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Adaptive control of infinite-dimensional systems without parameter estimation: an overview†

HARTMUT LOGEMANN
School of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY
AND
STUART TOWNLEY‡
Department of Mathematics, University of Exeter, North Park Road, Exeter EX4 4QE

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This paper contains an overview of adaptive control of infinite-dimensional systems without parameter estimation or identification. We describe the main problems and present the most comprehensive results. High-gain, low-gain and switching controllers are considered. The literature is discussed and a number of open problems are posed.

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1. Introduction

The purpose of this article is to give an overview of an approach to adaptive control of infinite-dimensional systems which is *universal* in that it does not involve parameter estimation or identification nor does it use any persistently exciting probing signals. The approach is in the spirit of the corresponding finite-dimensional theory due to Nussbaum [60], Byrnes & Willems [3], Willems & Byrnes [82] and Mårtensson [51, 52]. In this context the adaptive control problem can be described as follows: Given a set $\mathfrak{P}$ of plants/processes/systems with m inputs $(u_1, \ldots, u_m)^T = u$ and p outputs $(y_1, \ldots, y_p)^T = y$, the objective is to synthesize a single control law of the form

$$u(t) = F(t, g(t), y(t)), \quad \dot{g}(t) = H(t, g(t), y(t)), \quad g(t_0) \in \mathbb{R}^l$$

which guarantees that for every member of $\mathfrak{P}$ the resulting closed-loop system exhibits some prescribed dynamic behaviour (for example, attractivity or stability of an equilibrium or asymptotic output tracking of some reference signal). In the above equations, H should be interpreted as an adaptation rule driven by the plant output $y(t)$, while F should be considered as a feedback law which is adjusted to the particular plant (unknown to the controller) via the parameter vector $g(t)$.

The area of universal adaptive control has its origins in questioning to what extent the

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‡ Also with the Centre for Systems and Control Engineering, University of Exeter.

four 'classical' assumptions of adaptive control of linear systems (i.e. an upper bound of the McMillan degree of the plant is known, the relative degree of the plant is known, the sign of the high-frequency gain is known and the plant is minimum-phase) are necessary for the existence of stabilizing adaptive controllers. A systematic and rigorous analysis of this question began in the first half of the 1980s. The theory continues to develop with a current emphasis on universal adaptive control of nonlinear and infinite-dimensional systems. For a survey of the finite-dimensional theory, see Ilchmann [22]; see also the special issue of the *IMA J. of Math. Control & Information* edited by Ryan [66].

In this paper we give an overview of adaptive control of infinite-dimensional systems without parameter estimation. In particular, we intend to demonstrate how the well-posedness, admissibility and stabilizability results, which were derived primarily for tackling the linear quadratic control problem (see Pritchard & Salamon [64], Curtain *et al.* [8], and the recent regular linear systems framework of Weiss [76, 77, 78], find natural applications in formulating and proving the convergence of many adaptive algorithms for infinite-dimensional systems. We emphasize that the work summarized in this article reflects our own personal involvement in the development of the theory, which has benefited greatly from our awareness of, and involvement in, the development of abstract theories for infinite-dimensional systems introduced to capture both state-space and frequency-domain techniques. Whilst the theory of universal adaptive control for infinite-dimensional systems is by no means complete, it turns out that many of the finite-dimensional results do have counterparts for quite general classes of infinite-dimensional systems.

The article is organised as follows: Section 2 describes an approach to high-gain adaptive control of infinite-dimensional systems which is sufficiently general to encompass all other known approaches. Section 3 describes recent progress on the integral low-gain adaptive control problem for the class of regular systems, which—in a loose sense—is the inverse of the high-gain adaptive stabilization problem. Finally, in Section 4 we describe more general techniques from universal adaptive control which are applicable to classes of systems which are stabilizable and detectable, but not necessarily stabilizable by high-gain proportional output feedback or by a low-gain integrator. Here the main approach is to switch through the parameter space for a class of controllers in which a stabilizing controller is known to exist. An important role is played by stabilizability and detectability notions which have been developed recently for infinite-dimensional systems [8]. In each section we describe the main problems and present the most comprehensive results. The end of each section includes a note and references section, where a number of open problems requiring further research efforts are posed.

There have, of course, been attempts to generalize 'traditional' adaptive control algorithms to classes of infinite-dimensional systems: the papers by Kobayashi [29, 32] and by Wen & Balas [80, 81], deal with model-reference adaptive control for semigroup systems defined on Hilbert spaces. In [16], Fernández *et al.* it shows that some standard adaptive schemes designed for first order systems remain (locally) stable in the presence of 'small' input delays. Ortega *et al.* present in [61] a globally stable adaptive controller for scalar plants with one exactly known delay in the input. A 2-D-system approach is taken by Johannson in [26], where a class of delay-differential systems, with the delays consisting of a finite number of point delays, all having a rational relationship is considered. By utilizing algebraic methods, a standard model-reference adaptive controller for this class is proposed. In [21], Hong & Bentsman develop averaging techniques to prove the convergence and stability of a model-reference adaptive controller applied to a class of uncertain, linear parabolic partial differential equations of spatial dimension n. Finally, Duncan & Pasik-Duncan [13, 14] develop an adaptive control algorithm for classes of infinite-dimensional stochastic systems using linear quadratic ideas combined with an indirect self-tuning regulator approach.

Notation: For $\alpha \in \mathbb{R}$ define $\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\}$. Moreover, set

$$H^\infty(\mathbb{C}^{p\times m}) := \{f : \mathbb{C}_0 \to \mathbb{C}^{p\times m} \mid f \text{ is holomorphic and bounded}\}.$$

Let X and Y be Banach spaces. The set of all linear bounded operators from X to Y is denoted by $\mathcal{B}(X, Y)$. If A is a linear operator defined on X, we set

$$D(A) := \text{domain of } A\,, \quad \sigma(A) := \text{spectrum of } A\,, \quad \varrho(A) := \text{resolvent set of } A\,.$$

The Laplace transform is denoted by $\mathcal{L}$ or by the superscript " ^ ".
Finally, let f be a function defined on $[0, T)$, where $0 < T \leqslant \infty$, and let $\tau \in [0, T)$. We define

$$(\mathbf{P}_\tau f)(t) := \begin{cases} f(t) & \text{if } t \in [0, \tau] \\ 0 & \text{if } t > \tau \end{cases}.$$

2. Adaptive high-gain control

Results on (non-adaptive) high-gain stabilization for finite-dimensional systems are usually (at least in the single-input single-output case) derived using classical root-locus techniques. Although recently some progress in the development of root-locus ideas for infinite-dimensional systems has been made—see for example Byrnes *et al.* [2] and Rebarber & Townley [65]—a fairly complete parallel of the finite-dimensional root-locus theory for abstract infinite-dimensional systems is by no means available. Nevertheless, various well-known high-gain stabilization results extend to infinite-dimensional systems; see [2], Logemann & Owens [41] and Logemann & Zwart [48]. In this section we describe an input-output approach to adaptive high-gain control of infinite-dimensional systems which encompasses most of the results available in the literature. Since the approach is based on an external description of the plant, an abstract state-space representation of the plant is not required. However, non-zero initial conditions are taken into account by using 'initial-condition terms'.

System description

We assume that the plant is described externally by a transfer-function matrix $\mathbf{G}$ of size $m \times m$ which is meromorphic on $\mathbb{C}_0$ and satisfies

$$\left.\begin{aligned} \mathbf{G}^{-1}(s) \quad &= \quad sG^{-1} + \mathbf{H}(s)\,, \\ \text{where} \quad G \in \mathbb{R}^{m\times m}\,,\ \det(G) \quad &\neq 0 \quad \text{and} \quad \mathbf{H} \in H^\infty(\mathbb{C}^{m\times m})\,. \end{aligned}\right\} \tag{2.1}$$

Of course (2.1) is equivalent to

$$\mathbf{G}(s) = \left(I + \frac{1}{s}G\mathbf{H}(s)\right)^{-1} \frac{1}{s}G\,,$$

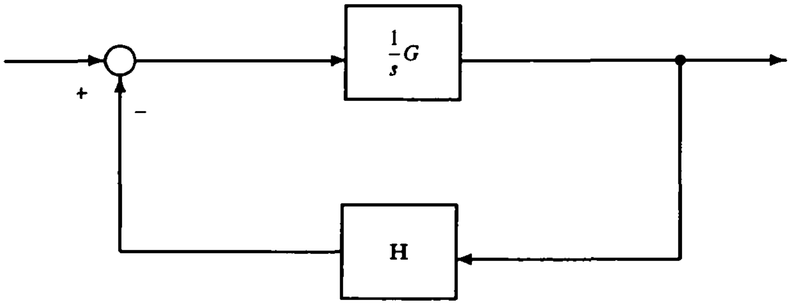

FIG. 1. G written as a feedback system

i.e. **G** is the feedback interconnection of the integrator $(1/s)G$ and the transfer-function matrix **H**; see Fig. 1. It has been shown by Logemann & Zwart [48] that condition (2.1) can be characterized in terms of the zeros and the high-frequency behaviour of **G**. More precisely, the following result holds true.

PROPOSITION 2.1 Let $\mathbf{G}(s)$ be a meromorphic transfer-function matrix of size $m \times m$ defined on a region $\Omega \supset \mathbb{C}_0^{cl}$. Then $\mathbf{G}^{-1}(s)$ admits a decomposition of the form (2.1) if and only if the following two conditions are satisfied

(i) The limit $G := \lim_{|s|\to\infty, s\in\mathbb{C}_0} \mathbf{G}(s)$ exists, $\det G \neq 0$ and

$$s\mathbf{G}(s) - G = O(\frac{1}{s}) \qquad \text{as } |s| \to \infty \text{ in } \mathbb{C}_0 . \tag{2.2}$$

(ii) $\mathbf{G}(s)$ has no zeros in $\mathbb{C}_0^{cl}$.

Note that condition (2.2) is a generalization of the relative-degree-one condition for finite-dimensional single-input single-output systems.

It is not difficult to show that, if (2.1) is satisfied and if $\sigma(G) \subset \mathbb{C}_0$, then the plant described by **G** can be stabilized by static output feedback of the form $u = -ky$, provided the feedback gain k is positive and sufficiently large. The following proposition which is a consequence of results in [41] makes this more precise.

PROPOSITION 2.2 Suppose that the transfer-function matrix **G** satisfies condition (2.1) and let the matrix $K \in \mathbb{R}^{m\times m}$ be such that $\sigma(GK) \subset \mathbb{C}_0$. Then there exists $k^* > 0$ such that for all $k > k^*$

$$\mathbf{G}(I + k\mathbf{G}K)^{-1} \in H^\infty(\mathbb{C}^{m\times m}),$$

i.e. for all $k > k^*$ the feedback $u = -kKy$ leads to a L^2-stable closed-loop system.

It is natural to seek ways of tuning the parameters k and K adaptively. This will be the topic of the next subsection.

In the following we shall assign an operator $\mathcal{H} : L^2(\mathbb{R}_+, \mathbb{C}^m) \to L^2(\mathbb{R}_+, \mathbb{C}^m)$ to the transfer-function matrix **H** by defining $\mathcal{H} := \mathcal{L}^{-1}\mathcal{M}_{\mathbf{H}}\mathcal{L}$, where $\mathcal{L}$ denotes the Laplace transform and $\mathcal{M}_{\mathbf{H}}$ denotes the multiplication by $\mathbf{H}$ on the Hardy space $H^2(\mathbb{C}^m)$. The operator $\mathcal{H}$ is linear, bounded, and shift-invariant (in the sense of Vidyasagar [73]). As a consequence $\mathcal{H}$ is causal (see [73]) and therefore has a unique causal extension to $L^2_{loc}(\mathbb{R}_+, \mathbb{C}^m)$. This extension will also be denoted by $\mathcal{H}$.

The function $\mathbf{G}$ satisfying (2.1) can be thought of as being the transfer-function matrix of

$$\dot{y} = G\,(u - (\mathcal{H}y + w))\ , \qquad y(0) = y_0 \in \mathbb{R}^m\ , \tag{2.3}$$

where $u \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ and $w \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ takes account of non-zero initial conditions in the system with transfer-function matrix $\mathbf{H}$. The initial value problem (2.3) is a special case of the following initial-value problem which will play an important role in this section. Consider

$$\dot{x}(t) = (\mathcal{S}x)(t) + f_1(t, x(t)) + f_2(t)\ , \qquad t \geqslant \alpha\ , \tag{2.4a}$$

$$x_{|[0,\alpha]} = x_0 \in C([0, \alpha], \mathbb{R}^n)\ ,\ , \tag{2.4b}$$

where $\alpha \geqslant 0$ and $\mathcal{S}$, f_1 and f_2 satisfy the following conditions:

(i) $\mathcal{S}$ maps $L^2_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ into itself, $\mathcal{S}(0) = 0$ and we assume that there exists $\lambda > 0$ such that $\|\mathbf{P}_t(\mathcal{S}x - \mathcal{S}x')\|_2 \leqslant \lambda \|\mathbf{P}_t(x - x')\|_2$ for all $x, x' \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ and for all $t \geqslant 0$, i.e. $\mathcal{S}$ is unbiased, causal and of finite incremental gain;

(ii) $f_1 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is such that $f_1(t, x)$ is continuous in t and locally Lipschitz continuous in x, uniformly in t on bounded intervals;

(iii) f_2 is in $L^1_{loc}(\mathbb{R}_+, \mathbb{R}^n)$.

Of course, if $\alpha = 0$ in (2.4b), then $C([0, \alpha], \mathbb{R}^n) = \mathbb{R}^n$. In order to define what we mean by a solution of the initial value problem (2.4), we have to give a meaning to $\mathcal{S}x$ if $x \in C([0, \beta), \mathbb{R}^n)$, where $\alpha < \beta \leqslant \infty$ (remember that $\mathcal{S}$ operates on functions whose domain of definition is $\mathbb{R}_+$). We set $(\mathcal{S}x)(t) = (\mathcal{S}\mathbf{P}_\tau x)(t)$ for $0 \leqslant t \leqslant \tau < \beta$. Since $\mathcal{S}$ is causal, this definition does not depend on the choice of τ. A *solution* of (2.4) on $[0, \beta)$ is a continuous function which is absolutely continuous on $[\alpha, \beta)$, satisfies (2.4a) a.e. on $[\alpha, \beta)$ and satisfies the initial condition (2.4b).

THEOREM 2.3 The initial-value problem (2.4) has a unique maximal solution. Precisely: there exists $\tau_{max} \in (\alpha, \infty]$ such that (2.4) has a unique solution x_{max} on $[0, \tau_{max})$ and moreover, if $\tau_{max} < \infty$, then there exists a sequence $t_i \in (0, \tau_{max})$, satisfying $\lim_{i\to\infty} t_i = \tau_{max}$, and such that

$$\lim_{i\to\infty} \|x_{max}(t_i)\| = \infty\,.$$

The above theorem has been proved by Logemann & Owens [42]. Similar results can be found in Gripenberg *et al.* [18], pp. 359, and Hinrichsen & Pritchard [20]. Theorem 2.3 implies in particular that the initial-value problem (2.3) has a unique solution for all $w \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, $u \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}^m)$, and $y_0 \in \mathbb{R}^m$.

Adaptive high-gain stabilization

In the following, we need a result from linear algebra which has been proved by Mårtensson (1986, 1991). For $m \geqslant 1$ we call a set $\mathfrak{U} \subset GL(m, \mathbb{R})$ *unmixing*, if for any $M \in GL(m, \mathbb{R})$ there is a $U \in \mathfrak{U}$ such that $\sigma(MU) \subset \mathbb{C}_0$.

PROPOSITION 2.4 For all $m \geqslant 1$, there exist unmixing sets of finite cardinality.

Unfortunately the cardinality of the unmixing sets constructed in [52, 54] is much larger than would be convenient for applications. In fact, hardly anything is known on the minimum cardinality of unmixing sets. However, for $m = 1$ the set $\{1, -1\}$ is obviously unmixing, while for $m = 2$ there exists an unmixing set of cardinality 6. It has been shown by Zhu [84] that $GL(3, \mathbb{R})$ can be unmixed by a set having cardinality 32.

In the following, let $\{K_1, ..., K_\ell\}$ be an unmixing set for $GL(m, \mathbb{R})$. Since (2.3) can be stabilized by high-gain feedback of the form $u(t) = -ky(t)$, provided that $\sigma(G) \subset \mathbb{C}_0$ and k is a sufficiently large positive number, it seems reasonable to consider the following adaptive control law, which was introduced by Byrnes & Willems [3] in a finite-dimensional state-space set-up:

$$u(t) = -k(t)K_{\Sigma(k(t))}y(t)\,, \tag{2.5a}$$

$$\dot{k}(t) = \|y(t)\|^2\,, \qquad k(0) = k_0 \in \mathbf{R}\,. \tag{2.5b}$$

In (2.5a) the function $\Sigma : \mathbb{R} \to \{1, ..., \ell\}$ is given by

$$\Sigma(\kappa) = \begin{cases} 1 \quad , & \kappa \in [-\kappa_1, \kappa_1) \\ i \quad , & \kappa \in [\kappa_{j\ell+i}, \kappa_{j\ell+i+1}) \cup [-\kappa_{j\ell+i+1}, -\kappa_{j\ell+i}) \ \text{ for some } \ j \in \mathbf{N}\,, \end{cases} \tag{2.6}$$

where the sequence (κ_j) is defined by

$$\kappa_{j+1} = \kappa_j^2\,, \quad \kappa_1 > 1\,. \tag{2.7}$$

Note that the gain $k(t)$ is monotonically increasing, and thus the function Σ ensures that $K_{\Sigma(k(t))}$ will hit some stabilizing gain matrix K_i if $k(t)$ diverges. The growth condition (2.7) captures the intuitive idea that the length of the intervals $[\kappa_j, \kappa_{j+1})$ should increase rapidly, in order to to enable the closed-loop system to settle down.

Although the closed-loop system given by (2.3) and (2.5) is of the form (2.4), Theorem 2.3 cannot be applied straight away in order to establish well-posedness of the closed loop, since the map $\mathbb{R} \to \{K_1, ..., K_\ell\}, \kappa \mapsto K_{\Sigma(\kappa)}$ is not continuous. However, Theorem 2.3 can be used to prove the following lemma; see Logemann & Ilchmann [38].

LEMMA 2.5 For each pair of initial conditions $(y_0, k_0) \in \mathbb{R}^m \times \mathbf{R}$ and for each $w \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ the closed-loop system given by (2.3) and (2.5) has a unique absolutely continuous solution (y, k) which can be extended to the right as long as it remains bounded.

Using the above well-posedness result, the following theorem on adaptive stabilization can be proved. It says that the control law (2.5) stabilizes any system of the form (2.3), or in other words (2.5) is a universal adaptive control law for this class of systems. The proof is based on a combination of ideas due to Byrnes & Willems [3] with a technical lemma which can be found in Ilchmann & Logemann [24], see [38] for the details.

THEOREM 2.6 The solution (y, k) of the closed-loop system given by (2.3) and (2.5) exists on $\mathbb{R}_+$ and has the following properties:

(i) $\lim_{t\to\infty} k(t)$ exists and is finite,
(ii) $y \in L^2(\mathbb{R}_+, \mathbb{R}^m) \cap L^\infty(\mathbb{R}_+, \mathbb{R}^m)$,

(iii) $\lim_{t\to\infty} y(t) = 0$.

It is not difficult to see that the sequence given by (2.7) can be replaced by any strictly increasing sequence (κ_j) satisfying $\lim_{j\to\infty} \kappa_{j+1}/\kappa_j = \infty$ [24, 67].

Next we formulate an open problem on the limiting closed-loop system. If the assumptions of Theorem 2.6 are satisfied, then $\lim_{t\to\infty} k(t) = k_\infty(w, y_0, k_0)$ exists and is finite. The linear system

$$\dot{\tilde{y}} = -G\left(k_\infty(w, y_0, k_0) K_{\Sigma(k_\infty(w,y_0,k_0))}\tilde{y} + \mathcal{H}\tilde{y} + \tilde{w}\right), \tag{2.8a}$$

$$\tilde{y}(0) = \tilde{y}_0 \in \mathbb{R}^m, \qquad \tilde{w} \in L^2(\mathbb{R}_+, \mathbb{R}^m) \tag{2.8b}$$

is called the *limit system* of the nonlinear closed-loop system given by (2.3) and (2.5). It is easy to see that (2.8) does not satisfy $\lim_{t\to\infty} \tilde{y}(t) = 0$ for arbitrary $(\tilde{w}, \tilde{y}_0) \in L^2(\mathbb{R}_+, \mathbb{R}^>) \times \mathbb{R}^>$. Indeed, consider the special case that $\mathcal{H} = 0$ and choose $w = 0$, $y_0 = 0$, and $k_0 = 0$ in (2.3) and (2.5). Since $k_\infty(0, 0, 0) = 0$, it follows that the solution $\tilde{y}$ of (2.8) is given by $\tilde{y}(t) = \tilde{y}_0 - G\int_0^t \tilde{w}(\tau)\, d\tau$, and hence $\tilde{y}(t)$ does in general not converge to 0 as $t \to \infty$. However, recent work of Townley [72] on adaptive stabilization of finite-dimensional systems leads us to the following problem.

Open problem. Let $k_0 \in \mathbb{R}$ be given. Does there exist an open and dense set $\mathfrak{J}(k_0) \subset L^2(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}^m$ such that the limit system (2.8) is stable in the sense that

$$\tilde{y} \in L^2 \cap L^\infty(\mathbb{R}_+, \mathbb{R}^m) \quad \text{and} \quad \lim_{t\to\infty} \tilde{y}(t) = 0 \quad \text{for all } (\tilde{w}, \tilde{y}_0) \in L^2(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}^m,$$

provided that $(w, y_0) \in \mathfrak{J}(k_0)$?

Finally, we close this subsection with some remarks on the single-input single-output (SISO) case where somewhat 'stronger' results can be proved. Consider the feedback law

$$u(t) = \mathcal{K}(g(t))y(t); \qquad \dot{g}(t) = y^2(t), \quad g(0) = g_0 \in \mathbb{R}, \tag{2.9}$$

where $\mathcal{K} : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies

$$\sup_{a>a_0} \frac{1}{a - a_0} \int_{a_0}^{a} \mathcal{K}(\gamma)\, d\gamma = +\infty, \qquad \inf_{a>a_0} \frac{1}{a - a_0} \int_{a_0}^{a} \mathcal{K}(\gamma)\, d\gamma = -\infty \tag{2.10}$$

for some $a_0 \in \mathbb{R}$. The condition (2.10) has its origin in Nussbaum's (1983) paper . Functions which satisfy (2.10) will be called *Nussbaum functions*. In the SISO case we can replace the discontinuous feedback (2.5a) by the smooth feedback law (2.9) to obtain a stabilizing controller. More precisely, we have the following result whose proof can be found in [42].

THEOREM 2.7 The solution (y, g) of the closed-loop system given by (2.3) and (2.9) exists on $\mathbb{R}_+$ and has the following properties

(i) $\lim_{t\to\infty} g(t)$ exists and is finite,
(ii) $y \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$,
(iii) $\lim_{t\to\infty} y(t) = 0$.

A Nussbaum function $\mathcal{K}$ is called *scaling-invariant* if the function $(\Xi_\alpha^\beta \circ \mathcal{K})\mathcal{K}$ is a Nussbaum function for all $\alpha, \beta > 0$, where Ξ_α^β is given by

$$\Xi_\alpha^\beta(\kappa) = \begin{cases} \alpha & \text{if} \quad \kappa > 0 \\ 0 & \text{if} \quad \kappa = 0 \\ \beta & \text{if} \quad \kappa < 0 \end{cases}$$

An example of a scaling-invariant Nussbaum function is given by

$$\mathcal{K}(\gamma) = \cos(\frac{\pi}{2}\gamma)\exp(\gamma^2)\,.$$

The concept of scaling invariance for Nussbaum functions was introduced in [42]. It is shown in [42] that Theorem 2.7 remains true when the plant **G** is subjected to a large class of sector-bounded static actuator and sensor nonlinearities, provided the Nussbaum function $\mathcal{K}$ is scaling invariant. The effect of actuator and sensor nonlinearities in the multi-input multi-output (MIMO) case is an interesting topic for future research.

Adaptive high-gain tracking

In this subsection we present a single controller such that the closed-loop system asymptotically tracks any reference trajectory r belonging to a precspecified finite-dimensional vector space for all plants of the form (2.1). Let ρ_i be real monic polynomials, $1 \leqslant i \leqslant m$, and set $\rho := (\rho_1, ..., \rho_m)^T$. The reference signals belong to the set given by

$$\mathfrak{R}_\rho := \left\{ r : \mathbb{R}_+ \to \mathbb{R}^m \mid \rho_i\left(\frac{d}{dt}\right) r_i \equiv 0, \;\; i = 1, ..., m \right\}.$$

The well-known internal model principle from linear control theory (see e.g. Vidyasagar [74: p. 294] and Wonham [83: p. 203] for the finite-dimensional case, and Callier & Desoer [4], Curtain [7] and Francis [17] for the infinite-dimensional case) suggests that the dynamics of the reference signals should be replicated in the loop via a precompensator. To this end set

$$p(s) = \operatorname{lcm}(s, \rho_1(s), ..., \rho_m(s)) ,$$

where we choose p to be monic. Moreover, let q by a monic polynomial which is Hurwitz and satisfies $\deg(q) = \deg(p)$. We define the precompensator $\mathbf{C}(s)$ containing the internal model to be

$$\mathbf{C}(s) = \frac{q(s)}{p(s)} I_m\,.$$

Let $\mathbf{G}_\mathrm{C}$ denote the precompensated plant, i.e. $\mathbf{G}_\mathrm{C}(s) = \mathbf{G}(s)\mathbf{C}(s)$. Now realize that, by (2.1),

$$\mathbf{G}_\mathrm{C}^{-1}(s) = \frac{p(s)}{q(s)}\left(sG^{-1} + \mathbf{H}(s)\right) = sG^{-1} + \mathbf{H}_\mathrm{C}(s)\,,$$

where

$$\mathbf{H}_\mathrm{C}(s) := s\left(\frac{p(s)}{q(s)} - 1\right)G^{-1} + \frac{p(s)}{q(s)}\mathbf{H}(s)\,,$$

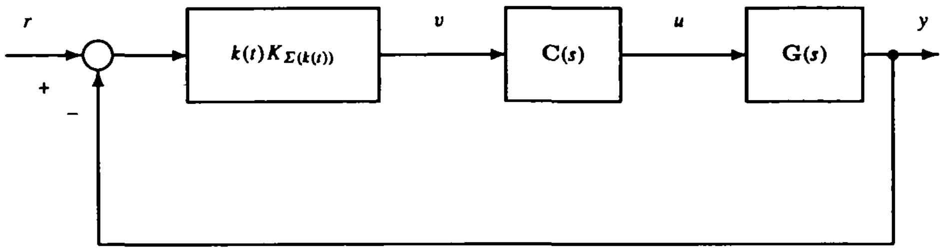

FIG. 2. High-gain adaptive tracing controller

belongs to $H^\infty(\mathbb{C}^{m\times m})$. The important point here is that the structural property (2.1) of the plant G remains invariant under precompensation by C(s). Consider the adaptive controller shown in Fig. 2 and formally given by

$$\hat{u}(s) = \mathrm{C}(s)\hat{v}(s), \tag{2.11a}$$

$$v(t) = k(t)K_{\Sigma(k(t))}(r(t) - y(t)), \tag{2.11b}$$

$$\dot{k}(t) = \|r(t) - y(t)\|^2, \qquad k(0) = k_0, \tag{2.11c}$$

where Σ is given by (2.6) and, as in the previous subsection, the matrices $K_1,\dots, K_\ell$ form an unmixing set for $GL(m, \mathbb{R})$.

Setting $\mathcal{H}_\mathrm{C} := \mathcal{L}^{-1}\mathcal{M}_{\mathrm{H_C}}\mathcal{L}$ ‡, we obtain the following time-domain description of the closed-loop system given by (2.3) and (2.11)

$$\dot{y}(t) = G\,(v(t) - \mathcal{H}_\mathrm{C}y)(t) - w_\mathrm{C}(t)), \quad y(0) = y_0, \; w_\mathrm{C} \in L^2(\mathbb{R}_+, \mathbb{R}^m), \tag{2.12a}$$

$$v(t) = k(t)K_{\Sigma(k(t))}\,(r(t) - y(t)), \tag{2.12b}$$

$$\dot{k}(t) = \|r(t) - y(t)\|^2, \quad k(0) = k_0, \tag{2.12c}$$

where w_C takes account of non-zero initial conditions.

The following result shows that (2.11) is a universal adaptive tracking controller for the class of systems given by (2.1).

THEOREM 2.8 The solution (y, k) of the closed-loop system (2.12) exists on $\mathbb{R}_+$ and has the following properties

(i) $\lim_{t\to\infty} k(t)$ exists and is finite,
(ii) $y - r \in L^2(\mathbb{R}_+, \mathbb{R}^m) \cap L^\infty(\mathbb{R}_+, \mathbb{R}^m)$,
(iii) $\lim_{t\to\infty}(y(t) - r(t)) = 0$.

The proof of the above result can be found in [83]. Note that, by construction, C(s) contains an integrator. This is required for a technical argument in the proof of Theorem 2.8. For a corresponding result on high-gain asymptotic disturbance rejection, see [38].

‡ As before, the unique causal extension of $\mathcal{H}_\mathrm{C}$ to $L^2_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ will be denoted by the same symbol $\mathcal{H}_\mathrm{C}$.

Application to retarded systems

Let $\mathbf{A} \in BV([a,b], \mathbb{R}^{n\times n})$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$ and consider the retarded system

$$\dot{x}(t) = \int_0^h d\mathbf{A}(\tau)x(t-\tau) + Bu(t)\,, \tag{2.13a}$$

$$y(t) = Cx(t)\,, \tag{2.13b}$$

$$x_{|[-h,0]} = x_0 \in C([-h,0], \mathbb{R}^n)\,. \tag{2.13c}$$

We assume that

$$\det(CB) \neq 0\,, \tag{2.14}$$

and

$$\det\begin{pmatrix} sI - \hat{\mathbf{A}}(s) & -B \\ C & 0 \end{pmatrix} \neq 0 \qquad \text{for all } s \in \mathbb{C}_0^{cl}\,, \tag{2.15}$$

where $\hat{\mathbf{A}}(s) := \int_0^h \exp(-s\tau)d\mathbf{A}(\tau)$ denotes the Laplace–Stieltjes transform of $\mathbf{A}$. The transfer-function matrix $\mathbf{G}(s)$ of (2.13) is given by

$$\mathbf{G}(s) = C(sI - \hat{\mathbf{A}}(s))^{-1}B\,.$$

The condition (2.15) is a generalization of the well-known finite-dimensional *minimum-phase* condition. It can be shown that (2.15) holds if and only if the following three conditions hold.

(i) The transfer-function matrix $\mathbf{G}(s)$ has no zeros in $\mathbb{C}_0^{cl}$,

(ii) $\operatorname{rank}(sI - \hat{\mathbf{A}}(s), B) = n$ for all $s \in \mathbb{C}_0^{cl}$,

(iii) $\operatorname{rank}\begin{pmatrix} sI - \hat{\mathbf{A}}(s) \\ C \end{pmatrix} = n$ for all $s \in \mathbb{C}_0^{cl}$.

Although each of the Theorems 2.6–2.8 is applicable to the class of retarded systems given by (2.13)–(2.15), we shall concentrate here on the application of the tracking result given in Theorem 2.8.

Let ρ, p and q be as in the previous subsection and let

$$\dot{\xi} = A_C\xi + B_C v\,, \qquad \xi(0) = \xi_0 \in \mathbb{R}^l\,, \tag{2.16a}$$

$$z = C_C\xi + I_m v \tag{2.16b}$$

be a stabilizable and detectable realization of $\mathbf{C}(s) = [q(s)/p(s)]I_m$. Let $r \in \mathfrak{R}_\rho$ and consider the closed-loop system given by (2.13), (2.16),

$$v(t) = k(t)K_{\Sigma(k(t))}(y(t) - r(t))\,, \tag{2.17a}$$

$$\dot{k}(t) = \|y(t) - r(t)\|^2\,, \qquad k(0) = k_0 \in \mathbb{R} \tag{2.17b}$$

and

$$u(t) = z(t)\,. \tag{2.18}$$

The following result shows that the universal adaptive controller presented in the previous subsection achieves asymptotic tracking and disturbance rejection for the class of retarded systems satisfying (2.14) and (2.15).

THEOREM 2.9 If (2.14) and (2.15) are satisfied, then for any $x_0 \in C([-h,0],\mathbb{R}^n)$, $\xi_0 \in \mathbb{R}^l$, $k_0 \in \mathbb{R}$ and $r \in \mathfrak{R}_\rho$ the closed-loop system given by (2.13) and (2.16)–(2.18) has the following properties:

(i) $\lim_{t\to\infty} k(t)$ exists and is finite,
(ii) $y - r \in L^2(\mathbb{R}_+,\mathbb{R}^m) \cap L^\infty(\mathbb{R}_+,\mathbb{R}^m)$,
(iii) $\lim_{t\to\infty}(y(t) - r(t)) = 0$,
(iv) $(x,\xi) \in L^\infty(\mathbb{R}_+,\mathbb{R}^{n+l})$, provided $r(\cdot)$ is bounded.

The proof of the above result can be found in [38]. It is based on the observation that the conditions (2.14) and (2.15) imply that

$$\mathbf{G}^{-1}(s) = s(CB)^{-1} + \mathbf{H}(s),$$

where $\mathbf{H} \in H^\infty(\mathbb{C}^{m\times m})$, i.e. $\mathbf{G}^{-1}(s)$ admits a decomposition of the form (2.1).

Theorem 2.9 remains true if the class of retarded systems given by (2.13)–(2.15) is replaced by a class of Volterra integrodifferential systems satisfying conditions similar to (2.14) and (2.15); see [38, 42].

Notes and references

There is a rich literature on adaptive high-gain control of finite-dimensional systems. To our knowledge, the first contributions in this tradition are due to Nussbaum [60], Willems & Byrnes [82] and Byrnes & Willems [3]. For a detailed treatment of finite-dimensional adaptive high-gain control, the reader is referred to Ilchmann's research monograph [23].

The first results on high-gain adaptive stabilization of infinite-dimensional systems were obtained by Dahleh & Hopkins [11], Kobayashi [31] and Byrnes [1]. The main result in Dahleh & Hopkins [11] (see also Dahleh [10]) is on the adaptive stabilization of a class of single-input single-output delay systems. It is contained as a special case in Theorem 2.7. Kobayashi [31] generalizes the result in [82] to a class infinite-dimensional systems described by semigroups on a Hilbert space. The fairly restrictive smoothness assumptions imposed on the plant in Kobayashi [31] can be relaxed as was shown by Logemann & Zwart [47]. More precisely, consider a system of the form

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{2.19}$$

evolving on a Banach space X, where A is the generator of a C_0-semigroup, $B \in \mathcal{B}(\mathbb{R}^m, X)$ and $C \in \mathcal{B}(X,\mathbb{R}^m)$. It is shown in [47] that, if

(i) (2.19) has no zeros in $\mathbb{C}_\alpha$ for some $\alpha < 0$ and $\det(CB) \neq 0$,
(ii) (2.19) is exponentially stabilizable,
(iii) $\operatorname{im} B \subset D(A)$,
(iv) $\operatorname{im} C^* \subset D(A^*)$,

then the transfer function $\mathbf{G}$ of (2.19) admits a decomposition of the form (2.1) and hence Theorems 2.6–2.7 can be applied. If A generates a holomorphic semigroup then only one of the assumptions (iii) and (iv) is needed. Notice that any common abstract model for the retarded system given by (2.13)–(2.15) does not satisfy the smoothness assumptions (iii) and (iv). Nevertheless, the transfer function of (2.13) satisfies condition (2.1). This leads us to the following open problem on high-gain stabilization of infinite-dimensional state-space systems.

Open problem. Suppose that (2.19) satisfies the assumptions (i) and (ii) and let $K \in GL(m, \mathbb{R})$ be such that $\sigma(CBK) \subset \mathbb{C}_0$. Does there exist $k^* > 0$ such that for all $k \geqslant k^*$ the application of the feedback $u = -kKy$ leads to an exponentially stable closed-loop system?

Byrnes [1] considers high-gain adaptive stabilization of abstract infinite-dimensional systems of the form (2.19) with bounded generator A. In this case the smoothness assumptions (iii) and (iv) are trivially satisfied and consequently the main result in [1] can be recovered as a special case of the results in [47]. In [2] Byrnes *et al.* proves high-gain stabilization results for a class of distributed-parameter systems which in general do not satisfy the generalized relative-degree-one condition (2.2). The problem of synthesizing adaptive high-gain controllers for this class deserves the attention of future research activities.

A modification of the adaptive stabilization scheme developed in [42] has been given by Logemann [34], presenting an algorithm which stabilizes a class of nonlinear retarded processes, with a prescribed rate of exponential decay (cf. also Logemann & Owens [44]).

Finally, Theorem 2.8 guarantees asymptotic tracking of any reference signal belonging to the kernel of a prespecified ordinary differential operator with constant coefficients. In a finite-dimensional set-up, Ilchmann & Ryan [25] have presented an adaptive high-gain strategy to control the output to track any reference signal in $W^{1,\infty}$, with tracking error asymptotic to a ball of arbitray prescibed radius. It is an interesting topic for future research to investigate to what extent this approach extends to the infinite-dimensional setting of this section.

3. Adaptive low-gain control

In Section 2 the emphasis was on the adaptive high-gain P-control of classes of uncertain unstable infinite-dimensional systems. In this section we focus on adaptive low-gain control of uncertain stable infinite-dimensional systems. The synthesis of low-gain I and PI controllers for uncertain stable plants has received considerable attention in the last 20 years. Let $\mathbf{G}$ be a stable proper rational transfer function matrix. The main existence result on robust (non-adaptive) low-gain I control of finite-dimensional systems says that, for any matrix K_0 satisfying $\sigma(\mathbf{G}(0)K_0) \subset \mathbb{C}_0$, there exists $k^* > 0$ such that for all $k \in (0, k^*)$ the controller $(1/s)kK_0$ stabilizes $\mathbf{G}$ and the resulting closed-loop system asymptotically tracks arbitrary constant reference signals. This result is essentially due to Davison [12] (see the subsection *Notes and references* for a short discussion of the literature). It is interesting to note that the low-gain control problem is, in a loose sense, the 'inverse' of the high-gain problem: in low-gain control we assume that the plant is stable and we impose an unmixing or invertibilty condition on the steady-state gain, whilst high-gain control schemes apply to minimum-phase systems whose high-frequency gain satisfies a suitable unmixing or invertibilty assumption.

There are two parts to the design of low-gain tracking controllers: choosing K_0 and tuning k. In this section we describe how this controller design approach, called 'tuning regulator theory' can be extended to a large class of infinite-dimensional systems, the class of so-called regular systems. Moreover, for this class we present nonlinear low-gain control laws which solve the tuning problem for k and K_0 adaptively. Regular systems encompass a large class of partial differential equations with boundary control and observation and functional differential equations with delays in the state, input and output variables and form the largest class of abstract infinite-dimensional systems for which there exist convenient representations both in state space and frequency domain.

Regular linear systems

A general class of m-input m-output continuous-time infinite-dimensional systems would be the *well-posed* systems as introduced by Salamon in [69]. The class of well-posed systems captures the systems theoretic properties of linearity, time-invariance and causality together with natural continuity properties of the input-to-state, state-to-output and input-to-output maps. Moreover, every well-posed system has a well-defined transfer function $\mathbf{G}(s)$. A *regular* system is a well-posed system satisfying the extra requirement that

$$\lim_{s\to\infty,\, s\in\mathbb{R}} \mathbf{G}(s) = D$$

exists. The theory of regular systems was developed by Weiss [75–78].

Let X be a real Hilbert space. Given an input function $u(\cdot) \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ the state of a regular linear system, with state space X, is described by

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X. \tag{3.1}$$

Here

- A is the generator of a C_0-semigroup $\mathbf{T}(t)$ on X,
- $B \in \mathcal{B}(\mathbb{R}^m, X_{-1})$ and X_{-1} is the completion of X with respect to $\|x\|_{X_{-1}} := \|(\beta I - A)^{-1}x\|_X$, where $\beta \in \varrho(A)$.

It is well-known that $\mathbf{T}(t)$ extends to a C_0-semigroup on X_{-1}. The generator of this semigroup is a bounded operator from X to X_{-1} which extends A. The extended semigroup and its generator will be denoted by the same symbols $\mathbf{T}(t)$ and A, respectively. Equality in (3.1) holds in X_{-1}.

Continuity of the input-to-state map is expressed by

$$\left\| \int_0^t \mathbf{T}(t-\tau)Bu(\tau)\,d\tau \right\|_X \leqslant b_t \|u(\cdot)\|_{L^2(0,t;\,\mathbb{R}^m)}, \tag{3.2}$$

where $b_t \geqslant 0$. If $x_0 \in X$, then the mild solution, given by

$$x(t) = \mathbf{T}(t)x_0 + \int_0^t \mathbf{T}(t-\tau)Bu(\tau)\,d\tau, \tag{3.3}$$

evolves continuously in X. The control operator B is called *bounded* if $B \in \mathcal{B}(\mathbb{R}^m, X)$.

To introduce an observation for (3.3), let X_1 denote the domain of A, as an operator defined on X, endowed with the graph norm. The semigroup $\mathbf{T}(t)$ restricts to a C_0-semigroup $\mathbf{T}(t)$ on X_1. The exponential growth bounds of $\mathbf{T}(t)$ are the same on all three spaces X_1,

X and X_{-1}. If $u(\cdot) = 0$ and $x_0 \in X_1$, then the output of a regular (or well-posed system) is given by

$$y(t) = C\mathbf{T}(t)x_0\,,$$

where the observation operator C is in $\mathcal{B}(X_1, \mathbb{R}^m)$. Continuity of the state-to-output map guarantees that for every $t \geqslant 0$ there exists $c_t \geqslant 0$ such that

$$\|C\mathbf{T}(\cdot)x\|_{L^2(0,t;\,\mathbb{R}^m)} \leqslant c_t\|x\|_X \qquad \text{for all } x \in X_1\,. \tag{3.4}$$

The observation operator C is called *bounded* if C can be extended to an operator in $\mathcal{B}(X, \mathbb{R}^m)$.

The continuity of the input-to-output map for a regular system combined with (3.4) gives an output $y(\cdot) \in L^2(0, T; \mathbb{R}^m)$ defined for almost all $t \geqslant 0$ by

$$y(t) = C_L x(t) + Du(t)\,. \tag{3.5}$$

Here C_L is the Lebesgue extension of C; see Weiss [75]. In particular, we have $X_1 \subset D(C_L)$. The following properties of C_L are consequences of regularity:

for each $x \in X$, $\mathbf{T}(t)x \in D(C_L)$ for a.e. $t \geqslant 0$,
$\operatorname{im}((sI - A)^{-1}B) \subset D(C_L)$ for all $s \in \varrho(A)$.

In the following, we denote the regular system given by (3.3) and (3.5) by Σ_{plant}. The transfer function $\mathbf{G}(s)$ of Σ_{plant} can be written as

$$\mathbf{G}(s) = C_L(sI - A)^{-1}B + D\,.$$

The operators A, B, C and D are called the *generating operators* of Σ_{plant}.

It follows from Salamon [70] that any function $\mathbf{G}(s)$ which is holomorphic and bounded in some right half-plane and for which $\lim_{s\to\infty,\, s\in\mathbb{R}} \mathbf{G}(s)$ exists, admits a regular state-space realization. Consequently, any function $\mathbf{G}(s)$ satisfying (2.1) is in fact the transfer function of a regular system.

Finally, consider the nonlinear system given by

$$\dot{g}(t) = \|v(t)\|^2\,, \quad g(0) = g_0 \in \mathbb{R}, \tag{3.6a}$$

$$w(t) = \mathcal{K}(g(t))v(t)\,, \quad t \geqslant 0\,, \tag{3.6b}$$

where $v \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ is the input and w denotes the output. The function $\mathcal{K} : \mathbb{R} \to \mathbb{R}$ is assumed to be locally Lipschitz. In the following we need a well-posedness result for the feedback interconnection of Σ_{plant} and (3.6). More precisely, consider the feedback system given by (3.3), (3.5), (3.6) and the interconnection equations

$$v = y\,, \quad u = -w\,.$$

The closed-loop equations for y and g then take the form

$$y(t) = C_L\mathbf{T}(t)x_0 - \{(\mathcal{L}^{-1}\mathbf{G}) \star [(\mathcal{K} \circ g)y]\}(t), \tag{3.7a}$$

$$g(t) = g_0 + \int_0^t \|y(\tau)\|^2\,d\tau. \tag{3.7b}$$

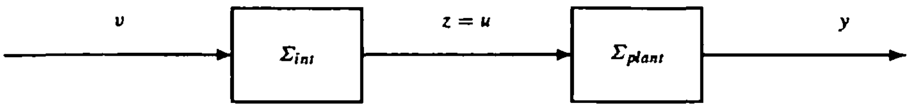

FIG. 3. Series connection $\tilde{\Sigma}$

Let $\tau \in (0, \infty]$. A function $(y, g) : [0, \tau) \to \mathbb{R}^m \times \mathbb{R}$ is called a *solution* of (3.7) on $[0, \tau)$ if

(i) $(y, g) \in L^2([0, \tau'], \mathbb{R}^m) \times AC([0, \tau'], \mathbb{R})$ for all $\tau' \in [0, \tau)$, where $AC([0, \tau'], \mathbb{R})$ denotes the absolutely continuous functions on $[0, \tau']$ with values in $\mathbb{R}$.

(ii) (y, g) satisfies (3.7) almost everywhere on $[0, \tau)$.

If (3.7) has a solution (y, g) on $[0, \tau)$, then the corresponding state trajectory of Σ_{plant} is given by

$$x(t) = \mathbf{T}(t)x_0 - \int_0^t \mathbf{T}(t - \tau)B\mathcal{K}(g(\tau))y(\tau)\,d\tau\,.$$

PROPOSITION 3.1 Suppose that $\mathcal{L}^{-1}\mathbf{G} \in L^1_{loc}(\mathbb{R}_+, \mathbf{R}^{m\times m})$. Then for any $(x_0, g_0) \in X \times \mathbb{R}$ there exists a maximal solution of (3.7). Precisely: there exists $\tau_{max} \in (0, \infty]$ such that (3.7) has a unique solution (y_{max}, g_{max}) on $[0, \tau_{max})$, and moreover

$$\tau_{max} < \infty \quad \Longrightarrow \quad \int_0^{\tau_{max}} \|y_{max}(t)\|^2\,dt = \infty\,.$$

The proof of Proposition 3.1 can be found in Logemann & Townley [45].

The above well-posedness result is sufficient for low-gain adaptive control of regular systems. Notice that it is restricted to systems whose impulse response satisfies the assumption $\mathcal{L}^{-1}\mathbf{G} \in L^1_{loc}(\mathbb{R}_+, \mathbf{R}^{m\times m})$ and to feedback laws given by (3.6). The investigations of more general well-posedness problems for the feedback connections of regular linear systems and and nonlinear controllers is an interesting topic for future research.

Non-adaptive low-gain control

In the following let Σ_{plant} be exponentially stable, i.e. $\mathbf{T}(t)$ is exponentially stable. Then it follows that Σ_{plant} is also L^2-stable, i.e. $\mathbf{G} \in H^\infty(\mathbb{C}^{m\times m})$. Let Σ_{int} denote the integrator described by

$$z(t) = z_0 + \int_0^t v(\tau)\,d\tau\,, \quad z_0 \in \mathbb{R}^m\,,$$

where $v \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ is the integrator input.

We will consider the series connection $\tilde{\Sigma}$ of Σ_{int} followed by Σ_{plant} shown in Fig. 3. It

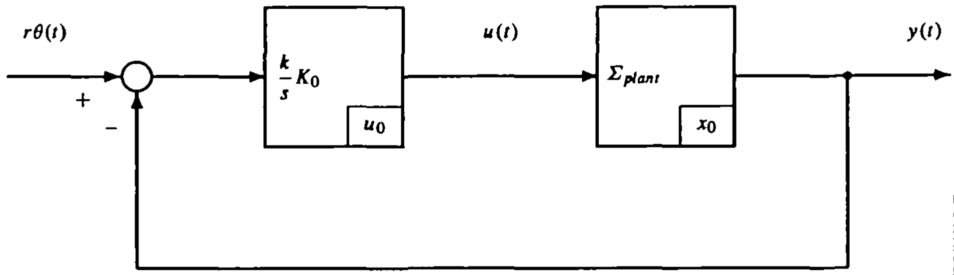

FIG. 4. Low-gain control system

is not difficult to show that $\tilde{\Sigma}$ is a regular system; see [45]. Clearly, the feedthrough $\tilde{D}$ of $\tilde{\Sigma}$ satisfies $\tilde{D} = 0$. The remaining generating operators of $\tilde{\Sigma}$ are given by

$$\tilde{A} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad \tilde{C} = (C_L, D),$$

where

$$D(\tilde{A}) = \{(x, u) \in D(C_L) \times \mathbb{R}^m \mid Ax + Bu \in X\}.$$

The C_0-semigroup generated by $\tilde{A}$ is denoted by $\tilde{\mathbf{T}}(t)$. If B is bounded, then it follows easily that $D(\tilde{A}) = D(A) \times \mathbf{R}^m$. Note that any unboundedness of B is absorbed into the unboundedness of $\tilde{A}$ and that hence the control operator $\tilde{B}$ of $\tilde{\Sigma}$ is bounded. Trivially, the function $\tilde{\mathbf{G}}(s) := (1/s)\mathbf{G}(s)$ is the transfer function of $\tilde{\Sigma}$.

In the following we apply static output feedback to $\tilde{\Sigma}$. Using results from [78] it is easy to see that for any feedback matrix $K \in \mathbb{R}^{m\times m}$, the resulting closed-loop system is again a regular system, which we denote by $\tilde{\Sigma}^K$. The corresponding C_0-semigroup and its generator are denoted by $\tilde{\mathbf{T}}^K(t)$ and $\tilde{A}^K$, respectively.

LEMMA 3.2 For every $K \in \mathbf{R}^{m\times m}$ the domain of the closed-loop generator $\tilde{A}^K$ is given by

$$D(\tilde{A}^K) = D(\tilde{A}) = \{(x, u) \in X \times \mathbb{R}^m \mid Ax + Bu \in X\}.$$

Consider the feedback system shown in Fig. 4, where $\theta(t)$ denotes the Heaviside step-function and u_0 denotes the initial state of the integrator. Since the output $y(t)$ depends on the initial states x_0 and u_0 we write $y(t) = y(t; (x_0, u_0))$. Moreover, we define the corresponding error by

$$e(t; (x_0, u_0)) = r\theta(t) - y(t; (x_0, u_0)).$$

THEOREM 3.3 Let $r \in \mathbf{R}^m$. Suppose that $\det \mathbf{G}(0) \neq 0$ and let $K_0 \in \mathbf{R}^{m\times m}$ be such that $\sigma(\mathbf{G}(0)K_0) \subset \mathbb{C}_0$. Then there exists $k^* > 0$ such that for any $k \in (0, k^*)$ the closed-loop semigroup $\tilde{\mathbf{T}}^{kK_0}(t)$ is exponentially stable and $e(\cdot\,; (x_0, u_0)) \in L^2(\mathbf{R}_+, \mathbf{R}^m)$ for all $(x_0, u_0) \in X \times \mathbf{R}^m$. Furthermore,

$$\lim_{t\to\infty} e(t; (x_0, u_0)) = 0 \quad \text{for all } (x_0, u_0) \in D(\tilde{A}).$$

If the observation operator C is bounded, then the above equation holds for all $(x_0, u_0) \in X \times \mathbb{R}^m$.

We close this subsection with a lemma which will be needed in the following to reformulate adaptive tracking problems as adaptive stabilization problems.

LEMMA 3.4 For any $r \in \mathbb{R}^m$ there exists $(x_r, u_r) \in D(\tilde{A})$ such that

$$\tilde{C}\tilde{\mathbf{T}}(t)(x_r, u_r) = r \quad \text{for all } t \geqslant 0 .$$

The proofs of Theorem 3.3 and Lemmas 3.2 and 3.4 can be found in [45].

Adaptive low-gain control of multivariable systems with sign-definite steady-state gain

In the previous subsection the main result was that, given a stable system with transfer function $\mathbf{G}$, if $\sigma(\mathbf{G}(0)K_0) \subset \mathbb{C}_0$, then the closed-loop semigroup $\tilde{\mathbf{T}}^{kK_0}(t)$ is exponentially stable for all sufficiently small $k > 0$. A natural problem is to seek ways of tuning the parameter k and selecting K_0 adaptively. How this adaptation is achieved depends strongly on which system information is available in the controller synthesis.

For Hermitian matrices $M, N \in \mathbb{C}^{m\times m}$, we write $M \prec N$ if $N - M$ is positive definite, and $M \succ N$ if $N - M$ is negative definite. Similarly, we write $M \preceq N$ if $N - M$ is positive semi-definite, and $M \succeq N$ if $N - M$ is negative semi-definite. Moreover, for a complex matrix M let M^H denote the conjugate transpose of M.

In this subsection we consider the adaptive low-gain control of systems with *sign-definite* steady-state gains $\mathbf{G}(0)$, that is where either $\mathbf{G}(0) \succ 0$ or $\mathbf{G}(0) \prec 0$. This situation arises most naturally in the single-input single-output case where we need to assume only that the steady-state gain is non-zero. A crucial tool is the following proposition which is of some interest in its own right.

PROPOSITION 3.5 Let $\mathbf{G}$ be the transfer function of an m-input m-ouput exponentially stable regular system and suppose that $\det \mathbf{G}(0) \neq 0$. For $k \in \mathbb{R}$ define

$$\tilde{\mathbf{G}}^k(s) = \tilde{\mathbf{G}}(s)(I + k\tilde{\mathbf{G}}(s))^{-1} = \frac{1}{s}\mathbf{G}(s)(I + \frac{k}{s}\mathbf{G}(s))^{-1} .$$

Under these conditions there exists $k^* > 0$ such that for all $k \in (0, k^*)$

$$\|\tilde{\mathbf{G}}^k\|_\infty = \frac{1}{k} \tag{3.8}$$

if and only if $\mathbf{G}(0) \succ 0$. Moreover, the claim remains true if we replace k by $-k$ in (3.8) and $\mathbf{G}(0) \succ 0$ by $\mathbf{G}(0) \prec 0$.

The proof of the above result can be found in [45].

The following theorem is the main result on adaptive low-gain control of systems with sign-definite steady-state gains.

THEOREM 3.6 Let Σ_{plant} be a m-input m-ouput exponentially stable regular system. Suppose that the transfer function $\mathbf{G}$ of Σ_{plant} is such that $\mathbf{G}(0)$ is sign-definite. Let $r\,\theta(t)$, $r \in \mathbb{R}^m$, be an arbitrary constant vector-valued reference signal, set

$$\mathcal{K}(\gamma) := (\log^\mu \gamma)\cos(\log^\nu \gamma) \tag{3.9}$$

and consider the control law

$$u(t) = u_0 + \int_0^t \mathcal{K}(g(\tau))e(\tau)\,d\tau\,, \tag{3.10a}$$

$$\dot{g}(t) = \|e(t)\|^2\,, \quad g(0) = g_0\,, \tag{3.10b}$$

where $e(t) = r - y(t)$ and $\mu \leqslant 0 < \nu$ and $\nu - 2\mu < 1$. Then for all $(x_0, u_0) \in X \times \mathbb{R}^m$ and $g_0 > 1$, where X denotes the state-space of Σ_{plant}, the following statements hold true

(i) $\lim_{t\to\infty} g(t) = g_\infty < \infty$,
(ii) $\|x(t)\|$ and $u(t)$ remain bounded as $t \to \infty$,
(iii) $e \in L^2(\mathbb{R}_+, \mathbb{R}^m)$.

Moreover, if $(x_0, u_0) \in D(\tilde{A})$, then

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} (y(t) - r) = 0\,. \tag{3.11}$$

If the observation operator C of Σ_{plant} is bounded, then (3.11) is true for all $(x_0, u_0) \in X \times \mathbb{R}^m$.

In the control law (3.10a) the integrator gain is given by $\mathcal{K}(g(t))$, where $g(t)$ is driven by the adaption law (3.10b). The controller is of low-gain nature in the sense that $\mathcal{K}(g(t)) \to 0$ as $g(t) \to \infty$. Moreover, $\mathcal{K}(\cdot)$ oscillates, enabling the controller to 'learn' the sign of $\mathbf{G}(0)$. Note that $\mathcal{K}(\cdot)$ is not a Nussbaum function in the sense of (2.10).

The above result is proved in [45]. The outline of the proof is as follows. By Proposition 3.1 the nonlinear closed-loop system has a unique solution. The next step is to use Lemma 3.4 to convert the tracking problem into a stabilization problem. Then the idea is to use Proposition 3.5 and the stability of the closed-loop semigroup for sufficiently small gains (cf. Theorem 3.3) to bound the possible growth in $g(t)$ and then to exploit the nature of the control law (3.10a) and (3.10b) to contradict the possible unboundedness of $g(t)$. Once the boundedness of $g(t)$ is established we have immediately that $e(\cdot) \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. The remainder of the proof then follows by exploiting the internal stability of Σ_{plant}.

Note that in Theorem 3.6 the tuning function $\mathcal{K}(\gamma) = (\log^\mu \gamma)\cos(\log^\nu \gamma)$ decays to 0 like a fractional power of $\log\gamma$ as $\gamma \to \infty$. In the finite-dimensional case, Cook [5] and Logemann & Townley [45] have shown that tuning functions $\mathcal{K}(\gamma)$ can be used which decay to 0 like a fractional power of γ. In the infinite-dimensional case, but with sign $\mathbf{G}(0)$ known, we can use tuning functions which decay to 0 like a fractional power, although more slowly than in the finite-dimensional case. More precisely, the following result holds.

PROPOSITION 3.7 Suppose that the conditions of Theorem 3.6 hold and that additionally $\mathbf{G}(0) \succ 0$. If

$$u(t) = u_0 + \int_0^t g^{-\nu}(\tau)e(\tau)\,d\tau\,,$$

$$\dot{g}(t) = \|e(t)\|^2\,, \quad g(0) = g_0 > 0\,,$$

and $0 < \nu < 1/2$, then the conclusions of Theorem 3.6 hold.

Sketch of Proof (see [45] for more details). It is sufficient to show that $g(\cdot)$ is bounded. Let $[0, \tau)$ be the maximal interval of existence. If $g(\cdot)$ is unbounded on $[0, \tau)$, then there exists $t_1 \geqslant 0$ such that, with $g_1 = g(t_1)$, $k_1 = g_1^{-\nu}$ is a stabilizing gain. For any $t \in (t_1, \tau)$ we have that, on $[t_1, t]$,

$$\mathbf{L}_{t_1}\mathbf{P}_t e = \tilde{\Psi}^{k_1}_{t-t_1}(\tilde{x}(t_1)) - \tilde{\mathbf{F}}^{k_1}_{t-t_1}(\mathbf{L}_{t_1}\mathbf{P}_t(\mathcal{K} \circ g - k_1)e)\,.$$

Here $\mathbf{L}_t$ and $\mathbf{P}_t$ are the left-shift and truncation operators respectively, that is $(\mathbf{L}_t f)(s) = f(t+s)$ and $(\mathbf{P}_t f)(s) = f(s)$ if $s \leqslant t$ and 0 otherwise. Moreover, $\tilde{\Psi}^{k_1}_{t-t_1}$ and $\tilde{\mathbf{F}}^{k_1}_{t-t_1}$ denote the state-to-output map and the input-to-output map at time $t - t_1$, respectively, of the closed-loop system obtained by applying static output feedback with gain k_1 to the series connection $\tilde{\Sigma}$ shown in Fig. 3.

We can assume that k_1 is small enough so that, using Proposition 3.5 and estimating, we obtain

$$\sqrt{g(t) - g_1} \leqslant c\, g^{\nu}(t)$$

for some $c > 0$ and all $t \in [t_1, \tau)$. This inequality clearly contradicts the unboundedness of $g(\cdot)$ and the assumption that $\nu < 1/2$.

For finite-dimensional systems Proposition 3.7 actually holds for all $\nu \in (0, 1)$, see [5, 45]. Whether or not this is also true for regular infinite-dimensional systems is a topic for future research.

Adaptive low-gain control of multivariable systems with sign-indefinite steady-state gain

In this subsection we consider the adaptive low-gain tracking problem, for regular systems with square $m \times m$ transfer functions $\mathbf{G}(s)$. In the previous subsection, under the assumption that $\mathbf{G}(0)$ is sign-definite, we could exploit the fact that for all gains k having the 'correct' sign and with $|k|$ sufficiently small, $\|\tilde{\mathbf{G}}^k\|_\infty = 1/|k|$ (see Proposition 3.5). If $\mathbf{G}(0)$ is sign-indefinite or even non-symmetric, then, again by Proposition 3.5, we no longer have this result.

To overcome this problem, we do not use a tuning function $\mathcal{K}$ reflecting the low-gain nature of the problem in the sense that $\lim_{\gamma\to\infty} \mathcal{K}(\gamma) = 0$, but resort instead to a gain which oscillates smoothly between 0 and 2 (in fact, 2 could be replaced by any positive number.)

We assume throughout that Σ_{plant} is a m-input m-output exponentially stable regular system. We will first consider the case when the spectrum of $\mathbf{G}(0)$ is unmixed in the sense that $\sigma(\mathbf{G}(0)) \subset \mathbb{C}_0$. As usual let $u(\cdot)$ and $y(\cdot)$ denote the plant input and plant output, respectively, and set $e(\cdot) = r - y(\cdot)$, where $r \in \mathbf{R}^m$ is a demand vector. Consider the control law given by

$$u(t) = u_0 + \int_0^t [1 + \cos(\log^{\nu} g(\tau))]e(\tau)\,d\tau\,, \quad \text{where } 0 < \nu < 1\,, \tag{3.12a}$$

$$\dot{g}(t) = \|e(t)\|^2\,, \quad g(0) = g_0\,. \tag{3.12b}$$

THEOREM 3.8 Assume that $\sigma(\mathbf{G}(0)) \subset \mathbb{C}_0$. Let $r \in \mathbf{R}^m$ be an arbitrary demand vector. If $u(t)$ is given by (3.12a), with gain adaptation (3.12b), then for each $(x_0, u_0) \in X \times \mathbf{R}^m$ and $g_0 > 1$ we have

(i) $\lim_{t\to\infty} g(t) = g_\infty < \infty$,
(ii) $\|x(t)\|$ and $\|u(t)\|$ remain bounded as $t \to \infty$,
(iii) $e \in L^2(\mathbb{R}_+, \mathbf{R}^m)$.

Moreover, if $(x_0, u_0) \in D(\tilde{A})$, then

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} (y(t) - r) = 0. \tag{3.13}$$

If the observation operator C is bounded, then (3.13) holds for all $(x_0, u_0) \in X \times \mathbf{R}^m$.

Specialized to the case where $\mathbf{G}(0) \succ 0$, it is natural to compare the control law in Theorem 3.8 to the one in Proposition 3.7. Intuitively, it should have advantages to use the controller in Proposition 3.7, since in this case the gain passes into the 'correct' parameter region once and remains there, whereas the gain in the controller in Theorem 3.8 may pass in and out of the 'correct' region several times before converging and small disturbances could lead to further cycles in the gain adaptation.

In Theorem 3.8 we assumed that $\sigma(\mathbf{G}(0)) \subset \mathbb{C}_0$. We now consider the case when we know only that $\det \mathbf{G}(0) \neq 0$. An application of Proposition 2.4 shows that there exists a finite set $\{K_1, ..., K_\ell\} \subset \mathbf{R}^{m\times m}$ so that given any real invertible $m \times m$ matrix M there exists $j \in \{1, 2, \ldots, \ell\}$ such that $\sigma(MK_j) \subset \mathbb{C}_0$. We now use this result in order to unmix the spectrum of $\mathbf{G}(0)$. Consider the feedback law

$$u(t) = u_0 + \int_0^t [1 + \cos(\log^\nu g(\tau))] K_{\Sigma(g(\tau))} e(\tau)\, d\tau, \tag{3.14}$$

combined with the adaptation rule (3.12b), where $0 < \nu < 1$, $g_0 \geqslant \exp(\sqrt[\nu]{2\pi})$ and

$$\Sigma(\gamma) = j \quad \text{if} \quad (2\pi)^{-1} \log^\nu \gamma \in [N\ell + j, N\ell + j + 1) \quad \text{for some} \quad N \in \mathbb{N}.$$

THEOREM 3.9 Assume that $\det \mathbf{G}(0) \neq 0$. Let $r \in \mathbf{R}^m$ be an arbitrary demand vector. If $u(t)$ is given by (3.14), with adaptation (3.12b), then for each $(x_0, u_0) \in X \times \mathbf{R}^m$ and $g_0 \geqslant \exp(\sqrt[\nu]{2\pi})$ we have

(i) $\lim_{t\to\infty} g(t) = g_\infty < \infty$,
(ii) $\|x(t)\|$ and $\|u(t)\|$ remain bounded as $t \to \infty$,
(iii) $e \in L^2(\mathbb{R}_+, \mathbb{R}^m)$.

Moreover, if $(x_0, u_0) \in D(\tilde{A})$, then (3.13) holds. If the control operator C is bounded, then (3.13) holds for all $(x_0, u_0) \in X \times \mathbf{R}^m$.

The proofs of Theorems 3.8 and 3.9 can be found in [45].

Notes and references

Non-adaptive low-gain control in a finite-dimensional setting has been considered by Davison [12] and Lunze [49] using state-space methods, and by Grosdidier *et al.* [19] and Morari [58] using frequency-domain methods. Applications of these and related results to industrial control problems can be found in Coppus *et al.* [6] and Lunze [50]. Pohjolainen [62, 63], Jussila & Koivo [27], Logemann & Owens [43] and Logemann *et al.* [37]

have extended the finite-dimensional tuning regulator results to various classes of infinite-dimensional systems.

The question how much integral action a stable finite-dimensional control system can tolerate was answered by Mustafa [59]. He derived a closed formula, in terms of a minimal state-space realization, for the *radius of integral controllability* of an integral controllable plant, i.e. a formula for the largest possible $k^* > 0$ such that $(k/s)I$ stabilizes the plant for all $k \in (0, k^*)$. The question whether there is a similar result in infinite dimensions leads to the following

Open problem. Is there a closed formula for the radius of integral controllability of a regular infinite-dimensional system in terms of its generating operators?

Low-gain universal adaptive controllers which achieve asymptotic tracking of constant reference signals for finite-dimensional linear plants have been presented by Cook [5] and by Miller & Davison [56, 57].‡ The controller given in [5] is smooth, while the control laws derived in [56, 57] are 'piecewise constant'. Cook's paper [5] is restricted to the single-input single-output case. The controller given in [57] satisfies a control input constraint. We mention that the main result in Cook [5] (at least as we understand it) relies on the Kalman–Yakubovich lemma. A straightforward extension of the approach in [5] to regular infinite-dimensional systems is not possible, since the existence of an appropriate infinite-dimensional version of the Kalman–Yakubovich lemma is a difficult open problem. The piecewise constant controllers presented in Miller & Davison [56, 57] seem unnecessarily complicated and do not generalize to the infinite-dimensional case either.

It is natural to add a proportional part to the integrator to produce an adaptive PI controller. If the proportional and integral gains are equal and the underlying plant is a Pritchard–Salamon system (see Section 4) without direct feedtrough, then the results of this section carry over with no changes. The problems of tuning the two gain parameters independently and of extending the results to the more general class of regular systems are challenges requiring further research. In the former we would need to find suitable two-parameter adaptation strategies while, in the latter, well-posedness problems for the nonlinear closed-loop system arise.

Non-adaptive sampled-data versions of tuning regulators for certain classes of infinite-dimensional systems were obtained by Kobayashi [30, 33] using state-space methods. Kobayashi's results were extended to the class of regular infinite-dimensional systems by Logemann and Townley [46] using a frequency-domain approach. Moreover, in the same paper they developed an approach to adaptive low-gain sampled-data control for regular systems. The adaptive tracking results in [46] guarantee not only asymptotic tracking at the sampling instants, but also in the sampling interval.

4. Adaptive controllers for classes of stabilizable and detectable systems

In this section we present results on adaptive stabilization which do not require the assumptions of minimum-phase and generalized relative-degree one, or open-loop stability

‡ Surprisingly, the low-gain adaptive tracking problem has received less attention than its high-gain counterpart; see Ilchmann [23], Logemann & Ilchmann [38], Ryan [68] and the references therein.

and invertibility of the steady-state gain, which played crucial roles in the adaptive high- and low-gain stabilization results of Sections 2 and 3, respectively. We present two main results. The first deals with smooth adaptive stabilization of systems which are stabilizable by scalar static output feedback. In a sense this result is a direct generalization of the results of Sections 2 and 3. The second result is the synthesis of switching controllers which adaptively stabilize fairly large classes of stabilizable and detectable systems. The results in this section apply to so-called Pritchard–Salamon systems—a class of systems which allows for unboundedness in the control and observation, but which is less general than the class of regular systems. The relationship between Pritchard–Salamon systems and regular systems is described more precisely in Emirsajlow *et al.* [15].

Pritchard–Salamon systems

As in Section 3 the system is formally represented by the abstract differential equation

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \tag{4.1a}$$

$$y(t) = Cx(t)\,. \tag{4.1b}$$

In this section we assume that there exist real Hilbert spaces $W \subset V$, with continuous dense injection, so that A is the generator of a strongly continuous semigroup, $\mathbf{T}(t)$, on both W and V (with A denoting the generator on V with domain $D_V(A)$) and that $B \in \mathcal{B}(\mathbb{R}^m, V)$ and $C \in \mathcal{B}(W, \mathbb{R}^p)$. In this sense, both B and C are unbounded.

We assume that the triple (A, B, C) forms a Pritchard–Salamon system; that is, for each $t > 0$ there exist positive constants b and c such that

(PS1) $\quad \|\int_0^t \mathbf{T}(t-\tau)Bu(\tau)\,d\tau\|_W \leqslant b\|u(\cdot)\|_{L^2(0,t;\mathbb{R}^m)} \quad$ for all $u(\cdot) \in W^{1,2}(0, t; \mathbb{R}^m)$,

(PS2) $\quad \|C\mathbf{T}(\cdot)x\|_{L^2(0,t;\mathbb{R}^p)} \leqslant c\|x\|_V \quad$ for all $x \in W$.

B is termed an *input-admissible* operator while C is an *output-admissible* operator. Note that if $\mathbf{T}(t)$ is exponentially stable on W, respectively V then (PS1), respectively (PS2), hold with $t = \infty$.

If $x_0 \in W$, then the mild solution of (4.1a), given by

$$x(t) = \mathbf{T}(t)x_0 + \int_0^t \mathbf{T}(t-\tau)Bu(\tau)\,d\tau\,, \tag{4.2}$$

evolves continuously in W, and the output given by (4.1b) defines a continuous function with values in $\mathbb{R}^p$. Most systems-theoretic properties of finite-dimensional systems extend to this class of infinite-dimensional systems. Three such properties of this class of systems which we require in this section are the well-posedness under arbitrary linear feedbacks; the equivalence of internal and external stability and, in the case of smooth adaptive stabilization, the well-posedness under locally Lipschitz nonlinear perturbations. This third property is particularly relevant for adaptive control. For a comprehensive treatment of this class of infinite-dimensional systems, in a linear quadratic or systems-theoretic setting, the reader is urged to consult Pritchard & Salamon [64] and Curtain *et al.* [8].‡ While the

‡ For duality theory and H^∞-optimal control of Pritchard–Salamon systems see also van Keulen [28].

Pritchard–Salamon class does include many examples of partial differential systems with boundary control and observation and of functional differential systems with delayed control and sensing action, it is less general than the class of regular systems, described in Section 3 in the context of low-gain adaptive control.

The following lemma, which can be found in [8], shows that the Pritchard–Salamon class is invariant under static output feedback.

LEMMA 4.1 For each $K \in \mathbb{R}^{m\times p}$ there exists a C_0-semigroup, denoted by $\mathbf{T}_{A+BKC}(t)$, defined on W and V, such that for each $x_0 \in V$, respectively $x_0 \in W$, $x(t) = \mathbf{T}_{A+BKC}(t)x_0$ is the unique solution, continuous in V, respectively W, of

$$x(t) = \mathbf{T}(t)x_0 + \int_0^t \mathbf{T}(t-\tau)BKCx(\tau)\,d\tau.$$

Moreover, $(A + BKC, B, C)$ is a Pritchard–Salamon system.

Let $\varrho(A)$ denote the resolvent set of A considered as an operator on the space V. The transfer function of (4.1) is given by

$$\mathbf{G}(s) = C(sI - A)^{-1}B\,,$$

which is defined for all $s \in \varrho(A)$. In the sequel we need the following stabilizability and detectability concepts.

DEFINITION 4.2 (i) (A, B) is *admissibly exponentially stabilizable* (*stabilizable* for short) if there exists an output-admissible feedback $F \in \mathcal{B}(W, \mathbb{R}^m)$ such $\mathbf{T}_{A+BF}(t)$ defines an exponentially stable semigroup on V and W.

(ii) (A, C) is *admissibly exponentially detectable* (*detectable* for short) if there exists an input-admissible operator $H \in \mathcal{B}(\mathbb{R}^p, V)$ such that $\mathbf{T}_{A+HC}(t)$ defines an exponentially stable semigroup on V and W.

(iii) (A, B, C) is *externally stabilizable* if there exists $K \in \mathbb{R}^{m\times p}$ such that the closed loop transfer function $\mathbf{G}(I + K\mathbf{G})^{-1} \in H^\infty(\mathbb{C}^{p\times m})$.

PROPOSITION 4.3 Assume that (A, B) is stabilizable and that (A, C) is detectable. Then $K \in \mathbb{R}^{m\times p}$ stabilizes (A, B, C) externally, if and only if $\mathbf{T}_{A+BKC}(t)$ is exponentially stable on W and V.

The above proposition has been proved in [8]. It is useful for adaptive stabilization of systems stabilizable by static output feedback. In the case that only stabilizability and detectability is imposed on the system to be controlled, then the natural approach is to use finite-dimensional dynamic output feedback. Let

$$\dot{z}(t) = Fz(t) + Gy(t)\,, \qquad z(t) \in \mathbb{R}^l\,, \tag{4.3a}$$

$$u(t) = Hz(t) + Ky(t) \tag{4.3b}$$

be a finite-dimensional compensator of order l. For $l = 0$, (4.3) is understood as $u(t) = Ky(t)$. If $l > 0$, then (4.3) can still be interpreted as static output feedback. More precisely, we regard the dynamic feedback as static feedback

$$\tilde{u}(t) = \tilde{K}\tilde{y}(t)\,, \tag{4.4}$$

applied to the augmented plant described formally by

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t), \tag{4.5a}$$

$$\tilde{y}(t) = \tilde{C}\tilde{x}(t), \tag{4.5b}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} K & H \\ G & F \end{bmatrix}. \tag{4.6}$$

It is clear that the triple $(\tilde{A}, \tilde{B}, \tilde{C})$ defines a Pritchard–Salamon system on the extended state spaces

$$\tilde{W} = W \oplus \mathbf{R}^l, \qquad \tilde{V} = V \oplus \mathbf{R}^l. \tag{4.7}$$

Note that the stabilizability and detectability notions introduced in Definition 4.2 will be inherited by the augmented system.

Let $\tilde{\mathbf{T}}(t)$ denote the strongly continuous semigroup on $\tilde{W}$ and $\tilde{V}$ generated by $\tilde{A}$. If we apply the compensator (4.3) to the plant (4.1), or equivalently, if we apply the static output feedback (4.4) to the augmented system (4.5), then by Lemma 4.1 there exists a strongly continuous semigroup $\tilde{\mathbf{T}}_{\text{cl}}(t)$ such that for all $\tilde{x}_0 \in \tilde{W}$ and all $t \geqslant 0$

$$\tilde{\mathbf{T}}_{\text{cl}}(t)\tilde{x}_0 = \tilde{\mathbf{T}}(t)\tilde{x}_0 + \int_0^t \tilde{\mathbf{T}}(t-\tau)\tilde{B}\tilde{K}\tilde{C}\tilde{\mathbf{T}}_{\text{cl}}(\tau)\tilde{x}_0 \, d\tau .$$

If for some choice of (4.3) the semigroup $\tilde{\mathbf{T}}_{\text{cl}}(t)$ is exponentially stable on W and V, then we say that (4.1) is *exponentially stabilizable by dynamic output feedback*. The next result shows that any stabilizable and detectable Pritchard–Salamon system can be stabilized by a finite-dimensional compensator.

PROPOSITION 4.4 If (4.1) is stabilizable and detectable (in the sense of Definition 4.2), then there exists $l \in \mathbf{N}$ and a compensator of the form (4.3) which exponentially stabilizes (4.1).

The above proposition follows from a combination of a number of well-known results on stabilizability and stabilization of Pritchard–Salamon systems; see Logemann [36] for details.

One of the basic tools in the area of adaptive stabilization without identification is the bounding of the state dynamics in terms of the initial state and the input and output, see Logemann & Mårtensson [39].

PROPOSITION 4.5 If (4.1) is exponentially detectable, then

(i) there exists positive constants c_0 and c_1 such that

$$\|x(t)\|_W^2 \leqslant c_0 \|x_0\|_W^2 + c_1 \left(\int_0^t \|y(\tau)\|^2 + \|u(\tau)\|^2 d\tau \right),$$

for all $t \geqslant 0$ and $u \in L^2_{loc}(\mathbf{R}_+; \mathbf{R}^m)$;

(ii) for each $x_0 \in W$, if $u \in L^2(\mathbf{R}_+, \mathbf{R}^m)$ produces an output $y \in L^2(\mathbf{R}_+, \mathbf{R}^p)$, then

$$\lim_{t \to \infty} \|x(t)\|_W = 0 .$$

Smooth stabilization of systems which are stabilizable by scalar static output feedback

In this subsection we will show that the simple control law

$$u(t) = \mathcal{K}(g(t))y(t), \qquad \dot{g}(t) = \|y(t)\|^2 \tag{4.8}$$

with

$$\mathcal{K}(\gamma) = (\log^{\mu}\gamma)\cos(\log^{\nu}\gamma), \quad \text{where } \mu,\ \nu > 0,\ 3\mu + \nu < 1, \tag{4.9}$$

stabilizes any exponentially stabilizable and detectable Pritchard–Salamon system which is externally stabilizable by scalar output feedback $u(t) = ky(t)$, for some $k \in \mathbb{R}$.‡ Notice that $\mathcal{K}$ is of the same form as the low-gain tuning function (3.9). However, the choice of the parameters μ and ν is different, and in fact, in the present case, $\mathcal{K}$ given by (4.9) is a Nussbaum function satisfying (2.10).

First we need a lemma on the existence and uniqueness of the solutions to the nonlinear closed-loop system given by (4.1) and (4.8).

LEMMA 4.6 Let $a \geqslant -\infty$, $K \in \mathbb{R}^{m\times p}$ and let $\mathcal{K} : (a, \infty) \to \mathbb{R}$ be a locally Lipschitz function. If $x_0 \in W$ and $g_0 \in (a, \infty)$, then there exists a unique solution $(x_{max}, g_{max}) \in C(0, \tau_{max}; W \times \mathbb{R})$ of

$$x(t) = \mathbf{T}(t)x_0 + \int_0^t \mathbf{T}(t-\tau)\mathcal{K}(g(\tau))BKCx(\tau)\,d\tau, \tag{4.10a}$$

$$g(t) = g_0 + \int_0^t \|Cx(\tau)\|^2 d\tau. \tag{4.10b}$$

on a maximal interval of existence $[0, \tau_{max})$, where $0 < \tau_{max} \leqslant \infty$. If $\tau_{max} < \infty$, then

$$\lim_{t \nearrow \tau_{max}} (\|x_{max}(t)\|_W + |g_{max}(t)|) = \infty.$$

The proof of the above result follows easily from lemma 4 in Logemann [35].

THEOREM 4.7 Suppose that (4.1) is stabilizable and detectable and that the function $\mathcal{K}$ is given by (4.9). If (4.1) is externally stabilized by the ouput feedback $K = kI$, for some $k \in \mathbb{R}$, then for all $x_0 \in W$ and $g_0 \in \mathbb{R}$, $g_0 > 1$, the solution $(x(t), g(t))$ of the closed-loop system given by (4.10a) and (4.10b) exists for all $t \geqslant 0$ and

(i) $\lim_{t\to\infty} g(t) = g_\infty < \infty$,
(ii) $\lim_{t\to\infty} \|x(t)\|_W = 0$.

In Townley [71] this result is proved using a Lyapunov approach under the extra assumption that $D_V(A) \subset W$, i.e. when the Pritchard–Salamon system is smooth. We give a sketch of an alternative proof which does not require the extra smoothness assumption.

‡ In one sense this result circumvents the 'adaptive analogue' of the open question posed at the end of Section 2, that is the tuning of the proportional gain in the case that high-gain stabilization of the system is possible, without a need to characterize and exploit any special features which might exist in this case.

Sketch of the proof of Theorem 4.7 Let $[0, \tau_{max})$ be the maximal interval of existence. From Proposition 4.5 and using (4.10b), which defines the adaptation of $g(\cdot)$, we can show that for $t \in [0, \tau_{max})$

$$\|x(t)\|_W^2 \leqslant c_0\|x_0\|_W^2 + c_1 \max_{\gamma \in [g_0, g(t)]} \left[1 + \mathcal{K}^2(\gamma)\right] (g(t) - g_0), \tag{4.11}$$

where $c_0, c_1 > 0$ are suitable constants. Define $\kappa \in \mathbb{R}$ by

$$\|\mathbf{G}(I + k\mathbf{G})^{-1}\|_\infty = 1/\kappa. \tag{4.12}$$

Seeking a contradiction, suppose that $g(t) \to \infty$ as $t \nearrow \tau_{max}$. Then it is easy to see from the form of $\mathcal{K}(\cdot)$ that there exists an increasing sequence $t_0, t_1, \ldots$ such that

$$\mathcal{K}(g(t_{2j})) = k - \kappa/2, \quad \mathcal{K}(g(t_{2j+1})) = k + \kappa/2, \tag{4.13}$$

with $\mathcal{K}(\gamma) \in (k - \kappa/2, k + \kappa/2)$ for $\gamma \in (g(t_{2j}), g(t_{2j+1}))$ and $g(t_j) \to \infty$. The change in $g(t)$ on $[t_{2j}, t_{2j+1}]$ is given by

$$g(t_{2j+1}) - g(t_{2j}) = \int_{t_{2j}}^{t_{2j+1}} \|Cx(\tau)\|^2 d\tau \leqslant 4\|C\mathbf{T}_{A+kBC}(\cdot)x(t_{2j})\|^2_{L^2(t_{2j}, t_{2j+1})}. \tag{4.14}$$

The estimate (4.14) is obtained by using (4.13) in a routine argument of small-gain type applied to the system $(A + kBC, B, C)$ on the interval $[t_{2j}, t_{2j+1})$, combined with the definition of κ as the inverse of the closed-loop gain, cf. (4.12). Combining (4.11) and (4.14), and using the specific form of $\mathcal{K}(\cdot)$, we have

$$g(t_{2j+1}) - g(t_{2j}) \leqslant \tilde{c}_0 \|x_0\|_W^2 + \tilde{c}_1 [1 + \log^{2\mu}(g(t_{2j}))](g(t_{2j}) - g_0) \tag{4.15}$$

for some positive constants $\tilde{c}_0$ and $\tilde{c}_1$. It is now easy to show, using again the specific form of $\mathcal{K}(\cdot)$ to estimate $g(t_{2j+1}) - g(t_{2j})$ in terms of $\log g(t_{2j})$ and $g(t_{2j})$, that the unboundedness of $g(\cdot)$ leads to a contradiction. Hence $g(\cdot)$ is bounded on $[0, \tau_{max})$. As a consequence, $y \in L^2(0, \tau_{max}; \mathbb{R}^m)$ and thus $\|x(\cdot)\|_W$ is bounded on $[0, \tau_{max})$. Using Lemma 4.6 we obtain that $\tau_{max} = \infty$ and $y \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ with $u(\cdot) = \mathcal{K}(g(\cdot))y(\cdot) \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. The remainder of the result follows from Proposition 4.5 (ii).

Adaptive stabilization by switching

In the seminal paper [51], Mårtensson proved that knowledge of the order of a stabilizing compensator is sufficient a priori information for the synthesis of adaptive stabilizers for finite-dimensional linear systems. In this section we describe the extensions of this basic result to infinite-dimensional systems. The complete details can be found in Logemann & Mårtensson [39].

Let $\mathfrak{C} = \{C_i\}_{i \in \mathbb{N}}$ be a countable set of controllers of the form (4.3), indexed by i. Of course $\mathfrak{C}$ could be obtained by cycling through a finite set of controllers, as would be the more realistic situation. By $u(\cdot) = Cy(\cdot)$ we mean the operator relationship between $u(\cdot)$ and $y(\cdot)$, for some initial condition $z(0)$, which is to be considered as a part of the operator C. Further, let (γ_i) be a sequence of real numbers, increasing towards infinity. We call a function $\Sigma : \mathbb{R} \to \mathbb{N}$ a *switching function* with *switching points* γ_i if for all $a \in \mathbb{R}$ it holds

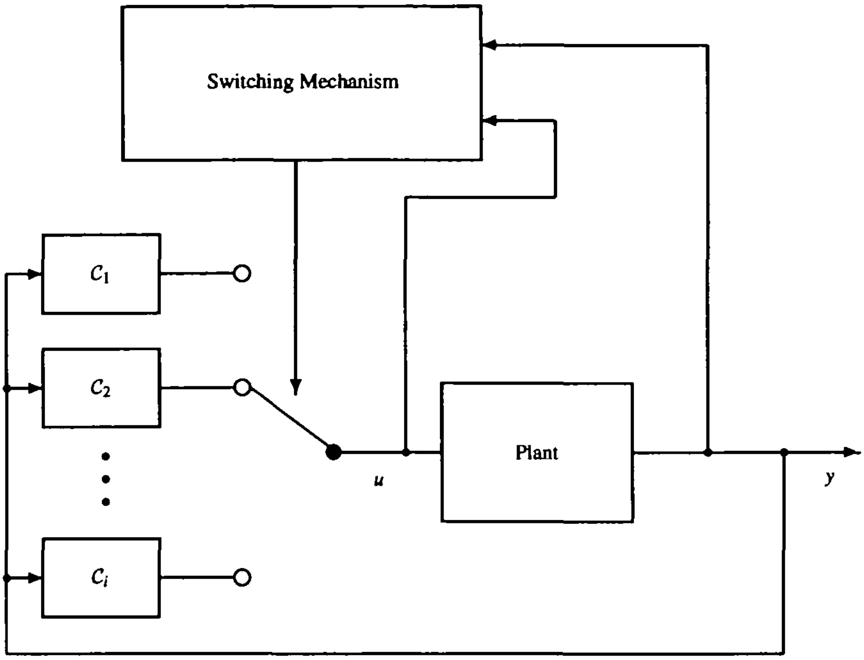

FIG. 5. Switching controller

that $\Sigma([a, \infty)) = \mathbb{N}$ and its discontinuity points are γ_i. For convenience, we require Σ to be right continuous.

The *switching controller* associated with $\mathfrak{C}$ and (4.1) is defined by

$$u = C_{\Sigma(g)} y, \tag{4.16a}$$

$$\dot{g} = \|y\|^2 + \|u\|^2, \quad g(0) = g_0. \tag{4.16b}$$

The above controller is illustrated in Fig. 5. We consider the switching sequence given by

$$\gamma_{j+1} = \gamma_j^2 \quad (j = 1, 2, \ldots), \qquad \gamma_1 > 1, \tag{4.17}$$

which is the same sequence as in (2.7). The control law (4.16) says that all controllers C_i are processing the plant output for all $t \geqslant 0$. Thus, unless all (or 'most') of C_i are memoryless, (4.16) is an infinite-dimensional controller. If all the C_i have a realization on a common state space $\mathbb{R}^l$, with a common initial condition $z(0)$, this difficulty can be avoided by considering each dynamic control as static ouput feedback $v = \tilde{K}\tilde{y}$ applied to the augmented plant (4.5). The algorithm becomes one of searching through a set of static output feedback gain matrices. Indeed we rewrite the controller (4.16) in the form

$$\tilde{u}(t) = \tilde{K}_{\Sigma(g(t))} \tilde{y}(t), \tag{4.18a}$$

$$\dot{g}(t) = \|\tilde{y}(t)\|^2 + \|\tilde{u}(t)\|^2, \quad g(0) = g_0, \tag{4.18b}$$

where $\tilde{K}_i$ is given by the expression for $\tilde{K}$ in (4.6) indexed by i. The mild form of the augmented plant (4.3) is given by

$$\tilde{x}(t) = \tilde{\mathbf{T}}(t)\tilde{x}_0 + \int_0^t \tilde{\mathbf{T}}(t-\tau)\tilde{B}\tilde{u}(\tau)\,d\tau\,, \quad \tilde{x}_0 \in \tilde{W}\,, \tag{4.19a}$$

$$\tilde{y}(t) = \tilde{C}\tilde{x}(t)\,, \tag{4.19b}$$

with notation as in (4.6) and (4.7).

A *solution* on $[0, a)$ of the closed-loop system given by (4.19) and (4.18) is a $\tilde{W} \times \mathbb{R}$-valued function $(\tilde{x}, g)$ such that $\tilde{x}$ is continuous in $\tilde{W}$, the function g is absolutely continuous and (4.19) and (4.18) are satisfied for almost every $t \in [0, a)$. It is not difficult to show that there exists a unique solution which can be continued to the right as long as it remains bounded.

Mårtensson [52, 53] proved a general theorem on adaptive stabilization of finite-dimensional linear systems by switching controllers. We present here the natural generalization to infinite-dimensional systems which was obtained in [39].

THEOREM 4.8 Assume that $\mathfrak{C} = \{C_i\}_{i\in\mathbf{N}}$ is a set of controllers of the type (4.3) (with a bound on the l's), with the property that there is a controller $C \in \mathfrak{C}$ which exponentially stabilizes (4.1). Then the controller (4.18), with γ_i given by (4.17), adaptivley stabilizes (4.19) in the sense that for each $\tilde{x}(0) \in \tilde{W}$ and $g(0) \in \mathbf{R}$ we have that $(\|\tilde{x}(t)\|_{\tilde{W}}, g(t)) \to (0, g_\infty)$ (where $g_\infty < \infty$) as $t \to \infty$.

Notes and references

The result in Theorem 4.7 can be extended, using the spectrum-unmixing techniques of Section 2, to construct adaptive controllers for systems which are stabilizable by kK_i, for some $k \in \mathbb{R}$ and $i \in \{1, ..., \ell\}$ where $\{K_1, ..., K_\ell\}$ is a fixed set of feedback matrices.

For a class of retarded systems, a result similar to Theorem 4.8 was proved by Dahleh [9]. However, one of the assumptions in [9] is that the plant is continuously initially observable, which is very restrictive (see [39] for a discussion). A discrete-time version of Theorem 4.8 can be found in Logemann & Mårtensson [40]. Since the adaptive control law in Theorem 4.8 is based on a piecewise constant adaptation of the gain, it seems likely that it can be extended to regular systems by using recent results of Weiss & Curtain [79] on dynamic stabilization of regular systems. However, the question whether the smooth controller in Theorem 4.7 extends to the class of regular systems leads to the following open problem.

Open problem. Is the smooth controller given by (4.8) and (4.9) an adaptive stabilizer for the class of regular systems, with zero feedthrough which are stabilizable by output feedback $u(t) = ky(t)$ for some $k \in \mathbb{R}$?

The main difficulty appears to be establishing the existence and uniqueness of solutions to the resulting closed-loop nonlinear equations.

Another interesting topic for future research is the question of how to use Theorem 4.8 for the synthesis of adaptive tracking controllers. A natural approach, of course, is to precompensate the plant using a suitable system containing an internal model of the dynamics of the reference signals and then to reformulate the tracking problem as a stabilization problem. To make this idea work, a general form of Lemma 3.4 is needed, i.e. a result which ensures that every reference signal can be 'realized' as the output of the uncontrolled precompensated plant by a suitable choice of the initial state.

In Section 2 we stated an open problem on the limiting behaviour of the closed-loop system in the context of smooth high-gain adaptive stabilization. An analogous open problem can be formulated for the adaptive switching controller in Theorem 4.8. Notice that the closed-loop dynamics of the evolution of $x(t)$ in (4.19a) are piecewise linear. Making use of this, Townley [71] has proved, in the case of finite-dimensional systems, that for all initial values $(x(0), g(0))$ in an open and dense set, the limiting closed-loop system resulting from one application of the adaptive switching controller (4.18) is exponentially stable. Whether the same is true in the infinite-dimensional Pritchard–Salamon set-up remains an open problem.

In a finite-dimensional setting Mårtensson [51] (see also Mårtensson & Polderman [55]) has shown that the switching controller (4.18) with piecewise constant gain adaptation can be replaced by a smooth control law. It is easily seen that this generalizes to Pritchard–Salamon systems. Finally, in the subsection on adaptive stabilization by switching we have exclusively considered stabilization by finite-dimensional controllers. Due to the progress in VLSI technology and, to a lesser extent, computer technology in general, a future exclusive emphasis on finite-dimensional stabilization seems unnatural.

REFERENCES

1. BYRNES, C. I., 1987. Adaptive stabilization of infinite-dimensional linear systems, pp. 1435–1440 in *Proc. 26th IEEE Conference on Decision and Control*, Los Angeles.
2. BYRNES, C. I., GILLIAM, D. S. & HE, J., 1994. Root-locus and boundary feedback design for a class of distributed parameter systems, *SIAM J. Control & Opt.* **32,** 1364–427.
3. BYRNES, C. I., & WILLEMS, J. C., 1984. Adaptive stabilization of multivariable linear systems, pp. 1574–1577 in *Proc. 23rd IEEE Conference on Decision and Control*, Las Vegas.
4. CALLIER, F. M., & DESOER, C. A., 1980. Stabilization, tracking and disturbance rejection in multivariable convolution systems, *Annales de la Société Scientifique de Bruxelles* **94,** 7–51.
5. COOK, P. A. 1992. Controllers with universal tracking properties, in *Proc. of Int. IMA Conf. on Control: Modelling, Computation, Information*, Manchester.
6. COPPUS, G. W. M., SHA, S. L., & WOOD, R. K., 1983. Robust multivariable control of a binary distillation column, *IEE Proceedings, Pt. D* **130,** 201–8.
7. CURTAIN, R. F., 1983. Tracking and regulation for distributed parameter systems, *Matematica Aplicada e Computacional* **2,** 199–218.
8. CURTAIN, R. F., LOGEMANN, H., TOWNLEY, S., & ZWART, H., 1994. Well-posedness, stabilizability and admissibility for Pritchard-Salamon systems, *Math. Systems, Estimation and Control* **4** (summary pp. 493–496; full electronic manuscript = 38pp; retrieval code 85404‡; full hardcopy to appear).
9. DAHLEH, M., 1988. Sufficient information for the adaptive stabilization of delay systems, *Systems & Control Letters* **11,** 357–63.
10. DAHLEH, M., 1989. Generalizations of Tychonov's theorem, with applications to adaptive control of SISO delay systems, *Systems & Control Letters* **13,** 421–7.
11. DAHLEH, M., & HOPKINS, JR. W. E., 1986. Adaptive stabilization of single-input single-output delay systems, *IEEE Trans. Auto. Control* **31,** 577–9.
12. DAVISON, E. J., 1976. Multivariable tuning regulators: the feedforward and robust control of a general servomechanism problem, *IEEE Trans. Auto. Control* **21,** 35–47.

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13. DUNCAN, T. E., & PASIK-DUNCAN, B., 1988. Adaptive control of linear delay time systems, *Stochastics* **24**, 45–74.
14. DUNCAN, T. E., & PASIK-DUNCAN, B., 1991. Adaptive control of linear stochastic evolution systems, *Stochastics and Stochastics Reports* **36**, 71–90.
15. EMIRSAJLOW, Z., PRITCHARD, A. J., & TOWNLEY, S., to appear. On structured perturbations for two classes of linear infinite-dimensional systems, *Dynamics and Control.*
16. FERNÁNDEZ, G., ORTEGA, R., & BEGOVICH, O., 1988. Conditions for preservation of stability of adaptive controllers for systems with unmodeled delay, *IEEE Trans. Auto. Control* **33**, 601–3.
17. FRANCIS, B. A., 1977. The multivariable servomechanism problem from the input-output viewpoint, *IEEE Trans. Auto. Control* **22**, 322–8.
18. GRIPENBERG, G., LONDEN, S.-O., & STAFFANS, O., 1990. *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge.
19. GROSDIDIER, P., MORARI, M., & HOLT, B. R., 1985. Closed-loop properties from steady-state gain information, *Ind. Eng. Chem. Fundam.* **24**, 221–35.
20. HINRICHSEN, D., & PRITCHARD, A. J., 1992. Destabilization by output feedback, *Differential and Integral Equations* **5**, 357–86.
21. HONG, K. S., & BENTSMAN, J., 1994. Direct adaptive control of parabolic systems: algorithm synthesis and convergence and stability analysis, *IEEE Trans. Auto. Control* **39**, 2018–33.
22. ILCHMANN, A., 1991. Non-identifier-based adaptive control of dynamical systems: a survey, *IMA J. of Math. Control & Information* **8**, 321–66.
23. ILCHMANN, A., 1993. *Non-Identifier-Based High-Gain Adaptive Control*, Springer-Verlag, London.
24. ILCHMANN, A., & LOGEMANN, H., 1992. High-gain adaptive stabilization of multivariable systems, *Systems & Control Letters* **18**, 355–364.
25. ILCHMANN, A., & RYAN, E. P., 1994. Universal λ-tracking for nonlinearly perturbed systems in the presence of noise, *Automatica* **30**, 337–346.
26. JOHANSSON, R., 1986. Estimation and direct adaptive control of delay-differential systems, *Automatica* **22**, 555–560.
27. JUSSILA, T. T., & KOIVO, H. N., 1987. Tuning of multivariable PI-controllers for unknown delay-differential systems, *IEEE Trans. Auto. Control* **32**, 364–368.
28. VAN KEULEN, B., 1993. *H^∞-Control for Distributed Parameter Systems*, Birkhäuser Verlag, Boston.
29. KOBAYASHI, T., 1984. A digital adaptive control law for a parabolic distributed parameter system, *Systems & Control Letters* **4**, 175–179.
30. KOBAYASHI, T., 1986. Design of robust multivariable tuning regulators for infinite-dimensional systems by discrete-time input-output data, *Int. J. Control* **44**, 83–90.
31. KOBAYASHI, T., 1987a. Global adaptive stabilization of infinite-dimensional systems, *Systems & Control Letters* **9**, 215–223.
32. KOBAYASHI, T., 1987b. Model reference adaptive control for spectral systems, *Int. J. Control* **46**, 1511–1523.
33. KOBAYASHI, T., 1988. A digital PI-controller for distributed parameter systems, *SIAM J. Control & Opt.* **26**, 1399–1414.
34. LOGEMANN, H., 1990. Adaptive exponential stabilization for a class of nonlinear retarded processes, *Math. Control, Signals, and Systems* **3**, 255–269.
35. LOGEMANN, H., 1991. Circle criteria, small-gain conditions and internal stability for infinite-dimensional systems, *Automatica* **27**, 677–690.
36. LOGEMANN, H., 1993. Stabilization and regulation of infinite-dimensional systems using coprime factorizations, pp. 102–139 in *Analysis and Optimization of Systems: State and Frequency-Domain Approaches to Infinite-Dimensional Systems*, edited by R.F. Curtain, A. Bensoussan and J.L. Lions, Springer-Verlag, Berlin.
37. LOGEMANN, H., BONTSEMA, J., & OWENS, D. H., 1988. Low-gain control of distributed parameter systems with unbounded control and observation, *Control Theory and Advanced Technology* **4**, 429–446.
38. LOGEMANN, H., & ILCHMANN, A., 1994. An adaptive servomechanism for a class of infinite-dimensional systems, *SIAM J. Control & Opt.* **32**, 917–936.

39. LOGEMANN, H., & MÅRTENSSON, B., 1992. Adaptive stabilization of infinite-dimensional systems, *IEEE Trans. Auto. Control* **37**, 1869–1883.
40. LOGEMANN, H., & MÅRTENSSON, B., 1994. Adaptive stabilization of discrete-time infinite-dimensional systems, pp. 767–768 in *Systems and Networks: Mathematical Theory and Applications, vol. II*, edited by U. Helmke, R. Mennicken and J. Saurer, Akademie Verlag, Berlin.
41. LOGEMANN, H., & OWENS, D. H., 1987. Robust high-gain feedback control of infinite-dimensional minimum-phase systems, *IMA J. Math. Control & Information* **4**, 195–220
42. LOGEMANN, H., & OWENS, D. H., 1988. Input-output theory of high-gain adaptive stabilization of infinite-dimensional systems with nonlinearities, *Int. J. Adaptive Control and Signal Processing* **2**, 193–216.
43. LOGEMANN, H., & OWENS, D. H., 1989. Low-gain control of unknown infinite-dimensional systems: a frequency-domain approach, *Dynamics and Stability of Systems* **4**, 13–29.
44. LOGEMANN, H., & OWENS, D. H., 1992. On high-gain adaptive stabilization of infinite-dimensional systems with nonlinearities, pp. 303–317 in *The Mathematics of Control Theory*, edited by D.H.Owens and N.K. Nichols, Oxford University Press, Oxford.
45. LOGEMANN, H., & TOWNLEY, S., 1997. Low-gain control of uncertain infinite-dimensional systems, Mathematics Preprint 94/13, University of Bath, 1994 † *SIAM J. Control & Opt.* **35**, 78–116.
46. LOGEMANN, H., & TOWNLEY, S., 1997. Discrete-time low-gain control of uncertain infinite-dimensional systems, Mathematics Preprint 95/06, University of Bath, 1995 ‡ *IEEE Trans. Auto. Control* **42**, 22–37.
47. LOGEMANN, H., & ZWART, H., 1991. Some remarks on adaptive stabilization of infinite-dimensional systems, *Systems & Control Letters* **16**, 199–207.
48. LOGEMANN, H., & ZWART, H., 1992. On robust PI-control of infinite-dimensional systems, *SIAM J. Control & Opt.* **30**, 573–593.
49. LUNZE, J., 1985. Determination of robust multivariable I-controllers by means of experiments and simulation, *Syst. Anal. Model. Simul.* **2**, 227–249.
50. LUNZE, J., 1987. Experimentelle Erprobung einer Einstellregel für PI-Mehrgrößenregler bei der Herstellung von Ammoniumnitrat-Harnstoff-Lösung, *Messen Steuern Regeln* **30**, 2–6.
51. MÅRTENSSON, B., 1985. The order of any stabilizing regulator is sufficient a priori information for adaptive stabilization, *Systems & Control Letters* **6**, 87–91.
52. MÅRTENSSON, B., 1986. *Adaptive Stabilization*, PhD thesis, Lund Institute of Technology, Dept. of Automatic Control.
53. MÅRTENSSON, B., 1987. Adaptive stabilization of multivariable linear systems, *Contemporary Mathematics* **68**, 191–225.
54. MÅRTENSSON, B., 1991. The unmixing problem, *IMA J. Math. Control & Information* **8**, 367–377.
55. MÅRTENSSON, B., & POLDERMAN, J. W., 1993. Correction and simplification to "The order of any stabilizing regulator is sufficient a priori information for adaptive stabilization", *Systems & Control Letters* **20**, 465–470.
56. MILLER, D. E., & DAVISON, E. J., 1989. The self-tuning robust servomechanism problem, *IEEE Trans. Auto. Control* **34**, 511–523.
57. MILLER, D. E., & DAVISON, E. J., 1993. An adaptive tracking problem with a control input constraint, *Automatica* **29**, 877–887.
58. MORARI, M., 1985. Robust stability of systems with integral control, *IEEE Trans. Auto. Control* **30**, 574–577.
59. MUSTAFA, D., 1994. How much integral action can a control system tolerate?, *Linear Algebra and its Applications* **205/206**, 965–970.
60. NUSSBAUM, R. D., 1983. Some remarks on a conjecture in parameter adaptive control, *Systems & Control Letters* **3**, 243–246.

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61. ORTEGA, R., & LOZANO, R., 1988. Globally stable adaptive controller for systems with delay, *Int. J. Control* **47,** 17–23.
62. POHJOLAINEN, S., 1982. Robust multivariable PI-controllers for infinite-dimensional systems, *IEEE Trans. Auto. Control* **27,** 17–30.
63. POHJOLAINEN, S., 1985. Robust controller for systems with exponentially stable strongly continuous semigroups, *J. Math. Analysis and Applications* **111,** 622–636.
64. PRITCHARD, A. J., & SALAMON, D., 1987. The linear-quadratic control problem for infinite-dimensional systems with unbounded input and output operators, *SIAM J. Control & Opt.* **25,** 121–144.
65. REBARBER, R., & TOWNLEY, S., 1995. Robustness and continuity of the spectrum for uncertain distributed parameter systems, *Automatica* **31,** 1533–1546.
66. RYAN, E. P., (Guest Editor). 1991. Special Issue on Universal Stabilizers and High-Gain Adaptive Control, *IMA J. of Math. Control & Information* **8,** 321–439.
67. RYAN, E. P., 1993. Adaptive stabilization of multi-input nonlinear systems, *Int. J. Robust and Nonlinear Control* **3,** 169–181.
68. RYAN, E. P., 1994. A nonlinear universal servomechanism, *IEEE Trans. Auto. Control* **39,** 753–761.
69. SALAMON, D., 1987. Infinite-dimensional linear systems with unbounded control and observation: a functional analytic approach, *Trans. Amer. Math. Soc.* **300,** 383–431.
70. SALAMON, D., 1989. Realization theory in Hilbert space, *Math. Systems Theory* **21,** 147–164.
71. TOWNLEY, S., 1995. Simple adaptive stabilization of output feedback stabilizable distributed parameter systems, *Dynamics and Control,* **5,** 107–123.
72. TOWNLEY, S., 1996. Topological aspects of universal adaptive stabilization, *SIAM J. Control & Opt.* **34,** 1044–70.
73. VIDYASAGAR, M., 1983. A note on time-invariance and causality, *IEEE Trans. Auto. Control* **28,** 929–931.
74. VIDYASAGAR, M., 1985. *Control systems synthesis: a factorization approach,* MIT Press, Cambridge, MA. operators,
75. WEISS, G., 1989a. Admissible observation operators for linear semigroups, *Israel J. Math.* **65,** 17–43.
76. WEISS, G., 1989b. The representation of regular linear systems on Hilbert spaces, pp. 401–416 in *Distributed Parameter Systems,* edited by F. Kappel, K. Kunisch & W. Schappacher, Birkhäuser Verlag, Basel.
77. WEISS, G., 1994a. Transfer functions of regular linear systems, part I: characterization of regularity, *Trans. Amer. Math. Soc.* **342,** 827–854.
78. WEISS, G., 1994b. Regular linear systems with feedback, *Math. Control, Signals, and Systems* **7,** 23–57.
79. WEISS, G., & CURTAIN, R. F., 1997. Dynamic stabilization of regular linear systems, *IEEE Trans. Auto. Control* **42,** 4–21.
80. WEN, J. T.-Y., & BALAS, M. J., 1986. Direct model reference adaptive control in infinite-dimensional Hilbert space, pp. 309–327 in *Adaptive and Learning Systems,* edited by K.S. Narendra, Plenum Press, New York.
81. WEN, J. T.-Y., & BALAS, M. J., 1989. Robust adaptive control in Hilbert space, *J. Math. Analysis and Applications* **143,** 1–26.
82. WILLEMS, J. C., & BYRNES, C. I., 1984. Global adaptive stabilization in the absence of information on the sign of the high frequency gain, pp. 49–57 in *Proc. INRIA Conf. on Analysis and Optimization of Systems,* Springer-Verlag, Berlin.
83. WONHAM, W. M., 1979. *Linear multivariable control: a geometric approach,* 2nd ed., Springer-Verlag, New York.
84. ZHU, X.-J., 1989. A finite spectrum unmixing set for $GL(3, \mathbb{R})$, pp. 403–410 in *Computation and Control* edited by K. Bowers & J. Lund, Birkhäuser Verlag, Boston.