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INPUT-TO-STATE STABILITY OF DISCRETE-TIME LUR'E SYSTEMS*

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Abstract. An input-to-state stability theory, which subsumes results of circle criterion type, is developed in the context of discrete-time Lur'e systems. The approach developed is inspired by the complexified Aizerman conjecture.

Key words. absolute stability, circle criterion, complexified Aizerman conjecture, input-to-state stability, Lur'e systems, stability radius

AMS subject classifications. 92D25, 93C10, 93C35, 93C55, 93D05, 93D10, 93D15

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1. Introduction.

In this paper, we link two well-established areas of nonlinear control theory: input-to-state stability (ISS) and absolute stability. The concept of ISS (for a general controlled nonlinear continuous-time system) appeared first in the paper [29], published in 1989. The theory of ISS, which has been developed since 1989, provides a natural stability framework for nonlinear systems with inputs, merging, in a sense, Lyapunov and input-output approaches to stability (see [7, 31] for overviews).

Classical absolute stability theory (see, for example, [11, 12, 22, 24, 26, 33, 35]), the origins of which go back to the late 1940s, is concerned with the analysis of systems of Lur'e type, that is, feedback interconnections of the form shown in Figure 1, consisting of a linear state-space system (A, B, C) in the forward path and a static sector-bounded nonlinearity f in the feedback path. Absolute stability theory seeks to conclude stability of the feedback system through the interplay or reciprocation of inherent frequency-domain properties of the linear component (A, B, C) and sector data for the nonlinearity f.

The present paper adopts a similar standpoint but differs from the classical absolute stability framework in three fundamental aspects: (i) in contrast with the majority of the relevant literature wherein the continuous-time case is treated, discrete-time systems are studied here; (ii) rather than focussing on global asymptotic stability of the unforced system ($v = 0$) or on input-output stability (in the l^2 or l^∞-sense), we address ISS issues here; (iii) our approach is inspired by the complexified Aizerman conjecture, a certain version of which is known to hold true in the continuous-time case (see [14, 15]).

In the continuous-time context, a number of results on ISS properties of Lur'e systems can be found in the literature; see [2, 16, 17, 18]. While a considerable amount of work on asymptotic stability properties of discrete-time Lur'e systems has been done (see, e.g., [1, 5, 9, 10, 11, 12, 23, 25] for stability results, [3, 13] for counterexamples to the discrete-time Kalman conjecture, and [8, 27], where it is shown that the Aizerman

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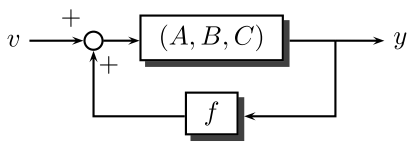

FIG. 1. *Lur'e system with linear component* (A, B, C)*, nonlinearity* f*, input* v*, and output* y*.*

conjecture holds for certain classes of nonnegative Lur'e systems), the theme of ISS criteria for discrete-time Lur'e systems in the spirit of absolute stability theory (that is, ISS criteria in terms of frequency-domain and sector conditions) remains largely unexplored. One of the few exceptions is [23], where an "exponential" ISS property is shown to hold in the context of observer synthesis for a class of discrete-time Lur'e systems.

The main result of this paper, Theorem 13, is a version of the circle criterion guaranteeing ISS. We consider the circle criterion from a perhaps unfamiliar but nevertheless intriguing point of view, namely, by relating it to a complexified version of the Aizerman conjecture. Balls of stabilizing gains (output feedback matrices) play a pivotal role in the "Aizerman version" of the circle criterion presented in Theorem 13, in contrast with classical versions of the circle criterion wherein positive-real and sector conditions are ubiquitous. In many situations, it is more intuitive to think in terms of balls of stabilizing gains. This point of view is partially inspired by classical results from the stability theory of linear multistep methods in numerical analysis: these results can be considered as Aizerman versions of the discrete-time circle criterion; see [4] and the references therein. Furthermore, we show that a more traditional version of the circle criterion, formulated in terms of positive-real and sector conditions, can be derived from the Aizerman version; see Corollary 16. The latter result shows that under conditions very similar to those of the classical circle criterion, the Lur'e system is ISS.

Initially, preceding the development of the ISS theory of the circle criterion, we prove an Aizerman version of the circle criterion for unforced Lur'e systems; see Theorem 9. This result offers a new perspective on the classical criterion and refines existing results. Furthermore, it allows us to demonstrate some of the finer points of our main ISS result, Theorem 13.

The paper is organized as follows. In section 2, we present a number of preliminary results which relate to the underlying linear system. Section 3 develops an Aizerman version of the circle criterion for unforced Lur'e systems, while in section 4 we present and prove versions of the circle criterion which apply to forced Lur'e systems and guarantee ISS. In section 5, we consider two applications of the ISS theory developed in section 4, namely, ISS with bias and environmental forcing in theoretical ecology. Finally, the proof of a technical lemma is relegated to the appendix (section 6).

Notation. Set $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. As usual, $\mathbb{F}^{p\times m}$ is the vector space of all matrices of format $m \times p$ with entries in $\mathbb{F}$. We set $\mathbb{F}^p := \mathbb{F}^{p\times 1}$. The field of rational functions with coefficients in $\mathbb{F}$ is denoted by $\mathbb{F}(z)$. For a matrix $M = (M_{ij}) \in \mathbb{F}^{m\times p}$, we define $M^* \in \mathbb{F}^{p\times m}$ by $(M^*)_{ij} := \overline{M_{ji}}$. If $\mathbb{F} = \mathbb{R}$, then M^* is simply the transposition of M. For a square matrix $M \in \mathbb{F}^{m\times m}$, $\sigma(M)$ denotes the spectrum of M, that is, the set of eigenvalues of M. The spectral radius of M is denoted by $\rho(M)$. We say that M is a Schur matrix (or that M is Schur) if $\rho(M) < 1$, or, equivalently, if M is asymptotically stable. If $M \in \mathbb{F}^{m\times m}$ is Schur, $W_1 \in \mathbb{F}^{m\times p}$ and $W_2 \in \mathbb{F}^{q\times m}$, then the structured complex stability radius of

M with respect to the weights W_1 and W_2 is defined by

$$r_{\mathbb{C}}(M; W_1, W_2) := \inf\{\|P\| : P \in \mathbb{C}^{p\times q} \text{ and } \rho(M + W_1 P W_2) \geq 1\},$$

where the operator norm is induced by the 2-norms in $\mathbb{C}^p$ and $\mathbb{C}^q$. If $M \in \mathbb{C}^{m\times m}$ is a square matrix, then we define its real part by

$$\operatorname{Re} M := \frac{1}{2}\big(M + M^*\big).$$

This is sometimes also known as the symmetric part of a matrix. For $K \in \mathbb{F}^{m\times p}$, set

$$\mathbb{B}_{\mathbb{F}}(K, r) := \{M \in \mathbb{F}^{m\times p} : \|K - M\| < r\}.$$

For $x \in \mathbb{F}^p$ and $Y \subset \mathbb{F}^p$, we define the *distance* $\operatorname{dist}(x, Y)$ of x to Y by $\operatorname{dist}(x, Y) := \inf\{\|x - y\| : y \in Y\}$. Furthermore, we define $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{E} := \{z \in \mathbb{C} : |z| > 1\}$. For $m, p \in \mathbb{N}$, the Hardy space $H^\infty_{p\times m}$ is the set of all holomorphic functions $\mathbf{H} : \mathbb{E} \to \mathbb{C}^{p\times m}$ such that $\|\mathbf{H}\|_{H^\infty} := \sup\{\|\mathbf{H}(z)\| : z \in \mathbb{E}\} < \infty$.

We will make use of the following classes of comparison functions:

$$\mathcal{K} := \{\alpha \in C([0,\infty)) : \alpha(0) = 0,\ \alpha \text{ is strictly increasing}\},$$
$$\mathcal{K}_\infty := \left\{\alpha \in \mathcal{K} : \lim_{s\to\infty} \alpha(s) = \infty\right\}.$$

Finally, we denote by $\mathcal{KL}$ the set of all functions $\beta : [0,\infty) \times \mathbb{N}_0 \to [0,\infty)$ with the following properties: if $\beta \in \mathcal{KL}$, then, for each fixed $t \in \mathbb{N}_0$, the function $\beta(\cdot, t) \in \mathcal{K}$ and, for each fixed $s \geq 0$, the function $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s, t) = 0$. We refer to [21] for more details on comparison functions.

2. The underlying linear system. Consider the following linear state-space systems:

$$x(t+1) = Ax(t) + Bu(t) + v(t),\ \ y(t) = Cx(t) \quad \forall\, t \in \mathbb{N}_0, \tag{2.1}$$

where $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{n\times m}$, and $C \in \mathbb{F}^{p\times n}$. Here u and v are input signals with values in $\mathbb{F}^m$ and $\mathbb{F}^n$, respectively. In sections 3 and 4, u will be generated by output feedback of the form $u = f(y)$, where $f : \mathbb{F}^p \to \mathbb{F}^m$, resulting in the feedback system

$$x(t+1) = Ax(t) + Bf(Cx(t)) + v(t) \tag{2.2}$$

with input v. Obviously, the (perhaps more familiar) feedback system

$$x(t+1) = Ax(t) + B\big(f(Cx(t)) + w(t)\big)$$

with $\mathbb{F}^m$-valued input w is a special case of (2.2) wherein $v(t) = Bw(t)$. It is convenient to set

$$\Sigma(m, n, p; \mathbb{F}) := \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m} \times \mathbb{F}^{p\times n}.$$

We say that $(A, B, C) \in \Sigma(m, n, p; \mathbb{F})$ is *canonical* (*semi-canonical*) if (A, B) is controllable (stabilizable) and (C, A) is observable (detectable). The *transfer function* $\mathbf{G}$ of $(A, B, C) \in \Sigma(m, n, p; \mathbb{F})$ is defined by

$$\mathbf{G}(z) = C(zI - A)^{-1}B.$$

The *behavior* of (2.1), denoted by $\mathcal{B}(A, B, C)$, is defined as the set of all trajectories $(v, u, x, y) \in (\mathbb{F}^n)^{\mathbb{N}_0} \times (\mathbb{F}^m)^{\mathbb{N}_0} \times (\mathbb{F}^n)^{\mathbb{N}_0} \times (\mathbb{F}^p)^{\mathbb{N}_0}$ satisfying (2.1). While this paper is not a contribution to a behavioral theory of Lur'e systems, we find that the concept of behaviors is convenient in the contexts of (i) formulating Lemmas 4 and 5 and Corollary 7, and (ii) relating trajectories of linear systems to those of associated linear and nonlinear feedback systems; see statements (1) and (2) of Lemmas 6 and 8.

2.1. A consequence of the bounded real lemma. In the following, a rational matrix function $\mathbf{H} \in \mathbb{F}(z)^{p\times m}$ is said to be *contractive* if $\|\mathbf{H}\|_{H^\infty} \le 1$. In systems and control theory, contractive rational functions are usually called *bounded real*. However, since we do not assume that $\mathbf{H}$ is real rational, the latter terminology would be potentially misleading in our context. In the square case (that is, $p = m$), contractive functions are closely related to positive real functions. Recall that a rational function $\mathbf{H} \in \mathbb{F}(z)^{m\times m}$ is said to be *positive real* if $\operatorname{Re}\mathbf{H}(z)$ is positive semi-definite for every $z \in \mathbb{E}$ which is not a pole of $\mathbf{H}$. It is a standard result that if $\mathbf{H}$ is positive real, then $\mathbf{H}$ is holomorphic in $\mathbb{E} \cup \{\infty\}$.

The following lemma is well-known.

LEMMA 1. *Let $M \in \mathbb{C}^{m\times m}$. Then $\operatorname{Re} M$ is positive semi-definite if, and only if, $-1 \notin \sigma(M)$ and $\|(I - M)(I + M)^{-1}\| \le 1$. Furthermore, $\operatorname{Re} M$ is positive definite if, and only if, $-1 \notin \sigma(M)$ and $\|(I - M)(I + M)^{-1}\| < 1$.*

The next result is an immediate consequence of Lemma 1.

COROLLARY 2. *Let $\mathbf{H} \in \mathbb{F}(z)^{m\times m}$. The following statements are equivalent:*
(1) *$\mathbf{H}$ is positive real.*
(2) *$-1 \notin \sigma(\mathbf{H}(z))$ for all $z \in \mathbb{E}$ and $(I - \mathbf{H})(I + \mathbf{H})^{-1}$ is contractive.*

Next, we state a version of the "bounded real lemma," which is convenient for our purposes.

LEMMA 3. *Let $(A, B, C) \in \Sigma(m, n, p; \mathbb{F})$ with transfer function $\mathbf{G}$. Assume that $\mathbf{G}$ is contractive and that either* (i) *(A, B, C) is canonical or* (ii) *(A, B, C) is semi-canonical and* $\min_{|z|=1} \|\mathbf{G}(z)\| < 1$. *Then there exist matrices L and W with entries in $\mathbb{F}$ and a positive semi-definite $P = P^* \in \mathbb{F}^{n\times n}$ such that*

$$A^*PA - P + C^*C = -L^*L, \quad A^*PB = -L^*W, \quad \text{and} \quad B^*PB = I - W^*W. \tag{2.3}$$

If (A, B, C) is canonical, then P is positive definite.

Proof. Assume that $\mathbf{G}$ is contractive. For a canonical triple (A, B, C) the result is very well-known; see, for example, [10, Lemma 3.1], where it is proved for real matrices. An inspection of the proof shows that the result extends to the complex case.

If (A, B, C) is semi-canonical and $\min_{|z|=1} \|\mathbf{G}(z)\| < 1$, then, by [34, Theorem 5.3], there exists a positive semi-definite $P = P^* \in \mathbb{F}^{n\times n}$ such that $I - B^*PB$ is positive definite and

$$A^*PA - P + A^*PB(I - B^*PB)^{-1}B^*PA + C^*C = 0.$$

Setting $W := (I - B^*PB)^{1/2} > 0$, it follows trivially that $W = W^*$ and $B^*PB = I - W^*W$. Furthermore, setting $L := -(W^*)^{-1}B^*PA$, we obtain the other two equations. We remark that [34, Theorem 5.3] is formulated for complex matrices. An inspection of the proof shows that the result remains valid over the field of real numbers. ∎

The following example shows that if (A, B, C) is semi-canonical and $\|\mathbf{G}(e^{i\omega})\| = 1$ for all $\omega \in [0, 2\pi)$, then the bounded-real equations (2.3) do not necessarily have a solution (L, W, P) with $P = P^*$.

Example.[1] Consider $(A, B, C) \in \Sigma(1, 2, 1; \mathbb{F})$ given by

$$A = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = (1, 1),$$

where $|\lambda| < 1$ and $\lambda \neq 0$. The pair (A, B) is not controllable and so (A, B, C) is not canonical. However, since A is Schur, (A, B) is stabilizable and (C, A) is detectable, and thus (A, B, C) is semi-canonical. The transfer function of (A, B, C) is $\mathbf{G}(z) = 1/z$ and consequently,

$$|\mathbf{G}(e^{i\omega})| = \|\mathbf{G}\|_{H^\infty} = 1 \quad \forall\, \omega \in [0, 2\pi).$$

A straightforward calculation shows that the bounded-real equations (2.3) do not have a solution (L, W, P) with $P = P^*$. ∎

We will now use Lemma 3 to obtain a quadratic form that will be used later to construct Lyapunov and ISS-Lyapunov functions for Lur'e systems.

LEMMA 4. *Let $(A, B, C) \in \Sigma(m, n, p; \mathbb{F})$ with transfer function $\mathbf{G} \in H^\infty_{p\times m}$ and let $r > 0$ be such that $r\|G\|_{H^\infty} \leq 1$. Assume that either* (i) *(A, B, C) is canonical or* (ii) *(A, B, C) is semi-canonical and $r \min_{|z|=1} \|\mathbf{G}(z)\| < 1$. Then there exists a constant $\kappa > 0$ and positive semi-definite matrix $P = P^* \in \mathbb{F}^{n\times n}$ such that the function $V : \mathbb{F}^n \to [0, \infty)$ defined by $V(\xi) = \langle P\xi, \xi\rangle$ satisfies*

$$V(x(t+1)) - V(x(t)) \leq -r^2 \|y(t)\|^2 + \|u(t)\|^2 + \kappa\|v(t)\|\big(\|v(t)\| + \|u(t)\| + \|x(t)\|\big) \tag{2.4}$$

for all $t \in \mathbb{N}_0$ and for all $(v, u, x, y) \in \mathcal{B}(A, B, C)$.

Moreover there exists a projection $\Pi : \mathbb{F}^n \to \mathbb{F}^n$ and $c > 0$ such that $\ker \Pi \subseteq \ker C$ *and $V(\xi) \geq c\|\Pi\xi\|^2$ for all $\xi \in \mathbb{F}^n$.*

Note that the condition $r \min_{|z|=1} \|\mathbf{G}(z)\| < 1$ is trivially satisfied if $r\|\mathbf{G}\|_{H^\infty} < 1$ and thus is only relevant in the case wherein $r\|\mathbf{G}\|_{H^\infty} = 1$.

Proof of Lemma 4. The rational matrix function $r\mathbf{G}$ is contractive and so, by Lemma 3 (applied to the system (A, rB, C)), there exists a positive semi-definite $Q = Q^*$ and matrices L and W such that

$$A^*QA - Q + C^*C = -L^*L, \quad A^*QB = -\frac{1}{r}L^*W \quad \text{and} \quad B^*QB = \frac{1}{r^2}(I - W^*W). \tag{2.5}$$

Define $U : \mathbb{F}^n \to [0, \infty)$ by $U(\xi) = \langle Q\xi, \xi\rangle$ and $T : \mathbb{F}^n \times \mathbb{F}^m \times \mathbb{F}^n \to \mathbb{F}$ by

$$T(\zeta_1, \zeta_2, \xi) = \langle Q\zeta_1, \zeta_1\rangle + 2\mathrm{Re}\,\langle B^*Q\zeta_1, \zeta_2\rangle + 2\mathrm{Re}\,\langle A^*Q\zeta_1, \xi\rangle.$$

Let $(v, u, x, y) \in \mathcal{B}(A, B, C)$. Then, invoking the bounded-real equations (2.5), we obtain

$$U(x(t+1)) - U(x(t)) = -\left\|Lx(t) + \frac{1}{r}Wu(t)\right\|^2 - \|y(t)\|^2 + \frac{1}{r^2}\|u(t)\|^2 + T(v(t), u(t), x(t)) \quad \forall\, t \in \mathbb{N}_0.$$

[1]The authors would like to thank Chris Guiver (Exeter) and Mark Opmeer (Bath), who constructed this example.

Hence, for all $t \in \mathbb{N}_0$,

$$U(x(t+1)) - U(x(t)) \leq -\|y(t)\|^2 + \frac{1}{r^2}\|u(t)\|^2 + \tilde{\kappa}\|v(t)\|\big(\|v(t)\| + \|u(t)\| + \|x(t)\|\big),$$

where $\tilde{\kappa} := \|Q\| \max(1, 2\|B\|, 2\|A\|)$. Setting $P := r^2 Q$, $\kappa := r^2\tilde{\kappa}$ and

$$V(\xi) := \langle P\xi, \xi\rangle = r^2 U(\xi) \quad \forall\, \xi \in \mathbb{F},$$

we see that (2.4) is satisfied.

To prove the existence of the projection Π, let $\xi \in V^{-1}(0)$ and note that, by the first equation in (2.5),

$$V(A\xi) + r^2\|C\xi\|^2 = -r^2\|L\xi\|^2,$$

implying that $C\xi = 0$, and consequently, $V^{-1}(0) \subset \ker C$. Let Π be the orthogonal projection onto $(\ker P)^\perp$ along $\ker P = V^{-1}(0)$. Then $\ker \Pi = V^{-1}(0) \subseteq \ker C$. Moreover, since $P = P\Pi$ and $P = P^*$, it follows that $V(\xi) = V(\Pi\xi)$ for all $\xi \in \mathbb{F}^n$. Finally, since $\ker\Pi = V^{-1}(0)$, the seminorm $\xi \mapsto \sqrt{V(\xi)}$ on $\mathbb{F}^n$ becomes a norm when restricted to $(\ker P)^\perp = \operatorname{im}\Pi$. Consequently, there exists a number $c > 0$ such that $V(\xi) = V(\Pi\xi) \geq c\|\Pi\xi\|^2$ for all $\xi \in \mathbb{F}^n$, completing the proof. ∎

2.2. Output injection and output feedback. The following lemma will turn out to be useful for the construction of ISS-Lyapunov functions for Lur'e systems.

LEMMA 5. *Let $(A, B, C) \in \Sigma(m, n, p; \mathbb{F})$ and assume that (C, A) is detectable. Then there exists a positive definite $P = P^* \in \mathbb{F}^{n\times n}$ and $\delta > 0$ such that the function $V : \mathbb{F}^n \to [0, \infty)$ defined by $V(\xi) = \langle P\xi, \xi\rangle$ satisfies*

$$V(x(t+1)) - V(x(t)) \leq -\delta\|x(t)\|^2 + \|y(t)\|^2 + \|u(t)\|^2 + \|v(t)\|^2$$

for all $t \in \mathbb{N}_0$ and all $(v, u, x, y) \in \mathcal{B}(A, B, C)$.

Proof. By detectability, we can choose an "output injection" matrix $H \in \mathbb{F}^{n\times m}$ such that $\sigma(A + HC) \subset \mathbb{D}$. Hence, by [15, Corollary 3.3.47], there exists a (unique) positive definite matrix $Q = Q^* \in \mathbb{F}^{n\times n}$ that solves the discrete-time Lyapunov equation

$$(A + HC)^* Q(A + HC) - Q = -I. \tag{2.6}$$

Define $U : \mathbb{F}^n \to [0, \infty)$ by $U(\xi) := \langle Q\xi, \xi\rangle$. Noting that, for all $(v, u, x, y) \in \mathcal{B}(A, B, C)$,

$$x(t+1) = Ax(t) + Bu(t) + v(t) = (A + HC)x(t) - Hy(t) + Bu(t) + v(t) \quad \forall\, t \in \mathbb{N}_0,$$

it follows from (2.6) that for all $(v, u, x, y) \in \mathcal{B}(A, B, C)$

$$U(x(t+1)) - U(x(t)) = -\|x(t)\|^2 + T(v(t), u(t), x(t), y(t)) \quad \forall\, t \in \mathbb{N}_0. \tag{2.7}$$

Here $T : \mathbb{F}^n \times \mathbb{F}^m \times \mathbb{F}^n \times \mathbb{F}^p \to \mathbb{R}$ is a map satisfying, for all $(\zeta_1, \zeta_2, \xi, \zeta_3) \in \mathbb{F}^n \times \mathbb{F}^m \times \mathbb{F}^n \times \mathbb{F}^p$,

$$T(\zeta_1, \zeta_2, \xi, \zeta_3) \leq \|\xi\| \sum_{j}^{3} l_j\|\zeta_j\| + \sum_{j,k=1}^{3} l_{jk}\|\zeta_j\|\|\zeta_k\|,$$

where the l_j and l_{jk}, $j,k = 1,\ldots,3$, are suitable positive constants. Since $2ab = 2(ca)(c^{-1}b) \le c^2a^2 + b^2/c^2$ for all real a, b and all nonzero c, it is clear that there exists $l > 0$ such that

$$T(\zeta_1,\zeta_2,\xi,\zeta_3) \le \frac{1}{2}\|\xi\|^2 + l\sum_{j}^{3}\|\zeta_j\|^2 \quad \forall\,(\zeta_1,\zeta_2,\xi,\zeta_3) \in \mathbb{F}^n \times \mathbb{F}^m \times \mathbb{F}^n \times \mathbb{F}^p.$$

Combining this with (2.7) yields that, for all $(v,u,x,y) \in \mathcal{B}(A,B,C)$,

$$U(x(t+1)) - U(x(t)) = -\frac{1}{2}\left\|x(t)\right\|^2 + l(\|y(t)\|^2 + \|u(t)\|^2 + \|v(t)\|^2) \quad \forall\, t \in \mathbb{N}_0.$$

Finally, setting $P := l^{-1}Q$ and $V(\xi) := \langle P\xi,\xi\rangle = l^{-1}U(\xi)$ shows that the claim holds with $\delta = 1/(2l)$. ∎

Let $(A,B,C) \in \Sigma(m,n,p;\mathbb{F})$, $K \in \mathbb{C}^{m\times p}$, and set $A_K := A + BKC$. Then $(A_K,B,C) \in \Sigma(m,n,p;\mathbb{F})$, and we denote the transfer function of (A_K,B,C) by $\mathbf{G}^K$, that is,

$$\mathbf{G}^K(z) = C(zI - A_K)^{-1}B = \mathbf{G}(z)(I - K\mathbf{G}(z))^{-1},$$

where $\mathbf{G}$ is the transfer function of (A,B,C). Set

$$\mathbb{S}_{\mathbb{C}}(\mathbf{G}) := \{K \in \mathbb{C}^{m\times p} : \mathbf{G}^K \in H^\infty_{p\times m}\},$$

the set of all (complex) output feedback matrices which stabilize (A,B,C) in the l^2-input-output sense. If (A,B,C) is semi-canonical, then

$$\mathbb{S}_{\mathbb{C}}(\mathbf{G}) = \{K \in \mathbb{C}^{m\times p} : \sigma(A_K) \subset \mathbb{D}\},$$

and so, $\mathbb{S}_{\mathbb{C}}(\mathbf{G})$ coincides with the set of all (complex) output feedback matrices which render the closed-loop system (A_K,B,C) asymptotically stable.

The following result collects simple properties of the feedback system (A_K,B,C) and its transfer function $\mathbf{G}^K$.

LEMMA 6. *Let $(A,B,C) \in \Sigma(m,n,p;\mathbb{C})$ with transfer function $\mathbf{G}$, and let $K \in \mathbb{C}^{m\times p}$. The following statements hold:*

(1) *$(v,u,x,y) \in \mathcal{B}(A,B,C)$ if, and only if, $(v,u-Ky,x,y) \in \mathcal{B}(A_K,B,C)$.*
(2) *$(v,u,x,y) \in \mathcal{B}(A_K,B,C)$ if, and only if, $(v,u+Ky,x,y) \in \mathcal{B}(A,B,C)$.*
(3) *$\mathbb{S}_{\mathbb{C}}(\mathbf{G}^K) = \mathbb{S}_{\mathbb{C}}(\mathbf{G}) - K := \{L - K : L \in \mathbb{S}_{\mathbb{C}}(\mathbf{G})\}$.*
(4) *For every $L \in \mathbb{C}^{m\times p}$, $(\mathbf{G}^K)^L = \mathbf{G}^{K+L}$.*
(5) *For $r > 0$, $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ if, and only if, $\|\mathbf{G}^K\|_{H^\infty} \le 1/r$.*
(6) *If $K \in \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, then*

$$\max\{r > 0 : \mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})\} = \frac{1}{\|\mathbf{G}^K\|_{H^\infty}}. \tag{2.8}$$

Proof. The proof of statements (1)–(4) is straightforward and is therefore left to the reader. Note that statement (5) is a consequence of statement (6). It remains to prove statement (6). To this end, note that, by statement (3), the identity (2.8) is equivalent to

$$\max\{r > 0 : \mathbb{B}_{\mathbb{C}}(0,r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G}^K)\} = \frac{1}{\|\mathbf{G}^K\|_{H^\infty}}. \tag{2.9}$$

We show that (2.9) holds. A trivial small-gain argument shows that if $r < 1/\|\mathbf{G}^K\|_{H^\infty}$, then $\mathbb{B}_{\mathbb{C}}(0,r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G}^K)$, implying that

$$\max\{r > 0 : \mathbb{B}_{\mathbb{C}}(0,r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G}^K)\} \geq \frac{1}{\|\mathbf{G}^K\|_{H^\infty}}.$$

To show that equality holds, we note that there exists z_0 in the closure of $\mathbb{E}$ (that is, $|z_0| \geq 1$) such that $\|\mathbf{G}^K(z_0)\| = \|\mathbf{G}^K\|_{H^\infty}$. Moreover, it is well-known that there exists a matrix $L \in \mathbb{C}^{m\times p}$ (of rank one) such that

$$\|L\| = \frac{1}{\|\mathbf{G}^K(z_0)\|} = \frac{1}{\|\mathbf{G}^K\|_{H^\infty}}$$

and $I - L\mathbf{G}^K(z_0)$ is singular. Consequently, $L \notin \mathbb{S}_{\mathbb{C}}(\mathbf{G}^K)$, and hence,

$$\max\{r > 0 : \mathbb{B}_{\mathbb{C}}(0,r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G}^K)\} \leq \frac{1}{\|\mathbf{G}^K\|_{H^\infty}},$$

completing the proof of statement (6). ∎

Statement (6) of Lemma 6 is closely related to the complex stability radius of $A_K = A + BKC$ with respect to the perturbation structure given by B and C: if (A,B,C) is semi-canonical and $K \in \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, then A_K is asymptotically stable, and it follows from a basic result on stability radii (see [15]) that $r_{\mathbb{C}}(A_K; B, C) = 1/\|\mathbf{G}^K\|_{H^\infty}$.

For $(A,B,C) \in \Sigma(m,n,p;\mathbb{F})$ with transfer function $\mathbf{G}$, $K \in \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ and $r > 0$ satisfying $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, we introduce the following assumption:

(A) Either (i) (A,B,C) is canonical or (ii) (A,B,C) is semi-canonical and $r\min_{|z|=1}\|\mathbf{G}^K(z)\| < 1$.

Note that, since $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, statement (5) of Lemma 6 guarantees that $r\|\mathbf{G}^K\|_{H^\infty} \leq 1$. We conclude that the condition $r\min_{|z|=1}\|\mathbf{G}^K(z)\| < 1$ is violated if, and only if, $\|\mathbf{G}^K(e^{i\omega})\| = \|\mathbf{G}^K\|_{H^\infty} = 1/r$ for all $\omega \in [0, 2\pi)$. Consequently, if $m = p$ ("square" case) and $\det \mathbf{G}(z) \not\equiv 0$, then $r\min_{|z|=1}\|\mathbf{G}^K(z)\| < 1$ if, and only if, $\underline{\sigma}(\mathbf{G}^{-1}(e^{i\omega}) - K) \not\equiv r$, where $\underline{\sigma}$ denotes the smallest singular value. If $m = p = 1$ (single-input single-output case), the latter condition means that the inverse Nyquist plot $\{1/\mathbf{G}(e^{i\omega}) : \omega \in [0, 2\pi)\}$ is *not* equal to the circle of radius r centered at K.

COROLLARY 7. *Let $(A,B,C) \in \Sigma(m,n,p;\mathbb{F})$ with transfer function $\mathbf{G}$, $K \in \mathbb{F}^{m\times p}$ and $r > 0$. Assume that $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ and* (A) *holds. Then there exists a constant $\kappa > 0$ a positive semi-definite matrix $P = P^* \in \mathbb{F}^{n\times n}$ such that the function $V : \mathbb{F}^n \to [0,\infty)$ defined by $V(\xi) = \langle P\xi, \xi\rangle$ satisfies*

$$\begin{aligned} V(x(t+1)) - V(x(t)) \leq -r^2\|y(t)\|^2 + \|u(t) - Ky(t)\|^2 \\ + \kappa\|v(t)\|\big(\|v(t)\| + \|u(t) - Ky(t)\| + \|x(t)\|\big) \end{aligned}$$

for all $t \in \mathbb{N}_0$ and for all trajectories $(v,u,x,y) \in \mathcal{B}(A,B,C)$.

Moreover there exists a projection $\Pi : \mathbb{F}^n \to \mathbb{F}^n$ and $c > 0$ such that $\ker \Pi \subseteq \ker C$ *and $V(\xi) \geq c\|\Pi\xi\|^2$ for all $\xi \in \mathbb{F}^n$*

Proof. Note first that if (A,B,C) is canonical (semi-canonical), then (A_K,B,C) is canonical (semi-canonical). Let $(v,u,x,y) \in \mathcal{B}(A,B,C)$. By statement (1) of Lemma 6, $(v, u - Ky, x, y) \in \mathcal{B}(A_K,B,C)$. Invoking statement (5) of Lemma 6, we have that $r\|\mathbf{G}^K\|_{H^\infty} \leq 1$, and an application of Lemma 4 to the system (A_K,B,C) yields the claim. ∎

3. Lur'e systems and the Aizerman version of the circle criterion. Applying feedback of the form $u = f(y) = f(Cx)$ to the system (2.1), where $f : \mathbb{F}^p \to \mathbb{F}^m$ is a nonlinearity, leads to the forced Lur'e system

$$x(t+1) = Ax(t) + Bf(Cx(t)) + v(t). \tag{3.1}$$

It is convenient to refer to (3.1) as the system (A, B, C, f). The *behavior* of (3.1), denoted by $\mathcal{B}(A, B, C, f)$, is defined as the set of all $(v, x) \in (\mathbb{F}^m)^{\mathbb{N}_0} \times (\mathbb{F}^n)^{\mathbb{N}_0}$ satisfying (3.1). Initially, we will be interested in the stability properties of the *unforced* Lur'e system (3.1) (that is, system (3.1) with $v = 0$), and we define

$$\mathcal{B}_0(A, B, C, f) = \{x \in (\mathbb{F}^n)^{\mathbb{N}_0} : (0, x) \in \mathcal{B}(A, B, C, f)\}.$$

An immediate consequence of these definitions of behaviors is the following lemma.

LEMMA 8. *Let $(A, B, C) \in \Sigma(m, n, p; \mathbb{F})$ and let $f : \mathbb{F}^p \to \mathbb{F}^m$ be a nonlinearity. The following statements hold:*

(1) *$(v, x) \in \mathcal{B}(A, B, C, f)$ if, and only if, $(v, f(Cx), x, Cx) \in \mathcal{B}(A, B, C)$.*
(2) *$x \in \mathcal{B}_0(A, B, C, f)$ if, and only if, $(0, f(Cx), x, Cx) \in \mathcal{B}(A, B, C)$.*

We now define three basic stability concepts for the unforced Lur'e system (3.1).

DEFINITION. *We say that the Lur'e system (A, B, C, f) is* globally stable *if there exists $c > 0$ such that*

$$\|x(t)\| \le c\,\|x(0)\| \quad \forall\, t \in \mathbb{N}_0,\ \ \forall\, x \in \mathcal{B}_0(A, B, C, f);$$

globally asymptotically stable *if it is globally stable and if*

$$\lim_{t\to\infty} x(t) = 0 \quad \forall\, x \in \mathcal{B}_0(A, B, C, f); \textit{and}$$

globally exponentially stable *if there exists $c > 0$ and $a \in (0, 1)$ such that*

$$\|x(t)\| \le ca^t\,\|x(0)\| \quad \forall\, t \in \mathbb{N}_0,\ \ \forall\, x \in \mathcal{B}_0(A, B, C, f).$$

We are now in the position to state and prove a version of the circle criterion, subequently referred to as the Aizerman version. This result shows in particular that, over the complex field, the following generalization of the Aizerman conjecture holds: if, for a linear system (A, B, C), there exists a matrix K and $r > 0$ such that, for all *complex* matrices F with $\|F - K\| < r$, the linear Lur'e system (A, B, C, F) is asymptotically stable (or, equivalently, $A + BFC$ is Schur), then the Lur'e system (A, B, C, f) is globally asymptotically stable for every continuous nonlinearity f satisfying $\|f(\xi) - K\xi\| < r\|\xi\|$ for all $\xi \ne 0$.

THEOREM 9 (circle criterion—Aizerman version). *Let $(A, B, C) \in \Sigma(m, n, p; \mathbb{F})$ with transfer function $\mathbf{G}$, $K \in \mathbb{F}^{m\times p}$, $r > 0$, and let $f : \mathbb{F}^p \to \mathbb{F}^m$ be a nonlinearity. Assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ and condition* (A) *is satisfied. Then the following statements hold:*

(1) *If*

$$\|f(\xi) - K\xi\| \le r\,\|\xi\| \quad \forall\, \xi \in \mathbb{F}^p, \tag{3.2}$$

then (A, B, C, f) is globally stable.

(2) *If f is continuous and*

$$\|f(\xi) - K\xi\| < r\,\|\xi\| \quad \forall\, \xi \in \mathbb{F}^p\backslash\{0\}, \tag{3.3}$$

then (A, B, C, f) is globally asymptotically stable.

(3) *If there exists* $\delta \in (0, r)$ *such that*

$$\|f(\xi) - K\xi\| \leq (r - \delta)\,\|\xi\| \quad \forall\, \xi \in \mathbb{F}^p, \tag{3.4}$$

then (A, B, C, f) *is globally exponentially stable.*

In view of the "gap" between conditions (3.3) and (3.4), it seems natural to consider an "intermediate" condition, namely, the existence of a function $\alpha \in \mathcal{K}_\infty$ such that

$$\|f(\xi) - K\xi\| \leq r\|\xi\| - \alpha(\|\xi\|) \quad \forall\, \xi \in \mathbb{F}^p. \tag{3.5}$$

One of the outcomes of the next section (section 4) is that if (3.5) holds with $\alpha \in \mathcal{K}_\infty$, then the Lur'e system (3.1) is ISS (provided that the linear system (A, B, C) satisfies the conditions of Theorem 9), but the unforced system may not be globally exponentially stable. Furthermore, an example in section 4 shows that condition (3.5) with $\alpha \in \mathcal{K}$ is too "weak" to guarantee ISS.

We remark that Theorem 9 can be "rephrased" in terms of positive-real and sector conditions; see Corollary 11. Furthermore, we note that statement (2) of Theorem 9 is reminiscent of a continuous-time stability radius result (complex Aizerman conjecture) by Hinrichsen and Pritchard; see [14, 15]. The proof below makes use of ideas from [14, 15].

Proof of Theorem 9. By Corollary 7, there exists a positive semi-definite matrix $P = P^* \in \mathbb{F}^{n\times n}$ such that the function $V : \mathbb{F}^n \to [0, \infty)$ given by $V(\xi) = \langle P\xi, \xi\rangle$ satisfies

$$V(x(t+1)) - V(x(t)) \leq -r^2\,\|y(t)\|^2 + \|u(t) - Ky(t)\|^2$$

for all $t \in \mathbb{N}_0$ and all $(0, u, x, y) \in \mathcal{B}(A, B, C)$. Invoking Lemma 8, we conclude that

$$\left.\begin{aligned} V(x(t+1) - V(x(t)) \leq -r^2\,\|Cx(t)\|^2 + \|f(Cx(t)) - KCx(t)\|^2 \\ \forall\, t \in \mathbb{N}_0,\ \ \forall\, x \in \mathcal{B}_0(A, B, C, f). \end{aligned}\right\} \tag{3.6}$$

Moreover, another application of Corollary 7 allows us to conclude that there exists a projection $\Pi : \mathbb{F}^n \to \mathbb{F}^n$ such that $\ker \Pi \subseteq \ker C$ and

$$c_1\,\|\Pi\xi\|^2 \leq V(\xi) \leq c_2\,\|\xi\|^2 \quad \forall\, \xi \in \mathbb{F}^n,$$

where c_1 and c_2 are suitable positive constants.

To prove statement (1), we note that, by (3.2) and (3.6), the function $t \mapsto V(x(t))$ is nonincreasing. Consequently, for every $x \in \mathcal{B}_0(A, B, C, f)$, we have that

$$c_1\,\|\Pi x(t)\|^2 \leq V(x(t)) \leq V(x(0)) \leq c_2\,\|x(0)\|^2 \quad \forall\, t \in \mathbb{N}_0.$$

As $\ker \Pi \subseteq \ker C$, it follows that $C\Pi = C$ and so, for every $x \in \mathcal{B}_0(A, B, C, f)$,

$$\|Cx(t)\| = \|C\Pi x(t)\| \leq \|C\|\,\|\Pi x(t)\| \leq \|C\|\sqrt{\frac{c_2}{c_1}}\,\|x(0)\| \quad \forall\, t \in \mathbb{N}_0.$$

Combining this with (3.2), we obtain that, for every $x \in \mathcal{B}_0(A, B, C, f)$,

$$\|f(Cx(t)) - KCx(t)\| \leq \rho\,\|x(0)\| \quad \forall\, t \in \mathbb{N}_0,$$

where $\rho := r\,\|C\|\,\sqrt{c_2/c_1}$. By hypothesis, $\mathbf{G}^K \in H^\infty_{p\times m}$, and it follows from assumption (A) that A_K is Schur. Furthermore, note that, by Lemma 8 and statement (1) of Lemma 6, for every $x \in \mathcal{B}_0(A,B,C,f)$, we have $(0, f(Cx) - KCx, x, Cx) \in \mathcal{B}(A_K, B, C)$. It follows that, for every $x \in \mathcal{B}_0(A,B,C,f)$,

$$\|x(t)\| \le c_3\|x(0)\| + c_4 \max_{0\le s\le t} \|f(Cx(s)) - KCx(s)\| \quad \forall\, t \in \mathbb{N}_0,$$

where c_3 and c_4 are suitable positive constants. Consequently, setting $c = c_3 + \rho c_4$, we have $\|x(t)\| \le c\|x(0)\|$ for all $t \in \mathbb{N}_0$ and all $x \in \mathcal{B}_0(A,B,C,f)$, completing the proof of statement (1).

We proceed to prove statement (2). Global stability follows of course from statement (1). We need to show global attractivity of 0. To this end, let $x \in \mathcal{B}_0(A,B,C,f)$. It is sufficient to show that

$$\lim_{t\to\infty} Cx(t) = 0. \tag{3.7}$$

Indeed, if (3.7) holds, then $f(Cx(t)) - KCx(t) \to 0$ as $t \to \infty$, which combined with the asymptotic stability of A_K and the fact that $(0, f(Cx) - KCx, x, Cx) \in \mathcal{B}(A_K, B, C)$ implies $x(t) \to 0$ as $t \to \infty$.

To establish (3.7), write $y = Cx$ and note that, by global stability, y is bounded. Consequently, the omega limit set Ω of y is nonempty and $\operatorname{dist}(y(t), \Omega) \to 0$ as $t \to \infty$. It remains to show that $\Omega = \{0\}$. To this end let $\xi \in \Omega$. Then there exists a sequence (t_k) in $\mathbb{N}_0$ such that $t_k \to \infty$ and $y(t_k) \to \xi$ as $k \to \infty$. Since $t \mapsto V(x(t))$ is a nonnegative nonincreasing function, the limit $\lim_{t\to\infty} V(x(t))$ exists. In particular $(V(x(t_k+1)) - V(x(t_k))) \to 0$ as $k \to \infty$. Invoking (3.6) for $t = t_k$, letting $k \to \infty$, and using the continuity of f yields the inequality

$$0 \le \|f(\xi) - K\xi\|^2 - r^2\,\|\xi\|^2 .$$

Together with (3.3) this implies $\xi = 0$, completing the proof of statement (2).

To prove statement (3), note that, by (3.4) and (3.6), we have, for every $x \in \mathcal{B}_0(A,B,C,f)$,

$$V(x(t+1)) - V(x(t)) \le (\delta^2 - 2\delta r)\,\|Cx(t)\|^2 \le -\delta^2\,\|Cx(t)\|^2 \quad \forall\, t \in \mathbb{N}_0.$$

Since, by assumption (A), (A,B,C) is detectable, Lemma 5 guarantees the existence of a positive-definite matrix $Q \in \mathbb{F}^{n\times n}$ and a constant $\varepsilon > 0$ such that the function $U : \mathbb{F}^n \to [0,\infty)$ satisfies

$$U(x(t+1)) - U(x(t)) \le -\varepsilon\,\|x(t)\|^2 + \|y(t)\|^2 + \|u(t)\|^2$$

for all $t \in \mathbb{N}_0$ and for all $(0,u,x,y) \in \mathcal{B}(A,B,C)$. Therefore, by Lemma 8,

$$\begin{aligned} U(x(t+1)) - U(x(t)) &\le -\varepsilon\,\|x(t)\|^2 + \|Cx(t)\|^2 + \|f(Cx(t))\|^2 \\ &\le -\varepsilon\,\|x(t)\|^2 + (1 + 2(r-\delta)^2 + 2\,\|K\|^2)\,\|Cx(t)\|^2 \end{aligned}$$

for all $t \in \mathbb{N}_0$ and for all $x \in \mathcal{B}_0(A,B,C,f)$. Setting $\mu := (1+2(r-\delta)^2+2\,\|K\|^2)/\delta^2 > 0$, the function $W : \mathbb{F}^n \to [0,\infty)$ defined by $W(\xi) := U(\xi) + \mu V(\xi) = \langle (Q+\mu P)\xi, \xi\rangle$ satisfies, for all $x \in \mathcal{B}_0(A,B,C,f)$,

$$W(x(t+1)) - W(x(t)) \le -\varepsilon\,\|x(t)\|^2 \quad \forall\, t \in \mathbb{N}_0.$$

Obviously, $Q + \mu P$ is positive definite and so, $\sqrt{W(\xi)}$ defines a norm on $\mathbb{F}^n$. Consequently, there exist positive constants c_1 and c_2 such that $c_1 \|\xi\|^2 \le W(\xi) \le c_2 \|\xi\|^2$ for all $\xi \in \mathbb{F}^n$. Hence, for every $x \in \mathcal{B}_0(A, B, C, f)$,

$$W(x(t+1)) \le \left(1 - \frac{\varepsilon}{c_2}\right) W(x(t)) \quad \forall\, t \in \mathbb{N}_0.$$

Setting $c := \sqrt{c_2/c_1}$ and $a := \sqrt{1 - \varepsilon/c_2}$, we obtain that, for every $x \in \mathcal{B}_0(A, B, C, f)$,

$$\|x(t)\| \le ca^t \|x(0)\| \quad \forall\, t \in \mathbb{N}_0,$$

completing the proof of statement (3). ∎

It is well-known that if the linear system (A, B, C) is asymptotically stable (that is, A is a Schur matrix), then there exists a "destabilizing" output feedback matrix $F \in \mathbb{C}^{m\times p}$ of minimal norm, that is, $A + BFC$ is not Schur (or equivalently, not asymptotically stable) and $\|F\| = r_{\mathbb{C}}(A; B, C)$. Note that, in general, F will be complex, even if (A, B, C) is real. The following result, which, somewhat surprisingly, does not seem to be available in the literature, shows that the application of a destabilizing output feedback matrix of minimal norm results in a marginally stable closed-loop system.

Corollary 10. *Let $(A, B, C) \in \Sigma(m, n, p; \mathbb{C})$ with A a Schur matrix and assume that $r_{\mathbb{C}}(A; B, C) < \infty$. Let $F \in \mathbb{C}^{m\times p}$ be such that $A + BFC$ is not Schur and $\|F\| = r_{\mathbb{C}}(A; B, C)$. Then $\rho(A+BFC) = 1$ and all $\lambda \in \sigma(A+BFC)$ with $|\lambda| = 1$ are semisimple.*

Proof. We proceed in two steps. We first prove the result under the assumption that (A, B, C) is canonical and then remove this assumption in the second step

Step 1. Assume that (A, B, C) is canonical. Thus, condition (A) is satisfied. Since $A + BFC$ is not Schur, it is clear that $\rho(A + BFC) \ge 1$. With $r := r_{\mathbb{C}}(A; B, C)$, we have that $\mathbb{B}(0, r) \subset \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, where $\mathbf{G}$ denotes the transfer function of (A, B, C). Defining $f : \mathbb{C}^p \to \mathbb{C}^m$ by $f(\xi) := F\xi$ for all $\xi \in \mathbb{C}^p$, an application of statement (1) of Theorem 9 (with $\mathbb{F} = \mathbb{C}$) shows that $\rho(A + BFC) \le 1$ and all $\lambda \in \sigma(A + BFC)$ with $|\lambda| = 1$ are semisimple.

Step 2. If (A, B, C) is not canonical, then there exists an invertible matrix $T \in \mathbb{C}^{n\times n}$ such that

$$T^{-1}AT = \begin{pmatrix} A_{11} & A_{12} & 0 \\ 0 & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{pmatrix}, \quad T^{-1}B = \begin{pmatrix} B_{11} \\ 0 \\ B_{31} \end{pmatrix}, \quad CT = (C_{11}, C_{12}, 0)\,,$$

where the triple (A_{11}, B_{11}, C_{11}) is canonical (see, for example, [30, Lemma 6.5.1]). Since A is Schur, the matrices A_{11}, A_{22} and A_{33} are Schur and, furthermore,

$$\mathbf{G}(z) = C(zI - A)^{-1}B = C_{11}(zI - A_{11})^{-1}B_{11}. \tag{3.8}$$

Now

$$T^{-1}(A + BFC)T = \begin{pmatrix} A_{11} + B_{11}FC_{11} & * & 0 \\ 0 & A_{22} & 0 \\ * & * & A_{33} \end{pmatrix}, \tag{3.9}$$

and so,

$$\sigma(A + BFC) = \sigma(A_{11} + B_{11}FC_{11}) \cup \sigma(A_{22}) \cup \sigma(A_{33}). \tag{3.10}$$

Hence, $A_{11}+B_{11}FC_{11}$ is not Schur and, by (3.8), $\|F\| = r_{\mathbb{C}}(A_{11}; B_{11}, C_{11})$. Therefore, by what has already been proved in Step 1, $\rho(A_{11}+B_{11}FC_{11}) = 1$ and all eigenvalues $\lambda \in \sigma(A_{11}+B_{11}FC_{11})$ with $|\lambda| = 1$ are semisimple. Since A_{22} and A_{33} are Schur, it now follows from (3.9) and (3.10) that $\rho(A+BFC) = 1$ and all $\lambda \in \sigma(A+BF)$ with $|\lambda| = 1$ are semisimple. ∎

We now use Theorem 9 to derive a version of the circle criterion which is formulated in terms of positive real and sector conditions.

COROLLARY 11. *Let* $(A, B, C) \in \Sigma(m, n, p; \mathbb{F})$ *with transfer function* $\mathbf{G}$, $K_1, K_2 \in \mathbb{F}^{m\times p}$ *and let* $f : \mathbb{F}^p \to \mathbb{F}^m$ *be a nonlinearity. Assume that* $\mathbf{H} := (I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1}$ *is positive real and that either* (i) (A, B, C) *is canonical or* (ii) (A, B, C) *is semi-canonical and there exists* $\theta \in [0, 2\pi)$ *such that* $\operatorname{Re}\mathbf{H}(e^{i\theta})$ *is positive definite. Then the following statements hold:*

(1) *If* $\ker(K_1 - K_2) = \{0\}$ *and*

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \le 0 \quad \forall\, \xi \in \mathbb{F}^p, \tag{3.11}$$

then (A, B, C, f) *is globally stable.*

(2) *If* f *is continuous and*

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle < 0 \quad \forall\, \xi \in \mathbb{F}^p\backslash\{0\}, \tag{3.12}$$

then (A, B, C, f) *is globally asymptotically stable.*

(3) *If there exists* $\delta > 0$ *such that*

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \le -\delta\|\xi\|^2 \quad \forall\, \xi \in \mathbb{F}^p, \tag{3.13}$$

then (A, B, C, f) *is globally exponentially stable.*

Statement (2) of Corollary 11 is reminiscent of classical absolute stability results as presented, for example, in [10, 11], where it is assumed that (A, B, C) is canonical. Note that, in [10, 11], global asymptotic stability is shown under the assumptions of strict positive realness[2] of $\mathbf{H} = (I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1}$ and sector-boundedness of f in the sense of (3.11).

Proof of Corollary 11. Setting

$$L := \frac{1}{2}(K_1 - K_2), \quad M := \frac{1}{2}(K_1 + K_2) \tag{3.14}$$

it follows that

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle = \|f(\xi) - M\xi\|^2 - \|L\xi\|^2 \quad \forall\, \xi \in \mathbb{F}^p. \tag{3.15}$$

Thus, if (3.12) or (3.13) holds, then $\ker L = \{0\}$, so in all statements (1)–(3) we have $\ker L = \{0\}$. Hence, L^*L is invertible, and $L^\sharp := (L^*L)^{-1}L^* \in \mathbb{F}^{p\times m}$ is a left-inverse of L. A routine calculation shows that

$$\mathbf{H} = (I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1} = I + 2L\mathbf{G}^{K_1} = I + 2\mathbf{F}, \quad \text{where } \mathbf{F} := L\mathbf{G}^{K_1}.$$

By hypothesis, $I + 2\mathbf{F}$ is positive real and therefore, invoking Corollary 2, we conclude that $\mathbf{F}(I + \mathbf{F})^{-1} = (I - \mathbf{H})(I + \mathbf{H})^{-1}$ is contractive. The identity

$$\mathbf{F}^{(-LL^\sharp)} = \mathbf{F}(I + LL^\sharp\mathbf{F})^{-1} = \mathbf{F}(I + L\mathbf{G}^{K_1})^{-1} = \mathbf{F}(I + \mathbf{F})^{-1}$$

[2] The transfer function $\mathbf{H}$ is said to be *strictly positive real* if all poles of $\mathbf{H}$ are in $\mathbb{D}$ and $\operatorname{Re}\mathbf{H}(e^{i\theta})$ is positive definite for all $\theta \in [0, 2\pi)$.

shows that $\mathbf{F}^{(-LL^\sharp)}$ is also contractive. Consequently, by statement (5) of Lemma 6,

$$\mathbb{B}_{\mathbb{C}}(-LL^\sharp, 1) \subset \mathbb{S}_{\mathbb{C}}(\mathbf{F}). \tag{3.16}$$

It is clear that $\mathbf{F}$ is the transfer function of (A_{K_1}, B, LC), where $A_{K_1} := A + BK_1C$. Note that if (A, B, C) is canonical (semi-canonical), then (A_{K_1}, B, LC) is canonical (semi-canonical). Furthermore, if there exists $\theta \in [0, 2\pi)$ such that $\operatorname{Re}\mathbf{H}(e^{i\theta})$ is positive definite, then, invoking Lemma 1, it follows that

$$\|\mathbf{F}^{(-LL^\sharp)}(e^{i\theta})\| = \|\mathbf{F}(e^{i\theta})(I + \mathbf{F}(e^{i\theta}))^{-1}\| = \|\big(I - \mathbf{H}(e^{i\theta})\big)\big((I + \mathbf{H}(e^{i\theta})\big)^{-1}\| < 1,$$

whence

$$\min_{|z|=1} \|\mathbf{F}^{(-LL^\sharp)}(z)\| < 1.$$

We now conclude that that assumption (A) holds in the context given by the linear system (A_{K_1}, B, LC), the feedback gain $K = -LL^\sharp$, and the radius $r = 1$. Furthermore, defining $g : \mathbb{F}^m \to \mathbb{F}^m$ by

$$g(\xi) := f(L^\sharp\xi) - K_1L^\sharp\xi \quad \forall\, \xi \in \mathbb{F}, \tag{3.17}$$

it follows that

$$\|g(\xi) + LL^\sharp\xi\| = \|f(L^\sharp\xi) - (K_1 - L)L^\sharp\xi\| = \|f(L^\sharp\xi) - ML^\sharp\xi\| \quad \forall\, \xi \in \mathbb{F}^m. \tag{3.18}$$

Hence, by (3.15),

$$\|g(\xi) + LL^\sharp\xi\|^2 = \operatorname{Re}\langle f(L^\sharp\xi) - K_1L^\sharp\xi, f(L^\sharp\xi) - K_2L^\sharp\xi\rangle + \|LL^\sharp\xi\|^2 \quad \forall\, \xi \in \mathbb{F}^m.$$

Now $LL^\sharp \in \mathbb{F}^{m\times m}$ is the orthogonal projection onto $\operatorname{im} L$ along $(\operatorname{im} L)^\perp$ and therefore $\|LL^\sharp\| = 1$. Consequently,

$$\|g(\xi) + LL^\sharp\xi\|^2 \le \|\xi\|^2 + \operatorname{Re}\langle f(L^\sharp\xi) - K_1L^\sharp\xi, f(L^\sharp\xi) - K_2L^\sharp\xi\rangle \quad \forall\, \xi \in \mathbb{F}^m. \tag{3.19}$$

We also note that

$$\mathcal{B}_0(A, B, C, f) = \mathcal{B}_0(A_{K_1}, B, LC, g). \tag{3.20}$$

The key step is now to apply Theorem 9 to the unforced Lur'e system (A_{K_1}, B, LC, g).

To prove statement (1), note that, by (3.11) and (3.19), $\|g(\xi) + LL^\sharp\xi\| \le \|\xi\|$ for all $\xi \in \mathbb{F}^p$. Since (3.16) holds, we may apply statement (1) of Theorem 9 to conclude global stability of (A_{K_1}, B, LC, g). Global stability of (A, B, C, f) follows from (3.20).

A similar argument will show that statement (2) holds, provided we can prove that

$$\|g(\xi) + LL^\sharp\xi\| < \|\xi\| \quad \forall\, \xi \in \mathbb{F}^m\backslash\{0\}. \tag{3.21}$$

For $\xi \notin \ker L^\sharp$, the above strict inequality follows from (3.12) and (3.19). Moreover, for $\xi \in \ker L^\sharp\backslash\{0\}$, the strict inequality also holds because for such ξ the left-hand side is equal to 0. Thus, (3.21) is satisfied.

We proceed to prove statement (3). Invoking (3.13) and (3.19), we see that

$$\|g(\xi) + LL^\sharp\xi\|^2 \le \|\xi\|^2 - \delta\|L^\sharp\xi\|^2 \quad \forall\, \xi \in \mathbb{F}^m.$$

Obviously, there exists $\lambda > 0$ such that

$$\|L^\sharp \xi\| \geq \lambda \|\xi\| \quad \forall\, \xi \in (\ker L^\sharp)^\perp, \tag{3.22}$$

and consequently,

$$\|g(\xi) + LL^\sharp \xi\|^2 \leq (1 - \delta\lambda^2)\|\xi\|^2 \quad \forall\, \xi \in (\ker L^\sharp)^\perp.$$

For arbitrary $\xi \in \mathbb{F}^m$, we have that $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \ker L^\sharp$ and $\xi_2 \in (\ker L^\sharp)^\perp$ and so,

$$\|g(\xi) + LL^\sharp \xi\| = \|g(\xi_2) + LL^\sharp \xi_2\| \leq \sqrt{1 - \delta\lambda^2}\|\xi_2\| \leq \sqrt{1 - \delta\lambda^2}\|\xi\|.$$

Invoking an argument similar to that used in the proof of statement (1) yields the claim. ∎

Since Theorem 9 plays a key role in the proof of Corollary 11 and since Theorem 9 was derived using Lyapunov theory, Corollary 11 ultimately rests on Lyapunov arguments. It is interesting to note that there is an alternative proof of statement (3) of Theorem 9, and hence of statement (3) of Corollary 11, based on small-gain and exponential weighting ideas [28]. (In a continuous-time setting, these ideas have been used in the proof of [17, Theorem 3.4].) This proof is more elementary and conceptually simpler than the proof based on Lyapunov arguments and, furthermore, generalizes to infinite-dimensional contexts. (See [16] for the continuous-time case.) However, we emphasize that the above comment is restricted to the proof of statements (3) of Theorem 9 and Corollary 11; it seems that for the proofs of the first two statements, Lyapunov theory is indispensable.

4. The circle criterion and ISS.

We say that the Lur'e system (3.1), determined by (A, B, C, f), is ISS if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma\left(\max_{0 \leq s \leq t} \|v(s)\|\right) \quad \forall\, (v, x) \in \mathcal{B}(A, B, C, f),\ \forall\, t \in \mathbb{N}_0. \tag{4.1}$$

The following corollary, an immediate consequence of [20, Lemma 3.5], shows that the existence of a so-called *ISS-Lyapunov function* implies ISS of (A, B, C, f).

COROLLARY 12. *Let $(A, B, C) \in \Sigma(m, n, p; \mathbb{F})$ and let $f : \mathbb{F}^p \to \mathbb{F}^m$ be a nonlinearity. If there exist a continuous function $V : \mathbb{F}^n \to [0, \infty)$ and $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, 3, 4$, such that*

$$V(x(t+1)) - V(x(t)) \leq -\alpha_1(\|x(t)\|) + \alpha_2(\|v(t)\|)\ \forall\, (v, x) \in \mathcal{B}(A, B, C, f),\ \forall\, t \in \mathbb{N}_0,$$

and $\alpha_3(\|\xi\|) \leq V(\xi) \leq \alpha_4(\|\xi\|)$ for all $\xi \in \mathbb{F}^n$, then the Lur'e system (A, B, C, f) is ISS. Furthermore, the functions β and γ in (4.1) depend only on the functions α_i, $i = 1, 2, 3, 4$.

A function V with the above properties is said to be an ISS-Lyapunov function for the Lur'e system (A, B, C, f). In [20], the underlying system is assumed to be real. Extensions to complex systems are, however, straightforward.

We now state the main result of this paper: an Aizerman version of the circle criterion guaranteeing ISS.

Theorem 13. *Let* $(A,B,C) \in \Sigma(m,n,p;\mathbb{F})$ *with transfer function* $\mathbf{G}$, $K \in \mathbb{F}^{m\times p}$, $r > 0$, *and* $\alpha \in \mathcal{K}_\infty$. *Assume that* $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ *and condition* (A) *holds. Then there exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *such that, for every nonlinearity* $f : \mathbb{F}^p \to \mathbb{F}^m$ *satisfying*

$$\|f(\xi) - K\xi\| \le r\,\|\xi\| - \alpha(\|\xi\|) \quad \forall\, \xi \in \mathbb{F}^p, \tag{4.2}$$

estimate (4.1) *holds. In particular, the Lur'e system* (A,B,C,f) *is ISS.*

In the special case $\alpha(s) = \delta s$ (for some positive δ), (4.2) is the same as (3.4). Therefore, under the conditions of statement (3) of Theorem 9, we not only have global exponential stability but also ISS. The example below shows that there exist (A,B,C,f), K, and r such that $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ and (4.2) holds for some nonlinear $\alpha \in \mathcal{K}_\infty$, implying ISS by Theorem 13, but (A,B,C,f) is not globally exponentially stable, and so, in particular, by statement (3) of Theorem 9, (4.2) fails to hold for any linear $\alpha(s) = \delta s$ with $\delta > 0$.

Example. Consider the one-dimensional system $(0,1,1)$ which has transfer function $\mathbf{G}(z) = 1/z$. Trivially, $(0,1,1)$ is canonical, and so, condition (A) holds. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(\xi) = \operatorname{sign}(\xi)\log(1+|\xi|)$ for $\xi \in \mathbb{R}$. Choosing $K = 0$ and $r = 1$, it is obvious that $\mathbb{B}_{\mathbb{C}}(K,r) = \mathbb{B}_{\mathbb{C}}(0,1) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$. Additionally, (4.2) holds with $\alpha \in \mathcal{K}_\infty$ given by $\alpha(s) = s - \log(1+s)$ for $s \ge 0$, and thus, by Theorem 13, the feedback system $(0,1,1,f)$ is ISS. Furthermore, since $f'(0) = 1$, it is clear that (4.2) is not satisfied for any $\alpha \in \mathcal{K}_\infty$ of the form $\alpha(s) = \delta s$ with $\delta > 0$. It is not difficult to show that the unforced feedback system $x(t+1) = f(x(t))$ is not globally exponentially stable. □

The following example shows that, in Theorem 13, the assumption that $\alpha \in \mathcal{K}_\infty$ cannot be replaced by the weaker assumption $\alpha \in \mathcal{K}$.

Example. Consider again the one-dimensional system $(0,1,1)$. Choose $K = 0$ and $r = 1$. Then $\mathbf{G}(z) = 1/z$ and $\mathbb{B}_{\mathbb{C}}(K,r) = \mathbb{B}_{\mathbb{C}}(0,1) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$. Let $f : \mathbb{R} \to \mathbb{R}$ be the deadzone nonlinearity given by

$$f(\xi) := \begin{cases} \xi + 1 & \text{if } \xi < -1, \\ 0 & \text{if } -1 \le |\xi| \le 1, \\ \xi - 1 & \text{if } \xi > 1. \end{cases}$$

Obviously, for every $\alpha \in \mathcal{K}$ such that $\alpha(s) \le \min\{s,1\}$ for all $s \ge 0$,

$$|f(\xi)| \le |\xi| - \alpha(|\xi|) \quad \forall\, \xi \in \mathbb{R}. \tag{4.3}$$

Note that there does not exist $\alpha \in \mathcal{K}_\infty$ such that (4.3) holds. Trivially, condition (A) is satisfied, and so, by statement (2) of Theorem 9, the Lur'e system $(0,1,1,f)$ is globally asymptotically stable. But $(0,1,1,f)$ is not ISS, because there exist bounded input signals v which lead to unbounded state signals x. For example, taking $v(t) = 2$ for all $t \in \mathbb{N}_0$ and $x(0) \ge 1$, we have $x(t+1) = x(t) + 1$ for all $t \in \mathbb{N}_0$, and thus $x(t) = x(0) + t$ for all $t \in \mathbb{N}_0$. □

We proceed to state a lemma on $\mathcal{K}_\infty$ functions which will facilitate the proof of Theorem 13.

Lemma 14. *Let* $\alpha \in \mathcal{K}_\infty$. *The following statements hold:*

(1) *There exists* $\gamma \in \mathcal{K}_\infty$ *such that*

$$s_1 s_2 \le s_1\alpha(s_1) + \gamma(s_2) \quad \forall s_1, s_2 \ge 0.$$

(2) *For every* $\varepsilon > 0$,

$$\alpha(s_1 + s_2) \le \alpha((1+\varepsilon)s_1) + \alpha((1+\varepsilon^{-1})s_2) \quad \forall\, s_1, s_2 \ge 0.$$

(3) *Define* $\tilde{\alpha} \in \mathcal{K}_\infty$ *by* $\tilde{\alpha}(s) := \sqrt{s}\,\alpha(\sqrt{s})$. *For every* $\varepsilon > 0$, *there exists* $\eta \in \mathcal{K}_\infty$ *such that*

$$\tilde{\alpha}(s_1 - s_2) \le \tilde{\alpha}((1+\varepsilon)s_1) - \eta(s_2) \quad \forall\, s_1 \ge s_2 \ge 0$$

and $\eta(s)/\sqrt{s} \to \infty$ *as* $s \to \infty$.

We relegate the proof of Lemma 14 to the appendix and proceed to prove Theorem 13.

Proof of Theorem 13. By Corollary 12, it suffices to construct an ISS-Lyapunov function for the Lur'e system (A, B, C, f). We do this by constructing two functions U and V and then showing that $U + V$ is an ISS-Lyapunov function.

By Lemma 5, there exists a positive definite matrix $Q = Q^* \in \mathbb{F}^{n\times n}$ and $\delta > 0$ such that the function $V_1 : \mathbb{F}^n \to [0,\infty)$ given by $V_1(\xi) = \langle Q\xi, \xi\rangle$ satisfies

$$V_1(x(t+1)) - V_1(x(t)) \le -\delta \|x(t)\|^2 + \|y(t)\|^2 + \|u(t)\|^2 + \|v(t)\|^2$$

for all $t \in \mathbb{N}_0$ and all $(v, u, x, y) \in \mathcal{B}(A, B, C)$. Set

$$c_1 := 1 + (r + \|K\|)^2,$$

and define $V_2 : \mathbb{F}^n \to [0,\infty)$ by $V_2(\xi) = V_1(\xi)/c_1$. Since, by Corollary 8, $(v, x) \in \mathcal{B}(A, B, C, f)$ if, and only if, $(v, f(Cx), x, Cx) \in \mathcal{B}(A, B, C)$, we can use (4.3) to estimate

$$\left.\begin{aligned} V_2(x(t+1)) - V_2(x(t)) \le -\delta_1 \|x(t)\|^2 + \|Cx(t)\|^2 + \|v(t)\|^2 \\ \forall\, t \in \mathbb{N}_0,\ \forall\,(v,x) \in \mathcal{B}(A,B,C,f) \end{aligned}\right\}, \tag{4.4}$$

where $\delta_1 := \delta/c_1$. Let $c_2 \ge \delta_1$ be such that $V_2(\xi) \le c_2\|\xi\|^2$ for all $\xi \in \mathbb{F}^n$ and choose $b > 1$ such that

$$a := b\left(1 - \frac{\delta_1}{2c_2}\right) < 1.$$

Note that $a \ge b/2 > 0$. Define $\alpha_1 \in \mathcal{K}_\infty$ by

$$\alpha_1(s) = \frac{r}{2}\sqrt{s}\,\alpha(\sqrt{s}) \quad \forall\, s \ge 0. \tag{4.5}$$

Statement (2) of Lemma 14 guarantees the existence of a number $k > 1$ such that

$$\alpha_1(as_1 + s_2) \le \alpha_1(s_1) + \alpha_1(ks_2) \quad \forall\, s_1, s_2 \ge 0. \tag{4.6}$$

Furthermore, by statement (3) of Lemma 14, there exists $\eta \in \mathcal{K}_\infty$ such that

$$\alpha_1(s_1 - s_2) \le \alpha_1(bs_1) - \eta(s_2) \quad \forall s_1 \ge s_2 \ge 0 \tag{4.7}$$

and

$$\frac{\eta(s)}{\sqrt{s}} \to \infty \quad \text{as } s \to \infty. \tag{4.8}$$

We define a function $U : \mathbb{F}^n \to [0,\infty)$ by $U(\xi) := \alpha_1(cV_2(\xi))$ for all $\xi \in \mathbb{F}^n$, where

$$c := \frac{1}{2bk}. \tag{4.9}$$

Now we use (4.4) and (4.7) to estimate

$$\begin{aligned} U(x(t+1)) \leq{}& \alpha_1\big(bc\big(V_2(x(t)) - \delta_1 \|x(t)\|^2/2 + \|Cx(t)\|^2 + \|v(t)\|^2\big)\big) \\ &- \eta\big(c\delta_1 \|x(t)\|^2/2\big) \quad \forall\, t \in \mathbb{N}_0,\ \forall\,(v,x) \in \mathcal{B}(A,B,C,f). \end{aligned}$$

Invoking (4.8), we may conclude that there exists $\mu \in \mathcal{K}_\infty$ such that

$$\eta(c\delta_1 s^2/2) \geq s\mu(s) \quad \forall\, s \geq 0.$$

Moreover, since $V_2(\xi) \leq c_2 \|\xi\|^2$ for all $\xi \in \mathbb{F}^n$, we have

$$b\left(V_2(\xi) - \frac{\delta_1}{2}\|\xi\|^2\right) \leq aV_2(\xi) \quad \forall\, \xi \in \mathbb{F}^n.$$

Therefore,

(4.10)
$$\left.\begin{aligned} U(x(t+1)) \leq \alpha_1\big(acV_2(x(t)) + bc\|Cx(t)\|^2 + bc\|v(t)\|^2\big) - \|x(t)\|\mu(\|x(t)\|) \\ \forall\, t \in \mathbb{N}_0,\ \forall\,(v,x) \in \mathcal{B}(A,B,C,f) \end{aligned}\right\}.$$

Using (4.6), (4.9), and the trivial inequality

$$\alpha_1(s_1 + s_2) \leq \alpha_1(2s_1) + \alpha(2s_2) \quad \forall\, s_1, s_2 \geq 0,$$

we obtain that, for all $t \in \mathbb{N}_0$ and all $(v,x) \in \mathcal{B}(A,B,C,f)$,

$$\begin{aligned} &\alpha_1\big(acV_2(x(t)) + bc\|Cx(t)\|^2 + bc\|v(t)\|^2\big) \\ &\quad\leq \alpha_1\big(cV_2(x(t))\big) + \alpha_1\big(bck\|Cx(t)\|^2 + bck\|v(t)\|^2\big) \\ &\quad\leq U(x(t)) + \alpha_1(\|Cx(t)\|^2) + \alpha_1(\|v(t)\|^2) \end{aligned}$$

for all $t \in \mathbb{N}_0$ and for all $(v,x) \in \mathcal{B}(A,B,C,f)$. Combined with (4.10) this yields

$$\left.\begin{aligned} U(x(t+1)) - U(x(t)) \leq \alpha_1\big(\|Cx(t)\|^2\big) + \alpha_1\big(\|v(t)\|^2\big) - \|x(t)\|\mu(\|x(t)\|) \\ \forall\, t \in \mathbb{N}_0,\ \forall\,(v,x) \in \mathcal{B}(A,B,C,f) \end{aligned}\right\}. \tag{4.11}$$

We shall now construct a quadratic form V with the property that $U + V$ is an ISS-Lyapunov function. By hypothesis, $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$. Invoking Corollary 7, there exists a positive semi-definite matrix $P = P^* \in \mathbb{F}^{n\times n}$ and a constant $\kappa > 0$ such that the function $V : \mathbb{F}^n \to \mathbb{R}_+$ given by $V(\xi) = \langle P\xi, \xi\rangle$ satisfies

$$\begin{aligned} V(x(t+1)) - V(x(t)) \leq{}& \|u(t) - Ky(t)\|^2 - r^2\|y(t)\|^2 \\ &+ \kappa\|v(t)\|\big(\|u(t) - Ky(t)\| + \|x(t)\| + \|v(t)\|\big) \end{aligned}$$

for all $t \in \mathbb{N}_0$ and for all $(v,u,x,y) \in \mathcal{B}(A,B,C)$. By Lemma 8, $(v, f(Cx), x, Cx) \in \mathcal{B}(A,B,C)$ if, and only if, $(v,x) \in \mathcal{B}(A,B,C,f)$. Hence, we have

$$\begin{aligned} V(x(t+1)) - V(x(t)) \leq{}& \|f(Cx(t)) - KCx(t)\|^2 - r^2\|Cx(t)\|^2 \\ &+ \kappa\|v(t)\|\big(\|f(Cx(t)) - KCx(t)\| + \|x(t)\| + \|v(t)\|\big) \end{aligned}$$

for all $t \in \mathbb{N}_0$ and all $(v,x) \in \mathcal{B}(A,B,C,f)$. Using (4.2), we estimate, for all $\xi \in \mathbb{F}^p$,

$$\begin{aligned} \|f(\xi) - K\xi\|^2 - r^2\|\xi\|^2 &\leq -2\alpha(\|\xi\|)r\|\xi\| + \alpha(\|\xi\|)^2 \\ &\leq -2\alpha(\|\xi\|)r\|\xi\| + \alpha(\|\xi\|)r\|\xi\| \\ &= -r\|\xi\|\,\alpha(\|\xi\|) \end{aligned}$$

and conclude that, for all $t \in \mathbb{N}_0$ and all $(v, x) \in \mathcal{B}(A, B, C, f)$,

$$V(x(t+1)) - V(x(t)) \leq -r \,\|Cx(t)\|\, \alpha(\|Cx(t)\|) + \kappa\|v(t)\|\big(r\|Cx(t)\| + \|x(t)\| + \|v(t)\|\big).$$

Statement (1) of Lemma 14 now guarantees the existence of a function $\gamma_1 \in \mathcal{K}_\infty$ such that

$$\kappa s_1 s_2 \leq \frac{1}{2} s_1 \alpha(s_1) + \gamma_1(s_2) \quad \forall\, s_1, s_2 \geq 0.$$

Thus, for all $t \in \mathbb{N}_0$ and for all $(v, x) \in \mathcal{B}(A, B, C, f)$,

$$\begin{aligned} V(x(t+1)) - V(x(t)) \leq {} & -\frac{r}{2} \,\|Cx(t)\|\, \alpha(\|Cx(t)\|) + r\gamma_1(\|v(t)\|) \\ & + \kappa\|v(t)\|\big(\|x(t)\| + \|v(t)\|\big). \end{aligned}$$

We obtain that, for all $t \in \mathbb{N}_0$ and for all $(v, x) \in \mathcal{B}(A, B, C, f)$,

$$V(x(t+1)) - V(x(t)) \leq -\alpha_1(\|Cx(t)\|^2) + \kappa\|x(t)\|\|v(t)\| + \gamma_2(\|v(t)\|),$$

where α_1 is given by (4.5) and $\gamma_2 \in \mathcal{K}_\infty$ is defined by $\gamma_2(s) = r\gamma_1(s) + \kappa s^2$. Yet another application of statement (1) of Lemma 14 shows that there exists $\gamma_3 \in \mathcal{K}_\infty$ such that

$$\kappa s_1 s_2 \leq \frac{1}{2} s_1 \mu(s_1) + \gamma_3(s_2) \quad \forall\, s_1, s_2 \geq 0,$$

and so,

$$\left.\begin{aligned} V(x(t+1)) - V(x(t)) \leq -\alpha_1(\|Cx(t)\|^2) + \|x(t)\|\mu(\|x(t)\|)/2 + \gamma_4(\|v(t\|) \\ \forall\, t \in \mathbb{N}_0, \forall\, (v, x) \in \mathcal{B}(A, B, C, f) \end{aligned}\right\}, \tag{4.12}$$

where $\gamma_4 := \gamma_2 + \gamma_3$.

Hence, setting $W := U + V$, it follows from (4.11) and (4.12) that, for all $t \in \mathbb{N}_0$ and all $(v, x) \in \mathcal{B}(A, B, C, f)$,

$$W(x(t+1)) - W(x(t)) \leq -\alpha_2(\|x(t)\|) + \alpha_3(\|v(t)\|),$$

where $\alpha_2(s) := s\mu(s)/2$ and $\alpha_3(s) := \alpha_1(s^2) + \gamma_4(s)$ for all $s \geq 0$. Obviously, α_2 and α_3 are in $\mathcal{K}_\infty$. Finally, let $c_3 > 0$ and $c_4 > 0$ be such that

$$V(\xi) \leq c_3\|\xi\|^2 \quad \text{and} \quad c_4\|\xi\|^2 \leq V_2(\xi) \qquad \forall\, \xi \in \mathbb{F}^n.$$

Defining $\mathcal{K}_\infty$ functions α_4 and α_5 by

$$\alpha_4(s) := \alpha_1(c_4 c s^2) \quad \text{and} \quad \alpha_5(s) := \alpha_1(c_2 c s^2) + c_3 s^2 \qquad \forall\, s \geq 0,$$

it follows that $\alpha_4(\|\xi\|) \leq W(\xi) \leq \alpha_5(\|\xi\|)$ for all $\xi \in \mathbb{F}^n$. We conclude that W is an ISS-Lyapunov function for the Lur'e system (A, B, C, f), and the proof is complete. ◻

The construction of the ISS-Lyapunov function $W = U + V$ in the above proof is inspired by a similar technique employed in [2] for a certain class of continuous-time Lur'e systems; however, the technical details and context in the current paper are very different to those in [2]. Novelties in our development include (i) control functions v the values of which are not required to be in the image of B and (ii) a number of results on $\mathcal{K}_\infty$ functions (in particular, statement (3) of Lemma 14 and Proposition 19) which are pivotal in showing that $W = U + V$ is indeed an ISS-Lyapunov function.

In the corollary below, Theorem 13 is expressed in the form of a "nonlinear small-gain" result.

COROLLARY 15. *Let* $(A, B, C) \in \Sigma(m, n, p; \mathbb{F})$ *with transfer function* $\mathbf{G}$*, let* $K \in \mathbb{S}_{\mathbb{C}}(\mathbf{G})$*, and let* $f : \mathbb{R}^p \to \mathbb{R}^m$ *be a nonlinearity. Assume that either* (i) (A, B, C) *is canonical or* (ii) (A, B, C) *is semi-canonical and* $\min_{|z|=1} \|\mathbf{G}^K(z)\| < \|\mathbf{G}^K\|_{H^\infty}$*. If there exists* $\alpha \in \mathcal{K}_\infty$ *such that*

$$\|\mathbf{G}^K\|_{H^\infty} \frac{\|f(\xi) - K\xi\|}{\|\xi\|} \leq 1 - \frac{\alpha(\|\xi\|)}{\|\xi\|} \quad \forall \xi \in \mathbb{R}^p,\ \xi \neq 0,$$

then the Lur'e system (A, B, C, f) *is input-to-state stable.*

Proof. Setting $r := 1/\|\mathbf{G}^K\|_{H^\infty}$, it follows from statement (5) of Lemma 6 that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$. Obviously, condition (A) is satisfied and an application of Theorem 13 yields the claim. ∎

We emphasize that Corollary 15 is not a special case of the general nonlinear small-gain theorems derived in [19, 20, 32].

We now reformulate Theorem 13 in terms of positive real and sector conditions.

COROLLARY 16. *Let* $(A, B, C) \in \Sigma(m, n, p; \mathbb{F})$ *with transfer function* $\mathbf{G}$*, let* K_1*,* $K_2 \in \mathbb{F}^{m\times p}$*, and let* $f : \mathbb{F}^p \to \mathbb{F}^m$ *be a nonlinearity. Assume that* $\mathbf{H} := (I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1}$ *is positive real and that either* (i) (A, B, C) *is canonical or* (ii) (A, B, C) *is semi-canonical and there exists* $\theta \in [0, 2\pi)$ *such that* $\operatorname{Re} \mathbf{H}(e^{i\theta})$ *is positive definite. If there exists* $\alpha \in \mathcal{K}_\infty$ *such that*

$$\operatorname{Re}\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi\rangle \leq -\alpha(\|\xi\|)\|\xi\| \quad \forall \xi \in \mathbb{F}^p, \tag{4.13}$$

then the Lur'e system (A, B, C, f) *is input-to-state stable.*

In the special case that $\alpha(s) = \delta s$ (for some positive δ), (4.13) is the same as (3.13). Therefore, under the conditions of statement (3) of Theorem 11, we not only have global exponential stability but also ISS.

Corollary 16 is a clear-cut ISS version of the circle criterion for a general class of multivariable discrete-time Lur'e systems: it shows that conditions very similar to those of the circle criterion guarantee ISS. While it is difficult to compare Corollary 16 with the ISS results for continuous-time Lur'e systems in [17] (where $p = m$ and $v(t)$ is in the image of B for $t \in \mathbb{N}_0$), it is clear that [17, Theorems 3.4 and 3.5] do not provide clear-cut ISS versions of the continuous-time multivariable circle criterion: in particular, (4.13) is considerably less restrictive than the corresponding condition in [17]. Moreover, it is even more difficult to compare Corollary 16 with the continuous-time result [2, Theorem 1], because there, an infinite sector condition is considered and the assumption on the underlying linear system is essentially equivalent to the positive realness of its transfer function, a scenario which is of no interest in our discrete-time setting, since the only strictly proper rational function which has the discrete-time positive real property is the zero function. Finally, we note that, in contrast to [2, 17], it is not assumed in Corollary 16 that the number of inputs is equal to the number of outputs.

Proof of Corollary 16. Define matrices L and M in $\mathbb{F}^{m\times p}$ by (3.14). Then (3.15) holds and thus, by (4.13),

$$\|f(\xi) - M\xi\|^2 \leq \|L\xi\|^2 - \alpha(\|\xi\|)\|\xi\| \quad \forall \xi \in \mathbb{F}^p. \tag{4.14}$$

In particular, $\ker L = \{0\}$. Hence, L^*L is invertible, and $L^\sharp := (L^*L)^{-1}L^* \in \mathbb{F}^{p\times m}$ is a left-inverse of L. Setting $A_{K_1} := A + BK_1C$, then, as in the proof of Corollary 11,

$\mathbf{F} := L\mathbf{G}^{K_1}$ is the transfer function of (A_{K_1}, B, LC), $\mathbb{B}_{\mathbb{C}}(-LL^\sharp, 1) \subset \mathbb{S}_{\mathbb{C}}(\mathbf{F})$, and assumption (A) holds in the context given by the linear system (A_{K_1}, B, LC), the feedback gain $K = -LL^\sharp$, and the radius $r = 1$. Defining $g : \mathbb{F}^m \to \mathbb{F}^m$ as in (3.17), we conclude

$$\mathcal{B}(A, B, C, f) = \mathcal{B}(A_{K_1}, B, LC, g). \tag{4.15}$$

It is sufficient to show that there exists $\gamma \in \mathcal{K}_\infty$ such that

$$\|g(\xi) + LL^\sharp\xi\| \le \|\xi\| - \gamma(\|\xi\|) \quad \forall\, \xi \in \mathbb{F}^m. \tag{4.16}$$

Indeed, if (4.16) holds, then an application of Theorem 13 to (A_{K_1}, B, LC, g) yields that this Lur'e system is ISS, and consequently, by (4.15), the Lur'e system (A, B, C, f) is also ISS.

We proceed to establish the existence of a $\mathcal{K}_\infty$ function γ such that (4.16) holds. Appealing to (3.18) and (4.14), we obtain

$$\|g(\xi) + LL^\sharp\xi\|^2 = \|f(L^\sharp\xi) - ML^\sharp\xi\|^2 \le \|LL^\sharp\xi\|^2 - \alpha(\|L^\sharp\xi\|)\|L^\sharp\xi\| \quad \forall\, \xi \in \mathbb{F}^p.$$

Let $\xi \in \mathbb{F}^m$ and decompose $\xi = \xi_1 + \xi_2$, where

$$\xi_1 \in \operatorname{im} L = (\ker L^*)^\perp = (\ker L^\sharp)^\perp \quad \text{and} \quad \xi_2 \in (\operatorname{im} L)^\perp = \ker L^* = \ker L^\sharp.$$

Then $\|LL^\sharp\xi\| = \|LL^\sharp\xi_1\| = \|\xi_1\|$. Letting $\lambda > 0$ be such that (3.22) holds, it follows that

$$\begin{aligned} \|g(\xi) + LL^\sharp\xi\|^2 &\le \|\xi_1\|^2 - \lambda\|\xi_1\|\alpha(\lambda\|\xi_1\|) \\ &= \|\xi\|^2 - \big(\lambda\|\xi_1\|\alpha(\lambda\|\xi_1\|) + \|\xi_2\|^2\big) \quad \forall\, \xi \in \mathbb{F}^m. \end{aligned} \tag{4.17}$$

Defining $\gamma \in \mathcal{K}_\infty$ by

$$\gamma(s) := \frac{1}{4}\min\{\lambda\,\alpha(\lambda s/2), s/2\} \quad \forall\, s \ge 0,$$

we have that

$$4s\gamma(2s) = \min\{\lambda s\,\alpha(\lambda s), s^2\} \quad \forall\, s \ge 0. \tag{4.18}$$

Now

$$\sqrt{s_1^2 + s_2^2}\,\gamma\left(\sqrt{s_1^2 + s_2^2}\right) \le (s_1 + s_2)\gamma(s_1 + s_2) \le 2s_1\gamma(2s_1) + 2s_2\gamma(2s_2) \quad \forall\, s_1, s_2 \ge 0,$$

and thus, by (4.18),

$$2\sqrt{s_1^2 + s_2^2}\,\gamma\left(\sqrt{s_1^2 + s_2^2}\right) \le \lambda s_1\,\alpha(\lambda s_1) + s_2^2 \quad \forall\, s_1, s_2 \ge 0.$$

This, in combination with (4.17), yields

$$\|g(\xi) + LL^\sharp\xi\|^2 \le \|\xi\|^2 - 2\|\xi\|\gamma(\|\xi\|) \le \big(\|\xi\| - \gamma(\|\xi\|)\big)^2 \quad \forall\, \xi \in \mathbb{F}^m,$$

showing that (4.16) holds and completing the proof. ∎

5. Applications. We describe two applications of the ISS results developed in section 4: (i) ISS with bias and (ii) "environmental" forcing in theoretical ecology.

5.1. ISS with bias. The following result shows that if, in Theorem 13, the nonlinearity f satisfies the condition (4.2) only in the complement of a compact set, then the Lur'e system still satisfies an estimate which is reminiscent of ISS.

COROLLARY 17. *Let* $(A, B, C) \in \Sigma(m, n, p; \mathbb{F})$ *with transfer function* $\mathbf{G}$. *Assume that* $K \in \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, *and condition* (A) *holds. Set* $r := 1/\|\mathbf{G}^K\|_{H^\infty}$ *and let* $\alpha \in \mathcal{K}_\infty$. *Then there exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *such that, for every nonlinearity* $f : \mathbb{F}^p \to \mathbb{F}^m$ *which is bounded on bounded sets and satisfies*

$$\|f(\xi) - K\xi\| \le r\|\xi\| - \alpha(\|\xi\|) \quad \text{for all sufficiently large } \|\xi\|, \tag{5.1}$$

we have

$$\|x(t)\| \le \beta(\|x(0)\|, t) + \gamma\left(\theta_f + \max_{0\le s\le t}\|v(s)\|\right) \quad (v, x) \in \mathcal{B}(A, B, C, f)\ \forall\, t \in \mathbb{N}_0, \tag{5.2}$$

where $\theta_f := \sup_{\xi\in\mathbb{F}^p} \operatorname{dist}(f(\xi), \mathbb{B}_{\mathbb{C}}(K\xi, r\|\xi\| - \alpha(\|\xi\|))$.

The number θ_f provides a natural measure of the extent of the violation of condition (4.2). Note that that θ_f is finite as follows from (5.1) in conjunction with the assumption that f is bounded on bounded sets. If (5.2) holds, then we say that the Lur'e system is *ISS with bias* and the number $\gamma(\theta_f)$ is sometimes called the *bias*. Note that, under the conditions of Corollary 17, the origin may or may not be an equilibrium of the unforced Lur'e system, and, if it is, then it may be unstable.

Proof of Corollary 17. Without loss of generality we may assume that $\alpha(s) < rs$ for all $s > 0$. Set $\rho(s) := rs - \alpha(s)$ for all $s \ge 0$. Then ρ is continuous, $\rho(0) = 0$, and $\rho(s) > 0$ for all $s > 0$. Define $\hat{f} : \mathbb{F}^p \to \mathbb{F}^m$ by

$$\hat{f}(\xi) := \begin{cases} f(\xi) - K\xi & \text{if } \|f(\xi) - K\xi\| \le \rho(\|\xi\|), \\ \dfrac{f(\xi) - K\xi}{\|f(\xi) - K\xi\|}\rho(\|\xi\|) & \text{if } \|f(\xi) - K\xi\| > \rho(\|\xi\|). \end{cases}$$

Define $\tilde{f} : \mathbb{F}^p \to \mathbb{F}^m$ by $\tilde{f}(\xi) = \hat{f}(\xi) + K\xi$. Note that $\tilde{f}(\xi) = f(\xi)$ for all $\xi \in \mathbb{F}^p$ such that $\|f(\xi) - K\xi\| \le \rho(\|\xi\|)$ and

$$\|\tilde{f}(\xi) - K\xi\| = \|\hat{f}(\xi)\| \le \rho(\|\xi\|) \quad \forall\, \xi \in \mathbb{F}^p. \tag{5.3}$$

Furthermore,

$$\|f(\xi) - \tilde{f}(\xi)\| = \|f(\xi) - K\xi - \hat{f}(\xi)\| = \operatorname{dist}(f(\xi) - K\xi, \mathbb{B}_{\mathbb{C}}(0, \rho(\|\xi\|))) \quad \forall\, \xi \in \mathbb{F}^p,$$

and so

$$\|f(\xi) - \tilde{f}(\xi)\| = \operatorname{dist}(f(\xi), \mathbb{B}_{\mathbb{C}}(K\xi, \rho(\|\xi\|))) \quad \forall\, \xi \in \mathbb{F}^p,$$

showing that

$$\sup_{\xi\in\mathbb{F}^p} \|f(\xi) - \tilde{f}(\xi)\| = \theta_f < \infty.$$

Let $(v, x) \in \mathcal{B}(A, B, C, f)$, set $w(t) = B(f(Cx(t)) - \tilde{f}(Cx(t)))$ for all $t \in \mathbb{N}_0$, and note that

$$x(t + 1) = Ax(t) + B\tilde{f}(Cx(t)) + w(t) + v(t) \quad \forall\, t \in \mathbb{N}_0.$$

Hence, $(v + w, x) \in \mathcal{B}(A, B, C, \tilde{f})$. Now, invoking statement (5) of Lemma 6, we see that $\mathbb{B}_{\mathbb{C}}(K, r) \subset \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, and so, by (5.3), Theorem 13 applies to the Lur'e system $(A, B, C, \tilde{f})$. Consequently, there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ (depending only on (A, B, C), K, and α) such that (5.2) holds. ∎

5.2. Environmental forcing in theoretical ecology. By way of motivation, consider the Beverton–Holt equation

$$y(t+1) = \frac{\rho y(t)}{1 + (\rho - 1)y(t)/\kappa}, \quad y(0) = y^0 \geq 0, \tag{5.4}$$

which, in theoretical ecology, is used to model the evolution of a single population y, where $y(t)$ is the size of the population at time t. The positive parameters ρ and κ are the *inherent growth rate* and the *carrying capacity*, respectively. The former is a characteristic of the population, determined by life cycle and demographic properties such as, for example, birth rates and survival rates, while the latter is a characteristic of the habitat or environment (e.g., resource availability, temperature, or humidity); see, for example, [6] and the references therein. If $\rho \leq 1$, the solution of the initial-value problem (5.4) converges to the equilibrium 0. If $\rho > 1$, then the positive equilibrium κ is globally asymptotically stable in the sense that it is stable and attracts every solution with positive initial value y^0.

In the following, in order to take into account fluctuations in the environment, we replace the constant κ by $\kappa(1 + k(t))$, where $k : \mathbb{N}_0 \to \mathbb{R}$ satisfies

$$-1 < \inf_{t \in \mathbb{N}_0} k(t) \leq \sup_{t \in \mathbb{N}_0} k(t) < \infty. \tag{5.5}$$

This leads to the environmentally forced Beverton–Holt equation

$$x(t+1) = \frac{\rho x(t)}{1 + (\rho - 1)x(t)/(1 + k(t))}, \quad x(0) = x^0 \geq 0, \tag{5.6}$$

where the parameter κ has been removed by the rescaling $x(t) := y(t)/\kappa$. Note that, if $k(t) \equiv 0$ and if $\rho > 1$, then $x_{\rm e} = 1$ is a globally asymptotically stable equilibrium in the sense that it is stable and attracts every solution x with $x(0) = x^0 > 0$. We will use the ISS theory developed in section 4 to analyze the robustness properties of the equilibrium $x_{\rm e} = 1$ with respect to the disturbance induced by k. This will be done in the context of a more general equation which contains the forced Beverton–Holt equation (5.6) as a special case. To this end, consider the equation

$$x(t+1) = g\big(x(t)/(1 + k(t))\big)x(t), \quad x(0) = x^0 \geq 0, \tag{5.7}$$

where k satisfies (5.5) and $g : (0, \infty) \to (0, \infty)$ is continuous and such that

$$g \text{ is strictly decreasing and } \limsup_{x \to \infty} g(x) < 1 \tag{E1}$$

and

$$\lim_{x \to 0} g(x)x = 0, \quad \lim_{x \to 0} g(x) =: g(0+) \in (1, \infty]. \tag{E2}$$

It follows from (E1) and (E2) that there exists a unique $x_{\rm e} > 0$ such that $g(x_{\rm e}) = 1$. If $k(t) \equiv 0$, then $x_{\rm e}$ is an equilibrium of (5.7).

Setting $f(x) := g(x)x$ for all $x \geq 0$, then $f(x_{\rm e}) = x_{\rm e}$, and we assume that the following sector condition holds:

$$\left| \frac{f(x) - f(x_{\rm e})}{x - x_{\rm e}} \right| = \left| \frac{f(x) - x_{\rm e}}{x - x_{\rm e}} \right| < 1 \quad \forall\, x > 0,\ x \neq x_{\rm e}. \tag{E3}$$

See Figure 2 for an illustration of the condition (E3).

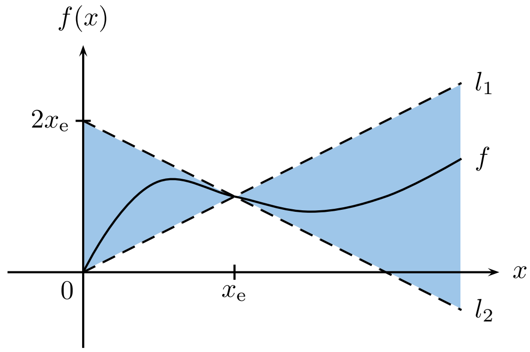

FIG. 2. *The graph of f is "sandwiched" between the lines $l_1(x) = x$ and $l_2(x) = 2x_{\mathrm{e}} - x$.*

We note that if (E1) and (E2) are satisfied, g is continuously differentiable, and $f'(x) \geq 0$ for all $x > 0$, then (E3) also holds. For the (rescaled) Beverton–Holt example,

$$g(x) = \frac{\rho}{1 + (\rho - 1)x}, \quad f(x) = g(x)x = \frac{\rho x}{1 + (\rho - 1)x},$$

it is obvious that if $\rho > 1$, then (E1) and (E2) hold, $x_{\mathrm{e}} = 1$, and (E3) follows via the above observation. A similar comment applies to the example

$$g(x) = \frac{1}{\sqrt{x}}, \quad f(x) = g(x)x = \sqrt{x}.$$

For the so-called Ricker nonlinearity,

$$g(x) = \rho e^{-cx}, \quad f(x) = g(x)x = \rho x e^{-cx}, \qquad \text{where } \rho \text{ and } c \text{ are positive parameters,}$$

it is again obvious that, for all $\rho > 1$ and $c > 0$, (E1) and (E2) hold. However, f is not monotone and (E3) does not follow from the argument used in the Beverton–Holt example. It can be shown (a calculus exercise) that (E3) holds if, and only if, $\rho \in (1, e^2]$.

Let $-1 < k^- < 0 < k^+ < \infty$ and let $I \subset (0, \infty)$ be a compact interval. For $k : \mathbb{N}_0 \to [k^-, k^+]$ define

$$\theta(k, I) := \sup\{\big|\big[g\big(x/(1 + k(t))\big) - g(x)\big]x\big| : x \in I,\, t \in \mathbb{N}_0\}. \tag{5.8}$$

Obviously, by continuity of g, $\theta(k, I) < \infty$ and $\theta(k, I) \to 0$ as $(\sup_{t\in\mathbb{N}_0} |k(t)|) \to 0$.

COROLLARY 18. *Assume that conditions* (E1)–(E3) *hold, let $k^- \in (-1, 0)$, $k^+ \in (0, \infty)$, and let $J \subset (0, \infty)$ be a compact interval. Then there exist a compact interval $I \subset (0, \infty)$, $\beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}$ such that, for every $x^0 \in J$ and every $k : \mathbb{N}_0 \to [k^-, k^+]$, the solution x of* (5.7) *satisfies $x(t) \in I$ for all $t \in \mathbb{N}_0$ and*

$$|x(t) - x_{\mathrm{e}}| \leq \beta(|x^0 - x_{\mathrm{e}}|, t) + \gamma\big(\theta(k, I)\big) \quad \forall\, t \in \mathbb{N}_0. \tag{5.9}$$

Proof. By (E1) and (E2), there exist positive x^- and x^+ such that $x^- < x^+$,

$$g(x^-/(1 + k^-)) = g(x^+/(1 + k^+)) = 1,$$
$$g(x/(1 + k)) < 1 \quad \forall\, x \in (x^+, \infty),\ \forall\, k \in [k^-, k^+] \tag{5.10}$$

and

$$g(x/(1+k)) > 1 \quad \forall\, x \in (0, x^-),\ \ \forall\, k \in [k^-, k^+]. \tag{5.11}$$

Setting

$$\mu^+ := \sup\{xg(x/(1+k)) : x \in [0, x^+],\, k \in [k^-, k^+]\}$$

and writing $J = [x_-^0, x_+^0]$, where $0 < x_-^0 < x_+^0$, we claim that

$$x(t) \le \max(\mu^+, x_+^0) =: \nu^+ \quad \forall\, t \in \mathbb{N}_0. \tag{5.12}$$

Note that $\mu^+ \ge x^+$ (since $g(x^+/(1+k^+)) = 1$) and so $\nu^+ \ge x^+$. Since $x(0) = x^0 \in J$, (5.12) holds for $t = 0$. Assume that the inequality (5.12) is valid for some $t = \tau \in \mathbb{N}_0$. Then, by (5.10),

$$x(\tau+1) = x(\tau)g\big(x(\tau)(1+k(\tau))\big) \le \begin{cases} \mu^+ & \text{if } x(\tau) \le x^+, \\ x(\tau) & \text{if } x(\tau) > x^+, \end{cases}$$

and so,

$$x(\tau+1) \le \max\big(\mu^+, x(\tau)\big) \le \max(\mu^+, x_+^0) = \nu^+,$$

showing that (5.12) is true.

Next, we claim that

$$x(t) \ge \min(\mu^-, x_-^0) =: \nu^- > 0 \quad \forall\, t \in \mathbb{N}_0, \tag{5.13}$$

where

$$\mu^- := \inf\{xg(x/(1+k)) : x \in [x^-, \nu^+],\, k \in [k^-, k^+]\} > 0.$$

Obviously, (5.13) is satisfied for $t = 0$. Assume that (5.13) holds for some $t = \tau \in \mathbb{N}_0$. Then, by (5.11),

$$x(\tau+1) = x(\tau)g\big(x(\tau)(1+k(\tau))\big) \ge \begin{cases} \mu^- & \text{if } x(\tau) \ge x^-, \\ x(\tau) & \text{if } x(\tau) < x^-, \end{cases}$$

and so,

$$x(\tau+1) \ge \min\big(\mu^-, x(\tau)\big) \ge \min(\mu^-, x_-^0) = \nu^-,$$

establishing (5.13).

Setting $I := [\nu^-, \nu^+] \subset (0, \infty)$, we have that, for every $x^0 \in J$ and every $k : \mathbb{N}_0 \to \mathbb{R}$ satisfying $k^- \le k(t) \le k^+$, $x(t) \in I$ for all $t \in \mathbb{N}_0$. It remains to show that there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that (5.9) holds. To this end, define

$$\tilde{f} : \mathbb{R} \to \mathbb{R},\ \ \xi \mapsto \begin{cases} f(\xi + x_{\mathrm{e}}) - x_{\mathrm{e}} & \text{if } \xi \ge -x_{\mathrm{e}} + \nu^-, \\ f(\nu^-) - x_{\mathrm{e}} & \text{if } \xi < -x_{\mathrm{e}} + \nu^-. \end{cases}$$

It follows from (E3) that $|\tilde{f}(\xi)| < |\xi|$ for all $\xi \in \mathbb{R}$, $\xi \ne 0$. Moreover, trivially, $|\xi| - |\tilde{f}(\xi)| \to \infty$ as $\xi \to -\infty$, and, by (E1), we also have that $|\xi| - |\tilde{f}(\xi)| \to \infty$ as $\xi \to +\infty$. Consequently, there exists $\alpha \in \mathcal{K}_\infty$ such that

$$|\tilde{f}(\xi)| \le |\xi| - \alpha(|\xi|) \quad \forall\, \xi \in \mathbb{R}. \tag{5.14}$$

Consider the one-dimensional linear system $(A, B, C) = (0, 1, 1)$, the transfer function of which is $\mathbf{G}(z) = 1/z$. Now $\|\mathbf{G}\|_{H^\infty} = 1$ and thus, combining statement (5) of

Lemma 6, (5.14), and Theorem 13, we conclude that the Lur'e system $(0, 1, 1, \tilde{f})$ is ISS. Hence there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$|\tilde{x}(t)| \leq \beta(|\tilde{x}(0)|, t) + \gamma\left(\max_{0\leq s\leq t} |v(s)|\right) \quad \forall\, (\tilde{x}, v) \in \mathcal{B}(0, 1, 1, \tilde{f}),\ \forall\, t \in \mathbb{N}_0. \tag{5.15}$$

Finally, let $x^0 \in J$ and $k : \mathbb{N}_0 \to \mathbb{R}$ be such that $k^- \leq k(t) \leq k^+$. Then the solution x of (5.7) satisfies $x(t) \in I = [\nu^-, \nu^+]$ for $t \in \mathbb{N}_0$. Defining $v : \mathbb{N}_0 \to \mathbb{R}$ by

$$v(t) := \big[g\big((x(t)/(1 + k(t))\big) - g(x(t))\big]x(t) \quad \forall\, t \in \mathbb{N}_0,$$

it follows that

$$|v(t)| \leq \theta(k, I) \quad \forall\, t \in \mathbb{N}_0 \tag{5.16}$$

with $\theta(k, I)$ given by (5.8). Obviously,

$$x(t + 1) = g\big(x(t)/(1 + k(t))\big)x(t) = f(x(t)) + v(t) \quad \forall\, t \in \mathbb{N}_0,$$

and so,

$$x(t + 1) - x_{\mathrm{e}} = \tilde{f}(x(t) - x_{\mathrm{e}}) + v(t) \quad \forall\, t \in \mathbb{N}_0.$$

This shows that $(x - x_{\mathrm{e}}, v) \in \mathcal{B}(0, 1, 1, \tilde{f})$, which, in view of (5.15) and (5.16), completes the proof. ∎

6. Appendix: Proof of Lemma 14. To facilitate the proof of statement (3) of Lemma 14, we state the following result.

Proposition 19. *Let $\alpha \in \mathcal{K}_\infty$ and $\varepsilon > 0$, assume that*

$$\lim_{s\to\infty} \big(\alpha\big((1 + \varepsilon)s\big) - \alpha(s)\big) = \infty, \tag{6.1}$$

and define $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\eta(s) := \inf_{\sigma\in[0,\infty)} \big[\alpha\big((1 + \varepsilon)(s + \sigma)\big) - \alpha(\sigma)\big] \quad \forall\, s \geq 0. \tag{6.2}$$

Then $\eta \in \mathcal{K}_\infty$ and

$$\alpha(s_1 - s_2) \leq \alpha\big((1 + \varepsilon)s_1\big) - \eta(s_2) \quad \forall s_1 \geq s_2 \geq 0. \tag{6.3}$$

It is not difficult to prove a partial converse of Proposition 19, namely, if, for given $\varepsilon > 0$ and $\alpha \in \mathcal{K}_\infty$, there exists $\eta \in \mathcal{K}_\infty$ such that (6.3) holds, then, for every $\delta > \varepsilon$,

$$\lim_{s\to\infty} \big(\alpha\big((1 + \delta)s\big) - \alpha(s)\big) = \infty.$$

This partial converse of Proposition 19 is not needed in the present paper, and therefore the proof is left to the interested reader.

We relegate the proof of Proposition 19 to the end of the appendix and proceed to prove Lemma 14.

Proof of Lemma 14. To prove statement (1), note that if $s_2 \leq \alpha(s_1)$, then $s_1 s_2 \leq s_1\alpha(s_1)$, and if $s_2 > \alpha(s_1)$, then $s_1 s_2 < s_2\alpha^{-1}(s_2)$. Defining $\gamma \in \mathcal{K}_\infty$ by $\gamma(s) := s\alpha^{-1}(s)$ for all $s \geq 0$, it follows that

$$s_1 s_2 \leq s_1\alpha(s_1) + \gamma(s_2) \quad \forall\, s_1, s_2 \geq 0.$$

As for statement (2), note that, for all $s_1, s_2 \geq 0$,

$$(s_1 + s_2) + \varepsilon(s_1 + s_2) = (1+\varepsilon)s_1 + \varepsilon(1+\varepsilon^{-1})s_2.$$

Hence, for all $s_1, s_2 \geq 0$,

$$s_1 + s_2 \leq (1+\varepsilon)s_1 \quad \text{or} \quad s_1 + s_2 \leq (1+\varepsilon^{-1})s_2.$$

Consequently,

$$\alpha(s_1+s_2) \leq \alpha\big((1+\varepsilon)s_1\big) + \alpha\big((1+\varepsilon^{-1})s_2\big) \quad \forall\, s_1, s_2 \geq 0.$$

Finally, to prove statement (3), let $\alpha \in \mathcal{K}_\infty$ and $\varepsilon > 0$. It is clear that $\tilde{\alpha} \in \mathcal{K}_\infty$ given by $\tilde{\alpha}(s) := \sqrt{s}\,\alpha(\sqrt{s})$ for all $s \geq 0$ satisfies

$$\lim_{s\to\infty} \big(\tilde{\alpha}\big((1+\varepsilon)s\big) - \tilde{\alpha}(s)\big) = \infty. \tag{6.4}$$

Now define $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\eta(s) := \inf_{\sigma\in[0,\infty)} \big[\tilde{\alpha}\big((1+\varepsilon)(s+\sigma)\big) - \tilde{\alpha}(\sigma)\big] \quad \forall\, s \geq 0,$$

which is (6.2) with α replaced by $\tilde{\alpha}$. It follows from Proposition 19 that $\eta \in \mathcal{K}_\infty$ and

$$\tilde{\alpha}(s_1 - s_2) \leq \tilde{\alpha}\big((1+\varepsilon)s_1\big) - \eta(s_2) \quad \forall s_1 \geq s_2 \geq 0.$$

It remains to show that $\eta(s)/\sqrt{s} \to \infty$ as $s \to \infty$. To this end, let (s_k) be a sequence in $\mathbb{R}_+$ such that $s_k \to \infty$. Invoking (6.4), we see that, for every $s \geq 0$,

$$\big[\tilde{\alpha}\big((1+\varepsilon)(s+\sigma)\big) - \tilde{\alpha}(\sigma)\big] \to \infty \quad \text{as } \sigma \to \infty.$$

Consequently, by continuity, for each $k \in \mathbb{N}$, there exists $\sigma_k \geq 0$ such that

$$\eta(s_k) = \tilde{\alpha}\big(1+\varepsilon)(s_k+\sigma_k)\big) - \tilde{\alpha}(\sigma_k),$$

and so,

$$\frac{\eta(s_k)}{\sqrt{s_k}} = \sqrt{\frac{(1+\varepsilon)(s_k+\sigma_k)}{s_k}}\,\alpha\Big(\sqrt{(1+\varepsilon)(s_k+\sigma_k)}\Big) - \sqrt{\frac{\sigma_k}{s_k}}\,\alpha(\sqrt{\sigma_k}).$$

Setting $\theta_k := \sigma_k/s_k$, we obtain

$$\begin{aligned}\frac{\eta(s_k)}{\sqrt{s_k}} &= \sqrt{(1+\varepsilon)(1+\theta_k)}\,\alpha\Big(\sqrt{(1+\varepsilon)(s_k+\sigma_k)}\Big) - \sqrt{\theta_k}\,\alpha(\sqrt{\sigma_k}) \\ &\geq \Big(\sqrt{(1+\varepsilon)(1+\theta_k)} - \sqrt{\theta_k}\Big)\,\alpha\Big(\sqrt{(1+\varepsilon)(s_k+\sigma_k)}\Big).\end{aligned}$$

In the last estimate, the first factor on the right-hand side is bounded away from 0, while the second factor goes to ∞ as $k \to \infty$. Consequently, $\eta(s_k)/\sqrt{s_k} \to \infty$ as $k \to \infty$, completing the proof of statement (3). ∎

It remains to prove Proposition 19.

Proof of Proposition 19. Assume that (6.1) holds. Define $\Delta := \{(s_1, s_2) \in \mathbb{R}\times\mathbb{R} : s_1 \geq s_2 \geq 0\}$ and consider the continuous function $g : \Delta \to \mathbb{R}_+$ given by

$$g(s_1, s_2) := \alpha\big((1+\varepsilon)s_1\big) - \alpha(s_1 - s_2) \quad \forall\, (s_1, s_2) \in \Delta.$$

The function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ given by (6.2) can be expressed as

$$\eta(s) = \inf_{\sigma\in[0,\infty)} g(s+\sigma, s).$$

It is obvious that, for $s_1 \geq s_2 \geq 0$,

$$\eta(s_2) \leq \alpha\big((1+\varepsilon)(s_2 + (s_1 - s_2))\big) - \alpha(s_1 - s_2) = \alpha\big((1+\varepsilon)s_1\big) - \alpha(s_1 - s_2),$$

and so (6.3) holds. We will now show that $\eta \in \mathcal{K}_\infty$. To this end note that, by (6.1), we have, for each fixed $s \geq 0$,

$$\lim_{\sigma\to\infty} g(\sigma, s) = \infty.$$

Therefore, by continuity of g, for each $s \geq 0$, the set

$$G(s) := \{\sigma \in \mathbb{R}_+ : g(s+\sigma, s) = \eta(s)\}$$

is nonempty and compact. For each $s \geq 0$, set $l(s) := \min G(s)$. In particular, we have that

$$\eta(s) = g(s + l(s), s) \quad \forall\, s \geq 0. \tag{6.5}$$

Since g has nonnegative values and $g(0,0) = 0$, we have that $\eta(0) = 0$. To show that η is strictly increasing, fix $s \geq 0$, $\delta > 0$ and set $a := l(s+\delta)$. Then

$$\eta(s+\delta) = \min_{\sigma\in[0,a]} g(s+\delta+\sigma, s+\delta) = \min_{\sigma\in[0,a]} \big[\alpha\big((1+\varepsilon)(s+\delta+\sigma)\big) - \alpha(\sigma)\big]. \tag{6.6}$$

Now, for every $\sigma \geq 0$,

$$\alpha\big((1+\varepsilon)(s+\delta+\sigma)\big) - \alpha(\delta+\sigma) = g(s+\delta+\sigma, s) \geq \eta(s),$$

and hence,

$$\alpha\big((1+\varepsilon)(s+\delta+\sigma)\big) - \alpha(\sigma) \geq \eta(s) + \alpha(\delta+\sigma) - \alpha(\sigma).$$

Consequently, by (6.6),

$$\eta(s+\delta) \geq \eta(s) + \min_{\sigma\in[0,a]} \big[\alpha(\delta+\sigma) - \alpha(\sigma)\big] > \eta(s),$$

where the second, strict, inequality follows because α is continuous and strictly increasing. Since $s \geq 0$ and $\delta > 0$ were arbitrary, we have now shown that η is strictly increasing.

We proceed to prove that $\eta(s) \to \infty$ as $s \to \infty$. Noting that

$$\begin{aligned}\lim_{s\to\infty} \eta(s) &= \lim_{s\to\infty} \inf_{\sigma\in[0,\infty)} \big[\alpha\big((1+\varepsilon)(s+\sigma)\big) - \alpha(\sigma)\big] \\ &\geq \lim_{s\to\infty} \inf_{\sigma\in[0,\infty)} \big[\alpha\big((1+\varepsilon)(s+\sigma)\big) - \alpha(s+\sigma)\big] \\ &= \liminf_{\sigma\to\infty} \big[\alpha((1+\varepsilon)\sigma) - \alpha(\sigma)\big],\end{aligned}$$

it follows from (6.1) that $\lim_{s\to\infty} \eta(s) = \infty$.

It remains to prove that η is continuous. To this end, let $s \geq 0$ and let (s_i) be a sequence in $\mathbb{R}_+$ such that $s_i \to s$ as $i \to \infty$. It is sufficient to show that

$$\limsup_{i\to\infty} \eta(s_i) \leq \eta(s) \leq \liminf_{i\to\infty} \eta(s_i). \tag{6.7}$$

Setting $\sigma_i := s_i + l(s) \geq s_i$, continuity of g and (6.5) guarantee that

$$\lim_{i\to\infty} g(\sigma_i, s_i) = g(s + l(s), s) = \eta(s).$$

Now $\eta(s_i) \leq g(\sigma_i, s_i)$ for all $i \in \mathbb{N}$, and thus

$$\limsup_{i\to\infty} \eta(s_i) \leq \limsup_{i\to\infty} g(\sigma_i, s_i) = \lim_{i\to\infty} g(\sigma_i, s_i) = \eta(s). \tag{6.8}$$

For $j \in \mathbb{N}$, set $\eta_j := \inf_{i\geq j} \eta(s_i)$. Obviously, for every $j \in \mathbb{N}$, there exists an integer $i_j \geq j$ such that $\eta(s_{i_j}) - \eta_j \leq 1/j$. Setting $z_j := s_{i_j}$, we have that

$$\lim_{j\to\infty} z_j = s \quad \text{and} \quad \lim_{j\to\infty} \eta(z_j) = \lim_{j\to\infty} \eta_j = \liminf_{i\to\infty} \eta(s_i). \tag{6.9}$$

Invoking (6.5), it follows that, for all $j \in \mathbb{N}$,

$$\eta(z_j) = g(l(z_j) + z_j, z_j) \geq \alpha\big((1+\varepsilon)(l(z_j) + z_j)\big) - \alpha(l(z_j) + z_j).$$

Boundedness of $(\eta(z_j))$ together with (6.1) implies that $(l(z_j))$ is bounded. Consequently, there exists a convergent subsequence $(l(z_{j_k}))$ with limit $\lambda \geq 0$. Appealing to (6.9), we have that

$$\lim_{k\to\infty} z_{j_k} = s \quad \text{and} \quad \lim_{k\to\infty} \eta\big(z_{j_k}\big) = \liminf_{i\to\infty} \eta(s_i),$$

and thus, by (6.5) and the continuity of g,

$$\liminf_{i\to\infty} \eta(s_i) = \lim_{k\to\infty} g\big(z_{j_k} + l(z_{j_k}), z_{j_k}\big) = g(s + \lambda, s) \geq \eta(s).$$

Together with (6.8) this shows that (6.7) holds, completing the proof. ∎

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