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INTEGRAL CONTROL OF INFINITE-DIMENSIONAL LINEAR SYSTEMS SUBJECT TO INPUT SATURATION*

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Abstract. Closing the loop around an exponentially stable single-input single-output regular linear system, subject to a globally Lipschitz and nondecreasing actuator nonlinearity and compensated by an integral controller, is shown to ensure asymptotic tracking of constant reference signals, provided that (a) the steady-state gain of the linear part of the plant is positive, (b) the positive integrator gain is sufficiently small, and (c) the reference value is feasible in a very natural sense. The class of actuator nonlinearities under consideration contains standard nonlinearities important in control engineering such as saturation and deadzone.

Key words. regular infinite-dimensional systems, integral control, actuator nonlinearities, input saturation, robust tracking, operator Riccati equations

AMS subject classifications. 93C10, 93C20, 93C25, 93D09, 93D10, 93D21

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1. Introduction.

The synthesis of low-gain integral (I) and proportional-plus-integral (PI) controllers for uncertain stable plants has received considerable attention in the last 20 years. The following principle is well known (see Davison [5], Lunze [20], and Morari [24]): closing the loop around a stable, finite-dimensional, continuous-time, single-input, single-output plant with transfer function $\mathbf{G}(s)$, compensated by a pure integral controller k/s (see Fig. 1.1), will result in a stable closed-loop system which achieves asymptotic tracking of arbitrary constant reference signals, provided that $|k|$ is sufficiently small and $k\mathbf{G}(0) > 0$. Therefore, if a plant is known to be stable and if the sign of $\mathbf{G}(0)$ is known (this information can be obtained from plant step response data), then the problem of tracking by low-gain integral control reduces to that of tuning the gain parameter k. Such a controller design approach ("tuning regulator theory" [5]) has been successfully applied in process control; see, for example, Coppus, Sha, and Wood [3] and Lunze [19].

An analogous result holds for finite-dimensional multivariable systems under suitable assumptions on $\mathbf{G}(0)$; see [5, 20] and [24]. Moreover, the result has been extended by Logemann, Bontsema, and Owens [13], Logemann and Owens [14], Logemann and Townley [17], Pohjolainen [27, 28], and Pohjolainen and Lätti [29] to various classes of (abstract) infinite-dimensional systems and by Jussila and Koivo [9] and Koivo and Pohjolainen [11] to differential delay systems. Furthermore, the problem of tuning the integrator gain adaptively has been addressed recently in a number of papers; see Cook [2] and Miller and Davison [22, 23] for the finite-dimensional case and Logemann and Townley [16, 17, 18] for the infinite-dimensional case.

In this paper we present results which show that the above principle remains true if the plant to be controlled is a single-input, single-output, infinite-dimensional, linear

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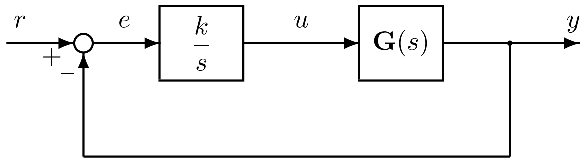

FIG. 1.1. *Low-gain control system.*

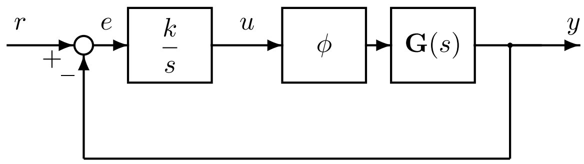

FIG. 1.2. *Low-gain control with input nonlinearity.*

system subject to an input nonlinearity (see Fig. 1.2). More precisely, we prove that, for an exponentially stable system with $\mathbf{G}(0) > 0$, there exists a number $K > 0$ such that, for all nondecreasing globally Lipschitz nonlinearities ϕ with Lipschitz constant λ and all $k \in (0, K/\lambda)$, the output $y(t)$ of the closed-loop system shown in Fig. 1.2 converges to r as $t \to \infty$, provided that $[\mathbf{G}(0)]^{-1}r \in \operatorname{clos}(\operatorname{im}\phi)$. The number K is the supremum of the set of all numbers $k > 0$ such that the function

$$1 + k\mathrm{Re}\,\frac{\mathbf{G}(s)}{s}$$

is positive real. The essence of our approach is to invoke a particular coordinate transformation and perform a Liapunov-type analysis on the transformed system. A parametrized operator Riccati equation plays a central role in the latter analysis, which further develops an idea presented in Townley and Kamstra [34].

The linear, infinite-dimensional part of the plant in Fig. 1.2 is assumed to be regular. The class of regular linear infinite-dimensional systems, introduced by Weiss [35, 36, 37, 38, 39], is rather general. It includes most distributed parameter systems and all time-delay systems (retarded and neutral) which are of interest in applications. Although there exist well-posed abstract infinite-dimensional systems which are not regular, the authors are of the opinion that any physically motivated, *well-posed*, linear, time-invariant control system is regular. We emphasize that our assumptions on the actuator nonlinearity allow for standard nonlinearities occurring in control engineering such as saturation and deadzone.

To our knowledge some of the results in this paper are new even for the finite-dimensional case. While Desoer and Lin [6] consider the low-gain tracking problem for a class of nonlinear finite-dimensional systems, their framework does not include input saturation.

The paper is organized as follows. Definitions and fundamental facts pertaining to regular systems are assembled in section 2. Section 3 contains the main result of the paper as outlined above. Examples and simulations illustrating our results are given in section 4. The proofs of three technical lemmas are given in the appendix.

Notation.

- For $\alpha \in \mathbb{R}$, set $\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \mathrm{Re}s > \alpha\}$.

- For $\alpha \in \mathbb{R}$ and H a Hilbert space, we define the exponentially weighted L^2-space $L^2_\alpha(\mathbb{R}_+, H) := \{f \in L^2_{loc}(\mathbb{R}_+, H) \mid f(\cdot)\exp(-\alpha\,\cdot) \in L^2(\mathbb{R}_+, H)\}$.
- If A is a linear operator, then the domain, spectrum, and resolvent set of A are denoted by $\mathrm{dom}\,(A)$, $\sigma(A)$, and $\varrho(A)$, respectively.
- The set of all linear bounded operators from H_1 to H_2 (where H_1, H_2 are Hilbert spaces) is denoted by $\mathcal{B}(H_1, H_2)$. We write $\mathcal{B}(H)$ for $\mathcal{B}(H, H)$.
- The Laplace transform is denoted by $\mathfrak{L}$.

2. Preliminaries on regular systems. In this section we give some background on well-posed linear systems; the reader is referred to Weiss [35, 36, 37, 38, 39] for full details.

First, we introduce some further notation. For any Hilbert space H and any $\tau \geq 0$, $\mathbf{R}_\tau$ denotes the right shift by τ on $L^2_{loc}(\mathbb{R}_+, H)$. The truncation operator $\mathbf{P}_\tau : L^2_{loc}(\mathbb{R}_+, H) \to L^2(\mathbb{R}_+, H)$ is given by

$$(\mathbf{P}_\tau u)(t) = \begin{cases} u(t) & \text{if} \quad t \in [0, \tau], \\ 0 & \text{if} \quad t > \tau. \end{cases}$$

For $u, v \in L^2_{loc}(\mathbb{R}_+, H)$ and $\tau \geq 0$, the *τ-concatenation* $u \overset{\tau}{\diamondsuit} v$ is defined by

$$u \overset{\tau}{\diamondsuit} v = \mathbf{P}_\tau u + \mathbf{R}_\tau v\,.$$

The fundamental concept of a well-posed linear system was introduced by Weiss [39]; an equivalent definition can be found in Salamon [33].

DEFINITION 2.1. *Let U, X, and Y be real Hilbert spaces. A* well-posed *linear system with* state-space X, input-space U, *and* output-space Y *is a quadruple* $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$, *where*

(1) $\mathbf{T} = (\mathbf{T}_t)_{t \geq 0}$ *is a C_0-semigroup of bounded linear operators on X,*

(2) $\mathbf{\Phi} = (\mathbf{\Phi}_t)_{t \geq 0}$ *is a family of bounded linear operators from $L^2(\mathbb{R}_+, U)$ to X such that*

$$\mathbf{\Phi}_{\tau+t}(u \overset{\tau}{\diamondsuit} v) = \mathbf{T}_t \mathbf{\Phi}_\tau u + \mathbf{\Phi}_t v$$

for all $u, v \in L^2(\mathbb{R}_+, U)$, and all $\tau, t \geq 0$,

(3) $\mathbf{\Psi} = (\mathbf{\Psi}_t)_{t \geq 0}$ *is a family of bounded linear operators from X to $L^2(\mathbb{R}_+, Y)$ such that*

$$\mathbf{\Psi}_{\tau+t} x_0 = \mathbf{\Psi}_\tau x_0 \overset{\tau}{\diamondsuit} \mathbf{\Psi}_t \mathbf{T}_\tau x_0$$

for all $x_0 \in X$ and all $\tau, t \geq 0$, and $\mathbf{\Psi}_0 = 0$,

(4) $\mathbf{F} = (\mathbf{F}_t)_{t \geq 0}$ *is a family of bounded linear operators from $L^2(\mathbb{R}_+, U)$ to $L^2(\mathbb{R}_+, Y)$ such that*

$$\mathbf{F}_{\tau+t}(u \overset{\tau}{\diamondsuit} v) = \mathbf{F}_\tau u \overset{\tau}{\diamondsuit} (\mathbf{\Psi}_t \mathbf{\Phi}_\tau u + \mathbf{F}_t v)\,,$$

$u, v \in L^2(\mathbb{R}_+, U)$ and all $\tau, t \geq 0$, and $\mathbf{F}_0 = 0$.

Let an input $u \in L^2_{loc}(\mathbb{R}_+, U)$ and an initial state $x_0 \in X$ be given. The state $x(t) = x(t; x_0, u)$ of $\mathbf{\Sigma}$ at time $t \geq 0$ and the output $y(\cdot) = y(\cdot\,; x_0, u)$ of $\mathbf{\Sigma}$ are defined by

$$x(t) = \mathbf{T}_t x_0 + \mathbf{\Phi}_t \mathbf{P}_t u\,, \tag{2.1}$$

$$\mathbf{P}_t y = \mathbf{\Psi}_t x_0 + \mathbf{F}_t \mathbf{P}_t u\,. \tag{2.2}$$

The state trajectory $x(\cdot)$ is a continuous function from $\mathbb{R}_+$ to X, and the output $y(\cdot)$ is in $L^2_{loc}(\mathbb{R}_+, Y)$.

We say that $\mathbf{\Sigma}$ is *exponentially stable* if the semigroup $\mathbf{T}$ is exponentially stable, i.e.,

$$\omega(\mathbf{T}) := \lim_{t\to\infty} \frac{1}{t} \log \|\mathbf{T}_t\| < 0\,.$$

If $\mathbf{\Sigma}$ is exponentially stable, then the operators $\mathbf{\Phi}_t$ and $\mathbf{\Psi}_t$ are uniformly bounded. It is clear that there exist unique operators $\mathbf{\Psi}_\infty : X \to L^2_{loc}(\mathbb{R}_+, Y)$ and $\mathbf{F}_\infty : L^2_{loc}(\mathbb{R}_+, U) \to L^2_{loc}(\mathbb{R}_+, Y)$ such that, for all $\tau \geq 0$,

$$\mathbf{\Psi}_\tau = \mathbf{P}_\tau \mathbf{\Psi}_\infty\,, \quad \mathbf{F}_\tau = \mathbf{P}_\tau \mathbf{F}_\infty\,.$$

It follows easily that $\mathbf{P}_\tau \mathbf{F}_\infty = \mathbf{P}_\tau \mathbf{F}_\infty \mathbf{P}_\tau$ for all $\tau \geq 0$, i.e., $\mathbf{F}_\infty$ is a *causal* operator. Moreover, if $\mathbf{\Sigma}$ is exponentially stable, then $\mathbf{\Psi}_\infty$ is a bounded operator from X into $L^2(\mathbb{R}_+, Y)$ and $\mathbf{F}_\infty$ maps $L^2(\mathbb{R}_+, U)$ boundedly into $L^2(\mathbb{R}_+, Y)$.

The generator of $\mathbf{T}$ is denoted by A. Let X_1 be the space dom$\,(A)$ endowed with the graph norm. The norm on X is denoted by $\|\cdot\|$, while $\|\cdot\|_1$ denotes the graph norm. Let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1} = \|(sI - A)^{-1}x\|$, where $s \in \varrho(A)$ is fixed. We have $X_1 \subset X \subset X_{-1}$, and the canonical injections are bounded and dense. The semigroup $\mathbf{T}$ can be restricted to a C_0-semigroup on X_1 and extended to a C_0-semigroup on X_{-1}. The exponential growth constant is the same on all three spaces. The generator on X_{-1} is an extension of A to X (which is bounded as an operator from X to X_{-1}). We shall use the same symbol $\mathbf{T}$ (respectively, A) for the original semigroup (respectively, its generator) and the associated restrictions and extensions. With this convention, we may write $A \in \mathcal{B}(X, X_{-1})$. Considered as a generator on X_{-1}, the domain of A is X.

By a representation theorem due to Salamon [33] (see also Weiss [37, 38]) there exist unique operators $B \in \mathcal{B}(U, X_{-1})$ and $C \in \mathcal{B}(X_1, Y)$ (the *control operator* and the *observation operator* of $\mathbf{\Sigma}$, respectively) such that, for all $t \geq 0$, $u \in L^2_{loc}(\mathbb{R}_+, U)$, and $x_0 \in X_1$,

$$\mathbf{\Phi}_t \mathbf{P}_t u = \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau \quad \text{and} \quad (\mathbf{\Psi}_\infty x_0)(t) = C \mathbf{T}_t x_0\ .$$

B is called *bounded* if $B \in \mathcal{B}(U, X)$ (and *unbounded* otherwise), whereas C is called *bounded* if it can be extended continuously to X (and *unbounded* otherwise). If $\mathbf{T}$ is exponentially stable, then there exist constants $\alpha, \beta > 0$ such that, for all $t \geq 0$, $u \in L^2(\mathbb{R}_+, U)$, and $x_0 \in X_1$,

$$\|\mathbf{\Phi}_t \mathbf{P}_t u\| = \left\| \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau \right\| \leq \alpha \|u\|_{L^2(0,t;\, U)}\,, \tag{2.3}$$

$$\|(\mathbf{\Psi}_\infty x_0)(\cdot)\|_{L^2(0,t;\, Y)} = \left(\int_0^t \|C \mathbf{T}_\tau x_0\|^2\, d\tau \right)^{1/2} \leq \beta \|x_0\|\,. \tag{2.4}$$

As in [38], the *Lebesgue extension* of C is defined by

$$C_L x_0 = \lim_{t\to 0} C \frac{1}{t} \int_0^t \mathbf{T}_\tau x_0\, d\tau\,,$$

where dom (C_L) is the set of all those $x_0 \in X$ for which the above limit exists. Clearly $X_1 \subset \operatorname{dom}(C_L) \subset X$ and, for any $x_0 \in X$, we have $\mathbf{T}_t x_0 \in \operatorname{dom}(C_L)$ for almost every (a.e.) $t \geq 0$. Furthermore,

$$(\mathbf{\Psi}_\infty x_0)(t) = C_L \mathbf{T}_t x_0 \quad \text{for a.e. } t \geq 0 .$$

It can be shown (see Weiss [36, 38]) that, if $\alpha > \omega(\mathbf{T})$, $x_0 \in X$, and $u \in L^2_\alpha(\mathbb{R}_+, U)$, then $\mathbf{\Psi}_\infty x_0 \in L^2_\alpha(\mathbb{R}_+, Y)$, $\mathbf{F}_\infty u \in L^2_\alpha(\mathbb{R}_+, Y)$, and there exists a unique holomorphic $\mathbf{G} : \mathbb{C}_{\omega(\mathbf{T})} \to \mathcal{B}(U, Y)$ such that, for all $s \in \mathbb{C}_\alpha$,

$$\mathbf{G}(s)(\mathfrak{L}u)(s) = [\mathfrak{L}(\mathbf{F}_\infty u)](s) .$$

In particular, $\mathbf{G}$ is bounded on $\mathbb{C}_\alpha$ for all $\alpha > \omega(\mathbf{T})$. The function $\mathbf{G}$ is called the *transfer function* of $\mathbf{\Sigma}$.

$\mathbf{\Sigma}$ and its transfer function $\mathbf{G}$ are said to be *regular* if, for any $u \in U$, the limit

$$\lim_{s \to \infty,\, s \in \mathbb{R}} \mathbf{G}(s) u = Du$$

exists. It follows, from the principle of uniform boundedness, that $D \in \mathcal{B}(U, Y)$. The operator D is called the *feedthrough operator* of $\mathbf{\Sigma}$. If $\mathbf{\Sigma}$ is regular, then for any $x_0 \in X$ and $u \in L^2_{loc}(\mathbb{R}_+, U)$ the functions $x(\cdot)$ and $y(\cdot)$, defined by (2.1) and (2.2), satisfy the equations

$$\dot{x}(t) = Ax(t) + Bu(t) , \quad x(0) = x_0 , \tag{2.5}$$

$$y(t) = C_L x(t) + Du(t) \tag{2.6}$$

for a.e. $t \geq 0$ (in particular $x(t) \in \operatorname{dom}(C_L)$ for a.e. $t \geq 0$). The derivative on the left-hand side of (2.5) has to be understood in X_{-1}. In other words, if we consider the initial value problem (2.5) in the space X_{-1}, then for any $x_0 \in X$ and $u \in L^2_{loc}(\mathbb{R}_+, U)$ the classical solution of (2.5) is given by the variation of parameters formula

$$x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau .$$

It has been demonstrated in [36] that if $\mathbf{\Sigma}$ is regular, then $(sI - A)^{-1} BU \subset \operatorname{dom}(C_L)$ for all $s \in \varrho(A)$ and the transfer function $\mathbf{G}$ can be expressed in the following way:

$$\mathbf{G}(s) = C_L (sI - A)^{-1} B + D \qquad \text{for all } s \in \mathbb{C}_{\omega(\mathbf{T})} ,$$

which is familiar from finite-dimensional systems theory. The operators A, B, C, and D are called the *generating operators* of $\mathbf{\Sigma}$.

The following lemma will be needed in section 3. Certainly, it should be well known. However, since we could not find it in the literature, we include the proof.

LEMMA 2.1. *Suppose that $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ is exponentially stable. Then the following statements hold:*

(1) *There exist $\alpha, \beta > 0$ such that, for any $x_0 \in X$ and any $u \in L^2(\mathbb{R}_+, U)$, the solution $x(\cdot)$ of the initial-value problem* (2.5) *satisfies*

$$\|x\|_{L^2(\mathbb{R}_+, X)} \leq \alpha \|u\|_{L^2(\mathbb{R}_+, U)} + \beta \|x_0\|.$$

(2) *If $u \in L^\infty(\mathbb{R}_+, U)$ and $\lim_{t \to \infty} u(t) = u_\infty$ exists, then for any $x_0 \in X$, $x(\cdot)$ defined by* (2.5) *satisfies*

$$\lim_{t \to \infty} \|x(t) + A^{-1} B u_\infty\| = 0 .$$

Proof. By the exponential stability we may assume, without loss of generality, that $x_0 = 0$. Consequently, we have $x(t) = \int_0^t \mathbf{T}_{t-\tau} Bu(\tau)\, d\tau$ for all $t \geq 0$. Let $H^2(\mathbb{C}_0, X)$ denote the usual Hardy space of holomorphic functions defined on $\mathbb{C}_0$ with values in X. Appealing to the Paley–Wiener theorem, statement (1) will follow if we can show that there exists $\alpha > 0$ such that, for all $u \in L^2(\mathbb{R}_+, U)$,

$$\|\mathfrak{L}x\|_{H^2(\mathbb{C}_0,X)} \leq \alpha \|\mathfrak{L}u\|_{H^2(\mathbb{C}_0,U)}\,. \tag{2.7}$$

To this end, set $\omega_0 := \omega(\mathbf{T})$ and recall from [35] that for any $\omega > \omega_0$ there exists $M_\omega > 0$ such that, for all $s \in \mathbb{C}_\omega$,

$$\|(sI - A)^{-1}B\|_{\mathcal{B}(U,X)} \leq \frac{M_\omega}{\sqrt{\operatorname{Re}\, s - \omega}}\,. \tag{2.8}$$

It is clear that $s \mapsto (sI-A)^{-1}B$ is a holomorphic $\mathcal{B}(U, X_{-1})$-valued function: using the resolvent identity, it follows that it is also holomorphic as a $\mathcal{B}(U, X)$-valued function. The Laplace transform $\mathfrak{L}x$ of x satisfies

$$(\mathfrak{L}x)(s) = (sI - A)^{-1}B(\mathfrak{L}u)(s) \quad \text{for all } s \in \mathbb{C}_{\omega_0}\,. \tag{2.9}$$

By hypothesis, $\omega_0 < 0$ and $\mathfrak{L}u \in H^2(\mathbb{C}_0, X)$. Therefore, choosing $\omega_1 \in (\omega_0, 0)$ and combining (2.8) and (2.9) we see that (2.7) holds with, for example, $\alpha = M_{\omega_1}/\sqrt{|\omega_1|}$. This establishes statement (1).

To prove statement (2), we proceed as follows. Choose $t^* > 0$ such that $\|\mathbf{T}_t\| \leq 1/2$ for all $t \geq t^*$, let (t_n) be a sequence of real numbers satisfying

$$t^* \leq t_{n+1} - t_n \leq 2t^*\,,$$

and set $\beta = \sup\{\|\mathbf{T}_t\| \,|\, 0 \leq t \leq 2t^*\}$. For $t \geq t_n$ we have

$$x(t) = \mathbf{T}_{t-t_n} x(t_n) + \int_{t_n}^t \mathbf{T}_{t-\tau} Bu(\tau)\, d\tau\,,$$

and so, by exponential stability, (2.3), and statement (1) above, there exists $\alpha > 0$ such that, for all $n \in \mathbb{N}$,

$$\|x(t)\| \leq \beta \|x(t_n)\| + \alpha\sqrt{2t^*}\|u\|_{L^\infty(t_n,t_{n+1})} \qquad \text{if } t \in [t_n, t_{n+1}] \tag{2.10}$$

and

$$\|x(t_{n+1})\| \leq \frac{1}{2}\|x(t_n)\| + \alpha\sqrt{2t^*}\|u\|_{L^\infty(t_n,t_{n+1})}\,. \tag{2.11}$$

We first consider the case when $u_\infty = 0$. Then

$$\lim_{n\to\infty} \|u\|_{L^\infty(t_n,t_{n+1})} = 0 \tag{2.12}$$

and (2.11) implies that

$$\lim_{n\to\infty} \|x(t_n)\| = 0\,. \tag{2.13}$$

Combining (2.10), (2.12), and (2.13) shows that $\lim_{t\to\infty} \|x(t)\| = 0$. Finally, if $u_\infty \neq 0$, then, by writing $u(t) = (u(t) - u_\infty) + u_\infty$, it is clear that it suffices to show that

$$\lim_{t\to\infty} \left\| \int_0^t \mathbf{T}_\tau Bu_\infty\, d\tau + A^{-1}Bu_\infty \right\| = 0\,. \tag{2.14}$$

Setting $z(t) = \int_0^t \mathbf{T}_\tau B u_\infty \, d\tau$ we have that

$$\lim_{t\to\infty} \|\dot{z}(t)\|_{-1} = \lim_{t\to\infty} \|\mathbf{T}_t B u_\infty\|_{-1} = 0\,. \tag{2.15}$$

The function $z(\cdot)$ is the classical solution of the initial-value problem $\dot{z}(t) = Az(t) + Bu_\infty$, $z(0) = 0$, considered in X_{-1}, and so we may write

$$z(\cdot) + A^{-1}Bu_\infty = A^{-1}\dot{z}(\cdot)\,. \tag{2.16}$$

Since $A^{-1} \in \mathcal{B}(X_{-1}, X)$, (2.14) follows from (2.15) and (2.16). □

3. Integral control in the presence of nonlinearities. In the following, let (A, B, C, D) be the generating operators of a linear, single-input, single-output regular system with state space X and transfer function $\mathbf{G}$. Suppose that the system is subject to an input nonlinearity ϕ, where $\phi : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz. Denoting the constant reference signal by r, an application of the integrator

$$u(t) = u_0 + k\int_0^t [r - C_L x(\tau) - D\phi(u(\tau))]\, d\tau\,,$$

where k is a real parameter (see Fig. 1.2), leads to the following nonlinear system of differential equations:

$$\dot{x} = Ax + B\phi(u)\,, \quad x(0) = x_0 \in X, \tag{3.1}$$

$$\dot{u} = k[r - C_L x - D\phi(u)]\,, \quad u(0) = u_0 \in \mathbb{R}\,. \tag{3.2}$$

For $a \in (0, \infty]$, a continuous function

$$[0, a) \to X \times \mathbb{R}\,, \quad t \mapsto (x(t), u(t))$$

is called a *solution* of (3.1)–(3.2) if $(x(\cdot), u(\cdot))$ is absolutely continuous as an $(X_{-1}\times\mathbb{R})$-valued function, $x(t) \in \operatorname{dom}(C_L)$ for a.e. $t \in [0, a)$, $(x(0), u(0)) = (x_0, u_0)$, and the differential equations (3.1) and (3.2) are satisfied a.e. on $[0, a)$. Of course, the derivative on the left-hand side on (3.1) has to be understood in X_{-1}.[1]

An application of a well-known result on abstract Cauchy problems (see Pazy [26, Thm. 2.4, p. 107]), shows that a continuous $(X \times \mathbb{R})$-valued function $(x(\cdot), u(\cdot))$ is a solution of (3.1)–(3.2) if and only if it satisfies the following integrated version of (3.1)–(3.2):

$$x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B\phi(u(\tau))\, d\tau\,, \tag{3.3}$$

$$u(t) = u_0 + k\int_0^t [r - C_L x(\tau) - D\phi(u(\tau))]\, d\tau\,. \tag{3.4}$$

The next result shows that (3.1)–(3.2) has a unique solution.

PROPOSITION 3.1. *For any pair $(x_0, u_0) \in X \times \mathbb{R}$ of initial conditions there exists a unique solution $(x(\cdot), u(\cdot))$ of (3.1)–(3.2) defined on a maximal interval $[0, a_{max})$. If $a_{max} < \infty$, then*

$$\limsup_{t\to a_{max}} \|(x(t), u(t))\| = \infty\,. \tag{3.5}$$

[1] Being a Hilbert space, $X_{-1} \times \mathbb{R}$ is reflexive. Hence any absolutely continuous $(X_{-1} \times \mathbb{R})$-valued function is a.e. differentiable and can be recovered from its derivative by integration; see [1, Thm. 3.1, p. 10].

If ϕ is globally Lipschitz, then $a_{max} = \infty$.

For the proof of the above result it will be useful to consider the following initial-value problem for u:

$$\dot{u} = k[r - \mathbf{\Psi}_\infty x_0 - \mathbf{F}_\infty \phi(u)] , \quad u(0) = u_0 . \tag{3.6}$$

Clearly, (3.6) [2] is obtained from (3.2) on noting that $C_L x(t) + D\phi(u(t)) = (\mathbf{\Psi}_\infty x_0)(t) + (\mathbf{F}_\infty \phi(u))(t)$. An absolutely continuous function $u : [0, a) \to \mathbb{R}$ is a *solution* of (3.6) if $u(0) = u_0$ and the differential equation in (3.6) is satisfied a.e. on $[0, a)$.

LEMMA 3.2. *Let $x_0 \in X$. For any initial condition $u_0 \in \mathbb{R}$ there exists a unique solution $u(\cdot)$ of* (3.6) *defined on a maximal interval $[0, a_{max})$. If $a_{max} < \infty$, then*

$$\limsup_{t \to a_{max}} |u(t)| = \infty . \tag{3.7}$$

If ϕ is globally Lipschitz, then $a_{max} = \infty$.

The proof of this lemma is relegated to the appendix.

Proof of Proposition 3.1. Let $u : [0, a_{max}) \to \mathbb{R}$ be the unique maximal solution of (3.6) (whose existence is guaranteed by Lemma 3.2), and *define* $x(\cdot)$ to be the unique solution of

$$\dot{x} = Ax + B\phi(u) , \quad x(0) = x_0 .$$

Then $(x(\cdot), u(\cdot))$ is the unique solution of equations (3.1)–(3.2), which satisfies equation (3.5) if $a_{max} < \infty$. Moreover, it follows trivially from Lemma 3.2 that $a_{max} = \infty$ if ϕ is globally Lipschitz. ◻

Henceforth, let $\mathcal{M}$ denote the set of all bounded measures on $[0, \infty)$. A measure $\mu \in \mathcal{M}$ can be written in the form

$$\mu(dt) = a(t)dt + \sum_{i=0}^{\infty} a_i \delta_{t_i}(dt) + \mu_s(dt) ,$$

where $a(\cdot) \in L^1(0, \infty)$, $\sum_{i=0}^{\infty} a_i \delta_{t_i}$, and μ_s, respectively, represent the absolutely continuous, the discrete, and the singular parts of μ. In particular, δ_{t_i} denotes the unit point mass at $t_i \geq 0$ and the a_i are real numbers such that $\sum_{i=0}^{\infty} |a_i| < \infty$.

Furthermore, for $\lambda > 0$, let $\mathcal{N}(\lambda)$ denote the set of all nondecreasing globally Lipschitz nonlinearities $\phi : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant λ. Finally, if $\mathbf{G}$ is holomorphic and bounded on $\mathbb{C}_\alpha$ for some $\alpha < 0$ and $\mathbf{G}(0) > 0$, then it is easy to show that

$$1 + k\text{Re}\,\frac{\mathbf{G}(s)}{s} \geq 0 \qquad \text{for all } s \in \mathbb{C}_0 \tag{3.8}$$

for all sufficiently small $k > 0$; see Lemma 3.10 in [17]. We define

$$K := \sup\{k > 0 \,|\, (3.8) \text{ holds}\} .$$

The main result of this section is the following theorem.

THEOREM 3.3. *Let $\lambda > 0$ and $\phi \in \mathcal{N}(\lambda)$. Assume that $\mathbf{T}_t$ is exponentially stable, $\mathbf{G}(0) > 0$, $k \in (0, K/\lambda)$, and $r \in \mathbb{R}$ is such that*

$$\phi_r := [\mathbf{G}(0)]^{-1} r \in \text{clos}\,(\text{im}\,\phi) . \tag{3.9}$$

If C is bounded, then for all $(x_0, u_0) \in X \times \mathbb{R}$ the unique solution $(x(\cdot), u(\cdot))$ of (3.1)–(3.2) *exists on $[0, \infty)$ and satisfies*

[2] Strictly speaking, to make sense of (3.6) we have to give a meaning to $\mathbf{F}_\infty v$ when v is a continuous function defined on a *finite* interval $[0, a)$ (recall that $\mathbf{F}_\infty$ operates on the space of locally square-integrable functions defined on the *infinite* interval $[0, \infty)$). This can easily be done using the causality of $\mathbf{F}_\infty$. Moreover, by slight abuse of notation, the expression $\phi(u)$ on the right-hand side of (3.6) denotes the function $t \mapsto \phi(u(t))$.

(1) $\lim_{t\to\infty} \phi(u(t)) = \phi_r$,
(2) $\lim_{t\to\infty} \|x(t) + A^{-1}B\phi_r\| = 0$,
(3) $\lim_{t\to\infty}(r - y(t)) = 0$, *where* $y(t) = Cx(t) + D\phi(u(t))$,
(4) *if* $\phi_r \in \operatorname{im}\phi$, *then*

$$\lim_{t\to\infty} \operatorname{dist}(u(t), \phi^{-1}(\phi_r)) = 0\,, \tag{3.10}$$

(5) *if* $\phi_r \in \operatorname{int}(\operatorname{im}\phi)$, *then* $u(\cdot)$ *is bounded.*

If C *is unbounded, then the statements* (1), (2), (4), *and* (5) *remain true provided that* $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}$ *and statement* (3) *remains true provided that* $x_0 \in \operatorname{dom}(A)$ *and* $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}$.

In particular, statement (4) says that $u(t)$ converges as $t \to \infty$ if the set $\phi^{-1}(\phi_r)$ is a singleton, which, in turn, is true if ϕ_r is not a critical value of ϕ.

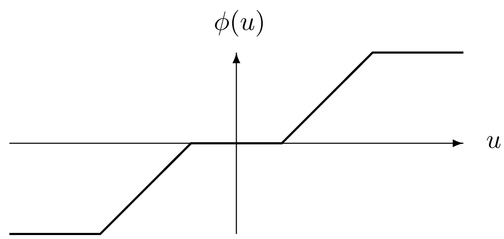

FIG. 3.1. *Nonlinearity with saturation and deadzone.*

The conditions imposed in Theorem 3.3 on ϕ are satisfied by saturation and deadzone nonlinearities and combinations of the two, as shown in Fig. 3.1. The assumption that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}$ is not very restrictive and seems to be satisfied in all practical examples of systems with H^∞-transfer functions (in applications one usually has $\mu_s = 0$). If C is unbounded and $x_0 \notin \operatorname{dom}(A)$, then statement (3) does not hold in general. However, in that case, as an inspection of the proof of Theorem 3.3 will show, the error $e(\cdot) = r - y(\cdot)$ admits a decomposition $e = e_1 + e_2$, where $e_1 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$ and e_2 is a continuous function satisfying $\lim_{t\to\infty} e_2(t) = 0$. Thus, while the error does not necessarily converge asymptotically to 0 as $t \to \infty$, it is small for large t in the sense that for all $\delta, \varepsilon > 0$ there exists $T > 0$ such that

$$\operatorname{meas}(\{t \geq T \mid |e(t)| \geq \delta\}) \leq \varepsilon\,,$$

where meas denotes the Lebesgue measure. In applying Theorem 3.3 it is important to know the constant K or at least a lower bound for K. In principle, K can be obtained from frequency-response experiments performed on the linear part of the plant; see [15] for details.

For the proof of Theorem 3.3 two lemmas are required, the proofs of which can be found in the appendix.

LEMMA 3.4. *Suppose that* $\mathbf{T}_t$ *is exponentially stable and* $\mathbf{G}(0) > 0$. *Define*

$$\mathbf{H}(s) = \frac{1}{s}(\mathbf{G}(s) - \mathbf{G}(0))\,.$$

If $0 < 2\kappa < K$, *then*

$$\|\mathbf{H}(1 + \kappa\mathbf{H})^{-1}\|_\infty < \frac{1}{\kappa} \tag{3.11}$$

and there exists $P \in \mathcal{B}(X)$, with $P = P^* \geq 0$ and such that the Riccati equation

$$\langle A_\kappa x_1, Px_2\rangle + \langle Px_1, A_\kappa x_2\rangle + \kappa^2\langle C_L x_1, C_L x_2\rangle + \langle (A^{-1}B)^* Px_1, (A^{-1}B)^* Px_2\rangle = 0 \tag{3.12}$$

is satisfied for all $x_1, x_2 \in \mathrm{dom}(A_\kappa) = \mathrm{dom}(A)$, where $A_\kappa := A - \kappa A^{-1}BC_L$.

LEMMA 3.5. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and (ε_n) be any sequence with $\varepsilon_n > 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$. Define the function $\phi^\diamond : \mathbb{R} \to \mathbb{R}$ by*

$$\phi^\diamond(\xi) = \limsup_{n\to\infty} \frac{\phi(\xi + \varepsilon_n) - \phi(\xi)}{\varepsilon_n}.$$

Then $\phi^\diamond \in L^\infty_{loc}(-\infty, \infty)$ ($\phi^\diamond \in L^\infty(-\infty, \infty)$ if ϕ is globally Lipschitz) and $\phi^\diamond \circ u$ is Lebesgue measurable for all Lebesgue measurable functions $u : [0, \infty) \to \mathbb{R}$. If u is absolutely continuous, so is $\phi \circ u$ and

$$\frac{d}{dt}(\phi \circ u)(t) = \phi^\diamond(u(t))\dot{u}(t) \quad \text{for a.e. } t \in [0, \infty).$$

Proof of Theorem 3.3. By Proposition 3.1, there exists a unique solution of (3.1)–(3.2) on $[0, \infty)$. We denote this solution by $(x(\cdot), u(\cdot))$ and introduce new variables by defining

$$z(t) := x(t) + A^{-1}B\phi(u(t)), \quad v(t) := \phi(u(t)) - \phi_r \quad \text{for all } t \geq 0.$$

By regularity it follows that $z(t) \in \mathrm{dom}\,(C_L)$ for a.e. $t \in [0, \infty)$. Moreover, by Lemma 3.5, $\dot{v}(t) = \phi^\diamond(u(t))\dot{u}(t)$ for a.e. $t \in [0, \infty)$. Therefore, an easy calculation yields

$$\dot{z} = Az - k\phi^\diamond(u)A^{-1}B(C_L z + \mathbf{G}(0)v), \quad z(0) = z_0 := x_0 + A^{-1}B\phi(u_0), \tag{3.13}$$

$$\dot{v} = -k\phi^\diamond(u)(C_L z + \mathbf{G}(0)v), \quad v(0) = v_0 := \phi(u_0) - \phi_r. \tag{3.14}$$

The derivative on the left-hand side of (3.13) and (3.14) has to be understood in X_{-1}. Notice that, since ϕ is nondecreasing, $\phi^\diamond(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. We observe that, while in these new variables we still have an unbounded operator $A^{-1}BC_L$, the operator $A^{-1}B$ is in $\mathcal{B}(\mathbb{R}, X)$. We will investigate the stability properties of (3.13) and (3.14) using a Liapunov approach.

Since $0 < k\lambda < K$, it follows that there exists $\mu > \lambda/2$ such that $0 < 2\mu k < K$, and therefore, by Lemma 3.4,

$$\|\mathbf{H}(1 + \mu k\mathbf{H})^{-1}\|_\infty < \frac{1}{\mu k}.$$

By the same lemma, the Riccati equation (3.12) with $\kappa = \mu k$ has a solution $P \in \mathcal{B}(X)$ satisfying $P = P^* \geq 0$. Set

$$\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & \mu k\mathbf{G}(0) \end{pmatrix},$$

and define

$$\tilde{A}_k = \begin{pmatrix} A - \mu kA^{-1}BC_L & -\mu kA^{-1}B\mathbf{G}(0) \\ -\mu kC_L & -\mu k\mathbf{G}(0) \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} A^{-1}B \\ 1 \end{pmatrix}, \quad \tilde{C} = (C_L \;\; \mathbf{G}(0)),$$

where $\mathrm{dom}(\tilde{A}_k) = \mathrm{dom}(A) \times \mathbb{R}$. The operator $\tilde{A}_k$ generates a C_0-semigroup. Using (3.12), it is easy to show that

$$\langle \tilde{A}_k \tilde{x}_1, \tilde{P}\tilde{x}_2\rangle + \langle \tilde{P}\tilde{x}_1, \tilde{A}_k\tilde{x}_2\rangle + \mu^2 k^2 \langle \tilde{C}\tilde{x}_1, \tilde{C}\tilde{x}_2\rangle + \langle \tilde{B}^*\tilde{P}\tilde{x}_1, \tilde{B}^*\tilde{P}\tilde{x}_2\rangle = 0 \tag{3.15}$$

is satisfied for all $\tilde{x}_1, \tilde{x}_2 \in \mathrm{dom}(\tilde{A}_k)$.

Setting $\tilde{z}(\cdot) = (z(\cdot), v(\cdot))$, (3.13) and (3.14) can be reformulated as

$$\dot{\tilde{z}} = \tilde{A}_k \tilde{z} + k(\mu - \phi^\diamond(u))\tilde{B}\tilde{C}\tilde{z}\,, \quad \tilde{z}(0) = \tilde{z}_0 := \begin{pmatrix} z_0 \\ v_0 \end{pmatrix}, \tag{3.16}$$

where the derivative on the left-hand side has to be understood in $X_{-1} \times \mathbb{R}$. For an intermediate step in the Liapunov analysis we need differentiability in $X \times \mathbb{R}$, and therefore, we will use an approximation argument. To this end let $T > 0$ be fixed but arbitrary, and choose $(w_n) \subset W^{1,2}(0, T; \mathbb{R})$ and $(\tilde{z}_0^n) \subset \mathrm{dom}(\tilde{A}_k)$ such that

$$\lim_{n\to\infty} \|k(\mu - \phi^\diamond(u))\tilde{C}\tilde{z} - w_n\|_{L^2(0,T)} = 0\,, \quad \lim_{n\to\infty} \|\tilde{z}_0 - \tilde{z}_0^n\|_{X\times\mathbb{R}} = 0\,. \tag{3.17}$$

Consider the system

$$\dot{\eta}(t) = \tilde{A}_k \eta(t) + \tilde{B} w_n(t)\,, \qquad \eta(0) = \tilde{z}_0^n\,. \tag{3.18}$$

$$\xi(t) = \tilde{C}\eta(t)\,. \tag{3.19}$$

The abstract initial-value problem (3.18) has a strong solution $\tilde{z}_n$ on $[0, T]$ in the sense that $\tilde{z}_n(0) = \tilde{z}_0^n$ and (3.18) is satisfied for a.e. $t \in [0, T]$ (see Pazy [26, Cor. 2.10, p. 109]). Using (3.17) we obtain

$$\lim_{n\to\infty} \|\tilde{z} - \tilde{z}_n\|_{L^2(0,T)} = 0\,; \qquad \lim_{n\to\infty} \|\tilde{z}(t) - \tilde{z}_n(t)\|_{X\times\mathbb{R}} = 0 \quad \text{for all } t \in [0, T]\,. \tag{3.20}$$

Setting $\xi_n(t) = \tilde{C}\tilde{z}_n(t)$, it follows from the regularity of (3.18) that

$$\lim_{n\to\infty} \|\tilde{C}\tilde{z} - \xi_n\|_{L^2(0,T)} = 0\,. \tag{3.21}$$

Differentiating the function

$$\tau \mapsto V_n(\tau) = \langle \tilde{z}_n(\tau), \tilde{P}\tilde{z}_n(\tau)\rangle$$

shows that, for a.e. $\tau \in [0, T]$,

$$\dot{V}_n(\tau) = \langle \tilde{z}_n(\tau), \tilde{P}\tilde{A}_k\tilde{z}_n(\tau)\rangle + \langle \tilde{A}_k\tilde{z}_n(\tau), \tilde{P}\tilde{z}_n(\tau)\rangle + 2\langle \tilde{B}w_n(\tau), \tilde{P}\tilde{z}_n(\tau)\rangle\,. \tag{3.22}$$

If $t \in [0, T]$, then integrating (3.22) from 0 to t, taking limits as $n \to \infty$, invoking (3.15), (3.17), (3.20), and (3.21), and setting

$$V(\tau) = \langle \tilde{z}(\tau), \tilde{P}\tilde{z}(\tau)\rangle$$

we obtain

$$V(t) - V(0) = -\int_0^t \mu^2 k^2 (\tilde{C}\tilde{z})^2 - \int_0^t (\tilde{B}^*\tilde{P}\tilde{z})^2 + 2\int_0^t \langle \tilde{B}k(\mu - \phi^\diamond(u))\tilde{C}\tilde{z}, \tilde{P}\tilde{z}\rangle\,.$$

Completing the square gives

$$V(t) - V(0) = -\int_0^t [\mu^2 k^2 - k^2(\phi^\diamond(u) - \mu)^2](\tilde{C}\tilde{z})^2 - \int_0^t [k(\phi^\diamond(u) - \mu)\tilde{C}\tilde{z} + \tilde{B}^*\tilde{P}\tilde{z}]^2\,,$$

and hence

$$V(t) - V(0) = -k^2 \int_0^t [2\mu\phi^\diamond(u) - (\phi^\diamond)^2(u)](\tilde{C}\tilde{z})^2 - \int_0^t [k(\phi^\diamond(u) - \mu)\tilde{C}\tilde{z} + \tilde{B}^*\tilde{P}\tilde{z}]^2 \,, \tag{3.23}$$

which holds for all $t \in [0, T]$. Since $T > 0$ was arbitrary, it follows that (3.23) holds for all $t \geq 0$. Therefore, using (3.23) and the definition of $\tilde{C}$,

$$k^2 \int_0^t (2\mu\phi^\diamond(u) - (\phi^\diamond)^2(u))(C_L z + \mathbf{G}(0)v)^2 \leq V(0) < \infty \quad \text{for all } t \geq 0\,. \tag{3.24}$$

Now recall that $2\mu > \lambda$ and $\|\phi^\diamond(u)\|_{L^\infty(\mathbb{R}_+)} \leq \lambda$, so that

$$2\mu\phi^\diamond(u) - (\phi^\diamond)^2(u) > \varepsilon(\phi^\diamond)^2(u)$$

for some $\varepsilon > 0$. Therefore, (3.24) gives

$$\varepsilon k^2 \int_0^t (\phi^\diamond)^2(u)(C_L z + \mathbf{G}(0)v)^2 \leq V(0) < \infty \qquad \text{for all } t \geq 0\,.$$

It follows that

$$\phi^\diamond(u)(C_L z + \mathbf{G}(0)v) \in L^2(\mathbb{R}_+)\,. \tag{3.25}$$

Using this in (3.13) and appealing to the fact that A, $A^{-1}B$, and C are the generating operators of a stable regular system we may conclude that

$$C_L z \in L^2(\mathbb{R}_+)\,. \tag{3.26}$$

Hence, by (3.25) and the boundedness of $\phi^\diamond(u)$,

$$\phi^\diamond(u)v \in L^2(\mathbb{R}_+), \tag{3.27}$$

and thus

$$(C_L z)\phi^\diamond(u)v \in L^1(\mathbb{R}_+)\,. \tag{3.28}$$

Using (3.24), (3.26)–(3.28), and the boundedness of $\phi^\diamond(u)$ it follows that

$$\phi^\diamond(u)v^2 \in L^1(\mathbb{R}_+)\,. \tag{3.29}$$

Multiplying (3.14) by $v(t)$, integrating, and then using (3.28) and (3.29) shows that

$$\lim_{t\to\infty} v^2(t) = v_0^2 + 2\lim_{t\to\infty}\int_0^t v\dot{v} = \nu$$

for some $\nu \in [0, \infty)$. By continuity of $v(\cdot)$ it follows that

$$\lim_{t\to\infty} v(t) = \sqrt{\nu} \qquad \text{or} \qquad \lim_{t\to\infty} v(t) = -\sqrt{\nu}\,.$$

In the following we distinguish two cases: bounded and unbounded observation.

Let us first consider the case of bounded C. In order to prove statement (1), we have to show that $\nu = 0$. Seeking a contradiction, suppose that $\nu > 0$. Assuming that $\lim_{t\to\infty} v(t) = \sqrt{\nu}$ (the case $\lim_{t\to\infty} v(t) = -\sqrt{\nu}$ can be dealt with in an entirely analogous fashion), we obtain that

$$\phi_\infty := \lim_{t\to\infty} \phi(u(t)) > \phi_r\,. \tag{3.30}$$

By Lemma 2.1, part (2), it follows that

$$\lim_{t\to\infty} \|x(t) + A^{-1}B\phi_\infty\| = 0\,. \tag{3.31}$$

Using the boundedness of C it follows from (3.2), (3.30), and (3.31) that

$$\lim_{t\to\infty} \dot{u}(t) = k(r + CA^{-1}B\phi_\infty - D\phi_\infty) = k\mathbf{G}(0)(\phi_r - \phi_\infty) < 0\,,$$

and so

$$\lim_{t\to\infty} u(t) = -\infty\,. \tag{3.32}$$

Since ϕ is nondecreasing we obtain

$$\phi_\infty = \lim_{t\to\infty} \phi(u(t)) = \inf(\operatorname{im}\phi) \le \phi_r\,,$$

contradicting (3.30). Therefore, $\nu = 0$, and consequently $\lim_{t\to\infty} \phi(u(t)) = \phi_r$, which is statement (1). Statement (2) follows now from Lemma 2.1, part (2), and statement (3) is a consequence of statements (1) and (2).

To prove statement (4), let $\phi_r \in \operatorname{im}\phi$. Seeking a contradiction, suppose that the claim is not true. Then there exists a sequence of positive numbers (t_n) with $\lim_{n\to\infty} t_n = \infty$ and $\varepsilon > 0$ such that

$$\operatorname{dist}(u(t_n), \phi^{-1}(\phi_r)) \ge \varepsilon\,. \tag{3.33}$$

If the sequence $(u(t_n))$ is bounded, we may assume, without loss of generality, that it converges to a finite limit u_∞. By continuity of ϕ and statement (1) we have that $\phi(u_\infty) = \phi_r$, and thus $u_\infty \in \phi^{-1}(\phi_r)$. This contradicts (3.33). So, suppose that $(u(t_n))$ is unbounded. Without loss of generality, we may then assume that $\lim_{n\to\infty} u(t_n) = \infty$. By monotonicity and statement (1) it follows that $\phi_r = \sup\phi$. Since $\phi_r \in \operatorname{im}\phi$ there exists ξ^* such that

$$\phi(\xi^*) = \phi_r = \sup\phi = \max\phi\,.$$

By monotonicity of ϕ we have

$$\phi(\xi) = \phi_r = \max\phi \qquad \text{for all } \xi \ge \xi^*\,.$$

In particular, we see that $u(t_n) \in \phi^{-1}(\phi_r)$ for all sufficiently large n, contradicting (3.33).

To prove statement (5) assume that $\phi_r \in \operatorname{int}(\operatorname{im}\phi)$. Again seeking a contradiction, suppose that the claim is not true. Then there exists a sequence of positive numbers (t_n) with $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} |u(t_n)| = \infty$. Without loss of generality, we may assume that $\lim_{n\to\infty} u(t_n) = \infty$. By monotonicity it then follows that

$$\phi_r = \lim_{n\to\infty} \phi(u(t_n)) = \sup\phi\,,$$

contradicting the hypothesis $\phi_r \in \operatorname{int}(\operatorname{im}\phi)$.

Now let us consider the case of unbounded C with $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}$. We will again be seeking a contradiction, and hence assume that $\nu > 0$. It is clear that (3.30) and (3.31) still hold. It only remains to show that (3.32) is also true in this case. To this end, write (3.2) in the form

$$\dot{u} = k[r - C_L \mathbf{T}_t x_0 - \mathfrak{L}^{-1}(\mathbf{G}) \star \phi(u)]\,. \tag{3.34}$$

Since $\lim_{t\to\infty} \phi(u(t)) = \phi_\infty$ and $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}$ it follows that $\lim_{t\to\infty}(\mathfrak{L}^{-1}(\mathbf{G}) \star \phi(u))(t) = \mathbf{G}(0)\phi_\infty$ (see [8, Thm. 6.1, part (ii), p. 96]). Therefore, by (3.30) there exists $\delta > 0$ and $T > 0$ such that

$$\mathbf{G}(0)\phi_r - (\mathfrak{L}^{-1}(\mathbf{G}) \star \phi(u))(t) \leq -\delta \qquad \text{for all } t \geq T\,. \tag{3.35}$$

Integrating (3.34) from T to t and using (3.35) gives

$$u(t) \leq u(T) + k\left[\int_T^t |C_L \mathbf{T}_\tau x_0|\, d\tau - \delta(t - T)\right]. \tag{3.36}$$

By exponential stability of $\mathbf{T}_t$ we have that the map $t \mapsto C_L \mathbf{T}_t x_0$ is in $L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$, and hence in $L^1(\mathbb{R}_+, \mathbb{R})$. As a consequence, (3.36) yields

$$\lim_{t\to\infty} u(t) = -\infty\,,$$

which is (3.32). Statements (2), (4), and (5) then follow as in the case of bounded C. Finally, write $y(t)$ in the form

$$y(t) = C_L \mathbf{T}_t x_0 + (\mathfrak{L}^{-1}(\mathbf{G}) \star \phi(u))(t)\,.$$

Under the assumption that $x_0 \in \operatorname{dom}(A)$ and $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}$, we obtain

$$\lim_{t\to\infty} y(t) = \mathbf{G}(0) \lim_{t\to\infty} \phi(u(t))\,.$$

Combining this with statement (1) yields statement (3). □

One of the conditions imposed in Theorem 3.3 is that $[\mathbf{G}(0)]^{-1} r \in \operatorname{clos}(\operatorname{im}\phi)$. The following proposition shows that this condition is necessary for solvability of the tracking problem.

Proposition 3.6. *Let $r \in \mathbb{R}$, and suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is continuous, $\mathbf{T}_t$ is exponentially stable, and $\mathbf{G}(0) \neq 0$. If there exist an initial condition $x_0 \in X$ and a continuous function $u : [0,\infty) \to \mathbb{R}$ such that $\phi(u(\cdot))$ is bounded and*

$$\lim_{t\to\infty}[C_L x(t) + D\phi(u(t))] = r\,,$$

where $x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B\phi(u(\tau))\, d\tau$, then $\phi_r = [\mathbf{G}(0)]^{-1} r \in \operatorname{clos}(\operatorname{im}\phi)$.

The proof of the above proposition requires some preparation. Recall the concept of an ω-limit point (and ω-limit set $\Omega(\psi)$) of a continuous function $\psi : [0,\infty) \to \mathbb{R}$. A point ψ^* is an *ω-limit point* of ψ if there exists an increasing sequence $(t_n) \subset [0,\infty)$ such that $t_n \to \infty$ and $\psi(t_n) \to \psi^*$ as $n \to \infty$. The set $\Omega(\psi)$ of all ω-limit points is the *ω-limit set* of ψ.

The following lemma is probably standard; however, we were unable to locate it in the literature and so include a proof for completeness.

LEMMA 3.7. *Let $\psi : [0,\infty) \to \mathbb{R}$ be continuous and bounded. Then*

$$\lim_{s\to 0,\, s>0}[s(\mathfrak{L}\psi)(s)] = \omega \quad \Longrightarrow \quad \omega \in \Omega(\psi)\,.$$

Proof. It suffices to prove the result in the case $\omega = 0$ (if $\omega \neq 0$, then simply replace ψ by $\psi_\omega : t \mapsto \psi(t) - \omega$). It is well known that $\Omega(\psi)$ is compact and is approached by $\psi(t)$ as $t \to \infty$ (see, for example, [10, p. 113]). Seeking a contradiction, suppose $0 \notin \Omega(\psi)$. Then there exists $\varepsilon > 0$ and $T > 0$ such that for all $t \geq T$, $|\psi(t)| \geq \varepsilon$. Since ψ is continuous, we may restrict our attention, without loss of generality, to the case $\psi(t) \geq \varepsilon$ for all $t \geq T$. Then, for all $s \in (0,\infty)$, we have

$$(\mathfrak{L}\psi)(s) = \int_0^\infty e^{-st}\psi(t)\,dt \geq \int_0^T e^{-st}\psi(t)\,dt + \varepsilon\int_T^\infty e^{-st}\,dt \tag{3.37}$$

$$= \int_0^T e^{-st}\psi(t)\,dt + \frac{\varepsilon e^{-sT}}{s}\,, \tag{3.38}$$

whence the contradiction

$$0 = \lim_{s\to 0,\, s>0} s(\mathfrak{L}\psi)(s) \geq \varepsilon > 0\,.$$

□

Proof of Proposition 3.6. For $\delta \in (0,\pi/2)$ define the open sector $\mathcal{S}(\delta) \subset \mathbb{C}_0$ by

$$\mathcal{S}(\delta) := \{\rho e^{i\alpha} \mid \rho \in (0,\infty),\, \alpha \in (-\delta,\delta)\}\,.$$

Setting $\psi(t) = \phi(u(t))$ and $y(t) = C_L x(t) + D\psi(t)$ we obtain

$$(\mathfrak{L}y)(s) = \mathbf{G}(s)(\mathfrak{L}\psi)(s) + C(sI - A)^{-1}x_0\,,$$

and so by the final-value theorem (see [7, Satz 34.2] or [25, Thm. 14, p. 95])

$$r = \lim_{t\to\infty} y(t) = \lim_{s\to 0,\, s\in\mathcal{S}(\delta)} s(\mathfrak{L}y)(s) = \lim_{s\to 0,\, s\in\mathcal{S}(\delta)} s\mathbf{G}(s)(\mathfrak{L}\psi)(s)\,.$$

Since $\mathbf{G}(0) \neq 0$ it follows using Lemma 3.7 that

$$\phi_r = [\mathbf{G}(0)]^{-1}r = \lim_{s\to 0,\, s\in\mathcal{S}(\delta)} s(\mathfrak{L}\psi)(s) \in \Omega(\psi) \subset \operatorname{clos}(\operatorname{im}\phi)\,. \qquad \square$$

A result similar to Proposition 3.6 was stated without proof by Miller and Davison [22] in a finite-dimensional context. However, their approach (as outlined by Miller [21]) does not extend to infinite-dimensional regular systems.

4. Example: Controlled diffusion process with output delay. Consider a diffusion process (with diffusion coefficient $a > 0$ and with Dirichlet boundary conditions), on the one-dimensional spatial domain $[0,1]$, with scalar nonlinear pointwise control action (applied at point $x_b \in (0,1)$ via a nonlinearity ϕ with Lipschitz constant $\lambda > 0$) and delayed (delay $h \geq 0$) pointwise scalar observation (output at point $x_c \in (0,1)$, $x_c \geq x_b$.). We formally write this single-input, single-output system as

$$z_t(t,x) = az_{xx}(t,x) + \delta(x - x_b)\phi(u(t)), \quad y(t) = z(t-h, x_c),$$
$$z(t,0) = 0 = z(t,1) \qquad \text{for all } t > 0\,.$$

For simplicity, we assume zero initial conditions as follows:

$$z(t,x) = 0 \qquad \text{for all } (t,x) \in [-h,0] \times [0,1]\,.$$

With input $\phi(u(\cdot))$ and output $y(\cdot)$, this example qualifies as a regular linear system with transfer function given by

$$\mathbf{G}(s) = \frac{e^{-sh} \sinh\left(x_b\sqrt{(s/a)}\right) \sinh\left((1-x_c)\sqrt{(s/a)}\right)}{a\sqrt{(s/a)}\sinh\sqrt{(s/a)}}\,.$$

In this case, a detailed analysis (see [15] for related investigations) yields

$$\begin{aligned} K &:= \sup\{k > 0 \mid (3.8) \text{ holds}\} \\ &= \frac{1}{|\mathbf{G}'(0)|} = \frac{6a^2}{x_b(1-x_c)(6ha + 1 - x_b^2 - (1-x_c)^2)}. \end{aligned}$$

Therefore, by Theorem 3.3, for each $k \in (0, K/\lambda)$, the integral control

$$u(t) = k\int_0^t [r - y(t)]\,dt$$

guarantees asymptotic tracking of every constant reference signal r satisfying

$$\frac{r}{\mathbf{G}(0)} = \frac{ar}{x_b(1-x_c)} \in \operatorname{clos}(\operatorname{im}\phi)\,.$$

For purposes of illustration, we adopt the following values:

$$a = 0.1, \quad x_b = \frac{1}{3}, \quad x_c = \frac{2}{3}, \quad h = 1, \quad r = 1.$$

We consider a nonlinearity ϕ of saturation type, defined as follows:

$$u \mapsto \phi(u) := \begin{cases} 1, & u \geq 1, \\ u, & u \in (0,1), \\ 0, & u \leq 0 \end{cases}$$

in which case $\lambda = 1$ and

$$K = \frac{243}{620}\ (\approx 0.3919).$$

For $r = 1$, we have

$$\frac{r}{\mathbf{G}(0)} = \frac{a}{x_b(1-x_c)} = 0.9 \in [0,1] = \operatorname{clos}(\operatorname{im}\phi)\,.$$

In each of the following three cases of admissible controller gains

$$\text{(i) } k = 0.39, \quad \text{(ii) } k = 0.26, \quad \text{(iii) } k = 0.13,$$

Fig. 4.1 depicts the output behavior of the system under integral control, while Fig. 4.2 depicts the corresponding control input. These figures were generated using SIMULINK Simulation Software within MATLAB wherein a truncated eigenfunction expansion, of order 10, was adopted to model the diffusion process.

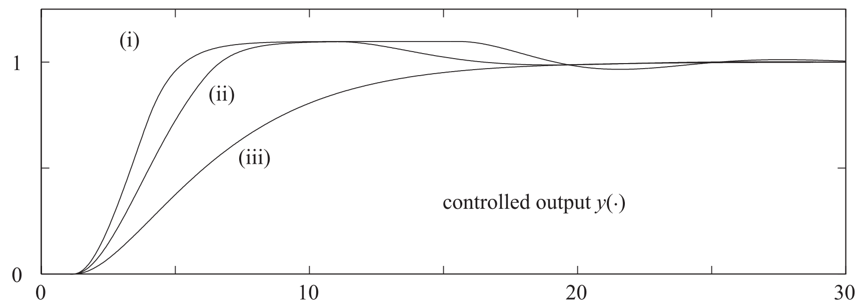

FIG. 4.1. *Controlled output.*

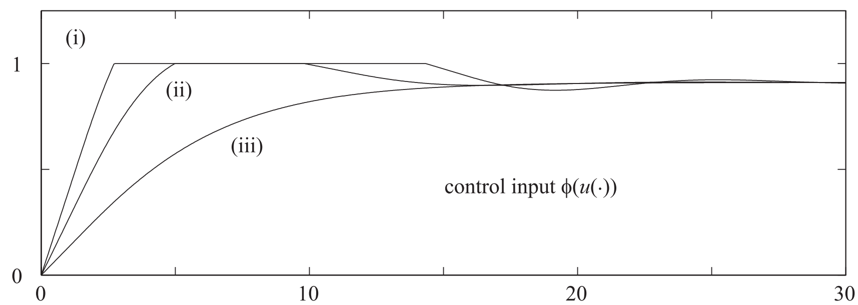

FIG. 4.2. *Control input.*

Appendix.

Proof of Lemma 3.2. In proving Lemma 3.2, we will study an initial-value problem which is slightly more general than (3.6). Let $\alpha \geq 0$, and let $w \in C([0,\alpha],\mathbb{R})$. Consider the initial-value problem

$$\dot{u}(t) = k[r - (\mathbf{\Psi}_\infty x_0)(t) - (\mathbf{F}_\infty \phi(u))(t)]\,, \quad t \geq \alpha\,, \tag{A.1}$$

$$u(t) = w(t)\,, \quad t \in [0,\alpha]\,. \tag{A.2}$$

LEMMA A.1. *Let $x_0 \in X$. For any initial function $w \in C([0,\alpha],\mathbb{R})$ there exists $\varepsilon > 0$ and a unique function $u \in C([0,\alpha+\varepsilon],\mathbb{R})$ with $u(t) = w(t)$ for all $t \in [0,\alpha]$ and such that u is absolutely continuous on $[\alpha,\alpha+\varepsilon]$ and* (A.1) *is satisfied for a.e. $t \in [\alpha,\alpha+\varepsilon]$.*

Proof. Without loss of generality, we may assume that $k = 1$. For $\delta > 0$ and $\eta > \|w\|_\infty$, define

$$\mathcal{C}_{\delta,\eta} = \{u \in C([0,\alpha+\delta],\mathbb{R}) \mid \ |u(t) - w(t)| \leq \eta \quad \text{if } 0 \leq t \leq \alpha\,; \quad |u(t) - w(\alpha)| \leq \eta \quad \text{if } \alpha \leq t \leq \alpha+\delta\}\,.$$

Choosing $\eta > \|w\|_\infty$ guarantees that $\mathcal{C}_{\delta,\eta}$ contains the zero function. Using the causality of $\mathbf{F}_\infty$, the boundedness of the operators $\mathbf{P}_t\mathbf{F}_\infty$, and the Lipschitz continuity of ϕ, it is clear that, for given numbers $\delta > 0$ and $\eta > \|w\|_\infty$, there exists $\lambda > 0$ such that, for all $\varepsilon \in (0,\delta]$ and all $u, v \in \mathcal{C}_{\varepsilon,\eta}$,

$$\int_\alpha^{\alpha+\varepsilon} |\mathbf{F}_\infty\phi(u) - \mathbf{F}_\infty\phi(v)|^2 \leq \lambda^2 \int_0^{\alpha+\varepsilon} |u-v|^2\,.$$

Using Hölder's inequality we obtain the estimate

$$\int_{\alpha}^{\alpha+\varepsilon} |\mathbf{F}_\infty \phi(u) - \mathbf{F}_\infty \phi(v)| \leq \lambda\sqrt{\varepsilon}\left(\int_0^{\alpha+\varepsilon} |u-v|^2\right)^{1/2}, \tag{A.3}$$

which holds for all $u, v \in \mathcal{C}_{\varepsilon,\eta}$, and all $\varepsilon \in (0, \delta]$. Moreover, if $v = 0$, then we may conclude that, for all $u \in \mathcal{C}_{\varepsilon,\eta}$ and all $\varepsilon \in (0, \delta]$,

$$\int_{\alpha}^{\alpha+\varepsilon} |\mathbf{F}_\infty \phi(u)| \leq \int_{\alpha}^{\alpha+\varepsilon} |(\mathbf{F}_\infty \phi(0))(\tau)|\, d\tau + \lambda\sqrt{\varepsilon}\left(\int_0^{\alpha+\varepsilon} |u|^2\right)^{1/2}. \tag{A.4}$$

Set $f(t) = r - (\mathbf{\Psi}_\infty x_0)(t)$, and choose $\rho > 0$ such that

$$\int_{\alpha}^{\alpha+\rho} (|f(\tau)| + |(\mathbf{F}_\infty \phi(0))(\tau)|)\, d\tau \leq \frac{\eta}{2}. \tag{A.5}$$

Now choose $\varepsilon > 0$ such that

$$\varepsilon \leq \delta, \quad \varepsilon \leq \rho, \quad \varepsilon < \frac{1}{\lambda}, \quad \varepsilon \leq \frac{1}{4(\alpha+\rho)}\left(\frac{\eta}{\lambda \max\{\|w\|_\infty, |w(\alpha)| + \eta\}}\right)^2. \tag{A.6}$$

Define the operator Γ by

$$\begin{aligned}
(\Gamma u)(t) &= w(t), \qquad 0 \leq t \leq \alpha, \\
(\Gamma u)(t) &= w(\alpha) + \int_\alpha^t f(\tau)\, d\tau - \int_\alpha^t (\mathbf{F}_\infty \phi(u))(\tau)\, d\tau, \qquad t \geq \alpha,
\end{aligned}$$

and set

$$\tilde{\mathcal{C}}_{\varepsilon,\eta} := \{u \in \mathcal{C}_{\varepsilon,\eta} \mid u(t) = w(t) \quad \text{if } 0 \leq t \leq \alpha\}.$$

Clearly, $\tilde{\mathcal{C}}_{\varepsilon,\eta}$ is a complete metric space, and the lemma follows if we can show that Γ is a contraction on $\tilde{\mathcal{C}}_{\varepsilon,\eta}$.

We first show that $\Gamma(\tilde{\mathcal{C}}_{\varepsilon,\eta}) \subset \tilde{\mathcal{C}}_{\varepsilon,\eta}$. Using (A.4)–(A.6) we obtain, for all $u \in \mathcal{C}_{\varepsilon,\eta}$ and all $t \in [\alpha, \alpha+\varepsilon]$,

$$\begin{aligned}
|(\Gamma u)(t) - w(\alpha)| &\leq \lambda\sqrt{\varepsilon}\left(\int_0^{\alpha+\varepsilon} |u(\tau)|^2\, d\tau\right)^{1/2} + \frac{\eta}{2} \\
&\leq \frac{\eta}{2} + \lambda\sqrt{\varepsilon(\alpha+\rho)}\max\{\|w\|_\infty, |w(\alpha)| + \eta\} \\
&\leq \eta,
\end{aligned}$$

which shows that $\Gamma(\tilde{\mathcal{C}}_{\varepsilon,\eta}) \subset \tilde{\mathcal{C}}_{\varepsilon,\eta}$. It remains to show that Γ is a contraction on $\tilde{\mathcal{C}}_{\varepsilon,\eta}$. To this end, let $u, v \in \tilde{\mathcal{C}}_{\varepsilon,\eta}$. Using (A.3) we obtain

$$\sup_{0 \leq \tau \leq \alpha+\varepsilon} |(\Gamma u)(\tau) - (\Gamma v)(\tau)| \leq \lambda\sqrt{\varepsilon}\left(\int_\alpha^{\alpha+\varepsilon} |u-v|^2\right)^{1/2} \leq \varepsilon\lambda \sup_{0 \leq \tau \leq \alpha+\varepsilon} |u(\tau) - v(\tau)|.$$

By (A.6) we have that $\varepsilon\lambda < 1$, showing that Γ is a contraction on $\tilde{\mathcal{C}}_{\varepsilon,\eta}$. ◻

Proof of Lemma 3.2. We proceed in several steps.

Step 1. Existence and uniqueness on a small interval.

An application of Lemma A.1 with $\alpha = 0$ shows that there exists an $\varepsilon > 0$ such that (3.6) has a unique solution on the interval $[0, \varepsilon)$.

Step 2. Extended uniqueness.

Let u_i be a solution of (3.6) on the interval $[0, a_i)$, $i = 1, 2$. We claim that $u_1(t) = u_2(t)$ for all $t \in [0, a)$, where $a = \min(a_1, a_2)$. Seeking a contradiction, assume that there exists $t \in (0, a)$ such that $u_1(t) \neq u_2(t)$. Defining

$$t^* = \inf\{t \in (0, a) \,|\, u_1(t) \neq u_2(t)\}\,,$$

it follows that $t^* > 0$ (by Step 1), $t^* < a$ (by assumption), and $u_1(t^*) = u_2(t^*)$ (by continuity of u_1 and u_2). Clearly, the initial-value problem

$$\begin{aligned} \dot u(t) &= k[r - (\mathbf{\Psi}_\infty x_0)(t) - (\mathbf{F}_\infty \phi(u))(t)]\,, \quad t \geq t^*\,, \\ u(t) &= u_1(t)\,, \quad t \in [0, t^*]\,, \end{aligned}$$

is solved by u_1 and u_2. This implies (by Lemma A.1) that there exists an $\varepsilon > 0$ such that $u_1(t) = u_2(t)$ for all $t \in [0, t^* + \varepsilon)$, which contradicts the definition of t^*.

Step 3. Continuation of solutions.

Let u be a solution of (3.6) on the interval $[0, a)$, $a < \infty$. In order to prove that u can be extended to a maximal solution (which satisfies (3.7) if $a_{max} < \infty$), it is sufficient to show that u can be continued to the right (beyond a) if u is bounded on $[0, a)$. Now $u(t) = (\Gamma u)(t)$ for all $t \in [0, a)$, where Γ is the operator defined in the proof of Lemma A.1 with $\alpha = 0$. It is clear that $\lim_{t \to a-} (\Gamma u)(t) = \gamma$ exists and is finite. Consequently, $\lim_{t \to a-} u(t) = \gamma$, and hence setting $u(a) = \gamma$ makes u into a continuous function on $[0, a]$. Finally, Lemma A.1 shows that the initial value problem

$$\begin{aligned} \dot v &= k[r - \mathbf{\Psi}_\infty x_0 - \mathbf{F}_\infty \phi(v)]\,, \quad t \geq a\,, \\ v(t) &= u(t)\,, \quad t \in [0, a]\,, \end{aligned}$$

has a unique solution u^* on $[0, a + \varepsilon)$ for some $\varepsilon > 0$. By the causality of the map $\mathbf{F}_\infty \phi$, the function u^* is a solution of (3.6) on $[0, a + \varepsilon)$, i.e., u^* is a continuation of u.

Step 4. Global existence if ϕ is globally Lipschitz.

Assume that ϕ is globally Lipschitz. Seeking a contradiction suppose that $a_{max} < \infty$. Let u be the solution of (3.6) defined on $[0, a_{max})$. Multiplying (3.6) by u and estimating we obtain that, for all $\tau \in [0, a_{max})$,

$$u(\tau)\dot u(\tau) \leq k[r^2 + (\mathbf{\Psi}_\infty x_0)^2(\tau) + u^2(\tau) + |(\mathbf{F}_\infty \phi(u))(\tau) u(\tau)|\,]\,. \tag{A.7}$$

Integrating (A.7) from 0 to t and combining the estimate

$$\int_0^t |(\mathbf{F}_\infty \phi(u))u| \leq \int_0^t |\mathbf{F}_\infty(\phi(u) - \phi(0))|\,|u| + \frac{1}{2}\left(\int_0^t (\mathbf{F}_\infty \phi(0))^2 + \int_0^t u^2\right)\,,$$

the Cauchy–Schwarz inequality, and the global Lipschitz property of ϕ, it can be readily shown that there exists positive constants α and β such that, for all $t \in [0, a_{max})$,

$$u^2(t) \leq \alpha + \beta \int_0^t u^2(\tau)\,d\tau\,.$$

An application of Gronwall's lemma then shows that $u^2(t) \leq \alpha e^{\beta t}$ for all $t \in [0, a_{max})$. Hence u is bounded on $[0, a_{max})$, which by Step 3 is in contradiction to the maximality of a_{max}. ∎

Proof of Lemma 3.4. Since $0 < 2\kappa < K$, it follows that there exists $\varepsilon > 0$ such that

$$1 + 2\kappa \mathrm{Re}\, \frac{\mathbf{G}(s)}{s} \geq \varepsilon \qquad \text{for all } s \in \mathbb{C}_0 .$$

Hence

$$1 + 2\kappa \mathrm{Re}\, \frac{\mathbf{G}(i\omega)}{i\omega} \geq \varepsilon \qquad \text{for all } \omega \in \mathbb{R},\ \omega \neq 0 ,$$

and thus

$$1 + 2\kappa \mathrm{Re}\mathbf{H}(i\omega) \geq \varepsilon \qquad \text{for all } \omega \in \mathbb{R} . \tag{A.8}$$

By considering

$$e^{-(1+2\kappa \mathrm{Re}\mathbf{H}(s))} = \left| e^{-(1+2\kappa \mathbf{H}(s))} \right| ,$$

applying the maximum modulus theorem, and using the fact that $\mathbf{H}(s) \to 0$ as $|s| \to \infty$ in $\mathbb{C}_0$, it follows from (A.8) that

$$1 + 2\kappa \mathrm{Re}\mathbf{H}(s) \geq \varepsilon \qquad \text{for all } s \in \mathbb{C}_0 .$$

Therefore, for all $s \in \mathbb{C}_0$,

$$\varepsilon + \kappa^2 \mathbf{H}(s)\bar{\mathbf{H}}(s) \leq (1 + \kappa \mathbf{H}(s))(1 + \kappa \bar{\mathbf{H}}(s)) .$$

Consequently, for all $s \in \mathbb{C}_0$,

$$\mathbf{H}(s)(1 + \kappa \mathbf{H}(s))^{-1} \bar{\mathbf{H}}(s)(1 + \kappa \bar{\mathbf{H}}(s))^{-1} < \frac{1}{\kappa^2} ,$$

yielding (3.11).

By using the identity $s(sI - A)^{-1} = A(sI - A)^{-1} + I$, we easily obtain

$$\mathbf{H}(s) = \frac{1}{s} (\mathbf{G}(s) - \mathbf{G}(0)) = C_L (sI - A)^{-1} A^{-1} B .$$

Consider the state-space system given by the triple $(A, A^{-1}B, C_L)$. For any $T > 0$, the input-to-state map of this system maps $L^2(0, T)$ boundedly into X_1. Consequently, the triple $(A, A^{-1}B, C_L)$ defines a Pritchard–Salamon system with respect to the spaces X_1 and X; see Curtain et al. [4] or Pritchard and Townley [31]. Now, (3.11) means in particular that the closed-loop system obtained from $\mathbf{H}$ by negative output feedback with gain κ is input-output stable. By the equivalence of input-output and exponential stability (see [4] or [32]), we may conclude that the semigroup generated by A_κ, with $0 < 2\kappa < K$, is exponentially stable. Moreover, combining Theorem 2.4 in Pritchard and Townley [30] (or, alternatively, Theorem 1 in Logemann [12]) and (3.11), it follows that the structured complex stability radius of A_κ with respect to the weightings $A^{-1}B$ and C_L is greater than κ. Therefore, an application of Proposition 1.5 in [31] shows that the Riccati equation (3.12) has a self-adjoint positive-semidefinite solution $P \in \mathcal{B}(X)$ such that (3.12) holds for all $x_1, x_2 \in \mathrm{dom}(A_\kappa)$. □

Proof of Lemma 3.5. It is clear that $\phi^\diamond \in L^\infty_{loc}(-\infty, \infty)$ if ϕ is locally Lipschitz and that $\phi^\diamond \in L^\infty(-\infty, \infty)$ if ϕ is globally Lipschitz. Moreover, as the limsup of a sequence of Borel functions, $\phi^\diamond$ is a Borel function. Consequently, $\phi^\diamond \circ u$ is Lebesgue measurable for all Lebesgue measurable functions u. Let u be absolutely continuous. Setting $v = \phi \circ u$, it follows from the Lipschitz continuity of ϕ and the absolute continuity of u that v is absolutely continuous. If $t \in \mathbb{R}$ is such that u is differentiable at t, then we have

$$v(t+h) - v(t) = \phi(u(t) + h\dot{u}(t)) - \phi(u(t)) + \phi(u(t+h)) - \phi(u(t) + h\dot{u}(t)) \,. \tag{A.9}$$

Moreover, by Lipschitz continuity of ϕ, there exists a constant $L > 0$ such that, for all sufficiently small $|h|$,

$$\left| \frac{1}{h} \left[\phi(u(t+h)) - \phi(u(t) + h\dot{u}(t)) \right] \right| \leq L \left| \frac{1}{h} [u(t+h) - u(t)] - \dot{u}(t) \right| \,. \tag{A.10}$$

Let $\mathcal{D} \subset \mathbb{R}$ be the set of all points t such that both u and v are differentiable at t. Then $\mathcal{D}$ is of full measure, and combining (A.9) and (A.10) yields

$$\lim_{h\to 0} \frac{1}{h}[v(t+h) - v(t)] = \lim_{h\to 0} \frac{1}{h}[\phi(u(t) + h\dot{u}(t)) - \phi(u(t))] \quad \text{for all } t \in \mathcal{D}.$$

Therefore, for every $t \in \mathcal{D}$,

$$\dot{v}(t) = 0 \qquad \text{if } \dot{u}(t) = 0 \,, \tag{A.11}$$

$$\begin{aligned} \dot{v}(t) &= \lim_{h\to 0} \frac{\phi(u(t) + h\dot{u}(t)) - \phi(u(t))}{h\dot{u}(t)} \, \dot{u}(t) \\ &= \phi'(u(t))\dot{u}(t) \qquad \text{if } \dot{u}(t) \neq 0 \,. \end{aligned} \tag{A.12}$$

In particular, if $t \in \mathcal{D}_0 := \{t \in \mathcal{D} \,|\, \dot{u}(t) \neq 0\}$, then ϕ is differentiable at $u(t)$. For $t \in \mathcal{D}_0$ we have, of course, $\phi^\diamond(u(t)) = \phi'(u(t))$, and thus it follows from (A.11) and (A.12) that

$$\dot{v}(t) = \phi^\diamond(u(t))\dot{u}(t) \qquad \text{for a.e. } t \in [0, \infty) \,. \qquad \square$$

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