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CONDITIONS FOR ROBUSTNESS AND NONROBUSTNESS OF THE STABILITY OF FEEDBACK SYSTEMS WITH RESPECT TO SMALL DELAYS IN THE FEEDBACK LOOP*

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Abstract. It has been observed that for many stable feedback control systems, the introduction of arbitrarily small time delays into the loop causes instability. In this paper we present a systematic frequency domain treatment of this phenomenon for distributed parameter systems. We consider the class of all matrix-valued transfer functions which are bounded on some right half-plane and which have a limit at $+\infty$ along the real axis. Such transfer functions are called regular. Under the assumption that a regular transfer function is stabilized by unity output feedback, we give sufficient conditions for the robustness and for the nonrobustness of the stability with respect to small time delays in the loop. These conditions are given in terms of the high-frequency behavior of the open-loop system. Moreover, we discuss robustness of stability with respect to small delays for feedback systems with dynamic compensators. In particular, we show that if a plant with infinitely many poles in the closed right half-plane is stabilized by a controller, then the stability is not robust with respect to delays. We show that the instability created by small delays is itself robust to small delays. Three examples are given to illustrate these results.

Key words. small time delays, robust stabilization, linear distributed parameter systems, regular transfer functions, dynamic stabilization

AMS subject classifications. 93C20, 93C25, 93D09, 93D15, 93D25

1. The main results.

Consider the linear feedback system shown in Fig. 1, where u is the input function and y is the output function, both $\mathbb{C}^m$-valued. $\mathbf{H}$ is the open-loop transfer function, with values in $\mathbb{C}^{m\times m}$, which we assume to be regular and in particular well posed. *Wellposedness* means that $\mathbf{H}$ is bounded on some right half-plane, and *regularity* means that, in addition, $\mathbf{H}$ has a limit at $+\infty$ along the real axis (see §2 for more detail on these concepts). The block with transfer function $e^{-\varepsilon s}$ represents a delay by ε, where $\varepsilon \geq 0$. The transfer function of the closed-loop system is given by

$$\mathbf{G}^{\varepsilon}(s) = \mathbf{H}(s)\left(I + e^{-\varepsilon s}\mathbf{H}(s)\right)^{-1}. \tag{1.1}$$

$\mathbf{G}^{\varepsilon}$ can be obtained from $\mathbf{G}^0$ by

$$\mathbf{G}^{\varepsilon}(s) = \mathbf{G}^0(s)\left[I - (1 - e^{-\varepsilon s})\mathbf{G}^0(s)\right]^{-1}. \tag{1.2}$$

To avoid possible complications with domains of transfer functions, we make the following convention: If a meromorphic function is defined on some right half-plane and can be extended meromorphically to a greater right half-plane, we will not make any distinction between the initial function and its extension. This will not lead to confusions.

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We say that $\mathbf{G}^\varepsilon$ is L^2-*stable* if $\mathbf{G}^\varepsilon \in H^\infty(\mathbb{C}^{m\times m})$; i.e., $\mathbf{G}^\varepsilon$ is a bounded analytic function on the open right half-plane $\mathbb{C}_0 = \{s \in \mathbb{C} \mid \text{Re } s > 0\}$. Indeed, as is well known, this property is equivalent to the one that $u \in L^2([0,\infty),\mathbb{C}^m)$ implies $y \in L^2([0,\infty),\mathbb{C}^m)$.

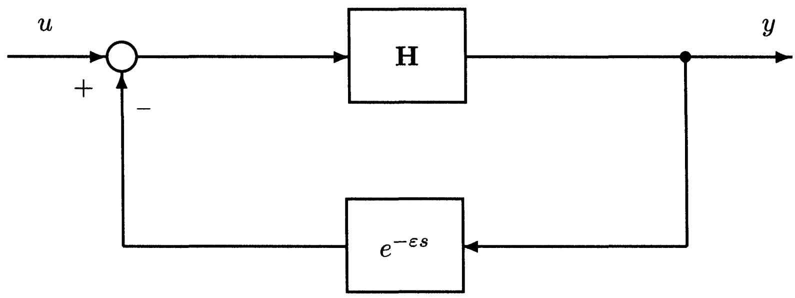

FIG. 1. *Feedback system with delay.*

In many engineering applications the aim is to stabilize a plant by a feedback controller. Here, stability may have various meanings—for example, exponential stability in the state space. We may think of $\mathbf{H}$ as the transfer function of the plant and the controller connected in cascade, and the stability of the corresponding closed-loop system implies that $\mathbf{G}^0$ is L^2-stable. However, stability might be lost if tiny (and often inevitable) delays are present in the feedback loop, leading to the feedback system shown in Fig. 1. Indeed, it might be that for arbitrarily small $\varepsilon > 0$, $\mathbf{G}^\varepsilon$ has poles in $\mathbb{C}_0$, which implies that the system cannot be stable in any reasonable state space sense either. Our aim in this paper is to find conditions on $\mathbf{H}$ (necessary and/or sufficient) for this phenomenon (observed by many authors) to happen.

We say that $\mathbf{G}^0$ is *robustly stable with respect to delays* if there is an $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0,\varepsilon_0]$, $\mathbf{G}^\varepsilon$ is L^2-stable. The absence of this property means that arbitrarily small destabilizing delays can be found for $\mathbf{G}^0$.

If the transfer function $\mathbf{H}$ is meromorphic on the half-plane $\mathbb{C}_0$, then we denote by $\mathfrak{P}_\mathbf{H}$ the (discrete) set of its poles in $\mathbb{C}_0$. (We say that p is a pole of $\mathbf{H}$ if p is a pole of at least one entry of $\mathbf{H}$.) We define

$$\gamma = \limsup_{\substack{|s|\to\infty \\ s\in\mathbb{C}_0\setminus\mathfrak{P}_\mathbf{H}}} r(\mathbf{H}(s)) , \tag{1.3}$$

where $r(\mathbf{H}(s))$ denotes the spectral radius of the matrix $\mathbf{H}(s)$. It might happen that $\gamma = \infty$—for example, if $\mathbf{H}$ is scalar and has an unbounded sequence of poles on the imaginary axis. If $\mathbf{G}^0$ is L^2-stable, then from the formula $\mathbf{H} = \mathbf{G}^0(I - \mathbf{G}^0)^{-1}$ it is not difficult to see that $\mathbf{H}$ is meromorphic on $\mathbb{C}_0$. Indeed, for s in the half-plane where $\mathbf{H}(s)$ was originally defined, $I - \mathbf{G}^0(s) = (I + \mathbf{H}(s))^{-1}$ so that $\det(I - \mathbf{G}^0(s))$ is not identically 0. Hence, if $\mathbf{G}^0$ is L^2-stable, then (1.3) makes sense. This fact is used implicitly in the statement of our main result, which is the following theorem.

THEOREM 1.1. *Let* $\mathbf{H}$ *be a* $\mathbb{C}^{m\times m}$*-valued regular transfer function and suppose that* $\mathbf{G}^0 = \mathbf{H}(I + \mathbf{H})^{-1}$ *is* L^2*-stable. Let* γ *be defined by* (1.3).

(i) *If* $\gamma < 1$, *then* $\mathbf{G}^0$ *is robustly stable with respect to delays.*

(ii) *If* $\gamma > 1$, *then* $\mathbf{G}^0$ *is not robustly stable with respect to delays.*

The proof of (i) is much easier than the proof of (ii) and is in §6. It is shown in the same section that (i) cannot be extended to multidelay perturbations. The proof of (ii) is very involved, so in order to present the ideas clearly, without multivariable technicalities, we first give the proof for $m = 1$ in §§3 and 4. The multivariable case is treated in §5. We were not able to prove a general result for the case $\gamma = 1$. However, trivial examples (e.g., $\mathbf{H}(s) \equiv I$) show that $\mathbf{G}^0$ will in general not be robustly stable with respect to delays if $\gamma = 1$.

In §7 we show that the instability created by a small delay in the closed loop is itself robust to small delays.

In §8 we discuss destabilization and robustness with respect to delays for systems with dynamic feedback. Let $\mathbf{P}$ and $\mathbf{K}$ be meromorphic functions on $\mathbb{C}_0$ of appropriate dimensions such that the products $\mathbf{PK}$ and $\mathbf{KP}$ exist. We say that $\mathbf{K}$ *stabilizes* $\mathbf{P}$ if

$$\begin{bmatrix} I & \mathbf{P} \\ -\mathbf{K} & I \end{bmatrix}^{-1}$$

is L^2-stable. Intuitively, this means that if we connect the plant $\mathbf{P}$ and the controller $\mathbf{K}$ in a feedback loop with two external inputs, then all the possible transfer functions in the loop are L^2-stable (see §8 for details).

Let us denote $\mathbf{K}_\varepsilon(s) = e^{-\varepsilon s}\mathbf{K}(s)$. We say that $\mathbf{K}$ *stabilizes* $\mathbf{P}$ *robustly with respect to delays* if there is an $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$ $\mathbf{K}_\varepsilon$ stabilizes $\mathbf{P}$. Intuitively, this means that the introduction of sufficiently small delays into the feedback loop mentioned earlier does not destroy its stability. By a corollary of Theorem 1.1 stated in §8, if $\mathbf{H} = \mathbf{PK}$ is regular and γ is defined by (1.3), then $\gamma < 1$ implies that $\mathbf{K}$ stabilizes $\mathbf{P}$ robustly, while $\gamma > 1$ implies that the opposite is true. Let $\mathbb{C}_0^{\mathrm{cl}}$ denote the closure of $\mathbb{C}_0$. Using the above mentioned corollary and a lemma of independent interest, we prove (still in §8) the following theorem.

THEOREM 1.2. *Let* $\mathbf{P}$ *and* $\mathbf{K}$ *be matrix-valued meromorphic functions on a right open half-plane containing* $\mathbb{C}_0^{\mathrm{cl}}$. *Assume that* $\mathbf{PK}$ *is regular and* $\mathbf{K}$ *stabilizes* $\mathbf{P}$. *If* $\mathbf{P}$ *has infinitely many poles in* $\mathbb{C}_0^{\mathrm{cl}}$, *then* $\mathbf{K}$ *does not stabilize* $\mathbf{P}$ *robustly with respect to delays.*

Thus, roughly speaking, if a plant with infinitely many poles in $\mathbb{C}_0^{\mathrm{cl}}$ is given, we cannot find a controller such that the resulting feedback system is robustly stable with respect to small delays in the loop.

In §9 we give three simple examples.

2. Preliminaries and discussion of earlier results. There are many examples in the literature of systems described by partial differential equations which are exponentially stabilized by a feedback but are destabilized by arbitrarily small time delays in the feedback loop. The first example of this sort appeared in Datko, Lagnese, and Polis [9], where a one-dimensional wave equation with boundary feedback was studied. The same phenomenon has been subsequently described in many other examples; see Datko [10], Desch, Hannsgen, Renardy, and Wheeler [13], Hannsgen, Renardy, and Wheeler [20], Bontsema and deVries [3] and Grimmer, Lenczewski, and Schappacher [19]. In a more abstract framework, this destabilization by small delays was demonstrated for classes of distributed parameter feedback systems in Datko [10], [11] and Desch and Wheeler [12]. In these classes of systems the open loop semigroup is unitary, and only one stabilizing feedback is considered for each given plant, typically a type of co-located control.

In contrast with the works referred to above, we will take a frequency domain approach here, which is not tied to a specific form for the stabilizing feedback. Our approach is similar in spirit to that in the considerably older paper of Barman, Callier, and Desoer [1]. In that paper, necessary and sufficient conditions were given for a class of single-input single-output (SISO) systems to be robustly stable with respect to delays. The results in [1] are limited by several restrictions, including the requirement that the open-loop transfer function has at most finitely many poles in the closed right half-plane. These results were applied to some systems described by partial differential equations in [3].

A more general class of perturbations of feedback systems, including small delays in the loop, were considered by Georgiou and Smith [17], [18] (see also Curtain [8]). Their concept of w-stability is considerably stronger than robust stability with respect to delays; it covers a large class of perturbations which represent high-frequency modelling uncertainties. The necessary and sufficient criterion for w-stability in [18] resembles our Corollaries 8.2 and 8.4, especially in the SISO case. For multiple-input multiple-output (MIMO) systems, there is a curious difference: the result for w-stability is in terms of the norms of the transfer functions, while our result for robustness with respect to delays is in terms of their spectral radius. The proof of destabilization results for w-stability is considerably easier than for robustness with respect to delays.

We will now recall some concepts and results needed in this paper. We will work with finite-dimensional input and output spaces, but we mention that these concepts and results have natural counterparts for Hilbert space-valued input/output functions, which means operator-valued transfer functions.

Let $\alpha \in \mathbb{R}$. We will use the notation

$$\mathbb{C}_\alpha = \{ s \in \mathbb{C} \mid \text{Re } s > \alpha \},$$

and $H^\infty_\alpha(\mathbb{C}^{p\times m})$ will denote the space of all bounded analytic $\mathbb{C}^{p\times m}$-valued functions on $\mathbb{C}_\alpha$. We write H^∞ for H^∞_0. The norm $\|\mathbf{G}\|_\infty$ of a function $\mathbf{G} \in H^\infty_\alpha$ is the supremum of $\|\mathbf{G}(s)\|$ over $\mathbb{C}_\alpha$, the matrix norm being defined as the greatest singular value. After the identification of functions with their meromorphic extensions, which was mentioned in §1, we have that

$$H^\infty_\alpha \subset H^\infty_\beta \quad \text{if} \quad \alpha \leq \beta.$$

DEFINITION 2.1. *A* well-posed $\mathbb{C}^{p\times m}$-valued transfer function *is an element of one of the spaces* $H^\infty_\alpha(\mathbb{C}^{p\times m})$.

Well-posed transfer functions correspond to shift invariant operators on $L^2_{loc}[0,\infty)$ with finite growth bound; see Weiss [32, §3]. In particular, the transfer function of any abstract linear system is well posed, as follows from [32, Prop. 4.1]. Conversely, for any well-posed transfer function $\mathbf{H}$ we can find an abstract linear system whose transfer function is $\mathbf{H}$, as follows from results in Salamon [29].

DEFINITION 2.2. *A well-posed matrix-valued transfer function* $\mathbf{H}$ *is* regular *if the limit* $\lim_{\lambda\to+\infty} \mathbf{H}(\lambda) = D$ *exists (where* λ *is real). Then* D *is the* feedthrough matrix *of* $\mathbf{H}$.

Practically all well-posed transfer functions of interest are regular. (In fact, it is a nontrivial exercise in complex analysis to construct an example of a well-posed and nonregular transfer function.) If the transfer function of an abstract linear system is regular (such systems are called regular), then the system has a simple and convenient state space representation, like finite-dimensional systems; see [32, §2].

For SISO systems, we will use the term *feedthrough value* instead of feedthrough matrix. We introduce a notation for angular domains in $\mathbb{C}$: for any number $\psi \in (0,\pi]$,

$$\mathcal{W}(\psi) = \left\{ re^{i\phi} \,\middle|\, r \in (0,\infty),\ \phi \in (-\psi,\psi) \right\}.$$

We will need the following simple fact about regular transfer functions.

PROPOSITION 2.3. *Let* $\mathbf{H}$ *be a regular matrix-valued transfer function, with feedthrough matrix* D. *Then for any* $\psi \in (0, \frac{\pi}{2})$,

$$\lim_{\substack{|s|\to\infty \\ s\in\mathcal{W}(\psi)}} \mathbf{H}(s) = D.$$

This follows from Duren [14, Thm. 1.3], after mapping the half-plane onto the unit disk. It follows also from the results in [32, §5], where Laplace transform techniques are used.

Remark 2.4. If $\mathbf{H}$ and $\mathbf{G}^\varepsilon$ are related as in (1.1) and $\varepsilon > 0$, then it is easy to see that $\mathbf{H}$ is well posed (regular) if and only if $\mathbf{G}^\varepsilon$ is well posed (regular). If $\mathbf{H}$ and $\mathbf{G}^0$ are both well posed, then one of them is regular if and only if the other is. Similar statements are true for operator-valued transfer functions but are more difficult to prove; see Weiss [33].

3. Nonrobustness: The SISO case with $|D| \leq 1$. In this section we prove (ii) of Theorem 1.1 for SISO systems and under the additional assumption that D, the feedthrough value of $\mathbf{H}$, satisfies $|D| \leq 1$. This situation is fairly typical for transfer functions of unstable vibrating systems. Since we can say slightly more than what is written in (ii) of Theorem 1.1, we restate the result.

THEOREM 3.1. *Let* $\mathbf{H}$ *be a regular SISO transfer function and, for any* $\varepsilon \geq 0$, *the function* $\mathbf{G}^\varepsilon$ *be defined by* (1.1). *Let* D *denote the feedthrough value of* $\mathbf{H}$. *Assume that*

(1) $\mathbf{G}^0 \in H^\infty$, *so that* γ *can be defined by* (1.3),

(2) $\gamma > 1$,

(3) $|D| \leq 1$.

Then there exist sequences (ε_n) *and* (p_n) *with*

$$\varepsilon_n > 0, \quad \varepsilon_n \to 0, \quad p_n \in \mathbb{C}_0, \quad |\mathrm{Im}\, p_n| \to \infty$$

and such that for any $n \in \mathbb{N}$, p_n *is a pole of* $\mathbf{G}^{\varepsilon_n}$.

Proof. From condition (1), using (1.1) we can see that the point -1 has a neighborhood which does not intersect the range of $\mathbf{H}$ (regarded as a meromorphic function on $\mathbb{C}_0$). This implies that there exist $\eta > 0$ and $\gamma_1 > 1$ such that the set

$$S_1 = \left\{ re^{i\phi} \;\middle|\; r \in [1, \gamma_1],\ |\phi - \pi| < \eta \right\}$$

does not intersect the range of $\mathbf{H}$:

$$\mathbf{H}(s) \notin S_1 \qquad \forall\, s \in \mathbb{C}_0. \tag{3.1}$$

Since we may choose γ_1 arbitrarily close to 1, by condition (2) we may assume

$$1 < \gamma_1 < \gamma. \tag{3.2}$$

The definition of γ and (3.2) enable us to show that there exists a sequence (z_n) in $\mathbb{C}_0$ with the following properties:

(a) $|z_n| \to \infty$ as $n \to \infty$;

(b) for any $n \in \mathbb{N}$,

$$|\mathbf{H}(z_n)| \geq \gamma_1;$$

(c) for any $n \in \mathbb{N}$, $\mathbf{H}$ is analytic on the ray

$$\Gamma_n = \{z_n + a \mid a \in [0, \infty)\}$$

(i.e., there are no poles on these rays).

By Proposition 2.3, for any $\psi \in (0, \frac{\pi}{2})$ there exists an $r_\psi > 0$ such that for any $s \in \mathcal{W}(\psi)$ with $|s| > r_\psi$ we have $|\mathbf{H}(s) - D| < \gamma_1 - 1$. Using that $|\mathbf{H}(s)| \leq |D| + |\mathbf{H}(s) - D|$ and condition (3), we get that, for s as above, $|\mathbf{H}(s)| < \gamma_1$. By property (b) it follows that the sequence (z_n) lies in the set $\{s \in \mathbb{C}_0 \mid |s| \leq r_\psi \text{ or } |\arg s| \geq \psi\}$. Here and in the rest of this proof, the argument of a (nonzero) complex number s is defined such that $\arg s \in (-\pi, \pi]$. By property (a), for n sufficiently large, $|z_n| \leq r_\psi$ is not possible, so that $|\arg z_n| \geq \psi$. Since the choice of $\psi \in (0, \frac{\pi}{2})$ was arbitrary, we conclude that

$$\lim_{n\to\infty} |\arg z_n| = \frac{\pi}{2}. \tag{3.3}$$

Together with property (a) this implies that

$$\lim_{n\to\infty} |\operatorname{Im} z_n| = \infty. \tag{3.4}$$

We may assume without loss of generality that for all $n \in \mathbb{N}$, Im $z_n > 0$. Indeed, if such a subsequence does not exist, then a similar argument can be made assuming that Im $z_n < 0$ for all $n \in \mathbb{N}$.

By property (c), $\mathbf{H}$ is continuous on each ray Γ_n, and by Proposition 2.3, $\mathbf{H}(s) \to D$ as $s \to \infty$ on Γ_n. Since $|D| \leq 1$ (by condition (3)), the numbers

$$a'_n = \max\{a \in [0, \infty) \mid |\mathbf{H}(z_n + a)| \geq \gamma_1\},$$
$$a''_n = \min\{a \in [a'_n, \infty] \mid |\mathbf{H}(z_n + a)| \leq 1\}$$

are well defined. (If $|D| = 1$, then it might happen that $a''_n = \infty$.) Put

$$z'_n = z_n + a'_n, \qquad z''_n = z_n + a''_n.$$

(It might happen that $z''_n = \infty$.) We will be looking for poles of $\mathbf{G}^\varepsilon$ in the open horizontal segments $(z'_n, z''_n) \subset \Gamma_n$.

From the definition of z'_n and z''_n, using property (c) we see that the image of $[z'_n, z''_n]$ through $\mathbf{H}$ is a curve Π_n contained in $\{s \in \mathbb{C} \mid 1 \leq |s| \leq \gamma_1\}$. The possibility that $z''_n = \infty$ is not disturbing since (by Proposition 2.3) $\mathbf{H}$ is continuous at infinity along the ray Γ_n. By (3.1) Π_n cannot enter S_1, so it is confined to the set

$$S_2 = \{re^{i\phi} \mid r \in [1, \gamma_1], \ |\phi| \leq \pi - \eta\};$$

see Fig. 2. Thus we have

$$\Pi_n \subset S_2 \qquad \forall\, n \in \mathbb{N}. \tag{3.5}$$

Using (1.1) we see that p is a pole of $\mathbf{G}^\varepsilon$ if and only if

$$e^{-\varepsilon p}\mathbf{H}(p) = -1.$$

A sufficient condition for this is

$$\log \mathbf{H}(p) - \varepsilon p = -i\pi, \tag{3.6}$$

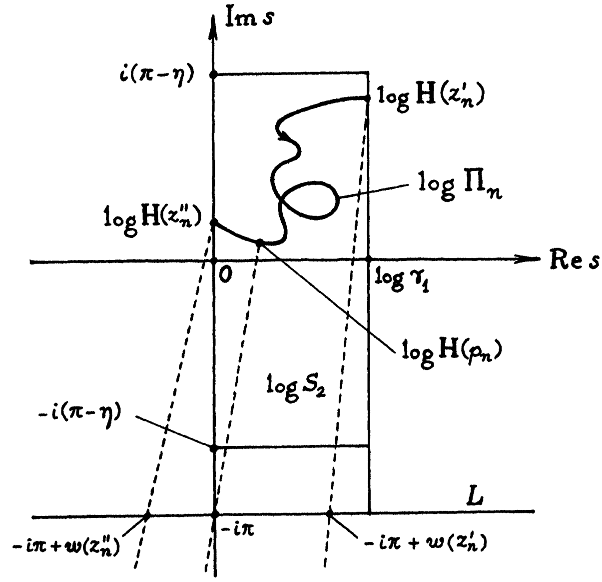

FIG. 2. *The sets S_1, S_2 and the curve Π_n.*

where we choose the branch of the logarithm to be $\log z = \log|z| + i\arg z$, with $\arg z \in (-\pi, \pi]$, as agreed earlier, and $\log|z| \in \mathbb{R}$.

For each $s \in \mathbb{C}_0$ with $\operatorname{Im} s > 0$, the ray $R(s) = \{\log \mathbf{H}(s) - \varepsilon s \mid \varepsilon \in [0, \infty)\}$ intersects the horizontal line $L = \{s \in \mathbb{C} \mid \operatorname{Im} s = -\pi\}$ in a point $w(s) - i\pi$. Indeed, for $\varepsilon = 0$ the corresponding point of $R(s)$ is above L, while for large $\varepsilon > 0$ the corresponding point of $R(s)$ is below L. Thus we can define the real-valued functions $w(s)$, $e(s)$ for each $s \in \mathbb{C}_0$ with $\operatorname{Im} s > 0$ such that $e(s) > 0$ and

$$\log \mathbf{H}(s) - e(s)s = w(s) - i\pi. \tag{3.7}$$

A simple computation shows that

$$e(s) = \frac{\arg \mathbf{H}(s) + \pi}{\operatorname{Im} s}, \tag{3.8}$$

$$w(s) = \log|\mathbf{H}(s)| - \frac{(\arg \mathbf{H}(s) + \pi)\operatorname{Re} s}{\operatorname{Im} s}. \tag{3.9}$$

CLAIM. *For all $n \in \mathbb{N}$ sufficiently large, there is a point $p_n \in (z'_n, z''_n)$ such that $w(p_n) = 0$.*

Figure 3 is intended to give an intuitive picture of this claim. In this figure we see the curve $\log \Pi_n$ which, according to (3.5), is contained in the rectangle $\log S_2$. It is assumed in the picture that z''_n is finite. The rays $R(z'_n)$, $R(z''_n)$, and $R(p_n)$ (dotted lines) and the horizontal line L are marked.

Proof of the claim. We define

$$w(\infty) = -\infty.$$

Then, as a $[-\infty, \infty)$-valued function, w is continuous on each segment $[z'_n, z''_n]$. Indeed, by (3.5) $\arg \mathbf{H}(s)$ has no jumps on such a segment. If z''_n is finite, then the continuity of w is clear from (3.9). If $z''_n = \infty$, then it is easy to see from (3.9) that $\lim_{s\to\infty} w(s) = -\infty$, where $s \in (z'_n, z''_n)$.

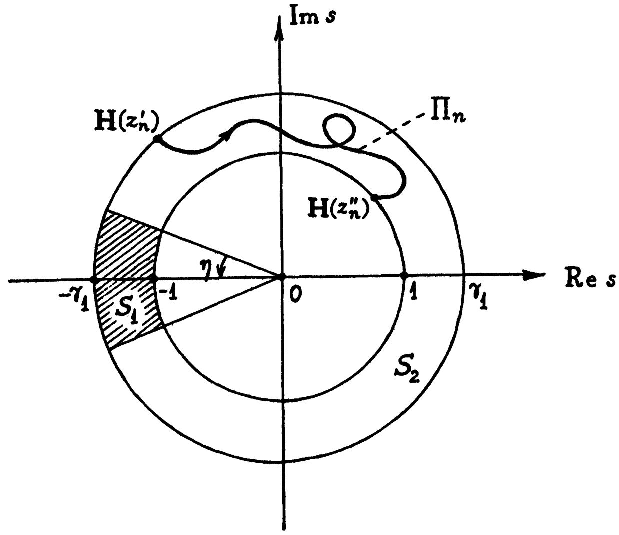

FIG. 3. *The claim (the existence of* p_n*).*

Next we show that for all $n \in \mathbb{N}$ sufficiently large,

$$w(z_n') > 0 \quad \text{and} \quad w(z_n'') < 0. \tag{3.10}$$

The sequence (z_n') shares with (z_n) properties (a) and (b) (also (c), but this is not needed now). By the exact same argument used to prove (3.3), and by the assumption $\operatorname{Im} z_n > 0$, we get that

$$\lim_{n\to\infty} \arg z_n' = \frac{\pi}{2}.$$

Hence

$$\lim_{n\to\infty} \frac{(\arg \mathbf{H}(z_n') + \pi)\operatorname{Re} z_n'}{\operatorname{Im} z_n'} = 0. \tag{3.11}$$

On the other hand, it is clear that

$$\frac{(\arg \mathbf{H}(z_n'') + \pi)\operatorname{Re} z_n''}{\operatorname{Im} z_n''} > 0. \tag{3.12}$$

By the definition of z_n' and z_n'', $|\mathbf{H}(z_n')| = \gamma_1 > 1$ and $|\mathbf{H}(z_n'')| = 1$. Therefore $\log|\mathbf{H}(z_n')| = \log \gamma_1 > 0$ and $\log|\mathbf{H}(z_n'')| = 0$. Combining this with (3.9), (3.11), and (3.12) we see that for all n sufficiently large, (3.10) holds.

Since $w(s)$ is continuous on $[z_n', z_n'']$, (3.10) implies that for all $n \in \mathbb{N}$ sufficiently large there exists $p_n \in (z_n', z_n'')$ such that $w(p_n) = 0$. (This is indicated in Fig. 3 by having the ray $R(p_n)$ go through $-i\pi$.) This completes the proof of the claim.

Returning to the main proof, we may now assume without loss of generality that for each $n \in \mathbb{N}$ there is a $p_n \in (z_n', z_n'')$ such that $w(p_n) = 0$. (If not, select an appropriate subsequence.) We denote $\varepsilon_n = e(p_n)$. By (3.7) (with $s = p_n$) we get that p_n and ε_n satisfy (3.6), so p_n is a pole of $\mathbf{G}^{\varepsilon_n}$. By (3.8), $\varepsilon_n > 0$. Since $\operatorname{Im} p_n = \operatorname{Im} z_n$, by (3.4) we have $|\operatorname{Im} p_n| \to \infty$. By (3.8) and (3.4) we have $\varepsilon_n \to 0$. □

4. Nonrobustness: The SISO case with $|D| > 1$. In this section we prove (ii) of Theorem 1.1 for SISO systems and under the additional assumption that D, the feedthrough value of $\mathbf{H}$, satisfies $|D| > 1$. Then obviously $\gamma > 1$, so that this does not have to be assumed. In fact, we will prove a stronger result, in which the assumption that $\mathbf{G}^0$ is L^2-stable is not needed. We do not even need that $\mathbf{H}$ should be meromorphic on $\mathbb{C}_0$.

THEOREM 4.1. *Let $\mathbf{H}$ be a regular SISO transfer function and, for any $\varepsilon \geq 0$, the function $\mathbf{G}^\varepsilon$ be defined by* (1.1). *Let D denote the feedthrough value of $\mathbf{H}$ and assume that*

$$|D| > 1.$$

Then there exist sequences (ε_n) and (p_n) with

$$\varepsilon_n > 0, \quad \varepsilon_n \to 0, \quad \operatorname{Re} p_n \to \infty, \quad \operatorname{Im} p_n \to \infty$$

and such that, for any $n \in \mathbb{N}$, p_n is a pole of $\mathbf{G}^{\varepsilon_n}$.

Proof. If D is not a negative real number, then we define the argument of any (nonzero) complex number s such that $\arg s \in (-\pi, \pi]$, as in the proof of Theorem 3.1. If D is negative then it is more convenient to change the definition such that $\arg s \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ (to avoid a jump discontinuity at D). The function log is defined by $\log s = \log|s| + i \arg s$ with $\log|s| \in \mathbb{R}$.

For any $r > 0$, B_r will denote the closed disk of radius r with center in D. Due to $|D| > 1$ and the way in which we have defined the function log, we can find a $\rho > 0$ such that (1) $|z| > 1$ for all $z \in B_\rho$ and (2) arg (and hence log) is continuous on B_ρ.

The simple inequalities

$$\min_{z \in B_\rho} (\arg z + \pi) > 0, \qquad \min_{z \in B_\rho} \log|z| > 0$$

enable us to find numbers $0 < \alpha < \beta$ such that for any $z \in B_\rho$

$$\log|z| - \frac{\arg z + \pi}{\alpha} < 0, \qquad \log|z| - \frac{\arg z + \pi}{\beta} > 0. \tag{4.1}$$

Let $\psi \in (0, \frac{\pi}{2})$ be such that $\beta < \operatorname{tg} \psi$. By Proposition 2.3, there exists a $\sigma > 0$ such that

$$\mathbf{H}(s) \in B_\rho \qquad \forall\, s \in \mathcal{W}(\psi) \cap \mathbb{C}_\sigma. \tag{4.2}$$

(The notation $\mathbb{C}_\sigma$ was introduced in §2.) Let (x_n) be a sequence of real numbers with $x_n > \sigma$ and such that $x_n \to \infty$. Define

$$z_n' = (1 + i\alpha)x_n, \qquad z_n'' = (1 + i\beta)x_n.$$

We will be looking for poles of $\mathbf{G}^\varepsilon$ in the open vertical segments (z_n', z_n'').

The remainder of this proof resembles the part of the proof of Theorem 3.1 which starts after (3.5), so we will be brief. Using (1.1) we see that p is a pole of $\mathbf{G}^\varepsilon$ if and only if $e^{-\varepsilon p}\mathbf{H}(p) = -1$. A sufficient condition for this is that (3.6) holds.

For each $s \in \mathbb{C}_0$ with $\operatorname{Im} s > 0$, the ray $R(s) = \{\log \mathbf{H}(s) - \varepsilon s \mid \varepsilon \in [0, \infty)\}$ intersects the horizontal line $L = \{s \in \mathbb{C} \mid \operatorname{Im} s = -\pi\}$ in a point $w(s) - i\pi$, as explained in the previous proof. Thus we can define the functions $w(s)$, $e(s)$ for each $s \in \mathbb{C}_0$ with $\operatorname{Im} s > 0$ such that $e(s) > 0$ and (3.7) holds. These functions are given by (3.8) and (3.9). The following claim is almost identical to the one in the proof of Theorem 3.1.

CLAIM. *For all $n \in \mathbb{N}$ there exists $p_n \in (z_n', z_n'')$ such that $w(p_n) = 0$.*

The proof is simpler in this case: By (4.2) and property (2) it is clear that w is continuous on each segment $[z_n', z_n'']$. Moreover, (3.9), (4.1), and (4.2) imply (3.10), from which the claim follows.

We return to the proof of the theorem. We denote $\varepsilon_n = e(p_n)$. By (3.7) (with $s = p_n$) we get that p_n and ε_n satisfy (3.6), so p_n is a pole of $\mathbf{G}^{\varepsilon_n}$. By the definition of the function $e(\cdot)$, $\varepsilon_n > 0$. We have $\operatorname{Re} p_n = x_n$ so that $\operatorname{Re} p_n \to \infty$. Since $\operatorname{Im} p_n > \alpha x_n$, we also have $\operatorname{Im} p_n \to \infty$. Now by (3.8) we have $\varepsilon_n \to 0$. □

5. Nonrobustness: The MIMO case. In this section we show that the results in §§3 and 4 extend to the multivariable case. In particular, we prove part (ii) of Theorem 1.1 for $m > 1$. In order to do this, it is convenient to state some preliminary facts and results. If $V \subset U \subset \mathbb{C}$, we say that V is a *discrete* set in U if V has no accumulation points in U. Let $\mathcal{H}_\alpha$ denote the ring of holomorphic functions defined on $\mathbb{C}_\alpha$ and $\mathcal{M}_\alpha$ denote the field of meromorphic functions on $\mathbb{C}_\alpha$. The vector spaces of $\mathbb{C}^{p\times m}$-valued holomorphic functions and of $\mathbb{C}^{p\times m}$-valued meromorphic functions on $\mathbb{C}_\alpha$ will be denoted by $\mathcal{H}_\alpha(\mathbb{C}^{p\times m})$ and $\mathcal{M}_\alpha(\mathbb{C}^{p\times m})$, respectively. It is clear that $\mathcal{H}_\alpha(\mathbb{C}^{p\times m}) = \mathcal{H}_\alpha{}^{p\times m}$ and $\mathcal{M}_\alpha(\mathbb{C}^{p\times m}) = \mathcal{M}_\alpha{}^{p\times m}$. A complex number $s_0 \in \mathbb{C}_\alpha$ is a *pole* of $\mathbf{H} \in \mathcal{M}_\alpha(\mathbb{C}^{p\times m})$ if and only if s_0 is a pole of at least one of the entries of $\mathbf{H}$.

In the following let $\mathbf{H}$ be in $\mathcal{M}_\alpha(\mathbb{C}^{m\times m})$. The set of all poles of $\mathbf{H}$ is denoted by $\mathfrak{P}_{\mathbf{H}}$. Moreover, define

$$\psi(\cdot\,,\lambda) := \det(\lambda I - \mathbf{H}(\cdot)) \in \mathcal{M}_\alpha[\lambda]\,,$$

where $\mathcal{M}_\alpha[\lambda]$ denotes the ring of polynomials over $\mathcal{M}_\alpha$. Since $\mathcal{M}_\alpha$ is a field, there exist a unique $\ell \in \mathbb{N}$ and unique monic irreducible polynomials $\psi_i(\cdot\,,\lambda) \in \mathcal{M}_\alpha[\lambda]$ such that

$$\psi(s,\lambda) = \prod_{i=0}^{\ell} \psi_i(s,\lambda)\,.$$

Let $\Delta_i(\cdot) \in \mathcal{M}_\alpha$ denote the discriminant of $\psi_i(\cdot\,,\lambda) \in \mathcal{M}_\alpha[\lambda]$. If $s_0 \notin \mathfrak{P}_{\mathbf{H}}$, then it is not difficult to show that the coefficients of the polynomials $\psi_i(\cdot\,,\lambda)$ are holomorphic in a sufficiently small neighborhood of s_0 (cf. Baumgärtel [2, p. 397]) and hence $\Delta_i(s_0)$ is the discriminant of $\psi_i(s_0,\lambda) \in \mathbb{C}[\lambda]$. Thus $\psi_i(s_0,\lambda)$ has only simple zeros if and only if $\Delta_i(s_0) \neq 0$; see, for example, Cohn [7, p. 175]. Since $\psi_i(\cdot\,,\lambda)$ is irreducible, it follows that $\Delta_i(s) \not\equiv 0$ and hence the set $\mathfrak{C}_{\mathbf{H}}$ of *critical points* of $\mathbf{H}$ defined by

$$\mathfrak{C}_{\mathbf{H}} := \left\{ s \in \mathbb{C}_\alpha \mid \prod_{i=0}^{\ell} \Delta_i(s) = 0 \right\}$$

is a discrete set in $\mathbb{C}_\alpha$. We shall need the following lemma from Forster [15, p. 52].

LEMMA 5.1. *Let $s_0 \in \mathbb{C}$, let $U \subset \mathbb{C}$ be an open neighborhood of s_0, and suppose that $c_1(s),\ldots,c_n(s)$ are holomorphic functions on U. If $\lambda_0 \in \mathbb{C}$ is a simple zero of the polynomial*

$$\lambda^n + c_1(s_0)\lambda^{n-1} \ldots + c_n(s_0) \in \mathbb{C}[\lambda],$$

then there exists an open neighborhood $V \subset U$ *of* s_0 *and a function* ξ *holomorphic on* V *such that* $\xi(s_0) = \lambda_0$ *and*

$$\xi^n(s) + c_1(s)\xi^{n-1}(s) \ldots + c_n(s) = 0 \quad \forall\, s \in V.$$

For $s_0 \in \mathbb{C}_\alpha$, $\phi \in [0, 2\pi)$, and $0 < a \le \infty$ set $\Gamma := \{s_0 + e^{i\phi}t \,|\, 0 \le t < a\}$. Hence Γ is a half-line ($a = \infty$) or a line segment ($a \ne \infty$) with initial point s_0.

PROPOSITION 5.2. *Suppose that* $\Gamma^{\rm cl} \subset \mathbb{C}_\alpha \setminus (\mathfrak{P}_{\mathbf{H}} \cup \mathfrak{C}_{\mathbf{H}})$. *If* $\lambda_0 \in \sigma(\mathbf{H}(s_0))$ *(the spectrum of* $\mathbf{H}(s_0)$*), then there exists a region* $V \subset \mathbb{C}_\alpha$ *satisfying* $\Gamma^{\rm cl} \subset V \subset \mathbb{C}_\alpha \setminus (\mathfrak{P}_{\mathbf{H}} \cup \mathfrak{C}_{\mathbf{H}})$ *and a function* ξ *holomorphic on* V *such that* $\xi(s_0) = \lambda_0$ *and*

$$\psi(s, \xi(s)) = \det(\xi(s)I - \mathbf{H}(s)) = 0 \quad \forall\, s \in V.$$

Moreover, if $a = \infty$*, then under the extra assumption that the limit*

$$\lim_{|s|\to\infty,\, s\in\Gamma} \mathbf{H}(s) = D \in \mathbb{C}^{m\times m}$$

exists, it follows that

$$\lim_{|s|\to\infty,\, s\in\Gamma} \xi(s) =: \xi_\infty \in \sigma(D).$$

The above proposition remains true if Γ is replaced by more general curves. However, for our purposes Proposition 5.2 is sufficient.

Proof. Let $\lambda_0 \in \sigma(\mathbf{H}(s_0))$. After a suitable renumbering we may assume that $\psi_0(s_0, \lambda_0) = 0$. Let us first consider the case when $a \ne \infty$. Define $\gamma(t) := s_0 + e^{i\phi}t$ for $t \in [0, a]$. It follows from the assumption that for any $t \in [0, a]$ the polynomial $\psi_0(\gamma(t), \lambda) \in \mathbb{C}[\lambda]$ has no multiple zeros. Let n denote the degree of $\psi_0(\cdot\,, \lambda)$ and $\lambda_t^1, \ldots, \lambda_t^n$ denote the n different simple zeros of $\psi_0(\gamma(t), \lambda) \in \mathbb{C}[\lambda]$. Moreover, for $t \in [0, a]$ let $\mathfrak{B}_t$ denote the set of all open balls B_t with center in $\gamma(t)$ such that $B_t \subset \mathbb{C}_\alpha \setminus (\mathfrak{P}_{\mathbf{H}} \cup \mathfrak{C}_{\mathbf{H}})$ and such that there exist n functions ξ_t^i holomorphic on B_t satisfying $\xi_t^i(\gamma(t)) = \lambda_t^i$,

$$\xi_t^i(s) \ne \xi_t^j(s) \;\; \forall\, s \in B_t\,, \forall\, i, j \in \underline{n},\; i \ne j, \;\text{ and }\; \psi_0(s, \xi_t^i(s)) = 0 \;\; \forall\, s \in B_t\,, \forall\, i \in \underline{n}\,,$$

where $\underline{n}$ denotes the set $\{1, \ldots, n\}$. By Lemma 5.1 the set $\mathfrak{B}_t$ is nonempty, and by setting $\hat{B}_t := \bigcup_{B_t \in \mathfrak{B}_t} B_t$ we obtain the maximal element of $\mathfrak{B}_t$. Denoting the radius of $\hat{B}_t$ by ϱ_t we claim that

$$\varrho := \inf_{t\in[0,a]} \varrho_t > 0\,. \tag{5.1}$$

Assume that (5.1) is not true. Then there exist numbers $t_j \in [0, a]$ such that

$$\lim_{j\to\infty} \varrho_{t_j} = 0\,. \tag{5.2}$$

Since $a < \infty$, we may assume without loss of generality that $\lim_{j\to\infty} t_j =: t^* \in [0, a]$ exists. By assumption $\gamma(t^*) \in \mathbb{C}_\alpha \setminus (\mathfrak{P}_{\mathbf{H}} \cup \mathfrak{C}_{\mathbf{H}})$ and hence $\varrho_{t^*} > 0$. Using $\lim_{j\to\infty} \gamma(t_j) = \gamma(t^*)$, we conclude that there exist $j_0 \in \mathbb{N}$ and $\delta > 0$ such that $B(\gamma(t_j), \delta) \subset \hat{B}_{t^*}$ for all $j \ge j_0$, where $B(\gamma(t_j), \delta)$ denotes the open ball of radius δ with center in $\gamma(t_j)$. But this implies that $B(\gamma(t_j), \delta) \in \mathfrak{B}_{t_j}$ for all $j \ge j_0$ and therefore $B(\gamma(t_j), \delta) \subset \hat{B}_{t_j}$ for all $j \ge j_0$ (by the maximality of $\hat{B}_{t_j}$). Thus we obtain that $\varrho_{t_j} \geq \delta$ for all $j \geq j_0$, which contradicts (5.2). It follows that (5.1) holds, and therefore there exists a largest number $m \in \mathbb{N}$ such that $m\varrho \leq a$. Setting $\tau_j := \varrho j$ it is clear that

$$B(\gamma(\tau_j), \varrho) \subset \hat{B}_{\tau_j} \text{ for } j = 0, \dots, m \text{ and } \Gamma^{cl} \subset \bigcup_{j=0,\dots,m} B(\gamma(\tau_j), \varrho).$$

Let $j_0 \in \underline{n}$ be such that $\lambda_{\tau_0}^{j_0} = \lambda_0^{j_0} = \lambda_0$, and set

$$\xi_0(s) := \xi_{\tau_0}^{j_0}(s) = \xi_0^{j_0}(s).$$

Now $S_0 := B(\gamma(0), \varrho) \cap B(\gamma(\tau_1), \varrho) \neq \emptyset$ is contained in $\mathbb{C}_\alpha \setminus (\mathfrak{P}_{\mathbf{H}} \cup \mathfrak{C}_{\mathbf{H}})$, and hence, for any $s \in S_0$, the polynomial $\psi_0(s, \lambda) \in \mathbb{C}[\lambda]$ has n different simple zeros, which are given by $\xi_{\tau_1}^1(s), \dots, \xi_{\tau_1}^n(s)$. On the other hand we have

$$\psi_0(s, \xi_0(s)) = 0 \quad \forall s \in S_0,$$

and hence there must exist $j_1 \in \underline{n}$ such that

$$\xi_0(s) = \xi_{\tau_1}^{j_1}(s) \quad \forall s \in S_0.$$

Setting

$$\xi_1(s) := \xi_{\tau_1}^{j_1}(s) \ \forall s \in B(\gamma(\tau_1), \varrho) \text{ and } S_1 := B(\gamma(\tau_1), \varrho) \cap B(\gamma(\tau_2), \varrho) \neq \emptyset,$$

the same argument can be used to conclude that there exists $j_2 \in \underline{n}$ such that

$$\xi_1(s) = \xi_{\tau_2}^{j_2}(s) \quad \forall s \in S_1.$$

Repeating the above argument shows that there exist $m+1$ holomorphic functions $\xi_j : B(\gamma(\tau_j), \varrho) \to \mathbb{C}$ $(j = 0, \dots, m)$ such that $\xi_{j+1}(s) = \xi_j(s)$ for all $s \in S_j := B(\gamma(\tau_j), \varrho) \cap B(\gamma(\tau_{j+1}), \varrho) \neq \emptyset$ $(j = 0, \dots, m-1)$. On the region

$$V := \bigcup_{j=0,\dots,m} B(\gamma(\tau_j), \varrho)$$

we define a function $\xi(s)$ by setting

$$\xi(s) := \xi_j(s) \text{ if } s \in B(\gamma(\tau_j), \varrho).$$

By construction ξ is a well-defined holomorphic function on V such that $\xi(s_0) = \lambda_0$ and $\det(\xi(s)I - \mathbf{H}(s)) = 0$ for all $s \in V$.

Let us now consider the case when $a = \infty$. For $j \in \mathbb{N}$ define $\Gamma_j := \{s_0 + e^{i\phi}t \,|\, 0 \leq t < j\}$. The above construction shows that there exist regions V_j satisfying $\Gamma_j^{\mathrm{cl}} \subset V_j \subset \mathbb{C}_\alpha \setminus (\mathfrak{P}_{\mathbf{H}} \cup \mathfrak{C}_{\mathbf{H}})$ and $V_j \subset V_{j+1}$ and functions ξ_j holomorphic on V_j such that $\xi_j(s_0) = \lambda_0$, $\psi_0(s, \xi_j(s)) = 0$ for all $s \in V_j$ and $\xi_{j+1}|_{V_j} = \xi_j$. Hence on $V := \bigcup_{j=1}^{\infty} V_i$ we obtain a well-defined holomorphic function $\xi(s)$ with the desired properties by setting

$$\xi(s) := \xi_j(s) \text{ if } s \in V_j.$$

Finally, the last statement in the proposition is a consequence of the fact that the eigenvalues of $\mathbf{H}(\gamma(t))$ are continuous in t; see Kato [21, p. 106]. □

The next result extends Theorem 3.1 to the multivarable case.

THEOREM 5.3. *Let* $\mathbf{H}(s)$ *be a* $\mathbb{C}^{m\times m}$*-valued regular transfer function and define for any* $\varepsilon \geq 0$

$$\mathbf{G}^{\varepsilon}(s) := \mathbf{H}(s)(I + e^{-\varepsilon s}\mathbf{H}(s))^{-1}. \tag{5.3}$$

Let $D \in \mathbb{C}^{m\times m}$ *denote the feedthrough matrix of* $\mathbf{H}$ *and assume the following:*

(1) $\mathbf{G}^0 \in H^\infty(\mathbb{C}^{m\times m})$,

(2) $\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} r(\mathbf{H}(s)) =: \gamma > 1$,

(3) $r(D) \leq 1$.

Then there exist sequences (ε_n) *and* (p_n) *with*

$$\varepsilon_n > 0\,,\ \ \varepsilon_n \to 0\,,\ \ p_n \in \mathbb{C}_0\,,\ \ |\mathrm{Im}\, p_n| \to \infty$$

and such that for any $n \in \mathbb{N}$, p_n *is a pole of* $\mathbf{G}^{\varepsilon_n}$.

Remark 5.4. (i) Since $r(\mathbf{H}(s))$ is not defined, if s is a pole of $\mathbf{H}$, condition (2) in the previous theorem should be formulated more precisely as

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0\setminus\mathfrak{P}_\mathbf{H}} r(\mathbf{H}(s)) =: \gamma > 1.$$

To simplify the notation, we make the convention that if $s \in \mathfrak{P}_\mathbf{H}$, then $r(\mathbf{H}(s)) = 0$.

(ii) Suppose that $\lim_{\mu\downarrow 0}\mathbf{H}(\mu + i\omega) =: \mathbf{H}(i\omega)$ exists for almost all $\omega \in \mathbb{R}$. (For example, this is the case if $\mathbf{H} \in H^\infty(\mathbb{C}^{m\times m})$.) Then condition (2) in Theorem 5.3 is satisfied if

$$\lim_{\varrho\to\infty} \operatorname{ess\,sup}\{r(\mathbf{H}(i\omega))\,|\,|\omega| > \varrho\} > 1. \tag{5.4}$$

Under the extra assumption that $\mathbf{H}(i\omega)$ is almost periodic, (5.4) holds if

$$\operatorname{ess\,sup}\{r(\mathbf{H}(i\omega)\,|\,\omega \in \mathbb{R}\} > 1.$$

For the proof of Theorem 5.3 we need the following simple lemma.

LEMMA 5.5. *Let the set* $A \subset \mathbb{C}_0$ *be discrete in* $\mathbb{C}_0$. *Then for any* $\varepsilon > 0$ *and* $\beta \in \mathbb{R}$, *there exists* $y_0 \in \mathbb{R}$ *such that*

$$|y_0 - \beta| \leq \varepsilon \quad \text{and} \quad \{x + iy_0\,|\,x > 0\} \cap A = \emptyset\,.$$

Proof. It is easy to see that A is countable, so we can choose $y_0 \in (\beta - \varepsilon, \beta + \varepsilon)$ such that $y_0 \notin \mathrm{Im}\, A$. □

Proof of Theorem 5.3. As in the SISO case it follows from the assumptions (1) and (2) that there exist constants $\eta > 0$ and $\gamma_1 \in (1, \gamma)$ such that the set

$$S_1 = \{re^{i\phi}\ |\ r \in [1, \gamma_1]\,,\ |\phi - \pi| < \eta\}$$

does not intersect $\sigma(\mathbf{H}(s))$ for all $s \in \mathbb{C}_0 \setminus \mathfrak{P}_\mathbf{H}$:

$$\sigma(\mathbf{H}(s)) \cap S_1 = \emptyset \quad \forall\, s \in \mathbb{C}_0 \setminus \mathfrak{P}_\mathbf{H}. \tag{5.5}$$

Since $\gamma_1 < \gamma$, by assumption (2) there exists a sequence (s_n) in $\mathbb{C}_0 \setminus \mathfrak{P}_\mathbf{H}$ such that $|s_n| \to \infty$ and $r(\mathbf{H}(s_n)) > \gamma_1$ for all $n \in \mathbb{N}$. It is obvious that we can find numbers $\delta_n \in (0,1)$ such that the vertical segment $J_n := [s_n - i\delta_n, s_n + i\delta_n]$ is contained in $\mathbb{C}_0 \setminus \mathfrak{P}_\mathbf{H}$ and

$$r(\mathbf{H}(s)) \geq \gamma_1 \ \ \forall\, s \in \bigcup_{n\in\mathbb{N}} J_n\,.$$

It follows from Lemma 5.5 that there exists $z_n \in J_n$ such that the ray

$$\Gamma_n = \{z_n + a \mid a \in [0, \infty)\}$$

does not intersect the set $\mathfrak{P}_{\mathbf{H}} \cup \mathfrak{C}_{\mathbf{H}}$:

$$\Gamma_n \cap (\mathfrak{P}_{\mathbf{H}} \cup \mathfrak{C}_{\mathbf{H}}) \neq \emptyset \quad \forall n \in \mathbb{N}. \tag{5.6}$$

Using assumption (3) it can be shown as in the SISO case that

$$\lim_{n\to\infty} |\operatorname{Im} z_n| = \infty. \tag{5.7}$$

Again, without loss of generality, we may assume that Im $z_n > 0$ for all $n \in \mathbb{N}$. By construction we have that $r(\mathbf{H}(z_n)) \geq \gamma_1$ and hence there exists $\lambda_n \in \sigma(\mathbf{H}(z_n))$ such that $|\lambda_n| \geq \gamma_1 > 1$. Since (5.6) holds, an application of Proposition 5.2 shows that for all $n \in \mathbb{N}$ there exists a region V_n satisfying $\Gamma_n \subset V_n \subset \mathbb{C}_0 \setminus (\mathfrak{P}_{\mathbf{H}} \cup \mathfrak{C}_{\mathbf{H}})$ and a function ξ_n holomorphic on V_n such that $\xi_n(z_n) = \lambda_n$ and

$$\xi_n(s) \in \sigma(\mathbf{H}(s)) \quad \forall s \in V_n.$$

Moreover, by Proposition 2.3

$$\lim_{|s|\to\infty,\, s\in\Gamma_n} \mathbf{H}(s) = D,$$

and hence we obtain by Proposition 5.2 that

$$\lim_{|s|\to\infty,\, s\in\Gamma_n} \xi_n(s) =: \xi_n^\infty \in \mathbb{C}$$

exists and $\xi_n^\infty \in \sigma(D)$. As a consequence of assumption (3) we have that $|\xi_n^\infty| \leq 1$. Therefore the extended real numbers

$$a_n' = \max\{a \in [0, \infty) \mid |\xi_n(z_n + a)| \geq \gamma_1\},$$
$$a_n'' = \min\{a \in [a_n', \infty] \mid |\xi_n(z_n + a)| \leq 1\}$$

are well defined. (If $r(D) = 1$, it might happen that $a_n'' = \infty$.) Setting

$$z_n' = z_n + a_n' \quad \text{and} \quad z_n'' = z_n + a_n''$$

(where $z_n'' = \infty$ is possible), we will be looking for poles of $\mathbf{G}^\varepsilon$ in (z_n', z_n''). Notice that a sufficient condition for $p \in (z_n', z_n'')$ to be a pole of $\mathbf{G}^\varepsilon$ is that

$$\log \xi_n(p) - \varepsilon p = -i\pi.$$

By (5.5) it follows that for all $n \in \mathbb{N}$

$$\xi_n([z_n', z_n'']) \subset S_2 := \{re^{i\phi} \mid r \in [1, \gamma_1],\ |\phi| \leq \pi - \eta\}. \tag{5.8}$$

It follows that $\log \xi_n$ and $\arg \xi_n$ (where log and arg are defined as in §2) are continuous functions on $[z_n', z_n'']$. For $s \in [z_n', z_n'']$ define

$$R_n(s) := \{\log \xi_n(s) - \varepsilon s \mid \varepsilon \in [0, \infty)\}.$$

Then for each $n \in \mathbb{N}$ and $s \in [z_n', z_n'']$ the ray $R_n(s)$ intersects the line $L = \{s \in \mathbb{C} \mid \operatorname{Im} s = -\pi\}$ in a point $w_n(s) - i\pi$. Thus we can define functions $w_n, e_n : [z_n', z_n''] \to \mathbb{R}$ such that $e_n(s) > 0$ for all $s \in [z_n', z_n'']$ and

$$\log \xi_n(s) - e_n(s)s = w_n(s) - i\pi \quad \forall s \in [z_n', z_n''], \ \forall n \in \mathbb{N}. \tag{5.9}$$

As in the proof of Theorem 3.1 it follows that for all sufficiently large n there exists $p_n \in (z_n', z_n'')$ such that $w_n(p_n) = 0$. Thus, by (5.9)

$$\log \xi_n(p_n) - e_n(p_n)p_n = -i\pi .$$

Finally, it is clear that

$$e_n(p_n) = \frac{\arg \xi_n(p_n) + \pi}{\operatorname{Im} p_n} . \tag{5.10}$$

The sequence $(\arg \xi_n(p_n))$ is bounded, and by (5.7), we have that $\operatorname{Im} p_n = \operatorname{Im} z_n \to \infty$ as $n \to \infty$. Therefore we obtain from (5.10) that $\lim_{n\to\infty} e_n(p_n) = 0$. Setting $\varepsilon_n = e_n(p_n)$ it follows that $\mathbf{G}^{\varepsilon_n}$ has a pole in $p_n \in \mathbb{C}_0$. □

Along the same lines a multivariable extension of Theorem 4.1 can be obtained. We state this result without proof.

THEOREM 5.6. *Let* $\mathbf{H}$ *be a* $\mathbb{C}^{m\times m}$*-valued regular transfer function, and for any* $\varepsilon \geq 0$ *let* $\mathbf{G}^\varepsilon$ *be defined by* (5.3). *Let* D *denote the feedthrough matrix of* $\mathbf{H}$*, and assume that*

$$r(D) > 1.$$

Then there exist sequences (ε_n) *and* (p_n) *with*

$$\varepsilon_n > 0, \ \varepsilon_n \to 0, \ \operatorname{Re} p_n \to \infty, \ \operatorname{Im} p_n \to \infty$$

and such that for any $n \in \mathbb{N}$, p_n *is a pole of* $\mathbf{G}^{\varepsilon_n}$.

Combining Theorem 5.3 and Theorem 5.6 yields part (ii) of Theorem 1.1.

6. Robustness of stability. So far we have proved only part (ii) of Theorem 1.1. In this section we conclude the proof. In fact, since for part (i) of Theorem 1.1 the wellposedness and regularity assumptions are not needed and we obtain additionally uniform boundedness for the matrices $\mathbf{G}^\varepsilon$, we restate our result.

THEOREM 6.1. *Suppose* $\mathbf{G}^0 \in H^\infty(\mathbb{C}^{m\times m})$ *and denote* $\mathbf{H} = \mathbf{G}^0(I - \mathbf{G}^0)^{-1}$ *(so that* $\mathbf{H} \in \mathcal{M}_0(\mathbb{C}^{m\times m})$*). If*

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} r(\mathbf{H}(s)) < 1, \tag{6.1}$$

then there exist numbers $\varepsilon_0 > 0$ *and* $M > 0$ *such that* $\mathbf{G}^\varepsilon(s) = \mathbf{H}(s)(I + e^{-\varepsilon s}\mathbf{H}(s))^{-1} \in H^\infty(\mathbb{C}^{m\times m})$ *and* $\|\mathbf{G}^\varepsilon\|_\infty \leq M$ *for all* $\varepsilon \in [0, \varepsilon_0]$.

Remark 6.2. Suppose that $\mathbf{H} \in H^\infty(\mathbb{C}^{m\times m})$. Then, as is well known,

$$\lim_{\mu\downarrow 0} \mathbf{H}(\mu + i\omega) =: \mathbf{H}(i\omega)$$

exists for almost all $\omega \in \mathbb{R}$. It is easy to show that $r(\mathbf{H}(s))$ is a subharmonic function on $\mathbb{C}_0$. Using standard results on subharmonic functions (see, for example, Narasimhan [25, p. 227]) it is not difficult to prove that

$$\sup\{r(\mathbf{H}(s)) \mid s \in \mathbb{C}_0\} = \operatorname{ess\,sup}\{r(\mathbf{H}(i\omega)) \mid \omega \in \mathbb{R}\}.$$

As a consequence, (6.1) will be satisfied if $\operatorname{ess\,sup}\{r(\mathbf{H}(i\omega)) \mid \omega \in \mathbb{R}\} < 1$.

The proof of Theorem 6.1 requires some preparation. If $\mathbf{H} \in \mathcal{M}_0(\mathbb{C}^{m\times m})$ and s_0 is a pole of $\mathbf{H}$, then trivial examples show that $r(\mathbf{H}(s))$ does not necessarily blow up as $s \to s_0$. However, the next lemma reveals that this phenomenon cannot occur if $\mathbf{H}(I+\mathbf{H})^{-1}$ is L^2-stable.

LEMMA 6.3. *Let $U \subset \mathbb{C}$, suppose $\mathbf{G}^0$ is bounded and holomorphic on U, and denote $\mathbf{H} = \mathbf{G}^0(I-\mathbf{G}^0)^{-1}$. If $\sup_{s\in U} r(\mathbf{H}(s)) < \infty$, then $\sup_{s\in U} \|\mathbf{H}(s)\| < \infty$.*

Proof. Assume the claim is not true; i.e., there exists a sequence (s_n) in U such that $\lim_{n\to\infty} \|\mathbf{H}(s_n)\| = \infty$. Using Cramer's rule and the boundedness of $\mathbf{G}^0$ on U, it follows that

$$\lim_{n\to\infty} \det(I - \mathbf{G}^0(s_n)) = 0. \tag{6.2}$$

Now $(\mathbf{G}^0(s_n))$ is a bounded sequence in $\mathbb{C}^{m\times m}$, and hence we may assume without loss of generality that $\lim_{n\to\infty} \mathbf{G}^0(s_n) =: D^0 \in \mathbb{C}^{m\times m}$ exists. From (6.2) it follows that $1 \in \sigma(D^0)$. This in turn implies that there exist eigenvalues $\lambda_n \in \sigma(\mathbf{H}(s_n))$ such that $\lim_{n\to\infty}[\lambda_n/(1+\lambda_n)] = 1$. But this leads to a contradiction, since the sequence (λ_n) is bounded by assumption. □

Proof of Theorem 6.1. *Step* 1: For $\varrho > 0$ set

$$\mathbb{C}_0^{\varrho} := \{s \in \mathbb{C}_0 \mid |s| \geq \varrho\}. \tag{6.3}$$

By (6.1) there exists $R > 0$ and $q \in (0,1)$ such that

$$r(\mathbf{H}(s)) \leq q < 1 \;\; \forall\, s \in \mathbb{C}_0^R. \tag{6.4}$$

Combining (6.4) and Lemma 6.3 shows that $\mathbf{H}(s)$ is bounded on $\mathbb{C}_0^R$.

Step 2: We claim that there exists a number $L > 0$ such that $\|\mathbf{G}^\varepsilon(s)\| \leq L$ for all $\varepsilon \geq 0$ and for all $s \in \mathbb{C}_0^R$. Suppose the claim is not true. Then, since by Step 1 $\mathbf{H}(s)$ is bounded on $\mathbb{C}_0^R$, it follows from Cramer's rule that there exists a sequence (s_n) in $\mathbb{C}_0^R$ and a sequence (ε_n) of nonnegative numbers such that

$$\lim_{n\to\infty} \det(I + e^{-\varepsilon_n s_n}\mathbf{H}(s_n)) = 0. \tag{6.5}$$

Now $(\mathbf{H}(s_n))$ is a bounded sequence in $\mathbb{C}^{m\times m}$ and $|e^{-\varepsilon_n s_n}| \leq 1$ for all $n \in \mathbb{N}$, and therefore (as in Step 1) we may assume without loss of generality that the limits $\lim_{n\to\infty} e^{-\varepsilon_n s_n} =: d$ and $\lim_{n\to\infty} \mathbf{H}(s_n) =: E$ exist. Using (6.4) and the fact that $|d| \leq 1$ we see that $r(dE) < 1$. On the other hand it follows from (6.5) that $-1 \in \sigma(dE)$, a contradiction.

Step 3: Choose $\varepsilon_0 > 0$ such that for any $s \in \mathbb{C}_0$ with $|s| \leq R$ and any $\varepsilon \in [0, \varepsilon_0]$

$$|1 - e^{-\varepsilon s}| \leq \frac{1}{2\|\mathbf{G}^0\|_\infty}\,.$$

The identity $\mathbf{G}^\varepsilon(s) = \mathbf{G}^0(s)(I - (1-e^{-\varepsilon s})\mathbf{G}^0(s))^{-1}$ shows that, for all s and ε as above, $\|\mathbf{G}^\varepsilon(s)\| \leq 2\|\mathbf{G}^0\|_\infty$.

Combining Steps 2 and 3, we obtain that $\mathbf{G}^\varepsilon \in H^\infty(\mathbb{C}^{m\times m})$ and $\|\mathbf{G}^\varepsilon\|_\infty \leq M$ for all $\varepsilon \in [0,\varepsilon_0]$, where $M := \max(L, 2\|\mathbf{G}^0\|_\infty)$. □

Theorem 6.1 deals with delay perturbations of the form $e^{-\varepsilon s}$. A natural question to ask is whether it remains true for *multidelay perturbations* of the form $\operatorname{diag}_{1\leq j\leq m}(e^{-\varepsilon^j s})$, where $\varepsilon^j \geq 0$, $j = 1, \ldots, n$. The answer is no, as the following example shows.

Example 6.4. Consider the transfer function $\mathbf{H}(s) \equiv D$, where D is given by

$$D = \begin{pmatrix} -1/2 & 1/4 \\ -1 & 1/2 \end{pmatrix}.$$

The matrix D is nilpotent; i.e., $\sigma(D) = \{0\}$ and hence $\mathbf{G}^0(s) = \mathbf{H}(s)(I + \mathbf{H}(s))^{-1} \equiv D(I + D)^{-1}$ belongs to $H^\infty(\mathbb{C}^{2\times 2})$. Moreover

$$\limsup_{|s|\to\infty, s\in\mathbb{C}_0} r(\mathbf{H}(s)) = r(D) = 0,$$

and thus, by Theorem 6.1, $\mathbf{G}^0$ is robustly stable with respect to delays. Setting $\Delta := \text{diag}(1, -1)$, a trivial computation shows that $\sigma(\Delta D) = \{-1, 0\}$. Therefore, defining $\varepsilon_n^1 := 2\pi/n$ and $\varepsilon_n^2 := \pi/n$, it follows that

$$\det(I + \text{diag}(e^{-in\varepsilon_n^1}, e^{-in\varepsilon_n^2})D) = \det(I + \Delta D) = 0 \quad \forall n \in \mathbb{N}.$$

As a consequence, for all $n \in \mathbb{N}$, $p_n = in$ is a pole of the closed-loop transfer function $\mathbf{G}^{\varepsilon_n}$ with multidelay $\varepsilon_n = (\varepsilon_n^1, \varepsilon_n^2)$, defined by

$$\mathbf{G}^{\varepsilon_n}(s) := \mathbf{H}(s)(I + \text{diag}(e^{-\varepsilon_n^1 s}, e^{-\varepsilon_n^2 s})\mathbf{H}(s))^{-1}.$$

In order to give a sufficient condition for robust stability in the presence of multidelay perturbations, set

$$\mathbf{\Delta} := \{\text{diag}_{1\le j\le m}(s_j) \mid s_j \in \mathbb{C}\} \subset \mathbb{C}^{m\times m}$$

and define the structured singular value $\mu_{\mathbf{\Delta}}(M)$ of $M \in \mathbb{C}^{m\times m}$ with respect to $\mathbf{\Delta}$ by

$$\mu_{\mathbf{\Delta}}(M) := \frac{1}{\min\{\|\Delta\| \mid \Delta \in \mathbf{\Delta},\ \det(I - M\Delta) = 0\}},$$

unless no $\Delta \in \mathbf{\Delta}$ makes $I - M\Delta$ singular, in which case $\mu_{\mathbf{\Delta}}(M) := 0$ (cf. Packard and Doyle [26]).

THEOREM 6.5. *Suppose* $\mathbf{G}^0 \in H^\infty(\mathbb{C}^{m\times m})$ *and denote* $\mathbf{H} = \mathbf{G}^0(I - \mathbf{G}^0)^{-1}$ *(so that* $\mathbf{H} \in \mathcal{M}_0(\mathbb{C}^{m\times m})$*). If*

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} \mu_{\mathbf{\Delta}}(\mathbf{H}(s)) < 1, \tag{6.6}$$

then there exist numbers $\delta > 0$ *and* $M > 0$ *such that*

$$\mathbf{G}^\varepsilon(s) := \mathbf{H}(s)(I + \text{diag}_{1\le j\le m}(e^{-\varepsilon^j s})\mathbf{H}(s))^{-1} \in H^\infty(\mathbb{C}^{m\times m}) \tag{6.7}$$

and $\|\mathbf{G}^\varepsilon\|_\infty \le M$ *for all* $\varepsilon = (\varepsilon^1, \ldots, \varepsilon^m) \in \mathbb{R}_+^m$ *satisfying* $\|\varepsilon\| < \delta$.

Using standard properties of structured singular values [26] the proof of Theorem 6.5 is a straightforward extension of the proof of Theorem 6.1 and is therefore left to the reader.

Condition (6.6) holds if

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} \|\mathbf{H}(s)\| < 1 \tag{6.8}$$

is satisfied. Equation (6.8) is not necessary for robustness with respect to multidelay perturbations, as the following simple example shows.

Example 6.6. Let h_1, h_2, and h_3 be in H^∞ with $\|h_1\|_\infty < 1$, $\|h_3\|_\infty < 1$ and

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} |h_2(s)| > 1 .$$

If we define

$$\mathbf{H} = \begin{pmatrix} h_1 & h_2 \\ 0 & h_3 \end{pmatrix},$$

then it is clear that $\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} \|\mathbf{H}(s)\| > 1$, so (6.8) is not satisfied. Since

$$\det(I + \mathrm{diag}(e^{-\varepsilon^1 s}, e^{-\varepsilon^2 s})\mathbf{H}(s)) = (1 + e^{-\varepsilon^1 s}h_1(s))(1 + e^{-\varepsilon^2 s}h_2(s)) \ \forall (\varepsilon^1, \varepsilon^2) \in \mathbb{R}_+^2 ,$$

it follows that, denoting $\varepsilon = (\varepsilon^1, \varepsilon^2) \in \mathbb{R}_+^2$,

$$\inf_{\varepsilon\in\mathbb{R}_+^2} \inf_{s\in\mathbb{C}_0} |\det(I + \mathrm{diag}(e^{-\varepsilon^1 s}, e^{-\varepsilon^2 s})\mathbf{H}(s))| > 0 .$$

Let $\mathbf{G}^\varepsilon$ be defined by (6.7). Using Cramer's rule we obtain that $\mathbf{G}^\varepsilon \in H^\infty(\mathbb{C}^{2\times 2})$ and $\sup_{\varepsilon\in\mathbb{R}_+^2} \|\mathbf{G}^\varepsilon\|_\infty < \infty$; in particular, we have robust stability with respect to multidelay perturbations.

It seems to be a difficult open problem whether the condition

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} \mu_{\boldsymbol{\Delta}}(\mathbf{H}(s)) > 1$$

implies lack of robustness with respect to small multidelay perturbations.

7. Robustness of instability. Given a transfer function $\mathbf{H}$ of size $m \times m$, we have shown in the previous sections that, under certain conditions, there exists a positive sequence (ε_n) with $\varepsilon_n \to 0$ such that the closed-loop transfer function $\mathbf{G}^{\varepsilon_n}$ has at least one pole in $\mathbb{C}_0$ for all $n \in \mathbb{N}$. In this section we show that this property is robust in the following sense: For any $n \in \mathbb{N}$ there exists $\delta_n \in (0, \varepsilon_n)$ such that for any $\varepsilon \in \mathbb{R}_+^m$ with $\varepsilon = (\varepsilon^1, \dots, \varepsilon^m) \in \bigcup_{n\in\mathbb{N}}(\varepsilon_n - \delta_n, \varepsilon_n + \delta_n)^m$, $\mathbf{G}^\varepsilon$ (defined by (6.7)) has a pole in $\mathbb{C}_0$.

In the following we shall need the notion of a right-coprime (or left-coprime) factorization of a matrix-valued meromorphic function.

LEMMA 7.1. *Suppose $\mathbf{H} \in \mathcal{M}_\alpha(\mathbb{C}^{p\times m})$. Then the following statements hold:*

(i) $\mathbf{H}$ *admits a right-coprime factorization over $\mathcal{H}_\alpha$; i.e., there exist matrices $N \in \mathcal{H}_\alpha(\mathbb{C}^{p\times m})$, $D, Y \in \mathcal{H}_\alpha(\mathbb{C}^{m\times m})$, and $X \in \mathcal{H}_\alpha(\mathbb{C}^{m\times p})$ such that*

$$\mathbf{H} = ND^{-1} \ \text{and} \ XN + YD = I .$$

The matrices N and D are unique up to multiplication from the right by a unimodular factor. A number $s_0 \in \mathbb{C}_\alpha$ is a pole of $\mathbf{H}$ *if and only if* $\det D(s_0) = 0$.

(ii) $\mathbf{H}$ *admits a left-coprime factorization over $\mathcal{H}_\alpha$; i.e., there exist matrices $\tilde{N} \in \mathcal{H}_\alpha(\mathbb{C}^{p\times m})$, $\tilde{D}, \tilde{Y} \in \mathcal{H}_\alpha(\mathbb{C}^{p\times p})$, and $\tilde{X} \in \mathcal{H}_\alpha(\mathbb{C}^{m\times p})$ with*

$$\mathbf{H} = \tilde{D}^{-1}\tilde{N} \ \text{and} \ \tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I .$$

The matrices $\tilde{N}$ and $\tilde{D}$ are unique up to multiplication from the left by a unimodular factor. A number $s_0 \in \mathbb{C}_\alpha$ is a pole of $\mathbf{H}$ *if and only if* $\det \tilde{D}(s_0) = 0$.

(iii) *If* $\mathbf{H} = ND^{-1}$ *is a right-coprime factorization over* $\mathcal{H}_\alpha$ *and* $\mathbf{H} = \tilde{D}^{-1}\tilde{N}$ *is a left-coprime factorization over* $\mathcal{H}_\alpha$, *then the zeros of* $\det D$ *and* $\det \tilde{D}$ *in* $\mathbb{C}_\alpha$ *coincide (counting multiplicities).*

Proof. It is well known that the ring $\mathcal{H}_\alpha$ is a Bézout domain; i.e., every finitely generated ideal is principal (see, for example, Narasimhan [25, p. 136]). Now $\mathcal{M}_\alpha$ is the quotient field of $\mathcal{H}_\alpha$, and statements (i) and (ii) follow from Vidyasagar [31, p. 330]. Statement (iii) is proved in [31, p. 76] for rational matrices. An inspection of the proof in [31] shows that it only utilizes the fact that the elementary divisor theorem holds for the ring of stable rational functions; i.e., any matrix with stable rational entries is equivalent to its Smith form. Since this is also true for the ring $\mathcal{H}_\alpha$ (see [25, p. 139]), it follows that the proof in [31] carries over to matrices with entries in $\mathcal{M}_\alpha$. □

If $f \in \mathcal{M}_0$ and $V \subset \mathbb{C}_0$ is compact, let $Z(f,V)$ denote the number of zeros of f in V, counting multiplicities. Moreover, if $\gamma : [0,1] \to \mathbb{C}$ is a closed curve and $a \in \mathbb{C} \setminus \gamma([0,1])$, we denote the winding number (index) of γ around a by $\mathrm{ind}(\gamma, a)$.

PROPOSITION 7.2. *Let* $\mathbf{H}$ *be in* $\mathcal{M}_0(\mathbb{C}^{m\times m})$ *and suppose that* $\mathbf{H}(s)(I+\mathbf{H}(s))^{-1}$ *has at least one pole in* $\mathbb{C}_0$. *Then there exists* $\delta > 0$ *such that* $\mathbf{G}^\eta$ *defined by*

$$\mathbf{G}^\eta(s) = \mathbf{H}(s)(I + \mathrm{diag}_{1\le j\le m}(e^{\eta^j s})\mathbf{H}(s))^{-1}$$

has at least one pole in $\mathbb{C}_0$ *for all* $\eta = (\eta^1, \ldots, \eta^m) \in \mathbb{C}^m$ *satisfying* $\|\eta\| < \delta$.

Proof. Let $\mathbf{H} = ND^{-1}$ be a right-coprime factorization over $\mathcal{H}_0$ and $s_0 \in \mathbb{C}_0$ be a pole of $\mathbf{H}(I+\mathbf{H})^{-1}$. Set $V := \{s \in \mathbb{C} \mid |s_0 - s| \le \varrho\}$ and choose $\varrho > 0$ such that

$$V \subset \mathbb{C}_0 \quad \text{and} \quad Z(\det(D+N), \partial V) \neq 0 \quad \forall s \in \partial V. \tag{7.1}$$

Let $\gamma_V : [0,1] \to \mathbb{C}$ be the continuous parametrization of ∂V given by $t \mapsto s_0 + \varrho e^{2\pi i t}$. For $\eta = (\eta^1, \ldots, \eta^m) \in \mathbb{C}^m$ set

$$N_\eta(s) := \mathrm{diag}_{1\le j\le m}(e^{\eta^j s})N(s)\,,$$
$$\Gamma_\eta(t) := \det[D(\gamma_V(t)) + N_\eta(\gamma_V(t))]\,.$$

It is clear that

$$\lim_{\eta\to 0}\left(\sup_{t\in[0,1]} |\Gamma_0(t) - \Gamma_\eta(t)|\right) = 0. \tag{7.2}$$

Now it follows from (7.1) that

$$\inf_{t\in[0,1]} |\Gamma_0(t)| > 0,$$

and therefore we may conclude, using (7.2), that there exists $\delta > 0$ such that

$$\inf_{t\in[0,1]} |\Gamma_\eta(t)| > 0 \ \text{ for all } \eta \in \mathbb{C} \text{ such that } \|\eta\| < \delta\,. \tag{7.3}$$

Choose $\eta \in \mathbb{C}^m$ with $\|\eta\| < \delta$, and define the map

$$\Lambda : [0,1]\times[0,1] \to \mathbb{C}\,, \ (t,\tau) \mapsto \Gamma_{\tau\eta}(t)\,.$$

Then Λ is continuous and, by (7.3), $0 \notin \Lambda([0,1]\times[0,1])$. Trivially, it holds that

$$\Lambda(t,0) = \Gamma_0(t) \ \ \forall t \in [0,1]\,,$$
$$\Lambda(t,1) = \Gamma_\eta(t) \ \ \forall t \in [0,1],$$

and furthermore we obtain for all $\tau \in [0,1]$ that

$$\Lambda(0,\tau) = \Gamma_{\tau\eta}(0) = \Gamma_{\tau\eta}(1) = \Lambda(1,\tau).$$

Thus we have shown that Γ_0 and Γ_η are homotopic in $\mathbb{C} \setminus \{0\}$, and therefore (cf. Rudin [27, Thm. 10.40]) it follows that

$$\operatorname{ind}(\Gamma_0, 0) = \operatorname{ind}(\Gamma_\eta, 0). \tag{7.4}$$

Using the principle of the argument we obtain

$$Z(\det(D+N_\eta), V) = \operatorname{ind}(\Gamma_\eta, 0) = \operatorname{ind}(\Gamma_0, 0) = Z(\det(D+N), V). \tag{7.5}$$

Now, $s_0 \in V \subset \mathbb{C}_0$ is a pole of $\mathbf{H}(I+\mathbf{H})^{-1}$ or equivalently $\det(D(s_0) + N(s_0)) = 0$, and thus, by (7.5)

$$Z(\det(D+N_\eta), V) = Z(\det(D+N), V) \neq 0. \tag{7.6}$$

It is easy to see that $\mathbf{G}^\eta = N(D+N_\eta)^{-1}$ is a right-coprime factorization over $\mathcal{H}_0$, and thus it follows from (7.6) that $\mathbf{G}^\eta$ has a pole in $V \subset \mathbb{C}_0$. □

Combining Proposition 7.2 and Theorem 5.3 we obtain the main result of this section, a "robust" version of Theorem 5.3.

THEOREM 7.3. *Let $\mathbf{H}(s)$ be a $\mathbb{C}^{m\times m}$-valued regular transfer function and, for $\varepsilon = (\varepsilon^1, \dots, \varepsilon^m) \in \mathbb{R}_+^m$, $\mathbf{G}^\varepsilon$ be given by (6.7). If the conditions (1)–(3) of Theorem 5.3 are satisfied, then there exist sequences (ε_n) and (δ_n) with $\varepsilon_n > 0$, $\varepsilon_n \to 0$, $\delta_n \in (0, \varepsilon_n)$ and such that $\mathbf{G}^\varepsilon$ has poles in $\mathbb{C}_0$ for all $\varepsilon \in \bigcup_{n\in\mathbb{N}} (\varepsilon_n - \delta_n, \varepsilon_n + \delta_n)^m$.*

It is clear that Theorem 5.6 can be strengthened in a similar way.

8. Dynamic output feedback. In this section we apply our results to systems with dynamic output feedback. In particular we show that—roughly speaking—for a plant with infinitely many unstable poles there does *not* exist any stabilizing (dynamic) output feedback compensator such that the stability of the closed-loop system is robust with respect to small delays.

DEFINITION 8.1. *If $\mathbf{P} \in \mathcal{M}_\alpha(\mathbb{C}^{p\times m})$ and $\mathbf{K} \in \mathcal{M}_\alpha(\mathbb{C}^{m\times p})$ for some $\alpha \in \mathbb{R}$, we say that* $\mathbf{K}$ stabilizes $\mathbf{P}$ *if* $\det(I + \mathbf{P}(s)\mathbf{K}(s)) \not\equiv 0$ *and*

$$\begin{pmatrix} I & \mathbf{P} \\ -\mathbf{K} & I \end{pmatrix}^{-1} \in H^\infty(\mathbb{C}^{(m+p)\times(m+p)}). \tag{8.1}$$

It follows from a well-known formula of Frobenius (see Gantmacher [16, p. 73]) that $\mathbf{K}$ stabilizes $\mathbf{P}$ if and only if $\det(I + \mathbf{P}(s)\mathbf{K}(s)) \not\equiv 0$ and the transfer function

$$F(\mathbf{P},\mathbf{K}) = \begin{pmatrix} \mathbf{K}(I+\mathbf{PK})^{-1} & -\mathbf{KP}(I+\mathbf{KP})^{-1} \\ \mathbf{PK}(I+\mathbf{PK})^{-1} & \mathbf{P}(I+\mathbf{KP})^{-1} \end{pmatrix} \tag{8.2}$$

is in $H^\infty(\mathbb{C}^{(m+p)\times(m+p)})$. Note that $F(\mathbf{P},\mathbf{K})$ is the transfer function from (u_1,u_2) to (y_1,y_2) of the feedback system shown in Fig. 4, if we take there $\varepsilon = 0$. If $\mathbf{K} \in H^\infty(\mathbb{C}^{m\times p})$, then $\mathbf{K}$ stablizes $\mathbf{P}$ if and only if $\mathbf{P}(I+\mathbf{KP})^{-1} \in H^\infty(\mathbb{C}^{p\times m})$. The next result follows trivially from Theorems 5.3 and 5.6.

COROLLARY 8.2. *Let $\mathbf{P}$ and $\mathbf{K}$ be matrix-valued transfer functions of size $p \times m$ and $m \times p$, respectively. Suppose that $\mathbf{PK}$ is regular, and for $\varepsilon \geq 0$ set $\mathbf{K}_\varepsilon(s) := e^{-\varepsilon s}\mathbf{K}(s)$ and define*

$$F^\varepsilon(\mathbf{P},\mathbf{K}) := \begin{pmatrix} \mathbf{K}(I+\mathbf{PK}_\varepsilon)^{-1} & -\mathbf{K}_\varepsilon\mathbf{P}(I+\mathbf{K}_\varepsilon\mathbf{P})^{-1} \\ \mathbf{PK}(I+\mathbf{PK}_\varepsilon)^{-1} & \mathbf{P}(I+\mathbf{K}_\varepsilon\mathbf{P})^{-1} \end{pmatrix}. \tag{8.3}$$

Then, if **K** *stabilizes* **P** *(i.e.,* $F^0(\mathbf{P},\mathbf{K}) \in H^\infty(\mathbb{C}^{(m+p)\times(m+p)})$*) and*

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} r(\mathbf{P}(s)\mathbf{K}(s)) = \limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} r(\mathbf{K}(s)\mathbf{P}(s)) > 1,$$

there exist sequences (ε_n) *and* (p_n) *with*

$$\varepsilon_n > 0\,,\ \ \varepsilon_n \to 0\,,\ \ p_n \in \mathbb{C}_0\,,\ \ |\mathrm{Im}\, p_n| \to \infty$$

and such that for any $n \in \mathbb{N}$, p_n *is a pole of* $\mathbf{PK}(I+\mathbf{PK}_{\varepsilon_n})^{-1}$ *and hence of the overall closed-loop transfer function* $F^{\varepsilon_n}(\mathbf{P},\mathbf{K})$.

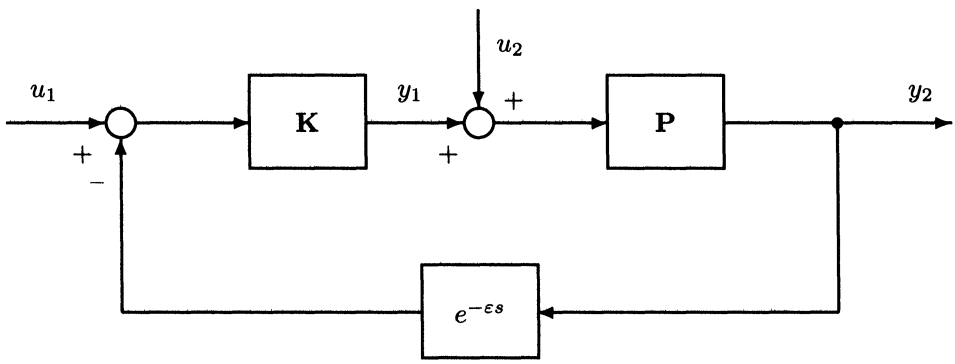

FIG. 4. *Feedback system with plant, compensator, and delay.*

The feedback system corresponding to $F^\varepsilon(\mathbf{P},\mathbf{K})$ is shown in Fig. 4; in particular we have that $(y_1,y_2)^T = F^\varepsilon(\mathbf{P},\mathbf{K})(u_1,u_2)^T$. It is clear that $F(\mathbf{P},\mathbf{K}_\varepsilon)$ is L^2-stable if $F^\varepsilon(\mathbf{P},\mathbf{K})$ is. Conversely, under the assumptions that **P** and **K** are well posed and **P** stabilizes **K**,[1] it is easy to show that $F^\varepsilon(\mathbf{P},\mathbf{K})$ is L^2-stable if $F(\mathbf{P},\mathbf{K}_\varepsilon)$ is.

In order to apply Theorem 6.1 to systems with dynamic output feedback we need the following lemma.

LEMMA 8.3. *For some* $\alpha \in \mathbb{R}$ *let* **P** *and* **K** *be in* $\mathcal{M}_\alpha(\mathbb{C}^{p\times m})$ *and* $\mathcal{M}_\alpha(\mathbb{C}^{m\times p})$*, respectively, and suppose that* **K** *stabilizes* **P**. *If* $U \subset \mathbb{C}_0$ *and*

$$\sup_{s\in U} \|\mathbf{P}(s)\| = \infty \ \ \textit{or} \ \ \sup_{s\in U} \|\mathbf{K}(s)\| = \infty\,,$$

then it follows that $\sup_{s\in U} \|\mathbf{P}(s)\mathbf{K}(s)\| = \infty$.

Proof. From the assumption that **K** stabilizes **P** it follows that the entries of **P** and **K** belong to the the quotient field of H^∞; i.e., they can be written as the fraction of two H^∞-functions. Moreover, it follows from Smith [30] that **P** and **K** both have right- and left-coprime factorizations over H^∞. This means in particular that there exist matrices $\tilde{N}_\mathbf{P}$, $\tilde{D}_\mathbf{P}$, $\tilde{X}_\mathbf{P}$, $\tilde{Y}_\mathbf{P}$, $N_\mathbf{K}$, $D_\mathbf{K}$, $X_\mathbf{K}$, and $Y_\mathbf{K}$ with entries in H^∞ satisfying

$$\mathbf{P} = \tilde{D}_\mathbf{P}^{-1}\tilde{N}_\mathbf{P}\,, \qquad \tilde{N}_\mathbf{P}\tilde{X}_\mathbf{P} + \tilde{D}_\mathbf{P}\tilde{Y}_\mathbf{P} = I,$$
$$\mathbf{K} = N_\mathbf{K}D_\mathbf{K}^{-1}\,, \qquad X_\mathbf{K}N_\mathbf{K} + Y_\mathbf{K}D_\mathbf{K} = I.$$

Moreover, since by assumption the closed-loop system is stable, it follows trivially that I stabilizes **PK**. Therefore, using again the result in Smith [30], we conclude that

[1] If **PK** is well posed and **P** stabilizes **K**, then Lemma 8.3 shows that **P** and **K** are well posed.

PK admits a right-coprime factorization over H^∞; i.e., there exist matrices N, D, X, and Y with entries in H^∞ such that

$$\mathbf{PK} = ND^{-1}, \quad XN + YD = I. \tag{8.4}$$

It is well known (see Vidyasagar [31, p. 364]) that closed-loop stability is equivalent to

$$\inf_{s \in \mathbb{C}_0} |\det(\tilde{D}_{\mathbf{P}}(s) D_{\mathbf{K}}(s) + \tilde{N}_{\mathbf{P}}(s) N_{\mathbf{K}}(s))| > 0. \tag{8.5}$$

Let us assume that $\sup_{s \in U} \|\mathbf{P}(s)\| = \infty$. Then there exists a sequence (s_n) in U such that $\lim_{n\to\infty} \|\mathbf{P}(s_n)\| = \infty$, and hence, using the boundedness of $\tilde{D}_{\mathbf{P}}(s)$ and $\tilde{N}_{\mathbf{P}}(s)$, we obtain

$$\lim_{n\to\infty} \det(\tilde{D}_{\mathbf{P}}(s_n)) = 0. \tag{8.6}$$

Realizing that

$$\begin{aligned} \det(\tilde{D}_{\mathbf{P}} D_{\mathbf{K}} + \tilde{N}_{\mathbf{P}} N_{\mathbf{K}}) &= \det(\tilde{D}_{\mathbf{P}}) \det(D_{\mathbf{K}}) \det(I + \mathbf{PK}) \\ &= \frac{\det(\tilde{D}_{\mathbf{P}}) \det(D_{\mathbf{K}})}{\det(D)} \det(D + N) \end{aligned} \tag{8.7}$$

and combining (8.5)–(8.7) we see that

$$\lim_{n\to\infty} \det(D(s_n)) = 0. \tag{8.8}$$

Moreover, using (8.4), it follows that

$$\det(X\mathbf{PK} + Y) = \frac{1}{\det(D)}. \tag{8.9}$$

Finally, (8.8), (8.9), and the boundedness of the matrices $X(s)$ and $Y(s)$ imply that $\lim_{n\to\infty} \|\mathbf{P}(s_n)\mathbf{K}(s_n)\| = \infty$ and thus $\sup_{s\in U} \|\mathbf{P}(s)\mathbf{K}(s)\| = \infty$. With a similar argument we can prove the claim if we assume that $\sup_{s\in U} \|\mathbf{K}(s)\| = \infty$. □

COROLLARY 8.4. *Let* $\mathbf{P} \in \mathcal{M}_\alpha(\mathbb{C}^{p\times m})$ *and* $\mathbf{K} \in \mathcal{M}_\alpha(\mathbb{C}^{m\times p})$ *for some* $\alpha \in \mathbb{R}$. *If* $F^0(\mathbf{P}, \mathbf{K}) \in H^\infty(\mathbb{C}^{(m+p)\times(m+p)})$ *and*

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} r(\mathbf{P}(s)\mathbf{K}(s)) < 1, \tag{8.10}$$

then there exist numbers $\varepsilon_0 > 0$ *and* $M > 0$ *such that* $F^\varepsilon(\mathbf{P}, \mathbf{K}) \in H^\infty(\mathbb{C}^{(m+p)\times(m+p)})$ *and* $\|F^\varepsilon(\mathbf{P}, \mathbf{K})\|_\infty \le M$ *for all* $\varepsilon \in [0, \varepsilon_0]$.

Proof. For $\varrho > 0$ let $\mathbb{C}_0^\varrho$ be defined by (6.3). Combining (8.10), the fact that $\mathbf{PK}(I + \mathbf{PK})^{-1} \in H^\infty(\mathbb{C}^{p\times p})$, and Lemma 6.3, we see that there exists numbers $R_1 > 0$ and $L_1 > 0$ such that

$$\|\mathbf{P}(s)\mathbf{K}(s)\| \le L_1 \;\; \forall\, s \in \mathbb{C}_0^{R_1}.$$

Hence we obtain using Lemma 8.3 that

$$\|\mathbf{P}(s)\| \le L_2 \text{ and } \|\mathbf{K}(s)\| \le L_2 \;\; \forall\, s \in \mathbb{C}_0^{R_2}, \tag{8.11}$$

where L_2 and R_2 are suitable positive constants. By Theorem 6.1 there exist numbers $\varepsilon_1 > 0$ and $M_1 > 0$ such that

$$\|\mathbf{PK}(I+\mathbf{PK}_\varepsilon)^{-1}\|_\infty \le M_1 \quad \forall \varepsilon \in [0, \varepsilon_1] \tag{8.12}$$

and so

$$\|(I+\mathbf{PK}_\varepsilon)^{-1}\|_\infty \le 1 + M_1 \quad \forall \varepsilon \in [0, \varepsilon_1] .$$

Therefore, and by (8.11), there exists $\tilde{M}_2 > 0$ such that

$$\|\mathbf{K}(s)(I+\mathbf{P}(s)\mathbf{K}_\varepsilon(s))^{-1}\| \le \tilde{M}_2 \quad \forall s \in \mathbb{C}_0^{R_2} , \ \forall \varepsilon \in [0, \varepsilon_1] . \tag{8.13}$$

Setting $\mathbf{L}_\varepsilon := \mathbf{K}(I+\mathbf{PK}_\varepsilon)^{-1}$, we have $\mathbf{L}_0 \in H^\infty(\mathbb{C}^{m\times p})$ and $\mathbf{PL}_0 \in H^\infty(\mathbb{C}^{p\times p})$. Choosing $\varepsilon_2 \in (0, \varepsilon_1]$ such that for any $s \in \mathbb{C}_0$ with $|s| \le R_2$ and any $\varepsilon \in [0, \varepsilon_2]$

$$|1 - e^{-\varepsilon s}| \le \frac{1}{2\|\mathbf{PL}_0\|_\infty}$$

and realizing that

$$\mathbf{L}_\varepsilon(s) = \mathbf{L}_0(s)[I - (1-e^{-\varepsilon s})\mathbf{P}(s)\mathbf{L}_0(s)]^{-1},$$

we obtain that for all s and ε as above

$$\|\mathbf{L}_\varepsilon(s)\| \le 2\|\mathbf{L}_0\|_\infty . \tag{8.14}$$

Combining (8.13) and (8.14) shows that

$$\|\mathbf{K}(I+\mathbf{PK}_\varepsilon)^{-1}\|_\infty \le M_2 \ \forall \varepsilon \in [0, \varepsilon_2] , \tag{8.15}$$

where $M_2 := \max(\tilde{M}_2, 2\|\mathbf{L}_0\|_\infty)$. Finally, using similar arguments, it can be shown that

$$\|\mathbf{K}_\varepsilon\mathbf{P}(I+\mathbf{K}_\varepsilon\mathbf{P})^{-1}\|_\infty \le M_3 \ \forall \varepsilon \in [0, \varepsilon_3] \tag{8.16}$$

and

$$\|\mathbf{P}(I+\mathbf{K}_\varepsilon\mathbf{P})^{-1}\|_\infty \le M_4 \ \forall \varepsilon \in [0, \varepsilon_4] , \tag{8.17}$$

where M_3, M_4, ε_3, and ε_4 are suitable positive numbers. The claim now follows from (8.12) and (8.15)–(8.17). □

Using Corollary 8.2 and Lemma 8.3 it is easy to give the proof of Theorem 1.2. More precisely, we prove the following result which is slightly stronger than Theorem 1.2.

THEOREM 8.5. *Let* $\mathbf{P} \in \mathcal{M}_\alpha(\mathbb{C}^{p\times m})$ *and* $\mathbf{K} \in \mathcal{M}_\alpha(\mathbb{C}^{m\times p})$ *for some* $\alpha \in \mathbb{R}$, *and suppose that* $\mathbf{PK}$ *is regular. Then, if* $\mathbf{K}$ *stabilizes* $\mathbf{P}$ *and* $\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} \|\mathbf{P}(s)\| = \infty$, *there exist sequences* (ε_n) *and* (p_n) *with*

$$\varepsilon_n > 0 , \ \varepsilon_n \to 0 , \ p_n \in \mathbb{C}_0 , \ |\mathrm{Im}\, p_n| \to \infty$$

and such that, for any $n \in \mathbb{N}$, p_n *is a pole of* $\mathbf{PK}(I+\mathbf{PK}_{\varepsilon_n})^{-1}$ *and hence of the overall closed-loop transfer function* $F^{\varepsilon_n}(\mathbf{P},\mathbf{K})$ *given by* (8.3).

Proof. Since $\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} \|\mathbf{P}(s)\| = \infty$ and $\mathbf{K}$ stabilizes $\mathbf{P}$, it follows from Lemma 8.3 that $\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} \|\mathbf{P}(s)\mathbf{K}(s)\| = \infty$. Now, by assumption, $\mathbf{PK}(I+$

$\mathbf{PK})^{-1} \in H^\infty(\mathbb{C}^{p\times p})$, and hence an application of Lemma 6.3 and Corollary 8.2 yields the claim. □

The following remark shows that for a large class of transfer functions which are bounded at high frequencies there always exists a stabilizing compensator such that the stability of the closed loop is robust with respect to small delays.

Remark 8.6. Define $\mathcal{T} := \bigcup_{\alpha<0} H^\infty_\alpha + \mathcal{R}_{spu}$, where $\mathcal{R}_{spu}$ denotes the ring of strictly proper totally unstable rational functions, i.e., $\mathcal{R}_{spu} := \{f \in \mathbb{C}(s) \mid f(\infty) = 0 \text{ and } f(s) \neq \infty \text{ for all } s \in \mathbb{C} \setminus \mathbb{C}_0^{\rm cl}\}$. Note that if $\mathbf{P} \in \mathcal{T}^{p\times m}$, then

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} \|\mathbf{P}(s)\| < \infty,$$

which implies in particular that $\mathbf{P}$ has at most finitely many poles in $\mathbb{C}_0^{\rm cl}$. The ring $\mathcal{T}$ contains the so-called Callier–Desoer ring of transfer functions (cf. Callier and Desoer [4],[5]). It is known that for any $\mathbf{P} \in \mathcal{T}^{p\times m}$ there exists a *strictly proper rational* compensator $\mathbf{K}$ such that $F(\mathbf{P},\mathbf{K}) \in H^\infty(\mathbb{C}^{(m+p)\times(m+p)})$; see Logemann [24] and the references therein. Combining this result with Corollary 8.4, it follows that for any $\mathbf{P} \in \mathcal{T}^{p\times m}$ there exists a compensator $\mathbf{K} \in \mathcal{T}^{m\times p}$ and a number $\varepsilon_0 > 0$ such that $F^\varepsilon(\mathbf{P},\mathbf{K}) \in H^\infty(\mathbb{C}^{(m+p)\times(m+p)})$ for all $\varepsilon \in [0,\varepsilon_0]$.

Remark 8.7. We claim that the conclusions of Theorem 8.5 do not remain true if the assumption $\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} \|\mathbf{P}(s)\| = \infty$ is replaced by the weaker assumption that there exist a sequence of poles of $\mathbf{P}$ in the open left half-plane going to ∞ tangentially along the imaginary axis. To this end let $\mathbf{P}$ be the transfer function of the following neutral system:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) - \dot{x}_2(t-h) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -a \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$
$$y(t) = x_2(t)\ ,$$

i.e.,

$$\mathbf{P}(s) = \frac{1}{s+1}\frac{1}{s(1-e^{-hs})+a},$$

where $a, h > 0$. It is shown in Logemann [23] that $\mathbf{P} \in H^\infty$. Trivially, for any compensator $\mathbf{K} \in H^\infty$ satisfying $\|\mathbf{PK}\|_\infty < 1$ the closed-loop transfer function $F^\varepsilon(\mathbf{P},\mathbf{K})$ is in $H^\infty(\mathbb{C}^{2\times 2})$ for all $\varepsilon \geq 0$. However, using Rouché's theorem, it is not difficult to show that there exist a sequence of poles $p_n \in \mathbb{C} \setminus \mathbb{C}_0^{\rm cl}$ of $\mathbf{P}$ and numbers $\ell_n \in \mathbb{N}$ with $\ell_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} |p_n - i\frac{2\pi}{h}\ell_n| = 0\,.$$

9. Examples.

In this section we illustrate Theorem 1.1 with three examples.

Example 9.1. In this example we analyze the robustness with respect to delays for a damped wave equation. For $x \in (0,1)$ and $t > 0$ we consider the following system:

$$w_{tt}(x,t) - w_{xx}(x,t) + 2aw_t(x,t) + a^2 w(x,t) = 0\,, \tag{9.1}$$

$$w(0,t) = 0\,, \qquad w_x(1,t) = u(t)\,, \tag{9.2}$$

$$y(t) = kw_t(1,t)\,. \tag{9.3}$$

We assume here that the viscous damping parameter a is nonnegative and the boundary damping parameter k is positive. It is known that the feedback control

$$u(t) = -y(t) \tag{9.4}$$

exponentially stabilizes the system (see, for instance, Chen [6]). Hence, if the transfer function of (9.1)–(9.3) is denoted by $\mathbf{H}$, then it follows that $\mathbf{H}(I+\mathbf{H})^{-1} \in H^\infty$. An easy computuation shows that $\mathbf{H}$ is given by

$$\mathbf{H}(s) = \frac{ks}{s+a}\left(\frac{1-e^{-2(s+a)}}{1+e^{-2(s+a)}}\right).$$

In Datko, Lagnese, and Polis [9] the robustness of the closed loop system (9.1)–(9.4) with respect to small delays was analyzed. We will obtain frequency domain versions of their results, using Theorem 1.1. We need to compute γ as defined by (1.3) for this system.

CLAIM.

$$\gamma = \limsup_{|s|\to\infty, s\in\mathbb{C}_0} |\mathbf{H}(s)| = k\frac{e^{2a}+1}{e^{2a}-1}\,.$$

Proof. The following simple estimates are clear for Re $s>0$:

$$|1-e^{-2(s+a)}| \le 1+e^{-2a}\,, \qquad |1+e^{-2(s+a)}| \ge 1-e^{-2a}\,, \qquad \left|\frac{s}{s+a}\right| \le 1\,.$$

These estimates show that $\gamma \le k(e^{2a}+1)/(e^{2a}-1)$. To obtain the opposite inequality, let $s_n = (1/n) + i(2n+1)\pi/2$ for $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \mathbf{H}(s_n) = \lim_{n\to\infty} \frac{ks_n}{s_n+a}\frac{1+e^{-2/n}e^{-2a}}{1-e^{-2/n}e^{-2a}} = k\frac{e^{2a}+1}{e^{2a}-1}\,.$$

This shows that $\gamma \ge k(e^{2a}+1)/(e^{2a}-1)$, completing the proof of the claim. □

Let us apply Theorem 1.1 to this system. We consider two cases.

Case 1: $k \ge 1$. In this case $\gamma > 1$ for any $a \ge 0$, so the transfer function $\mathbf{H}(s)(I+e^{-\varepsilon s}\mathbf{H}(s))^{-1}$ has poles in $\mathbb{C}_0$ for arbitrarily small $\varepsilon > 0$.

Case 2: $k < 1$. In this case $\gamma > 1$ if and only if

$$a < \frac{1}{2}\ln\frac{1+k}{1-k}\,.$$

If a satisfies this estimate, then the same conclusion as in case 1 holds. When

$$a > \frac{1}{2}\ln\frac{1+k}{1-k}\,,$$

the delayed feedback system is L^2-stable for all sufficiently small delays.

Example 9.2. We consider the following first-order neutral system:

$$\dot{x}(t) - a\dot{x}(t-h) = -bx(t) + u(t)\,,$$
$$y(t) = c\dot{x}(t-h)\,.$$

Here $a \geq 1$, $b > 0$, $c \in \mathbb{R}$, and $h > 0$. We consider the feedback $u(t) = -y(t)$, so the free dynamics of the closed loop are described by

$$\dot{x}(t) + (c-a)\dot{x}(t-h) = -bx(t)\,.$$

This system is exponentially stable if $|c-a| < 1$. The open-loop transfer function is

$$\mathbf{H}(s) = \frac{sce^{-hs}}{s(1-ae^{-sh})+b}.$$

$\mathbf{H}$ is clearly well posed and regular with feedthrough 0. If $a > 1$, then the equation $1-ae^{-sh} = 0$ has a zero at $s = \log(a)/h$, which is in $\mathbb{C}_0$. Hence, by a result in Salamon [28, p. 160], the characteristic equation $s(1-ae^{-sh}) + b = 0$ has infinitely many zeros in $\mathbb{C}_0$. (This follows also directly from the periodicity of $1-ae^{-hs}$ and an application of Rouché's theorem.) As a consequence $\gamma = \infty > 1$, so the closed-loop stability is destroyed by arbitrarily small delays. If $a = 1$, then the equation $1 - e^{-sh} = 0$ has a zero at $s = 0$. It is easy to see that $\mathbf{H}(s)$ has no poles in $\mathbb{C}_0^{\mathrm{cl}}$. However, as shown in Logemann [23], we have that $\limsup_{\omega\to\infty} |\mathbf{H}(i\omega)| = \infty$. Hence $\gamma = \infty > 1$, and so the closed-loop system is not robustly stable with respect to delays.

Example 9.3. In this example the input space and the output space are $\mathbb{R}^2$. We consider two coupled vibrating strings, one with spatial extent $0 \leq x \leq 1$ and the other with spatial extent $1 \leq x \leq 2$. Each string satisfies the damped wave equation

$$w_{tt}(x,t) - w_{xx}(x,t) + 2aw_t(x,t) + a^2w(x,t) = 0\,, \quad x \in (0,1)\cup(1,2)\,,$$

where the viscous damping parameter $a \geq 0$. At the linkage we assume the displacement is continuous, so

$$w(1^-,t) = w(1^+,t)\,,$$

and we set the discontinuity of the vertical tension force equal to a control variable:

$$w_x(1^-,t) - w_x(1^+,t) = u_1(t)\,.$$

We take the right endpoint fixed, and at the left endpoint we set the vertical tension force equal to another control variable, leading to

$$w(2,t) \;=\; 0\,, \qquad w_x(0,t) = u_2(t)\,.$$

We take one observation proportional to the velocity at the linkage, and the other observation negatively proportional to the velocity at the left endpoint, leading to

$$y_1(t) \;=\; k_1w_t(1,t)\,, \qquad y_2(t) \;=\; -k_2w_t(0,t)\,,$$

for $k_1, k_2 \geq 0$. Let $u(t) = [u_1(t), u_2(t)]^T$ and $y(t) = [y_1(t), y_2(t)]^T$. The transfer function $\mathbf{H}$ for this system can be computed to be

$$\mathbf{H}(s) \;=\; \frac{s}{s+a}\begin{pmatrix} -\dfrac{k_1}{2}\dfrac{e^{-4(s+a)}-1}{e^{-4(s+a)}+1} & k_1\dfrac{e^{-(s+a)}(e^{-2(s+a)}-1)}{e^{-4(s+a)}+1} \\ k_2\left(\dfrac{e^{-(s+a)}(e^{-2(s+a)}-1)}{e^{-4(s+a)}+1}\right) & k_2\left(\dfrac{1-e^{-4(s+a)}}{e^{-4(s+a)}+1}\right) \end{pmatrix}.$$

TABLE 1
Values of γ for given values of a, k_1, and k_2.

(k_1, k_2)	$a = .1$	$a = .25$	$a = .5$	$a = 3$	$a = 10$
(.1, .1)	.7562	.3146	.1755	.1005	.1
(.1, .5)	2.7790	1.1714	.6881	.5003	.5
(.1, 1)	5.3112	2.2513	1.3426	1.0090	1
(.5, .1)	1.7647	.7348	.4120	.2508	.25
(.5, .5)	3.7810	1.5730	.8777	.5024	.5
(.5, 1)	6.3051	2.6337	1.4992	1.0016	1
(1, .1)	3.0278	1.2673	.7274	.5006	.5
(1, .5)	5.0410	2.0957	1.1633	.5351	.5003
(1, 1)	7.5621	3.1461	1.7553	1.0049	1

Clearly, $\mathbf{H}$ is regular with feedthrough matrix

$$D = \begin{pmatrix} k_1/2 & 0 \\ 0 & k_2 \end{pmatrix}.$$

It is not hard to show that for any values of $a \geq 0$, $k_1 \geq 0$, $k_2 \geq 0$, $a + k_1 + k_2 > 0$ the closed-loop transfer function $\mathbf{H}(I + \mathbf{H})^{-1}$ is in $H^\infty(\mathbb{C}^{2\times 2})$. In the case when $a = 0$ this follows also from the fact that the closed-loop semigroup is shown in Liu, Huang, and Chen [22] to be exponentially stable.

There are some values of k_1, k_2, and a where no further computation needs to be done in order to apply the results in the preceeding sections. If $k_1 > 2$ or $k_2 > 1$, the spectral radius of D is greater than 1, so Theorem 5.6 implies that there exists $\varepsilon_n \downarrow 0$ such that $\mathbf{G}^{\varepsilon_n}(s) = \mathbf{H}(s)(I + e^{-\varepsilon_n s}\mathbf{H}(s))^{-1}$ has poles $p_n \in \mathbb{C}_0$ such that the real and imaginary parts of p_n go to infinity as n goes to infinity. Another simple case is when $a = 0$ and $k_1 + k_2 > 0$. In this case $\mathbf{G}^0$ is stable and $\mathbf{H}$ has poles at $s = \pi i(1 + 2n)/4$ for all integers n. Thus we obtain from Lemma 6.3 that $\gamma = \limsup_{|s|\to\infty,\, s\in\mathbb{C}_0} r(\mathbf{H}(s)) = \infty$, and hence, by Theorem 1.1, $\mathbf{G}^0$ is not robustly stable with respect to delays.

In the case where $a > 0$, $0 < k_1 < 2$, $0 < k_2 < 1$ we need to compute γ. First note that γ is the same for $\tilde{\mathbf{H}}(s) := ((s + a)/s)\mathbf{H}(s)$ as it is for $\mathbf{H}(s)$. To compute the spectral radius of $\tilde{\mathbf{H}}(s)$, we need to compute the eigenvalues of $\tilde{\mathbf{H}}(s)$. These are found to be

$$\eta_\pm(k_1, k_2, s, a) = -\frac{k_1 + 2k_2}{4}\frac{1 - e^{4(s+a)}}{1 + e^{4(s+a)}} \pm \frac{1}{4(1 + e^{4(s+a)})}\sqrt{g(k_1, k_2, s, a)}\,,$$

where

$$\begin{aligned} g(k_1, k_2, s, a) = {} & (k_1 - 2k_2)^2 + 16k_1k_2e^{2(s+a)} - 2e^{4(s+a)}(k_1^2 + 12k_1k_2 + 4k_2^2) \\ & + 16k_1k_2e^{6(s+a)} + e^{8(s+a)}(k_1 - 2k_2)^2. \end{aligned}$$

Since $\eta_\pm(k_1, k_2, s + \pi i, a) = \eta_\pm(k_1, k_2, s, a)$ and $\tilde{\mathbf{H}}$ is in $H^\infty(\mathbb{C}^{2\times 2})$, we obtain, using Remark 6.2, that

$$\gamma = \limsup_{|s|\to\infty, s\in\mathbb{C}_0} r(\tilde{\mathbf{H}}(s)) = \sup_{s\in\mathbb{C}_0} r(\tilde{\mathbf{H}}(s)) = \sup_{\omega\in\mathbb{R}} r(\tilde{\mathbf{H}}(i\omega)) = \sup_{0\leq\omega\leq\pi} r(\tilde{\mathbf{H}}(i\omega))\,.$$

Thus, computing γ is a fairly straightfoward numerical problem. Using Mathematica, we obtain Table 1, giving values of γ for some values of k_1, k_2, and a. As we see from the table, the possibility of robustness increases as a increases and decreases as

k_1 and k_2 increase. Note that the last column, with $a = 10$, is almost the same as that obtained by taking the limit of γ as $a \to \infty$, which is easily computed to be $\max\{k_2, k_1/2\}$. Thus, for large values of the viscous damping coefficient a, robustness is determined in a simple way by k_1 and k_2.

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