Adaptive integral control of time-delay systems

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Abstract: It is well known that if the steady-state gain G(0) of a stable lumped system, with transfer function G(*s*), is positive, then compensating the system by an integral controller *k*/*s*, where *k* is a gain parameter, leads to a stable closed-loop system which achieves tracking of arbitrary constant reference signals, provided that the gain parameter *k* is positive and sufficiently small. It is also well known that this result extends to certain classes of differential-delay and distributed parameter systems. The authors derive an adaptive version of the above result for the class of stable lumped systems with output delay, i.e. they show that the gain parameter *k* can be tuned adaptively, so that tracking is achieved for any system of this class. The resulting adaptive tracking controller is not based on system identification or parameter estimation algorithms, nor is the injection of probing signals required.

1 Introduction

In this paper we consider adaptive integral control of time delay systems. We emphasise that this paper is tutorial in the sense that it gives a self-contained treatment of a result which is a special case of a much more general theory presented in Logemann and Townley [1]. However, in [1] the approach is based on functional analysis and the theory of regular infinite-dimensional systems, whereas in this paper we use standard engineering mathematics.

The synthesis of low-gain I and PI-controllers for uncertain stable plants has received considerable attention in the last 20 years. For a stable single-input, single-output, lumped parameter system with positive steady-state gain, the main existence result on robust low-gain I-control states that there exists $k^* > 0$ such that for all $k \in (0, k^*)$, the controller *k*/*s* is stabilising and the resulting closed-loop system asymptotically tracks arbitrary constant reference signals. This result has been proved by Davison [2] and Lunze [3] using time-domain methods, and by Grosdidier *et al.* [4] and Morari [5] using frequency-domain methods (see also [3], chapter 10, and [6], pp. 362). Methods for tuning the integral gain *k* by means of experiments and simulations have been developed and discussed in many places, we only mention [2, 3, 7], and the papers by Owens and Chotai [8] and Penttinen and Koivo [9]. Such a controller design approach (called 'tuning regulator theory' [2]) has been successfully applied in process control, see, e.g. Coppus *et al.* [10] and Lunze [7].

The tuning regulator result mentioned above has been extended by Jussila and Koivo [11] and Koivo and Pohjolainen [12] to differential delay systems, and by Logemann and Owens [13], and Pohjolainen [14], to certain classes of distributed parameter systems.

If the plant uncertainty is large and/or if reliable step response data is not available then the gain parameter *k* has to be tuned adaptively. Low-gain universal adaptive I-controllers which achieve asymptotic tracking of constant reference signals for stable lumped plants have been presented by Cook [15] and by Miller and Davison [16, 17]. By 'universal' we mean that the controllers are not based on system identification or parameter estimation algorithms. The controller given in [15] is smooth, while the control laws derived in [16, 17] are 'piecewise-constant'. The controller given in [16] satisfies a control input constraint.

In this paper we consider the problem of low-gain I-control, and especially adaptive I-control, for stable lumped systems with output delay, which we will call stable time-delay systems.

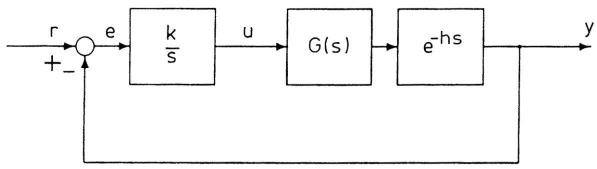

Fig. 1 *Low-gain control with output delay*

2 Non-adaptive integral control

As usual, let H^∞ denote the set of all functions F(*s*) which are analytic and bounded on the open right-half plane Re $s > 0$. If a transfer function F is in H^∞, then we define the H^∞-gain of F to be

$$\|\mathbf{F}\|_\infty := \sup_{\mathrm{Re}\, s>0} |\mathbf{F}(s)|. \qquad (1)$$

The H^∞ transfer functions we will be dealing with in the following will have the extra property that they are analytic in an open right-half plane of the form Re $s > -\varepsilon$ for some $\varepsilon > 0$. For such transfer functions F an important property is that the H^∞-gain of F is given by the maximal distance of the Nyquist diagram of F to the origin, i.e.

$$\|\mathbf{F}\|_\infty = \sup_{-\infty<\omega<\infty} |\mathbf{F}(j\omega)|. \qquad (2)$$

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For the rest of this paper let G(s) be a stable strictly proper real-rational transfer function (this means in particular that G is in H^∞). We say that the feedback system shown in Fig. 1 is input-output stable if the closed-loop transfer function

$$\mathbf{H}_k(s) = \frac{k}{s}\mathbf{G}(s)e^{-hs}\left(1 + \frac{k}{s}\mathbf{G}(s)e^{-hs}\right)^{-1} \quad (3)$$

has no poles in the closed-right halfplane, equivalently if H_k is in H^∞.

Proposition 1: If G(s) is a stable strictly proper real-rational transfer function with G(0) > 0 (i.e. the steady-state gain is positive), then there exists $k^* > 0$ (depending on h) such that for all $k \in (0, k^*)$ the closed-loop transfer function H_k is in H^∞, i.e. for such k the feedback system shown in Fig. 1 is input-output stable.

Proof: Clearly, under the above assumptions on G(s), $H_k(0) = 1$, so that the closed-loop transfer function H_k has no poles in the closed-right half plane if and only if the function

$$\mathbf{L}_k(s) = 1 + \frac{k}{s}\mathbf{G}(s)e^{-hs} \quad (4)$$

has no zeros in the closed right-half plane. For $|s|$ small, a series expansion of $L_k(s)$ at $s = 0$ yields

$$\mathbf{L}_k(s) = 1 + \frac{k}{s}\mathbf{G}(0) + kO(1) \quad \text{as } s \to 0 \quad (5)$$

Since G(0) > 0, an inspection of eqn. 5 shows that we can find $k_1 > 0$ and $\varrho > 0$ such that

$$|\mathbf{L}_k(s)| > 0$$
$$\text{for all } k \in (0, k_1), \text{ all } s \text{ with } |s| < \varrho \text{ and Re } s \geq 0. \quad (6)$$

Moreover, since the expression $G(s)e^{-hs}/s$ is bounded for s with $|s| > \varrho$ and Re $s \geq 0$, we can find $k_2 > 0$ such that

$$|\mathbf{L}_k(s)| > 0$$
$$\text{for all } k \in (0, k_2), \text{ all } s \text{ with } |s| \geq \varrho \text{ and Re } s \geq 0. \quad (7)$$

Combining eqns. 6 and 7 and choosing $k^* = \min(k_1, k_2)$ completes the argument. □

Proposition 1, combined with a straigthforward application of the final value theorem from Laplace transforms, yields the following corollary.

Corollary 1: If G(s) is a stable strictly proper real-rational transfer function with G(0) > 0, then there exists $k^* > 0$ (depending on h) such that for all $k \in (0, k^*)$ the output y of the closed-loop system shown in Fig. 1 asymptotically tracks any constant reference r.

The following proposition shows that the H^∞-gain of the feedback system shown in Fig. 1 is equal to 1 for all sufficiently small $k > 0$. This result will be used as the main tool in Section 3, but it is of some interest in its own right.

Proposition 2: If G(s) is a stable strictly proper real-rational transfer function with G(0) > 0, then there exists $k^{**} > 0$ (depending on h) such that for all $k \in (0, k^{**})$

$$\|\mathbf{H}_k\|_\infty = \sup_{\text{Re } s\geq 0}\left|\frac{k}{s}\mathbf{G}(s)e^{-hs}\left(1 + \frac{k}{s}\mathbf{G}(s)e^{-hs}\right)^{-1}\right| = 1. \quad (8)$$

Proof: Clearly

$$\mathbf{H}_k(s) = \left(\frac{s}{k\mathbf{G}(s)e^{-hs}} + 1\right)^{-1}. \quad (9)$$

Using the fact that G(0) > 0, a simple calculation, shows that the radius of curvature ϱ_0 of the inverse Nyquist diagram of $kG(s)e^{-hs}/s$, i.e. of the curve

$$N_k = \left\{\frac{j\omega}{k\mathbf{G}(j\omega)e^{-jh\omega}} \mid \omega \in \mathbb{R}\right\} \quad (10)$$

at $0 \in$ C is given by

$$\varrho_0 = 1/2k|\mathbf{G}'(0) - h\mathbf{G}(0)|. \quad (11)$$

Clearly, $\varrho_0 \to \infty$ as $k \downarrow 0$. Moreover, the imaginary axis is the tangent of N_k at 0. Let k^* be the constant of Proposition 1. It follows that there exists $k^{**} \in (0, k^*]$, so that for all $k \in (0, k^{**})$, no points inside the circle centred at (–1, 0) with radius 1 lie on N_k. In particular, $\|H_k\|_\infty = |H_k(0)| = 1$, so that eqn. 8 holds. □

So far we have considered the feedback system shown in Fig. 1 from an input-output point of view. If Fig. 1 describes a system with internal dynamics, then we must also consider the effects of non-zero initial conditions. This we will do in the following. To this end let $A \in R^{n\times n}$ be asymptotically stable, $b, c \in R^n$ and set $G(s) = c^T(sI - A)^{-1}b$. The closed-loop system shown in Fig. 1 can then be described by

$$\dot{x}(t) = Ax(t) + bu(t), \quad (12)$$

$$\dot{u}(t) = k[r - c^T x(t-h)]. \quad (13)$$

In the following we will be interested in the internal stability of the above system, and therefore we will assume that $r = 0$. Then eqns. 12 and 13 can be expressed as

$$\begin{pmatrix}\dot{x}\\ \dot{u}\end{pmatrix}(t) = \begin{pmatrix}A & b\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\ u\end{pmatrix}(t) + \begin{pmatrix}0 & 0\\ -kc^T & 0\end{pmatrix}\begin{pmatrix}x\\ u\end{pmatrix}(t-h). \quad (14)$$

Eqn. 14 is a so called delay-differential equation. There is a well-established mathematical theory for delay-differential equations (sometimes also called functional differential equations), see for example the books by Driver [18] and by Hale and Verduyn Lunel [19]. An immediate mathematical question is, what kind of initial condition should one use in order to obtain a unique solution to eqn. 14. A moment's thought shows that the specification of an initial function on the interval [–h, 0] is the most natural answer. This means we require

$$\begin{pmatrix}x\\ u\end{pmatrix} = \phi(t), \quad t \in [-h, 0] \quad (15)$$

for a given function ϕ. The fundamental solution X(t) of eqn. 14 is a function which is defined for all $t \in [-h, \infty)$, its values are $(n + 1) \times (n + 1)$-matrices, it satisfies the matrix differential-delay equation

$$\dot{\mathbf{X}}(t) = \begin{pmatrix}A & b\\ 0 & 0\end{pmatrix}\mathbf{X}(t) + \begin{pmatrix}0 & 0\\ -kc^T & 0\end{pmatrix}\mathbf{X}(t-h) \quad (16)$$

for all $t \geq 0$ and, finally, it satisfies the initial condition

$$\mathbf{X}(t) = \begin{cases}0 & \text{if } -h \leq t < 0\\ I & \text{if } t = 0\end{cases}. \quad (17)$$

The characteristic equation of eqn. 14 is given by

$$\mathbf{q}_k(s) := \det\left[sI - \begin{pmatrix}A & b\\ -ke^{-hs}c^T & 0\end{pmatrix}\right] = 0. \quad (18)$$

Notice that $q_k(s)$ is not a polynomial, but it is a so called quasi-polynomial in s. If all the zeros of $q_k(s)$ have negative real parts, then, as is the case for ordinary differential equations, the system eqn. 14 is exponentially stable in the sense that any solution of this equation decays exponentially, or equivalently that there exist numbers $M \geq 1$ and $\mu > 0$ such that

$$\|\mathbf{X}(t)\| \leq Me^{-\mu t}, \tag{19}$$

where $\|\cdot\|$ denotes any matrix norm (for example the largest singular value).

We now show, using the input-output result Proposition 1, that for all sufficiently small $k > 0$ the system in eqn. 14 is exponentially stable.

Corollary 2: Let $G(s) = c^T(sI - A)^{-1}b$, where $A \in \mathbb{R}^{n\times n}$ is an asymptotically stable matrix, and suppose that $G(0) = -c^T A^{-1}b > 0$. Then there exists $k^* > 0$ (the same k^* as in Proposition 1) such that for all $k \in (0, k^*)$ the delay-differential system of eqn. 14 is exponentially stable.

Proof: We need to show that for any sufficiently small $k > 0$, all the zeros of q_k have negative real parts. To this end note that

$$\begin{aligned}\mathbf{q}_k(s) &= \det\begin{pmatrix} sI - A & -b \\ ke^{-hs}c^T & s\end{pmatrix} \\ &= \det(sI - A)(s + k\mathbf{G}(s)e^{-hs}) \\ &= s\det(sI - A)\mathbf{L}_k(s). \end{aligned}\tag{20}$$

By assumption, all the zeros of $\det(sI - A)$ have negative real parts. From Proposition 1 we know that there exists $k^* > 0$ such that for all $k \in (0, k^*)$, $L_k(s)$ has no zeros in the closed right-half plane. Moreover, for $s = 0$ we have that $q_k(0) = (-1)^n kG(0) \det A \neq 0$. Combining these arguments with eqn. 20 shows that for all $k \in (0, k^*)$, all the zeros of $q_k(s)$ have negative real parts. □

The tuning regulator result for stable time-delay systems given in Corollary 2 is not new. It can be found in [11] and [12]. For the sake of completeness, and in order to make the paper self-contained, we have included the short proof. Note that our development is different to the approaches in [11] and [12] in that we first give an input-output version of the tuning regulator result in Proposition 1 (which is needed in the proof of the important Proposition 2) and then apply it to obtain the corresponding internal version in Corollary 2.

3 Adaptive integral control

In Section 2 we saw that tracking of constant reference signals can be achieved for stable time-delay systems with positive steady-state gain by integral control action if the integral gain parameter is positive and sufficiently small (Proposition 1). If reliable step-response data is available, then there are techniques for off-line tuning of the integral gain parameter so as to achieve the required tracking, see [11, 12]. However, if the step-response data is unreliable, or it is difficult to obtain a sufficiently good model, then an adaptive on-line tuning mechanism for the integral gain parameter is required. In this section, the main result gives a simple adaptation rule for the on-line tuning of the gain. The main tool in proving this adaptive I-control result is Proposition 2, where we showed that if the integral gain is sufficiently small, then the closed-loop H^∞-gain is equal to one.

To formulate this adaptive I-control result, consider the system

$$\dot{x}(t) = Ax(t) + bu(t), \tag{21}$$

$$y(t) = c^T x(t - h), \tag{22}$$

where $A \in \mathbb{R}^{n\times n}$ and $b, c \in \mathbb{R}^n$. Clearly, setting $G(s) = c^T(sI - A)^{-1}b$, the transfer function of eqns. 21 and 22 is given by $G(s)e^{-hs}$.

Theorem 1: Fix a set point $r \in \mathbb{R}$ and suppose that A is asymptotically stable and $G(0) > 0$. Define the input $u(t)$ in eqn. 21 by

$$\dot{u}(t) = \gamma^{-p}(t)(r - c^T x(t - h)), \tag{23}$$

$$\dot{\gamma}(t) = (r - c^T x(t - h))^2, \tag{24}$$

and let

$$\begin{pmatrix} x(t) \\ u(t)\end{pmatrix} = \phi(t), \quad -h \leq t \leq 0; \quad \gamma(t) = \gamma_0 \tag{25}$$

be the initial conditions for the closed-loop nonlinear delay-differential system given by eqns. 21, 23 and 24.

If $p \in (0, 1/2)$, then for every continuous initial function ϕ and $\gamma_0 > 0$ we have

(i) $\lim_{t\to\infty} x(t) = x_r := -A^{-1}bG(0)^{-1}r$;

(ii) $\lim_{t\to\infty} u(t) = u_r := G(0)^{-1}r$;

(iii) $\lim_{t\to\infty} \gamma(t) = \gamma_\infty < \infty$ and

(iv) $\lim_{t\to\infty} y(t) = r$.

Statements (i) and (ii) simply mean that the internal variables describing the system converge, (iii) states that the adaptation parameter converges, and (iv), being most important, states that the plant output asymptotically tracks the given set-point.

Proof of Theorem 1: Let $z(t) = x(t) - x_r$ and $v(t) = u(t) - u_r$. For any $k \in \mathbb{R}$ we can rewrite the closed-loop equations eqns. 21 and 22 in the form of a forced differential-delay equation

$$\begin{aligned}\begin{pmatrix}\dot{z}(t) \\ \dot{v}(t)\end{pmatrix} &= \begin{pmatrix} A & b \\ 0 & 0\end{pmatrix}\begin{pmatrix} z(t) \\ v(t)\end{pmatrix} - \begin{pmatrix} 0 \\ 1\end{pmatrix} kc^T z(t - h) \\ &\quad + \begin{pmatrix} 0 \\ 1\end{pmatrix}(k - \gamma^{-p}(t))c^T z(t - h)\end{aligned}\tag{26}$$

Here k is an ‘artificial’ gain parameter whose value will be specified later. For convenience let

$$\tilde{x}(t) = \begin{pmatrix} z(t) \\ v(t)\end{pmatrix}. \tag{27}$$

We observe that conclusions (i) and (ii) above, for $x(t)$ and $u(t)$ become $\lim_{t\to\infty} \tilde{x}(t) = 0$.

Using the variation of parameters formula for a delay system with a forcing term (see for example [19, 20]) we have, for any $T \geq 0$, that

$$\begin{aligned}\tilde{x}(t) &= \xi(t - T; \tilde{x}_T, 0) \\ &\quad + \int_T^t \Xi_k(t - \tau)\begin{pmatrix} 0 \\ 1\end{pmatrix}(k - \gamma^{-p}(\tau))\,[\,c^T \quad 0\,] \\ &\quad \times \tilde{x}(\tau - h)d\tau\end{aligned}\tag{28}$$

Here $\xi(t; \tilde{x}_T, 0)$ denotes the solution of the unforced linear system

$$\dot{\xi}(t) = \begin{pmatrix} A & b \\ 0 & 0\end{pmatrix}\xi(t) - \begin{pmatrix} 0 \\ 1\end{pmatrix} k\,[\,c^T \quad 0\,]\,\xi(t-h) \tag{29}$$

starting at time 0 with initial function

$$\tilde{x}_T(t) = \tilde{x}(t + T), \quad -h \leq t \leq 0 \tag{30}$$

and $\Xi_k(t)$ is the fundamental solution of eqn. 29.

Now let k be sufficiently small so that, in the sense of Proposition 2, the closed loop H^∞-gain is one, i.e.

$$\|\mathbf{H}_k\|_\infty = 1, \tag{31}$$

where $H_k(s)$ is given by eqn. 3. We first show that statement (iii) holds. Since $\gamma(t)$ is non-decreasing, this will follow if we can prove that $\gamma(t)$ is bounded. The claims (i), (ii) and (iv) will then follow easily. Now, either $\gamma^p(t)$ is always greater than or equal to k for all $t \geq 0$ so that $\gamma(t)$ is bounded, or else we can choose $T \geq 0$ in eqn. 28 (fixed from now on) so that $\gamma^p(t) \leq k$ for all $t \geq T$. We will show that $\gamma(t)$ is also bounded in the latter case.

Denoting the error by $e(t)$, we have that

$$e(t) = r - c^T x(t-h) = c^T z(t-h) = [\, c^T \quad 0\,]\, \tilde{x}(t-h). \tag{32}$$

Hence, by eqn. 28

$$\begin{aligned} e(t+h) = &\; [\, c^T \quad 0\,]\, \xi(t-T; \tilde{x}_T, 0) \\ &+ \int_T^t [\, c^T \quad 0\,]\, \Xi_k(t-\tau) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\times (k - \gamma^{-p}(\tau)) e(\tau) d\tau. \end{aligned} \tag{33}$$

Consider the map

$$\mathcal{M}_k : w(\cdot) \mapsto \int_0^{\cdot} [\, c^T \quad 0\,]\, \Xi_k(\cdot - \tau) \begin{pmatrix} 0 \\ 1 \end{pmatrix} w(\tau) d\tau. \tag{34}$$

Of course, $\mathcal{M}_k$ can be considered as the input-output operator of a linear system. By taking Laplace transforms (denoted by $\mathcal{L}$) we obtain the corresponding transfer function

$$\begin{aligned} [\mathcal{L}(\mathcal{M}_k w)](s) &= \frac{\mathbf{G}(s)}{s} \left(1 + \frac{k}{s} \mathbf{G}(s) e^{-hs} \right)^{-1} [\mathcal{L} w](s) \\ &= \frac{1}{k} e^{hs} \mathbf{H}_k(s) [\mathcal{L} w](s). \end{aligned} \tag{35}$$

Since $\|H_k\|_\infty = 1$, we obtain

$$\begin{aligned} \sup_{\mathrm{Re}\, s > 0} \left| \frac{1}{k} e^{hs} \mathbf{H}_k(s) \right| &= \sup_{-\infty < \omega < \infty} \left| \frac{1}{k} e^{j\omega h} \mathbf{H}_k(j\omega) \right| \\ &= \frac{1}{k} \|\mathbf{H}_k\|_\infty \\ &= \frac{1}{k}. \end{aligned} \tag{36}$$

If $\int_0^\infty w^2(\tau)\, d\tau < \infty$, then it follows from a combination of eqn. 35 considered for $s = j\omega$, eqn. 36 and Plancherel's theorem from Fourier transforms that

$$\int_0^\infty (\mathcal{M}_k w)^2(\tau) d\tau \leq \frac{1}{k^2} \int_0^\infty w^2(\tau) d\tau. \tag{37}$$

Defining

$$w_{[T,t]}(\theta) = \begin{cases} w(\theta) & \text{if } T \leq \theta \leq t \\ 0 & \text{if } t \notin [T,t] \end{cases} \tag{38}$$

it follows that

$$\begin{aligned} &\int_T^t \left(\int_T^\theta [\, c^T \quad 0\,]\, \Xi_k(\theta - \tau) \begin{pmatrix} 0 \\ 1 \end{pmatrix} w(\tau) d\tau \right)^2 d\theta \\ &= \int_T^t (\mathcal{M}_k(w_{[T,t]}))^2(\theta) d\theta \\ &\leq \int_0^\infty (\mathcal{M}_k(w_{[T,t]}))^2(\theta) d\theta. \end{aligned} \tag{39}$$

Hence, by eqn. 37

$$\begin{aligned} &\int_T^t \left(\int_T^\theta [\, c^T \quad 0\,]\, \Xi_k(\theta - \tau) \begin{pmatrix} 0 \\ 1 \end{pmatrix} w(\tau) d\tau \right)^2 d\theta \\ &\leq \frac{1}{k^2} \int_T^t w^2(\theta) d\theta. \end{aligned} \tag{40}$$

Moreover, we know from the exponential stability of the delay-differential system eqn. 29 that

$$|\, [\, c^T \quad 0\,]\, \xi(t-T; \tilde{x}_T, 0)| \leq M e^{-\mu t} \tag{41}$$

for some M, $\mu > 0$, where M depends on $\tilde{x}_T$ and c.

Hence, estimation in eqn. 33, using eqns. 40 and 41, gives

$$\begin{aligned} &\left(\int_T^t e^2(\theta + h) d\theta \right)^{1/2} \\ &\leq \frac{M}{\sqrt{2\mu}} + \frac{1}{k} (k - \gamma^{-p}(t)) \left(\int_T^t e^2(\theta) d\theta \right)^{1/2}. \end{aligned} \tag{42}$$

Here we used the fact that $\gamma^p(t)$ is non-increasing as a function of t and that, by our choice of T, $\gamma^p(t) \leq k$ for all $t \geq T$. Now, by eqn. 24

$$\sqrt{\gamma(t+h) - \gamma(T+h)} = \left(\int_T^t e^2(\theta + h) d\theta \right)^{1/2}, \tag{43}$$

and therefore eqn. 42 becomes

$$\begin{aligned} &\sqrt{\gamma(t+h) - \gamma(T+h)} \\ &\leq \frac{M}{\sqrt{2\mu}} + \frac{1}{k} (k - \gamma^{-p}(t)) \sqrt{\gamma(t) - \gamma(T)}. \end{aligned} \tag{44}$$

Since $\gamma(t)$ is non-decreasing, on squaring both sides of eqn. 44 and re-arranging, we obtain

$$\begin{aligned} &\frac{1}{k^2} (2k\gamma^{-p}(t) - \gamma^{-2p}(t)) \gamma(t) \\ &\leq \gamma(T+h) + M^2 + \frac{2}{k} M (k - \gamma^{-p}(t)) \sqrt{\gamma(t) - \gamma(T)} \\ &- \frac{1}{k^2} (k - \gamma^{-p}(t))^2 \gamma(T). \end{aligned} \tag{45}$$

Estimating all T-dependent terms (recall T is fixed) we arrive at

$$(2k\gamma^{-p}(t) - \gamma^{-2p}(t)) \gamma(t) \leq N(1 + \sqrt{\gamma(t)}), \tag{46}$$

where $N > 0$ is some suitably chosen constant (depending only on T, $\tilde{x}_T$, γ_0 and k). Since $p \in (0, 1/2)$ it follows that $\gamma(t)$ is bounded. Indeed, we have that

$$2k\gamma(t) \leq \gamma^{1-p}(t) + N[\gamma^p(t) + \gamma^{\frac{1}{2}+p}(t)], \tag{47}$$

so that the left hand side grows with $\gamma(t)$, while the right hand side grows like a fractional power $\gamma^q(t)$, where $q \in (0, 1)$.

We have so far shown that $\gamma(t)$ is bounded. It follows using eqn. 24 that $\int_0^\infty e^2(\tau)\, d\tau < \infty$. Hence combining eqns. 28 and 41 and the exponential decay of the fundamental solution $\Xi_k(t)$, it follows that $\lim_{t\to\infty} \tilde{x}(t) = 0$. Here we used the well known fact that the convolution of two square integrable functions tends to zero as $t \to \infty$. By definition of $\tilde{x}(t)$, it follows that statements (i) and (ii) hold. Statement (iv) follows easily from statement (i).

We note that in the square, multivariable case, if the steady-state gain $G(0)$ is positive definite, then the simple modification

$$\dot{u}(t) = \gamma^{-p}(t)(r - Cx(t-h)), \tag{48}$$

$$\dot{\gamma}(t) = \|r - Cx(t-h)\|^2 \qquad (49)$$

of eqns. 23 and 24, achieves asymptotic tracking of arbitrary reference vectors. In eqn. 49, $\|\cdot\|$ denotes the Euclidean norm.

4 Example

We illustrate the results of Section 3 by a simple second order time-delay system. In eqns. 13 and 14 suppose that

$$A = \begin{pmatrix} 0 & 1 \\ -10 & -11 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 10 \\ 0 \end{pmatrix} \qquad (50)$$

so that

$$\mathbf{G}(s) = c^T(sI - A)^{-1}b = \frac{1}{(s+1)(0.1s+1)}. \qquad (51)$$

If $h = 0$, then $\dot{u} = -ky$ is stabilising for all $k \in (0, 11)$. Using a stability window analysis, see Walton and Marshall [20], we can compute for each h, the largest $k(h)$ so that $\dot{u} = -ky$ is stabilising for all $k \in (0, k(h))$. In Fig. 2, $k(h)$ is plotted against h.

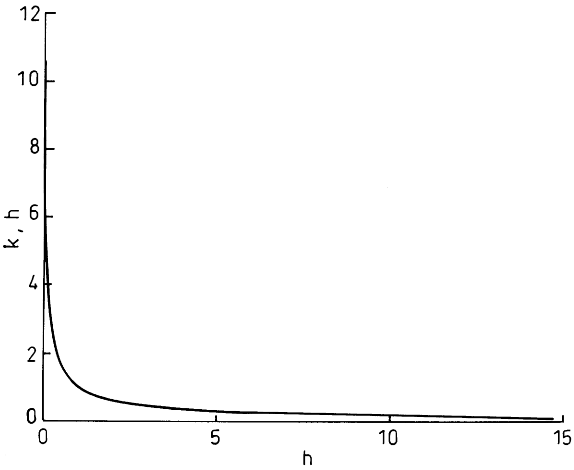

Fig.2 *Allowable gain k(h) as a function of h*

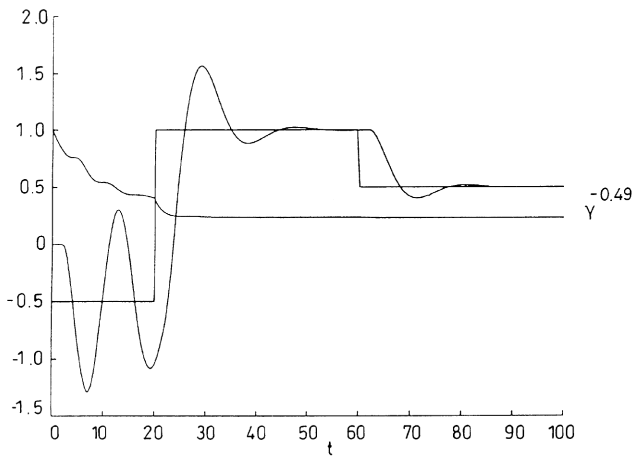

Fig.3 *Reference, output and integral gain with adaptive gain $\gamma^{-0.49}$*

Recall that the proof of Theorem 1 relies on the fact that eqn. 8 holds for all $k \in (0, k^{**})$, for some small enough $k^{**} > 0$. In this simple case we can show that the largest value for k^{**} is $10/(11h + 10)$, see Logemann, Ryan and Townley [21].

Let $H(t)$ denote the Heaviside step function. We choose a reference signal $r(t) = -0.5H(t) + 1.5H(t - 20) - 0.5H(t - 60)$. It is clear that the same adaptive algorithm allows any reference signal which has finitely many steps. Fig. 3 shows plots of the reference signal $r(t)$, the output $y(t)$ and the adaptive gain $\gamma^{-p}(t)$ against t for eqns. 23 and 24 with $p = 0.49$, when $h = 2$, $x(0) = 0$, $u(0) = 0$ and $y(t) = 0$ for $t < 0$, whilst Fig. 4 shows a plot of the input $u(t)$ and, for purposes of comparison, the required steady state input. In this case, the stability window analysis shows that any integrator gain in $(0, 0.588)$ is stabilising and eqn. 8 holds if $k \in (0, 0.3125)$. In the simulation the limiting value, $\gamma^{-0.49}(\infty)$, of the adaptive gain is 0.2296. Notice also that initially the adaptive gain equals 1 which is not a stabilising value and that the closed-loop response is poor until the adaptive-gain is below 0.3125, the value of gain for which the important property in eqn. 8 holds.

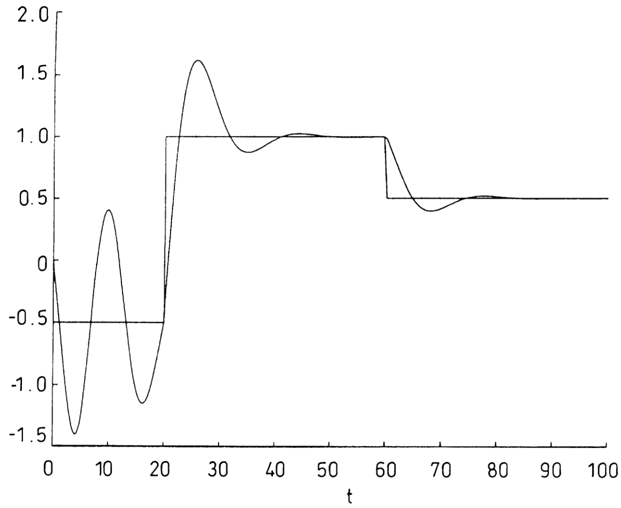

Fig.4 *Input and steady-state input with adaptive gain $\gamma^{-0.49}$*

5 Notes, comments and concluding remarks

In Section 3 we presented a simple adaptive I-controller which, when applied to any stable system with output delay, achieves asymptotic tracking of constant references, provided that the steady-state gain is positive. The main emphasis of this paper was to present a mathematically rigorous convergence proof for the adaptation, based only on elementary mathematical techniques. Of course, we only addressed the problem of tuning the integrator gain. In a real control engineering application this would be just one aspect amongst many others. Other aspects to be considered would include the effects of disturbances, input saturation and speeding up the transient response by adding a proportional part to the controller. Output disturbances could result in ever decreasing integrator gain, so leading to an unacceptably slow transient behaviour. In practice, this problem could be overcome by including a reset mechanism in the gain adaptation. The issue of input saturation is dealt with in Logemann and Ryan [22]. A proportional part could be added to the controller quite easily if more plant information was available. For example, if an upper bound M for the H^∞-gain of the open-loop plant is known, then we could replace eqn. 23 by

$$u(t) = u(0) + \int_0^t \gamma^{-p}(\tau)(r - y(\tau))d\tau + k_P(r - y(t)),$$

where $|k_P|M < 1$. However, the above strategy does not address the problem of how to tune both the integrator and proportional gains simultaneously. This problem of improving, adaptively, the transient performance is an interesting topic for future research.

In this paper we considered lumped systems with output delays. These are widely used in chemical process control. More elaborate models, taking into consideration the distributed nature of chemical processes, would naturally give rise to more general classes of infinite-dimensional or distributed parameter systems. However, it is likely that certain salient features would be retained in these more complicated models, namely that the system is stable and the steady-state gain is positive (or non-zero). In [1], we considered low-gain control of a general class of infinite-dimensional systems — so called regular linear systems (see Weiss [23]). Starting from an abstract point of view this class of systems takes into account various natural properties that any linear, time-invariant model should possess, such as continuous dependence on data (input data and initial conditions). Lumped systems with output delay are included as a special case, as are many systems described by partial differential equations. The main result in [1] states that if a process is described by a multivariable square stable regular linear system with positive-definite steady-state gain, but which is otherwise completely unknown, then asymptotic tracking of arbitrary constant reference vectors can be achieved by using the adaptive I-controller (eqns. 48 and 49). In Logemann and Townley [24] we have considered sampled-data versions of this adaptive I-controller. Indeed, if y_n denotes a sampled version of the continuous-time plant output (obtained by some natural sampling operation), then the input

$$u(t) = u_n, \quad t \in [n\tau, (n+1)\tau), \tag{52}$$

where u_n is given by the discrete-time adaptive I-controller

$$\begin{aligned} u_{n+1} &= \gamma_n^{-p}(r - y_n), \\ \gamma_{n+1} &= \gamma_n + \|r - y_n\|^2, \end{aligned} \tag{53}$$

when applied to any multivariable square stable regular system with positive steady-state gain, achieves continuous-time asymptotic tracking of arbitrary constant reference vectors.

Both papers [1] and [24] rely very much on functional analysis and the theory of abstract linear systems so that their contribution, for a control engineering audience, can be obscured by the level of abstraction. We hope that the self-contained treatment in this paper of a more concrete case, bridges the gap.

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