[2] G. Basile and G. Marro, *Controlled and Conditioned Invariants in Linear System Theory*. Englewood Cliffs, NJ: Prentice-Hall, 1992.
[3] S. P. Bhattacharyya, "Disturbance rejection in linear systems," *Int. J. Syst.*, vol. 5, no. 7, pp. 633–637, 1974.
[4] M. Bonilla, M. Malabre, and M. Fonseca, "On the approximation of nonproper control laws," *Int. J. Contr.*, vol. 68, no. 4, pp. 775–796, 1997.
[5] M. Bonilla, M. Malabre, and P. Acosta, "Disturbance decoupling with stability using PID control laws," in *Proc. 33rd IEEE Conf. Decision and Contr.*, Lake Buena Vista, FL, Dec. 14–16, 1994, vol. 1, pp. 883–884.
[6] S. Jaffe and N. Karcanias, "Matrix pencil characterization of almost (A, B)—Invariant subspaces: A classification of geometric concepts," *Int. J. Contr.*, vol. 33, no. 1, pp. 51–93, 1981.
[7] A. S. Morse, "Structural invariants of linear multivariable systems," *SIAM J. Contr. Optimization*, vol. 11, no. 3, pp. 446–465, 1973.
[8] H. L. Trentelman, "Almost invariant subspaces and high gain feedback," Ph.D. dissertation, Dept. Mathematics Computing Science, Eindhoven University of Technology, The Netherlands, May 31, 1985.
[9] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 3rd ed. New York: Springer-Verlag, 1985.
[10] J. C. Willems, "Almost invariant subspaces: An approach to high gain feedback design—Part 1: Almost controlled invariant subspaces (—Part 2: Almost conditionally invariant subspaces)," *IEEE Trans. Automat. Contr.*, vol. AC-26 (AC-27), pp. 235–252 (1071–1084), 1981 (1982).
[11] ——, "Feedforward control, PID control laws, and almost invariant subspaces," *Syst. Contr. Lett.*, vol. 1, no. 4, pp. 277–282, 1982.

Integral Control of Linear Systems with Actuator Nonlinearities: Lower Bounds for the Maximal Regulating Gain

H. Logemann, E. P. Ryan, and S. Townley

***Abstract*— Closing the loop around an exponentially stable single-input/single-output regular linear system, subject to a globally Lipschitz and nondecreasing actuator nonlinearity and compensated by an integral controller, is known to ensure asymptotic tracking of constant reference signals, provided that: 1) the steady-state gain of the linear part of the plant is positive; 2) the positive integrator gain is sufficiently small; and 3) the reference value is feasible in a very natural sense. Here lower bounds are derived for the maximal regulating gain for various special cases including systems with nonovershooting step-response and second-order systems with a time-delay in the input or output. The lower bounds are given in terms of open-loop frequency/step response data and the Lipschitz constant of the nonlinearity, and are hence readily obtainable.**

***Index Terms*—Actuator nonlinearities, infinite-dimensional systems, integral control, input saturation, monotone step-response, robust tracking, systems with time-delay.**

I. Introduction

The synthesis of low-gain integral (I) and proportional-plus-integral (PI) controllers for uncertain stable plants has received considerable attention in the last 20 years. The following principle is well known (see Davison [4], Lunze [10] and Morari [11]): closing the loop around a stable finite-dimensional continuous-time single-input/single-output plant, with transfer function $\boldsymbol{G}(s)$, compensated by a pure integral controller k/s (see Fig. 1), will result in a stable closed-loop system which achieves asymptotic tracking of arbitrary constant reference signals, provided that $|k|$ is sufficiently small and $k\boldsymbol{G}(0) > 0$. Therefore, if a plant is known to be stable and if the sign of $\boldsymbol{G}(0)$ is known (this information can be obtained from plant step response data), then the problem of tracking by low-gain integral control reduces to that of tuning the gain parameter k. Such a controller design ("tuning regulator theory" [4]) has been successfully applied in process control; see, for example, Coppus *et al.* [3] and Lunze [9]. The approach has been extended to various classes of infinite-dimensional systems; see Logemann and Townley [7] and [8] and the references therein. Furthermore, the problem of tuning the integrator gain adaptively has been addressed recently in a number of papers (again we refer the reader to [7] and [8] and the references therein).

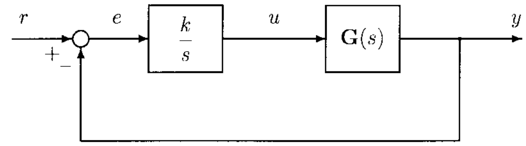

Fig. 1. Low-gain control system.

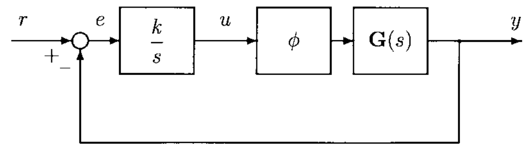

Fig. 2. Low-gain control with input nonlinearity.

In a recent paper, Logemann *et al.* [6] have proved that the above principle remains true if the plant to be controlled is a single-input/single-output regular infinite-dimensional linear system subject to an input nonlinearity (see Fig. 2). More precisely, it is shown in [6] that for an exponentially stable system with $\boldsymbol{G}(0) > 0$, there exists $K > 0$ such that for all nondecreasing globally Lipschitz nonlinearities ϕ with Lipschitz constant λ and all $k \in (0, K/\lambda)$, the output $y(t)$ of the closed-loop system shown in Fig. 2 converges to r as $t \to \infty$, provided that $[\boldsymbol{G}(0)]^{-1}r \in \operatorname{clos}(\operatorname{im}\phi)$. In particular, K is the supremum of the set of all $k > 0$ such that the function

$$1 + k \operatorname{Re} \frac{\boldsymbol{G}(s)}{s}$$

is positive real. In this paper, we show that K can be obtained from frequency and step-response experiments performed on the linear part of the plant. Moreover, we present an easily obtainable lower bound for K. For a number of special cases, we show that $K = 1/|\boldsymbol{G}'(0)|$; determination of $|\boldsymbol{G}'(0)|$ (and hence of K), in principle, requires only frequency and step-response data. In particular, the latter formula for K applies to systems with nonovershooting step-response and a class of second-order systems with a time-delay in the input or output. We remark that, in the finite-dimensional and linear case, Mustafa [12] has recently derived a formula for the smallest $k > 0$ such that the closed-loop system shown in Fig. 1 is unstable: this formula is in terms of a minimal realization of $\boldsymbol{G}$ and hence requires exact knowledge of the system.

Manuscript received June 23, 1997; revised December 1, 1997. Recommended by Associate Editor, S. Weiland. This work was supported by the Human Capital and Mobility programme under Project CHRX-CT93-0402 and NATO under Grant CRG 950179.

H. Logemann and E. P. Ryan are with the Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, U.K.

S. Townley is with the Department of Mathematics, University of Exeter, Exeter EX4 4QE, U.K. and the Centre for Systems and Control, Engineering, School of Engineering, University of Exeter, U.K.

Publisher Item Identifier S 0018-9286(99)03019-6.

0018–9286/99$10.00 © 1999 IEEE

II. Preliminaries

Let $\mathbb{R}_+ := [0, \infty)$ and, for $\alpha \in \mathbb{R}$, set $\mathbb{C}_\alpha := \{s \in \mathbb{C} | \operatorname{Re} s > \alpha\}$. The algebra of all holomorphic and bounded functions on $\mathbb{C}_\alpha$ is denoted by H_α^∞. If $f \in H_\alpha^\infty$ for some $\alpha < 0$, we define $\|f\|_\infty = \sup_{s \in \mathbb{C}_0} |f(s)|$ and, as is well-known, we have $\|f\|_\infty = \sup_{\omega \in \mathbb{R}} |f(i\omega)|$. The Hardy space, of order two, of holomorphic functions defined on $\mathbb{C}_\alpha$ is denoted by H_α^2. Let $L_\alpha^2(\mathbb{R}_+)$ denote the space of all locally square-integrable functions f such that the weighted function $t \mapsto f(t)e^{-\alpha t}$ is in $L^2(\mathbb{R}_+)$. Moreover, let $M(\mathbb{R}_+)$ denote the set of all bounded Borel measures on $\mathbb{R}_+$. For $\alpha \in \mathbb{R}$, let $M_\alpha(\mathbb{R}_+)$ denote the set of all locally bounded Borel measures μ on $\mathbb{R}_+$ such that $e^{-\alpha t}\mu(dt)$ belongs to $M(\mathbb{R}_+)$. The Laplace transform is denoted by $\mathcal{L}$. If $\mu \in M_\alpha(\mathbb{R}_+)$, then $\mathcal{L}(\mu) \in H_\alpha^\infty$ and, moreover, $\mathcal{L}(\mu)(s)$ exists and is continuous on the *closed* right-half plane $\operatorname{Re} s \geq \alpha$ (see [5] for details).

In the following, we will use some concepts from the theory of linear *regular* infinite-dimensional systems. For a comprehensive treatment of regular systems, see Weiss [14], [15] and the references therein. For a treatment of regular systems specific to low-gain control, the reader is referred to [6] and [7]. We remark that most linear distributed parameter systems and time-delay systems arising in control engineering fall within the framework of regular systems. Let (A, B, C, D) be the generating operators of a linear single-input/single-output regular system with state space X, a Hilbert space. Let $\boldsymbol{T}_t$ denote the strongly continuous semigroup generated by A, let C_L denote the so-called Lebesgue extension of C, and let $\boldsymbol{G}(s)$ denote the transfer function of (A, B, C, D). We mention that a transfer function $\boldsymbol{H}$ has a regular state-space realization if and only if $\boldsymbol{H} \in H_\alpha^\infty$ for some $\alpha \in \mathbb{R}$ and $\lim_{\xi \to \infty, \xi \in \mathbb{R}} \boldsymbol{H}(\xi)$ exists and is finite. Such transfer functions are called *regular*. Suppose that the linear regular system generated by (A, B, C, D) is subject to an input nonlinearity ϕ. We assume that $\phi \in N(\lambda)$, where $N(\lambda)$ denotes the set of all nondecreasing globally Lipschitz nonlinearities $f: \mathbb{R} \to \mathbb{R}$ with Lipschitz constant λ. Denoting the constant reference signal by r and the output of the system by y, an application of the integrator

$$\begin{aligned} u(t) &= u_0 + k \int_0^t [r - y(\tau)]\, d\tau \\ &= u_0 + k \int_0^t [r - C_L x(\tau) - D\phi(u(\tau))]\, d\tau \end{aligned}$$

where k is a real parameter (see Fig. 2), leads to the following nonlinear system of differential equations:

$$\dot{x} = Ax + B\phi(u) \quad x(0) = x_0 \in X \tag{1a}$$

$$\dot{u} = k[r - C_L x - D\phi(u)] \quad u(0) = u_0 \in \mathbb{R}. \tag{1b}$$

If, for some $\alpha < 0$, $\boldsymbol{G} \in H_\alpha^\infty$ (this is true if $\boldsymbol{T}_t$ is exponentially stable or if $\mathcal{L}^{-1}(\boldsymbol{G}) \in M_\alpha(\mathbb{R}_+)$) and $\boldsymbol{G}(0) > 0$, then it is not difficult to show that

$$1 + k \operatorname{Re} \frac{\boldsymbol{G}(s)}{s} \geq 0 \quad \text{for all } s \in \mathbb{C}_0 \tag{2}$$

for all sufficiently small $k > 0$ (see Lemma 3.1 below). We define

$$K := \sup\{k > 0 | (2) \text{ holds}\}. \tag{3}$$

Clearly, (2) holds for all $k \in [0, K)$. Moreover, if $K < \infty$, then the supremum in (3) is attained. It is easy to construct examples for which $K = \infty$, as is the case in Example 3.5, part 1).

The following tuning regulator result was proved in [6].

Theorem 2.1: Let $\lambda > 0$ and $\phi \in N(\lambda)$. Assume that $\boldsymbol{T}_t$ is exponentially stable, $\boldsymbol{G}(0) > 0, k \in (0, K/\lambda)$, and $r \in \mathbb{R}$ is such that

$$\phi_r := [\boldsymbol{G}(0)]^{-1} r \in \operatorname{clos}(\operatorname{im} \phi).$$

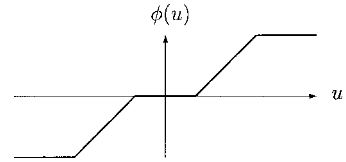

Fig. 3. Nonlinearity with saturation and deadzone.

If C is bounded, then for all $(x_0, u_0) \in X \times \mathbb{R}$, the unique solution $(x(\cdot), u(\cdot))$ of (1) exists on $[0, \infty)$ and satisfies:

1) $\lim_{t \to \infty} \phi(u(t)) = \phi_r$;
2) $\lim_{t \to \infty} \|x(t) + A^{-1}B\phi_r\| = 0$;
3) $\lim_{t \to \infty} (r - y(t)) = 0$, where $y(t) = Cx(t) + D\phi(u(t))$;
4) if $\phi_r \in \operatorname{im} \phi$, then

$$\lim_{t \to \infty} \operatorname{dist}(u(t), \phi^{-1}(\phi_r)) = 0$$

5) if $\phi_r \in \operatorname{int}(\operatorname{im} \phi)$, then $u(\cdot)$ is bounded.

If C is unbounded, then the statements 1)–5) remain true provided that $\mathcal{L}^{-1}(\boldsymbol{G}) \in M(\mathbb{R}_+)$ and x_0 is in the domain of A.

In particular, 4) states that $u(t)$ converges as $t \to \infty$ if the set $\phi^{-1}(\phi_r)$ is a singleton, which, in turn, is true if ϕ_r is not a critical value of ϕ. The conditions imposed in Theorem 2.1 on ϕ are satisfied by saturation and deadzone nonlinearities and combinations of the two, as shown in Fig. 3. The assumption that $\mathcal{L}^{-1}(\boldsymbol{G}) \in M(\mathbb{R}_+)$ is not very restrictive and seems to be satisfied in all practical examples of systems with H^∞-transfer functions. Generally, a measure $\mu \in M(\mathbb{R}_+)$ can be written in the form

$$\mu(dt) = a(t)\, dt + \sum_{j=0}^{\infty} a_j \delta_{t_j}(dt) + \mu_s(dt)$$

where $a(\cdot) \in L^1(\mathbb{R}_+), \Sigma_{j=0}^\infty a_j \delta_{t_j}$ and μ_s, respectively, represent the absolutely continuous, the discrete, and the singular parts of μ. In particular, δ_{t_j} denotes the unit point mass at $t_j \geq 0$ and the a_j are real numbers such that $\Sigma_{j=0}^\infty |a_j| < \infty$. In most applications one has $\mu_s = 0$.

For the application of Theorem 2.1, especially in process control, it is important to develop formulas or lower bounds for K in terms of easily obtainable open-loop data, such as Nyquist diagrams and step-response data. This development will be addressed in the next section.

III. Estimation and Determination of K

We shall invoke one or both of the following two assumptions where appropriate:

A1) $\boldsymbol{G} \in H_\alpha^\infty$ for some $\alpha < 0$

A2) $\overline{\boldsymbol{G}}(\overline{s}) = \boldsymbol{G}(s)$ for all $s \in \mathbb{C}_0$.

We mention that A2) is satisfied for all systems with real parameters. For $k > 0$ set

$$G_k(s) = \frac{k}{s}\boldsymbol{G}(s)\left(1 + \frac{k}{s}\boldsymbol{G}(s)\right)^{-1}.$$

Clearly, $\boldsymbol{G}_k$ is the transfer function of the feedback system obtained by applying the integral controller k/s to $\boldsymbol{G}$.

The next result is a trivial consequence of [7, Lemma 3.10].

Lemma 3.1: Assume that A1) holds and that $\boldsymbol{G}(0) > 0$. Then (2) holds for all sufficiently small $k > 0$ (and so $K > 0$). Moreover, for given $k > 0$, (2) holds if and only if $||\boldsymbol{G}_{k/2}||_\infty = 1$.

In Corollary 3.2, we give a graphical characterization of the number K. To this end, let $k > 0$ and define D_k to be the open disc in the complex plane of radius k and with centre $(-k, 0)$, i.e., $D_k = \{s \in \mathbb{C} | |s + k| < k\}$. The inverse Nyquist curve of $\boldsymbol{G}(s)/s$ given by

$$N = \left\{ \frac{i\omega}{\boldsymbol{G}(i\omega)} \middle| \omega \in \mathbb{R} \right\}.$$

Corollary 3.2: Assume that A1) holds and that $\boldsymbol{G}(0) > 0$. Then, $N \cap D_k = \emptyset$ for all sufficiently small $k > 0$, and

$$0 < \sup\{k > 0 | N \cap D_k = \emptyset\} = K/2.$$

Proof: Setting $\tilde{\boldsymbol{G}}_k(s) = (s/\boldsymbol{G}(s) + k)^{-1}$, it follows that $\boldsymbol{G}_k(s) = k\tilde{\boldsymbol{G}}_k(s)$. Clearly, for any $k > 0$, we have

$$||\boldsymbol{G}_k||_\infty = 1 \Leftrightarrow ||\tilde{\boldsymbol{G}}_k||_\infty = 1/k \Leftrightarrow N \cap D_k = \emptyset.$$

By Lemma 3.1, $K/2 = \sup\{k > 0 | ||\boldsymbol{G}_k||_\infty = 1\}$, and therefore the claim follows from the above equivalences. □

It will turn out to be convenient to introduce the following auxiliary transfer function

$$\boldsymbol{E}(s) := \frac{1}{s}(\boldsymbol{G}(s) - \boldsymbol{G}(0)).$$

The above definition makes sense for all $s \neq 0$ for which $\boldsymbol{G}(s)$ is defined. If $\boldsymbol{G}(s)$ is holomorphic at zero (which is the case if A1) is satisfied), then we set $\boldsymbol{E}(0) = \boldsymbol{G}'(0)$.

Lemma 3.3: Assume that A1) holds and that $\boldsymbol{G}(0) > 0$ and let $k > 0$. Then the following statements are equivalent:

1) $1 + k \operatorname{Re}(\boldsymbol{G}(s)/s) \geq 0$ for all $s \in \mathbb{C}_0$;
2) $1 + k \operatorname{Re}(\boldsymbol{G}(i\omega)/i\omega) \geq 0$ for all $\omega \in \mathbb{R}\backslash\{0\}$;
3) $1 + k \operatorname{Re} \boldsymbol{E}(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$;
4) $1 + k \operatorname{Re} \boldsymbol{E}(s) \geq 0$ for all $s \in \mathbb{C}_0$.

Proof: Trivially, 1) implies 2), and since $\boldsymbol{G}(0)$ is real, 2) implies 3). In order to show that 4) follows from 3), assume that 3) holds. By considering

$$e^{-(1+k \operatorname{Re} \boldsymbol{E}(s))} = |e^{-(1+k\boldsymbol{E}(s))}|$$

applying the maximum modulus theorem and using the fact that $\boldsymbol{E}(s) \to 0$ as $|s| \to \infty$ in $\mathbb{C}_0$, it then follows that

$$1 + k \operatorname{Re} \boldsymbol{E}(s) \geq 0 \quad \text{for all } s \in \mathbb{C}_0$$

which is (4). Finally, since $\boldsymbol{G}(0) > 0$ we have that $\operatorname{Re}(\boldsymbol{G}(0)/s) > 0$ for all $s \in \mathbb{C}_0$, and therefore (1) is implied by (4). □

The following corollary provides a lower bound and an upper bound for K in terms of the transfer function $\boldsymbol{E}$.

Corollary 3.4: Assume that A1) holds and that $\boldsymbol{G}(0) > 0$. Then

$$\frac{1}{\sup_{s \in \mathbb{C}_0} |\operatorname{Re} \boldsymbol{E}(s)|} \leq K \leq \begin{cases} 1/|\operatorname{Re} \boldsymbol{E}(0)|, & \text{if } \operatorname{Re} \boldsymbol{E}(0) < 0 \\ \infty, & \text{if } \operatorname{Re} \boldsymbol{E}(0) \geq 0 \end{cases}. \tag{4}$$

Proof: For $k > 0$ we have that

$$1 + k \operatorname{Re} \boldsymbol{E}(s) \geq 1 - k \sup_{s \in \mathbb{C}_0} |\operatorname{Re} \boldsymbol{E}(s)| \quad \text{for all } s \in \mathbb{C}_0.$$

Combining this with Lemma 3.3 and the definition of K yields the first inequality in (4). Moreover, using Lemma 3.3 again, it follows from the definition of K that $1 + K \operatorname{Re} \boldsymbol{E}(0) \geq 0$. If $\operatorname{Re} \boldsymbol{E}(0) < 0$, we may conclude that $K \leq 1/|\operatorname{Re} \boldsymbol{E}(0)|$, yielding the second inequality in (4).

The following examples show that if $\operatorname{Re} \boldsymbol{E}(0) = \operatorname{Re} \boldsymbol{G}'(0) \geq 0$, then cases of finite K and infinite K can occur.

Example 3.5:

1) Consider $\boldsymbol{G}(s) = (2s+1)/(s+1)$. Obviously, $\boldsymbol{G} \in H_\alpha^\infty$ for all $\alpha \in (-1, 0)$, and $\boldsymbol{G}(0) = \boldsymbol{G}'(0) = 1 > 0$. An easy calculation yields

$$\operatorname{Re} \frac{\boldsymbol{G}(i\omega)}{i\omega} = \frac{1}{1 + \omega^2}$$

showing that $K = \infty$.

2) As a second example consider $\boldsymbol{G}(s) = (s + 1)/(s + 2)^2$. Then $\boldsymbol{G} \in H_\alpha^\infty$ for all $\alpha \in (-2, 0), \boldsymbol{G}(0) = 1/4 > 0$, and $\boldsymbol{G}'(0) = 0$. Since

$$\operatorname{Re} \frac{\boldsymbol{G}(i\omega)}{i\omega} = \frac{-\omega^2}{(4 - \omega^2)^2 + 16\omega^2}$$

we see that $K < \infty$. □

In the following we introduce a condition which will guarantee that $K = 1/|\boldsymbol{G}'(0)|$. To this end, let $\sigma(\cdot)$ denote the *step-response* of the regular system (A, B, C, D) and define the *step-response error* $\varepsilon(\cdot)$ by

$$\varepsilon(t) = \sigma(t) - \boldsymbol{G}(0)$$

with Laplace transforms given by $[\mathcal{L}(\sigma)](s) = \boldsymbol{G}(s)/s$ and $[\mathcal{L}(\varepsilon)](s) = \boldsymbol{E}(s)$, respectively. Under the assumption that $\mathcal{L}^{-1}(\boldsymbol{G}) \in M(\mathbb{R}_+)$ it follows trivially that $\lim_{t\to\infty} \varepsilon(t) = 0$. This is in general not true under Assumption A1). However, we can prove the following lemma.

Lemma 3.6: If A1) holds, then there exists $\alpha < 0$ such that $\varepsilon \in L_\alpha^2(\mathbb{R}_+)$.

Proof: Choose $\alpha < 0$ such that $\boldsymbol{G} \in H_\alpha^\infty$. Then, $\boldsymbol{E} \in H_\alpha^2$, and by a well-known theorem of Paley and Wiener $\varepsilon = \mathcal{L}^{-1}(\boldsymbol{E}) \in L_\alpha^2(\mathbb{R}_+)$. □

If A2) is satisfied then the step-response error is real-valued and we say that the system satisfies the *no-overshoot condition* if $\varepsilon(t) \leq 0$ for almost all $t \in \mathbb{R}_+$. We say that the step-response $\sigma(\cdot)$ is *essentially nondecreasing* if there exists a nondecreasing function $\tilde{\sigma}(\cdot)$ such that $\sigma(t) = \tilde{\sigma}(t)$ for almost all $t \in \mathbb{R}_+$. If $\mathcal{L}^{-1}(\boldsymbol{G}) \in L^1(\mathbb{R}_+)$, then the step-response $\sigma(\cdot)$ is continuous, and hence $\sigma(\cdot)$ is essentially nondecreasing if and only if $\sigma(\cdot)$ is nondecreasing. However, if $\mathcal{L}^{-1}(\boldsymbol{G}) \notin L^1(\mathbb{R}_+)$, then $\sigma(\cdot)$ might be discontinuous, and consequently $\sigma(\cdot)$ might be essentially nondecreasing, but not nondecreasing. If A1) and A2) hold, then it follows from Lemma 3.6 that systems with an essentially nondecreasing step-response satisfy the no-overshoot condition. We mention that systems with monotone step-responses have received some attention in the robust control literature; see, e.g., Åström [1]. For an early, rigorous paper on systems with monotone step-responses, see Zemanian [16].

Corollary 3.7: Assume that A1) and A2) hold and that $\boldsymbol{G}(0) > 0$. If the system satisfies the no-overshoot condition, then $\boldsymbol{G}'(0) \leq 0$ and $K = 1/|\boldsymbol{G}'(0)|$ (where we define $1/0 = \infty$).

Proof: By the no-overshoot condition, we obtain for $s \in \mathbb{C}_0$

$$\begin{aligned} -\boldsymbol{G}'(0) = -\boldsymbol{E}(0) &= -\int_0^\infty \varepsilon(\tau)\, d\tau \\ &= \int_0^\infty |\varepsilon(\tau)|\, d\tau \geq |\boldsymbol{E}(s)| \geq |\operatorname{Re} \boldsymbol{E}(s)|. \end{aligned}$$

By A2), $\boldsymbol{E}(0) \in \mathbb{R}$, and thus the above inequality leads to

$$-\operatorname{Re} \boldsymbol{E}(0) = -\boldsymbol{G}'(0) = \sup_{s \in \mathbb{C}_0} |\operatorname{Re} \boldsymbol{E}(s)| = ||\boldsymbol{E}||_\infty. \tag{5}$$

Therefore, in particular, $\boldsymbol{G}'(0) \leq 0$. If $\boldsymbol{G}'(0) < 0$, then the claim follows from (5) and Corollary 3.4. If $\boldsymbol{G}'(0) = 0$, then by (5), $\boldsymbol{E}(s) \equiv 0$, and so $\boldsymbol{G}(s) \equiv \boldsymbol{G}(0)$, which in turn implies that $K = \infty$. □

Remark 3.8: The quantity $\boldsymbol{E}(0) = \boldsymbol{G}'(0)$ plays an important role in Corollaries 3.4 and 3.7. The following remarks show that this quantity can be obtained from step as well as frequency-response data. We assume that A1) and A2) are satisfied.

1) By Lemma 3.6, $\varepsilon \in L^1(\mathbb{R}_+)$, and hence $\boldsymbol{G}'(0) = \boldsymbol{E}(0) = \int_0^\infty \varepsilon(\tau)\, d\tau$, i.e., $\boldsymbol{G}'(0)$ is equal to the area enclosed between the graphs of $t \mapsto \sigma(t)$ and $t \mapsto \boldsymbol{G}(0)$.
2) The curvature κ of N at zero is given by $\kappa = 2|\boldsymbol{G}'(0)|$ (this follows from a straightforward calculation which is left to the reader), hence $|\boldsymbol{G}'(0)|$ can be obtained from the inverse Nyquist diagram of $\boldsymbol{G}(s)/s$. □

Example 3.9: Assume that $\boldsymbol{G}$ satisfies A1) and A2), $\boldsymbol{G}(0) > 0$, and the no-overshoot condition holds. Then the same is true for the transfer function

$$H(s) = \boldsymbol{G}(s) \sum_{n=0}^{\infty} \gamma_n e^{-h_n s} \tag{6}$$

where $\gamma_n, h_n \geq 0$ and

$$0 < \sum_{n=0}^{\infty} \gamma_n e^{\delta h_n} < \infty \quad \text{for some } \delta > 0.$$

By Corollary 3.7, the constant K (for $\boldsymbol{H}$) is then given by

$$K = 1/|\boldsymbol{H}'(0)| = \left(\sum_{n=0}^{\infty} \gamma_n |\boldsymbol{G}'(0) - h_n \boldsymbol{G}(0)| \right)^{-1}.$$

Note that $\boldsymbol{H}$ is regular if $\boldsymbol{G}$ is. Let us consider two specific examples.

1) Consider the following first-order system with time delay

$$\boldsymbol{H}(s) = \frac{e^{-hs}}{1 + \tau s}$$

where $h \geq 0$ and $\tau > 0$. Then, by the above remarks, $\boldsymbol{H}$ satisfies the no-overshoot condition and hence

$$K = 1/|\boldsymbol{H}'(0)| = 1/(h + \tau).$$

2) Consider

$$\boldsymbol{H}(s) = \frac{e^{-\alpha s}}{(1 + \tau s)(1 - \gamma e^{-\beta s})}$$

where $\alpha, \beta \geq 0, \tau > 0$, and $\gamma \in (0, 1)$. The above transfer function has been used to model a heat circulation process; see Blanchini [2]. In this application, $\gamma = 1 - \eta$, where η is a heat exchange efficiency index (which by definition is positive and smaller than one).

Setting $\gamma_n = \gamma^n, h_n = \alpha + n\beta$, and $\boldsymbol{G}(s) = 1/(1 + \tau s), \boldsymbol{H}$ can be written in the form (6). Hence

$$K = \frac{1}{|\boldsymbol{H}'(0)|} = \frac{(1 - \gamma)^2}{(\alpha + \tau)(1 - \gamma) + \gamma\beta}.$$

□

Example 3.10: Consider a diffusion process (with diffusion coefficient $a > 0$ and with Dirichlet boundary conditions) on the one-dimensional spatial domain $I = [0, 1]$, with scalar pointwise control action (applied at point $x_b \in I$) and pointwise scalar observation (output at point $x_c \in I, x_c \geq x_b$). We formally write this single-input/single-output system as

$$z_t(t, x) = a z_{xx}(t, x) + \delta(x - x_b) u(t), \qquad y(t) = z(t, x_c)$$
$$z(t, 0) = 0 = z(t, 1), \qquad \text{for all } t > 0.$$

The transfer function of this system is given by

$$\boldsymbol{G}(s) = \frac{\sinh(x_b\sqrt{s/a}) \sinh((1 - x_c)\sqrt{s/a})}{a\sqrt{s/a} \sinh \sqrt{s/a}}.$$

Since $\boldsymbol{G} \in H_\alpha^\infty$ for any $\alpha > -a\pi^2$ and $\lim_{\xi \to -\infty, \xi \in \mathbb{R}} \boldsymbol{G}(\xi) = 0$, the transfer function $\boldsymbol{G}$ is regular. Clearly, a positive injection of heat at x_b produces a nonnegative response at x_c, and so the step-response is nondecreasing. Therefore, the no-overshoot condition is satisfied, and thus by Corollary 3.7

$$K = \frac{1}{|\boldsymbol{G}'(0)|} = \frac{6a^2}{x_b(1 - x_c)(1 - x_b^2 - (1 - x_c)^2)}.$$

□

If A1) and A2) are satisfied, then Corollary 3.7 shows that the no-overshoot condition is sufficient for the formula $K = 1/|\boldsymbol{G}'(0)|$ to hold. However, not surprisingly, the no-overshoot condition is not necessary for the validity of the latter formula. The next result identifies a class of second-order systems with time-delay for which $K = 1/|\boldsymbol{G}'(0)|$, but which may have overshoot.

Proposition 3.11: Let $\boldsymbol{G}(s) = e^{-sh}/(s^2 + as + b)$, where $a, b > 0$ and $h \geq 0$. If

$$a^2 \geq 2b - \frac{b^2 h}{a + bh} \tag{7}$$

then $K = 1/|\boldsymbol{G}'(0)| = b^2/(a + bh)$.

Note that the right-hand side of (7) is decreasing as a function of h. In particular, if

$$a^2 \geq 2b \tag{8}$$

then (7) is satisfied for all $h \geq 0$, and consequently the formula $K = 1/|\boldsymbol{G}'(0)|$ holds independently of the length of the delay h. Clearly, condition (8) is satisfied if and only if the poles of $\boldsymbol{G}(s)$ belong to the sector $\{s \in \mathbb{C} | 3\pi/4 \leq \arg s \leq 5\pi/4\}$.

Proof of Proposition 3.11: In view of Lemma 3.3, it suffices to prove that the function

$$\omega \mapsto f(\omega) := -\mathrm{Re}\, \boldsymbol{E}(i\omega) = \frac{p(\omega)}{q(\omega)}$$

attains its maximum at $\omega = 0$, where, for convenience, we have introduced

$$p(\omega) := \begin{cases} a \cos \omega h + (b - \omega^2) \sin(\omega h)/\omega, & \omega \neq 0 \\ a + bh, & \omega = 0 \end{cases}$$

and

$$q(\omega) := (b - \omega^2)^2 + a^2\omega^2.$$

We will first show that $f(\omega) \leq f(0)$ for all ω with $\omega^2 \leq b$. Let $0 \leq \omega^2 \leq b$. Then $p(\omega) \leq a + (b - \omega^2)h$ and so

$$f(\omega) \leq g(\omega) := \frac{(a + bh) - \omega^2 h}{q(\omega)}.$$

By direct calculation

$$g'(\omega) = \frac{2\omega[h\omega^4 - 2\omega^2(a + bh) - hb^2 - (a + bh)(a^2 - 2b)]}{q^2(\omega)}.$$

Using (7)

$$hb^2 + (a + bh)(a^2 - 2b) \geq hb^2 - hb^2 = 0$$

and so we may conclude that

$$\omega g'(\omega) \leq 2\omega^4[h\omega^2 - 2(a + bh)]/q^2(\omega)$$
$$\leq 0 \quad \text{for all } \omega \in [-\sqrt{b}, \sqrt{b}].$$

Therefore

$$f(\omega) \leq g(\omega) \leq g(0) = f(0) \quad \text{for all } \omega \in [-\sqrt{b}, \sqrt{b}].$$

We complete the proof by showing that, for every $n \in \mathbb{N}, f(\omega) < f(0)$ for all ω with $nb \leq \omega^2 \leq (n + 1)b$.

Let $n \in \mathbb{N}$ and let $\omega \in \mathbb{R}$ be such that

$$nb \leq \omega^2 \leq (n+1)b.$$

Then

$$\begin{aligned} p(\omega) &< a + (\omega^2 - b)h = a + nbh - ((n+1)b - \omega^2)h \\ &\leq a + nbh \leq n(a + bh) \end{aligned}$$

and [again using (7)]

$$q(\omega) = b^2 + \omega^4 + (a^2 - 2b)\omega^2 \geq b^2 + \omega^4 - b\omega^2.$$

Since $nb \leq \omega^2 \leq (n+1)b$, we have $-b\omega^2 \geq -\omega^4/n$ and

$$q(\omega) \geq b^2 + \frac{(n-1)\omega^4}{n} \geq b^2(n^2 - n + 1) \geq nb^2.$$

We may now conclude that

$$f(\omega) < \frac{a + bh}{b^2} = f(0).$$

This completes the proof. □

Example 3.12: Consider

$$\boldsymbol{G}(s) = \frac{e^{-hs}}{s^2 + as + b}$$

where $h \geq 0$ and $a, b > 0$. Suppose that

$$4b > a^2 \geq 2b.$$

Then, by Proposition 3.11, it follows that

$$K = 1/|\boldsymbol{G}'(0)| = b^2/(a + bh) \quad \text{for all } h \geq 0.$$

Note that $\boldsymbol{G}$ does not satisfy the no-overshoot condition. Indeed, since $4b > a^2$, the maximum overshoot is $e^{-a\pi/\sqrt{4b-a^2}}/b > 0$; see for example [13, p. 191]. □

References

[1] K. J. Åström, "A robust sampled regulator for stable systems with monotone step responses," *Automatica*, vol. 16, pp. 313–315, 1980.

[2] F. Blanchini, private communication, 1996.

[3] G. W. M. Coppus, S. L. Sha, and R. K. Wood, "Robust multivariable control of a binary distillation column," *Proc. Inst. Elec. Eng.*, vol. 130, Pt. D, pp. 201–208, 1983.

[4] E. J. Davison, "Multivariable tuning regulators: The feedforward and robust control of a general servomechanism problem," *IEEE Trans. Automat. Contr.*, vol. 21, pp. 35–47, 1976.

[5] G. Gripenberg, S.-O. Londen, and O. Staffans, *Volterra Integral and Functional Equations*. Cambridge, U.K.: Cambridge Univ. Press, 1990.

[6] H. Logemann, E. P. Ryan, and S. Townley, "Integral control of infinite-dimensional linear systems subject to input saturation," *SIAM J. Contr. Optimiz.*, vol. 36, pp. 1940–1961, 1998.

[7] H. Logemann and S. Townley, "Low-gain control of uncertain regular linear systems," *SIAM J. Contr. Optimiz.*, vol. 35, pp. 78–116, 1997.

[8] ——, "Discrete-time low-gain control of uncertain infinite-dimensional systems," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 22–37, 1997.

[9] J. Lunze, "Experimentelle Erprobung einer Einstellregel für PI-Mehrgrößenregler bei der Herstellung von Ammoniumnitrat-Harnstoff-Lösung, *Messen Steuern Regeln*, vol. 30, pp. 2–6, 1987.

[10] ——, *Robust Multivariable Feedback Control*. London, U.K.: Prentice Hall, 1988.

[11] M. Morari, "Robust stability of systems with integral control," *IEEE Trans. Automat. Contr.*, vol. 30, pp. 574–577, 1985.

[12] D. Mustafa, "How much integral action can a control system tolerate?," *Linear Algebra its Appl.*, vol. 205/206, pp. 965–970, 1994.

[13] C. E. Rohrs, J. L. Melsa, and D. G. Schultz, *Linear Control Systems*. New York: McGraw-Hill, 1993.

[14] G. Weiss, "Transfer functions of regular linear systems, Part I: Characterization of regularity," *Trans. Amer. Math. Soc.*, vol. 342, pp. 827–854, 1994.

[15] ——, "The representation of regular linear systems on Hilbert spaces," *Distributed Parameter Systems*, F. Kappel, K. Kunisch, and W. Schappacher, Eds. Basel: Birkhäuser Verlag, 1989, pp. 401–416.

[16] A. H. Zemanian, "The properties of pole and zero locations for nondecreasing step responses," *Trans. Amer. Inst. Elect. Eng. Part I: Communication and Electronics*, vol. 79, pp. 421–426, 1960.