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PDEs with Distributed Control and Delay in the Loop: Transfer Function Poles, Exponential Modes and Robustness of Stability*

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In this paper we consider a large class of partial differential equations (PDEs) in one space dimension with distributed control and with a time-delay in the feedback loop. We analyse the relationship between the poles of the closed-loop transfer function and the exponential modes of the underlying retarded PDE in order to derive internal stability properties from external ones. Our approach is based on a combination of input–output methods and modal analysis. We give a number of sufficient conditions for robustness/non-robustness of closed-loop modal stability with respect to delays. The theory is illustrated by two examples.

Keywords: Distributed control; Partial differential equations; Robustness; Small time delays; Stability; Transfer functions

1. Introduction

The literature on robustness and lack of robustness of distributed parameter systems with respect to small delays in the feedback loop can be roughly divided into two types. Those papers dealing with individual partial differential equations (PDEs) focus on the existence, or lack of existence, of exponentially growing modes (see for instance Datko [3], Datko et al. [4] and Desch and Wheeler [6]. Those papers considering input–output systems in the frequency domain deal with input–output stability (see for instance Barman et al. [1] and Logemann et al. [11].

In Logemann and Rebarber [10] we examined the relationship between the external concept of *spectral stability* and the internal concept of *modal stability* for a large class of boundary control systems in one space dimension (allowing for in-span control). Spectral stability simply means that the transfer function of the system has no poles in the closed right-half plane, whilst model stability means that any exponential solutions (modes) of the form $e^{st}\phi(x)$ must satisfy Re $s < 0$. It was shown in [10] that the frequency-domain results in [10,11] on robustness/non-robustness of spectral stability translate to results on robustness/non-robustness of modal stability. This paper addresses similar questions, but instead of boundary control systems we consider here systems with distributed control action, where it is more complicated to unravel the relationship between modes and poles. One reason that distributed control is of interest is that for certain problems the control action is more accurately modelled by a function with 'small' support than by a δ-function formulation. The underlying class of PDEs for the control systems under consideration consists of linear PDEs of spatial dimension 1, where on different parts of the space interval different partial differential equations are satisfied. The coefficients may

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depend on the spatial variable. The boundary conditions are general enough to allow all natural coupling conditions.

In order to relate modal stability of the PDE to the spectral stability of the transfer function, a relationship between the modes of a PDE and the poles of the associated transfer function is needed. As in the case of the boundary control systems considered previously in [10], we show in this paper that a pole s_0 of the transfer function leads to a mode with exponent s_0 of the corresponding free dynamics. However, this does not follow in a straightforward way from [10], but requires a different analysis. Of course, as in finite dimensions, it is possible that a mode will not appear as a pole of the associated transfer function, since any possible effect of the mode on the output might be annihilated by the observation and control operators. However, it will be shown that for any system in our class the exponents of the unstable modes coincide with the unstable poles of the transfer function, provided the closed-loop system (without delay) is modally stable. This result will in turn be applied to prove that for a large class of systems robustness/non-robustness of spectral stability implies robustness/non-robustness of modal stability. In Moyer and Rebarber [12] the relationship between robustness of spectral stability and robustness of modal stability was studied for a special class of parabolic systems in more than one space variable.

The paper is organised as follows. In Section 2 we introduce the class of systems under consideration. In Section 3 we show, for the open-loop as well as for the closed-loop system with delay, that the exponents of exponential modes can be characterised as the zeros of certain holomorphic functions. The relationship between the exponents of the system modes and the transfer function poles is analysed in Section 4. In Section 5 we briefly describe the input–output robustness/non-robustness results that we need from [11], and in Corollaries 5.3 and 5.4 we give the related results for robustness/non-robustness of modal stability for the class of systems described in Section 2. Finally, in Section 6 we present two examples illustrating the results in the previous sections.

1.1. Notation and Terminology

For any $\alpha \in \mathbb{R}$, let $\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \text{Re } s > \alpha\}$. We define

$$H_- := \{f : \Omega_f \to \mathbb{C} | \Omega_f \text{ open}, \Omega_f \supset \mathbb{C}_0^{cl} \text{ and } f \text{ holomorphic}\}$$

The field of all meromorphic functions on $\mathbb{C}_\alpha$ is denoted by M_α, while H_α^∞ denotes the algebra of all bounded holomorphic functions defined on $\mathbb{C}_\alpha$. We write H^∞ for H_0^∞. If $f \in M_\alpha$ and $g \in M_\beta$, where $\alpha < \beta$, and if $f(s) = g(s)$ for all $s \in \mathbb{C}_\beta$, then we shall identify f and g.

Let $\Omega \subset \mathbb{C}$. A function $\mathbf{H} : \Omega \to \mathbb{C}$ is called a *transfer function* if $\mathbb{C}_\alpha \subset \Omega$ for some $\alpha \in \mathbb{R}$ and $\mathbf{H}|_{\mathbb{C}_\alpha} \in M_\alpha$. A transfer function $\mathbf{H}$ is called *well-posed* if $\mathbf{H} \in H_\alpha^\infty$ for some $\alpha \in \mathbb{R}$. Moreover, a transfer function $\mathbf{H}$ is called *regular* if it is well-posed and if the limit $\lim_{\xi \to +\infty} \mathbf{H}(\xi) = D$ exists (where $\xi \in \mathbb{R}$). The number D is called the *feedthrough* of $\mathbf{H}$. If $\mathbf{H}$ is not well-posed, we say that it is *ill-posed.*

2. PDEs with Distributed Control and Delay in the Feedback Loop

In the following we introduce a class of controlled and observed linear partial differential equations in one space dimension with coefficients which may depend upon the spatial variable. On different parts of the space interval different PDEs are satisfied. They are coupled via the boundary conditions at the in-span points. The control action is assumed to be distributed.

We suppose that the space variable x belongs to some closed interval $[a,b]$. Without loss of generality we assume that $[a,b] = [0,1]$. Let $\lambda \in \mathbb{N}$, and $\{x_l\}_{l=1}^{\lambda} \subset (0,1)$, where $x_1 < x_2 < \ldots < x_\lambda$. These numbers determine a decomposition of (0,1) into $\lambda + 1$ open intervals $\{I_k\}_{k=0}^{\lambda}$. Let $\iota \in \mathbb{N}$. For $j = 0, \ldots, \iota$ and $k = 0, \ldots, \lambda$, let p_j^k be polynomials, let a_j^k be continuous functions on I_k^{cl}, and let $b_j^k \in L^2(0,1;\mathbb{C})$. Let D_x denote differentiation with respect to x, and consider the controlled PDE

$$\left.\begin{aligned} &\sum_{j=0}^{\iota} a_j^k(x) p_j^k(D_x) \frac{\partial^j w}{\partial t^j}(x,t) = \sum_{j=0}^{\iota} b_j^k(x) \frac{\partial^j u}{\partial t^j}(t), \\ &x \in I_k, t > 0 \end{aligned}\right\} \tag{2.1}$$

Set $n_k := \max_{0 \le j \le \iota} \deg p_j^k$ and let $p_k(x,s)$ be the coefficient of $D_x^{n_k}$ in the expression

$$\sum_{j=0}^{\iota} a_j^k(x) s^j p_j^k(D_x)$$

where s is a complex variable. We introduce the following assumption:

(**A1**) There exists an open set $\Omega \supset \mathbb{C}_0^{cl}$ such that for any $k = 0, \ldots, \lambda$

$$p_k(x,s) \neq 0 \qquad \text{for all x} \in I_k^{cl}, \text{ s} \in \Omega \qquad \square$$

This condition guarantees that when the Laplace transform of the PDE in I_k is taken, the resulting ordinary differential equation is not degenerate.

To define boundary operators for the PDE (2.1), we note that, by (**A1**), the PDE has spatial order n_k in I_k, so we need

$$n := \sum_{k=0}^{\lambda} n_k$$

boundary conditions. While the boundary for $\cup_{k=0}^{\lambda} I_k$ is $\{x_l\}_{l=1}^{\lambda} \cup \{0,1\}$, for the purpose of defining boundary conditions each x_j should be represented by x_j^+ and x_j^-. This allows coupling conditions (for example $D_x w(0.5^-,t) = D_x w(0.5^+,t)$) and in-span control. Therefore, we consider the boundary set to be $\{x_l^-\}_{l=1}^{\lambda} \cup \{x_l^+\}_{l=1}^{\lambda} \cup \{0,1\}$, which we rename as $\{z_l\}_{l=1}^{\mu}$, where $\mu = 2(\lambda + 1)$. For any piecewise continuous function $f: [0,1] \rightarrow \mathbb{C}$ and for $l = 1, \ldots, \lambda$ we define

$$f(x_l^-) := \lim_{x \nearrow x_l} f(x), \qquad f(x_l^+) := \lim_{x \searrow x_l} f(x)$$

so that $f(z_l)$ is a well-defined complex number for all $l = 1, \ldots, \mu$.

Let $q_{l,j}^i$ be polynomials for $i = 1, \ldots, n$, $j = 0, \ldots, \iota$, $l = 1, \ldots, \mu$. We define boundary operators B_i on solutions $w(x,t)$ of (2.1) by

$$(B_i w)(t) = \sum_{l=1}^{\mu} \sum_{j=0}^{\iota} q_{l,j}^i (D_x) \frac{\partial^j}{\partial t^j} w(z_l,t) \tag{2.2}$$

We need to impose bounds on the order of the spatial derivatives in (2.2). In particular, we do not wish to take spatial derivatives at the boundary of I_k which are of an order larger than $n_k - 1$. To this end it is useful to introduce the function κ: $\{1, \ldots, \mu\} \rightarrow \{0, \ldots, \lambda\}$ given by

$$\kappa(l) := \begin{cases} 0 & \text{if } z_l = 0 \\ \lambda & \text{if } z_l = 1 \\ l_0 & \text{if } z_l = x_{l_0}^+ \\ l_0 - 1 & \text{if } z_l = x_{l_0}^- \end{cases}$$

We assume that for any $i = 1, \ldots, n$, $j = 0, \ldots, \iota$, $1 = 1, \ldots, \mu$

$$\deg q_{l,j}^i \leq n_{\kappa(l)} - 1$$

The boundary operators (2.2) are sufficiently general to allow higher-order differential equations at the boundary, as in the 'hybrid systems' in Littman and Markus [9]. We consider the following boundary conditions for the PDE (2.1):

$$(B_i w)(t) = 0, \qquad i = 1, \ldots, n \tag{2.3}$$

The observation operator for the system will be quite general, since we will allow for distributed and point observation. For $j = 0, \ldots, \iota$, $l = 1, \ldots, \mu$, let $r_{l,j}$ be polynomials, let $f_j \in L^2(0,1; \mathbb{C})$, and define the observation operator by

$$(Cw)(t) = \sum_{l=1}^{\mu} \sum_{j=0}^{\iota} r_{l,j} (D_x) \frac{\partial^j}{\partial t^j} w(z_l,t) + \sum_{j=0}^{\iota} \int_0^1 f_j(x) \frac{\partial^j}{\partial t^j} w(x,t)\, dx$$

where we assume that

$$\deg r_{l,j} \leq n_{\kappa(l)} - 1$$

The observation $y(t)$ is then given by

$$y(t) = (Cw)(t) \tag{2.4}$$

We refer to the observed distributed control system given by (2.1), (2.3) and (2.4) as the *open-loop system.* In the following it will be denoted by (**OLS**).

Application of output feedback of the form $u(t) = v(t) - y(t - \varepsilon)$ leads to

$$\sum_{j=0}^{\iota} a_j^k(x) p_j^k (D_x) \frac{\partial^j w}{\partial t^j} (x,t) = \sum_{j=0}^{\iota} b_j^k(x) \frac{\partial^j v}{\partial t^j}(t) - \sum_{j=0}^{\iota} b_j^k(x) \frac{\partial^j}{\partial t^j} (Cw)(t - \varepsilon), \qquad x \in I_k,\ t > 0 \tag{2.5}$$

where $\varepsilon \geq 0$ is a time-delay and $v(t)$ denotes the input of the feedback system. We refer to the system given by (2.5), (2.3) and (2.4) as the *closed-loop system with delay* ε. In the following it will be denoted by (**CLS**$_\varepsilon$). If $\varepsilon = 0$, then we will call (**CLS**$_0$) the *undelayed closed-loop system.* If $u(t) \equiv 0$ (resp. $v(t) \equiv 0$), then we refer to (**OLS**) (resp. (**CLS**$_\varepsilon$)) as the *uncontrolled* open-loop system (resp. *uncontrolled* closed-loop system with delay ε).

3. Existence of Exponential Modes

We will be looking for exponential solutions of the form

$$\left. \begin{aligned} & w(x,t) = e^{st}\phi(x), \qquad \text{where } s \in \mathbb{C}, \\ & \phi \in L^2(0,1);\mathbb{C}),\ \phi \not\equiv 0 \end{aligned} \right\} \tag{3.1}$$

for the uncontrolled open-loop and closed-loop systems. Such a solution will be called a *mode* of (**OLS**) (resp. (**CLS**$_\varepsilon$)). The complex number s is called the *exponent* of the mode.

Suppose that (3.1) is a solution of the uncontrolled system (**OLS**). Since ϕ will in general depend on s, we write $\phi(x) = \phi(x,s)$. Clearly, $\phi(x,s)$ satisfies

$$\left(\sum_{j=0}^{\iota} a_j^k(x)s^j p_j^k(D_x)\right)\phi(x,s) = 0, \qquad x \in I_k \tag{3.2}$$

Using assumption (**A1**), we see that for every $s \in \Omega \supset \mathbb{C}_0^{cl}$ and every $k = 0, \ldots, \lambda$, (3.2) is an ordinary differential equation of order n_k on I_k^{cl}. Let $\{e_j^k\}_{j=1}^{n_k}$ be a basis of $\mathbb{C}^{n_k}$ and let $\{\phi_j^k(\cdot,s)\}_{j=1}^{n_k}$ be solutions of (3.2) on I_k^{cl} satisfying

$$\left(\phi_j^k(x_k,s),\ D_x\phi_j^k(x_k,s), \ldots, D_x^{n_k-1}\phi_j^k(x_k,s)\right) = e_j^k,$$
$$j = 1, \ldots, n_k;\ s \in \Omega$$

where $x_0 := 0$. Clearly, for any $k = 0, \ldots, \lambda$ and any $s \in \Omega$, the functions $\phi_j^k(\cdot,s)$ are linearly independent, and hence span the solution space of (3.2) on I_k^{cl}. In particular, for every $s \in \Omega$, $\phi_j^k(\cdot,s) \in L^2(I_k^{cl},\mathbb{C})$, and every solution of (3.2) in I_k can be written in the form

$$\sum_{j=1}^{n_k} A_j^k(s)\phi_j^k(x,s), \qquad x \in I_k$$

for some coefficients $A_j^k(s)$. For $k = 0, \ldots, \lambda$ and $j = 1, \ldots, n_k$, let us define

$$\Phi_j^k(x,s) := \begin{cases} \phi_j^k(x,s) & \text{for } x \in I_k^{cl} \\ 0 & \text{for } x \in [0,1] \setminus I_k^{cl} \end{cases}$$

It will be convenient to rename these n vectors as $\{\Phi_p(x,s)\}_{p=1}^n$. It is clear that any solution of (3.2) can be written as

$$\phi(x,s) = \sum_{p=1}^{n} A_p(s)\Phi_p(x,s)\,,\quad x \in \bigcup_{k=0}^{\lambda} I_k\,,$$

$$\phi(z_l,s) = \sum_{p=1}^{n} A_p(s)\Phi_p(z_l,s),\quad l = 1, \ldots \mu$$

for some coefficients $A_p(s)$.

Setting

$$(\hat{B}_i\Phi_p)(s) = \sum_{l=1}^{\mu}\sum_{j=0}^{\iota} s^j q_{l,j}^i(D_x)\Phi_p(z_l,s) \tag{3.3}$$

we introduce the $n \times n$ matrix

$$\mathbf{\Delta}(s) := \left((\hat{B}_i\Phi_p)(s)\right), \qquad i,p = 1, \ldots, n \tag{3.4}$$

By Lemma 4.1 in [10], $\Delta(s)$ is holomorphic on Ω. Thus the following result can be obtained in the same way as proposition 4.2 in [10].

Proposition 3.1. Suppose that (**A1**) is satisfied and let $s_0 \in \Omega$. Then (**OLS**) has a mode with exponent s_0 if and only if $\det \mathbf{\Delta}(s_0) = 0$.

In order to prove a similar result for the closed-loop system (**CLS**$_\varepsilon$), suppose that

$$w(x,t) = e^{st}\phi(x), \qquad \text{where}$$
$$\phi \in L^2(0,1;\mathbb{C}),\ \phi \not\equiv 0$$

is a solution of the uncontrolled system (**CLS**$_\varepsilon$). Again we will indicate the dependence of ϕ on s explicitly by writing $\phi(x) = \phi(x,s)$. We then obtain from (2.5) with $v(t) \equiv 0$

$$\left(\sum_{j=0}^{\iota} a_j^k(x)s^j p_j^k(D_x)\right)\phi(x,s)$$
$$= -e^{-\varepsilon s}b^k(x,s)(\hat{C}\phi)(s), \qquad x \in I_k \tag{3.5}$$

where

$$b^k(x,s) := \sum_{j=0}^{\iota} b_j^k(x)s^j$$

and $(\hat{C}\phi)(s)$ is defined by

$$(\hat{C}\phi)(s) := \sum_{l=1}^{\mu}\sum_{j=0}^{\iota} s^j r_{l,j}(D_x)\phi(z_l,s)$$
$$+ \sum_{j=0}^{\iota} s^j \int_0^1 f_j(x)\phi(x,s)dx \tag{3.6}$$

For each $s \in \Omega$, let $\mathbf{\Psi}(\cdot,s)$ be the unique function satisfying

$$\left(\sum_{j=0}^{\iota} a_j^k(x)s^j p_j^k(Dx)\right)\Psi(x,s) = b^k(x,s), \qquad x \in I_k,$$
$$D_x^j\Psi(x_k,s) = 0, \qquad j = 0, \ldots, n_k - 1 \tag{3.7}$$

Invoking (3.7) and using the fact that $\Sigma_{p=1}^n A_p(s)\Phi_p(\cdot,s)$ is the general solution of the homogeneous equation (3.2), the solution $\phi(x,s)$ of (3.5) can be written in the form

$$\phi(x,s) = \sum_{p=1}^{n} A_p(s)\Phi_p(x,s) + a(s)\Psi(x,s) \tag{3.8}$$

where $a(s) = -e^{-\varepsilon s}(\hat{C}\phi)(s)$ and the $A_p(s) \in \mathbb{C}$ are suitable coefficients. The boundary conditions (2.3) become

$$\left.\begin{array}{l}\sum_{p=1}^{n} A_p(s)(\hat{B}_i\Phi_p)(s) + a(s)(\hat{B}_i\Psi)(s) = 0,\\ \qquad i = 1, \ldots, n\end{array}\right\} \quad (3.9)$$

where $(\hat{B}_i\Psi)(s)$ is defined as in (3.3) with Φ_p replaced by Ψ. Plugging (3.8) into (3.5), noting that Φ_p satisfies (3.2) and using (3.7), we obtain

$$b^k(x,s)\left(\sum_{p=1}^{n} A_p(s)(\hat{C}\Phi_p)(s) + a(s)\left((\hat{C}\Psi)(s) + e^{\varepsilon s}\right)\right) = 0,\ x \in I_k, s \in \Omega \quad (3.10)$$

where $(\hat{C}\Phi_p)$ (s) and $(\hat{C}\Psi)$ (s) are defined as in (3.6) with ϕ replaced by Φ_p and Ψ, respectively. Let

$$\mathbf{c} := (\hat{C}\Phi_1, \ldots, \hat{C}\Phi_n)^T,$$
$$\mathbf{b} := -(\hat{B}_1\Psi, \ldots, \hat{B}_n\Psi)^T$$

and

$$\mathbf{\Delta}_\varepsilon(s) := \begin{pmatrix} \mathbf{\Delta}(s) & -\mathbf{b}(s) \\ \mathbf{c}^T(s) & (\hat{C}\Psi)(s) + e^{\varepsilon s} \end{pmatrix} \quad (3.11)$$

where $\mathbf{\Delta}$ is given by (3.4). We are now in the position to prove the following result.

Proposition 3.2. Suppose that (**A1**) is satisfied and let $s_0 \in \Omega$. Then ($\mathbf{CLS}_\varepsilon$) has a mode with exponent s_0 if and only if $\det\mathbf{\Delta}_\varepsilon(s_0) = 0$.

Proof. Suppose that ($\mathbf{CLS}_\varepsilon$) has a mode with exponent s_0. Then using the above notation, $e^{s_0 t}\phi(x,s_0)$ is a solution of ($\mathbf{CLS}_\varepsilon$), where $\phi(\cdot,s_0) \not\equiv 0$.

Case 1. Assume that there exists $k_0 \in \{0, \ldots, \lambda\}$ and $\xi_0 \in I_{k_0}$ such that $b^{k_0}(\xi_0, s_0) \neq 0$. Then, by (3.10)

$$\sum_{p=1}^{n} A_p(s_0)\,(\hat{C}\Phi_p)\,(s_0) + a(s_0)\left((\hat{C}\Psi)(s_0) + e^{\varepsilon s_0}\right) = 0 \quad (3.12)$$

Setting $A(s) := (A_1(s), \ldots, A_n(s))^T$ and $\tilde{A}(s) := (A^T(s), a(s))^T$, then a combination of (3.9) and (3.12) yields that $\mathbf{\Delta}_\varepsilon(s_0)\tilde{A}(s_0) = 0$. Since $\phi(\cdot,s_0) \not\equiv 0$, it follows from (3.8) that $\tilde{A}(s_0) \neq 0$ and hence that $\det\mathbf{\Delta}_\varepsilon(s_0) = 0$.

Case 2. Now assume that $b^k(x,s_0) = 0$ for all $x \in I_k$ and for all $k \in \{0, \ldots, \lambda\}$. Then $\Psi(x,s_0) = 0$ for all $x \in [0,1]$ and hence $\mathbf{b}(s_0) = 0$. Moreover, by (3.9)

$$\sum_{p=1}^{n} A_p(s_0)(\hat{B}_i\Phi_p)(s_0) = 0, \qquad i = 1, \ldots, n \quad (3.13)$$

Since $\Psi(\cdot,s_0) \equiv 0$ and $\phi(\cdot,s_0) \not\equiv 0$, (3.8) yields that $A(s_0) = (A_1(s_0), \ldots, A_n(s_0))^T \neq 0$. So, combining (3.13) with (3.4), we see that $\det\mathbf{\Delta}(s_0) = 0$. Now

$$\mathbf{\Delta}_\varepsilon(s_0) = \begin{pmatrix} \mathbf{\Delta}(s_0) & 0 \\ \mathbf{c}^T(s_0) & e^{\varepsilon s_0} \end{pmatrix}$$

and thus $\det\mathbf{\Delta}_\varepsilon(s_0) = e^{\varepsilon s_0}\det\mathbf{\Delta}(s_0) = 0$.

Conversely, suppose that $\det\mathbf{\Delta}_\varepsilon(s_0) = 0$ for some $s_0 \in \Omega$. Then there exists $\tilde{A}(s_0) = (A^T(s_0), a(s_0))^T \in \mathbb{C}^{n+1}$, $\tilde{A}(s_0) \neq 0$, such that $\mathbf{\Delta}_\varepsilon(s_0)\tilde{A}(s_0) = 0$. Writing $A(s_0) = (A_1(s_0), \ldots, A_n(s_0))^T$ and defining $\phi(x,s_0)$ by (3.8), we obtain

$$\begin{aligned} -e^{-\varepsilon s_0}&(\hat{C}\phi)(s_0) \\ &= -e^{-\varepsilon s_0}[\mathbf{c}^T(s_0)A(s_0) + a(s_0)(\hat{C}\Psi)(s_0)] \\ &= -e^{-\varepsilon s_0}[-e^{\varepsilon s_0}a(s_0)] \\ &= a(s_0) \end{aligned} \quad (3.14)$$

where we used the fact that $\mathbf{\Delta}_\varepsilon(s_0)\tilde{A}(s_0) = 0$ implies that

$$\mathbf{c}^T(s_0)A(s_0) + a(s_0)\left((\hat{C}\Psi)(s_0) + e^{\varepsilon s_0}\right) = 0$$

Now

$$\left(\sum_{j=0}^{\iota} a_j^k(x)s_0^j p_j^k(D_x)\right)\phi(x,s_0) = a(s_0)b^k(x,s_0),\quad x \in I_k$$

and so, using (3.14) it follows that $\phi(x,s_0)$ satisfies (3.5) for $s = s_0$. Finally, since $\mathbf{\Delta}_\varepsilon(s_0)\tilde{A}(s_0) = 0$, it is clear that (3.9) is satisfied for $s = s_0$. Thus the function $w(x,t) := e^{s_0 t}\phi(x,s_0)$ is a solution of ($\mathbf{CLS}_\varepsilon$), and hence ($\mathbf{CLS}_\varepsilon$) has a mode with exponent s_0. □

4. Transfer Function Poles and Exponential Modes

We shall need the following assumption:

(**A2**) $\qquad \det\mathbf{\Delta}(s) \not\equiv 0$ □

For a given $s_0 \in \Omega$, let $\mathscr{E}_{s_0}$ denote the vector space of all functions ψ: $(\cup_{k=0}^{\lambda} I_k) \times [0,\infty) \to \mathbb{C}$ of the form $\psi(x,t) = e^{s_0 t}\tilde{\psi}(x)$, where $\tilde{\psi}$: $\cup_{k=0}^{\lambda} I_k \to \mathbb{C}$ solves the ordinary differential equation (3.2) for $s = s_0$. It is easy to see that assumption (**A2**) is equivalent to (**A2′**). There exists $s_0 \in \Omega$ such that the restricted boundary operators $B_i|_{\mathscr{E}_{s_0}}$ are linearly independent, i.e. if $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are such that $\sum_{i=1}^{n} \alpha_i B_i\phi = 0$ for all $\phi \in \mathscr{E}_{s_0}$, than $\alpha_i = 0$ for all $i = 1, \ldots, n$.

If (**A2**) is satisfied, then the function

$$\mathbf{H}(s) := \mathbf{c}^T(s)\mathbf{\Delta}^{-1}(s)\mathbf{b}(s) + (\hat{C}\Psi)(s) \tag{4.1}$$

is well-defined. While the functions $\mathbf{c}(s)$ and $\Delta(s)$ depend on the choice of the basis $\{\phi_j^k\}_{j=1}^{n_k}$ of the solution space of (3.2) on I_k^{cl}, it is easy to prove that the product of $\mathbf{c}^T(s)\mathbf{\Delta}^{-1}(s)$ does not. If (**A1**) is satisfied, then it follows as in the proof of Lemma 4.1 in [10] that $\hat{C}\Psi \in H_-$ and that $\mathbf{c}$ and $\mathbf{b}$ have all their entries in H_-. Moreover, we know that the entries of $\mathbf{\Delta}$ are in H_-. As a consequence, $\mathbf{H}$ is a transfer function in the sense of Section 1 (see Section 1.1), and we say that $\mathbf{H}$ is the transfer function of (**OLS**). Setting

$$\left.\begin{aligned} \mathbf{D} &:= \begin{pmatrix} \mathbf{\Delta} & 0_{n\times 1} \\ 0_{1\times n} & 1 \end{pmatrix}, \quad \tilde{\mathbf{N}} := \begin{pmatrix} b \\ 1 \end{pmatrix}, \\ \mathbf{N} &:= (\mathbf{c}^T, \hat{C}\Psi) \end{aligned}\right\} \tag{4.2}$$

we can write

$$\mathbf{H}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)\tilde{\mathbf{N}}(s) \tag{4.3}$$

In order to show that $\mathbf{H}$ admits the usual dynamical interpretation, let $u(\cdot)$ be a sufficiently smooth Laplace transformable input function with zero initial conditions, i.e. $(d^ju/dt^j)(0) = 0$ for all $j = 0, \ldots, \iota - 1$. Denoting the corresponding solution of (**OLS**) with zero initial conditions by $w(x,t;u)$, we see that $\hat{w}(x,s;u)$ satisfies the ordinary differential equation

$$\left(\sum_{j=0}^{\iota} a_j^k(x)s^j p_j^k(D_x)\right)\phi(x,s) = b^k(x,s)\hat{u}(s), \qquad x \in I_k$$

and hence $\hat{w}(x,s;u)$ can be written in the form

$$\hat{w}(x,s;u) = \sum_{p=1}^{n} A_p(s)\Phi_p(x,s) + \hat{u}(s)\Psi(x,s) \tag{4.4}$$

The boundary conditions (2.3) then lead to

$$\sum_{p=1}^{n} A_p(s)(\hat{B}_i\Phi_p)(s) + \hat{u}(s)(\hat{B}_i\Psi)(s) = 0, \qquad i = 1, \ldots, n$$

This is equivalent to

$$\mathbf{\Delta}(s)A(s) = \mathbf{b}(s)\hat{u}(s) \tag{4.5}$$

Where $A(s) = (A_1(s), \ldots, A_n(s))^T$. Using (4.4), the Laplace transform of the observation (2.4) can be expressed as follows:

$$\hat{y}(s) = \mathbf{c}^T(s)A(s) + (\hat{C}\Psi)(s)\hat{u}(s)$$

Therefore, using (4.5) and the fact that $\det\mathbf{\Delta}(s) \not\equiv 0$ by (**A2**), we obtain

$$\hat{y}(s) = (\mathbf{c}^T\Delta^{-1}(s)\mathbf{b}(s) + (\hat{C}\Psi)(s))\hat{u}(s) = \mathbf{H}(s)\hat{u}(s)$$

Using (4.3) and Lemma 2.4 in [10], the transfer function $\mathbf{G}_\varepsilon(s)$ of ($\mathbf{CLS}_\varepsilon$) is given by

$$\begin{aligned} \mathbf{G}_\varepsilon(s) &= \mathbf{H}(s)(1 + e^{-\varepsilon s}\mathbf{H}(s))^{-1} \\ &= \mathbf{N}(s)(\mathbf{D}(s) + e^{\varepsilon s}\tilde{\mathbf{N}}(s)\mathbf{N}(s))^{-1}\tilde{\mathbf{N}}(s) \end{aligned} \tag{4.6}$$

In contrast to the results for boundary control systems given in [10], it is not true that $\mathbf{\Delta}_\varepsilon(s) = \mathbf{D}(s) + e^{-\varepsilon s}\tilde{\mathbf{N}}(s)\mathbf{N}(s)$. However, we can show the following.

Theorem 4.1. Suppose that (**A1**) holds. Then the zeros of the functions $\det\mathbf{\Delta}_\varepsilon(s)$ and $\det(\mathbf{D}(s) + e^{-\varepsilon s}\mathbf{N}(s)\mathbf{N}(s))$ in Ω coincide (counting multiplicities).

Proof. Using (4.2) we obtain that

$$\begin{aligned} &\mathbf{D}(s) + e^{-\varepsilon s}\tilde{\mathbf{N}}(s)\mathbf{N}(s) \\ &= \begin{pmatrix} \Delta(s) + e^{-\varepsilon s}\mathbf{b}(s)\mathbf{c}^T(s) & e^{-\varepsilon s}(\hat{C}\Psi)(s)\mathbf{b}(s) \\ e^{-\varepsilon s}\mathbf{c}^T(s) & 1 + e^{-\varepsilon s}(\hat{C}\Psi)(s) \end{pmatrix} \end{aligned} \tag{4.7}$$

For any matrices M_1, M_2, M_3, M_4, where M_1 is square and M_4 is square and invertible,

$$\begin{aligned} &\det\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \\ &= \det M_4 \det(M_1 - M_2M_4^{-1}M_3) \end{aligned} \tag{4.8}$$

see, for example, Gantmacher [7], p. 46. Setting $h_\varepsilon(s) := 1 + e^{-\varepsilon s}(\hat{C}\Psi)(s)$ we distinguish between two cases.

Case 1. If $h_\varepsilon(s) \not\equiv 0$, then an application of (4.7) and (4.8) yields, after simplication

$$\begin{aligned} \det(\mathbf{D}(s) + e^{-\varepsilon s}\tilde{\mathbf{N}}(s)\mathbf{N}(s)) &= \frac{1}{h_\varepsilon^{n-1}(s)} \\ &\det(h_\varepsilon(s)\Delta(s) + e^{-\varepsilon s}\mathbf{b}(s)\mathbf{c}^T(s)) \end{aligned} \tag{4.9}$$

Combining (3.11) and (4.8) we obtain

$$\begin{aligned} \det\Delta_\varepsilon(s) &= ((\hat{C}\Psi)(s) + e^{\varepsilon s})\det\left(\mathbf{\Delta}(s) + \frac{1}{(\hat{C}\Psi)(s) + e^{\varepsilon s}}\mathbf{b}(s)\mathbf{c}^T(s)\right) \\ &= e^{\varepsilon s}h_\varepsilon(s)\det\left(\Delta(s) + \frac{e^{-\varepsilon s}}{h_\varepsilon(s)}\mathbf{b}(s)\mathbf{c}^T(s)\right) \end{aligned} \tag{4.10}$$

Comparing (4.9) and (4.10) shows that

$$\det\mathbf{\Delta}_\varepsilon(s) = e^{\varepsilon s}\det\left(\mathbf{D}(s) + e^{-\varepsilon s}\tilde{\mathbf{N}}(s)\mathbf{N}(s)\right) \tag{4.11}$$

Case 2. If $h_\varepsilon(s) \equiv 0$, then it follows that

$$(\hat{C}\Psi)(s) = -e^{\varepsilon s} \tag{4.12}$$

Let $\varepsilon_n > 0$ be such that $\lim_{n\to\infty} \varepsilon_n = \varepsilon$ and $\varepsilon_n \neq \varepsilon$

for all $n \in \mathbb{N}$. Clearly, by (4.12), $h_{\varepsilon_n}(s) \not\equiv 0$ for all $n \in \mathbb{N}$, and thus, by Case 1

$$\det \mathbf{\Delta}_{\varepsilon_n}(s) = e^{\varepsilon_n s} \det (\mathbf{D}(s) + e^{-\varepsilon_n s}\tilde{\mathbf{N}}(s)\mathbf{N}(s))$$

Letting $n \to \infty$ we obtain by continuity that (4.11) holds true. □

5. Modal Stability and Small Delays: Robustness Issues

Let $\mathbf{H}$ be a transfer function and consider the feedback system shown in Fig. 1, where u is the input function, y is the output function and the block with transfer function $e^{-\varepsilon s}$ represents a delay of length $\varepsilon \geq 0$.

If $1 + e^{-\varepsilon s}\mathbf{H}(s) \not\equiv 0$, then the function $\mathbf{G}_\varepsilon$ defined by

$$\mathbf{G}_\varepsilon(s) = \frac{\mathbf{H}(s)}{1 + e^{-\varepsilon s}\mathbf{H}(s)}$$

is a transfer function, the so-called closed-loop transfer function of the feedback system shown in Fig. 1. We say that $\mathbf{G}_\varepsilon$ is *L^2-stable* if $\mathbf{G}_\varepsilon \in H^\infty$. If $\mathbf{G}_\varepsilon \in H_-$, then $\mathbf{G}_\varepsilon$ is called *spectrally stable*.

For a transfer function $\mathbf{H}$, let $\mathfrak{p}_{\mathbf{H}}$ denote the set of its poles. We define

$$\gamma(\mathbf{H}) := \limsup_{|s| \to \infty, s \in \mathbb{C}_0 \setminus \mathfrak{p}_{\mathbf{H}}} |\mathbf{H}(s)|$$

The following destabilisation result for regular transfer functions was proved by Logemann et al. [11].

Theorem 5.1. Let $\mathbf{H}$ be a transfer function and suppose that $\mathbf{H}$ is regular with feedthrough D. Then the following statements hold true:

(i) If $\mathbf{G}_0$ is L^2-stable and if $\gamma(\mathbf{H}) > 1$, then there exist sequences (ε_n) and (p_n) with

$$\varepsilon_n > 0, \lim_{n\to\infty} \varepsilon_n = 0, p_n \in \mathbb{C}_0, \lim_{n\to\infty} |\text{Im}\, p_n| = \infty$$

and such that for any $n \in \mathbb{N}$, p_n is a pole of $\mathbf{G}_{\varepsilon_n}$.

(ii) If $|D| > 1$, then there exist sequences (ε_n) and (p_n) with

$$\varepsilon_n > 0, \lim_{n\to\infty} \varepsilon_n = 0, \quad p_n \in \mathbb{C}_0,$$
$$\lim_{n\to\infty} \text{Re}\, p_n = \infty, \quad \lim_{n\to\infty} |\text{Im}\, p_n| = \infty$$

and such that for any $n \in \mathbb{N}$, p_n is a pole of $\mathbf{G}_{\varepsilon_n}$.

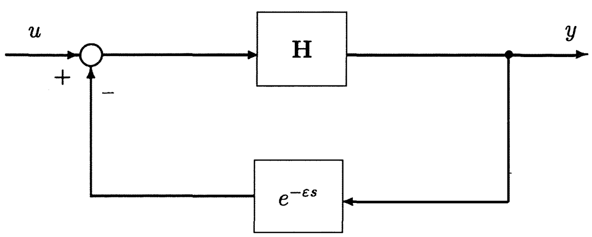

Fig. 1. Feedback system with delay.

There are many partial differential equation models of physical and technical systems which have transfer functions which are not well-posed and hence, in particular, are not regular. In [10], a result similar to part (ii) of Theorem 5.1 has been proved for a large class of ill-posed transfer functions, and applications to boundary control systems were given in the same paper. Roughly speaking, since distributed control is bounded in a state-space sense, systems with distributed control have regular transfer functions, unless the observation is highly unbounded in a way that is not typically seen in the literature on control of PDEs. For that reason, and since the two examples given in Section 6 have regular transfer functions, we have restricted our attention in Theorem 5.1 to the regular case.

The following robustness result complements Theorem 5.1.

Theorem 5.2. Let $\mathbf{H}$ be a transfer function and suppose that $\mathbf{G}_0$ is L^2-stable (spectrally stable). If $\gamma(\mathbf{H}) < 1$, then there exists $\varepsilon^* > 0$ such that $\mathbf{G}_\varepsilon$ is L^2-stable (spectrally stable) for all $\varepsilon \in (0, \varepsilon^*)$.

The part of the theorem which relates to L^2-stability is proved in [11], Theorem 6.1. Robustness of spectral stability can be shown in a similar way and is therefore left to the reader.

In the following we shall apply the above results to the system ($\mathbf{CLS}_\varepsilon$). We call a mode $w(x,t) = e^{st}\phi(x)$ of ($\mathbf{CLS}_\varepsilon$) *stable* if Re $s < 0$, otherwise the mode is called *unstable*. We say that the system $(\mathbf{CLS})_\varepsilon$) is *modally stable* if all the modes of ($\mathbf{CLS}_\varepsilon$) are stable. This is the kind of internal stability which is considered in most of the literature on robust stabilisation of PDEs [3–6,8].

Combining Proposition 3.2, Theorem 4.1, Theorem 5.1 and Eq. (4.6), we obtain the following corollary.

Corollary 5.3. Assume that (**A1**) and (**A2**) hold and suppose that $\mathbf{H}$ given by (4.1) (or equivalently, by (4.3)) is regular. Then the following statements hold true:

(i) If $\mathbf{G}_0$ is L^2-stable and if $\gamma(\mathbf{H}) > 1$, then there exist sequences (ε_n) and (s_n) with

$$\varepsilon_n > 0, \lim_{n\to\infty} \varepsilon_n = 0, s_n \in \mathbb{C}_0,$$

$$\lim_{n\to\infty} |\mathrm{Im}\, s_n| = \infty$$

and such that for any $n \in \mathbb{N}$, $\det \mathbf{\Delta}_{\varepsilon_n}(s_n) = 0$, i.e. for any $n \in \mathbb{N}$ the closed-loop system $(\mathbf{CLS}_{\varepsilon_n})$ has a mode with exponent s_n.

(ii) If $|D| > 1$, then there exist sequences (ε_n) and (s_n) with

$$\varepsilon_n > 0, \lim_{n\to\infty} \varepsilon_n = 0, s_n \in \mathbb{C}_0,$$

$$\lim_{n\to\infty} \mathrm{Re}\ s_n = \infty, \lim_{n\to\infty} |\mathrm{Im} s_n| = \infty$$

and such that for any $n \in \mathbb{N}$, $\det \mathbf{\Delta}_{\varepsilon_n}(s_n) = 0$, i.e. for any $n \in \mathbb{N}$ the closed-loop system $(\mathbf{CLS}_{\varepsilon_n})$ has a mode with exponent s_n.

Turning now to robustness of modal stability, we note that if there exists $\varepsilon^* > 0$ such that $\mathbf{G}_\varepsilon$ is spectrally stable for all $\varepsilon \in [0,\varepsilon^*)$, we cannot immediately conclude modal stability of $(\mathbf{CLS}_\varepsilon)$ for all such ε. However, the following result shows that modal stability can be obtained by using Theorem 5.2 and results from [10].

Corollary 5.4. Assume that (**A1**) and (**A2**) hold and suppose that $(\mathbf{CLS}_0)$ is modally stable. If $\mathbf{H}(s)$ given by (4.1) (or equivalently, by (4.3)) satisfies $\gamma(\mathbf{H}) < 1$, then there exists $\varepsilon^* > 0$ such that $(\mathbf{CLS}_\varepsilon)$ is modally stable for all $\varepsilon \in (0,\varepsilon^*)$.

Proof. Since $(\mathbf{CLS}_0)$ is modally stable, it follows from Theorem 4.1 that

$$\det(\mathbf{D}(s) + \tilde{\mathbf{N}}(s)\mathbf{N}(s)) \neq 0 \qquad \text{for all } s \in \mathbb{C}_0^{cl} \tag{5.1}$$

Thus $\mathbf{G}_0$ is spectrally stable by (4.6). An application of Theorem 5.2 shows that there exists $\varepsilon^* > 0$ such that

$$\mathbf{G}_\varepsilon \text{ is spectrally stable for all } \varepsilon \in (0,\varepsilon^*) \tag{5.2}$$

Moreover, using (5.1), we obtain from Lemma 2.4 in [10] that the triple $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ is bi-coprime (in the sense of [10]). Setting $\mathbf{D}_\varepsilon(s) = \mathbf{D}(s) + e^{-\varepsilon s}\tilde{\mathbf{N}}(s)\mathbf{N}(s)$, it follows from the same result that the triple $(\mathbf{N}, \mathbf{D}_\varepsilon, \tilde{\mathbf{N}})$ is bi-coprime. Combining (5.2) with proposition 2.3 in [10] shows that $\det \mathbf{D}_\varepsilon(s) \neq 0$ for all $s \in \mathbb{C}_0^{cl}$ and hence, by Theorem 4.1, $(\mathbf{CLS}_\varepsilon)$ is modally stable for all $\varepsilon \in (0,\varepsilon^*)$. □

6. Examples

In this section we illustrate our results by two simple examples. The first is an example of robustness of modal stability, while the second is an example of lack of robustness of modal stability. In both cases we specialise to distributed control action approximating point or boundary control.

Example 1

In this example we study the heat equation in a rod made of two different materials, with homogeneous Neumann boundary conditions, localised distributed control input, and 'co-located' output. We show that an application of unity output feedback yields a modally stable closed-loop system which remains stable in the presence of small delay in the feedback loop.

Let $\chi_{[x_1,x_2]}$ represent the characteristic function for the interval $[x_1,x_2]$. Consider the system given by

$$w_t(x,t) = a_1 w_{xx}(x,t) + b(x)u(t), \quad a_1 > 0, x \in (0,\xi), t > 0 \tag{6.1}$$

$$w_t(x,t) = a_2 w_{xx}(x,t) + b(x)u(t), \quad a_2 > 0, x \in (\xi,1), t > 0 \tag{6.2}$$

$$w_x(0,t) = w_x(1,t) = 0, \quad w(\xi^-,t) = w(\xi^+,t),$$
$$a_1 w_x(\xi^-,t) = a_2 w_x(\xi^+,t) \tag{6.3}$$

where $\xi \in (0,1)$ and

$$b = \frac{1}{\delta}\chi_{[x_1,x_2]}, \qquad 0 \le x_1 < x_2 \le 1, \ \delta = x_2 - x_1 \tag{6.4}$$

The last condition in (6.3) arises because the flux coming into $x = \xi$ should be equal to the flux leaving $x = \xi$. If x_1 is close to x_2, then the corresponding distributed control is an approximation of boundary or in-span point control, and is perhaps more accurate than the more typical δ-function formulation for such control. We introduce the observation

$$y(t) = \frac{k}{\delta}\int_0^1 b(x)w(x,t)dx, \qquad k > 0 \tag{6.5}$$

It is easy to show that the above system satisfies the assumptions (**A1**) and (**A2**).

We shall show that the feedback $u(t) = -y(t)$ is exponentially stabilising, so, in particular, the corresponding closed-loop system is modally stable. Using the results in Section 5 we shall then prove that modal stability is robust with respect to small delays in the loop. To this end, it will be easiest to analyse (6.1)–(6.5) in a state-space setting. Let $X = L^2(0,1)$ and let $\|\cdot\|$ denote the usual L^2-norm. Define

$$(\mathcal{A}z)(x) = \begin{cases} a_1 z_{xx}(x) & \text{for } x \in (0,\xi) \\ a_2 z_{xx}(x) & \text{for } x \in (\xi,1) \end{cases}$$

with

$$\begin{aligned}\operatorname{dom}(\mathcal{A}) = \{z \mid z|_{(0,\xi)} \in H^2(0,\xi), z|_{(\xi,1)} \in H^2(\xi,1),\\ z_x(0) = z_x(1) = 0, z(\xi^-) = (z\xi^+), a_1 z_x(\xi^-) = a_2 z_x(\xi^+)\}\end{aligned}$$

It is easy to check that $\mathcal{A}$ is self-adjoint,

$$\langle \mathcal{A}z,z\rangle \le 0 \tag{6.6}$$

for all $z \in \operatorname{dom}(\mathcal{A})$, and that $\mathcal{A}$ has compact resolvent. It follows in particular that $\mathcal{A}$ generates a C_0-semigroup. It is also clear that 0 is an eigenvalue of $\mathcal{A}$ with normalised eigenvector $e(x) \equiv 1$.

Let $\mathcal{B} : \mathbb{R} \to X$ be given by $\mathcal{B}u = b(\cdot)u$ and let $z(t) = w(\cdot,t)$. The system (6.1)–(6.5) is equivalent to the following abstract system with state space X:

$$\dot z(t) = \mathcal{A}z(t) + \mathcal{B}u(t) \tag{6.7}$$

$$y(t) = k\mathcal{B}^*z(t) \tag{6.8}$$

The transfer function can be written as

$$\mathbf{H}(s) = k\mathcal{B}^*(sI - \mathcal{A})^{-1}\mathcal{B}$$

Since $b \in X$, $\mathcal{B}$ is a bounded operator into X, and hence it is easy to verify that $\mathbf{H}(s)$ is analytic in $\mathbb{C}_0^{cl} \setminus \{0\}$ and regular, and that

$$\gamma(\mathbf{H}) = 0 \tag{6.9}$$

If $u(t) = -y(t)$, (6.7) becomes

$$\dot z(t) = (\mathcal{A} - k\mathcal{B}\mathcal{B}^*)z(t)$$

We show that the C_0-semigroup generated by $\mathcal{A} - k\mathcal{B}\mathcal{B}^*$ is exponentially stable. To this end, note that $\mathcal{A} - k\mathcal{B}\mathcal{B}^*$ is self-adjoint and

$$\langle(\mathcal{A} - k\mathcal{B}\mathcal{B}^*)z,z\rangle = \langle\mathcal{A}z,z\rangle - k|\mathcal{B}^*z|^2, \quad \text{for all z} \in \operatorname{dom}(\mathcal{A}) \tag{6.10}$$

Set $X_u = \{\alpha e | \alpha \in \mathbb{R}\}$ and $X_s = X_u^\perp$. If $z \in X_u$, then $z = \alpha e$ for some $\alpha \in \mathbb{R}$, and thus

$$\begin{aligned} k|B^*z|^2 &= k\frac{\alpha^2}{\delta^2}\left(\int_0^1 b(x)\,dx\right)^2 \\ &= k\frac{\alpha^2}{\delta^2}(x_2 - x_1)^2 \\ &= m_1\|z\|^2, \quad \text{for } z \in X_u \end{aligned} \tag{6.11}$$

where $m_1 = k\delta^{-2}\ (x_2 - x_1)^2$. Moreover, for some $m_2 > 0$

$$\langle\mathcal{A}z,z\rangle \le -m_2\|z\|^2, \quad \text{for } z \in X_s \cap \operatorname{dom}(\mathcal{A}) \tag{6.12}$$

Combining (6.6), (6.10), (6.11) and (6.12), we see that, for some $m_3 > 0$

$$\langle(\mathcal{A} - k\mathcal{B}\mathcal{B}^*)z,z\rangle \le -m_3\|z\|^2, \quad \text{for all } z \in \operatorname{dom}(\mathcal{A})$$

from which we may conclude that the C_0-semigroup generated by $\mathcal{A} - k\mathcal{B}\mathcal{B}^*$ is exponentially stable. Therefore, the closed-loop system is modally stable when there is no delay in the feedback loop. Combining Corollary 5.4 and (6.9) shows that there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0,\varepsilon^*)$ the feedback $u(t) = -y(t - \varepsilon)$ produces a modally stable closed-loop system.

Example 2

In this example we describe systems with distributed and co-located control and observation for which negative unity output feedback leads to an input–output stable closed-loop system, but arbitrarily small delays result in exponentially growing modes. Let $\mathcal{A}$ be the generator of a strongly continuous semigroup on a real or complex Hilbert space X, with inner product $\langle\cdot,\cdot\rangle$, and suppose that $\mathcal{A}$ is dissipative, i.e.

$$\operatorname{Re}\langle\mathcal{A}x,x\rangle \le 0, \quad \text{for all } x \in \operatorname{dom}(\mathcal{A}) \tag{6.13}$$

Let $b \in X$, and consider the control system on X described by

$$\dot z(t) = \mathcal{A}z(t) + bu(t), \quad y(t) = k\langle z(t),b\rangle \tag{6.14}$$

where $k > 0$. The transfer function of (6.14) is

$$\mathbf{H}(s) = k\langle(sI - \mathcal{A})^{-1}b,b\rangle$$

Since $b \in X$, $\mathbf{H}(s)$ is clearly regular with feedthrough 0.

Lemma 6.1. The transfer function $\mathbf{H}(s)$ is holomorphic in $\mathbb{C}_0$

$$\operatorname{Re}\mathbf{H}(s) \ge 0, \quad \text{for all } s \in \mathbb{C}_0 \tag{6.15}$$

and

$$\frac{\mathbf{H}}{1 + \mathbf{H}} \in H^\infty \tag{6.16}$$

Proof. Note first that, by (6.13), $\mathbb{C}_0$ is contained in the resolvent set of A. It follows that $\mathbf{H}$ is holomorphic in $\mathbb{C}_0$. For $s \in \mathbb{C}_0$, set $\psi(s) = (sI - \mathcal{A})^{-1}b \in \text{dom}(\mathcal{A})$. Then

$$\begin{aligned}\text{Re } \mathbf{H}(s) &= \text{Re } \langle b,\psi(s)\rangle = \text{Re } \langle (sI - \mathcal{A})\psi(s),\psi(s)\rangle \\ &= (\text{Re } s)\langle \psi(s),\psi(s)\rangle - \text{Re } \langle \mathcal{A}\psi(s),\psi(s)\rangle \\ &\geq 0\end{aligned}$$

using (6.13) and the fact that Re $s > 0$.

To prove (6.16), observe that, by (6.15), we have

$$|1 + \mathbf{H}(s)| \geq 1 + \text{Re } \mathbf{H}(s) > 1, \text{ for all } s \in \mathbb{C}_0$$

and hence

$$\frac{\mathbf{H}}{1+\mathbf{H}} = 1 - \frac{1}{1+\mathbf{H}} \in H^\infty \qquad \square$$

Lemma 6.1 means that the feedback $u(t) = -y(t)$ stabilises (6.14) in an L^2-input–output sense. If (6.14) is the abstract formulation of a system in the form (**OLS**) (see Section 2) satisfying (**A1**) and (**A2**) and if

$$\gamma(\mathbf{H}) > 1 \tag{6.17}$$

then Corollary 5.3 applies, so there exist arbitrarily small delays ε such that the feedback $u(t) = -y(t-\varepsilon)$ leads to a closed-loop system with exponentially growing modes.

For instance, suppose that the following two conditions hold:

(C1) $\mathcal{A}$ has an orthonormal basis of eigenvectors $\{\Phi_j\}$ with associated eigenvalues $\{\lambda_j\}$, where Re $\lambda_j = 0$ and $\lim_{j\to\infty} |\lambda_j| = \infty$;

(C2) $b = \Sigma\beta_j\Phi_j$, where infinitely many of the β_js are non-zero.

Then

$$\mathbf{H}(s) = \sum \frac{|\beta_j|^2}{(s-\lambda_j)} \qquad \text{for Re } s > 0$$

and it is easy to see that (6.13) and (6.17) hold. Related conditions for lack of time-delay robustness are given in Datko and You [5]. Of course, conditions (C1) and (C2) above are far from necessary for (6.13) and (6.17) to hold, but they are often easily checkable, as in the following specific example.

Consider a Euler–Bernoulli beam with hinged boundary conditions on the left end and sliding boundary conditions on the right end, and co-located observation.

$$w_{tt}(x,t)+w_{xxxx}(x,t)=(1/\delta)\chi_{[x_1,x_2]}(x)u(t), \; x \in (0,1), \; t \geq 0 \tag{6.18}$$

$$w(0,t) = w_{xx}(0,t) = w_x(1,t) = w_{xxx}(1,t) = 0 \tag{6.19}$$

$$y(t) = \frac{k}{\delta}\int_{x_1}^{x_2} w_t(x,t)dx, \qquad k > 0 \tag{6.20}$$

where $0 \leq x_1 < x_2 \leq 1$ and $\delta = x_2 - x_1$. This system is of the form (**OLS**) satisfying (**A1**) and (**A2**).

If $x_2 = 1$ and x_1 is close to 1, then b is an approximation to the delta function at 1, so this system may be thought of as an approximation of the boundary control system

$$w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \quad x \in (0,1), \, t \geq 0 \tag{6.21}$$

$$w(0,t) = w_{xx}(0,t) = w_x(1,t) = 0, \; w_{xxx}(1,t) = u(t) \tag{6.22}$$

$$y(t) = kw_t(1,t), \quad k > 0 \tag{6.23}$$

It is well-known that (6.21)–(6.23) is exponentially stabilised by $u(t) = -y(t)$ (see Chen et al. [2]) and that this stabilisation is not robust with respect to delays (as was first shown for a similar system in Datko [3]). The system (6.18)–(6.20) is not exponentially stabilised by $u(t) = -y(t)$, since both the input and observation operators are bounded (see, for instance, Russell [13]). However, we will show that $u(t) = -y(t)$ L^2-stabilises (6.18)–(6.20), and that arbitrarily small delays in the feedback loop lead to exponentially growing modes.

To put (6.18)–(6.20) in an appropriate state space setting, define the operator $\mathcal{A}_0$ on $L^2(0,1)$ by

$$\begin{aligned}&\mathcal{A}_0 z = D_x^4 z, \\ &\text{dom}(\mathcal{A}_0) = \{z \in H^4(0,1) \mid \\ &\qquad z(0) = D_{xx}z(0) = D_x z(1) = D_{xxx}z(1) = 0\}\end{aligned}$$

The operator $\mathcal{A}_0$ is self-adjoint on $L^2(0,1)$ and positive semi-definite, i.e.

$$\langle \mathcal{A}_0 z,z\rangle \geq 0, \qquad \text{for all } z \in \text{dom}(\mathcal{A}_0)$$

Let

$$X = \text{dom}(\mathcal{A}_0^{1/2}) \oplus L^2(0,1)$$

with inner product

$$\langle [z_1,z_2]^T,[v_1,v_2]^T\rangle = \langle \mathcal{A}_0^{1/2} z_1, \mathcal{A}_0^{1/2} v_1\rangle_{L^2(0,1)} + \langle z_2,v_2\rangle_{L^2(0,1)}$$

Define

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -\mathcal{A}_0 & 0 \end{pmatrix}$$

with domain

$$\text{dom}(\mathcal{A}) = \text{dom}(\mathcal{A}_0) \oplus \text{dom}(\mathcal{A}_0^{1/2})$$

It is well-known that $\mathcal{A}$ is the generator of a C_0-semigroup on X. Let $\eta_j = -(\pi/2) + \pi j$ and $\phi_j(x) =$

$\sqrt{2}\sin(\eta_j x)$. Then $\{\eta_j^4\}_{j=1}^{\infty}$ are the eigenvalues of $\mathcal{A}_0$, $\{\phi_j\}_{j=1}^{\infty}$ are the associated eigenvectors, and $\{\phi_j\}_{j=1}^{\infty}$ is an orthonormal basis for $L^2(0,1)$. It is easy to check that if

$$\Phi_{\pm j} := \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\eta_j^2}\phi_j \\ \pm i\phi_j \end{bmatrix}$$

then $\Phi_{\pm j}$ is an eigenvector of $\mathcal{A}$ with associated eigenvalue $\lambda_{\pm j} := \pm i\eta_j^2$, and that $\{\Phi_{\pm j}\}_{j=1}^{\infty}$ is an orthonormal basis for X. Hence $\mathcal{A}$ satisfies condition (C1) above.

Let

$$b := \begin{bmatrix} 0 \\ \frac{1}{\delta}\chi_{[x_1,x_2]} \end{bmatrix} \in X$$

Then system (6.18)–(6.20) is equivalent to

$$\dot{z}(t) = \mathcal{A}z(t) + bu(t), \qquad y(t) = k\langle z(t),b\rangle \tag{6.24}$$

where $z(t) = [w(\cdot,t),\ w_t(\cdot,t)]^T$. We then see from Lemma 6.1 that the feedback $u(t) = -y(t)$ stabilises (6.18)–(6.20) in the L^2-input–output sense, i.e. $\mathbf{H}/(1+\mathbf{H}) \in H^{\infty}$, where $\mathbf{H}(s)$ denotes the transfer function of (6.18)–(6.20) (which, of course, is also the transfer function of (6.24)). We can write

$$b = \sum_{j=1}^{\infty} (\beta_j\Phi_j + \beta_{-j}\Phi_{-j}) \tag{6.25}$$

where

$$\beta_{\pm j} = \langle b, \Phi_{\pm j}\rangle = \frac{\mp i}{\delta\sqrt{2}}\int_{x_1}^{x_2}\phi_j(x)dx$$

$$= \frac{\pm i}{\eta_j\delta}(\cos(\eta_j x_2) - \cos(\eta_j x_1)) \tag{6.26}$$

If x_2 is not a rational multiple of x_1, it is very easy to show that $\cos(\eta_j x_2) - \cos(\eta_j x_1) \neq 0$ for any j. More generally, we have the following lemma.

Lemma 6.2. There are no three consecutive integer values of j such that $\cos(\eta_j x_1) - \cos(\eta_j x_2) = 0$ for all three.

This lemma, combined with (6.25) and (6.26), shows that b satisfies condition (C2), so that $\gamma(\mathbf{H}) = \infty$, and hence, by Corollary 5.3, there exist arbitrarily small $\varepsilon > 0$, such that the feedback $u(t) = -y(t-\varepsilon)$ applied to system (6.18)–(6.20) results in exponentially growing modes.

Proof of Lemma 6.2. Suppose $\cos(\eta_j x_1) - \cos(\eta_j x_2) = 0$ for three consecutive integers j, $j+1$ and $j+2$. Then there exist integers, k_0, k_1 and k_2 and choices of $+$ or $-$ such that

$$(-(\pi/2) + \pi j)x_2 = \pm(-(\pi/2) + \pi j)x_1 + 2\pi k_0 \tag{6.27}$$

$$(-(\pi/2) + \pi(j+1))x_2 = \pm(-(\pi/2) + \pi(j+1))x_1 + 2\pi k_1 \tag{6.28}$$

$$(-(\pi/2) + \pi(j+2))x_2 = \pm(-(\pi/2) + \pi(j+2))x_1 + 2\pi k_2 \tag{6.29}$$

At least two of these equations have the same choice of $+$ or $-$.

Case 1. (6.27) and (6.28) have the same choice of sign.

In this case subtract (6.27) from (6.28) and divide by π to obtain

$$x_2 = \pm x_1 + 2(k_1 - k_0).$$

Since $0 \le x_1 < x_2 \le 1$ and $(k_1 - k_0)$ is an integer, this is impossible.

Case 2. (6.28) and (6.29) have the same choice of sign.

This case proceeds exactly like Case 1.

Case 3. (6.27) and (6.29) have the same choice of sign.

In this case subtract (6.27) from (6.29) and divide by 2π to obtain

$$x_2 = \pm x_1 + (k_2 - k_0).$$

Hence $x_2 \mp x_1$ must be an integer. Since $0 \le x_1 < x_2 \le 1$, it follows that $x_2 \mp x_1 = 1$. If $x_2 - x_1 = 1$, then $x_2 = 1$ and $x_1 = 0$, which contradicts (6.27), since $1/2 + j \neq 2k_0$. If $x_2 + x_1 = 1$, then plugging $x_2 = 1 - x_1$ into (6.27) also easily leads to a contradiction.

Therefore, all three cases lead to a contradiction, so there do not exist three consecutive integers j for which $\cos(\eta_j x_1) - \cos(\eta_j x_2) = 0$, proving the lemma. □

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