is treated in section 5, we have to resort to tuning functions which oscillate smoothly between 0 and an arbitary positive number.

We illustrate our results by a number of examples and simulations in section 6.

Notation.

- For $\alpha \in \mathbb{R}$ set $\mathbb{C}_\alpha := \{s \in \mathbb{C} \,|\, \operatorname{Re} s > \alpha\}$.
- For $\alpha \in \mathbb{R}$ and H a Hilbert space we define the exponentially weighted L^2-space $L^2_\alpha(\mathbb{R}_+, H) := \{f \in L^2_{loc}(\mathbb{R}_+, H) \,|\, f(\cdot)\exp(-\alpha\,\cdot) \in L^2(\mathbb{R}_+, H)\}$.
- If A is a linear operator, then the domain, spectrum, and resolvent set of A are denoted by $D(A)$, $\sigma(A)$, and $\varrho(A)$, respectively.
- The Laplace transform is denoted by $\mathbb{L}$.

2. Preliminaries on abstract linear systems. In this section we give some background on abstract linear systems. Apart from Proposition 2.4 almost all the results are due to Weiss [44], [45], [46], [47], [48], [49].

First we introduce some notation. For any Hilbert space H and any $\tau \geq 0$, $\mathbf{R}_\tau$ and $\mathbf{L}_\tau$ will denote the right-shift by τ and the left-shift by τ on $L^2_{loc}(\mathbb{R}_+, H)$, respectively. The truncation operator $\mathbf{P}_\tau : L^2_{loc}(\mathbb{R}_+, H) \to L^2(\mathbb{R}_+, H)$ is given by

$$(\mathbf{P}_\tau u)(t) = \begin{cases} u(t) & \text{if} \quad t \in [0, \tau], \\ 0 & \text{if} \quad t > \tau. \end{cases}$$

For $u, v \in L^2_{loc}(\mathbb{R}_+, H)$ and $\tau \geq 0$, the *τ-concatenation* $u \overset{\tau}{\diamondsuit} v$ is defined by

$$u \overset{\tau}{\diamondsuit} v = \mathbf{P}_\tau u + \mathbf{R}_\tau v\,.$$

The following concept was introduced by Weiss [46]. An equivalent definition can be found in Salamon [39].

DEFINITION 2.1. *Let U, X, and Y be real Hilbert spaces. An* abstract linear system *with* state-space X, input-space U, *and* output-space Y *is a quadruple* $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$, *where*

(i) $\mathbf{T} = (\mathbf{T}_t)_{t \geq 0}$ *is a C_0-semigroup of bounded linear operators on X;*

(ii) $\mathbf{\Phi} = (\mathbf{\Phi}_t)_{t \geq 0}$ *is a family of bounded linear operators from $L^2(\mathbb{R}_+, U)$ to X such that*

$$\mathbf{\Phi}_{\tau+t}(u \overset{\tau}{\diamondsuit} v) = \mathbf{T}_t \mathbf{\Phi}_\tau u + \mathbf{\Phi}_t v$$

for all $u, v \in L^2(\mathbb{R}_+, U)$ and all $\tau, t \geq 0$;

(iii) $\mathbf{\Psi} = (\mathbf{\Psi}_t)_{t \geq 0}$ *is a family of bounded linear operators from X to $L^2(\mathbb{R}_+, Y)$ such that*

$$\mathbf{\Psi}_{\tau+t} x_0 = \mathbf{\Psi}_\tau x_0 \overset{\tau}{\diamondsuit} \mathbf{\Psi}_t \mathbf{T}_\tau x_0$$

for all $x_0 \in X$ and all $\tau, t \geq 0$, and $\mathbf{\Psi}_0 = 0$;

(iv) $\mathbf{F} = (\mathbf{F}_t)_{t \geq 0}$ *is a family of bounded linear operators from $L^2(\mathbb{R}_+, U)$ to $L^2(\mathbb{R}_+, Y)$ such that*

$$\mathbf{F}_{\tau+t}(u \overset{\tau}{\diamondsuit} v) = \mathbf{F}_\tau u \overset{\tau}{\diamondsuit} (\mathbf{\Psi}_t \mathbf{\Phi}_\tau u + \mathbf{F}_t v)\,,$$

$u, v \in L^2(\mathbb{R}_+, U)$ and all $\tau, t \geq 0$, and $\mathbf{F}_0 = 0$.

It follows easily from the definition that $\mathbf{\Phi}_0 = 0$ and that for any $\tau \geq 0$, $x_0 \in X$, and $u \in L^2_{loc}(\mathbb{R}_+, U)$

$$(\mathbf{\Psi}_\tau x_0)(t) = (\mathbf{F}_\tau u)(t) = 0 \quad \text{for a.e. } t \geq \tau\,.$$

Let an input $u \in L^2_{loc}(\mathbb{R}_+, U)$ and an initial state $x_0 \in X$ be given. The state $x(t) = x(t; x_0, u)$ of $\mathbf{\Sigma}$ at time $t \geq 0$ and the output $y(\cdot) = y(\cdot\,; x_0, u)$ of $\mathbf{\Sigma}$ are defined by

$$x(t) = \mathbf{T}_t x_0 + \mathbf{\Phi}_t \mathbf{P}_t u\,, \tag{2.1a}$$

$$\mathbf{P}_t y = \mathbf{\Psi}_t x_0 + \mathbf{F}_t \mathbf{P}_t u\,. \tag{2.1b}$$

The state trajectory $x(\cdot)$ is continuous from $\mathbb{R}_+ \to X$, and the output $y(\cdot)$ is in $L^2_{loc}(\mathbb{R}_+, Y)$. Furthermore, if $t \geq \tau \geq 0$, then the functions $x(\cdot)$ and $y(\cdot)$ defined by (2.1) satisfy

$$x(t) = \mathbf{T}_{t-\tau} x(\tau) + \mathbf{\Phi}_{t-\tau} \mathbf{L}_\tau \mathbf{P}_t u\,, \tag{2.2a}$$

$$\mathbf{L}_\tau \mathbf{P}_t y = \mathbf{\Psi}_{t-\tau} x(\tau) + \mathbf{F}_{t-\tau} \mathbf{L}_\tau \mathbf{P}_t u\,. \tag{2.2b}$$

The equations (2.2) express the *time-invariance* of $\mathbf{\Sigma}$. They follow in a straightforward way from Definition 2.1. We say that $\mathbf{\Sigma}$ is *exponentially stable* if the semigroup $\mathbf{T}$ is exponentially stable, i.e.,

$$\omega(\mathbf{T}) := \lim_{t \to \infty} \frac{1}{t} \log \|\mathbf{T}_t\| < 0\,.$$

It is clear that there exist unique operators $\mathbf{\Psi}_\infty : X \to L^2_{loc}(\mathbb{R}_+, Y)$ and $\mathbf{F}_\infty : L^2_{loc}(\mathbb{R}_+, U) \to L^2_{loc}(\mathbb{R}_+, Y)$ such that for all $\tau \geq 0$

$$\mathbf{\Psi}_\tau = \mathbf{P}_\tau \mathbf{\Psi}_\infty \,, \quad \mathbf{F}_\tau = \mathbf{P}_\tau \mathbf{F}_\infty \,.$$

The generator of $\mathbf{T}$ is denoted by A. Let X_1 be the space $D(A)$ endowed with the graph norm, and let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1} = \|(\lambda I - A)^{-1})x\|$, where $\lambda \in \varrho(A)$ is fixed. We have $X_1 \subset X \subset X_{-1}$ and the canonical injections are bounded and dense. The semigroup $\mathbf{T}$ can be restricted to a C_0-semigroup on X_1 and extended to a C_0-semigroup on X_{-1}. The exponential growth constant is the same on all three spaces. The generator on X_1 is the restriction of A to $D(A^2)$, and the generator on X_{-1} is an extension of A to X (which is bounded as an operator from X to X_{-1}). We shall use the same symbols for the original semigroup and its generator and the corresponding restrictions and extensions.

By a representation theorem due to Salamon [39] (see also Weiss [44], [45]) there exist unique operators $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$ (the *control operator* and the *observation operator* of $\mathbf{\Sigma}$, respectively) such that for all $t \geq 0$, all $u \in L^2_{loc}(\mathbb{R}_+, U)$, and all $x_0 \in X_1$

$$\mathbf{\Phi}_t \mathbf{P}_t u = \int_0^t \mathbf{T}_{t-\xi} B u(\xi)\, d\xi \quad \text{and} \quad (\mathbf{\Psi}_\infty x_0)(t) = C \mathbf{T}_t x_0 \ .$$

B is called *bounded* if $B \in \mathcal{L}(U, X)$ (and *unbounded* otherwise), whereas C is called *bounded* if it can be extended continuously to X (and *unbounded* otherwise).

The *Lebesgue extension* of C was introduced in [45] and is defined by

$$C_L x_0 = \lim_{t \to 0} C \frac{1}{t} \int_0^t \mathbf{T}_\xi x_0 \, d\xi \,,$$

where $D(C_L)$ is equal to the set of all those $x_0 \in X$ for which the above limit exists. Clearly $X_1 \subset D(C_L) \subset X$, and for any $x_0 \in X$ we have that $\mathbf{T}_t x_0 \in D(C_L)$ for almost every $t \geq 0$. Furthermore,

$$(\mathbf{\Psi}_\infty x_0)(t) = C_L \mathbf{T}_t x_0 \quad \text{for a.e. } t \geq 0 \,.$$

Let Ω be a subset of $\mathbb{C}$. A function $\mathbf{H} : \Omega \to \mathcal{L}(U, Y)$ is called *well posed* if there exists $\alpha \in \mathbb{R}$ such that $\mathbb{C}_\alpha \subset \Omega$ and $\mathbf{H}$ is holomorphic and bounded on $\mathbb{C}_\alpha$. It can be shown (see Weiss [47]) that if $\alpha > \omega(\mathbf{T})$ and if $u \in L^2_\alpha(\mathbb{R}_+, U)$, then $\mathbf{F}_\infty u \in L^2_\alpha(\mathbb{R}_+, Y)$ and there exists a unique well-posed function $\mathbf{G} : \mathbb{C}_{\omega(\mathbf{T})} \to \mathcal{L}(U, Y)$ such that

$$\mathbf{G}(s)(\mathbb{L}u)(s) = [\mathbb{L}(\mathbf{F}_\infty u)](s) \quad \forall\, s \in \mathbb{C}_\alpha \,.$$

In particular, $\mathbf{G}$ is holomorphic on $\mathbb{C}_{\omega(\mathbf{T})}$ and bounded on $\mathbb{C}_\alpha$ for all $\alpha > \omega(\mathbf{T})$. The function $\mathbf{G}$ is called the *transfer function* of $\mathbf{\Sigma}$. Conversely, due to a result by Salamon [39], any well-posed function can be realized by an abstract linear system in the sense of Definition 2.1.

The following lemma will be needed in section 3. Certainly, it should be well known. However, since we could not find it in the literature, we include the proof.

LEMMA 2.2. *Suppose that* $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ *is exponentially stable. For any* $x_0 \in X$ *and any* $u \in L^2(\mathbb{R}_+, U)$*, the functions* $x(\cdot)$ *and* $y(\cdot)$ *defined by* (2.1) *satisfy*

$$x \in L^2(\mathbb{R}_+, X)\,, \quad y \in L^2(\mathbb{R}_+, Y)\,.$$

Proof. Since $x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\xi} Bu(\xi)\, d\xi$, it follows from the exponential stability of $\mathbf{T}$ that $x \in L^2(\mathbb{R}_+, X)$ if and only if the function $\bar{x} : t \mapsto \int_0^t \mathbf{T}_{t-\xi} Bu(\xi)\, d\xi$ is in $L^2(\mathbb{R}_+, X)$. Let $H^2(\mathbb{C}_0, X)$ denote the usual Hardy space of holomorphic functions defined on $\mathbb{C}_0$ with values in X. Appealing to the Paley–Wiener theorem, it follows that $\bar{x} \in L^2(\mathbb{R}_+, X)$ if we can show that $\mathbb{L}\bar{x} \in H^2(\mathbb{C}_0, X)$. To this end set $\omega_0 := \omega(\mathbf{T})$ and recall from [48] that for any $\omega > \omega_0$ there exists $M_\omega > 0$ such that

$$\|(sI - A)^{-1} B\|_{\mathcal{L}(U,X)} \leq \frac{M_\omega}{\sqrt{\operatorname{Re} s - \omega}} \quad \forall\, s \in \mathbb{C}_\omega\,. \tag{2.3}$$

(In particular, $(sI - A)^{-1} B \in \mathcal{L}(U, X)$ for all $s \in \mathbb{C}_\omega$.) Moreover, it is routine to show that the function

$$\mathbb{C}_{\omega_0} \to \mathcal{L}(U, Y)\,, \quad s \mapsto (sI - A)^{-1} B$$

is holomorphic. Finally, the Laplace transform of $\bar{x}$ is given by

$$(\mathbb{L}\bar{x})(s) = (sI - A)^{-1} B (\mathbb{L}u)(s) \quad \forall\, s \in \mathbb{C}_{\omega_0}\,, \tag{2.4}$$

and by hypothesis, $\omega_0 < 0$ and $\mathbb{L}u \in H^2(\mathbb{C}_0, X)$. Therefore, combining (2.3) and (2.4) we obtain that $\mathbb{L}\bar{x} \in H^2(\mathbb{C}_0, X)$.

In order to prove that $y \in L^2(\mathbb{R}_+, Y)$, write y in the form

$$y = \mathbf{\Psi}_\infty x_0 + \mathbf{F}_\infty u.$$

Using the remarks preceding the lemma, it follows from the hypothesis that $\mathbf{F}_\infty u \in L^2(\mathbb{R}_+, Y)$. It remains to show that $\mathbf{\Psi}_\infty x_0 \in L^2(\mathbb{R}_+, Y)$. By the exponential stability of $\mathbf{T}$ it follows in a straightforward way from condition (iii) in Definition 2.1 that there exists a constant $\gamma > 0$ such that

$$\|\mathbf{\Psi}_\tau x_0\|_{L^2(\mathbb{R}_+, Y)} \leq \gamma \|x_0\| \quad \forall\, \tau \geq 0 \;\; \forall\, x_0 \in X\,.$$

Hence

$$\|\mathbf{P}_\tau \mathbf{\Psi}_\infty x_0\|_{L^2(\mathbb{R}_+, Y)} = \|\mathbf{\Psi}_\tau x_0\|_{L^2(\mathbb{R}_+, Y)} \leq \gamma \|x_0\| \quad \forall\, \tau \geq 0\,,\ \forall\, x_0 \in X\,,$$

which implies that $\mathbf{\Psi}_\infty x_0 \in L^2(\mathbb{R}_+, Y)$. ∎

$\mathbf{\Sigma}$ and its transfer function $\mathbf{G}$ are called *regular* if for any $u \in U$ the limit

$$\lim_{s \to \infty,\, s \in \mathbb{R}} \mathbf{G}(s) u = Du$$

exists. It follows from the principle of uniform boundedness that $D \in \mathcal{L}(U, Y)$. The operator D is called the *feedthrough operator* of $\mathbf{\Sigma}$. If $\mathbf{\Sigma}$ is regular, then for any $x_0 \in X$ and $u \in L^2_{loc}(\mathbb{R}_+, U)$, the functions $x(\cdot)$ and $y(\cdot)$, defined by (2.1), satisfy the equations

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \tag{2.5a}$$

$$y(t) = C_L x(t) + Du(t) \tag{2.5b}$$

for a.e. $t \geq 0$ (in particular $x(t) \in D(C_L)$ for a.e. $t \geq 0$). The derivative on the left-hand side of (2.5a) has of course to be understood in X_{-1}. Moreover, as has been shown in [47], if $\mathbf{\Sigma}$ is regular, then $(sI - A)^{-1} BU \subset D(C_L)$ for all $s \in \varrho(A)$ and the transfer function $\mathbf{G}$ can be expressed in the following way:

$$\mathbf{G}(s) = C_L (sI - A)^{-1} B + D \quad \forall\, s \in \mathbb{C}_{\omega(\mathbf{T})}\,,$$

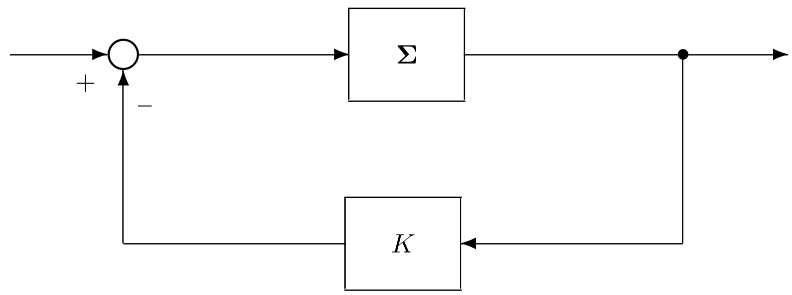

FIG. 2.1. *Static output feedback.*

which is familiar from finite-dimensional systems theory. The operators A, B, C, and D are called the *generating operators* of $\mathbf{\Sigma}$.

Finally, we review some of the results on static output feedback for abstract linear systems which have been recently obtained by Weiss [49]. Consider the feedback system shown in Figure 2.1.

An operator $K \in \mathcal{L}(Y,U)$ is called an *admissible feedback operator* for $\mathbf{\Sigma}$ if $I + K\mathbf{G}$ has a well-posed inverse, i.e., if there exists a well-posed transfer function $\mathbf{J}$ such that

$$\mathbf{J}(s)(I + K\mathbf{G}(s)) = (I + K\mathbf{G}(s))\mathbf{J}(s) = I \quad \forall\, s \in \mathbb{C}_\alpha$$

for some $\alpha \in \mathbb{R}$. It is easy to see that $I + K\mathbf{G}$ has a well-posed inverse if and only if $I + \mathbf{G}K$ has. If $\mathbf{\Sigma}$ is regular and if $K \in \mathcal{L}(Y,U)$ is an admissible feedback operator for $\mathbf{\Sigma}$, then $I + DK$ (and hence also $I + KD$) is left invertible. In particular, if U or Y is finite-dimensional, then $I + DK$ (and hence also $I + KD$) is invertible.

The next result shows that if K is an admissible feedback operator for $\mathbf{\Sigma}$, then there exists a unique abstract linear system $\mathbf{\Sigma}^K$ representing the feedback system shown in Figure 2.1.

THEOREM 2.3. *Let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ be an abstract linear system, let $\mathbf{G}$ denote its transfer function and let $K \in \mathcal{L}(Y,U)$ be an admissible feedback operator for $\mathbf{\Sigma}$. Then the following statements are true:*

(i) *There exists a unique abstract linear system $\mathbf{\Sigma}^K = (\mathbf{T}^K, \mathbf{\Phi}^K, \mathbf{\Psi}^K, \mathbf{F}^K)$ such that, when we denote*

$$\mathbf{\Sigma}_\tau = \begin{pmatrix} \mathbf{T}_\tau & \mathbf{\Phi}_\tau \\ \mathbf{\Psi}_\tau & \mathbf{F}_\tau \end{pmatrix}, \qquad \mathbf{\Sigma}_\tau^K = \begin{pmatrix} \mathbf{T}_\tau^K & \mathbf{\Phi}_\tau^K \\ \mathbf{\Psi}_\tau^K & \mathbf{F}_\tau^K \end{pmatrix}$$

$(\tau \geq 0)$, *we have*

$$\mathbf{\Sigma}_\tau^K = \mathbf{\Sigma}_\tau - \mathbf{\Sigma}_\tau \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \mathbf{\Sigma}_\tau^K \quad \textit{and} \quad \mathbf{\Sigma}_\tau = \mathbf{\Sigma}_\tau^K + \mathbf{\Sigma}_\tau^K \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \mathbf{\Sigma}_\tau \quad \forall\, \tau \geq 0\,. \tag{2.6}$$

The transfer function $\mathbf{G}^K$ of $\mathbf{\Sigma}^K$ is given by $\mathbf{G}^K = \mathbf{G}(I + K\mathbf{G})^{-1}$. Moreover, $L \in \mathcal{L}(Y,U)$ is an admissible feedback operator for $\mathbf{\Sigma}^K$ if and only if $K+L$ is an admissible feedback operator for $\mathbf{\Sigma}$. If this is the case, then

$$(\mathbf{\Sigma}^K)^L = \mathbf{\Sigma}^{K+L}\,. \tag{2.7}$$

(ii) *Under the extra assumptions that* $\mathbf{\Sigma}$ *is regular and that* $I + DK$ *is invertible, it follows that* $\mathbf{\Sigma}^K$ *is regular, and the generating operators* A^K, B^K, C^K, *and* D^K *of* $\mathbf{\Sigma}^K$ *are given by*

$$A^K = A - BK(I+DK)^{-1}C_L\,,\ C^K = (I+DK)^{-1}C_L\,,\ B^K = B(I+KD)^{-1}\,,$$
$$\text{and} \quad D^K = (I+DK)^{-1}D\,,$$

where $D(A^K) = \{x \in D(C_L) \,|\, (A - BK(I+DK)^{-1}C_L)x \in X\}$.

For $x_0 \in X$ and $u \in L^2_{loc}(\mathbb{R}_+, U)$ define the functions $x(\cdot)$ and $y(\cdot)$ by (2.1). The second equation in (2.6) then implies for $t \geq 0$

$$x(t) = \mathbf{T}_t^K x_0 + \mathbf{\Phi}_t^K \mathbf{P}_t(Ky + u), \tag{2.8a}$$

$$\mathbf{P}_t y = \mathbf{\Psi}_t^K x_0 + \mathbf{F}_t^K \mathbf{P}_t(Ky + u)\,. \tag{2.8b}$$

Moreover, for $t \geq \tau \geq 0$ we have that

$$x(t) = \mathbf{T}_{t-\tau}^K x(\tau) + \mathbf{\Phi}_{t-\tau}^K \mathbf{L}_\tau \mathbf{P}_t(Ky + u), \tag{2.9a}$$

$$\mathbf{L}_\tau \mathbf{P}_t y = \mathbf{\Psi}_{t-\tau}^K x(\tau) + \mathbf{F}_{t-\tau}^K \mathbf{L}_\tau \mathbf{P}_t(Ky + u)\,. \tag{2.9b}$$

The above formulas (2.8) and (2.9) will turn out to be very useful in sections 4 and 5.

Finally, consider the nonlinear system given by

$$\dot\gamma(t) = \|v(t)\|^2\,, \quad k(0) = k_0 \in \mathbb{R}, \tag{2.10a}$$

$$w(t) = \mathcal{K}(\gamma(t))v(t)\,, \quad t \geq 0\,, \tag{2.10b}$$

where $v \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ is the input and w denotes the output. The function $\mathcal{K} : \mathbb{R} \to \mathbb{R}$ is assumed to be locally Lipschitz.

For sections 4 and 5 we need a well-posedness result for the feedback interconnection of $\mathbf{\Sigma}$ and (2.10). More precisely, consider the feedback system given by (2.1), (2.10), and the interconnection equations

$$v = y\,, \quad u = -w$$

(where, of course, we assume that $U = Y = \mathbb{R}^m$). The closed-loop equations for y and γ then take the following form:

$$y(t) = (\mathbf{\Psi}_\infty x_0)(t) - (\mathbf{F}_\infty \mathcal{K}(\gamma)y)(t)\,, \tag{2.11a}$$

$$\gamma(t) = \gamma_0 + \int_0^t \|y(\xi)\|^2\, d\xi\,. \tag{2.11b}$$

Let $\tau \in (0, \infty]$. A function $(y, \gamma) : [0, \tau) \to \mathbb{R}^m \times \mathbb{R}$ is called a *solution* of (2.11) on $[0, \tau)$ if

(i) $(y, \gamma) \in L^2([0, \tau'], \mathbb{R}^m) \times AC([0, \tau'], \mathbb{R})$ for all $\tau' \in [0, \tau)$, where $AC([0, \tau'], \mathbb{R})$ denotes the real-valued absolutely continuous functions defined on $[0, \tau']$.

(ii) (y, γ) satisfies (2.11) almost everywhere on $[0, \tau)$.

If (2.11) has a solution (y, γ) on $[0, \tau)$, then the corresponding state trajectory of $\mathbf{\Sigma}$ is given by

$$x(t) = \mathbf{T}_t x_0 - \mathbf{\Phi}_t(\mathbf{P}_t \mathcal{K}(\gamma)y) \quad \forall\, t \in [0, \tau)\,.$$

PROPOSITION 2.4. *Suppose that* $U = Y = \mathbb{R}^m$ *and that* $\mathbb{L}^{-1}\mathbf{G} \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}^{m \times m})$. *Then for any* $(x_0, \gamma_0) \in X \times \mathbb{R}$ *there exists a maximal solution of* (2.11). *To be more*

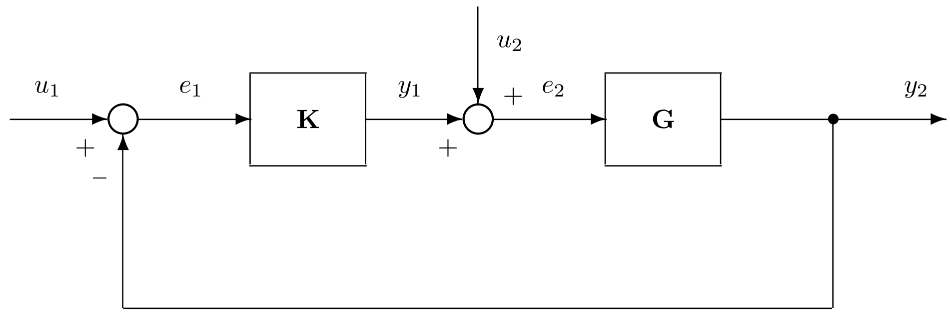

FIG. 3.1. *Closed-loop system* $\mathcal{F}(\mathbf{G},\mathbf{K})$.

precise, there exists $\tau_{max} \in (0,\infty]$ *such that* (2.11) *has a unique solution* (y_{max},γ_{max}) *on* $[0,\tau_{max})$*, and moreover*

$$\tau_{max} < \infty \quad \Longrightarrow \quad \int_0^{\tau_{max}} \|y_{max}(t)\|^2 \, dt = \infty \; .$$

The proof of Proposition 2.4 is given in the appendix.

3. Nonadaptive low-gain control. For $\alpha \in \mathbb{R}$ let $\mathcal{M}_\alpha$ denote the field of all meromorphic functions defined on $\mathbb{C}_\alpha$. The algebra of all bounded holomorphic functions defined on $\mathbb{C}_\alpha$ will be denoted by H^∞_α. The symbol H^2_α stands for the vector space of all holomorphic functions $f : \mathbb{C}_\alpha \to \mathbb{C}$ such that $\sup_{\xi>\alpha} \int_{-\infty}^{\infty} |f(\xi+\imath\omega)|^2 \, d\omega < \infty$. Moreover, we define

$$\mathcal{M}_- := \bigcup_{\alpha<0} \mathcal{M}_\alpha \, , \quad H^\infty_- := \bigcup_{\alpha<0} H^\infty_\alpha \, , \quad H^2_- := \bigcup_{\alpha<0} H^2_\alpha \, .$$

Let $\mathbf{G} \in \mathcal{M}_-^{m\times m}$ and $\mathbf{K} \in \mathcal{M}_-^{m\times m}$ be square transfer-function matrices, and consider the feedback system shown in Figure 3.1, which will be denoted by $\mathcal{F}(\mathbf{G},\mathbf{K})$. We shall call the feedback system $\mathcal{F}(\mathbf{G},\mathbf{K})$ input-output stable if every transfer function $u_i \mapsto y_j$ that occurs around the loop has all its entries in H^∞_-. More precisely, we make the following definition.

DEFINITION 3.1. *Let* $\mathbf{G} \in \mathcal{M}_-^{m\times m}$ *and* $\mathbf{K} \in \mathcal{M}_-^{m\times m}$*. The feedback system* $\mathcal{F}(\mathbf{G},\mathbf{K})$ *is called* input-output stable *if* $\det(I + \mathbf{G}(s)\mathbf{K}(s)) \not\equiv 0$ *and*

$$F(\mathbf{G},\mathbf{K}) := \begin{pmatrix} \mathbf{K}(I+\mathbf{G}\mathbf{K})^{-1} & -\mathbf{K}\mathbf{G}(I+\mathbf{K}\mathbf{G})^{-1} \\ \mathbf{G}\mathbf{K}(I+\mathbf{G}\mathbf{K})^{-1} & \mathbf{G}(I+\mathbf{K}\mathbf{G})^{-1} \end{pmatrix} \in H_-^{\infty\; 2m\times 2m} \, .$$

We say that $\mathbf{K}$ stabilizes $\mathbf{G}$ *if* $\mathcal{F}(\mathbf{G},\mathbf{K})$ *is input-output stable.*

Note that the above concept of input-output stability is stronger than L^2-stability, which is equivalent to $F(\mathbf{G},\mathbf{K}) \in H_0^{\infty\; 2m\times 2m}$. However, Definition 3.1 has the advantage that it guarantees the analyticity of the closed-loop transfer function on $\mathbb{C}_\alpha$ for some $\alpha < 0$, a property which will be needed in the following.

Remark 3.2. (i) It is trivial that $\mathbf{K}$ stabilizes $\mathbf{G}$ if and only if $\mathbf{G}$ stabilizes $\mathbf{K}$.

(ii) Let $\mathcal{Q}(H^\infty_-)$ denote the quotient field of H^∞_-, i.e., $\mathcal{Q}(H^\infty_-) = \{n/d \,|\, n, d \in H^\infty_-,\, d(s) \not\equiv 0\}$. If $\mathcal{F}(\mathbf{G},\mathbf{K})$ is input-output stable, then $\mathbf{G} \in \mathcal{Q}(H^\infty_-)^{\,m\times m}$ and $\mathbf{K} \in \mathcal{Q}(H^\infty_-)^{\,m\times m}$.

(iii) If $\mathbf{G} \in H_-^{\infty\; m\times m}$, then $\mathcal{F}(\mathbf{G},\mathbf{K})$ is input-output stable if and only if $\det(I + \mathbf{G}(s)\mathbf{K}(s)) \not\equiv 0$ and $\mathbf{K}(I+\mathbf{G}\mathbf{K})^{-1}$ is in $H_-^{\infty\; m\times m}$.

(iv) A *left coprime factorization* of $\mathbf{G}$ over H^∞_- is a pair $(\mathbf{D}, \mathbf{N}) \in H^{\infty\ m\times m}_- \times H^{\infty\ m\times m}_-$ such that $\det \mathbf{D} \not\equiv 0$, $\mathbf{G} = \mathbf{D}^{-1}\mathbf{N}$ and there exist $\mathbf{X}, \mathbf{Y} \in H^{\infty\ m\times m}_-$ satisfying $\mathbf{DX}+\mathbf{NY} = I$. *Right coprime factorizations* over H^∞_- are defined in an analogous way. It follows from Smith [40] that $\mathbf{G}$ and $\mathbf{K}$ admit left and right coprime factorizations over H^∞_- if $\mathcal{F}(\mathbf{G}, \mathbf{K})$ is input-output stable.

PROPOSITION 3.3. *Let* $\mathbf{G} \in \mathcal{M}^{m\times m}_-$ *and* $\mathbf{K} \in \mathcal{M}^{m\times m}_-$. *If* $\mathbf{K}$ *stabilizes* $\mathbf{G}$ *and if*

$$\lim_{\operatorname{Re} s\to\infty} \mathbf{K}(s) = 0\,,$$

then $\mathbf{G}$ *is well posed.*

Proof. By Remark 3.2 (ii) we have that $\mathbf{G}, \mathbf{K} \in \mathcal{Q}(H^\infty_-)^{m\times m}$, and hence, by Remark 3.2 (iv), there exists a right coprime factorization $(\mathbf{N_G}, \mathbf{D_G})$ of $\mathbf{G}$ over H^∞_- and a left coprime factorization $(\mathbf{D_K}, \mathbf{N_K})$ of $\mathbf{K}$ over H^∞_-. By a standard result in fractional representation theory (cf. Vidyasagar, Schneider, and Francis [42]) the input-output stability of the closed-loop system is equivalent to

$$\inf_{s\in\mathbb{C}^{cl}_0} |\det[\mathbf{N_K}(s)\mathbf{N_G}(s) + \mathbf{D_K}(s)\mathbf{D_G}(s)]| > 0\,. \tag{3.1}$$

Seeking a contradiction, suppose that $\mathbf{G}$ is not well posed. Then there exists a sequence $(s_n)_{n\in\mathbb{N}} \subset \mathbb{C}^{cl}_0$ with $\lim_{n\to\infty} \operatorname{Re} s_n = \infty$ and such that $\lim_{n\to\infty} \|\mathbf{G}(s_n)\| = \infty$. As a consequence

$$\lim_{n\to\infty} \det \mathbf{D_G}(s_n) = 0\,. \tag{3.2}$$

On the other hand $\lim_{n\to\infty} \mathbf{K}(s_n) = 0$, and hence

$$\lim_{n\to\infty} \mathbf{N_K}(s_n) = 0\,. \tag{3.3}$$

Combining (3.2) and (3.3) shows that

$$\lim_{n\to\infty} \det[\mathbf{N_K}(s_n)\mathbf{N_G}(s_n) + \mathbf{D_K}(s_n)\mathbf{D_G}(s_n)] = 0\,,$$

contradicting (3.1). □

Since in this paper we will be mainly concerned with controllers of the form $\mathbf{K}(s) = (1/s)\Gamma$, where $\Gamma \in \mathbb{R}^{m\times m}$, the following definition will turn out to be useful.

DEFINITION 3.4. *A transfer function matrix* $\mathbf{G} \in \mathcal{M}^{m\times m}_-$ *is called* integral stabilizable *if there exists* $\Gamma \in \mathbb{R}^{m\times m}$ *such that the controller* $\mathbf{K}(s) = (1/s)\Gamma$ *stabilizes* $\mathbf{G}$. *If the extra condition*

$$[\mathbf{GK}(I + \mathbf{GK})^{-1}](0) = I \tag{3.4}$$

is satisfied, then $\mathbf{G}$ *is called* integral controllable.

A controller of the form $(1/s)\Gamma$ is called an *integrator*. It is a trivial consequence of Proposition 3.3 that if a transfer-function matrix in $\mathcal{M}^{m\times m}_-$ is integral stabilizable, then necessarily it is well posed.

In the following let $\theta(\cdot)$ denote the Heaviside step function, i.e.,

$$\theta(t) = \begin{cases} 1 & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t \le 0, \end{cases}$$

As usual, convolution will be denoted by $\star$. The next result shows that condition (3.4) is closely related to the asymptotic tracking of constant reference signals.

PROPOSITION 3.5. *Suppose that* $\mathbf{G} \in \mathcal{M}_{-}^{m\times m}$ *is integral stabilizable, and let* $\mathbf{K}(s) = (1/s)\Gamma$, *where* $\Gamma \in \mathbb{R}^{m\times m}$, *be a stabilizing integrator. Then*

$$\lim_{t\to\infty}[\mathbb{L}^{-1}(\mathbf{GK}(I+\mathbf{GK})^{-1}) \star \theta r](t) = r$$

for all $r \in \mathbb{R}^m$ *if and only if* (3.4) *holds.*

For the proof of the above proposition we need the following lemma, which is a special case of the main result in Mossaheb [27].

LEMMA 3.6. *Suppose that* h *is a holomorphic function defined on* $\mathbb{C}_\alpha$ *such that the function* $s \mapsto sh(s)$ *is in* H_α^∞. *Then there exists a measurable function* $f : \mathbb{R}_+ \to \mathbb{C}$ *with* $f(\cdot)\exp(-\beta\,\cdot) \in L^1(\mathbb{R}_+, \mathbb{C})$ *for all* $\beta > \alpha$ *and such that*

$$(\mathbb{L}f)(s) = h(s) \quad \forall\, s \in \mathbb{C}_\alpha\,.$$

Proof of Proposition 3.5. By assumption we have that

$$\mathbf{H} := (I+\mathbf{GK})^{-1}\mathbf{G} = \mathbf{G}(I+\mathbf{KG})^{-1} \in H_{-}^{\infty\; m\times m}\,,$$

and hence

$$s[\mathbf{GK}(I+\mathbf{GK})^{-1}](s) = s[(I+\mathbf{GK})^{-1}\mathbf{GK}](s) = \mathbf{H}(s)\Gamma \in H_{-}^{\infty\; m\times m}\,.$$

Thus, by Lemma 3.6

$$\mathbb{L}^{-1}[\mathbf{GK}(I+\mathbf{GK})^{-1}] \in L^1(\mathbb{R}_+, \mathbb{C}^{m\times m})\,.$$

Therefore

$$\begin{aligned}\lim_{t\to\infty}[\mathbb{L}^{-1}(\mathbf{GK}(I+\mathbf{GK})^{-1}) \star \theta r](t) &= \lim_{t\to\infty}\left(\int_0^t [\mathbb{L}^{-1}(\mathbf{GK}(I+\mathbf{GK})^{-1})](\tau)\,d\tau\right) r \\ &= [\mathbf{GK}(I+\mathbf{GK})^{-1}](0)r\,,\end{aligned}$$

which yields the claim. □

The next result gives a necessary condition for integral controllability. It shows that an integral controllable transfer function does not have any transmission zeros at 0.

PROPOSITION 3.7. *Suppose that* $\mathbf{G} \in \mathcal{M}_{-}^{m\times m}$ *is integral controllable. Then there exists a left coprime factorization* $(\mathbf{D}, \mathbf{N})$ *of* $\mathbf{G}$ *over* H_{-}^{∞}, *and the numerator* $\mathbf{N}$ *in any such factorization satisfies*

$$\det \mathbf{N}(0) \neq 0\,.$$

Proof. It follows from Remark 3.2 (iv) that there exists a left coprime factorization $(\mathbf{D}, \mathbf{N})$ of $\mathbf{G}$ over H_{-}^{∞}. Let $\Gamma \in \mathbb{R}^{m\times m}$ be such that $\mathbf{K}(s) = (1/s)\Gamma$ stabilizes $\mathbf{G}$ and (3.4) is satisfied. Define

$$\mathbf{H} := \mathbf{GK}(I+\mathbf{GK})^{-1}\,, \quad \Delta := \lim_{s\to 0}[\mathbf{K}(I+\mathbf{GK})^{-1}](s)\,.$$

Then $\mathbf{DH} = \mathbf{NK}(I+\mathbf{GK})^{-1}$. Moreover, letting $s \to 0$ and using (3.4) yield $\mathbf{D}(0) = \mathbf{N}(0)\Delta$. Since $\mathbf{D}$ and $\mathbf{N}$ are left coprime over H_{-}^{∞}, it follows that

$$\operatorname{rank} \mathbf{N}(0)(\Delta, I) = \operatorname{rank}\,[\mathbf{D}(0), \mathbf{N}(0)] = m\,.$$

Therefore $\operatorname{rank} \mathbf{N}(0) = m$, and hence $\det \mathbf{N}(0) \neq 0$. □

The following theorem is the main input-output result of this section.

THEOREM 3.8. *Suppose that* $\mathbf{G} \in H_-^{\infty\, m\times m}$ *and that* $\mathbf{G}(0)$ *is real. Then* $\mathbf{G}$ *is integral controllable if and only if*

$$\det \mathbf{G}(0) \neq 0\,. \tag{3.5}$$

If (3.5) *holds, then there exists* $\Gamma_0 \in \mathbb{R}^{m\times m}$ *such that*

$$\sigma(\mathbf{G}(0)\Gamma_0) \subset \mathbb{C}_0\,, \tag{3.6}$$

and for any $\Gamma_0 \in \mathbb{R}^{m\times m}$ *satisfying* (3.6)*, there exists* $k^* > 0$ *such that for all* $k \in (0, k^*)$

$$F(\mathbf{G}, \mathbf{K}_k) \in H_-^{\infty\, 2m\times 2m} \quad \text{and} \quad [\mathbf{G}\mathbf{K}_k(I + \mathbf{G}\mathbf{K}_k)^{-1}](0) = I\,, \tag{3.7}$$

where $\mathbf{K}_k(s) := (1/s)k\Gamma_0$*. Moreover, setting* $\mathbf{E}_k(s) = (1/s)(I + \mathbf{G}\mathbf{K}_k)^{-1}(s)$*, we have that* $\mathbf{E}_k \in H_-^{2\; m\times m}$ *for all* $k \in (0, k^*)$.

The result shows in particular that there exist low-gain integral controllers which achieve stability and asymptotic tracking of constant reference signals. Since for constant reference signals $r\theta(t)$, the error signal $e(t)$ of the feedback system is given by $(\mathbb{L}e)(s) = \mathbf{E}_k(s)r$, it follows from the last statement of Theorem 3.8 via the Paley–Wiener theorem that $e \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ for all $k \in (0, k^*)$. In order to apply Theorem 3.8, we have to know only that the plant is stable and that (3.5) holds. Estimates of G_0 of $\mathbf{G}(0)$ can be obtained from step response data. An obvious choice for the gain matrix Γ_0 is $\Gamma_0 = G_0^{-1}$. Once a Γ_0 satisfying (3.6) has been found, the solution of the tracking problem reduces to the tuning of the gain parameter k.

Proof of Theorem 3.8. The necessity of (3.5) for integral controllability follows from Proposition 3.7 and from the hypothesis that $\mathbf{G} \in H_-^{\infty\, m\times m}$. In order to prove sufficiency, define $\Gamma_0 := \mathbf{G}^{-1}(0)$. Then, trivially, (3.6) is satisfied. Moreover, as in Logemann and Owens [15, pp. 17, 18], it can be shown that there exists a number $k^* > 0$ such that for all $k \in (0, k^*)$ the controller $\mathbf{K}_k$ stabilizes $\mathbf{G}$, i.e.,

$$F(\mathbf{G}, \mathbf{K}_k) \in H_-^{\infty\, 2m\times 2m}\,.$$

Next observe that by the invertibility of $k\mathbf{G}(0)\Gamma_0$

$$\lim_{s\to 0}[\mathbf{G}\mathbf{K}_k(I + \mathbf{G}\mathbf{K}_k)^{-1}](s) = \lim_{s\to 0} \mathbf{G}(s)k\Gamma_0(sI + k\mathbf{G}(s)\Gamma_0)^{-1} = I\,,$$

which yields the second equation in (3.7). Finally, $\mathbf{E}_k = k^{-1}\Gamma_0^{-1}\mathbf{K}_k(I + \mathbf{G}\mathbf{K}_k)^{-1}$, and therefore $\mathbf{E}_k \in H_-^{\infty\, m\times m}$ for all $k \in (0, k^*)$. Since for all such k the transfer function matrix $(I + \mathbf{G}\mathbf{K}_k)^{-1}$ is in $H_-^{\infty\, m\times m}$ as well, we see that $\mathbf{E}_k \in H_-^{2\; m\times m}$ for all $k \in (0, k^*)$. □

For Hermitian matrices $M, N \in \mathbb{C}^{m\times m}$, in the following we write $M \prec N$ if $N - M$ is positive definite and $M \succ N$ if $N - M$ is negative definite. Similarly, we write $M \preceq N$ if $N - M$ is positive semidefinite and $M \succeq N$ if $N - M$ is negative semidefinite. Moreover, for a complex matrix M let M^H denote the conjugate transpose of M.

The next result will be an important tool in section 4, although it is interesting in its own right.

PROPOSITION 3.9. *Let* $\mathbf{G} \in H_-^{\infty\, m\times m}$ *and suppose that* $\det \mathbf{G}(0) \neq 0$*. Setting* $\tilde{\mathbf{G}}(s) := (1/s)\mathbf{G}(s)$ *and using the notation of Theorem* 2.3 *we write*

$$\tilde{\mathbf{G}}^k(s) = \tilde{\mathbf{G}}(s)(I + k\tilde{\mathbf{G}}(s))^{-1} = \frac{1}{s}\mathbf{G}(s)\left(I + \frac{k}{s}\mathbf{G}(s)\right)^{-1}, [3]$$

[3] By slight abuse of notation we write $\tilde{\mathbf{G}}^k$ instead of $\tilde{\mathbf{G}}^{kI}$.

where $k \in \mathbb{R}$. Under these conditions there exists $k^* > 0$ such that for all $k \in (0, k^*)$

$$\|\tilde{\mathbf{G}}^k\|_\infty = \frac{1}{k} \tag{3.8}$$

if and only if $\mathbf{G}(0) \succ 0$. Moreover, the claim remains true if we replace k with $-k$ in (3.8) *and $\mathbf{G}(0) \succ 0$ by $\mathbf{G}(0) \prec 0$.*

As usual, the H^∞-norm in (3.8) is defined to be the supremum over $\mathbb{C}_0$ of $\sigma_{max}(\tilde{\mathbf{G}}^k(s))$, the largest singular value of $\tilde{\mathbf{G}}^k(s)$. For the single-input single-output case it follows that if $\mathbf{G}(0) \neq 0$ and if $\mathbf{G}(0) \in \mathbb{R}$, then there exists $k^* > 0$ such that $\|\tilde{\mathbf{G}}^k\|_\infty = 1/|k|$ for all $k \in \mathbb{R}$ satisfying $|k| \in (0, k^*)$ and $k\mathbf{G}(0) > 0$.

Proposition 3.9 is an immediate consequence of the following lemma.

LEMMA 3.10. *Let $\mathbf{G} \in H_-^{\infty\, m\times m}$. Using the notation of Proposition* 3.9*, the following statements hold:*

(i) *Suppose that* $\det \mathbf{G}(0) \neq 0$ *and $k \neq 0$. Then* (3.8) (*with k replaced by $|k|$*) *is true if and only if $I + k\tilde{\mathbf{G}}(s) + k\tilde{\mathbf{G}}^H(s) \succeq 0$ for all $s \in \mathbb{C}_0$.*

(ii) *There exists $k^* > 0$ such that $I + k\tilde{\mathbf{G}}(s) + k\tilde{\mathbf{G}}^H(s) \succeq 0$ for all $s \in \mathbb{C}_0$ and for all $k \in (0, k^*)$ if and only if $\mathbf{G}(0) \succeq 0$.*

Note that if $\mathbf{G}(s) \in \mathbb{R}^{m\times m}$ for all $s \in (0, \infty)$, then $I + k\tilde{\mathbf{G}}(s) + k\tilde{\mathbf{G}}^H(s) \succeq 0$ for all $s \in \mathbb{C}_0$ if and only if $(1/2)I + k\tilde{\mathbf{G}}(s)$ is positive real.

Proof of Lemma 3.10. (i) By assumption, $\mathbf{G}^{-1}(0)$ exists, and thus $\sigma_{max}(\tilde{\mathbf{G}}^k(0)) = 1/k$. Therefore (3.8) holds if and only if

$$\sigma_{max}(\tilde{\mathbf{G}}^k(s)) \le \frac{1}{k} \quad \forall\, s \in \mathbb{C}_0\,,$$

or equivalently

$$(I + k\tilde{\mathbf{G}}(s))^{-1}\tilde{\mathbf{G}}(s)\tilde{\mathbf{G}}^H(s)(I + k\tilde{\mathbf{G}}^H(s))^{-1} \preceq \frac{1}{k^2}I \quad \forall\, s \in \mathbb{C}_0\,,$$

or equivalently

$$k^2\tilde{\mathbf{G}}(s)\tilde{\mathbf{G}}^H(s) \preceq (I + k\tilde{\mathbf{G}}(s))(I + k\tilde{\mathbf{G}}^H(s)) \quad \forall\, s \in \mathbb{C}_0\,,$$

which in turn is equivalent to the positive semidefiniteness of $I + k\tilde{\mathbf{G}}(s) + k\tilde{\mathbf{G}}^H(s)$ for all $s \in \mathbb{C}_0$.

(ii) Since $\mathbf{G}$ is holomorphic at 0, we can write

$$\mathbf{G}(s) = \mathbf{G}(0) + \sum_{i=1}^{\infty} G_i s^i\,, \tag{3.9}$$

where $G_i \in \mathbb{C}^{m\times m}$ and the power series in (3.9) converges in some disc Δ_ε centred at 0 and with radius $\varepsilon > 0$. Consequently,

$$I + k\tilde{\mathbf{G}}(s) + k\tilde{\mathbf{G}}^H(s) = I + \frac{k}{s}\mathbf{G}(0) + \frac{k}{\bar{s}}\mathbf{G}^H(0) + k\mathbf{H}(s) \quad \forall\, s \in \Delta_\varepsilon\,, \tag{3.10}$$

where

$$\mathbf{H}(s) := \sum_{i=1}^{\infty} G_i s^{i-1} + \sum_{i=1}^{\infty} G_i^H \bar{s}^{\,i-1}\,.$$

Moreover, since $\tilde{\mathbf{G}}(s)$ is bounded on $\mathbb{C}_0 \setminus \Delta_\varepsilon$, there exists $k_1 > 0$ such that

$$I + k\tilde{\mathbf{G}}(s) + k\tilde{\mathbf{G}}^H(s) \succeq 0 \quad \forall\, s \in \mathbb{C}_0 \setminus \Delta_\varepsilon\,,\ \forall\, k \in (0, k_1)\,. \tag{3.11}$$

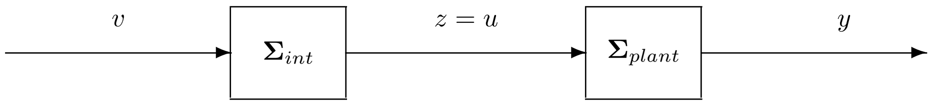

FIG. 3.2. *Cascade $\tilde{\mathbf{\Sigma}}$ with input v and output y.*

Suppose first that $\mathbf{G}(0) \succeq 0$. Then, using (3.10) and the boundedness of $\mathbf{H}(s)$ on Δ_ε, it follows that there exists $k_2 > 0$ such that

$$I + k\tilde{\mathbf{G}}(s) + k\tilde{\mathbf{G}}^H(s) \succeq 0 \quad \forall\, s \in \mathbb{C}_0 \cap \Delta_\varepsilon\,,\ \forall\, k \in (0, k_2)\,. \tag{3.12}$$

Setting $k^* := \min(k_1, k_2)$ we obtain from (3.11) and (3.12) that

$$I + k\tilde{\mathbf{G}}(s) + k\tilde{\mathbf{G}}^H(s) \succeq 0 \quad \forall\, s \in \mathbb{C}_0\,,\ \forall\, k \in (0, k^*)\,. \tag{3.13}$$

Conversely, suppose that (3.13) holds. Then, by (3.10), we obtain for any $\xi \in \mathbb{C}^m$ that

$$2\mathrm{Re}\left\langle \xi, \frac{k}{s}\mathbf{G}(0)\xi \right\rangle + \|\xi\|^2 + k\langle \xi, \mathbf{H}(s)\xi\rangle \geq 0 \quad \forall\, s \in \mathbb{C}_0 \cap \Delta_\varepsilon\,,\ \forall\, k \in (0, k^*)\,,$$

and hence it follows that for all $s \in \mathbb{C}_0 \cap \Delta_\varepsilon$ and all $k \in (0, k^*)$

$$\frac{2k}{|s|^2}\left(\mathrm{Re}\, s\, \mathrm{Re}\,\langle \xi, \mathbf{G}(0)\xi\rangle - \mathrm{Im}\, s\, \mathrm{Im}\,\langle \xi, \mathbf{G}(0)\xi\rangle\right) + \|\xi\|^2 + k\langle \xi, \mathbf{H}(s)\xi\rangle \geq 0\,.$$

Therefore, using the boundedness of $\mathbf{H}(s)$ on Δ_ε, we may conclude that for all $\xi \in \mathbb{C}^m$, $\mathrm{Im}\,\langle \xi, \mathbf{G}(0)\xi\rangle = 0$ and $\mathrm{Re}\,\langle \xi, \mathbf{G}(0)\xi\rangle \geq 0$, which in turn implies that $\mathbf{G}(0) \succeq 0$. □

In the following we will apply Theorem 3.8 to regular linear state space-systems. Since this additional assumption of regularity does not exclude any physically motivated well-posed system, the following results are as general as can be expected. For the rest of the section let $\mathbf{\Sigma}_{plant} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ be an exponentially stable abstract linear regular system with generating operators (A, B, C, D), state space X, input space $U = \mathbb{R}^m$, output space $Y = \mathbb{R}^m$, and transfer function $\mathbf{G}$. Clearly, by exponential stability, $\mathbf{G} \in H_-^{\infty\ m\times m}$. If $u \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ denotes the input and $x_0 \in X$ denotes the initial state, then the state $x(\cdot)$ and the output $y(\cdot)$ are given by (2.1). Moreover, let $\mathbf{\Sigma}_{int}$ denote the integrator described by

$$z(t) = z_0 + \int_0^t v(\tau)\, d\tau\,, \quad z_0 \in \mathbb{R}^m\,,$$

where $v \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ is the integrator input.

We will consider the series connection $\tilde{\mathbf{\Sigma}}$ of $\mathbf{\Sigma}_{int}$ followed by $\mathbf{\Sigma}_{plant}$ with input v and output y (cf. Figure 3.2).

In order to show that $\tilde{\mathbf{\Sigma}}$ is again an abstract linear regular system, we introduce an extra external input $w \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ and consider the cascade interconnection $\hat{\mathbf{\Sigma}}$ with input (w, v) and output (y, z) obtained by setting $u = z + w$ (cf. Figure 3.3).

We claim that $\hat{\mathbf{\Sigma}}$ is an abstract linear regular system. To this end consider the parallel interconnection $\mathbf{\Sigma}_{par}$ of $\mathbf{\Sigma}_{int}$ and $\mathbf{\Sigma}_{plant}$ shown in Figure 3.4.

Clearly, $\mathbf{\Sigma}_{par}$ is an abstract linear regular system, and the matrix J given by

$$J = \begin{pmatrix} 0 & -I \\ 0 & 0 \end{pmatrix}$$

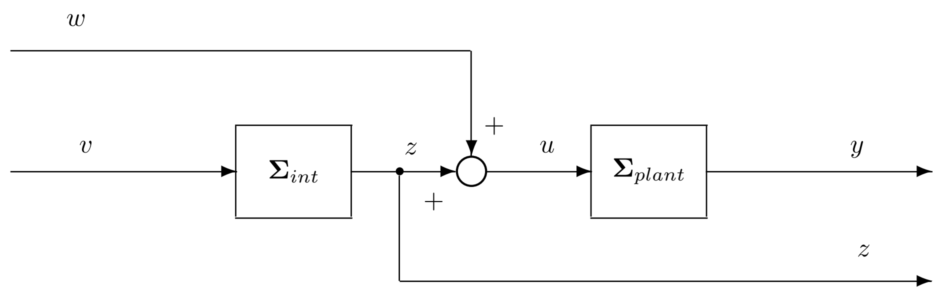

FIG. 3.3. *Cascade* $\hat{\mathbf{\Sigma}}$ *with input* (w, v) *and output* (y, z).

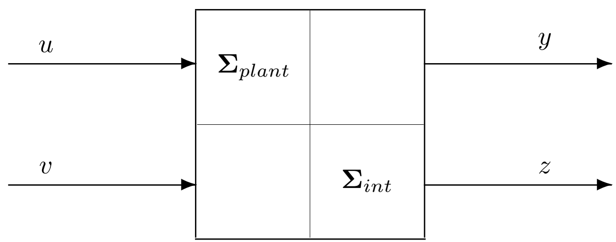

FIG. 3.4. *Parallel interconnection* $\mathbf{\Sigma}_{par}$.

is an admissible feedback operator for $\mathbf{\Sigma}_{par}$. Using the notation of section 2, we have that $\hat{\mathbf{\Sigma}} = (\mathbf{\Sigma}_{par})^J$, and hence it follows from Theorem 2.3 that $\hat{\mathbf{\Sigma}}$ is an abstract linear regular system. Writing $\hat{\mathbf{\Sigma}} = (\hat{\mathbf{T}}, \hat{\mathbf{\Phi}}, \hat{\mathbf{\Psi}}, \hat{\mathbf{F}})$, we see that $\tilde{\mathbf{\Sigma}} = (\tilde{\mathbf{T}}, \tilde{\mathbf{\Phi}}, \tilde{\mathbf{\Psi}}, \tilde{\mathbf{F}})$, where

$$\tilde{\mathbf{T}} = \hat{\mathbf{T}}\,, \quad \tilde{\mathbf{\Phi}} = \hat{\mathbf{\Phi}}\begin{pmatrix} 0 \\ I \end{pmatrix}, \quad \tilde{\mathbf{\Psi}} = (I, 0)\hat{\mathbf{\Psi}}\,, \quad \tilde{\mathbf{F}} = (I, 0)\hat{\mathbf{F}}\begin{pmatrix} 0 \\ I \end{pmatrix}.$$

Therefore $\tilde{\mathbf{\Sigma}}$ is an abstract linear regular system whose state, input, and output spaces are given by $X \times \mathbb{R}^m$, $U = \mathbb{R}^m$, and $Y = \mathbb{R}^m$, respectively. Denoting the generating operators of $\tilde{\mathbf{\Sigma}}$ by $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ it follows from Theorem 2.3 (ii) that

$$\tilde{A} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad \tilde{C} = (C_L, D), \quad \tilde{D} = 0, \tag{3.14}$$

where the domain $D(\tilde{A})$ of $\tilde{A}$ is given by

$$D(\tilde{A}) = \{(x, u) \in D(C_L) \times \mathbb{R}^m \mid Ax + Bu \in X\}\,.$$

If B is bounded, then it follows easily that $D(\tilde{A}) = D(A) \times \mathbb{R}^m$. Note that any unboundedness of B is absorbed into the unboundedness of $\tilde{A}$ and hence the control operator $\tilde{B}$ of $\tilde{\mathbf{\Sigma}}$ is bounded. Trivially, the function $\tilde{\mathbf{G}}(s) := (1/s)\mathbf{G}(s)$ is the transfer function of $\tilde{\mathbf{\Sigma}}$.

LEMMA 3.11. *Every* $\Gamma \in \mathbb{R}^{m\times m}$ *is an admissible feedback operator for* $\tilde{\mathbf{\Sigma}}$ *and* (*using the notation of section* 2) *we have that for all* $\Gamma \in \mathbb{R}^{m\times m}$

$$D(\tilde{A}^\Gamma) = D(\tilde{A}) = \{(x, u) \in X \times \mathbb{R}^m \mid Ax + Bu \in X\}\,. \tag{3.15}$$

Proof. Since $\tilde{\mathbf{G}}(s) = (1/s)\mathbf{G}(s)$ and $\mathbf{G} \in H^\infty_\alpha$ for some $\alpha < 0$, it follows from section 2 that any $\Gamma \in \mathbb{R}^{m\times m}$ is an admissible feedback operator for $\tilde{\mathbf{\Sigma}}$.

We show first that the second equality in (3.15) holds. It is clear that

$$D(\tilde{A}) \subset \{(x,u) \in X \times \mathbb{R}^m \,|\, Ax + Bu \in X\} =: \mathcal{D}\,,$$

and it only remains to prove that $\mathcal{D} \subset D(\tilde{A})$. To this end define

$$W := D(A) + (\lambda I - A)^{-1} B\mathbb{R}^m\,,$$

where $\lambda \in \varrho(A)$. Since $D(A) \subset D(C_L)$ and, by regularity, $(\lambda I - A)^{-1}B\mathbb{R}^m \subset D(C_L)$, it follows that $W \subset D(C_L)$.

Let $(x,u) \in \mathcal{D}$. Then $\xi := (\lambda I - A)x - Bu \in X$, and hence

$$x = (\lambda I - A)^{-1}\xi + (\lambda I - A)^{-1}Bu \in W\,.$$

It follows that $x \in D(C_L)$ and therefore $(x,u) \in D(\tilde{A})$.

In order to show that the first equality in (3.15) is true, recall from section 2 that

$$\tilde{A}^{\Gamma}(x,u) = (\tilde{A} - \tilde{B}\Gamma\tilde{C}_L)(x,u)$$

for all $(x,u) \in D(\tilde{A}^{\Gamma})$, where $D(\tilde{A}^{\Gamma})$ is given by

$$D(\tilde{A}^{\Gamma}) = \{(x,u) \in D(\tilde{C}_L) \,|\, (\tilde{A} - \tilde{B}\Gamma\tilde{C}_L)(x,u) \in X \times \mathbb{R}^m\}\,.$$

Moreover, using (3.14), we see that for all $(x,u) \in D(\tilde{A}^{\Gamma})$

$$\tilde{A}^{\Gamma}(x,u) = (Ax + Bu, -\Gamma\tilde{C}_L(x,u))\,.$$

This shows that

$$D(\tilde{A}^{\Gamma}) = \{(x,u) \in D(\tilde{C}_L) \,|\, Ax + Bu \in X\}\,. \tag{3.16}$$

Since $D(\tilde{C}_L) \subset X \times \mathbb{R}^m$, it follows from (3.16) that $D(\tilde{A}^{\Gamma}) \subset \mathcal{D} = D(\tilde{A})$.

To prove that $D(\tilde{A}) \subset D(\tilde{A}^{\Gamma})$, let $(x,u) \in D(\tilde{A})$. Then $(x,u) \in D(\tilde{C}_L)$ and $Ax + Bu \in X$, and hence, by (3.16), $(x,u) \in D(\tilde{A}^{\Gamma})$. $\square$

In the following we endow $D(\tilde{A}^{\Gamma})$ with its graph norm. The resulting complete space will be denoted by $\tilde{X}_1^{\Gamma}$.

PROPOSITION 3.12. *Let $\Gamma \in \mathbb{R}^{m\times m}$ and suppose that $\det\Gamma \neq 0$. If the integrator $\mathbf{K}(s) = (1/s)\Gamma$ stabilizes $\mathbf{G}$ (in the sense of Definition 3.1), then the following statements hold:*

(i) *The closed-loop semigroup $\tilde{\mathbf{T}}^{\Gamma}$ is exponentially stable.*

(ii) *$\tilde{C}^{\Gamma} = \tilde{C}$ and there exist $M > 0$ and $\omega > 0$ such that for all $(x_0,u_0) \in D(\tilde{A})$*

$$\|\tilde{C}\tilde{\mathbf{T}}_t^{\Gamma}(x_0,u_0)\| \le Me^{-\omega t}\|(x_0,u_0)\|_{\tilde{X}_1^{\Gamma}} \quad \forall\, t \ge 0\,.$$

If the observation operator C is bounded, then for any $(x_0,u_0) \in X \times \mathbb{R}^m$

$$\|\tilde{C}\tilde{\mathbf{T}}_t^{\Gamma}(x_0,u_0)\| \le Me^{-\omega t}\|(x_0,u_0)\|_{X\times\mathbb{R}^m} \quad \forall\, t \ge 0\,.$$

Proof. (i) The semigroup $\tilde{\mathbf{T}}^{\Gamma}$ describes the dynamics of the feedback system shown in Figure 3.5. Note that the state of $\mathbf{\Sigma}_{int}$ and the input of $\mathbf{\Sigma}_{plant}$ are identical. Therefore we denote both by the same symbol $u(\cdot)$.

The state $(x(t),u(t)) \in X \times \mathbb{R}^m$ at time $t \ge 0$ is given by

$$(x(t),u(t)) = \tilde{\mathbf{T}}_t^{\Gamma}(x_0,u_0)\,,$$

where $(x_0,u_0) := (x(0),u(0)) \in X \times \mathbb{R}^m$. Defining

$$y_0(t) := C_L\mathbf{T}_t x_0\,, \quad t \ge 0\,,$$

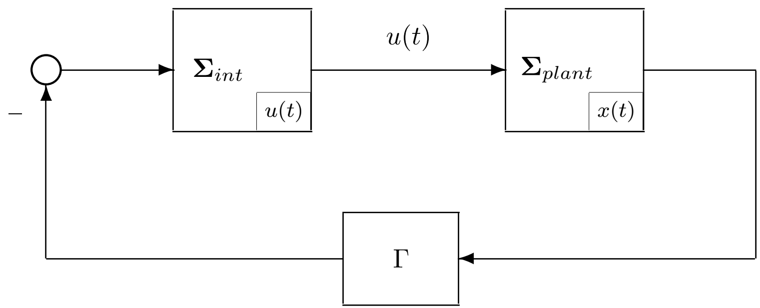

FIG. 3.5. *Internal dynamics of the closed loop.*

it follows from the exponential stability of $\mathbf{T}$ that $y_0 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. The Laplace transform of $u(\cdot)$ is then given by

$$(\mathbb{L}u)(s) = \frac{1}{s}u_0 - \mathbf{K}(s)[(\mathbb{L}y_0)(s) + \mathbf{G}(s)(\mathbb{L}u)(s)];$$

cf. Figure 3.5. It follows that

$$\mathbb{L}u = (I + \mathbf{KG})^{-1}\mathbf{K}\Gamma^{-1}u_0 - (I + \mathbf{KG})^{-1}\mathbf{K}\mathbb{L}y_0 \,. \tag{3.17}$$

By assumption the closed-loop system is input-output stable, and so $(I + \mathbf{KG})^{-1}\mathbf{K}$, $(I + \mathbf{KG})^{-1} \in H_-^{\infty\ m\times m}$. Using the fact that $\mathbf{K}(s) = (1/s)\Gamma$ we see that

$$(I + \mathbf{KG})^{-1}\mathbf{K} \in H_-^{2\ m\times m}\,,$$

and thus, since $\mathbb{L}y_0 \in H_0^{2\ m}$, we obtain from (3.17) that $\mathbb{L}u \in H_0^{2\ m}$. Hence, by the Paley–Wiener theorem, $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. Moreover, $\mathbf{\Sigma}_{plant}$ is exponentially stable and driven by u, and therefore by Lemma 2.2, $x \in L^2(\mathbb{R}_+, X)$. Thus, we see that for all $(x_0, u_0) \in X \times \mathbb{R}^m$

$$t \mapsto \tilde{\mathbf{T}}_t^{\Gamma}(x_0, u_0) \in L^2(\mathbb{R}_+, X \times \mathbb{R}^m)\,.$$

By a well-known result on the stability of C_0-semigroups (cf. Pazy [30, p. 116]) it follows that the semigroup $\tilde{\mathbf{T}}^{\Gamma}$ is exponentially stable.

(ii) Since $\tilde{D} = 0$, it follows from Theorem 2.3 (ii) that

$$\tilde{C}^{\Gamma}(x, u) = \tilde{C}_L(x, u) \qquad \forall\, (x, u) \in D(\tilde{A}^{\Gamma})\,.$$

An application of Lemma 3.11 shows that $\tilde{C}^{\Gamma} = \tilde{C}$.

Let $(x_0, u_0) \in D(\tilde{A})$. Then, by Lemma 3.11, $(x_0, u_0) \in \tilde{X}_1^{\Gamma}$. By part (i) the semigroup $\tilde{\mathbf{T}}^{\Gamma}$ is exponentially stable on $\tilde{X} = X \times \mathbb{R}^m$, and hence it is also exponentially stable on $\tilde{X}_1^{\Gamma}$. Since $\tilde{C}^{\Gamma} \in \mathcal{L}(\tilde{X}_1^{\Gamma}, \mathbb{R}^m)$, it follows from the above that $\tilde{C} \in \mathcal{L}(\tilde{X}_1^{\Gamma}, \mathbb{R}^m)$ as well. As a consequence there exist $M, \omega > 0$ such that

$$\|\tilde{C}\tilde{\mathbf{T}}_t^{\Gamma}(x_0, u_0)\| \le Me^{-\omega t}\|(x_0, u_0)\|_{\tilde{X}_1^{\Gamma}} \quad \forall\, t \ge 0\,.$$

The last statement of part (ii) follows from the fact that the boundedness of the observation operator C implies the boundedness of the observation operator $\tilde{C}$. $\square$

Remark 3.13. Part (i) of Proposition 3.12 shows that in our special situation (i.e., the plant is exponentially stable and the controller is an integrator) input-output stability implies exponential stability. Using a result by Rebarber [37], it can be

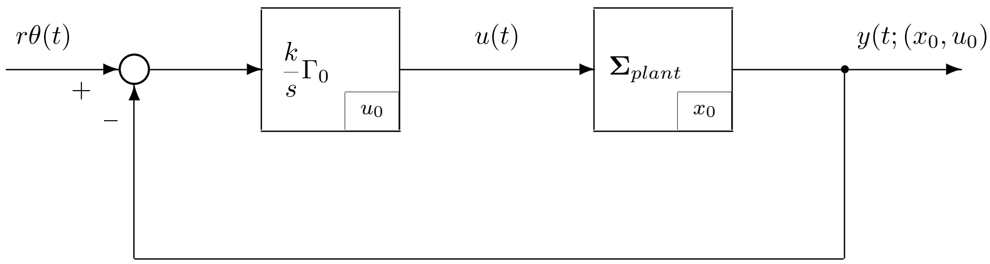

FIG. 3.6. *Low-gain control system.*

shown (Weiss [50]) that under suitable stabilizability and detectability assumptions the feedback interconnection of any two linear regular systems is exponentially stable if it is input-output stable. Since this result is not yet available in the literature (not even in form of a preprint), we have included a proof of Proposition 3.12 (i).

We are now in the position to prove the main result of this section, an internal version of Theorem 3.8 which applies to abstract linear regular state-space systems. Consider the feedback system in Figure 3.6, where $r \in \mathbb{R}^m$, $\Gamma_0 \in \mathbb{R}^{m\times m}$, $k > 0$, and $(x_0, u_0) \in X \times \mathbb{R}^m$. The output $y(\cdot\,;(x_0,u_0))$ can be written in the form

$$y(t;(x_0,u_0)) = \tilde{C}_L^{k\Gamma_0}\tilde{\mathbf{T}}_t^{k\Gamma_0}(x_0,u_0) + y(t;(0,0))\,. \tag{3.18}$$

Moreover, we define the corresponding error by

$$e(t;(x_0,u_0)) = r\theta(t) - y(t;(x_0,u_0))\,.$$

THEOREM 3.14. *Let $r \in \mathbb{R}^m$. Suppose that* $\det \mathbf{G}(0) \neq 0$ *and let $\Gamma_0 \in \mathbb{R}^{m\times m}$ be such that $\sigma(\mathbf{G}(0)\Gamma) \subset \mathbb{C}_0$. Then there exists $k^* > 0$ such that for any $k \in (0,k^*)$ the closed-loop semigroup $\tilde{\mathbf{T}}^{k\Gamma_0}$ is exponentially stable and $e(\cdot\,;(x_0,u_0)) \in L^2(\mathbb{R}_+,\mathbb{R}^m)$ for all $(x_0,u_0) \in X \times \mathbb{R}^m$. Furthermore,*

$$\lim_{t\to\infty} e(t;(x_0,u_0)) = 0 \quad \forall\,(x_0,u_0) \in D(\tilde{A})\,.$$

If the observation operator C is bounded, then the above equation holds for all $(x_0,u_0) \in X \times \mathbb{R}^m$.

Remark 3.15. If $(x_0,u_0) \notin D(\tilde{A})$, then in general $e(t) := e(t;(x_0,u_0))$ will not converge to 0 as $t \to \infty$. (In fact $e(\cdot)$ does not even make sense pointwise.) However, by Theorem 3.14, we still have that $e \in L^2(\mathbb{R}_+,\mathbb{R}^m)$, which implies that $e(t)$ *converges to* 0 *in measure* as $t \to \infty$ in the sense that for any $\varepsilon > 0$ and any $\delta > 0$ there exists $T = T(\varepsilon,\delta) > 0$ such that

$$\boldsymbol{\lambda}(\{t \in [\tau,\infty) \,|\, |e(t)| > \delta\} < \varepsilon \quad \forall\,\tau \geq T\,,$$

where $\boldsymbol{\lambda}$ denotes the Lebesgue measure.

Proof of Theorem 3.14. As in Theorem 3.8 we set $\mathbf{E}_k(s) = (1/s)(I+\mathbf{G}\mathbf{K}_k)^{-1}(s)$. By Theorem 3.8 there exists a $k^* > 0$ such that for all $k \in (0,k^*)$ the compensator $\mathbf{K}_k(s) = (1/s)k\Gamma_0$ stabilizes $\mathbf{G}$, and furthermore

$$\mathbf{E}_k \in H_-^{2\ m\times m} \quad \text{and} \quad [\mathbf{G}\mathbf{K}_k(I+\mathbf{G}\mathbf{K}_k)^{-1}](0) = I \qquad \forall\,k \in (0,k^*)\,. \tag{3.19}$$

In particular it follows from Proposition 3.12(i) that the semigroup $\tilde{\mathbf{T}}^{k\Gamma_0}$ is exponentially stable for all $k \in (0,k^*)$. Moreover, we have that

$$e(\cdot\,;(0,0)) = \mathbb{L}^{-1}(\mathbf{E}_k r)\,, \quad y(\cdot\,;(0,0)) = \mathbb{L}^{-1}[\mathbf{G}\mathbf{K}_k(I+\mathbf{G}\mathbf{K}_k)^{-1} \star \theta r]\,,$$

and therefore we obtain, using (3.19) and Proposition 3.5,

$$e(\cdot\,;(0,0)) \in L^2(\mathbb{R}_+,\mathbb{R}^m) \quad \text{and} \quad \lim_{t\to\infty} e(t;(0,0)) = 0\,, \tag{3.20}$$

provided that $k \in (0,k^*)$. Since the function

$$y_0(t;(x_0,u_0)) := \tilde{C}_L^{k\Gamma_0}\tilde{\mathbf{T}}_t^{k\Gamma_0}(x_0,u_0)$$

is the output of an exponentially stable regular system, it follows from Lemma 2.2 that $y_0(\cdot\,;(x_0,u_0)) \in L^2(\mathbb{R}_+,\mathbb{R}^m)$ for all $(x_0,u_0) \in X\times\mathbb{R}^m$ and all $k \in (0,k^*)$. Now, by (3.18),

$$e(t;(x_0,u_0)) = e(t;(0,0)) - y_0(t;(x_0,u_0))\,,$$

and thus, using (3.20), we obtain

$$e(\cdot\,;(x_0,u_0)) \in L^2(\mathbb{R}_+,\mathbb{R}^m) \quad \forall\,(x_0,u_0)\in X\times\mathbb{R}^m\,,$$

provided that $k \in (0,k^*)$. Finally, let $(x_0,u_0) \in D(\tilde{A})$. Then, by Proposition 3.12(ii), we may conclude that

$$\lim_{t\to\infty} y_0(t;(x_0,u_0)) = \lim_{t\to\infty} \tilde{C}_L^{k\Gamma_0}\tilde{\mathbf{T}}_t^{k\Gamma_0}(x_0,u_0) = \lim_{t\to\infty} \tilde{C}\tilde{\mathbf{T}}_t^{k\Gamma_0}(x_0,u_0) = 0\,. \tag{3.21}$$

Using (3.18), (3.20), and (3.21) we obtain that

$$\lim_{t\to\infty} e(t;(x_0,u_0)) = 0\,. \tag{3.22}$$

It follows from Proposition 3.12 (ii) that (3.22) holds for all $(x_0,u_0)\in X\times\mathbb{R}^m$ if the observation operator C is bounded. ∎

We close this section with a lemma which will be needed in section 4 in order to reformulate adaptive tracking problems as adaptive stabilization problems.

LEMMA 3.16. *For any $r\in\mathbb{R}^m$ there exists $(x_r,u_r)\in D(\tilde{A})$ such that*

$$\tilde{C}\tilde{\mathbf{T}}_t(x_r,u_r) = r \quad \forall\,t\ge 0\,.$$

Proof. For given $r\in\mathbb{R}^m$ define

$$x_r := -A^{-1}B\mathbf{G}^{-1}(0)r\,, \quad u_r := \mathbf{G}^{-1}(0)r\,.$$

Then $(x_r,u_r)\in X\times\mathbb{R}^m$, and moreover $Ax_r+Bu_r=0$. It follows that $(x_r,u_r)\in\{(x,u)\in X\times\mathbb{R}^m\,|\,Ax+Bu\in X\} = D(\tilde{A})$, and by (3.14), $\tilde{A}(x_r,u_r)=0$. We therefore easily conclude that $\tilde{\mathbf{T}}_t(x_r,u_r)=(x_r,u_r)$ for all $t\ge 0$. Finally, since $\mathbf{G}(0)=D-C_LA^{-1}B$, we see that for all $t\ge 0$

$$\tilde{C}\tilde{\mathbf{T}}_t(x_r,u_r) = C_Lx_r+Du_r = (\mathbf{G}(0)-D)\mathbf{G}^{-1}(0)r + D\mathbf{G}^{-1}(0)r = r\,. \qquad ∎$$

4. Adaptive low-gain control of multivariable systems with sign-definite steady-state gain. In this section we consider the adaptive low-gain control of systems with *sign-definite* steady-state gains $\mathbf{G}(0)$, that is where either $\mathbf{G}(0)\succ 0$ or $\mathbf{G}(0)\prec 0$. This situation arises most naturally in the single-input single-output case where we need to assume only that the steady-state gain is nonzero.[4] In the multivariable case the situation of significance is when the steady-state gain is positive definite (see Propositions 4.4 and 4.6).

[4]Of course, we also need that $\mathbf{G}(0)$ is real. This will always be the case if $\mathbf{G}$ is the transfer function of a regular system, which is real by definition.

Consider the control law given by

$$\dot{u}(t) = \mathcal{K}(\gamma(t))e(t)\,, \quad u(0) = u_0\,, \tag{4.1a}$$

$$\dot{\gamma}(t) = \|e(t)\|^2\,, \quad \gamma(0) = \gamma_0 > a \geq -\infty\,, \tag{4.1b}$$

where $\mathcal{K} : (a, \infty) \to \mathbb{R}$ is locally Lipschitz. In the following $\mathcal{K}$ will be called a *tuning function*. Choosing $a = 0$ and

$$\mathcal{K}(\gamma) = \sin(\gamma^q)/\gamma^p\,, \quad 0 < q < p < 1 - q, \tag{4.2}$$

Cook [1] has shown that (4.1) is a universal adaptive, low-gain tracking controller for the class of single-input single-output, exponentially stable, finite-dimensional, linear systems with transfer function $\mathbf{G}$, input function $u(\cdot)$, output function $y(\cdot)$, and constant reference signal $r\theta(t), r \in \mathbb{R}$, in the sense that (i) $e(t) = (r - y(t)) \to 0$ as $t \to \infty$ and (ii) state and input functions remain bounded, independently of initial data, provided that $\mathbf{G}(0) \neq 0$. It is also shown in [1] that if $\mathbf{G}(0) > 0$, then $\mathcal{K}$ in (4.2) can be replaced by $\mathcal{K}(\gamma) = \gamma^{-p}$, $0 < p < 1$. The main tool in [1] is the fact that the return difference function is positive real for all k small enough and of the correct sign. It is clear, using Lemma 3.10, that these results extend to the multivariable case provided that $\mathbf{G}(0)$ is sign-definite.

In this section we prove that with different, suitably chosen tuning functions $\mathcal{K}$, these results extend to the case when the system is infinite-dimensional, regular, and exponentially stable. However, first we give alternative proofs of the finite-dimensional results in [1].

The finite-dimensional case. Our approach is based on Proposition 3.9, i.e., the fact that the H^∞-norm of the closed-loop transfer function $\tilde{\mathbf{G}}^k$ equals $1/|k|$ for all small enough k of the correct sign, and on the connection between this result and the existence of solutions to certain algebraic Riccati equations which arise in the characterization of the complex stability radius given in Hinrichsen and Pritchard [6]. We note that whilst neither this approach based on the algebraic Riccati equation nor the approach based on positive realness of the return difference equation and associated Lur'e equations extends to general regular systems, Proposition 3.9 will remain a crucial tool in the infinite-dimensional case.

LEMMA 4.1. *There exists $k^* > 0$ such that for any k with $|k| < k^*$ and $k\mathbf{G}(0) \succ 0$ the Riccati equation*

$$(\tilde{A} - k\tilde{B}\tilde{C})^T Z + Z(\tilde{A} - k\tilde{B}\tilde{C}) - k^2\tilde{C}^T\tilde{C} - Z\tilde{B}\tilde{B}^T Z = 0, \tag{4.3}$$

where $\tilde{A}, \tilde{B}$, and $\tilde{C}$ are given by (3.14)*, has a unique solution $\tilde{P}_k = \tilde{P}_k^T \preceq 0$.*

Proof. An application of Theorem 3.8 and Proposition 3.9 shows the existence of a constant $k^* > 0$ such that for any k with $|k| < k^*$ and $k\mathbf{G}(0) \succ 0$ the matrix $\tilde{A} - k\tilde{B}\tilde{C}$ is exponentially stable and $\|\tilde{\mathbf{G}}^k\|_\infty = 1/|k|$. Therefore the existence of a unique $\tilde{P}_k = \tilde{P}_k^T \preceq 0$ satisfying (4.3) is guaranteed by Hinrichsen and Pritchard [6, pp. 107–109]. □

The above lemma can now be used to give an alternative proof of the main result in [1].

THEOREM 4.2. *Let*

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad x(0) = x_0 \in \mathbb{R}^n\,, \tag{4.4a}$$

$$y(t) = Cx(t) + Du(t) \tag{4.4b}$$

be any finite-dimensional, m-input m-output, exponentially stable system with sign-definite steady-state gain $\mathbf{G}(0)$. Moreover, let $\mathcal{K} : (a, \infty) \to \mathbb{R}$, where $a \geq -\infty$, be locally Lipschitz and bounded with $\mathcal{K} \in L^2(b, \infty; \mathbb{R})$ for some $b > a$ and such that

$$\liminf_{\gamma \to \infty} \int_b^\gamma \mathcal{K}(\xi)\, d\xi = -\infty\,, \quad \limsup_{\gamma \to \infty} \int_b^\gamma \mathcal{K}(\xi)\, d\xi = +\infty\,. \tag{4.5}$$

If $r\theta(t)$, $r \in \mathbb{R}^m$, is any constant reference signal and $u(t)$ is defined by (4.1), *with $e(t) = r - y(t)$, then for each $\gamma_0 > a$, $x_0 \in \mathbb{R}^n$, and $u_0 \in \mathbb{R}^m$ the following statements hold:*

(i) $\lim_{t\to\infty} \gamma(t) = \gamma_\infty < \infty$*;*
(ii) $\|x(t)\|$ *and* $\|u(t)\|$ *remain bounded as* $t \to \infty$*;*
(iii) $\lim_{t\to\infty} y(t) = r$*.*

Proof. The first step is to realize the reference signal $r\theta$ as an unforced motion of the series connection of the integrator $1/s$ followed by (4.4). By Lemma 3.16, applied in this simple finite-dimensional context, there exists $(x_r, u_r) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $r = \tilde{C}e^{\tilde{A}t}(x_r, u_r)$ for all $t \geq 0$. It follows that

$$e(t) = r - y(t) = \tilde{C}\tilde{x}(t)\,, \tag{4.6}$$

where $\tilde{x}(t)$ is given by

$$\tilde{x}(t) = e^{\tilde{A}t}(x_r, u_r) - (x(t), u(t))\,. \tag{4.7}$$

Clearly, $\tilde{x}(\cdot)$ satisfies

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) - \mathcal{K}(\gamma(t))\tilde{B}\tilde{C}\tilde{x}(t) \tag{4.8}$$

$$= (\tilde{A} - k\tilde{B}\tilde{C})\tilde{x}(t) - (\mathcal{K}(\gamma(t) - k)\tilde{B}e(t)\,, \tag{4.9}$$

where $k \in \mathbb{R}$ is arbitrary. Now the right-hand sides of (4.1b) and (4.8) are locally Lipschitz in $\tilde{x}$ and γ so that $\tilde{x}(t)$ and $\gamma(t)$ are uniquely determined on a maximal interval of existence—say, $[0, \tau)$. We now invoke Lemma 4.1 and define

$$V(t) = -\langle \tilde{x}(t), \tilde{P}_k\tilde{x}(t)\rangle,$$

where $\tilde{P}_k = \tilde{P}_k^T \preceq 0$ is the unique solution of (4.3), with $|k|$ small enough and $k\mathbf{G}(0) \succ 0$. Differentiating V along solutions of (4.1b) and (4.8) gives

$$\begin{aligned}\dot{V} &= -k^2\|\tilde{C}\tilde{x}\|^2 - \|\tilde{B}^T\tilde{P}_k\tilde{x}\|^2 - 2(\mathcal{K}(\gamma) - k)\langle \tilde{C}\tilde{x}, \tilde{B}^T\tilde{P}_k\tilde{x}\rangle \\ &= -(k^2 - (\mathcal{K}(\gamma) - k)^2)\|\tilde{C}\tilde{x}\|^2 - \|(\mathcal{K}(\gamma) - k)\tilde{C}\tilde{x} + \tilde{B}^T\tilde{P}_k\tilde{x}\|^2 \\ &\leq \mathcal{K}(\gamma)(\mathcal{K}(\gamma) - 2k)\|\tilde{C}\tilde{x}\|^2\,.\end{aligned}$$

Integrating this inequality from t_0 to t, where $0 \leq t_0 < t < \tau$, and using (4.1b) and (4.6) yield

$$-\infty < -V(t_0) \leq V(t) - V(t_0) \leq \int_{\gamma(t_0)}^{\gamma(t)} \mathcal{K}(\xi)(\mathcal{K}(\xi) - 2k)\, d\xi\,. \tag{4.10}$$

Seeking a contradiction, assume that $\lim_{t\to\tau} \gamma(t) = \infty$. Then, using (4.5) and exploiting the assumption that $\mathcal{K} \in L^2(b, \infty; \mathbb{R})$ we obtain

$$\lim_{n\to\infty} \int_{\gamma(t_0)}^{\gamma(t_n)} \mathcal{K}(\xi)(\mathcal{K}(\xi) - 2k)\, d\xi = -\infty$$

for some sequence $(t_n)_{n\in\mathbb{N}}$ with $\gamma(t_0) = b$ and $\lim_{n\to\infty} t_n = \tau$. Since this contradicts (4.10), it follows that $\gamma(t)$ is bounded on $[0, \tau)$ and consequently $\tau = \infty$, which establishes (i).

In order to prove statements (ii) and (iii), choose k in (4.9) such that $\tilde{A} - k\tilde{B}\tilde{C}$ is exponentially stable (this is possible by Theorem 3.8). Trivially, by (i), $e \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, and so it follows from the boundedness of $\mathcal{K}$ that the forcing term on the right-hand side of (4.9) is in $L^2(\mathbb{R}_+, \mathbb{R}^m)$. Therefore $\tilde{x}(t)$ is the state of an exponentially stable system driven by an L^2-input, and consequently $\lim_{t\to\infty} \tilde{x}(t) = 0$. Statements (ii) and (iii) follow now from (4.7) and (4.6), respectively. □

Remark 4.3. Whilst the property of symmetry for a general $m\times m$ matrix is nongeneric in that symmetry is destroyed by arbitrarily small perturbations, symmetry of $\mathbf{G}(0)$ is a direct consequence of, for example,

$$A = A^T\,, \quad B = C^T, \quad \text{and} \quad D = D^T\,.$$

If additionally, $D \succeq 0$, then positive definiteness of $\mathbf{G}(0)$ follows, since A is exponentially stable and $\mathbf{G}(0)$ is invertible.

It is not difficult to show that the function given in (4.2) satisfies the conditions imposed on $\mathcal{K}$ in Theorem 4.2. Notice that in general these conditions do not imply that $\lim_{\gamma\to\infty} \mathcal{K}(\gamma) = 0$.

Proposition 4.4. *Suppose* $\mathbf{G}(0) \succ 0$. *With the tuning function* $\mathcal{K}(\gamma) = \gamma^{-p}$, $0 < p < 1$, *and* $\gamma_0 > 0$ *statements* (i)–(iii) *of Theorem* 4.2 *hold.*

Proof. The proof is the same as that of Theorem 4.2 up to (4.10). By the special choice of $\mathcal{K}$, (4.10) implies that $\gamma(\cdot)$ is bounded. The remainder of the proof is the same as that of Theorem 4.2. □

In Proposition 4.4 we may replace γ^{-p} by any function $\mathcal{K}$ which satisfies

$$\int_{\gamma_0}^{\infty} \mathcal{K}(\xi)(\mathcal{K}(\xi) - 2k)d\xi = -\infty$$

for some stabilizing gain $k > 0$.

The infinite-dimensional case. For the rest of this paper we will let $\mathbf{\Sigma}_{plant} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ be an exponentially stable regular system with transfer function $\mathbf{G}$. Let A, B, C, and D denote the generating operators of $\mathbf{\Sigma}_{plant}$. As in section 3 we denote the series connection of the integrator $1/s$ followed by $\mathbf{\Sigma}_{plant}$ by $\tilde{\mathbf{\Sigma}} = (\tilde{\mathbf{T}}, \tilde{\mathbf{\Phi}}, \tilde{\mathbf{\Psi}}, \tilde{\mathbf{F}})$. It was shown in section 3 that the system $\tilde{\mathbf{\Sigma}}$ is regular. Let $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ denote the corresponding generating operators (trivially, $\tilde{D} = 0$), and let $\tilde{\mathbf{G}}(s) = (1/s)\mathbf{G}(s)$ denote the transfer function of $\tilde{\mathbf{\Sigma}}$.

We were not able to extend the proofs of Theorem 4.2 and Proposition 4.4 to the infinite-dimensional setting outlined in section 2. The problem is caused by the fact that Lemma 4.1 does not hold in the infinite-dimensional case, unless very strong and unnatural controllability assumptions are imposed. As already mentioned in the introduction, the approach in Cook [1] does not carry over to infinite-dimensional systems either. Nevertheless, it will turn out that in the infinite-dimensional situation we can still use tuning functions $\mathcal{K}$ satisfying $\mathcal{K}(\gamma) \to 0$ as $\gamma \to \infty$.

Theorem 4.5. *Let* $\mathbf{\Sigma}_{plant}$ *be a* m*-input* m*-output exponentially stable regular system given by* (2.1). *Suppose that the transfer function* $\mathbf{G}$ *of* $\mathbf{\Sigma}_{plant}$ *is such that* $\mathbf{G}(0)$ *is sign definite. Let* $r\,\theta(t)$, $r \in \mathbb{R}^m$, *be an arbitrary constant vector-valued reference signal, and consider the control law*

$$u(t) = u_0 + \int_0^t \log^{-p}\gamma(\xi)\cos(\log^q \gamma(\xi))e(\xi)\,d\xi\,, \tag{4.11}$$

$$\dot{\gamma}(t) = \|e(t)\|^2 \,, \quad \gamma(0) = \gamma_0 \,, \tag{4.12}$$

where $e(t) = r - y(t)$ *and* $p \geq 0$, $q > 0$, *and* $q + 2p < 1$. *Then for all* $(x_0, u_0) \in X \times \mathbb{R}^m$ *and* $\gamma_0 > 1$, *where* X *denotes the state space of* $\mathbf{\Sigma}_{plant}$, *the following statements hold true:*

(i) $\lim_{t\to\infty} \gamma(t) = \gamma_\infty < \infty$;
(ii) $\|x(t)\|$ *and* $\|u(t)\|$ *remain bounded as* $t \to \infty$;
(iii) $e(\cdot) \in L^2(\mathbb{R}_+, \mathbb{R}^m)$.

Moreover, if $(x_0, u_0) \in D(\tilde{A})$, *then*

$$\lim_{t\to\infty} y(t) = r \,. \tag{4.13}$$

If the observation operator C *of* $\mathbf{\Sigma}_{plant}$ *is bounded, then* (4.13) *is true for all* $(x_0, u_0) \in X \times \mathbb{R}^m$.

Proof. We assume throughout the proof that $p > 0$. The case $p = 0$ can be proven using the techniques in the proof of Theorem 5.1. The first step is to convert the tracking problem ($r \neq 0$) into a stabilization problem ($r = 0$). By Lemma 3.16 there exists $(x_r, u_r) \in D(\tilde{A})$ so that $r = \tilde{C}\tilde{\mathbf{T}}_t(x_r, u_r)$ for all $t \geq 0$. Therefore, setting $\mathcal{K}(\gamma) = \log^{-p}\gamma \cos(\log^q \gamma)$ and using (4.11), it follows that

$$e = r\theta - y = \tilde{\mathbf{\Psi}}_\infty(x_r - x_0, u_r - u_0) - \tilde{\mathbf{F}}_\infty(\mathcal{K}(\gamma)e) \,. \tag{4.14}$$

The nonlinear closed-loop system given by (4.14) and (4.12) is in a form so that Proposition 2.4 is applicable. Let $[0, \tau)$ be the maximal interval of existence for solutions (e, γ) of (4.14) and (4.12) as guaranteed by Proposition 2.4. We know that $\tau < \infty$ only if $\lim_{t\to\tau} \gamma(t) = \infty$. We will prove that $\gamma(t)$ is bounded on $[0, \tau)$.

Let $(\rho_i)_{i\in\mathbb{N}}$, with $\rho_0 \geq \gamma_0$, be a strictly increasing sequence converging to ∞ and satisfying

$$\operatorname{sign}(\mathbf{G}(0)) \cos(\log^q \rho_{2i}) = 1 \quad \text{and} \quad \mathcal{K}(\rho_{2i+1}) = \mathcal{K}(\rho_{2i})/2 \,, \quad i = 0, 1, 2, \ldots \,,$$

where $\operatorname{sign}(\mathbf{G}(0)) = \pm 1$, depending on whether $\mathbf{G}(0)$ is positive or negative definite. Choosing ρ_0 sufficiently large, it follows from Theorem 3.8 that the numbers

$$k_i := \mathcal{K}(\rho_{2i})$$

are stabilizing gains for $\tilde{\mathbf{G}}(s) = (1/s)\mathbf{G}(s)$; i.e., the integrators k_i/s stabilize $\mathbf{G}$ in the sense of Definition 3.1. Note that $(\rho_i)_{i\in\mathbb{N}}$ can be chosen so that

$$|\mathcal{K}(\gamma)| \in (|k_i|/2, |k_i|) \quad \text{and} \quad k_i\mathcal{K}(\gamma) > 0 \qquad \forall\, \gamma \in (\rho_{2i}, \rho_{2i+1})$$

and that $|k_i| \searrow 0$ as $i \to \infty$. Moreover, by applying Proposition 3.9 we can always choose ρ_0 sufficiently large so that

$$\|\tilde{\mathbf{G}}^{k_i}\|_\infty = \frac{1}{|k_i|} \tag{4.15}$$

for all i.

Seeking a contradiction, suppose that $\gamma(t)$ is unbounded on $[0, \tau)$. Then we can find a sequence of times $t_0 < t_1 < \cdots < \tau$ with

$$\gamma(t_i) = \rho_i.$$

We now use these observations combined with estimates we obtain from contraction-mapping–type arguments. Using (2.9b) on each interval $[t_{2i}, t_{2i+1}]$ we can write the error $e(\cdot)$ as

$$\mathbf{L}_{t_{2i}}\mathbf{P}_{t_{2i+1}}e = \tilde{\mathbf{\Psi}}^{k_i}_{t_{2i+1}-t_{2i}}(\tilde{x}(t_{2i})) - \tilde{\mathbf{F}}^{k_i}_{t_{2i+1}-t_{2i}}(\mathbf{L}_{t_{2i}}\mathbf{P}_{t_{2i+1}}(\mathcal{K}(\gamma) - k_i)e) \,, \text{[5]} \tag{4.16}$$

[5] By slight abuse of notation we write $\tilde{\mathbf{\Psi}}^{k_i}_{t_{2i+1}-t_{2i}}$ instead of $\tilde{\mathbf{\Psi}}^{k_i I}_{t_{2i+1}-t_{2i}}$, etc.

where

$$\tilde{x}(t) = \tilde{\mathbf{T}}_t(x_r - x_0, u_r - u_0) - \tilde{\mathbf{\Phi}}_t(\mathbf{P}_t \mathcal{K}(\gamma) e)\,.$$

By using (2.8a) we can express $\tilde{x}(t)$ as

$$\tilde{x}(t) = \tilde{\mathbf{T}}_t^{k_0}(x_r - x_0, u_r - u_0) - \tilde{\mathbf{\Phi}}_t^{k_0}(\mathbf{P}_t(\mathcal{K}(\gamma) - k_0)e)\,. \tag{4.17}$$

Using (2.7) and (2.8b), with $u = 0$ and $K = k_0 - k_i$, we obtain

$$\tilde{\mathbf{\Psi}}_t^{k_i} z = \tilde{\mathbf{\Psi}}_t^{k_0} z - \tilde{\mathbf{F}}_t^{k_0}((k_i - k_0)\tilde{\mathbf{\Psi}}_t^{k_i} z) \quad \forall\, t \geq 0\,,\ \ \forall\, z \in X \times \mathbb{R}^m\,. \tag{4.18}$$

Now for all $t \in [t_{2i}, t_{2i+1}]$ we have

$$|\mathcal{K}(\gamma(t)) - k_i| \leq \frac{|k_i|}{2}.$$

Moreover, $\|\tilde{\mathbf{F}}_\infty^{k_i}\| = \|\tilde{\mathbf{G}}^{k_i}\|_\infty$, and hence it follows from (4.15) that

$$\|\tilde{\mathbf{F}}_{t_{2i+1}-t_{2i}}^{k_i}\| \leq \|\tilde{\mathbf{F}}_\infty^{k_i}\| = \frac{1}{|k_i|}\,, \qquad \text{whilst} \quad \|\tilde{\mathbf{F}}_t^{k_0}\| \leq \|\tilde{\mathbf{F}}_\infty^{k_0}\| = \frac{1}{|k_0|}\,.$$

Therefore integrating (4.16) from 0 to $t_{2i+1} - t_{2i}$ and taking estimates we have

$$\|e\|_{L^2(t_{2i},t_{2i+1})} \leq \frac{1}{1 - \|\tilde{\mathbf{F}}_\infty^{k_i}\|\,\|\mathcal{K}(\gamma) - k_i\|_{L^\infty(t_{2i},t_{2i+1})}} \|\tilde{\mathbf{\Psi}}_{t_{2i+1}-t_{2i}}^{k_i}(\tilde{x}(t_{2i}))\|_{L^2(0,t_{2i+1}-t_{2i})}$$

$$\leq 2\|\tilde{\mathbf{\Psi}}_{t_{2i+1}-t_{2i}}^{k_i}(\tilde{x}(t_{2i}))\|_{L^2(0,t_{2i+1}-t_{2i})}\,. \tag{4.19}$$

Since $\tilde{\mathbf{G}}^{k_i} \in H_-^{\infty\ m\times m}$, an application of Theorem 3.14 yields that the closed-loop semigroup $\tilde{\mathbf{T}}^{k_i}$ is exponentially stable. It follows that $\tilde{\mathbf{\Psi}}_\infty^{k_i} z \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ for all $z \in X \times \mathbb{R}^m$. As a consequence, integrating in (4.18) from 0 to ∞ and taking estimates gives

$$\|\tilde{\mathbf{\Psi}}_\infty^{k_i}(\tilde{x}(t_{2i}))\|_{L^2(0,\infty)} \leq \frac{1}{1 - \|\tilde{\mathbf{F}}_\infty^{k_0}\||k_0 - k_i|} \|\tilde{\mathbf{\Psi}}_\infty^{k_0}(\tilde{x}(t_{2i}))\|_{L^2(0,\infty)}$$

$$= \frac{k_0}{k_i}\|\tilde{\mathbf{\Psi}}_\infty^{k_0}(\tilde{x}(t_{2i}))\|_{L^2(0,\infty)}\,. \tag{4.20}$$

Combining (4.19) and (4.20) and using the definition of $\gamma(t)$, we obtain

$$\sqrt{\rho_{2i+1} - \rho_{2i}} = \|e\|_{L^2(t_{2i},t_{2i+1})} \leq 2\frac{k_0}{k_i}\|\tilde{\mathbf{\Psi}}_\infty^{k_0}\tilde{x}(t_{2i})\|_{L^2(0,\infty)} \leq \frac{c_0}{|k_i|}\|\tilde{x}(t_{2i})\|\,, \tag{4.21}$$

where $c_0 > 0$ is a constant obtained from the exponential stability of the semigroup $\tilde{\mathbf{T}}^{k_0}$. Setting $t = t_{2i}$ in (4.17) and taking estimates yield

$$\|\tilde{x}(t_{2i})\| \leq c_1 + c_2\sqrt{\rho_{2i} - \gamma_0} \tag{4.22}$$

for suitable constants $c_1 > 0$ and $c_2 > 0$. Combining (4.21) and (4.22) and using the fact that $k_i = \mathcal{K}(\rho_{2i})$, we have

$$\sqrt{\rho_{2i+1} - \rho_{2i}} \leq \frac{c_0}{|\mathcal{K}(\rho_{2i})|}(c_1 + c_2\sqrt{\rho_{2i} - \gamma_0})\,. \tag{4.23}$$

Now, by the mean value theorem, there exists $\xi_{2i} \in (\rho_{2i}, \rho_{2i+1})$ such that

$$-\frac{1}{\mathcal{K}'(\xi_{2i})} = \frac{\rho_{2i+1} - \rho_{2i}}{\mathcal{K}(\rho_{2i}) - \mathcal{K}(\rho_{2i+1})} = 2\,\frac{\rho_{2i+1} - \rho_{2i}}{\mathcal{K}(\rho_{2i})}$$

so that (4.23) becomes

$$\sqrt{\frac{-\mathcal{K}^3(\rho_{2i})}{2\mathcal{K}'(\xi)}} \le c_0(c_1 + c_2\sqrt{\rho_{2i} - \gamma_0})\,. \tag{4.24}$$

Using the specific form of $\mathcal{K}$ we have

$$\mathcal{K}'(\xi) = -\frac{p\cos(\log^q \xi) + q\log^q \xi \sin(\log^q \xi)}{\xi \log^{1+p} \xi}\,,$$

which on substituting in (4.24) and rearranging yields

$$1 \le 2\,\left[c_0(c_1 + c_2\sqrt{\rho_{2i} - \gamma_0})\right]^2 \frac{1}{|\mathcal{K}(\rho_{2i})|^3}\,\frac{|p\cos(\log^q \xi_{2i}) + q\log^q \xi_{2i}\sin(\log^q \xi_{2i})|}{\xi_{2i}\log^{1+p}\xi_{2i}}\,. \tag{4.25}$$

Using the fact that $\mathcal{K}(\rho_{2i}) = |\log^{-p}\rho_{2i}|$ and gathering dominant terms in (4.25) lead to

$$1 \le c_3\left(p(\log\rho_{2i})^{2p-1} + q(\log\rho_{2i})^{2p+q-1}\right) \tag{4.26}$$

for some constant $c_3 > 0$. But $p, q > 0$ and $q + 2p < 1$ so that the right-hand side of (4.26) approaches zero for $\rho_{2i} \to \infty$, which is in contradiction to (4.26). Hence $\gamma(\cdot)$ is bounded, which establishes statements (i) and (iii). Boundedness of $\tilde{x}(t)$ and therefore part (ii) follows directly from (4.17), the exponential stability of $\tilde{\mathbf{T}}^{k_0}$, and statements (i) and (iii).

To prove the last statement in the theorem let $(x_0, u_0) \in D(\tilde{A})$. Then $\tilde{x}_0 := (x_r - x_0, u_r - u_0) \in D(\tilde{A})$, and from (4.14) and (2.8b) we obtain

$$e(t) = \tilde{C}^{k_0}\tilde{\mathbf{T}}_t^{k_0}\tilde{x}_0 - (\tilde{\mathbf{F}}_\infty^{k_0}[(\mathcal{K}(\gamma) - k_0)e])(t) \quad \forall\, t \ge 0\,. \tag{4.27}$$

By Lemma 3.11, $\tilde{x}_0 \in D(\tilde{A}^{k_0})$, and hence it follows from the exponential stability of the semigroup $\tilde{\mathbf{T}}^{k_0}$ that the first term on the right-hand side of (4.27) tends exponentially to 0 as $t \to \infty$. In order to show that the second term converges to 0 as $t \to \infty$ set $v(t) = (\mathcal{K}(\gamma(t)) - k_0)e(t)$, and realize that, by statements (i) and (iii), $v \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. Clearly,

$$(\mathbb{L}(\tilde{\mathbf{F}}_\infty^{k_0}v))(s) = \tilde{\mathbf{G}}(s)(1 + k_0\tilde{\mathbf{G}}(s))^{-1}(\mathbb{L}v)(s) = \tilde{\mathbf{G}}^{k_0}(s)(\mathbb{L}v)(s)\,.$$

Since $|k_0|$ is sufficiently small, it follows from Theorem 3.8 that $\tilde{\mathbf{G}}^{k_0} \in H_-^{2\ m\times m}$. (Note that using the notation in Theorem 3.8 we have $\tilde{\mathbf{G}}^{k_0} = \mathbf{G}\mathbf{E}_{k_0}$.) Therefore, by the Paley–Wiener theorem, $\tilde{\mathbf{F}}_\infty^{k_0}$ is a convolution operator with a matrix-valued kernel whose entries are L^2-functions. Now it is well known that the convolution of two functions belonging to $L^2(\mathbb{R}_+, \mathbb{R})$ converges to 0 as $t \to \infty$, and hence $\lim_{t\to\infty}(\tilde{\mathbf{F}}_\infty^{k_0}v)(t) = 0$. Finally, if C is bounded, then $\tilde{C}$ is bounded, i.e., $\tilde{C} \in \mathcal{L}(X \times \mathbb{R}^m, \mathbb{R}^m)$. Furthermore, by Proposition 3.12, $\tilde{C}^{k_0} = \tilde{C}$, and therefore the first term on the right-hand side of (4.27) converges (exponentially) to 0 as $t \to \infty$ for all $(x_0, u_0) \in X \times \mathbb{R}^m$. ∎

Note that the condition $(x_0, u_0) \in D(\tilde{A})$ in statement (iv) of Theorem 4.5 required in proving that $\lim_{t\to 0} e(t) = 0$ is system dependent. This is a little disturbing since, from the outset, we assume that the specific system to be controlled is unknown. However, in most cases, the initial states will be sufficiently smooth so that the condition $(x_0, u_0) \in D(\tilde{A})$ is satisfied. Note that if $(x_0, u_0) \notin D(\tilde{A})$, then $e(\cdot)$ will in general not make sense pointwise and cannot be expected to converge to 0 in the usual sense (see, however, Remark 3.15).

Note that in the infinite-dimensional case the tuning function $\mathcal{K}(\gamma)$ decays to 0 like a fractional power of $\log \gamma$ as $\gamma \to \infty$, whereas in the finite-dimensional case it decays to 0 like a fractional power of γ. However, in the case when it is known that $\mathbf{G}(0) \succ 0$, we can use tuning functions which decay to 0 like a fractional power, although more slowly than in the finite-dimensional case.

PROPOSITION 4.6. *Suppose that the conditions of Theorem* 4.5 *hold and that additionally* $\mathbf{G}(0) \succ 0$. *If*

$$u(t) = u_0 + \int_0^t \gamma^{-p}(\xi) e(\xi)\, d\xi \,, \tag{4.28}$$

$$\dot{\gamma}(t) = \|e(t)\|^2 \,, \quad \gamma(0) = \gamma_0 > 0 \,, \tag{4.29}$$

and $0 < p < \frac{1}{2}$, *then the conclusions of Theorem* 4.5 *hold.*

Proof. It is sufficient to show that $\gamma(\cdot)$ is bounded. Let $[0, \tau)$ be the maximal interval of existence. If $\gamma(\cdot)$ is unbounded on $[0, \tau)$, then there exists $t_1 \geq 0$ such that with $\gamma_1 = \gamma(t_1)$, $k_1 = \gamma_1^{-p}$ is a stabilizing gain. For any $t \in (t_1, \tau)$ we have, as in the proof of Theorem 4.5, that on $[t_1, t]$

$$\mathbf{L}_{t_1} \mathbf{P}_t e = \tilde{\mathbf{\Psi}}^{k_1}_{t-t_1}(\tilde{x}(t_1)) - \tilde{\mathbf{F}}^{k_1}_{t-t_1}(\mathbf{L}_{t_1} \mathbf{P}_t (\mathcal{K}(\gamma) - k_1) e) \,. \tag{4.30}$$

We can assume that k_1 is small enough so that, using Proposition 3.9 and estimating, we obtain

$$\sqrt{\gamma(t) - \gamma_1} \leq c \gamma^p(t)$$

for some $c > 0$ and all $t \in [t_1, \tau)$. This inequality clearly contradicts the unboundedness of $\gamma(\cdot)$ and the assumption that $p < 1/2$. ∎

The condition $\mathbf{G}(0) \succ 0$ is satisfied for a large class of exponentially stable infinite-dimensional systems with self-adjoint generator A, co-located control and observation and positive semidefinite feedthrough (cf. Remark 4.3).

5. Adaptive low-gain control of multivariable systems with sign-indefinite steady-state gain. In this section we consider the adaptive low-gain tracking problem, for stable regular systems with square $m \times m$ transfer functions $\mathbf{G}(s)$ and invertible steady-state gain. In section 4, under the assumption that $\mathbf{G}(0)$ is sign definite, we could exploit the fact that for all gains k having the "correct" sign and with $|k|$ sufficiently small, $\|\tilde{\mathbf{G}}^k\|_\infty = 1/|k|$ (see Proposition 3.9). If $\mathbf{G}(0)$ is sign indefinite or even nonsymmetric, then, again by Proposition 3.9, we no longer have this result. To overcome this problem we do not use a tuning function $\mathcal{K}$ reflecting the low-gain nature of the problem in the sense that $\lim_{\gamma \to \infty} \mathcal{K}(\gamma) = 0$ but instead resort to a gain which oscillates smoothly between 0 and 2. (In fact, 2 could be replaced by any positive number δ.)

As in the previous sections let $u(\cdot)$ and $y(\cdot)$ denote the plant input and plant output, respectively, and set $e(\cdot) = r - y(\cdot)$, where $r \in \mathbb{R}^m$ is a demand vector. Modulo certain technicalities involving "spectrum unmixing" of $\mathbf{G}(0)$ (to be made precise) we show that

$$u(t) = u_0 + \int_0^t [1 + \cos(\log^q \gamma(\xi))] e(\xi)\, d\xi \,, \quad \text{where } 0 < q < 1 \,, \tag{5.1}$$

$$\dot{\gamma}(t) = \|e(t)\|^2\,, \quad \gamma(0) = \gamma_0, \tag{5.2}$$

is a universal adaptive tracking controller.

We assume throughout that $\mathbf{\Sigma}_{plant}$ is an m-input m-output, exponentially stable, regular system given by (2.1). We will consider two cases. In the first one we assume that the spectrum of $\mathbf{G}(0)$ is unmixed in the sense that $\sigma(\mathbf{G}(0)) \subset \mathbb{C}_0$. In the second case the a priori knowledge about $\mathbf{G}(0)$ guarantees only that $\mathbf{G}(0)$ is invertible.

THEOREM 5.1. *Assume that $\sigma(\mathbf{G}(0)) \subset \mathbb{C}_0$. Let $r \in \mathbb{R}^m$ be an arbitrary demand vector. If $u(t)$ is given by* (5.1)*, with gain adaptation* (5.2)*, then for each $(x_0, u_0) \in X \times \mathbb{R}^m$ and $\gamma_0 > 1$ we have*

(i) $\lim_{t\to\infty} \gamma(t) = \gamma_\infty < \infty$;
(ii) $\|x(t)\|$ *and* $\|u(t)\|$ *remain bounded as* $t \to \infty$;
(iii) $e(\cdot) \in L^2(\mathbb{R}_+, \mathbb{R}^m)$.

Moreover, if $(x_0, u_0) \in D(\tilde{A})$, then

$$\lim_{t\to\infty} y(t) = r\,. \tag{5.3}$$

If the observation operator C is bounded, then (5.3) *holds for all $(x_0, u_0) \in X \times \mathbb{R}^m$.*

In the proof of this result we do not have to be so careful with the estimates, since we need only to work in a neighborhood of a stabilizing integral gain and do not need to account for the possibility of the feedback gain approaching 0.

Proof. The first step is to convert the tracking problem ($r \neq 0$) into a stabilization problem ($r = 0$). Let $r \in \mathbb{R}^m$, $(x_0, u_0) \in X \times \mathbb{R}^m$ be given and set $\mathcal{K}(\gamma) := 1 + \cos(\log^q \gamma(t))$. By Lemma 3.16 there exists $\tilde{x}_0 \in X \times \mathbb{R}^m$ such that

$$e = \tilde{\mathbf{\Psi}}_\infty \tilde{x}_0 - \tilde{\mathbf{F}}_\infty(\mathcal{K}(\gamma)e)\,. \tag{5.4}$$

Moreover, if $(x_0, u_0) \in D(\tilde{A})$, then $\tilde{x}_0 \in D(\tilde{A})$. The closed-loop system given by (5.4) and (5.2) is in a form so that Proposition 2.4 is applicable.

By Theorem 3.8 there exists $k \in (0,1)$ for which $\tilde{\mathbf{G}}^k \in H_-^{\infty\; m\times m}$. Consequently, by Theorem 3.14, $\tilde{\mathbf{T}}^k$ is an exponentially stable semigroup on $X \times \mathbb{R}^m$. As in the sign-definite case, seeking a contradiction, suppose that $\gamma(t)$ is unbounded on the maximal interval of existence $[0, \tau)$. To this end choose $\varepsilon \in (0, k)$ such that $k + \varepsilon < 1$ and let $(\rho_i)_{i\in\mathbb{N}}$ be a sequence with

$$\rho_i \nearrow \infty\,, \quad \rho_0 \geq \gamma_0\,, \quad \mathcal{K}(\rho_{2i}) = k - \varepsilon\,, \quad \mathcal{K}(\rho_{2i+1}) = k + \varepsilon$$

and such that

$$\mathcal{K}(\gamma) \in (k - \varepsilon, k + \varepsilon) \quad \forall\, \gamma \in (\rho_{2i}, \rho_{2i+1})\,.$$

Exploiting the unboundedness of $\gamma(t)$ we can find a sequence of times $t_0 < t_1 < \cdots < \tau$ so that $\gamma(t_i) = \rho_i$. Using (2.9b) we obtain

$$\mathbf{L}_{t_{2i}}\mathbf{P}_{t_{2i+1}} e = \tilde{\mathbf{\Psi}}^k_{t_{2i+1}-t_{2i}} \tilde{x}(t_{2i}) - \tilde{\mathbf{F}}^k_{t_{2i+1}-t_{2i}}(\mathbf{L}_{t_{2i}}\mathbf{P}_{t_{2i+1}}(\mathcal{K}(\gamma) - k)e)\,, \tag{5.5}$$

where

$$\tilde{x}(t) = \tilde{\mathbf{T}}_t \tilde{x}_0 - \tilde{\mathbf{\Phi}}_t \mathbf{P}_t \mathcal{K}(\gamma) e\,.$$

Integrating from 0 to $t_{2i+1} - t_{2i}$ in (5.5) and taking estimates yield

$$\begin{aligned} \|e\|_{L^2(t_{2i}, t_{2i+1})} &\leq \frac{1}{1 - \|\tilde{\mathbf{F}}^k_\infty\| \|\mathcal{K}(\gamma) - k\|_{L^\infty(t_{2i}, t_{2i+1})}} \|\tilde{\mathbf{\Psi}}^k_\infty \tilde{x}(t_{2i})\|_{L^2(0,\infty)} \\ &\leq c_0 \|\tilde{x}(t_{2i})\| \end{aligned} \tag{5.6}$$

for some suitable $c_0 > 0$, provided that ε is small enough (for example, $\|\tilde{\mathbf{F}}^k_\infty\|\varepsilon = 1/2$).

Applying the input-state variation of parameters formula (2.8a) to $\tilde{\mathbf{\Sigma}}$ with $K = kI$ and $u = \mathcal{K}(\gamma)e$ it follows from the exponential stability of $\tilde{\mathbf{T}}^k$ and (5.2) that

$$\|\tilde{x}(t_{2i})\| \leq c_1 + c_2\sqrt{\rho_{2i} - \gamma_0} \tag{5.7}$$

for some constants $c_1 > 0$ and $c_2 > 0$. Combining (5.6) and (5.7) we have

$$\sqrt{\rho_{2i+1} - \rho_{2i}} \leq c_0(c_1 + c_2\sqrt{\rho_{2i} - \gamma_0})\,. \tag{5.8}$$

Clearly,

$$\rho_{2i+1} - \rho_{2i} = 2\varepsilon/\mathcal{K}'(\xi_{2i})$$

for some $\xi_{2i} \in (\rho_{2i}, \rho_{2i+1})$. Combining this with (5.8) leads to

$$1 \leq \frac{1}{2\varepsilon}[c_0(c_1 + c_2\sqrt{\rho_{2i} - \gamma_0})]^2\mathcal{K}'(\xi_{2i})\,. \tag{5.9}$$

Now

$$\mathcal{K}'(\xi) = -q\sin(\log^q \xi)(\log^{q-1}\xi)/\xi,$$

and $0 < q < 1$, and we see that the right-hand side of (5.9) converges to 0 as $i \to \infty$, which yields a contradiction. It follows that $\gamma(\cdot)$ is bounded, showing that (i) and (iii) hold true. The remaining claims follow readily, using the same techniques as in the proof of Theorem 4.5. ∎

Specialized to the case when $\mathbf{G}(0) \succ 0$, it is natural to compare the control law in Theorem 5.1 to the one in Proposition 4.6. Intuitively it should be advantageous to use the controller in Proposition 4.6, since in this case the gain passes rapidly into the "correct" parameter region once and remains there, whereas the gain in the controller in Theorem 5.1 oscillates slowly and may pass in and out of the "correct" region several times before converging. Moreover, small output disturbances could lead to further cycles in the gain adaptation.

In Theorem 5.1 we assumed that $\sigma(\mathbf{G}(0)) \subset \mathbb{C}_0$. We now consider the case when we know only that $\det \mathbf{G}(0) \neq 0$. In the context of high-gain adaptive stabilization Mårtensson [21], [22] has shown that there exists a finite set $\{\Gamma_1, \ldots, \Gamma_\ell\}$ so that given any invertible $m \times m$ matrix M there exists $\nu \in \{1, 2, \ldots, \ell\}$ such that $\sigma(M\Gamma_\nu) \subset \mathbb{C}_0$. We now use this result in order to unmix the spectrum of $\mathbf{G}(0)$. Consider the feedback law

$$u(t) = u_0 + \int_0^t [1 + \cos(\log^q \gamma(\xi))]\Gamma_{S(\gamma(\xi))}e(\xi)\,d\xi\,, \tag{5.10}$$

where $0 < q < 1$ and

$$S(\gamma) = j \quad \text{if} \quad (2\pi)^{-1}\log^q \gamma \in [p\ell + j, p\ell + j + 1) \ \text{ for some } \ p \in \mathbb{N}.$$

Note that the feedback gain matrix in (5.10) is piecewise smooth but discontinuous whenever $(2\pi)^{-1}\log^q \gamma$ takes on integer values, so Proposition 2.4 is no longer valid. However, these discontinuities in the gain are easily handled by a minor modification to the proof of Proposition 2.4.

THEOREM 5.2. *Assume that* $\det \mathbf{G}(0) \neq 0$. *Let* $r \in \mathbb{R}^m$ *be an arbitrary demand vector. If* $u(t)$ *is given by* (5.10), *with adaptation* (5.2), *then for each* $(x_0, u_0) \in X \times \mathbb{R}^m$ *and* $\gamma_0 > \exp(\sqrt[q]{2\pi})$ *we have*[6]

[6]Note that $S(\gamma)$ is defined only for $\gamma \geq \exp(\sqrt[q]{2\pi})$.

(i) $\lim_{t\to\infty}\gamma(t) = \gamma_\infty < \infty$;
(ii) $\|x(t)\|$ *and* $\|u(t)\|$ *remain bounded as* $t \to \infty$;
(iii) $e(\cdot) \in L^2(\mathbb{R}_+, \mathbb{R}^m)$.

Moreover, if $(x_0, u_0) \in D(\tilde{A})$*, then* (5.3) *holds. If the observation operator* C *is bounded, then* (5.3) *holds for all* $(x_0, u_0) \in X \times \mathbb{R}^m$*.*

Proof. Let $\nu \in \{1, 2, \ldots, \ell\}$ be such that $\sigma(\mathbf{G}(0)\Gamma_\nu) \in \mathbb{C}_0$. By Theorem 3.8 there exists $k \in (0,1)$ such that the integrator $(k/s)\Gamma_\nu$ stabilizes $\mathbf{G}$. Consequently, by Theorem 3.14, the semigroup $\tilde{\mathbf{T}}^{k\Gamma_\nu}$ is exponentially stable. As in the proof of Theorem 5.1 set $\mathcal{K}(\gamma) = 1 + \cos(\log^q \gamma)$. By Lemma 3.16 there exists $\tilde{x}_0 \in X \times \mathbb{R}^m$ such that

$$e = \tilde{\mathbf{\Psi}}_\infty \tilde{x}_0 - \tilde{\mathbf{F}}_\infty(\mathcal{K}(\gamma)\Gamma_{S(\gamma)} e)\,. \tag{5.11}$$

Let $[0, \tau)$ be the maximal interval of existence for the solution (e, γ) of the closed-loop system given by (5.11) and (5.2). Seeking a contradiction, suppose that $\lim_{t\to\tau}\gamma(t) = \infty$. Choose $\varepsilon \in (0, k)$ such that $\varepsilon + k < 1$. Then there exists a sequence $0 \le t_0 < t_1 < \cdots < \tau$ with

$$\mathcal{K}(\gamma(t_{2i})) = k - \varepsilon\,, \quad \mathcal{K}(\gamma(t_{2i+1})) = k + \varepsilon$$

and such that

$$\mathcal{K}(\gamma(t)) \in (k - \varepsilon, k + \varepsilon) \quad \text{and} \quad S(\gamma(t)) = \nu \qquad \forall\, t \in [t_{2i}, t_{2i+1}]\,.$$

As in the proof of Theorem 5.1, we can use (2.9b) to obtain

$$\mathbf{L}_{t_{2i}}\mathbf{P}_{t_{2i+1}} e = \tilde{\mathbf{\Psi}}^{k\Gamma_\nu}_{t_{2i+1}-t_{2i}} \tilde{x}(t_{2i}) - \tilde{\mathbf{F}}^{k\Gamma_\nu}_{t_{2i+1}-t_{2i}}(\mathbf{L}_{t_{2i}}\mathbf{P}_{t_{2i+1}}(\mathcal{K}(\gamma) - k)\Gamma_\nu e)\,.$$

The remainder of the proof follows closely that of Theorem 5.1 and is omitted. ∎

The control law given by (5.10) and (5.2) depends crucially on the unmixing set $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_\ell\}$. Clearly, if $m = 1$, then $\{1, -1\}$ is an unmixing set. For the case $m = 2$ an unmixing set of cardinality 6 is given in Mårtensson [21], [22]. Zhu [51] has constructed an unmixing set having cardinality 32 for the case $m = 3$. Unfortunately, the cardinality of the unmixing sets given by the general construction in [22] is far too large than would be convenient for applications.

6. Examples and simulations. The results of sections 3–5 apply to the general class of regular linear systems. For the purpose of illustration we consider two simple examples: finite-dimensional systems with output delays and a damped wave equation in a single spatial variable with boundary control and observation. In all of the simulations we used Simulink in Matlab. Note that the reference signals to be tracked are stepped, with nonzero step time.

Example 6.1. Systems with output delays:

We consider a class

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad y(t) = Cx(t-h) \tag{6.1}$$

of systems with output delay, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $h > 0$. The system (6.1) can be represented as a so-called Pritchard–Salamon system with state space $\mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n)$; see, e.g., Pritchard and Salamon [35, 36]. Since Pritchard–Salamon systems are regular in the sense of section 2, it follows that the results of sections 3–5 can be applied to (6.1), provided that $\sigma(A) \subset \mathbb{C}_0$ and $\det CA^{-1}B \neq 0$. We consider three particular cases.

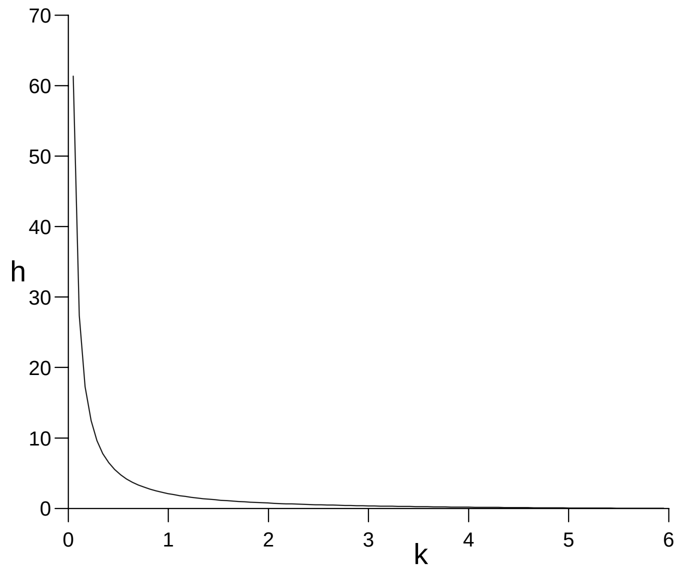

FIG. 6.1. *Tolerable delay as a function of k.*

(a) $m = 1$, $n = 2$, and

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = (1, 0)\,.$$

If $h = 0$, then $\dot{u} = -ky$ stabilizes (6.1) for all $k \in (0, 6)$. Using a stability window analysis (Walton and Marshall [43]), we can compute for each $k \in (0, 6)$ the range of $h \in (0, h(k))$ for which $\dot{u} = -ky$ stabilizes (6.1). In Figure 6.1, $h(k)$ is plotted against k for k in the range $(0, 6)$. Figure 6.2 shows a plot of $y(t)$, $r(t)$, and $\mathcal{K}(\gamma(t))$ against t for (4.28) with $p = 0.4$ when $h = 4$, $x(0) = (-1 \;\; 3)^T$, $u(0) = -1$, and $y(t) = -4$ for $t < 0$. Note in this case that the integrator gain can take values in $(0, 0.6)$ and that $\mathcal{K}(\gamma(\infty)) = 0.07$.

(b) We now consider two cases with $m = 2$, $n = 3$. In the first case $\mathbf{G}(0)$ is sign definite and in the second $\mathbf{G}(0)$ is sign indefinite.

(i) In this example we take

$$A = \begin{pmatrix} -1 & 0 & -2 \\ 0 & -1 & -3 \\ -2 & -3 & -14 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

so that

$$\mathbf{G}(0) = \begin{pmatrix} 7 & 6 \\ 6 & 11 \end{pmatrix} \succ 0\,.$$

We assume that this knowledge of the sign of the steady-state gain is available and use (4.28) with $p = 0.15$.

Figure 6.3 shows plots of $y(t)$, $r(t)$, and $\mathcal{K}(\gamma(t))$ for the case $h = 1$ with $y(\cdot) = 0$ on $[-1, 0)$, $x(0) = (0.4, 0.3, 0.25)^T$, and $u(0) = (1.5, 1)^T$, with the reference signal $r(t) = \theta(t)(5, 0)^T + \theta(t - 20)(5, 3)^T$.

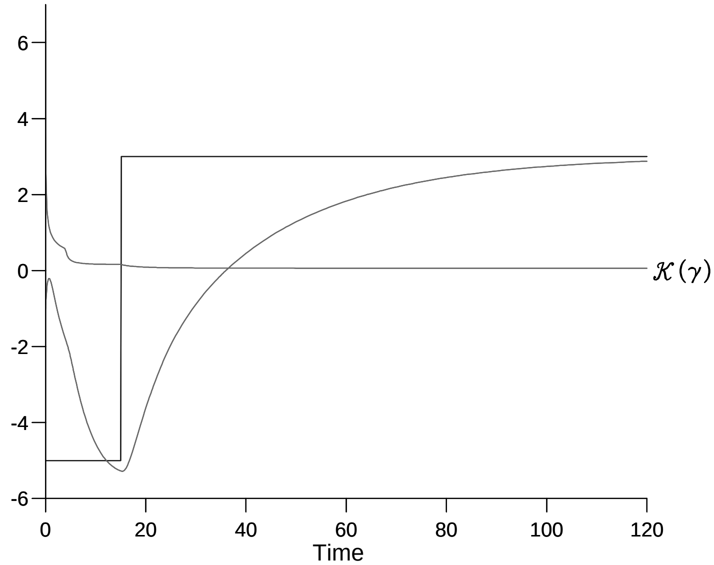

FIG. 6.2. *Simulation with* $\mathcal{K}(\gamma) = \gamma^{-0.4}$.

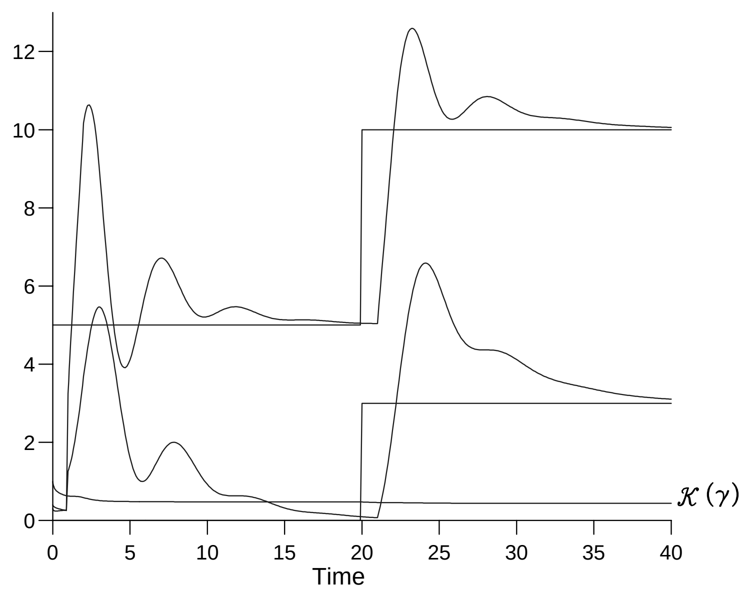

FIG. 6.3. *Simulation with* $\mathcal{K}(\gamma) = \gamma^{-0.15}$.

(ii) In this example we take

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

so that

$$\mathbf{G}(0) = \begin{bmatrix} 3 & 0.1667 \\ 0 & 1 \end{bmatrix}.$$

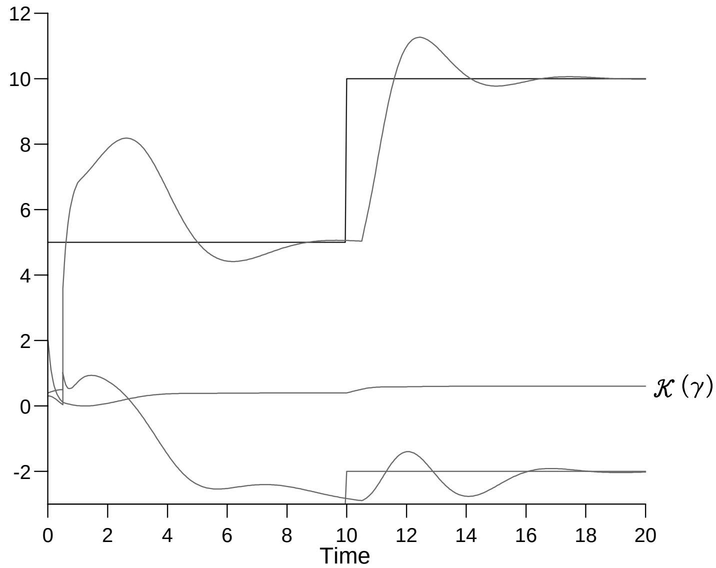

FIG. 6.4. *Simulation with* $\mathcal{K}(\gamma) = 1 + \cos(\log^{0.95}\gamma)$.

Clearly $\sigma(\mathbf{G}(0)) \subset \mathbb{C}_0$. We assume that this knowledge is available and use (5.1) with $q = 0.95$.

Figure 6.4 shows plots of $y(t)$, $r(t)$, and $\mathcal{K}(\gamma(t))$ for the case $h = 0.5$ with $y(\cdot) = 0$ on $[-0.5, 0)$, $x(0) = (0.4, 0.3, 0.25)^T$ and $u(0) = (1.5, 1)^T$ with the reference signal $r(t) = \theta(t)(5, -3)^T + \theta(t - 10)(5, 1)^T$.

Example 6.2. A wave equation with boundary control and observation: We consider the damped wave equation

$$(6.2) \qquad \frac{\partial^2 w}{\partial t^2}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) - 2a\frac{\partial w}{\partial t}(z,t) - a^2 w(z,t)\,, \qquad t > 0\,,\ z \in (0,1)\,,$$

with boundary conditions

$$w(0,t) = 0\,, \quad \frac{\partial w}{\partial z}(1,t) = u(t)$$

and boundary observation

$$y(t) = \frac{\partial w}{\partial t}(1,t) + bw(1,t)\,,$$

where $a > 0$ and $b \neq 0$. This system has a regular, exponentially stable realization on the state space

$$X = \{x = [x_1, x_2]^T \in H^1[0,1] \oplus L^2[0,1] \mid x_1(0) = 0\}.$$

Moreover, $\mathbf{G}(s) = \frac{s+b}{s+a}\frac{\sinh(s+a)}{\cosh(s+a)}$ so that $\mathbf{G}(0) = \frac{b\sinh(a)}{a\cosh(a)} \neq 0$. We assume that $a = \frac{1}{2}\log 0.3$ and $b = 0.3$. For purposes of illustration we assume that sign $(\mathbf{G}(0))$ is unknown so that we use (4.11) with $p = 0$ and $q = 0.9$ and the initial conditions are equal to zero.

Figure 6.5 shows $y(t)$, $r(t)$, and $\mathcal{K}(\gamma(t))$, whilst Figure 6.6 shows $y(t)$, $r(t)$, and $\mathcal{K}(\gamma(t))$ when the sign of $\mathbf{G}(0)$ is switched. Note that whilst (6.2) gives a partial differential equation realization of $\mathbf{G}(s)$, for the simulations we exploited the fact

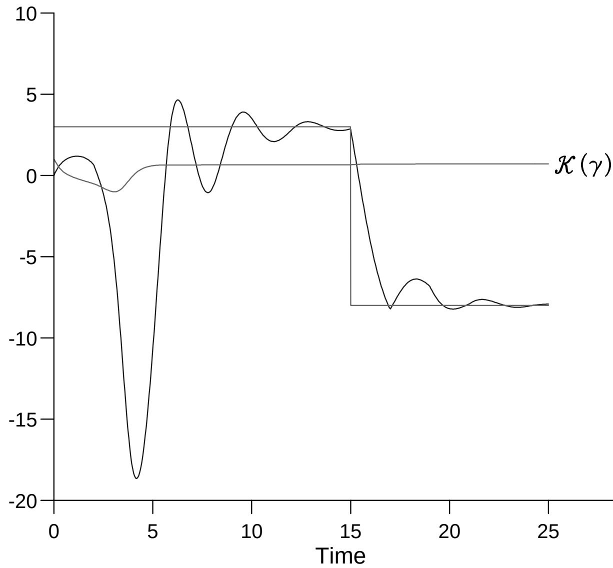

FIG. 6.5. *Simulation with* $\mathcal{K}(\gamma) = \cos(\log^{0.9}\gamma)$ *and* $\mathbf{G}(0) > 0$.

that the input-ouput behavior of (6.2), with zero initial conditions, is the same as that for the series connection of $\frac{s+b}{s+a}$ with the functional difference equation

$$y(t) = -e^{-2a}y(t-2) + u(t) - e^{-2a}u(t-2)\,. \tag{6.3}$$

The system given by (6.3) is easily realized using Simulink in Matlab.

In comparing Figures 6.5 and 6.6, we note that in the former, the gain function $\mathcal{K}(\gamma)$ undergoes two switches in sign before reaching a positive limit and in the latter switches sign only once before reaching a negative limit. The simulations are consistent with the fact that $\mathbf{G}(0) > 0$ in Figure 6.5 and $\mathbf{G}(0) < 0$ in Figure 6.6.

7. Concluding remarks. In this paper we have obtained results on nonadaptive and adaptive low-gain control of square regular systems for tracking step reference signals. It is possible to extend some of the results to nonsquare systems and sinusoidal reference signals. Finally, in [16] we have obtained discrete-time versions of the results in sections 3 and 4, with applications to sampled-data control of regular systems.

Appendix.

Proof of Proposition 2.4. For $a < b \leq \infty$ we define $L(a,b) := L^2(a,b;\mathbb{R}^m) \times L^\infty(a,b;\mathbb{R})$ and $L_{loc}(a,\infty) := L^2_{loc}(a,\infty;\mathbb{R}^m) \times L^\infty_{loc}(a,\infty;\mathbb{R})$. We define a norm on $L(a,b)$ by setting $\|(f_1,f_2)\|_{(a,b)} := \|f_1\|_{L^2(a,b)} + \|f_2\|_{L^\infty(a,b)}$. In order to prove Proposition 2.4 we shall first consider an initial value problem which contains (2.11) as a special case.

Let $T \geq 0$, $(y^0,\gamma^0) \in L_{loc}(T,\infty)$ and $(f,g) \in L(0,T)$ be given, and suppose that $F \in L^1_{loc}(\mathbb{R}_+,\mathbb{R}^{m\times m})$ and $\mathcal{K} : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function. For $\tau > T$ define the operator $\mathbf{N}_\tau : L(0,\tau) \to L(0,\tau)$ by

$$\mathbf{N}_\tau \begin{pmatrix} y \\ \gamma \end{pmatrix}(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \quad t \in [0,T]\,, \tag{A.1a}$$

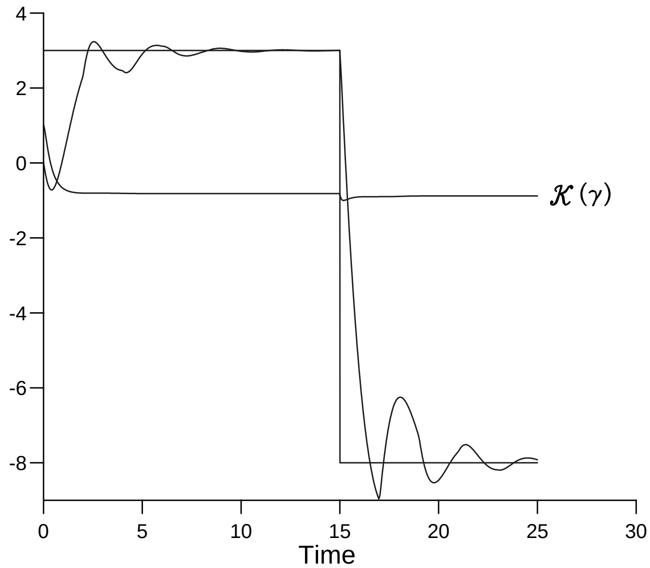

FIG. 6.6. *Simulation with* $\mathcal{K}(\gamma) = \cos(\log^{0.9}\gamma)$ *and* $\mathbf{G}(0) < 0$.

$$\text{(A.1b)}\quad \mathbf{N}_\tau \begin{pmatrix} y \\ \gamma \end{pmatrix}(t) = \begin{pmatrix} y^0(t) \\ \gamma^0(t) \end{pmatrix} + \int_0^t \begin{pmatrix} F(t-\xi) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{K}(\gamma(\xi))y(\xi) \\ \|y(\xi)\|^2 \end{pmatrix} d\xi\,, \quad t \geq T\,.$$

For $\rho > 0$ and $\tau > T$, let $B_{\rho,\tau}$ denote the closed ball in $L(T,\tau)$ of radius ρ with center in $(y^0|_{[T,\tau]}, \gamma^0|_{[T,\tau]} + \|f\|^2_{L^2(0,T)})$. Finally define

$$\mathfrak{M}_{\rho,\tau} := \{(y,\gamma) \in L(0,\tau) \,|\, (y,\gamma)|_{[T,\tau]} \in B_{\rho,\tau},\ (y,\gamma)|_{[0,T]} = (f,g)\}\,.$$

Endowed with the metric

$$d[(y_1,\gamma_1),(y_2,\gamma_2)] = \|(y_1-y_2,\gamma_1-\gamma_2)\|_{(T,\tau)} = \|(y_1-y_2,\gamma_1-\gamma_2)\|_{(0,\tau)}\,,$$

$\mathfrak{M}_{\rho,\tau}$ becomes a complete metric space.

The following lemma will be the key tool for the proof of Proposition 2.4.

LEMMA A.1. *Let* $\rho \in (0,1/2)$. *Then there exists a* $T^* > T$ *such that for all* $\tau \in (T,T^*)$ *the operator* $\mathbf{N}_\tau$ *is a contraction on* $\mathfrak{M}_{\rho,\tau}$, *i.e.,* (i) $\mathbf{N}_\tau\mathfrak{M}_{\rho,\tau} \subset \mathfrak{M}_{\rho,\tau}$ *and* (ii) *there exists* $\delta_\tau \in (0,1)$ *such that for all* $(y_1,\gamma_1),(y_2,\gamma_2) \in \mathfrak{M}_{\rho,\tau}$

$$\|\mathbf{N}_\tau(y_1,\gamma_1) - \mathbf{N}_\tau(y_2,\gamma_2)\|_{(T,\tau)} \leq \delta_\tau\|(y_1,\gamma_1)-(y_2,\gamma_2)\|_{(T,\tau)}\,.$$

In particular, for all τ *as above,* $\mathbf{N}_\tau$ *has a unique fixed point in* $\mathfrak{M}_{\rho,\tau}$.

Proof. Let $\mathbf{\Pi}_i$, $i = 1,2$, denote the operator on $L(0,\tau)$ defined by $\mathbf{\Pi}_i(f_1,f_2) = f_i$, and let $\tau^* > T$ be fixed.

(i) Setting $\eta(t) := \int_0^T F(t-\xi)\mathcal{K}(g(\xi))f(\xi)\,d\xi$, it is clear that $\eta \in L^2_{loc}(\mathbb{R}_+,\mathbb{R}^m)$. For all $\tau \in (T,\tau^*)$ and all $(y,\gamma) \in \mathfrak{M}_{\rho,\tau}$ it follows that

$$\begin{aligned}
&\|\mathbf{\Pi}_1\mathbf{N}_\tau(y,\gamma) - y^0\|^2_{L^2(T,\tau)} \\
&= \int_T^\tau \|\eta(t) + \int_0^t (\mathbf{P}_{\tau-T}F)(t-\xi)[(I-\mathbf{P}_T)\mathcal{K}(\gamma)y](\xi)\,d\xi\|^2\,dt \\
&\leq 2\left(\|\eta\|^2_{L^2(T,\tau)} + \left(\int_0^\tau \|(\mathbf{P}_{\tau-T}F)(\xi)\|\,d\xi\right)^2 \int_T^\tau \|\mathcal{K}(\gamma(\xi))y(\xi)\|^2\,d\xi\right)
\end{aligned}$$

$$\leq 2\left(\|\eta\|^2_{L^2(T,\tau)} + K^2\left(\int_0^{\tau-T}\|F(\xi)\|\,d\xi\right)^2\left(\rho^2+\int_T^\tau\|y^0(\xi)\|^2\,d\xi\right)\right), \tag{A.2}$$

where $K>0$ is such that $|\mathcal{K}(\kappa)|\leq K$ for all $\kappa\in\mathbb{R}$ with $|\kappa|\leq\rho+\|\gamma^0\|_{L^\infty(T,\tau^*)}+\|f\|^2_{L^2(0,T)}$. It follows from (A.2) that there exists $T_1\in(T,\tau^*)$ such that for all $(y,\gamma)\in\mathfrak{M}_{\rho,\tau}$ and for all $\tau\in(T,T_1)$

$$\|\mathbf{\Pi}_1\mathbf{N}_\tau(y,\gamma)-y^0\|^2_{L^2(T,\tau)}\leq\frac{\rho^2}{4}\,. \tag{A.3}$$

Moreover, we have that for $(y,\gamma)\in\mathfrak{M}_{\rho,\tau}$

$$\|\mathbf{\Pi}_2\mathbf{N}_\tau(y,\gamma)-\gamma^0(\cdot)-\|f\|^2_{L^2(0,T)}\|_{L^\infty(T,\tau)}=\int_T^\tau\|y(\xi)\|^2\,d\xi\leq\rho^2+\|y^0\|^2_{L^2(T,\tau)}\,. \tag{A.4}$$

Since $\rho<\frac{1}{2}$, it follows that $\rho^2<\rho/2$, and hence we obtain by using (A.4) that there exists $T_2>T$ such that for all $(y,\gamma)\in\mathfrak{M}_{\rho,\tau}$ and for all $\tau\in(T,T_2)$

$$\|\mathbf{\Pi}_2\mathbf{N}_\tau(y,\gamma)-\gamma^0(\cdot)-\|f\|^2_{L^2(0,T)}\|_{L^\infty(T,\tau)}<\frac{\rho}{2}\,. \tag{A.5}$$

Combining (A.3) and (A.5), we see that

$$\mathbf{N}_\tau\mathfrak{M}_{\rho,\tau}\subset\mathfrak{M}_{\rho,\tau}\quad\forall\,\tau\in(T,\min(T_1,T_2))\,. \tag{A.6}$$

(ii) For any $\tau\in(T,\tau^*)$ and any $(y_1,\gamma_1),(y_2,\gamma_2)\in\mathfrak{M}_{\rho,\tau}$ the following estimates hold:

$$\begin{aligned}
&\|\mathbf{\Pi}_1\mathbf{N}_\tau(y_1,\gamma_1)-\mathbf{\Pi}_1\mathbf{N}_\tau(y_2,\gamma_2)\|^2_{L^2(T,\tau)}\\
&\quad=\int_0^\tau\left(\int_0^t(\mathbf{P}_{T-\tau}F)(t-\xi)(\mathcal{K}(\gamma_1(\xi))y_1(\xi)-\mathcal{K}(\gamma_2(\xi))y_2(\xi))\,d\xi\right)^2dt\\
&\quad\leq\left(\int_0^\tau\|(\mathbf{P}_{\tau-T}F)(\xi)\|\,d\xi\right)^2\int_0^\tau\|\mathcal{K}(\gamma_1(\xi))y_1(\xi)-\mathcal{K}(\gamma_1(\xi))y_2(\xi)\\
&\qquad\qquad+\mathcal{K}(\gamma_1(\xi))y_2(\xi)-\mathcal{K}(\gamma_2(\xi))y_2(\xi)\|^2\,d\xi\\
&\quad\leq 2\left(\int_0^{\tau-T}\|F(\xi)\|\,d\xi\right)^2\left(K^2\int_T^\tau\|y_1(\xi)-y_2(\xi)\|^2\,d\xi\right.\\
&\qquad\qquad\left.+\;L^2\left(\int_0^\tau\|y_2(\xi)\|^2\,d\xi\right)\|\gamma_1-\gamma_2\|^2_{L^\infty(T,\tau)}\right),
\end{aligned} \tag{A.7}$$

where we have chosen $K>0$ and $L>0$ in such a way that for all real numbers κ, κ_1 and κ_2 with $|\kappa|,|\kappa_1|,|\kappa_2|\leq\max(\|g\|_{L^\infty(0,T)},\rho+\|\gamma^0\|_{L^\infty(T,\tau^*)}+\|f\|^2_{L^2(0,T)})$

$$\mathcal{K}(\kappa)\leq K\quad\text{and}\quad|\mathcal{K}(\kappa_1)-\mathcal{K}(\kappa_2)|\leq L|\kappa_1-\kappa_2|\,.$$

Realizing that

$$\int_0^\tau\|y_2(\xi)\|^2\,d\xi\leq\|f\|^2_{L^2(0,T)}+\|y^0\|^2_{L^2(T,\tau)}+\rho^2\,,$$

it follows from (A.7) that there exists $M > 0$ such that for all $\tau \in (T, \tau^*)$ and all $(y_1, \gamma_1), (y_2, \gamma_2) \in \mathfrak{M}_{\rho,\tau}$

$$\|\mathbf{\Pi}_1 \mathbf{N}_\tau(y_1, \gamma_1) - \mathbf{\Pi}_1 \mathbf{N}_\tau(y_2, \gamma_2)\|^2_{L^2(T,\tau)} \leq M \left(\int_0^{\tau - T} \|F(\xi)\|\, d\xi \right)^2 (\|y_1 - y_2\|^2_{L^2(T,\tau)} + \|\gamma_1 - \gamma_2\|^2_{L^\infty(T,\tau)}).$$

Defining

$$\delta'_\tau := \sqrt{M} \int_0^{\tau - T} \|F(\xi)\|\, d\xi\,, \tag{A.8}$$

we obtain that for all $\tau \in (T, \tau^*)$ and all $(y_1, \gamma_1), (y_2, \gamma_2) \in \mathfrak{M}_{\rho,\tau}$

$$\|\mathbf{\Pi}_1 \mathbf{N}_\tau(y_1, \gamma_1) - \mathbf{\Pi}_1 \mathbf{N}_\tau(y_2, \gamma_2)\|_{L^2(T,\tau)} \leq \delta'_\tau (\|y_1 - y_2\|_{L^2(T,\tau)} + \|\gamma_1 - \gamma_2\|_{L^\infty(T,\tau)})\,. \tag{A.9}$$

Furthermore, we have that for all $(y_1, \gamma_1), (y_2, \gamma_2) \in \mathfrak{M}_{\rho,\tau}$

$$\begin{aligned} &\|\mathbf{\Pi}_2 \mathbf{N}_\tau(y_1, \gamma_1) - \mathbf{\Pi}_2 \mathbf{N}_\tau(y_2, \gamma_2)\|_{L^\infty(T,\tau)} \\ &\qquad = \sup_{t \in [T,\tau]} \left| \int_T^t \|y_1(\xi)\|^2\, d\xi - \int_T^t \|y_2(\xi)\|^2\, d\xi \right| \\ &\qquad \leq \int_T^\tau (\|y_1(\xi)\| + \|y_2(\xi)\|) \|y_1(\xi) - y_2(\xi)\|\, d\xi \\ &\qquad \leq (\|y_1\|_{L^2(T,\tau)} + \|y_2\|_{L^2(T,\tau)}) \|y_1 - y_2\|_{L^2(T,\tau)} \\ &\qquad \leq 2(\rho + \|y^0\|_{L^2(T,\tau)}) \|y_1 - y_2\|_{L^2(T,\tau)}\ . \end{aligned} \tag{A.10}$$

Setting

$$\delta''_\tau := 2(\rho + \|y^0\|_{L^2(T,\tau)})\,, \tag{A.11}$$

it follows from (A.10) that for all $(y_1, \gamma_1), (y_2, \gamma_2) \in \mathfrak{M}_{\rho,\tau}$

$$\|\mathbf{\Pi}_2 \mathbf{N}_\tau(y_1, \gamma_1) - \mathbf{\Pi}_2 \mathbf{N}_\tau(y_2, \gamma_2)\|_{L^\infty(T,\tau)} \leq \delta''_\tau \|y_1 - y_2\|_{L^2(T,\tau)}\,. \tag{A.12}$$

Clearly, since $\rho < \frac{1}{2}$ and by (A.8) and (A.11), there exists $T_3 \in (T, \tau^*)$ such that $\delta_\tau := \max(\delta'_\tau, \delta''_\tau) < 1$ for all $\tau \in (T, T_3)$. Setting $T^* = \min(T_1, T_2, T_3)$, we see that $T^* > T$, $\delta_\tau < 1$ for all $\tau \in (T, T^*)$, and moreover, by (A.6), (A.9), and (A.12), we have that for all $\tau \in (T, T^*)$ and all $(y_1, \gamma_1), (y_2, \gamma_2) \in \mathfrak{M}_{\rho,\tau}$

$$\mathbf{N}_\tau \mathfrak{M}_{\rho,\tau} \subset \mathfrak{M}_{\rho,\tau}\,, \quad \|\mathbf{N}_\tau(y_1, \gamma_1) - \mathbf{N}_\tau(y_2, \gamma_2)\|_{(T,\tau)} \leq \delta_\tau \|(y_1, \gamma_1) - (y_2, \gamma_2)\|_{(T,\tau)}\,.$$

Finally, it follows from Banach's contraction mapping theorem that for all τ as above $\mathbf{N}_\tau$ has a unique fixed point in $\mathfrak{M}_{\rho,\tau}$. ∎

Proof of Proposition 2.4. We proceed in several steps.

Step 1 (existence and uniqueness on a small interval). An application of Lemma A.1 to the case where $T = 0$, $y^0 = \mathbf{\Psi}_\infty x_0$, $\gamma^0(t) \equiv \gamma_0$ and $F = -\mathbb{L}^{-1}\mathbf{G}$ shows that for all sufficiently small $\tau > 0$ the operator $\mathbf{N}_\tau$ has a unique fixed point in $\mathfrak{M}_{\rho,\tau}$ and hence there exists $\tau^* > 0$ such that (2.11) has a unique solution (y^*, γ^*) on $[0, \tau^*)$.

Step 2 (continuation of solutions). If $\|y^*\|_{L^2(0,\tau^*)} = \infty$, then $\tau_{max} = \tau^*$ and $(y_{max}, \gamma_{max}) = (y^*, \gamma^*)$, and we are finished. Thus, let us suppose that $\|y^*\|_{L^2(0,\tau^*)} <$

∞. We claim that then the solution (y^*, γ^*) can be extended beyond τ^*. To this end we apply Lemma A.1 to the case where $T = \tau^*$, $(f, g) = (y^*, \gamma^*)$, $y^0 = (\mathbf{\Psi}_\infty x_0)|_{[\tau^*,\infty)}$, $\gamma^0(t) \equiv \gamma_0$, and $F = -\mathbb{L}^{-1}\mathbf{G}$. It follows that there exist $\tau^{**} > \tau^*$ and $(y^{**}, \gamma^{**}) \in L_{(0,\tau^{**})}$ such that $(y^{**}, \gamma^{**})|_{[0,\tau^*]} = (y^*, \gamma^*)$, and moreover (y^{**}, γ^{**}) solves (2.11) on $[0, \tau^{**})$.

Step 3 (extended uniqueness). Let (y_1, γ_1) and (y_2, γ_2) be two solutions of (2.11) on $[0, \tau_1)$ and $[0, \tau_2)$, respectively, where $\tau_2 \geq \tau_1 > 0$. We claim that

$$(y_2(t), \gamma_2(t)) = (y_1(t), \gamma_1(t)) \quad \text{for a.e. } t \in [0, \tau_1)\,. \tag{A.13}$$

For $\tau \in [0, \tau_1)$ define

$$\Omega_\tau := \{t \in [0, \tau] \,|\, (y_1(t), \gamma_1(t)) \neq (y_2(t), \gamma_2(t))\}\,,$$

and set

$$\hat{\tau} := \inf\{\tau \in [0, \tau_1) \,|\, \boldsymbol{\lambda}(\Omega_\tau) > 0\}\,,$$

where $\boldsymbol{\lambda}$ denotes the Lebesgue measure. It is clear that (A.13) is equivalent to $\hat{\tau} = \tau_1$. Seeking a contradiction, assume that $\hat{\tau} < \tau_1$. Let $t_n \in (0, \hat{\tau})$ with $\lim_{n\to\infty} t_n = \hat{\tau}$. (Recall that by Step 1, $\hat{\tau} > 0$.) Obviously,

$$\Omega_{\hat{\tau}} \setminus \{\hat{\tau}\} = \bigcup_{n\in\mathbb{N}} \Omega_{t_n}\,.$$

Now $\boldsymbol{\lambda}(\Omega_{t_n}) = 0$ for all $n \in \mathbb{N}$, and thus $\boldsymbol{\lambda}(\Omega_{\hat{\tau}}) = 0$, which in turn implies that for a.e. $t \in [0, \hat{\tau}]$

$$(y_1(t), \gamma_1(t)) = (y_2(t), \gamma_2(t)) =: (\hat{y}(t), \hat{\gamma}(t))\,.$$

An application of Lemma A.1 to the case where $T = \hat{\tau}$, $(f, g) = (\hat{y}, \hat{\gamma})$, $y^0 = (\mathbf{\Psi}_\infty x_0)|_{[\hat{\tau},\infty)}$, $\gamma^0(t) \equiv \gamma_0$, and $F = -\mathbb{L}^{-1}\mathbf{G}$ shows that there exists $t^* \in (\hat{\tau}, \tau_1)$ such that the operator $\mathbf{N}_{t^*}$ has a unique fixed point in $\mathfrak{M}_{\rho,t^*}$. Since the restrictions of (y_1, γ_1) and of (y_2, γ_2) to $[0, t^*]$ are both fixed points of $\mathbf{N}_{t^*}$, we see that $(y_1, \gamma_1)|_{[0,t^*]} = (y_2, \gamma_2)|_{[0,t^*]}$, which is in contradiction to the definition of $\hat{\tau}$.

Step 4 (existence of a maximal solution). Define

$$\mathcal{T} := \{\tau > 0 \,|\, (2.11) \text{ has a solution on } [0, \tau)\}\,.$$

Set $\tau_{max} := \sup \mathcal{T}$ and let $\tau_n \in \mathcal{T}$ be such that $\tau_n \nearrow \tau_{max}$ as $n \to \infty$. Let (y_n, γ_n) denote the unique (by Step 3) solution of (2.11) on $[0, \tau_n)$. Using Step 3 again it is clear that $(y_n, \gamma_n)_{[0,\tau_m]} = (y_m, \gamma_m)$ for all $m, n \in \mathbb{N}$ with $n > m$. Therefore, we obtain a well-defined function (y_{max}, γ_{max}) on $[0, \tau_{max})$ by setting

$$(y_{max}(t), \gamma_{max}(t)) = (y_n(t), \gamma_n(t)) \quad \text{if } t \in [0, \tau_n)\,.$$

By construction (y_{max}, γ_{max}) is a solution of (2.11) on $[0, \tau_{max})$, which, by Step 3, is unique. Finally, it follows from Step 2 and the definition of τ_{max} that

$$\tau_{max} < \infty \quad \Longrightarrow \quad \int_0^{\tau_{max}} \|y_{max}(\xi)\|^2 \, d\xi = \infty\,. \qquad \square$$

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