Volterra Functional Differential Equations: Existence, Uniqueness, and Continuation of Solutions

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Abstract. The initial-value problem for a class of Volterra functional differential equations—of sufficient generality to encompass, as special cases, ordinary differential equations, retarded differential equations, integro-differential equations, and hysteretic differential equations—is studied. A self-contained and elementary treatment of this over-arching problem is provided, in which a unifying theory of existence, uniqueness, and continuation of solutions is developed. As an illustrative example, a controlled differential equation with hysteresis is considered.

1. INTRODUCTION. Initial-value problems for systems of differential equations permeate many areas of mathematics: such problems arise naturally in modelling the evolution of dynamical processes in economics, engineering, and the physical and biological sciences. As a starting point, consider an *ordinary differential equation* in $\mathbb{R}^N$:

$$u'(t) = f(t, u(t)), \tag{1.1}$$

where f is some suitably regular function, and the independent variable t carries the connotation of "time." Loosely speaking, such a formulation is appropriate in applications wherein the forward-time, or future, behaviour of the process under investigation depends only on the current state $u(t)$ (at current time t) and, in particular, is independent of its past $u(s)$, $s < t$, and future $u(s)$, $s > t$, states. Adopting the standpoint that the processes under investigation are "real-world" phenomena, it is reasonable to assume independence with respect to future states (and this we do throughout, via an assumption of *causality* or *non-anticipativity*); however, there are many situations wherein the process may "remember" the past and so its future behaviour depends, not only on the current state, but also on its past states. The simplest example of such dependence on the past is a differential equation with a *point delay* of length $h > 0$:

$$u'(t) = f(t, u(t), u(t-h)). \tag{1.2}$$

System (1.2) may be embedded in the class of *retarded differential equations* of the form

$$u'(t) = f(t, u_t), \tag{1.3}$$

where, for a continuous function u on some interval I containing $[-h, 0]$ and $t \in I$ with $t \geq 0$, u_t denotes the continuous function on $[-h, 0]$ defined by $u_t(s) := u(t+s)$.

Another commonly encountered class of systems which exhibit dependence on the past is comprised of *integro-differential equations* of the form

$$u'(t) = \int_0^t k(t, s) g(s, u(s))\, \mathrm{d}s. \tag{1.4}$$

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As a final class of systems with memory, consider *hysteretic differential equations* of the form

$$u'(t) = f(t, u(t), (H(u))(t)), \tag{1.5}$$

where H is a hysteresis operator (that is, an operator which is causal and rate independent in a sense to be made precise in due course). As a prototype, consider a scalar nonlinear mechanical system with hysteretic restoring force

$$y''(t) + g(t, y(t), y'(t))y'(t) + (P(y))(t) = 0,$$

where P is the play or backlash operator, the action of which is captured in Figure 1, wherein $z = P(y)$ (we will return later to such an operator, with full details).

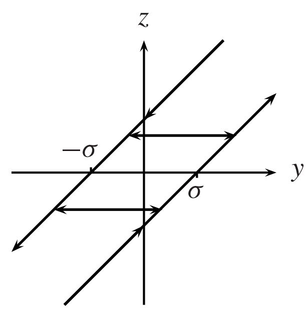

Figure 1. Play hysteresis.

Writing $u(t) = (u_1(t), u_2(t)) := (y(t), y'(t))$ and defining $f : \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}$ by

$$f(t, v, w) = f(t, (v_1, v_2), w) := (v_2, -g(t, v_1, v_2)v_2 - w),$$

this system takes the form (1.5) with $H(u) = P(u_1)$.

Notwithstanding an outward appearance of diversity, the above examples (1.1)–(1.5) can be subsumed (as we shall see) in a common formulation, expressed as an initial-value problem for a functional differential equation of the form

$$u'(t) = (F(u))(t), \quad t \geq 0, \quad u|_{[-h,0]} = \varphi, \tag{1.6}$$

with $h \geq 0$ and φ continuous. The operator F is assumed to be *causal* or *non-anticipative* (loosely speaking, F is causal if, whenever functions u and v are such that their values $u(t)$ and $v(t)$ coincide up to $t = \tau$, $(F(u))(t)$ and $(F(v))(t)$ also coincide up to $t = \tau$: a precise definition is contained in hypothesis (H1) in Section 3 below). This paper provides an elementary, self-contained, and tutorial treatment of this over-arching problem: a unifying theory of existence, uniqueness, and continuation of solutions is developed which, when specialized, applies in the context of each of the systems (1.1)–(1.5) outlined above.

For clarity of exposition, proofs of only the main results (viz., Proposition 3.5, Theorem 3.6, Theorem 3.7, and Corollary 3.8) are contained in the main body of the text: proofs of auxiliary technical propositions and lemmas are provided in the appendix.

We close the introduction with a brief list of some standard references: for ordinary differential equations, see [**1, 15**]; for functional equations with causal operators, see [**3, 4, 8**]; for retarded differential equations of the form (1.3), see [**5, 6, 9**]; for integro-differential equations of the form (1.4), see [**3, 8**]; for systems with hysteresis of the form (1.5), see [**2, 12**].

Notation. The vector space of continuous functions defined on an interval I with values in $\mathbb{R}^N$ is denoted by $C(I)$. If I is compact, then, endowed with the norm

$$\|u\|_{C(I)} := \sup\{\|u(t)\| : t \in I\}, \tag{1.7}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^N$, $C(I)$ is a Banach space. For $u \in C(I)$, define $\operatorname{gr} u$, the *graph* of u, by $\operatorname{gr} u := \{(t, u(t)) : t \in I\} \subset \mathbb{R} \times \mathbb{R}^N$. Finally, if I is an interval, then $I_+ := I \cap \mathbb{R}_+$, where $\mathbb{R}_+ := [0, \infty)$.

2. PRELIMINARIES: ABSOLUTELY CONTINUOUS FUNCTIONS AND THE SOBOLEV SPACE $W^{1,1}$. Let $a < b$. A function $u: [a, b] \to \mathbb{R}^N$ is said to be *absolutely continuous* if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for any finite collection of disjoint open intervals $(a_1, b_1), \ldots, (a_n, b_n)$ contained in $[a, b]$,

$$\sum_{i=1}^{n}(b_i - a_i) \le \delta \quad \Rightarrow \quad \sum_{i=1}^{n} \|u(b_i) - u(a_i)\| \le \varepsilon.$$

The importance of the concept of absolute continuity stems from the fact that absolutely continuous functions are precisely the functions for which the fundamental theorem of calculus (in the context of Lebesgue integration) is valid: a function $u : [a, b] \to \mathbb{R}^N$ is absolutely continuous if, and only if, u is differentiable at almost every $t \in [a, b]$, $u' \in L^1[a, b]$, and $u(t) = u(a) + \int_a^t u'(s)\,ds$ for all $t \in [a, b]$; see, for example, [**7, 11**]. We define

$$W^{1,1}[a, b] := \{u: [a, b] \to \mathbb{R}^N \mid u \text{ is absolutely continuous}\}.$$

It is well known that, endowed with the norm $\|u\|_{W^{1,1}} := \|u\|_{L^1} + \|u'\|_{L^1}$, the space $W^{1,1}[a, b]$ is complete, and hence a Banach space. Usually, $W^{1,1}[a, b]$ is referred to as a Sobolev space. An alternative, and sometimes more convenient, norm on $W^{1,1}[a, b]$ is the BV-norm (where BV stands for bounded variation) defined by

$$\|u\|_{BV} := \|u(a)\| + \|u'\|_{L^1}.$$

The total variation of a function $u: [a, b] \to \mathbb{R}^N$ is given by $\sup\{\sum_{k=1}^n \|u(t_k) - u(t_{k-1})\| : n \in \mathbb{N}, a = t_0 < t_1 < \cdots < t_n = b\}$ and, if this quantity is finite, u is said to be of bounded variation. As is well known, an absolutely continuous function u is of bounded variation and its total variation is equal to $\|u'\|_{L^1}$: this fact motivates the name "BV-norm."

Proposition 2.1. *There exist $k_1, k_2 > 0$ such that*

$$k_1\|u\|_{W^{1,1}} \le \|u\|_{BV} \le k_2\|u\|_{W^{1,1}} \quad \forall\, u \in W^{1,1}[a, b],$$

that is, the norms $\|\cdot\|_{W^{1,1}}$ and $\|\cdot\|_{BV}$ are equivalent. Consequently, $W^{1,1}[a, b]$ is complete with respect to the BV-norm.

The proof of Proposition 2.1 can be found in the appendix. For an arbitrary (not necessarily compact) interval $I \subset \mathbb{R}$ and arbitrary $p \in [1, \infty]$, we define

$$L^p_{\mathrm{loc}}(I) := \{u: I \to \mathbb{R}^N \mid u|_{[a,b]} \in L^p[a, b] \text{ for all } a, b \in I, a < b\}$$

and

$$W^{1,1}_{\mathrm{loc}}(I) := \{u\colon I \to \mathbb{R}^N \mid u|_{[a,b]} \in W^{1,1}[a,b] \text{ for all } a, b \in I, a < b\}.$$

Clearly, if I is compact, then $L^p_{\mathrm{loc}}(I) = L^p(I)$ and $W^{1,1}_{\mathrm{loc}}(I) = W^{1,1}(I)$.

3. VOLTERRA FUNCTIONAL DIFFERENTIAL EQUATIONS. The focus of our study is an initial-value problem of the form

$$u'(t) = (F(u))(t), \quad t \ge 0; \quad u|_{[-h,0]} = \varphi \in C[-h,0], \quad \mathrm{gr}\,\varphi \subset \Delta, \tag{3.1}$$

with $h \ge 0$ and $\Delta \subset [-h,\infty) \times \mathbb{R}^N$, and where F is an operator acting on suitable spaces of functions u defined on intervals of the form $[-h,\eta]$ or $[-h,\eta)$ (where $\eta > 0$) and with $\mathrm{gr}\,u \subset \Delta$. Throughout, we assume that

- Δ is open relative to $[-h,\infty) \times \mathbb{R}^N$, that is, there exists an open set $D \subset \mathbb{R}^{N+1}$ such that $\Delta = D \cap ([-h,\infty) \times \mathbb{R}^N)$;
- $\{u \in C[-h,0]\colon\ \mathrm{gr}\,u \subset \Delta\} \ne \emptyset$.

The following convention is adopted: in the case $h = 0$, $C[-h,0]$ should be interpreted as $\mathbb{R}^N$ and the second of the above assumptions is equivalent to

$$\Delta_0 := \{x \in \mathbb{R}^N\colon\ (0,x) \in \Delta\} \ne \emptyset.$$

We proceed to describe the nature of the operator F in (3.1). For each interval I (possibly singleton), we define

$$W_\Delta(I) := \{u \in C(I)\colon\ u|_{I_+} \in W^{1,1}_{\mathrm{loc}}(I_+),\ \mathrm{gr}\,u \subset \Delta\}.$$

Note that

$$W_\Delta[-h,0] = \{u \in C[-h,0]\colon\ \mathrm{gr}\,u \subset \Delta\},$$

and, by convention, in the case $h = 0$ we have $W_\Delta(\{0\}) = \Delta_0$. As will shortly be precisely defined, a solution of (3.1) is a function in $W_\Delta(I)$ for some interval I of the form $[-h,\eta]$ (where $0 < \eta < \infty$) or $[-h,\eta)$ (where $0 < \eta \le \infty$). In the following, $\mathcal{I}$ denotes the set of all such intervals I with the property that $W_\Delta(I) \ne \emptyset$. Thus, $\mathcal{I}$ contains all possible domains of solutions of the functional differential equation. From the assumptions imposed on Δ, it follows that $[-h,\alpha] \in \mathcal{I}$ for all sufficiently small $\alpha > 0$, or equivalently,

$$T := \sup\{\alpha > 0\colon\ W_\Delta[-h,\alpha] \ne \emptyset\} > 0.$$

We are now in a position to make precise the nature of the operator F in (3.1) and the concept of a solution of the initial-value problem. We assume that, for every $I \in \mathcal{I}$, the operator F (in general, nonlinear) maps $W_\Delta(I)$ to $L^1_{\mathrm{loc}}(I_+)$. In particular, the domain of F is $\cup_{I\in\mathcal{I}} W_\Delta(I)$ and the range of F is contained in $\cup_{I\in\mathcal{I}} L^1_{\mathrm{loc}}(I_+)$. We say that $u\colon I \to \mathbb{R}^N$ is a *solution* of (3.1) (on the interval I) if $I \in \mathcal{I}$, $u \in W_\Delta(I)$, $u|_{[-h,0]} = \varphi$, and u satisfies the differential equation in (3.1) for almost every $t \in I_+$.

If $w \in W_\Delta[-h,\alpha]$ is a solution, then it is natural to ask if this solution can be extended to a solution u on $[-h,\beta]$ with $\beta > \alpha$. We will be especially interested in extensions u which are "well behaved" in the sense that, for given $\gamma > 0$, they satisfy

$\int_\alpha^\beta \|u'(s)\|\,\mathrm{d}s \le \gamma$ (so that, in particular, $\|u(s) - w(\alpha)\| \le \gamma$ for all $s \in [\alpha, \beta]$). This and other technical reasons lead to the consideration of a suitable space of extensions of functions $w \in W_\Delta[-h, \alpha]$. Specifically, let $0 \le \alpha < T$, $\beta > \alpha$, $\gamma > 0$, and $w \in W_\Delta[-h, \alpha]$, and define

$$\mathcal{W}(w; \alpha, \beta, \gamma) := \\ \left\{ u \in C[-h, \beta] \colon\ u|_{[-h,\alpha]} = w,\ u|_{[\alpha,\beta]} \in W^{1,1}[\alpha, \beta],\ \int_\alpha^\beta \|u'(s)\|\,\mathrm{d}s \le \gamma \right\}.$$

It is clear that, for all $u \in \mathcal{W}(w; \alpha, \beta, \gamma)$, $u|_{[0,\beta]} \in W^{1,1}[0, \beta]$. We equip the space $\mathcal{W}(w; \alpha, \beta, \gamma)$ with the metric μ given by

$$\mu(u, v) = \int_\alpha^\beta \left\| u'(s) - v'(s) \right\| \mathrm{d}s.$$

A routine argument, invoking Proposition 2.1, yields the following lemma (see appendix for details).

Lemma 3.1. *For every $\alpha \in [0, T)$, $\beta > \alpha$, $\gamma > 0$, and $w \in W_\Delta[-h, \alpha]$, the metric space $\mathcal{W}(w; \alpha, \beta, \gamma)$ is complete.*

Observe that some elements of $\mathcal{W}(w; \alpha, \beta, \gamma)$ may not be in $W_\Delta[-h, \beta]$: such elements do not qualify as candidate solutions extending the solution w. This observation motivates the following definition. Given $\alpha \in [0, T)$ and $w \in W_\Delta[-h, \alpha]$, we define

$$A(w; \alpha) := \{(\beta, \gamma) \in (\alpha, T) \times (0, \infty) \colon \mathcal{W}(w; \alpha, \beta, \gamma) \subset W_\Delta[-h, \beta]\}.$$

It is clear that $(\beta, \gamma) \in (\alpha, T) \times (0, \infty)$ is in $A(w; \alpha)$ if and only if $\operatorname{gr} u \subset \Delta$ for all $u \in \mathcal{W}(w; \alpha, \beta, \gamma)$. Important properties of the set $A(w; \alpha)$ are given in the following lemma, the proof of which is relegated to the appendix.

Lemma 3.2. *If $\alpha \in [0, T)$ and $w \in W_\Delta[-h, \alpha]$, then the following statements hold.*

(1) *The set $A(w; \alpha)$ is nonempty.*
(2) *If $(\beta, \gamma) \in A(w; \alpha)$, then $(b, c) \in A(w; \alpha)$ for all $b \in (\alpha, \beta]$ and all $c \in (0, \gamma]$.*

If $\alpha \in [0, T)$, $w \in W_\Delta[-h, \alpha]$, $(\beta, \gamma) \in A(w; \alpha)$, $b \in (\alpha, \beta]$, and $c \in (0, \gamma]$, then it follows from Lemma 3.2 that $\mathcal{W}(w; \alpha, b, c) \subset W_\Delta[-h, b]$, implying that $F(u)$ is well defined for all $u \in \mathcal{W}(w; \alpha, b, c)$. This fact will be used freely throughout.

We assemble the following hypotheses on F which will be variously invoked in the theory developed below.

(H1) *Causality:* if $I, J \in \mathfrak{I}$, then, for all $\tau \in (I_+ \cap J_+) \setminus \{0\}$, all $u \in W_\Delta(I)$, and all $v \in W_\Delta(J)$,

$$u|_{[-h,\tau]} = v|_{[-h,\tau]} \quad \Rightarrow \quad F(u)|_{[0,\tau]} = F(v)|_{[0,\tau]}.$$

(H2) *Local Lipschitz-type condition:* for every $\alpha \in [0, T)$ and every function $w \in W_\Delta[-h, \alpha]$, there exists $\lambda \in (0, 1)$ and $(\beta, \gamma) \in A(w; \alpha)$ such that

$$\|F(u) - F(v)\|_{L^1[\alpha,\beta]} \le \lambda \mu(u, v) \quad \forall\, u, v \in \mathcal{W}(w; \alpha, \beta, \gamma). \tag{3.2}$$

(H3) *Integrability condition:* for every $I \in \mathfrak{I}$ and every $u \in W_\Delta(I)$ such that $\overline{\operatorname{gr} u}$ is compact and contained in Δ, the function $F(u)$ is integrable, that is, $F(u) \in L^1(I_+)$.

The causality condition in (H1) is also referred to as *non-anticipativity* or as the *Volterra property*. Furthermore, in the literature, the term *Volterra operator* is sometimes used for a causal (or non-anticipative) operator. If (H1) holds, then the differential equation in (3.1) is often referred to as a *Volterra functional differential equation* or an *abstract Volterra integro-differential equation* (see, for example, [**3, 8**]). The essence of (H3) is that it encompasses all $I \in \mathfrak{I}$ (including noncompact intervals): if I is compact, then obviously, even without (H3) being satisfied, $F(u) \in L^1(I_+)$ for all $u \in W_\Delta(I)$, since $L^1_{\mathrm{loc}}(I_+) = L^1(I_+)$.

The following lemma records a particular consequence of hypotheses (H1) and (H2), which will be invoked in the later analysis. A proof of the lemma is provided in the appendix.

Lemma 3.3. *Assume that* (H1) *and* (H2) *hold. Then, for every $\alpha \in [0, T)$ and every $w \in W_\Delta[-h, \alpha]$, there exist $\lambda \in (0, 1)$ and $(\beta, \gamma) \in A(w; \alpha)$ such that, for every $b \in (\alpha, \beta]$ and every $c \in (0, \gamma]$,*

$$\|F(u) - F(v)\|_{L^1[\alpha,b]} \le \lambda \int_\alpha^b \|u'(s) - v'(s)\| \,\mathrm{d}s, \quad \forall\, u, v \in \mathcal{W}(w; \alpha, b, c).$$

On first encounter, it may seem that, in hypothesis (H2), the requirement that $\lambda < 1$ is quite restrictive. We proceed to show that this is not the case. To this end, assume that, for every $I \in \mathfrak{I}$, $F(W_\Delta(I)) \subset L^p_{\mathrm{loc}}(I_+)$ for some $p \in (1, \infty]$ and consider the following hypothesis.

(H2′) For every $\alpha \in [0, T)$ and every $w \in W_\Delta[-h, \alpha]$, there exists $\rho > 0$ and $(\beta, \gamma) \in A(w; \alpha)$ such that

$$\|F(u) - F(v)\|_{L^p[\alpha,\beta]} \le \rho\mu(u, v) \quad \forall\, u, v \in \mathcal{W}(w; \alpha, \beta, \gamma). \tag{3.3}$$

Note that, in (H2′), the Lipschitz constant ρ is not required to be smaller than 1.

Proposition 3.4. *Assume that there exists $p \in (1, \infty]$ such that $F(W_\Delta(I)) \subset L^p_{\mathrm{loc}}(I_+)$ for every $I \in \mathfrak{I}$. If* (H1) *and* (H2′) *are satisfied, then* (H2) *holds.*

A proof of this proposition is contained in the appendix.

In order to study the problems of existence, uniqueness, and continuation of solutions, it is convenient to consider the following initial-value problem which is slightly more general than (3.1):

$$u'(t) = (F(u))(t),\ t \ge \alpha; \quad u|_{[-h,\alpha]} = \psi \in W_\Delta[-h, \alpha],\ \alpha \in [0, T). \tag{3.4}$$

Trivially, the original initial-value problem (3.1) can be recovered from (3.4) by setting $\alpha = 0$.

We say that $u : I \to \mathbb{R}^N$ is a *solution* of (3.4) (on the interval I) if $I \in \mathfrak{I}$ with $\sup I > \alpha$, $u \in W_\Delta(I)$, $u|_{[-h,\alpha]} = \psi$, and u satisfies the differential equation in (3.4) for almost every $t \in I \cap [\alpha, \infty)$.

We now arrive at the first of three core results.

Proposition 3.5. *Assume that the operator F satisfies* (H1) *and* (H2)*, and let $\alpha \in [0, T)$ and $\psi \in W_\Delta[-h, \alpha]$. There exists $\eta > \alpha$ such that* (3.4) *has precisely one solution u on the interval $[-h, \eta]$; moreover, for every $I \in \mathfrak{I}$ such that $\alpha < \sup I \leq \eta$, the function $u|_I$ is the only solution of* (3.4) *on I.*

Proof. Let $\alpha \in [0, T)$ and $\psi \in W_\Delta[-h, \alpha]$. We proceed in two steps.

Step 1. Existence and uniqueness in $\mathcal{W}(\psi; \alpha, \beta, \gamma)$ for small $\beta - \alpha > 0$. First, we convert (3.4) into an integral equation. To this end, for each $\beta \in (\alpha, T)$, define an operator G_β on $W_\Delta[-h, \beta]$ by

$$(G_\beta(u))(t) := \begin{cases} \psi(t), & t \in [-h, \alpha] \\ \psi(\alpha) + \int_\alpha^t (F(u))(s)\, \mathrm{d}s, & t \in (\alpha, \beta]. \end{cases}$$

It is clear that $u \in W_\Delta[-h, \beta]$ is a solution of (3.4) if and only if $G_\beta(u) = u$. We claim that there exist $\beta^* > \alpha$ and $\gamma > 0$ such that (3.4) has a solution $u \in \mathcal{W}(\psi; \alpha, \beta^*, \gamma)$, and moreover, for every $\beta \in (\alpha, \beta^*]$, $u|_{[-h,\beta]}$ is the only solution of (3.4) in $\mathcal{W}(\psi; \alpha, \beta, \gamma)$. Invoking the completeness of $\mathcal{W}(\psi; \alpha, \beta, \gamma)$ (guaranteed by Lemma 3.1) and the contraction-mapping theorem, it is sufficient to show that there exist $(\beta^*, \gamma) \in A(w; \alpha)$ and $\lambda \in (0, 1)$ such that

$$G_\beta(\mathcal{W}(\psi; \alpha, \beta, \gamma)) \subset \mathcal{W}(\psi; \alpha, \beta, \gamma) \quad \forall\, \beta \in (\alpha, \beta^*] \tag{3.5}$$

and, moreover,

$$\mu(G_\beta(u), G_\beta(v)) \leq \lambda\mu(u, v) \quad \forall\, u, v \in \mathcal{W}(\psi; \alpha, \beta, \gamma), \quad \forall\, \beta \in (\alpha, \beta^*]. \tag{3.6}$$

We proceed to establish (3.5) and (3.6).

Using Lemma 3.3, we conclude that there exist $\lambda \in (0, 1)$ and $(\beta^\sharp, \gamma) \in A(w; \alpha)$ such that, for every $\beta \in (\alpha, \beta^\sharp]$,

$$\|F(u) - F(v)\|_{L^1[\alpha,\beta]} \leq \lambda\mu(u, v) \quad \forall\, u, v \in \mathcal{W}(\psi; \alpha, \beta, \gamma). \tag{3.7}$$

Let $\beta \in (\alpha, \beta^\sharp]$. To show that $G_\beta(\mathcal{W}(\psi; \alpha, \beta, \gamma)) \subset \mathcal{W}(\psi; \alpha, \beta, \gamma)$, let $u \in \mathcal{W}(\psi; \alpha, \beta, \gamma)$ and note that, by the definition of G_β, we have that $G_\beta(u) \in C[-h, \beta]$, $(G_\beta(u))|_{[-h,\alpha]} = \psi$, and $(G_\beta(u))|_{[\alpha,\beta]} \in W^{1,1}[\alpha, \beta]$. It remains to show that $\int_\alpha^\beta \|(G_\beta(u))'(s)\|\, \mathrm{d}s \leq \gamma$. To this end, define a function $\tilde{\psi} : [-h, \beta] \to \mathbb{R}^N$ by

$$\tilde{\psi}(t) = \begin{cases} \psi(t), & t \in [-h, \alpha] \\ \psi(\alpha), & t \in (\alpha, \beta]. \end{cases}$$

Clearly, $\tilde{\psi} \in \mathcal{W}(\psi; \alpha, \beta, \gamma)$. By Lemma 3.2, $(\beta, \gamma) \in A(w; \alpha)$, so that $\tilde{\psi} \in W_\Delta[-h, \beta]$. Consequently, $F(\tilde{\psi})$ is well defined, and, furthermore, invoking (3.7),

$$\begin{aligned} \int_\alpha^\beta \|(G_\beta(u))'(s)\|\, \mathrm{d}s &\leq \int_\alpha^\beta \|(F(u))(s) - (F(\tilde{\psi}))(s)\|\, \mathrm{d}s + \int_\alpha^\beta \|(F(\tilde{\psi})(s)\|\, \mathrm{d}s \\ &\leq \lambda \int_\alpha^\beta \|u'(s) - \tilde{\psi}'(s)\|\, \mathrm{d}s + \int_\alpha^\beta \|(F(\tilde{\psi}))(s)\|\, \mathrm{d}s. \end{aligned}$$

Since $\tilde{\psi}'(s) = 0$ for all $s \in [\alpha, \beta]$, we obtain

$$\int_\alpha^\beta \|(G_\beta(u))'(s)\| \, \mathrm{d}s \le \lambda\gamma + \int_\alpha^\beta \|(F(\tilde{\psi}))(s)\| \, \mathrm{d}s.$$

Now choose $\beta^* \in (\alpha, \beta^\sharp]$ such that

$$\int_\alpha^{\beta^*} \|(F(\tilde{\psi}))(s)\| \, \mathrm{d}s \le \gamma(1 - \lambda).$$

Then $\int_\alpha^\beta \|(G_\beta(u))'(s)\| ds \le \gamma$ for every $\beta \in (\alpha, \beta^*]$, so that (3.5) follows. Furthermore, (3.6) is an immediate consequence of (3.7) (in which $\lambda < 1$) and the fact that, for all $\beta \in (\alpha, \beta^*]$,

$$\mu(G_\beta(u), G_\beta(v)) = \int_\alpha^\beta \|(F(u))(s) - (F(v))(s)\| \, \mathrm{d}s \quad \forall\, u, v \in \mathcal{W}(\psi; \alpha, \beta, \gamma).$$

Step 2. Uniqueness in $W_\Delta[-h, \beta]$ for small $\beta - \alpha > 0$. By Step 1, there exists $\beta^* > \alpha$ such that (3.4) has a solution $u \in \mathcal{W}(\psi; \alpha, \beta^*, \gamma)$, and moreover, for every $\beta \in (\alpha, \beta^*]$, $u|_{[-h,\beta]}$ is the only solution of (3.4) in $\mathcal{W}(\psi; \alpha, \beta, \gamma)$. Choosing $\eta \in (\alpha, \beta^*]$ such that

$$\int_\alpha^\eta \|u'(s)\| \, \mathrm{d}s < \gamma, \tag{3.8}$$

let $I \in \mathcal{I}$ be such that $\alpha < \sup I \le \eta$ and let $v \in W_\Delta(I)$ be a solution of (3.4). Setting $\sigma := \sup I$, we claim that

$$\int_\alpha^\sigma \|v'(s)\| \, \mathrm{d}s \le \gamma. \tag{3.9}$$

Seeking a contradiction, suppose that this is not true. Then there exists $\tau \in (\alpha, \sigma)$ such that

$$\int_\alpha^\tau \|v'(s)\| \, \mathrm{d}s = \gamma. \tag{3.10}$$

In particular, $v \in \mathcal{W}(w; \alpha, \tau, \gamma)$, so that, by Step 1, $v_{[-h,\tau]} = u|_{[-h,\tau]}$. Consequently, by (3.8), $\int_\alpha^\tau \|v'(s)\| \, \mathrm{d}s < \gamma$, contradicting (3.10). We conclude that (3.9) holds, and thus, for every $\beta \in (\alpha, \sigma)$, $v \in \mathcal{W}(w; \alpha, \beta, \gamma)$. Invoking Step 1 again, we obtain that $v_{[-h,\beta]} = u|_{[-h,\beta]}$ for every $\beta \in (\alpha, \sigma)$, implying that $v = u|_I$. ∎

We now use Proposition 3.5 to prove the following result relating to the initial-value problem (3.1).

Theorem 3.6. *Assume that F satisfies* (H1) *and* (H2).

(1) *The initial-value problem* (3.1) *has a solution $u\colon [-h, \eta] \to \mathbb{R}^N$ for sufficiently small $\eta > 0$.*

(2) *Given an interval $I \in \mathcal{I}$, there exists at most one solution $u\colon I \to \mathbb{R}^N$ of the initial-value problem* (3.1).

Proof. (1) The claim follows immediately from an application of Proposition 3.5 with $\alpha = 0$.

(2) Let $I \in \mathfrak{I}$. Suppose that u_1 and u_2 are solutions of (3.1) defined on I. Define

$$\tau := \sup\{t \in I_+ : u_1(s) = u_2(s) \;\; \forall s \in [0, t]\}.$$

Note that an application of Proposition 3.5 with $\alpha = 0$ shows that $\tau > 0$. It is sufficient to show that $\tau = \sup I = \sup I_+$. Seeking a contradiction, suppose that $\tau < \sup I$. Defining $\psi : [-h, \tau] \to \mathbb{R}^N$ by

$$\psi(t) = \begin{cases} \varphi(t), & -h \le t \le 0 \\ u_1(t), & 0 < t \le \tau, \end{cases}$$

it is clear that $\psi \in W_\Delta[-h, \tau]$ and

$$u_1(t) = u_2(t) = \psi(t) \quad \forall t \in [-h, \tau].$$

Applying Proposition 3.5 again, now with $\alpha = \tau$, yields the existence of an $\varepsilon > 0$ such that $u_1(t) = u_2(t)$ for all $t \in [0, \tau + \varepsilon]$, contradicting the definition of τ. ■

Let $I \in \mathfrak{I}$ and assume that $u \in W_\Delta(I)$ is a solution of the initial-value problem (3.1). We say that the interval I is a *maximal interval of existence*, and u is a *maximally defined solution*, if u does not have a proper extension which is also a solution of (3.1), that is, there does not exist $\tilde{I} \in \mathfrak{I}$ and a solution $\tilde{u} \in W_\Delta(\tilde{I})$ of (3.1) such that $I \subset \tilde{I}$, $I \neq \tilde{I}$, and $u(t) = \tilde{u}(t)$ for all $t \in I$.

Theorem 3.7. *Assume that F satisfies* (H1) *and* (H2). *Then there exists a unique maximally defined solution u of* (3.1). *The associated maximal interval of existence is of the form $[-h, \tau)$, where $0 < \tau \le \infty$. If $\tau < \infty$ and, in addition, F satisfies* (H3), *then, for every compact set $\Gamma \subset \Delta$ and every $\sigma \in (0, \tau)$, there exists $t \in (\sigma, \tau)$ such that $(t, u(t)) \notin \Gamma$.*

If F satisfies (H1)–(H3) and if $u\colon [-h, \tau) \to \mathbb{R}^N$ is the unique maximally defined solution of (3.1), then the last assertion of the above theorem implies the following two statements:

(i) if $\tau < \infty$, then there exists a sequence (t_n) in $[0, \tau)$ such that $t_n \to \tau$ as $n \to \infty$ and at least one of the following two properties holds:
(a) $\lim_{n\to\infty} \|u(t_n)\| = \infty$, (b) $(t_n, u(t_n))$ approaches $\partial\Delta$ as $n \to \infty$;

(ii) if u is bounded and $\overline{\operatorname{gr} u} \subset \Delta$, then $\tau = \infty$.

Proof of Theorem 3.7. Set

$$\tau := \sup\{\eta \ge 0\colon \text{there exists a solution of (3.1) on } [-h, \eta]\}$$

By Theorem 3.6, $\tau > 0$, and, for every $\eta \in (0, \tau)$, there exists a unique solution $u^\eta \in W_\Delta[-h, \eta]$ of (3.1). We define $u : [-h, \tau) \to \mathbb{R}^N$ by

$$u(t) = u^\eta(t) \quad -h \le t \le \eta, \quad \text{for every } \eta \in (0, \tau).$$

It follows from statement (2) of Theorem 3.6 that u is well defined. Moreover, it is clear that $u \in W_\Delta[-h, \tau)$ and that u solves (3.1). We claim that u is a maximally defined solution on the maximal interval of existence $[-h, \tau)$. If $\tau = \infty$, then there is nothing to prove. So without loss of generality assume that $\tau < \infty$. Seeking a contradiction, suppose that there exists an extension $\tilde{u}$ of u, $\tilde{u} \in W_\Delta[-h, \tilde{\tau}]$ with $\tilde{\tau} \geq \tau$, and such that $\tilde{u}$ solves (3.1). From the definition of τ it follows immediately that $\tilde{\tau} = \tau$. An application of Proposition 3.5, with $\alpha = \tau = \tilde{\tau}$ and $\psi = \tilde{u}$, shows that there exist $\tau^* > \tau$ and a (proper) extension $u^* \in W_\Delta[-h, \tau^*]$ of $\tilde{u}$ which also solves (3.1); this contradicts the definition of τ. Uniqueness of u follows from statement (2) of Theorem 3.6.

Now assume that $\tau < \infty$ and F satisfies (H3). Seeking a contradiction, suppose there exist a compact set $\Gamma \subset \Delta$ and $\sigma \in (0, \tau)$ such that $(t, u(t)) \in \Gamma$ for all $t \in (\sigma, \tau)$. Then

$$\overline{\operatorname{gr} u} \subset \{(t, u(t)) \colon t \in [-h, \sigma]\} \cup \Gamma,$$

and consequently, $\overline{\operatorname{gr} u}$ is compact and contained in Δ. Invoking (H3), together with the identity

$$u(t) = u(0) + \int_0^t (F(u))(s)\,\mathrm{d}s, \quad \forall\, t \in [0, \tau),$$

shows that the limit $l := \lim_{t \uparrow \tau} u(t)$ exists. Obviously, $(\tau, l) \in \overline{\operatorname{gr} u}$, and thus, $(\tau, l) \in \Delta$. This implies that the function $\tilde{u} : [-h, \tau] \to \mathbb{R}^N$ defined by

$$\tilde{u}(t) = \begin{cases} u(t), & -h \leq t < \tau \\ l, & t = \tau \end{cases}$$

is in $W_\Delta[-h, \tau]$, is a solution of (3.1), and is a proper extension of the solution u. This contradicts the fact that u is a maximally defined solution. ∎

Example: A Controlled Differential Equation with Hysteresis. Consider a forced system with forcing input (control) v subject to hysteresis H:

$$my''(t) + cy'(t) + ky(t) + (H(v))(t) = 0, \quad y(0) = y^0,\ y'(0) = y^1. \tag{3.11}$$

In a mechanical context, $y(t)$ represents displacement at time $t \in \mathbb{R}_+$, $m > 0$ and c are the mass and the damping constant, and k is a linear spring constant: the function v is interpreted as a control (which is open to choice and may be generated by feedback of y) and the operator H models hysteretic actuation. Such hysteretic effects arise in, for example, micro-positioning control problems using piezo-electric actuators or smart actuators, as investigated in, for example, [**13**]; general treatments of hysteresis phenomena can be found in, for example, [**2**], [**12**], and [**14**]. We deem an operator $H \colon C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ to be a hysteresis operator if it is both causal and rate independent. By *rate independence* we mean that $H(y \circ \zeta) = H(y) \circ \zeta$ for every $y \in C(\mathbb{R}_+)$ and every time transformation $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ (that is, a continuous, nondecreasing, and surjective function).

The control objective is to generate the input function v in such a way that the displacement $y(t)$ tends, as $t \to \infty$, to some desired value $r \in \mathbb{R}$. In view of this objective, it is natural to seek to generate the input by feedback of the error $y(t) - r$. Proportional-integral-derivative (PID) feedback control action is ubiquitous in control theory and practice and takes the form

$$v(t) = k_p(y(t) - r) + k_i \int_0^t (y(\tau) - r)\ \mathrm{d}\tau + k_d y'(t), \tag{3.12}$$

the control objective being reduced to that of determining the parameter values $k_p, k_i, k_d \in \mathbb{R}$ so as to cause the variable y to approach asymptotically the prescribed constant value r. This methodology, applied in the context of (3.11), is depicted in Figure 2. Introducing the variables

$$u_1(t) = \int_0^t (y(\tau) - r)\,\mathrm{d}t, \quad u_2(t) = y(t) - r, \quad u_3(t) = y'(t),$$

the feedback system given by (3.11) and (3.12) may be expressed as

$$\left.\begin{array}{l} u_1'(t) = u_2(t), \;\; u_2'(t) = u_3(t), \\ mu_3' + cu_3(t) + k(u_2(t) + r) + (H(k_i u_1 + k_p u_2 + k_d u_3))(t) = 0, \\ u_1(0) = 0, \; u_2(0) = y^0 - r, \; u_3(0) = y^1. \end{array}\right\} \tag{3.13}$$

Therefore, central to any study (see [**10**], for example) of the efficacy of the PID feedback structure is the initial-value problem (3.13) which we proceed to show is subsumed by (3.1). Writing $u := (u_1, u_2, u_3)$ and defining an operator F by

$$F(u) = \big(u_2,\, u_3,\, -cm^{-1}u_3 - km^{-1}(u_2 + r) - m^{-1}H(k_i u_1 + k_p u_2 + k_d u_3)\big), \tag{3.14}$$

we see that the initial-value problem (3.13) may be expressed as

$$u' = F(u), \quad u(0) = \varphi := (0,\, y^0 - r,\, y^1), \tag{3.15}$$

which has the form (3.1) with $h = 0$ and $\Delta = \mathbb{R}_+ \times \mathbb{R}^3$. In particular, $\mathcal{I}$ consists of all intervals of the form $[0, \eta]$ (where $0 < \eta < \infty$) or $[0, \eta)$ (where $0 < \eta < \infty$). Clearly, for every $I \in \mathcal{I}$, the operator F maps $W_\Delta(I) = W^{1,1}_{\mathrm{loc}}(I)$ to $L^1_{\mathrm{loc}}(I)$.

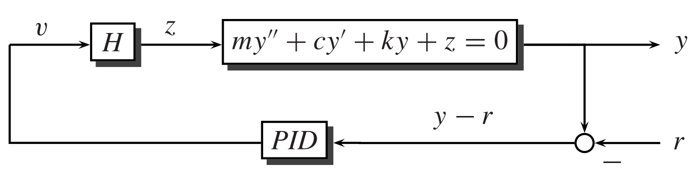

Figure 2. System (3.11) under PID control action.

Many commonly-encountered hysteresis operators H satisfy a global Lipschitz condition in the sense that there exists a Lipschitz constant $L > 0$ such that

$$\sup_{t \in \mathbb{R}_+} |(H(y_1))(t) - (H(y_2))(t)| \le L \sup_{t \in \mathbb{R}_+} |y_1(t) - y_2(t)| \quad \forall\, y_1, y_2 \in C(\mathbb{R}_+).$$

Corollary 3.8. *Assume that the hysteresis operator H satisfies a global Lipschitz condition. Then the operator F given by* (3.14) *satisfies* (H1)–(H3) *and, for every initial condition φ, the equation* (3.15) *has a unique maximally defined solution with maximal interval of existence $\mathbb{R}_+$.*

Proof. Since H is a hysteresis operator, H is causal, whence causality of F follows. Therefore hypothesis (H1) holds. The global Lipschitz condition for H clearly implies that F is globally Lipschitz and hence, (H2) is satisfied. Moreover, it follows easily from the rate independence of H that $H(0)$ (where 0 denotes the zero function on $\mathbb{R}_+$) is a constant function, implying that the function $F(0)$ is constant. Denoting this constant by c and using the global Lipschitz property of F, a routine argument yields

$$\|(F(u))(t)\| \leq c + L_F\|u(t)\| \quad \forall\, u \in W^{1,1}_{\mathrm{loc}}[0,\eta), \quad \forall\, \eta \in (0,\infty), \tag{3.16}$$

where L_F denotes the Lipschitz constant of F. It follows that (H3) also holds. Consequently, Theorem 3.7 applies. Let $u : [0, \tau) \to \mathbb{R}^3$ be a maximally defined solution of (3.15). To complete the proof, we have to show that $\tau = \infty$. Seeking a contradiction, suppose that $\tau < \infty$. Since $\Delta = \mathbb{R}_+ \times \mathbb{R}^3$, it then follows from Theorem 3.7 that u is unbounded. Integrating (3.15) from 0 to t and invoking (3.16) leads to

$$\|u(t)\| \leq \|u(0)\| + ct + L_F \int_0^t \|u(s)\|\,\mathrm{d}s \quad \forall\, t \in [0,\tau).$$

An application of Gronwall's lemma (see, for example, [**1**, Lemma 6.1]) now yields that

$$\|u(t)\| \leq \big(\|u(0)\| + c\tau\big)e^{L_F\tau} \quad \forall\, t \in [0,\tau),$$

which is in contradiction to the unboundedness of u. ■

There follows an illustrative example of hysteresis with the requisite Lipschitz property.

Play and Prandtl Hysteresis. A basic hysteresis operator is the *play* operator (already alluded to in the introduction). A detailed discussion of the play operator (also called the *backlash* operator) can be found in, for example, [**2, 12, 14**]. Intuitively, the play operator describes the input-output behaviour of a simple mechanical play between two mechanical elements as shown in Figure 3, where the input y is the position of the vertical component of element I and the output z is the the position of the midpoint of element II. The resulting input-output diagram is shown in Figure 1. The output value $z(t)$ at time $t \in \mathbb{R}_+$ depends not only on the input value $y(t)$ but also on the past history of the input. To aid in the characterization of this dependence, it is convenient to restrict initially to piecewise monotone input functions y. We seek an operator $P_{\sigma,\zeta}$ such that, given a piecewise monotone function y, the corresponding output function is given by $z = P_{\sigma,\zeta}(y)$. Here the parameter $\zeta \in \mathbb{R}$ plays the role of an "initial state," determining the initial output value $z(0) \in [y(0) - \sigma, y(0) + \sigma]$. To give a formal definition of the play operator, let $\sigma \in \mathbb{R}_+$ and introduce the function $p_\sigma : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\begin{aligned} p_\sigma(v_1, v_2) &:= \max\big\{v_1 - \sigma,\ \min\{v_1 + \sigma, v_2\}\big\} \\ &= \begin{cases} v_1 - \sigma, & \text{if } v_2 < v_1 - \sigma \\ v_2, & \text{if } v_2 \in [v_1 - \sigma, v_1 + \sigma] \\ v_1 + \sigma, & \text{if } v_2 > v_1 + \sigma. \end{cases} \end{aligned}$$

Let $C_{\mathrm{pm}}(\mathbb{R}_+)$ denote the space of continuous piecewise monotone functions defined on $\mathbb{R}_+$. For all $\sigma \in \mathbb{R}_+$ and $\zeta \in \mathbb{R}$, define the operator $P_{\sigma,\zeta} : C_{\mathrm{pm}}(\mathbb{R}_+) \to C(\mathbb{R}_+)$ by

$$(P_{\sigma,\zeta}(y))(t) = \begin{cases} p_\sigma(y(0), \zeta) & \text{for } t = 0, \\ p_\sigma(y(t), (P_{\sigma,\zeta}(y))(t_i)) & \text{for } t_i < t \leq t_{i+1},\ i = 0, 1, 2, \ldots, \end{cases}$$

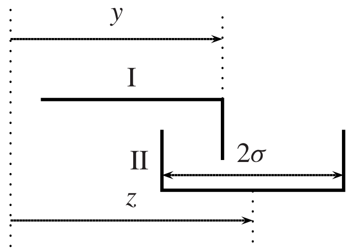

Figure 3. Mechanical play.

where $0 = t_0 < t_1 < t_2 < \cdots$, $\lim_{n\to\infty} t_n = \infty$, and u is monotone on each interval $[t_i, t_{i+1}]$. It is not difficult to show that the definition is independent of the choice of the partition (t_i). It is well known that $P_{\sigma,\zeta}$ extends to a hysteresis operator on $C(\mathbb{R}_+)$, the so-called play operator, which we shall denote by the same symbol $P_{\sigma,\zeta}$; furthermore, $P_{\sigma,\zeta}$ is globally Lipschitz with Lipschitz constant $L = 1$ (see [**2**] for details).

We are now in a position to model more complex hysteretic effects (displaying, for example, nested hysteresis loops) by using the play operator as a basic building block. To this end, let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a compactly supported and globally Lipschitz function with Lipschitz constant 1 and let $m \in L^1(\mathbb{R}_+)$. The operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$(\mathcal{P}_\xi(y))(t) = \int_0^\infty (P_{\sigma,\,\xi(\sigma)}(y))(t)m(\sigma)\,\mathrm{d}\sigma \quad \forall\, y \in C(\mathbb{R}_+), \quad \forall\, t \in \mathbb{R}_+,$$

is called a *Prandtl* operator. It is clear that $\mathcal{P}_\xi$ is a hysteresis operator (this follows from the fact that $P_{\sigma,\,\xi(\sigma)}$ is a hysteresis operator for every $\sigma \geq 0$). Moreover, $\mathcal{P}_\xi$ is gobally Lipschitz with Lipschitz constant $L = \|m\|_{L^1}$ (see [**2**]). For $\xi = 0$ and $m = \mathbb{I}_{[0,5]}$ (where $\mathbb{I}_{[0,5]}$ denotes the indicator function of the interval $[0, 5]$), the Prandtl operator is illustrated in Figure 4.

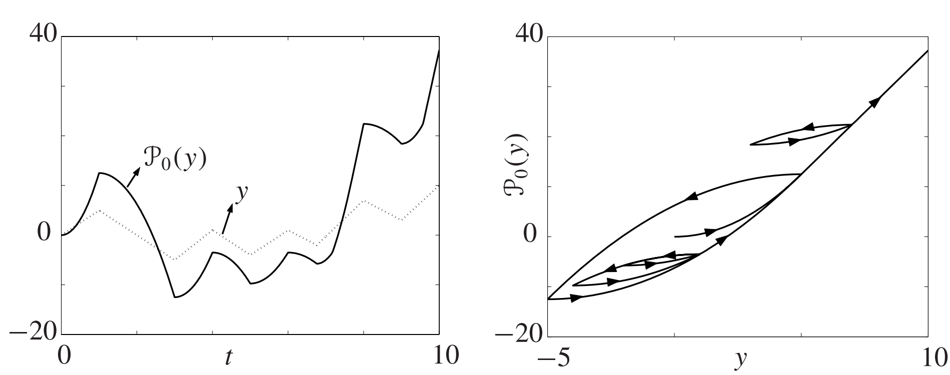

Figure 4. Example of Prandtl hysteresis.

4. RETARDED DIFFERENTIAL EQUATIONS. Let $h \geq 0$ and let I be an interval containing $[-h, 0]$. For $u \in C(I)$ and $t \in I_+$, $u_t \in C[-h, 0]$ is defined by $u_t(s) := u(t+s)$ for all $s \in [-h, 0]$. Let $\Delta \subset [-h, \infty) \times \mathbb{R}^N$ be a relatively open set with the property that the set $\{u \in C[-h, 0] \colon \operatorname{gr} u \subset \Delta\}$ is nonempty. As before, let $\mathcal{J}$ be the set of all intervals I of the form $[-h, \eta]$ (with $0 < \eta < \infty$) or $[-h, \eta)$ (with $0 < \eta \leq \infty$) such that $W_\Delta(I) \neq \emptyset$ and recall that $T := \sup\{\alpha > 0 : W_\Delta[-h, \alpha] \neq \emptyset\} > 0$. Define

$$\mathcal{D}_\Delta := \{(t, u) \in \mathbb{R}_+ \times C[-h, 0] \colon (t + s, u(s)) \in \Delta \ \forall s \in [-h, 0]\}.$$

It is readily verified that $\mathcal{D}_\Delta$ is nonempty and is open relative to $\mathbb{R}_+ \times C[-h, 0]$, where $C[-h, 0]$ is endowed with the topology induced by the supremum norm (1.7). Moreover, for $I \in \mathcal{I}$ and $u \in C(I)$, we have

$$\operatorname{gr} u \subset \Delta \quad \Leftrightarrow \quad (t, u_t) \in \mathcal{D}_\Delta \ \forall t \in I_+.$$

If Δ is a Cartesian product, that is, $\Delta = [-h, \alpha) \times G$, where $0 < \alpha \leq \infty$ and $G \subset \mathbb{R}^N$ is open, then $\mathcal{D}_\Delta$ is also a Cartesian product, namely $\mathcal{D}_\Delta = [0, \alpha) \times C([-h, 0], G)$, where $C([-h, 0], G)$ is the subset of all functions in $C[-h, 0]$ with values in G.

Consider the initial-value problem

$$u'(t) = f(t, u_t), \quad u|_{[-h,0]} = \varphi \in C[-h, 0], \quad \operatorname{gr} \varphi \subset \Delta, \tag{4.1}$$

where $f \colon \mathcal{D}_\Delta \to \mathbb{R}^N$. By a *solution* of (4.1), we mean a function $u \in W_\Delta(I)$, with $I \in \mathcal{I}$, such that $u|_{[-h,0]} = \varphi$ and $u'(t) = f(t, u_t)$ for almost every $t \in I_+$. We impose the following hypotheses on f.

(RDE1) For every $(t, w) \in \mathcal{D}_\Delta$ there exist a relatively open interval $I \subset \mathbb{R}_+$ containing t, an open ball $B \subset C[-h, 0]$ containing w, and a function $l \in L^1(I)$ such that $I \times B \subset \mathcal{D}_\Delta$ and

$$\|f(s, u) - f(s, v)\| \leq l(s)\, \|u - v\|_{C[-h,0]} \quad \forall s \in I, \ \forall u, v \in B.$$

(RDE2) For every $I \in \mathcal{I}$ and every $u \in W_\Delta(I)$ such that $\overline{\operatorname{gr} u}$ is a compact subset of Δ, the function $I_+ \to \mathbb{R}^N$, $t \mapsto f(t, u_t)$ is in $L^1(I_+)$.

For each $I \in \mathcal{I}$, define the operator F on $W_\Delta(I)$ by

$$(F(v))(t) := f(t, v_t) \quad \forall t \in I_+.$$

Let $I \in \mathcal{I}$ and $v \in W_\Delta(I)$ be arbitrary. Let $\tau > 0$ be such that $J := [-h, \tau] \subset I$ and write $u := v|_J$. Then $\overline{\operatorname{gr} u}$ is a compact subset of Δ and so, by (RDE2), the function $F(u)$ is in $L^1(J_+)$. It follows that the function $F(v)$ is in $L^1_{\mathrm{loc}}(I_+)$. Therefore, for each $I \in \mathcal{I}$, F maps $W_\Delta(I)$ to $L^1_{\mathrm{loc}}(I_+)$ and, in view of (RDE2), F satisfies the integrability hypothesis (H3). It is evident that the causality hypothesis (H1) is also valid. We proceed to show that F satisfies the local Lipschitz hypothesis (H2). Let $\alpha \in [0, T)$ and $w \in W_\Delta[-h, \alpha]$ be arbitrary. By (RDE1), there exist a relatively open interval $I \subset \mathbb{R}_+$ containing α, an open ball $B \subset C[-h, 0]$ containing w_α, and $l \in L^1(I)$ such that $I \times B \subset \mathcal{D}_\Delta$ and

$$\|f(t, x) - f(t, y)\| \leq l(t) \|x - y\|_{C[-h,0]} \quad \forall t \in I, \ \forall x, y \in B, \tag{4.2}$$

A routine argument (see appendix for details) then yields the following.

Lemma 4.1. *There exists $\gamma > 0$ such that*

$$\left.\begin{aligned} &\lambda := \textstyle\int_\alpha^{\alpha+\gamma} l(t)\, \mathrm{d}t < 1, \\ &v_t \in B \quad \forall v \in \mathcal{W}(w; \alpha, \alpha + \gamma, \gamma), \ \forall t \in [\alpha, \alpha + \gamma]. \end{aligned}\right\} \tag{4.3}$$

By Lemma 4.1, there exists $\gamma > 0$ such that (4.3) holds. Set $\beta := \alpha + \gamma$ and let $u, v \in \mathcal{W}(w; \alpha, \beta, \gamma)$ be arbitrary. Then, by (4.2) and (4.3), we have

$$\begin{aligned}\|(F(u))(t) - (F(v))(t)\| &= \|f(t, u_t) - f(t, v_t)\| \\ &\le l(t)\|u_t - v_t\|_{C[-h,0]} \le l(t)\|u - v\|_{C[\alpha,\beta]} \\ &\le l(t)\int_\alpha^\beta \|u'(s) - v'(s)\|\,\mathrm{d}s \;\forall\, t \in [\alpha, \beta].\end{aligned}$$

Integrating from α to $\beta = \alpha + \gamma$ yields

$$\|F(u) - F(v)\|_{L^1[\alpha,\beta]} \le \lambda \int_\alpha^\beta \|u'(s) - v'(s)\|\,\mathrm{d}s.$$

Therefore, the local Lipschitz hypothesis (H2) holds.

We may now infer that, if (RDE1) and (RDE2) hold, then the assertions of Theorems 3.6 and 3.7 are valid in the context of the initial-value problem (4.1).

5. ORDINARY DIFFERENTIAL EQUATIONS. Let $G \subset \mathbb{R}_+ \times \mathbb{R}^N$ be a relatively open set with $G_0 := \{x \in \mathbb{R}^N : (0, x) \in G\} \neq \emptyset$. In this section, we consider the initial-value problem for an ordinary differential equation of the form

$$u'(t) = f(t, u(t)), \quad u(0) = u^0 \in G_0, \tag{5.1}$$

where $f : G \to \mathbb{R}^N$. Let $\mathcal{I}$ be the set of all intervals I of the form $[0, \alpha]$ (with $0 < \alpha < \infty$) or $[0, \alpha)$ (with $0 < \alpha \le \infty$) such that $W_G(I) = \{u \in W^{1,1}_{\mathrm{loc}}(I) : \operatorname{gr} u \subset G\} \neq \emptyset$. By a *solution* of (5.1), we mean a function $u \in W_G(I)$, with $I \in \mathcal{I}$, such that $u(0) = u^0$ and $u'(t) = f(t, u(t))$ for almost all $t \in I$. We impose the following hypothesis on f.

(ODE) For every $(t, z) \in G$, there exists a relatively open interval $I \subset \mathbb{R}_+$ containing t, an open ball $B \subset \mathbb{R}^N$ containing z, and a function $l \in L^1(I)$ such that $I \times B \subset G$,

$$\|f(s, x) - f(s, y)\| \le l(s)\|x - y\| \quad \forall\, s \in I,\ \forall\, x, y \in B,$$

and, moreover, the function $I \to \mathbb{R}^N$, $s \mapsto f(s, x)$ is measurable for all $x \in B$ and is in $L^1(I)$ for some $x \in B$.

The following proposition shows that, under this hypothesis, the initial-value problem (5.1) is subsumed by the theory of retarded differential equations developed in the previous section specialized to the situation $h = 0$, in which case $C[-h, 0] = \mathbb{R}^N$, $\Delta = \mathcal{D}_\Delta = G$, and the initial-value problems (4.1) and (5.1) coincide.

Proposition 5.1. *If* (ODE) *is satisfied, then* (RDE1) *and* (RDE2) *hold with* $h = 0$.

Proposition 5.1 (a proof of which may be found in the appendix), together with the conclusion of Section 4, imply that, if hypothesis (ODE) is satisfied, then Theorems 3.6 and 3.7 apply to the initial-value problem (5.1).

6. INTEGRO-DIFFERENTIAL EQUATIONS. In this section, we apply the theory developed in Section 3 to initial-value problems associated with integro-differential equations (also called Volterra integro-differential equations), that is,

$$u'(t) = \int_0^t k(t, s)g(s, u(s))\,\mathrm{d}s, \ \ t \ge 0; \quad u(0) = u^0. \tag{6.1}$$

Here $k \in L^p_{\mathrm{loc}}(J \times J, \mathbb{R}^{N\times M})$ and $g : G \to \mathbb{R}^M$, where $p \in [1, \infty]$, $J = [0, a)$ for some a satisfying $0 < a \leq \infty$, $\mathbb{R}^{N\times M}$ is the set of all $N \times M$ matrices with real entries, and $G \subset \mathbb{R}_+ \times \mathbb{R}^N$ is relatively open. It is assumed that $G_0 := \{x \in \mathbb{R}^N : (0, x) \in G\} \neq \emptyset$ and $u^0 \in G_0$.

To apply the theory developed in Section 3 to the initial-value problem (6.1), we set $h := 0$ and $\Delta := (J \times \mathbb{R}^N) \cap G$. Note that $\Delta_0 = G_0$ and, moreover, since $h = 0$, $I_+ = I$ for all $I \in \mathfrak{I}$. Trivially, if $I \in \mathfrak{I}$ is such that $W_\Delta(I) \neq \emptyset$, then $I \subset J$ and $W_\Delta(I) = W_G(I)$. By a *solution* of (6.1), we mean a function $u \in W_\Delta(I)$, with $I \in \mathfrak{I}$, such that $u(0) = u^0$ and $u'(t) = \int_0^t k(t, s)g(s, u(s))\,\mathrm{d}s$ for almost every $t \in I$.

Defining, for each $I \in \mathfrak{I}$, the operator F on $W_\Delta(I)$ by

$$(F(v))(t) = \int_0^t k(t, s)g(s, v(s))\,\mathrm{d}s \quad \forall\, t \in I, \tag{6.2}$$

the initial-value problem (6.1) can be written in the form (3.1).

Set $q := p/(p - 1)$. We impose the following hypothesis on g.

(IDE) For every $(t, z) \in G$, there exist a relatively open interval $I \subset \mathbb{R}_+$ containing t, an open ball $B \subset \mathbb{R}^N$ containing z, and a nonnegative function $l \in L^q(I)$ such that $I \times B \subset G$, the function $I \to \mathbb{R}^M$, $s \mapsto g(s, x)$ is measurable for all $x \in B$ and is in $L^q(I)$ for some $x \in B$, and moreover,

$$\|g(s, x) - g(s, y)\| \leq l(s)\|x - y\| \quad \forall\, s \in I, \quad \forall\, x, y \in B. \tag{6.3}$$

Proposition 6.1. *Assume that* (IDE) *holds and that F is given by* (6.2). *Then, for all $I \in \mathfrak{I}$, F maps $W_\Delta(I)$ to $L^1_{\mathrm{loc}}(I)$ and F satisfies* (H1), (H2), *and* (H3).

Proposition 6.1, a proof of which may be found in the appendix, implies that if hypothesis (IDE) is satisfied, then Theorems 3.6 and 3.7 apply to the initial-value problem (6.1).

7. APPENDIX.

Proof of Proposition 2.1. Let $u \in W^{1,1}[a, b]$. Since $u(t) = u(a) + \int_a^t u'(s)\,\mathrm{d}s$ for all $t \in [a, b]$, it is clear that

$$\|u\|_\infty := \max_{t\in[a,b]} \|u(t)\| \leq \|u\|_{BV}.$$

Consequently,

$$\|u\|_{W^{1,1}} \leq (b - a)\|u\|_\infty + \|u'\|_{L^1} \leq \big((b - a) + 1\big)\|u\|_{BV},$$

and so, setting $k_1 := \big((b - a) + 1\big)^{-1}$, it follows that $k_1\|u\|_{W^{1,1}} \leq \|u\|_{BV}$ for all $u \in W^{1,1}[a, b]$.

Furthermore, denoting the components of u by u_j, the mean value theorem for integrals guarantees the existence of $c_j \in [a, b]$ such that

$$u_j(c_j) = \frac{1}{b - a}\int_a^b u_j(s)\,\mathrm{d}s \quad j = 1, \ldots, N.$$

Consequently, for $j = 1, \dots, N$,

$$|u_j(a)| \leq |u_j(a) - u_j(c_j)| + |u_j(c_j)| \leq \int_a^b |u_j'(s)|\,\mathrm{d}s + \frac{1}{b-a}\int_a^b |u_j(s)|\,\mathrm{d}s,$$

and thus, by a routine calculation,

$$\|u(a)\| = \sqrt{\sum_{j=1}^{N} |u_j(a)|^2} \leq \frac{(1+b-a)\sqrt{N}}{b-a}\|u\|_{W^{1,1}}.$$

Therefore, setting $k_2 := 1 + (1+b-a)\sqrt{N}/(b-a)$, it follows that $\|u\|_{BV} \leq k_2\|u\|_{W^{1,1}}$ for all $u \in W^{1,1}[a,b]$. ■

Proof of Lemma 3.1. It is clear that μ is a metric on $\mathcal{W}(w;\alpha,\beta,\gamma)$. Let (u_n) be a Cauchy sequence in $\mathcal{W}(w;\alpha,\beta,\gamma)$ and set $v_n = u_n|_{[\alpha,\beta]}$. Then $v_n \in W^{1,1}[\alpha,\beta]$, $v_n(\alpha) = w(\alpha)$, and

$$\|v_n - v_m\|_{BV} = \int_\alpha^\beta \|u_n'(s) - u_m'(s)\|\,\mathrm{d}s = \mu(u_n, u_m),$$

showing that (v_n) is a Cauchy sequence in $W^{1,1}[\alpha,\beta]$ with respect to the norm $\|\cdot\|_{BV}$. By Proposition 2.1, $W^{1,1}[\alpha,\beta]$ is complete with respect to the BV-norm and so there exists $v \in W^{1,1}[\alpha,\beta]$ such that

$$\|v(\alpha) - w(\alpha)\| + \int_\alpha^\beta \|v'(s) - v_n'(s)\|\,\mathrm{d}s = \|v - v_n\|_{BV} \to 0 \quad \text{as } n \to \infty.$$

Thus, $v(\alpha) = w(\alpha)$, $\int_\alpha^\beta \|v'(s) - v_n'(s)\|\,\mathrm{d}s \to 0$ as $n \to \infty$, and $\int_\alpha^\beta \|v'(s)\|\,\mathrm{d}s \leq \gamma$. Therefore, defining

$$u(t) := \begin{cases} w(t), & t \in [-h,\alpha] \\ v(t), & t \in (\alpha,\beta], \end{cases}$$

it follows that $u \in \mathcal{W}(w;\alpha,\beta,\gamma)$ and $\mu(u,u_n) \to 0$ as $n \to \infty$, completing the proof. ■

Proof of Lemma 3.2. (1) Since $(\alpha, w(\alpha)) \in \Delta$, it follows from the assumptions imposed on Δ that there exists $\beta > \alpha$ and $\gamma > 0$ such that $[\alpha,\beta] \times B_\gamma \subset \Delta$, where B_γ denotes the closed ball of radius $\gamma > 0$ centered at $w(\alpha) \in \mathbb{R}^N$. We claim that $(\beta,\gamma) \in A(w;\alpha)$. To this end, let $u \in \mathcal{W}(w;\alpha,\beta,\gamma)$. Then,

$$\|u(t) - w(\alpha)\| = \|u(t) - u(\alpha)\| \leq \int_\alpha^\beta \|u'(s)\|\,\mathrm{d}s \leq \gamma \quad \forall\, t \in [\alpha,\beta].$$

Consequently, $(t, u(t)) \in [\alpha,\beta] \times B_\gamma \subset \Delta$ for all $t \in [\alpha,\beta]$, implying that $\operatorname{gr} u \subset \Delta$. Since $u \in \mathcal{W}(w;\alpha,\beta,\gamma)$ was arbitrary, it follows that $(\beta,\gamma) \in A(w;\alpha)$.

(2) Let $(\beta,\gamma) \in A(w;\alpha)$, $b \in (\alpha,\beta]$, and $c \in (0,\gamma]$, and let $u \in \mathcal{W}(w;\alpha,b,c)$. It is sufficient to show that $\operatorname{gr} u \subset \Delta$. To this end, define an extension $\tilde{u} : [-h,\beta] \to \mathbb{R}^N$ of u by

$$\tilde{u}(t) := \begin{cases} u(t), & t \in [-h,b] \\ u(b), & t \in (b,\beta]. \end{cases}$$

Clearly, $\tilde{u} \in \mathcal{W}(w; \alpha, \beta, \gamma)$, and hence, by the admissibility of (α, β), $\operatorname{gr} \tilde{u} \subset \Delta$. Since $\tilde{u}$ is an extension of u, we have that $\operatorname{gr} u \subset \operatorname{gr} \tilde{u}$, and so $\operatorname{gr} u \subset \Delta$. ■

Proof of Lemma 3.3. Let $\alpha \in [0, T)$ and $w \in W_\Delta[-h, \alpha]$. Then, by (H2), there exist $\lambda \in (0, 1)$ and $(\beta, \gamma) \in A(w; \alpha)$ such that (3.2) holds. Let $b \in (\alpha, \beta]$, $c \in (0, \gamma]$, and $u, v \in \mathcal{W}(w; \alpha, b, c)$. Define $\tilde{u}, \tilde{v} \in \mathcal{W}(w; \alpha, \beta, \gamma)$ by

$$\tilde{u}(t) := \begin{cases} u(t), & t \in [-h, b] \\ u(b), & t \in (b, \beta], \end{cases} \qquad \tilde{v}(t) := \begin{cases} v(t), & t \in [-h, b] \\ v(b), & t \in (b, \beta]. \end{cases}$$

By (H1), $F(\tilde{u})|_{[0,b]} = F(u)|_{[0,b]}$ and $F(\tilde{v})|_{[0,b]} = F(v)|_{[0,b]}$, so that, by (3.2),

$$\|F(u) - F(v)\|_{L^1[\alpha,b]} \leq \|F(\tilde{u}) - F(\tilde{v})\|_{L^1[\alpha,\beta]} \leq \lambda \int_\alpha^\beta \|\tilde{u}'(s) - \tilde{v}'(s)\| \, \mathrm{d}s.$$

The claim now follows, since $\int_\alpha^\beta \|\tilde{u}'(s) - \tilde{v}'(s)\| \, \mathrm{d}s = \int_\alpha^b \|u'(s) - v'(s)\| \, \mathrm{d}s$. ■

Proof of Proposition 3.4. Let $\alpha \in [0, T)$ and $w \in W_\Delta[-h, \alpha]$. Since (H2′) is satisfied, there exists $\rho > 0$ and $(\beta, \gamma) \in A(w; \alpha)$ such that (3.3) holds. Invoking (H1), the argument employed in the proof of Lemma 3.3 applies *mutatis mutandis* to yield that, for every $b \in (\alpha, \beta]$,

$$\|F(u) - F(v)\|_{L^p[\alpha,b]} \leq \rho \int_\alpha^b \|u'(s) - v'(s)\| \, \mathrm{d}s \quad \forall\, u, v \in \mathcal{W}(w; \alpha, b, \gamma).$$

Hence, setting $q := p/(p-1) \in [1, \infty)$, it follows from Hölder's inequality that, for every $b \in (\alpha, \beta]$ and all $u, v \in \mathcal{W}(w; \alpha, b, \gamma)$,

$$\begin{aligned} \|F(u) - F(v)\|_{L^1[\alpha,b]} &\leq (b-\alpha)^{1/q} \|F(u) - F(v)\|_{L^p[\alpha,b]} \\ &\leq \rho(b-\alpha)^{1/q} \int_\alpha^b \|u'(s) - v'(s)\| \, \mathrm{d}s. \end{aligned}$$

Choosing $b \in (\alpha, \beta]$ sufficiently close to α, so that $\rho(b-\alpha)^{1/q} < 1$, and noting that, by Lemma 3.2, $(b, \gamma) \in A(w; \alpha)$, we conclude that (H2) holds with $\lambda = \rho(b-\alpha)^{1/q}$. ■

Proof of Lemma 4.1. To show that there exists $\gamma > 0$ such that (4.3) holds, recall that $\alpha \in [0, T)$, $w \in W_\Delta[-h, \alpha]$, $I \subset \mathbb{R}_+$ is a relatively open interval containing α, and $B \subset C[-h, 0]$ is an open ball containing w_α. Without loss of generality we may assume that B is centered at w_α. Let ρ denote the radius of B.

By Lemma 3.2, the set $A(w; \alpha)$ is nonempty. Let $(\beta^*, \gamma^*) \in A(w; \alpha)$ and define $\tilde{w} \in C[-h, \beta^*]$ by

$$\tilde{w}(t) := \begin{cases} w(t), & t \in [-h, \alpha] \\ w(\alpha), & t \in (\alpha, \beta^*]. \end{cases}$$

By uniform continuity of w, there exists $\delta > 0$ such that

$$s, t \in [-h, \alpha], \quad |s - t| < \delta \implies \|w(s) - w(t)\| < \frac{\rho}{2}.$$

Now choose $\gamma > 0$ sufficiently small so that

$$\gamma < \min\{\beta^* - \alpha, \gamma^*, \delta, \rho/2\}, \quad [\alpha, \alpha + \gamma] \subset I, \quad \lambda := \int_\alpha^{\alpha+\gamma} l(t)\,\mathrm{d}t < 1$$

and define $\beta := \alpha + \gamma$. By Lemma 3.2, $(\beta, \gamma) \in A(w; \alpha)$. Let $t \in [\alpha, \beta]$ and $v \in \mathcal{W}(w; \alpha, \beta, \gamma)$ be arbitrary. Observe that

$$s \in [-h, 0], \quad t + s \le \alpha \implies |(\alpha + s) - (t + s)| = t - \alpha \le \beta - \alpha = \gamma < \delta$$

and so

$$s \in [-h, 0],\ t + s \le \alpha \implies \|w(\alpha + s) - v(t + s)\| = \|w(\alpha + s) - w(t + s)\| < \frac{\rho}{2}.$$

Furthermore,

$$s \in [-h, 0], \quad t + s > \alpha \implies |(\alpha + s) - \alpha| = |s| < t - \alpha \le \beta - \alpha = \gamma < \delta$$

and so, for $s \in [-h, 0]$ such that $t + s > \alpha$ it follows that

$$\begin{aligned}
\|w(\alpha + s) - v(t + s)\| &\le \|w(\alpha + s) - \tilde{w}(t + s)\| + \|\tilde{w}(t + s) - v(t + s)\| \\
&= \|w(\alpha + s) - w(\alpha)\| + \Big\|w(\alpha) - v(\alpha) - \int_\alpha^{t+s} v'(\sigma)\ \mathrm{d}\sigma\Big\| \\
&< \frac{\rho}{2} + \int_\alpha^\beta \|v'(\sigma)\|\,\mathrm{d}\sigma \le \frac{\rho}{2} + \gamma < \frac{\rho}{2} + \frac{\rho}{2} = \rho.
\end{aligned}$$

Thus,

$$s \in [-h, 0], \ \ t + s > \alpha \implies \|w(\alpha + s) - v(t + s)\| < \rho.$$

We have now shown that

$$\|w(\alpha + s) - v(t + s)\| < \rho \quad \forall\, v \in \mathcal{W}(w; \alpha, \beta, \gamma),\ \forall\, s \in [-h, 0],\ \forall\, t \in [\alpha, \beta].$$

Consequently, $v_t \in B$ for all $v \in \mathcal{W}(w; \alpha, \alpha + \gamma, \gamma)$ and for all $t \in [\alpha, \alpha + \gamma]$, completing the proof. ■

To facilitate the proofs of Propositions 5.1 and 6.1, we state and prove the following lemma.

Lemma 7.1. *Let $G \subset \mathbb{R}_+ \times \mathbb{R}^N$ be a relatively open set and let $1 \le q \le \infty$. Assume that $g\colon G \to \mathbb{R}^M$ is such that, for every $(t, z) \in G$, there exist a relatively open interval $I \in \mathbb{R}_+$ containing t, an open ball $B \subset \mathbb{R}^N$ containing z, and a function $l \in L^q(I)$ such that $I \times B \subset G$,*

$$\|g(s, x) - g(s, y)\| \le l(s)\|x - y\| \quad \forall\, s \in I,\ \forall\, x, y \in B, \tag{7.1}$$

and the function $I \to \mathbb{R}^M$, $s \mapsto g(s, x)$ is measurable for all $x \in B$ and is in $L^q(I)$ for some $x \in B$. Then the following statements hold.

(1) *If $I \subset \mathbb{R}_+$ is an interval and $v \in C(I)$ is such that* gr $v \subset G$*, then the function $I \to \mathbb{R}^M$, $s \mapsto g(s, v(s))$ is measurable.*

(2) *For every* $(t, z) \in G$, *there exist a relatively open interval* $I \in \mathbb{R}_+$ *containing* t, *an open ball* $B \subset \mathbb{R}^N$ *containing* z, *and a function* $b \in L^q(I)$ *such that* $I \times B \subset G$ *and* $\|g(s, x)\| \leq b(s)$ *for all* $(s, x) \in I \times B$.

(3) *If* $I \subset \mathbb{R}_+$ *is an interval and* $v \in C(I)$ *is such that* $\overline{\operatorname{gr} v}$ *is a compact subset of* G, *then the function* $I \to \mathbb{R}^M$, $s \mapsto g(s, v(s))$ *is in* $L^q(I)$.

Proof. (1) It is sufficient to show that, for every compact subinterval $K \subset I$, the function $K \to \mathbb{R}^M$, $s \mapsto g(s, v(s))$ is measurable. To this end, let $K \subset I$ be a compact subinterval. Then $\operatorname{gr}(v|_K)$ is compact and contained in G. It follows from the hypothesis that there exist finitely many relatively open intervals $I_1, \ldots, I_m$ in $\mathbb{R}_+$ and open balls $B_1, \ldots, B_m$ in $\mathbb{R}^N$ such that $\operatorname{gr}(v|_K) \subset \cup_{i=1}^m I_i \times B_i$ and, for $i = 1, \ldots, m$, $v(I_i) \subset B_i$, $I_i \times B_i \subset G$, and the function $I_i \to \mathbb{R}^M$, $s \mapsto g(s, x)$ is measurable for every $x \in B_i$. Setting $K_i := I_i \cap K$ it follows that $K = \cup_{i=1}^m K_i$. Let $i \in \{1, \ldots, m\}$ be arbitrary, let $J_1, \ldots, J_n$ be any finite disjoint family of subintervals of K_i such that $K_i = \cup_{j=1}^n J_j$, and let $x_1, \ldots, x_n \in B_i$. Since $K_i \to \mathbb{R}^M$, $s \mapsto g(s, x_j)$ is measurable for each j, it follows that the function

$$K_i \to \mathbb{R}^M, \; s \mapsto \sum_{j=1}^n g(s, x_j)\mathbb{I}_{J_j}(s) \tag{7.2}$$

is measurable, where $\mathbb{I}_{J_j}$ denotes the characteristic function of J_j. Since, for fixed $s \in K_i$, the function $B_i \to \mathbb{R}^M$, $x \mapsto g(s, x)$ is continuous and $v|_{K_i}$ is continuous with $v(K_i) \subset B_i$, it follows that the function $K_i \to \mathbb{R}^M$, $s \mapsto g(s, v(s))$ is the pointwise limit of functions of the form (7.2) and hence is measurable. Since $i \in \{1, \ldots, m\}$ is arbitrary and $K = \cup_{i=1}^m K_i$, the function $K \to \mathbb{R}^M$, $s \mapsto g(s, v(s))$ is measurable.

(2) Let $(t, z) \in G$. Then there exist a relatively open interval $I \subset \mathbb{R}_+$ containing t, an open ball $B \subset \mathbb{R}^N$ containing z, and $l \in L^q(I)$ such that $I \times B \subset G$, (7.1) holds, and, moreover, the function $I \to \mathbb{R}^M$, $s \mapsto g(s, y)$ is in $L^q(I)$ for some $y \in B$. Therefore,

$$\begin{aligned} \|g(s, x)\| &\leq \|g(s, x) - g(s, y)\| + \|g(s, y)\| \leq l(s)\|x - y\| + \|g(s, y)\| \\ &\leq l(s) \sup_{x \in B} \|x - y\| + \|g(s, y)\| =: b(s) \quad \forall\, s \in I, \; \forall\, x \in B. \end{aligned}$$

Since $l \in L^q(I)$ and $g(\cdot, y) \in L^q(I)$, it follows that $b \in L^q(I)$.

(3) By compactness of $\overline{\operatorname{gr} v}$ and statement (2), there exist finitely many relatively open intervals $I_1, \ldots, I_m$ in $\mathbb{R}_+$, open balls $B_1, \ldots, B_m$ in $\mathbb{R}^N$, and functions $b_i \in L^q(I_i)$, $i = 1, \ldots, m$, such that $I_i \times B_i \subset G$, $i = 1, \ldots, m$, $\overline{\operatorname{gr} v} \subset \cup_{i=1}^m I_i \times B_i$, and $\|g(s, x)\| \leq b_i(s)$ for all $(s, x) \in I_i \times B_i$, $i = 1, \ldots, m$. Defining $\tilde{b}_i : I \to \mathbb{R}_+$ by

$$\tilde{b}_i(s) := \begin{cases} b_i(s), & s \in I_i \cap I \\ 0, & s \in I \backslash I_i, \end{cases}$$

it follows that $\|g(s, x)\| \leq \sum_i \tilde{b}_i(s)$ for all $(s, x) \in \overline{\operatorname{gr} v}$. Therefore, since $\sum_i \tilde{b}_i \in L^q(I)$ and since the function $s \mapsto g(s, v(s))$ is measurable (by statement (1)), the result follows. ∎

Proof of Proposition 5.1. We make the following connections with the notation of Section 4: for $h = 0$, we set $C[-h, 0] = \mathbb{R}^N$ and $\Delta = \mathcal{D}_\Delta = G$. Then the initial-value problems (4.1) and (5.1) coincide. Furthermore note that, for all $I \in \mathcal{I}$, $I_+ = I$.

By (ODE), we see that (RDE1) holds (with $h = 0$). It remains only to show that (RDE2) also holds (with $h = 0$). Let $I \in \mathcal{I}$ and $v \in W_G(I)$ be such that $\overline{\operatorname{gr} v}$ is a compact subset of G. In view of (ODE), we see that the hypotheses of Lemma 7.1 hold with $M = N$, $q = 1$, and $g = f$. By assertion (3) of Lemma 7.1, we may infer that $I \to \mathbb{R}^N$, $t \mapsto f(t, v(t))$ is in $L^1(I)$. Therefore, (RDE2) holds. ■

Proof of Proposition 6.1. In view of (IDE), we see that the hypotheses of Lemma 7.1 hold. Let $I \in \mathcal{I}$ and $v \in W_\Delta(I)$. Invoking statement (3) of Lemma 7.1, we conclude that the function $s \mapsto g(s, v(s))$ is in $L^q_{\mathrm{loc}}(I)$. It follows now from a routine argument based on Fubini's theorem (see, for example, [**7, 11**]) that $F(v) \in L^1_{\mathrm{loc}}(I)$. Similarly, under the additional assumption that $\overline{\operatorname{gr} v}$ is compact and contained in Δ, statement (3) of Lemma 7.1 guarantees that the function $s \mapsto g(s, v(s))$ is in $L^q(I)$ and the same routine argument based on Fubini's theorem yields $F(v) \in L^1(I)$, showing that (H3) is valid. Moreover, it is trivial that (H1) holds, and therefore, it only remains to show that (H2) is satisfied. To this end, let $\alpha > 0$, $w \in W_\Delta[0, \alpha]$, and $(\beta_0, \gamma_0) \in A(w; \alpha)$. Then $(\alpha, w(\alpha)) \in \Delta \subset G$ and, by (IDE), there exist $\beta_1 \in (\alpha, \beta_0]$, an open ball $B \subset \mathbb{R}^N$, and $l \in L^q[\alpha, \beta_1]$ such that $w(\alpha) \in B$, $[\alpha, \beta_1] \times B \subset \Delta \subset G$, and

$$\|g(s, x) - g(s, y)\| \le l(s)\|x - y\| \quad \forall\, s \in [\alpha, \beta_1], \quad \forall\, x, y \in B.$$

Let B_γ denote the closed ball of radius γ centered at $w(\alpha)$. Choose $\beta \in (\alpha, \beta_1]$ and $\gamma \in (0, \gamma_0]$ such that

$$\lambda := \int_\alpha^\beta \int_0^t \|k(s, t)\| l(s)\, \mathrm{d}s\, \mathrm{d}t < 1 \qquad \text{and} \qquad B_\gamma \subset B.$$

Let $u, v \in \mathcal{W}(w; \alpha, \beta, \gamma)$. Then, $u(s), v(s) \in B_\gamma \subset B$ for all $s \in [\alpha, \beta]$, and, moreover,

$$\begin{aligned}
\|F(u) - F(v)\|_{L^1[\alpha,\beta]} &\le \int_\alpha^\beta \int_0^t \|k(s, t)\| \|g(s, u(s)) - g(s, v(s))\|\, \mathrm{d}s\, \mathrm{d}t \\
&\le \int_\alpha^\beta \int_0^t \|k(s, t)\| l(s) \|u(s) - v(s)\|\, \mathrm{d}s\, \mathrm{d}t \\
&\le \lambda \sup_{s \in [\alpha,\beta]} \|u(s) - v(s)\| \le \lambda \int_\alpha^\beta \|u'(s) - v'(s)\|\, \mathrm{d}s,
\end{aligned}$$

completing the proof. ■

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REFERENCES

1. H. Amann, *Ordinary Differential Equations*, Walter de Gruyter, Berlin, 1990.
2. M. Brokate and J. Sprekels, *Hysteresis and Phase Transitions*, Springer, New York, 1996.
3. C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, Cambridge, 1991.
4. ———, *Functional Equations with Causal Operators*, Taylor & Francis, London, 2002.
5. O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H.-O. Walther, *Delay Equations: Functional-, Complex-, and Nonlinear Analysis*, Springer, New York, 1995.
6. R. D. Driver, *Ordinary and Delay Differential Equations*, Springer, New York, 1977.
7. G. B. Folland, *Real Analysis*, 2nd ed., John Wiley, New York, 1999.

8. G. Gripenberg, S.-O. Londen, and O. Staffans, *Volterra Integral Equations and Functional Equations*, Cambridge University Press, Cambridge, 1990.
9. J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer, New York, 1993.
10. B. Jayawardhana, H. Logemann, and E. P. Ryan, PID control of second-order systems with hysteresis, *Internat. J. Control* **81** (2008) 1331–1342. `doi:10.1080/00207170701772479`
11. A. N. Kolmogorov and S. V. Fomin, *Introductory Real Analysis*, Dover, New York, 1975.
12. M. A. Krasnosel'skii and A. V. Pokrovskii, *Systems with Hysteresis*, Springer, Berlin, 1989.
13. X. Tan, J. S. Baras, and P. S. Krishnaprasad, Control of hysteresis in smart actuators with application to micro-positioning, *Systems Control Lett.* **54** (2005) 483–492. `doi:10.1016/j.sysconle.2004.09.013`
14. A. Visintin, *Differential Models of Hysteresis*, Springer, Berlin, 1994.
15. W. Walter, *Ordinary Differential Equations*, Springer, New York, 1998.

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