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The Effect of Small Time-Delays on the Closed-Loop Stability of Boundary Control Systems*

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Abstract. It has been observed that for many stable feedback systems, the introduction of arbitrarily small time-delays into the loop causes instability. In this paper we present a systematic treatment of this phenomenon for a large class of boundary control systems which allows for in-span control. Our approach is based on a combination of input–output methods and modal analysis. We give a number of sufficient conditions for robustness/nonrobustness of closed-loop input–output stability with respect to delays. Our framework includes a large class of ill-posed systems, i.e., systems whose open-loop transfer function is unbounded on any right half-plane. We then analyze the relationship between the poles of the transfer function and the exponential modes of the underlying boundary-value problem to derive internal stability properties from external ones.

Key words. Small time-delays, Stability, Robustness, Partial differential equations, Boundary control, In-span control, Transfer functions.

1. Introduction

It is well known that for many stable feedback systems the introduction of small delays into the feedback loop can cause instability. While this phenomenon even occurs in finite-dimensional feedback systems, it becomes considerably more complicated in infinite dimensions due to the rich high-frequency behaviour exhibited by many distributed parameter systems. In the last decade various authors have presented specific partial differential equations (PDEs), or classes of related PDEs, which are exponentially stabilized by some feedback, but destabilized when arbitrarily small delays are introduced into the feedback loop, see, for example, [4]–[7], [9], and [11]. More precisely, it was shown that there exists a sequence of delays $\varepsilon_n > 0$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that the feedback system with delay ε_n in the

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loop has an unstable mode. By "mode" we mean a solution of the form $e^{s_0 t}\varphi$, where $\varphi \not\equiv 0$ is a function of the space variable. A mode is called unstable if the exponent s_0 satisfies $\operatorname{Re} s_0 \geq 0$. We say that a system described by PDEs is modally stable if it has no unstable modes. While the above papers are based on PDE and related techniques, the problem of robustness/nonrobustness in the presence of small delays has also been investigated using frequency-domain approaches, see [1] and [15]. Closely related to these papers is the work by Georgiou and Smith [10]. However, their concept of w-stability is considerably stronger than robust stability with respect to small delays; it covers a large class of perturbations which represent high-frequency modelling uncertainties including small time-delays.

In the frequency-domain a linear time-invariant system is described by a transfer function matrix $\mathbf{H}$. Applying negative unity feedback in the presence of a delay $\varepsilon \geq 0$ in the loop leads to the closed-loop transfer function $\mathbf{G}_\varepsilon(s) = \mathbf{H}(s)(I + e^{-\varepsilon s}\mathbf{H}(s))^{-1}$. A transfer function is called spectrally stable if it is holomorphic in the closed right half-plane and it is called L^2-stable if it is holomorphic and bounded in the open right half-plane. If $\mathbf{G}_0$ is spectrally stable or L^2-stable, then either form of stability is called robust with respect to small delays if it is preserved for all sufficiently small delays, i.e., if there exists $\varepsilon^* > 0$ such that $\mathbf{G}_\varepsilon$ is stable for all $\varepsilon \in (0, \varepsilon^*)$. In [15] conditions are given (in terms of the high-frequency behaviour of $\mathbf{H}$) for robustness and nonrobustness of L^2-stability if the open-loop transfer function $\mathbf{H}$ is in the class of regular transfer functions, defined in Section 3. This class contains all well-posed transfer functions which are relevant in the applications to PDEs and functional differential equations,[1] where well-posedness means that $\mathbf{H}$ is bounded on some right half-plane. While the frequency-domain approach in [15] is quite general in the sense that it is not tied to specific classes of PDEs and boundary conditions or specific feedback laws, the results obtained are external in the sense that the conclusions are in terms of $\mathbf{G}_\varepsilon$, and not in terms of the solutions of the PDE.

The purpose of this paper is twofold. The results in [15] apply to regular transfer functions. However, examples in the PDE literature (in particular, see [5] and [9]) seem to indicate that systems with open-loop transfer functions which are ill-posed (that is, not well-posed) are even more likely to be destabilized by delays than well-posed systems. Thus the first goal in this paper is to present destabilization results for systems with ill-posed open-loop transfer functions. We find that if the open-loop transfer function is ill-posed, then $\mathbf{G}_\varepsilon$ is not well-posed, and hence not L^2-stable, for all $\varepsilon > 0$. Moreover, in Theorem 3.7 we identify a class of ill-posed transfer functions for which there exist delays $\varepsilon_n \to 0$ and complex numbers s_n with $\operatorname{Re} s_n \to \infty$ such that $\mathbf{G}_{\varepsilon_n}$ has a pole at s_n.

The second goal of this paper is to identify a large class of multivariable systems described by PDEs for which results about robustness/nonrobustness of spectral stability translate to results about robustness/nonrobustness of modal stability. The class we consider consists of linear PDEs of spatial dimension 1, where on

[1] For an application of the frequency-domain results in [15] to neutral functional differential equations, see [16].

different parts of the space interval different PDEs are satisfied. The coefficients may depend on the spatial variable. The boundary conditions are general enough to allow all natural coupling conditions and in-span control, including some dynamic control. It is not assumed that the controlled, observed system has any useful (A, B, C, D) state–space representation, and in fact many systems in our class do not. Moreover, we do not even assume that the free dynamics are described by a strongly continuous semigroup. In order to relate modal stability of the PDE to the spectral stability of the transfer function, a relationship between the modes of a PDE and the poles of the associated transfer function is needed. A natural relationship, expected for systems which have a state–space representation, is that a pole s_0 of the transfer function leads to a mode with exponent s_0 of the corresponding free dynamics and we find this to be the case for our class of systems. Of course, as in finite dimensions, it is possible that a mode will not appear as a pole of the associated transfer function, since any possible effect of the mode on the output might be anihilated by the observation and control operators. However, the results in Section 2 will be used to show that for any system in our class the exponents of the unstable modes and the unstable poles of the transfer function coincide, provided the closed-loop system (without delay) is modally stable. This result will in turn be applied to prove that for a large class of systems robustness of spectral stability implies robustness of modal stability.

The paper is organized as follows. Section 2 contains preliminaries about factorizations of matrix-valued meromorphic functions, which are needed later in the paper. In Section 3 we prove a number of robustness and destabilization results for ill-posed systems in a frequency-domain setting. We emphasize that most of the results in Section 3 cover not only single-delay perturbations but also multidelay perturbations of the form $\operatorname{diag}(e^{-\varepsilon_1 s}, \ldots, e^{-\varepsilon_m s})$. In Section 4 we discuss robustness/nonrobustness of modal stability for a class of systems described by PDEs with boundary and in-span control. Finally, in Section 5 we present three examples illustrating the results in the previous sections.

2. Coprime and Bi-Coprime Factorizations of Matrix-Valued Meromorphic Functions

Let $\alpha \in \mathbb{R}$. We use the notation $\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\}$. Furthermore, we define

$$\mathscr{H}_- := \{f\colon \Omega_f \to \mathbb{C} \mid \Omega_f \text{ open, } \Omega_f \supset \mathbb{C}_0^{cl} \text{ and } f \text{ holomorphic}\},$$

$$\mathscr{M}_- := \{f\colon \Omega_f \to \mathbb{C} \mid \Omega_f \text{ open, } \Omega_f \supset \mathbb{C}_0^{cl} \text{ and } f \text{ meromorphic}\}.$$

Let $m, l \in \mathbb{N}$ and let $\mathbf{H} \in \mathscr{M}_-^{m \times l}$. A number $p \in \mathbb{C}_0^{cl}$ is called a *pole* of $\mathbf{H}$, if p is a pole of at least one entry of $\mathbf{H}$. We say that $\mathbf{H} \in \mathscr{M}_-^{m \times m}$ is *invertible* if $\det \mathbf{H}(s) \not\equiv 0$. The inverse $\mathbf{H}^{-1}$ is then in $\mathscr{M}_-^{m \times m}$ and is given by Cramer's rule, i.e., $\mathbf{H}^{-1} = (1/\det \mathbf{H}) \operatorname{adj}(\mathbf{H})$, where $\operatorname{adj}(\mathbf{H})$ denotes the adjugate of $\mathbf{H}$. Obviously, $\mathbf{H}$ is invertible if and only if $\det \mathbf{H}(s_0) \neq 0$ for some $s_0 \in \mathbb{C}_0^{cl}$.

A matrix $\mathbf{H} \in \mathscr{H}_-^{m \times m}$ is called *unimodular* if it is invertible and $\mathbf{H}^{-1}$ belongs to $\mathscr{H}_-^{m \times m}$ as well. Clearly, $\mathbf{H}$ is unimodular if and only if $\det \mathbf{H}(s) \neq 0$ for all $s \in \mathbb{C}_0^{cl}$. Two matrices $\mathbf{N} \in \mathscr{H}_-^{m \times l}$ and $\mathbf{D} \in \mathscr{H}_-^{l \times l}$ are called *right-coprime* if there exist

$\mathbf{X} \in \mathscr{H}_-^{l \times m}$ and $\mathbf{Y} \in \mathscr{H}_-^{l \times l}$ such that $\mathbf{XN} + \mathbf{YD} = I$. Similarly, two matrices $\tilde{\mathbf{N}} \in \mathscr{H}_-^{m \times l}$ and $\tilde{\mathbf{D}} \in \mathscr{H}_-^{m \times m}$ are called *left-coprime* if there exist $\tilde{\mathbf{X}} \in \mathscr{H}_-^{l \times m}$ and $\tilde{\mathbf{Y}} \in \mathscr{H}_-^{m \times m}$ such that $\tilde{\mathbf{N}}\tilde{\mathbf{X}} + \tilde{\mathbf{D}}\tilde{\mathbf{Y}} = I$.

The following lemma can be found in [16].

Lemma 2.1. *Suppose* $\mathbf{H} \in \mathscr{M}_-^{m \times l}$. *Then the following statements hold*:

(i) $\mathbf{H}$ *admits a right-coprime factorization* $(\mathbf{N}, \mathbf{D})$ *over* $\mathscr{H}_-$, *i.e., there exist right-coprime matrices* $\mathbf{N} \in \mathscr{H}_-^{m \times l}$ *and* $\mathbf{D} \in \mathscr{H}_-^{l \times l}$ *such that* $\det \mathbf{D}(s) \not\equiv 0$ *and* $\mathbf{H} = \mathbf{N}\mathbf{D}^{-1}$. *The matrices* $\mathbf{N}$ *and* $\mathbf{D}$ *are unique up to multiplication from the right by a unimodular factor. A number* $p \in \mathbb{C}_0^{cl}$ *is a pole of* $\mathbf{H}$ *if and only if* $\det \mathbf{D}(p) = 0$.

(ii) $\mathbf{H}$ *admits a left-coprime factorization* $(\tilde{\mathbf{D}}, \tilde{\mathbf{N}})$ *over* $\mathscr{H}_-$, *i.e., there exist left-coprime matrices* $\tilde{\mathbf{D}} \in \mathscr{H}_-^{m \times m}$ *and* $\tilde{\mathbf{N}} \in \mathscr{H}_-^{m \times l}$ *such that* $\det \tilde{\mathbf{D}}(s) \not\equiv 0$ *and* $\mathbf{H} = \tilde{\mathbf{D}}^{-1}\tilde{\mathbf{N}}$. *The matrices* $\tilde{\mathbf{D}}$ *and* $\tilde{\mathbf{N}}$ *are unique up to multiplication from the left by a unimodular factor. A number* $p \in \mathbb{C}_0^{cl}$ *is a pole of* $\mathbf{H}$ *if and only if* $\det \tilde{\mathbf{D}}(p) = 0$.

(iii) *If* $(\mathbf{N}, \mathbf{D})$ *is a right-coprime factorization of* $\mathbf{H}$ *over* $\mathscr{H}_-$ *and if* $(\tilde{\mathbf{D}}, \tilde{\mathbf{N}})$ *is a left-coprime factorization of* $\mathbf{H}$ *over* $\mathscr{H}_-$, *then the zeros of* $\det \mathbf{D}$ *and* $\det \tilde{\mathbf{D}}$ *in* $\mathbb{C}_0^{cl}$ *coincide* (*counting multiplicities*).

For $h \in \mathscr{H}_-$ and $s \in \mathbb{C}_0^{cl}$ we define

$$\operatorname{zero}_s(h) = \begin{cases} 0 & \text{if} \quad h(s) \neq 0, \\ \text{multiplicity of } s & \text{if} \quad h(s) = 0. \end{cases}$$

Let $\mathbf{H} \in \mathscr{M}_-^{m \times l}$ and let $(\mathbf{N}, \mathbf{D})$ be a right-coprime factorization of $\mathbf{H}$ over $\mathscr{H}_-$. For $s \in \mathbb{C}_0^{cl}$ we define

$$\operatorname{pole}_s(\mathbf{H}) := \operatorname{zero}_s(\det \mathbf{D}).$$

If $p \in \mathbb{C}_0^{cl}$ is a pole of $\mathbf{H}$, then we define its *multiplicity* to be $\operatorname{pole}_p(\mathbf{H})$. Lemma 2.1 shows that we obtain the same concept if in the definition of $\operatorname{pole}_s(\mathbf{H})$ the matrix $\mathbf{D}$ is replaced by the "denominator" $\tilde{\mathbf{D}}$ of a left-coprime factorization of $\mathbf{H}$.

Suppose that $\mathbf{H} \in \mathscr{M}_-^{m \times l}$, $\mathbf{N} \in \mathscr{H}_-^{m \times n}$, $\mathbf{D} \in \mathscr{H}_-^{n \times n}$, and $\tilde{\mathbf{N}} \in \mathscr{H}_-^{n \times l}$, where $n \in \mathbb{N}$. The triple $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ is called *bi-coprime* if $\mathbf{N}$ and $\mathbf{D}$ are right-coprime and if $\tilde{\mathbf{N}}$ and $\mathbf{D}$ are left-coprime. We say that the triple $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ is a *bi-coprime factorization* of $\mathbf{H}$ over $\mathscr{H}_-$ if $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ is bi-coprime, $\det \mathbf{D}(s) \not\equiv 0$, and $\mathbf{H} = \mathbf{N}\mathbf{D}^{-1}\tilde{\mathbf{N}}$. Let $s_0 \in \mathbb{C}_0^{cl}$. If

$$\operatorname{rank}(\mathbf{N}^T(s_0), \mathbf{D}^T(s_0)) = \operatorname{rank}(\tilde{\mathbf{N}}(s_0), \mathbf{D}(s_0)) = n, \tag{2.1}$$

then we say that $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ satisfies the *generalized Hautus conditions* in s_0.

Remark 2.2. It is not difficult to show that $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ satisfies the generalized Hautus conditions in $s_0 \in \mathbb{C}_0^{cl}$ if and only if there exist an open neighbourhood U of s_0 and matrices $\mathbf{X}$, $\mathbf{Y}$, $\tilde{\mathbf{X}}$, and $\tilde{\mathbf{Y}}$ whose entries are holomorphic on U and such that the Bezout identities $\mathbf{X}(s)\mathbf{N}(s) + \mathbf{Y}(s)\mathbf{D}(s) = I$ and $\tilde{\mathbf{N}}(s)\tilde{\mathbf{X}}(s) + \mathbf{D}(s)\tilde{\mathbf{Y}}(s) = I$ hold for all $s \in U$. In fact, while it is trivial that the Hautus conditions are necessary for the solvability of the Bezout equations, sufficiency follows from the fact that the elementary divisor theorem holds for holomorphic matrices, i.e., any holomorphic matrix is equivalent to its Smith form (see p. 139 of [18]). Moreover, the same argument applies globally, and thus $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ is bi-coprime if and only if it satisfies the generalized Hautus conditions in s for all $s \in \mathbb{C}_0^{cl}$.

Proposition 2.3. *Let* $\mathbf{N} \in \mathscr{H}_-^{m\times n}$, $\mathbf{D} \in \mathscr{H}_-^{n\times n}$, *and* $\tilde{\mathbf{N}} \in \mathscr{H}_-^{n\times l}$, *suppose that* $\det \mathbf{D}(s) \not\equiv 0$ *and set* $\mathbf{H} = \mathbf{N}\mathbf{D}^{-1}\tilde{\mathbf{N}}$. *If* (2.1) *holds for some* $s_0 \in \mathbb{C}_0^{cl}$, *then*

$$\operatorname{pole}_{s_0}(\mathbf{H}) > 0 \quad \Leftrightarrow \quad \operatorname{zero}_{s_0}(\det \mathbf{D}) > 0. \tag{2.2}$$

In particular, if the triple $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ *is bi-coprime, then* (2.2) *holds for all* $s_0 \in \mathbb{C}_0^{cl}$.

The above proposition follows from a considerably stronger result by Pandolfi [20] which says that, for any $s_0 \in \mathbb{C}_0^{cl}$,

$$\operatorname{pole}_{s_0}(\mathbf{H}) = \operatorname{zero}_{s_0}(\det \mathbf{D}) \quad \Leftrightarrow \quad (2.1) \text{ holds.}$$

For sake of completeness we give a simple direct proof of Proposition 2.3 which does not rely on Pandolfi's result.

Proof of Proposition 2.3. Clearly, if $\operatorname{pole}_{s_0}(\mathbf{H}) > 0$, then $\operatorname{zero}_{s_0}(\det \mathbf{D}) > 0$. Conversely assume that $\operatorname{zero}_{s_0}(\det \mathbf{D}) > 0$. By Remark 2.2 there exist an open neighbourhood U of s_0 and holomorphic matrices $\mathbf{X}$, $\mathbf{Y}$, $\tilde{\mathbf{X}}$, and $\tilde{\mathbf{Y}}$ on U such that

$$\mathbf{X}(s)\mathbf{N}(s) + \mathbf{Y}(s)\mathbf{D}(s) = I \quad \text{and} \quad \tilde{\mathbf{N}}(s)\tilde{\mathbf{X}}(s) + \mathbf{D}(s)\tilde{\mathbf{Y}}(s) = I, \qquad \forall s \in U. \tag{2.3}$$

By the first equation, $\mathbf{X}\mathbf{N}\mathbf{D}^{-1} + \mathbf{Y} = \mathbf{D}^{-1}$, and it follows that

$$\operatorname{pole}_{s_0}(\mathbf{N}\mathbf{D}^{-1}) > 0. \tag{2.4}$$

Multiplying the second equation in (2.3) with $\mathbf{N}\mathbf{D}^{-1}$ from the left we obtain $\mathbf{H}\tilde{\mathbf{X}} + \mathbf{N}\tilde{\mathbf{Y}} = \mathbf{N}\mathbf{D}^{-1}$. Combining this with (2.4) shows that $\operatorname{pole}_{s_0}(\mathbf{H}) > 0$. ■

Finally, we prove a simple lemma which will be needed in Section 4.

Lemma 2.4. *Let* $\mathbf{N} \in \mathscr{H}_-^{m\times n}$, $\mathbf{D} \in \mathscr{H}_-^{n\times n}$, $\tilde{\mathbf{N}} \in \mathscr{H}_-^{n\times l}$, *and* $\mathbf{F} \in \mathscr{H}_-^{l\times m}$, *suppose that* $\det \mathbf{D}(s) \not\equiv 0$ *and set* $\mathbf{H} = \mathbf{N}\mathbf{D}^{-1}\tilde{\mathbf{N}}$. *Then the following statements are true*:

(i) *The matrix* $I + \mathbf{F}\mathbf{H}$ *is invertible if and only if* $\mathbf{D} + \tilde{\mathbf{N}}\mathbf{F}\mathbf{N}$ *is invertible. If this is the case, then*

$$\mathbf{H}(I + \mathbf{F}\mathbf{H})^{-1} = \mathbf{N}(\mathbf{D} + \tilde{\mathbf{N}}\mathbf{F}\mathbf{N})^{-1}\tilde{\mathbf{N}}. \tag{2.5}$$

(ii) *Let* $s_0 \in \mathbb{C}_0^{cl}$. *The triple* $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ *satisfies the generalized Hautus conditions in* s_0 *if and only if* $(\mathbf{N}, \mathbf{D} + \tilde{\mathbf{N}}\mathbf{F}\mathbf{N}, \tilde{\mathbf{N}})$ *does.*

Statement (ii) *implies that if* $\det(\mathbf{D}(s_0) + \tilde{\mathbf{N}}(s_0)\mathbf{F}(s_0)\mathbf{N}(s_0)) \neq 0$, *then* $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ *satisfies the generalized Hautus conditions in* s_0. *In particular, if* $\det(\mathbf{D}(s) + \tilde{\mathbf{N}}(s)\mathbf{F}(s)\mathbf{N}(s)) \neq 0$ *for all* $s \in \mathbb{C}_0^{cl}$, *then* $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ *is bi-coprime.*

Proof. (i) If $I + \mathbf{F}\mathbf{H}$ is invertible, then it is not difficult to show that $\mathbf{D}^{-1}(I - \tilde{\mathbf{N}}(I + \mathbf{F}\mathbf{H})^{-1}\mathbf{F}\mathbf{N}\mathbf{D}^{-1})$ is the inverse of $\mathbf{D} + \tilde{\mathbf{N}}\mathbf{F}\mathbf{N}$. Conversely, if $\mathbf{D} +$

$\tilde{\mathbf{N}}\mathbf{F}\mathbf{N}$ is invertible, then $I - \mathbf{F}\mathbf{N}(\mathbf{D} + \tilde{\mathbf{N}}\mathbf{F}\mathbf{N})^{-1}\tilde{\mathbf{N}}$ is the inverse of $I + \mathbf{F}\mathbf{H}$. The proof of formula (2.5) is straightforward and is left to the reader.

(ii) Let $x \in \mathbb{C}^n$. The claim follows from the following two equivalences

$$\begin{pmatrix}\mathbf{N}(s_0)\\ \mathbf{D}(s_0)\end{pmatrix} x = 0 \quad\Leftrightarrow\quad \begin{pmatrix}\mathbf{N}(s_0)\\ \mathbf{D}(s_0) + \tilde{\mathbf{N}}(s_0)\mathbf{F}(s_0)\mathbf{N}(s_0)\end{pmatrix} x = 0,$$

$$x^T(\tilde{\mathbf{N}}(s_0), \mathbf{D}(s_0)) = 0 \quad\Leftrightarrow\quad x^T(\tilde{\mathbf{N}}(s_0), \mathbf{D}(s_0) + \tilde{\mathbf{N}}(s_0)\mathbf{F}(s_0)\mathbf{N}(s_0)) = 0. \qquad \blacksquare$$

The above lemma is purely algebraic. If has, however, a control interpretation: If $\mathbf{H}$ is the transfer function of a linear system and if $\mathbf{F}$ is the transfer function of a feedback law, then (2.5) is the corresponding closed-loop transfer function. Moreover, in many cases the stability of the feedback system is determined by the zeros of $\det(\mathbf{D} + \tilde{\mathbf{N}}\mathbf{F}\mathbf{N})$. In Section 4 and 5 we combine Proposition 2.3 and Lemma 2.4 to relate external and modal stability.

3. Robustness and Nonrobustness with Respect to Small Delays in the Frequency-Domain

Let $\alpha \in \mathbb{R}$. The field of all meromorphic function on $\mathbb{C}_\alpha$ is denoted by $\mathscr{M}_\alpha$, while $\mathscr{H}_\alpha$ denotes the algebra of all holomorphic functions on $\mathbb{C}_\alpha$. H_α^∞ denotes the algebra of all bounded holomorphic functions defined on $\mathbb{C}_\alpha$. We write H^∞ for H_0^∞. If $f \in \mathscr{M}_\alpha$ and $g \in \mathscr{M}_\beta$, where $\alpha < \beta$, and if $f(s) = g(s)$ for all $s \in \mathbb{C}_\beta$, then we identify f and g. Consequently, we have

$$\mathscr{M}_\alpha \subset \mathscr{M}_\beta, \quad \mathscr{H}_\alpha \subset \mathscr{H}_\beta, \quad H_\alpha^\infty \subset H_\beta^\infty \qquad \text{if} \quad \alpha < \beta.$$

Let $\Omega \subset \mathbb{C}$. A function $\mathbf{H}\colon \Omega \to \mathbb{C}^{m\times m}$ is called a ($\mathbb{C}^{m\times m}$-valued) *transfer function*[2] if there exists $\alpha \in \mathbb{R}$ such that $\mathbb{C}_\alpha \subset \Omega$ and $\mathbf{H}|_{\mathbb{C}_\alpha} \in \mathscr{M}_\alpha^{m\times m}$.

Let $\mathbf{H}$ be a transfer function and for $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m) \in [0, \infty)^m$ set

$$\mathbf{E}_{\vec{\varepsilon}}(s) := \operatorname{diag}_{1\le j\le m}(e^{-\varepsilon_j s}).$$

Consider the feedback system shown in Fig. 3.1, where u is the input function, y is the output function, and the block with transfer function $\mathbf{E}_{\vec{\varepsilon}}(s)$ represents a delay by $\varepsilon_j \ge 0$ in the the jth feedback loop, $j = 1, \ldots, m$. If $\det(I + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s)) \not\equiv 0$, then the function $\mathbf{G}_{\vec{\varepsilon}}$ defined by

$$\mathbf{G}_{\vec{\varepsilon}}(s) := \mathbf{H}(s)(I + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s))^{-1} \tag{3.1}$$

is a transfer function, the so-called closed-loop transfer function of the feedback system shown in Fig. 3.1.

Definition 3.1. $\mathbf{G}_{\vec{\varepsilon}}$ is called *L^2-stable* if $\mathbf{G}_{\vec{\varepsilon}} \in (\mathbf{H}^\infty)^{m\times m}$. $\mathbf{G}_{\vec{\varepsilon}}$ is called *spectrally stable* if $\mathbf{G}_{\vec{\varepsilon}} \in \mathscr{H}_-^{m\times m}$.

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[2] We assume that all transfer functions are square. If we want to apply feedback and the plant $\mathbf{H}$ is not square, we use a compensator $\mathbf{K}$ such that $\mathbf{HK}$ is square, and absorb $\mathbf{K}$ into $\mathbf{H}$.

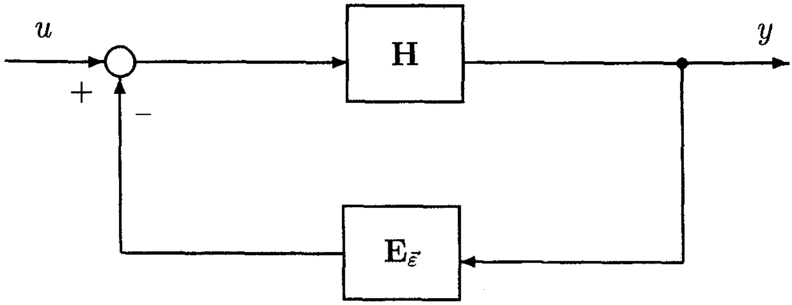

Fig. 3.1. Feedback system with delay.

It is well known that L^2-stability of $\mathbf{G}_{\vec{\varepsilon}}$ is equivalent to the existence of a constant $K > 0$ such that

$$\|y\|_{L^2(0,\infty)} \leq K\|u\|_{L^2(0,\infty)}, \qquad \forall u \in L^2(0, \infty; \mathbb{C}^m),$$

where u and y are related as in Fig. 3.1. Moreover, notice that $\mathbf{G}_{\vec{\varepsilon}}$ is spectrally stable if and only if $\mathbf{G}_{\vec{\varepsilon}} \in \mathscr{M}_{-}^{m \times m}$ and $\mathbf{G}_{\vec{\varepsilon}}$ has no poles in $\mathbb{C}_0^{cl}$.

It is easy to see that if $\mathbf{G}_{\vec{\varepsilon}_0}$ is L^2-stable for some $\vec{\varepsilon}_0 \in [0, \infty)^m$, then $\mathbf{H} \in \mathscr{M}_0^{m \times m}$. Similarly, if there exists $\vec{\varepsilon}_0 \in [0, \infty)^m$ such that $\mathbf{G}_{\vec{\varepsilon}_0}$ is spectrally stable, then $\mathbf{H} \in \mathscr{M}_{-}^{m \times m}$.

Definition 3.2. Let $\mathbf{H}$ be a transfer function. $\mathbf{H}$ is called *well-posed* if $\mathbf{H} \in (H_\alpha^\infty)^{m \times m}$ for some $\alpha \in \mathbb{R}$. Moreover, $\mathbf{H}$ is called *regular* if it is well-posed and if the limit $\lim_{\xi \to +\infty} \mathbf{H}(\xi) = D$ exists (where $\xi \in \mathbb{R}$). The matrix D is called the *feedthrough matrix* of $\mathbf{H}$. If $\mathbf{H}$ is not well-posed, we say that it is *ill-posed*.

Well-posed and regular transfer functions play an important role in the theory of abstract infinite-dimensional control systems, see [22] and [23].

Nonrobustness Results

For a transfer function $\mathbf{H}$ let $\mathfrak{P}_\mathbf{H}$ denote the set of its poles. We define

$$\gamma(\mathbf{H}) := \limsup_{|s| \to \infty,\, s \in \mathbb{C}_0 \setminus \mathfrak{P}_\mathbf{H}} r(\mathbf{H}(s)), \tag{3.2}$$

where $r(\mathbf{H}(s))$ denotes the spectral radius of $\mathbf{H}(s)$. As usual, the spectrum of $\mathbf{H}(s)$ is denoted by $\sigma(\mathbf{H}(s))$.

The following destabilization result for regular transfer functions was proved by Logemann and Townley [16]. We write $\mathbf{G}_\varepsilon$ for $\mathbf{G}_{\vec{\varepsilon}}$ if the components of $\vec{\varepsilon}$ satisfy $\varepsilon_i = \varepsilon \geq 0$ for all $i = 1, \ldots, m$.

Theorem 3.3. *Let* $\mathbf{H}$ *be a transfer function and suppose that* $\mathbf{H}$ *is regular with feedthrough matrix* D. *Then the following statements hold true*:

(i) *If* $\mathbf{G}_0$ *is* L^2*-stable, and if* $\gamma(\mathbf{H}) > 1$, *then there exist sequences* (ε_n) *and* (p_n) *with*

$$\varepsilon_n > 0, \qquad \lim_{n\to\infty} \varepsilon_n = 0, \qquad p_n \in \mathbb{C}_0, \qquad \lim_{n\to\infty} |\mathrm{Im}\, p_n| = \infty,$$

and such that, for any $n \in \mathbb{N}$, p_n *is a pole of* $\mathbf{G}_{\varepsilon_n}$.

(ii) *If* $r(D) > 1$, *then there exists sequences* (ε_n) *and* (p_n) *with*

$$\varepsilon_n > 0, \qquad \lim_{n\to\infty} \varepsilon_n = 0, \qquad p_n \in \mathbb{C}_0, \qquad \lim_{n\to\infty} \mathrm{Re}\, p_n = \infty, \qquad \lim_{n\to\infty} |\mathrm{Im}\, p_n| = \infty,$$

and such that, for any $n \in \mathbb{N}$, p_n *is a pole of* $\mathbf{G}_{\varepsilon_n}$.

There are many PDE models of physical and technical systems which have transfer functions which are not well-posed. In the following we show that if $\mathbf{H}$ is not well-posed, then the robustness properties of $\mathbf{G}_0$ with respect to small delays are extremely poor (as has been observed in [9] for a class of systems governed by skew-adjoint generators), in fact they are worse than in the regular case. First we show that if $\mathbf{H}$ is not well-posed, then $\mathbf{G}_{\vec{\varepsilon}}$ is not well-posed (and hence in particular is not L^2-stable) for any $\vec{\varepsilon} \in (0, \infty)^m$.

Proposition 3.4. *Suppose that* $\mathbf{H}$ *is a transfer function. If* $\mathbf{H}$ *is well-posed, then* $\mathbf{G}_{\vec{\varepsilon}}$ *is well-posed for all* $\vec{\varepsilon} \in (0, \infty)^m$. *Conversely, if* $\mathbf{G}_{\vec{\varepsilon}_0}$ *is well-posed for some* $\vec{\varepsilon}_0 \in (0, \infty)^m$, *then* $\mathbf{H}$ *is well-posed.*

Proof. If $\mathbf{H}$ is well-posed, then it is clear that for any $\vec{\varepsilon} \in (0, \infty)^m$ the transfer function $\mathbf{G}_{\vec{\varepsilon}}$ is also well-posed, since $\|\mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s)\| < 1$ in $\mathbb{C}_\alpha$ for large enough α. Conversely, assume that $\mathbf{G}_{\vec{\varepsilon}_0}$ is well-posed for some $\vec{\varepsilon}_0 \in (0, \infty)^m$. Then the transfer function $(I - \mathbf{G}_{\vec{\varepsilon}_0}\mathbf{E}_{\vec{\varepsilon}_0})^{-1}$ is well-posed, and so is $\mathbf{H} = (I - \mathbf{G}_{\vec{\varepsilon}_0}\mathbf{E}_{\vec{\varepsilon}_0})^{-1}\mathbf{G}_{\vec{\varepsilon}_0}$. ∎

The following corollary is a trivial consequence of Proposition 3.4.

Corollary 3.5. *Let* $\mathbf{H}$ *be a transfer function. If* $\mathbf{H}$ *is not well-posed, then* $\mathbf{G}_{\vec{\varepsilon}} \notin (H^\infty)^{m\times m}$ *for all* $\vec{\varepsilon} \in (0, \infty)^m$.

The corollary says in particular that if $\mathbf{H}$ is not well-posed and if $\mathbf{G}_0$ is L^2-stable, then any positive time-delay will destroy the L^2-stability of the undelayed closed-loop system. Note that Corollary 3.5 does not ensure the existence of poles of $\mathbf{G}_{\vec{\varepsilon}}$ in the open right half-plane. The following simple example shows that there exist transfer functions which are not well-posed and which have the property that the delayed closed-loop transfer function is spectrally stable for all sufficiently small delays.

Example 3.6. Consider the transfer function $\mathbf{H}(s) = 2e^s$ which clearly is not well-posed. The delayed closed-loop transfer function

$$\mathbf{G}_\varepsilon(s) = \frac{\mathbf{H}(s)}{1 + e^{-\varepsilon s}\mathbf{H}(s)} = \frac{2e^s}{1 + 2e^{(1-\varepsilon)s}}$$

is in $\mathscr{H}_-$ for any $\varepsilon \in (0, 1)$.

The next theorem shows that for a large class of ill-posed transfer functions arbitrarily small delays lead to closed-loop poles with arbitrarily large real parts. In order to state the theorem we introduce some more notation. For $\delta \in (0, \pi]$ define the open sector $\mathscr{S}(\delta)$ by

$$\mathscr{S}(\delta) := \{\lambda e^{\iota\psi} \mid \lambda \in (0, \infty), \psi \in (-\delta, \delta)\},$$

and for $\psi_0 \in [-\pi, \pi)$ set

$$\mathscr{S}(\psi_0, \delta) := e^{\iota\psi_0}\mathscr{S}(\delta).$$

Theorem 3.7. *Let* $\mathbf{H}$ *be a transfer function and assume the following*:

(1) *There exist* $\theta \in (0, \pi/2)$ *and* $\alpha > 0$ *such that* $\mathbf{H}(s)$ *is holomorphic in* $\mathscr{S}(\theta) \cap \mathbb{C}_\alpha$.
(2) *There exist numbers* $\rho > \alpha$, $v > 0$, $\psi_0 \in [-\pi, \pi)$, $\delta \in (0, \theta)$, *and* $\eta \in (0, \pi/2)$ *such that*

$$\lim_{|s|\to\infty, s\in\mathscr{S}(\delta)} r(\mathbf{H}(s)) = \infty, \tag{3.3}$$

$$r(\mathbf{H}(s)) \le |s|^v, \qquad \forall s \in \mathscr{S}(\delta) \cap \mathbb{C}_\rho, \tag{3.4}$$

$$\sigma(\mathbf{H}(s)) \subset \mathbb{C}\backslash\mathscr{S}(\psi_0, \eta), \qquad \forall s \in \mathscr{S}(\delta) \cap \mathbb{C}_\rho. \tag{3.5}$$

Then there exist sequences (ε_n) *and* (p_n) *with*

$$\varepsilon_n > 0, \qquad \varepsilon_n \to 0, \qquad p_n \in \mathbb{C}_0, \qquad \operatorname{Im} p_n \to \infty, \qquad \operatorname{Re} p_n \to \infty$$

and such that, for any $n \in \mathbb{N}$, p_n *is a pole of* $\mathbf{G}_{\varepsilon_n}$.

Condition (3.3) guarantees that $\mathbf{H}$ is not well-posed. If $\mathbf{G}_0$ is L^2-stable, then it follows from Lemma 6.3 in [16] that

$$\lim_{|s|\to\infty, s\in\mathscr{S}(\delta)} \|\mathbf{H}(s)\| = \infty \quad\Rightarrow\quad \lim_{|s|\to\infty, s\in\mathscr{S}(\delta)} r(\mathbf{H}(s)) = \infty.$$

Therefore Theorem 3.7 remains true if $r(\mathbf{H}(s))$ is replaced by $\|\mathbf{H}(s)\|$, provided that $\mathbf{G}_0$ is L^2-stable. Condition (3.5) says that the spectrum of $\mathbf{H}(s)$ does not spiral around the origin as s moves in a sector of sufficiently small angle.

Proof of Theorem 3.7. Let $\mathbf{H}(s)$ be of size $m \times m$ and let $s \in \mathscr{S}(\theta) \cap \mathbb{C}_\alpha$. Moreover, let ℓ be the largest nonnegative integer such that $\det(\lambda I - \mathbf{H}(s))$ can be written in the form

$$\det(\lambda I - \mathbf{H}(s)) = \lambda^\ell(\lambda^{m-\ell} + h_{m-\ell-1}(s)\lambda^{m-\ell-1} + \cdots + h_0(s)),$$

where the h_i are holomorphic functions defined on $\mathscr{S}(\theta) \cap \mathbb{C}_\alpha$. By (3.3), $\ell \le m - 1$, and by the maximality of ℓ we have that $h_0(s) \not\equiv 0$. Clearly, $\ell > 0$ if and only if $\det \mathbf{H}(s) \equiv 0$. For $s \in \mathscr{S}(\theta) \cap \mathbb{C}_\alpha$ define

$$\mathbf{h}_s(\lambda) := \lambda^{m-\ell} + h_{m-\ell-1}(s)\lambda^{m-\ell-1} + \cdots + h_0(s), \qquad \zeta(\mathbf{h}_s) := \{\lambda \in \mathbb{C} \mid \mathbf{h}_s(\lambda) = 0\}.$$

Obviously, for given $s_0 \in \mathscr{S}(\theta) \cap \mathbb{C}_\alpha$ we have

$$0 \in \zeta(\mathbf{h}_{s_0}) \quad\Leftrightarrow\quad h_0(s_0) = 0. \tag{3.6}$$

Moreover,

$$\zeta(\mathbf{h}_s) \subset \sigma(\mathbf{H}(s)), \qquad \forall s \in \mathscr{S}(\theta) \cap \mathbb{C}_\alpha. \tag{3.7}$$

Set $\Psi_0 := \{\lambda e^{\imath\psi_0} \mid \lambda \geq 0\}$ and let arg: $\mathbb{C}\backslash\Psi_0 \to (\psi_0, \psi_0 + 2\pi)$ be a continuous branch of the argument on $\mathbb{C}\backslash\Psi_0$. Then, by (3.5), $\arg \lambda$ is well defined for any $\lambda \in \sigma(\mathbf{H}(s))\backslash\{0\}$, where $s \in \mathscr{S}(\delta) \cap \mathbb{C}_\rho$. For all such s we define

$$A_s := \{\arg \lambda \mid \lambda \in \sigma(\mathbf{H}(s))\backslash\{0\}\}.$$

In the following let $\beta \in (0, \tan \delta)$. By (3.3) there exists $R \geq \rho$ such that $A_s \neq \varnothing$ for all $s \in \mathscr{S}(\delta) \cap \mathbb{C}_R$, and

$$\log r(\mathbf{H}(s)) - \frac{\sup A_s + 3\pi}{\beta} > 0, \qquad \forall s \in \mathscr{S}(\delta) \cap \mathbb{C}_R. \tag{3.8}$$

For a, $b \in \mathbb{C}$, let $[a, b]$ denote the closed segment in the complex plane with endpoints a and b. We define $\mu := \inf_{s \in \mathscr{S}(\delta) \cap \mathbb{C}_R}(\inf A_s + 3\pi) > 0$ and

$$\alpha_n := \frac{\mu}{2\nu \log(\sqrt{1 + \beta^2}x_n)}, \tag{3.9}$$

where (x_n) is a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = \infty$. We choose the x_n such that $x_n > \max(\alpha, R)$ and $\alpha_n < \beta$ for all $n \in \mathbb{N}$ and

$$h_0(s) \neq 0, \qquad \forall s \in [x_n, x_n + \imath\beta x_n], \quad \forall n \in \mathbb{N}. \tag{3.10}$$

Note that by the choice of β and x_n

$$x_n + \imath\alpha_n x_n \in [x_n, x_n + \imath\beta x_n] \subset \mathscr{S}(\delta) \cap \mathbb{C}_R, \qquad \forall n \in \mathbb{N}.$$

Using (3.4) and (3.9) we obtain that, for all $s \in [x_n, x_n + \imath\beta x_n]$,

$$\begin{aligned} \log r(\mathbf{H}(s)) - \frac{\inf A_s + 3\pi}{\alpha_n} &\leq \log |s|^\nu - \frac{\mu}{\alpha_n} \\ &\leq \nu \log(\sqrt{1 + \beta^2}x_n) - 2\nu \log(\sqrt{1 + \beta^2}x_n) \\ &< 0. \end{aligned} \tag{3.11}$$

Setting $z_n' := (1 + \imath\alpha_n)x_n$ and $z_n'' := (1 + \imath\beta)x_n$ we have that

$$[z_n', z_n''] \subset [x_n, x_n + \imath\beta x_n] \subset \mathscr{S}(\delta) \cap \mathbb{C}_R, \qquad \forall n \in \mathbb{N},$$

and, moreover,

$$\frac{\operatorname{Re} z_n'}{\operatorname{Im} z_n'} = \frac{1}{\alpha_n}, \qquad \frac{\operatorname{Re} z_n''}{\operatorname{Im} z_n''} = \frac{1}{\beta}.$$

Hence it follows from (3.11) and (3.8) that

$$\log r(\mathbf{H}(z_n')) - \frac{(\inf A_{z_n'} + 3\pi) \operatorname{Re} z_n'}{\operatorname{Im} z_n'} < 0, \qquad \forall n \in \mathbb{N}, \tag{3.12}$$

$$\log r(\mathbf{H}(z_n'')) - \frac{(\sup A_{z_n''} + 3\pi) \operatorname{Re} z_n''}{\operatorname{Im} z_n''} > 0, \qquad \forall n \in \mathbb{N}. \tag{3.13}$$

An application of Proposition 5.2 in [16] shows that there exists a holomorphic function λ_n defined on an open set $U_n \supset [z_n', z_n'']$ such that $|\lambda_n(z_n'')| = r(\mathbf{H}(z_n''))$ and $\lambda_n(s) \in \zeta(\mathbf{h}_s)$ for all $s \in U_n$.[3] A combination of (3.6) and (3.10) shows that $\lambda_n(s) \neq 0$ for all $s \in U_n$, provided we choose U_n small enough. It then follows from (3.12) and (3.13) that

$$\log|\lambda_n(z_n')| - \frac{(\arg \lambda(z_n') + 3\pi) \operatorname{Re} z_n'}{\operatorname{Im} z_n'} < 0, \qquad \forall n \in \mathbb{N}, \tag{3.14}$$

$$\log|\lambda_n(z_n'')| - \frac{(\arg \lambda(z_n'') + 3\pi) \operatorname{Re} z_n''}{\operatorname{Im} z_n''} > 0, \qquad \forall n \in \mathbb{N}. \tag{3.15}$$

We show that there exist sequences (ε_n) in $(0, \infty)$ and (p_n) in $\mathbb{C}_0$ with $\varepsilon_n \downarrow 0$ and $p_n \in [z_n', z_n'']$ and such that

$$\log \lambda_n(p_n) - \varepsilon_n p_n = -3\imath\pi, \qquad \forall n \in \mathbb{N}, \tag{3.16}$$

where we choose the branch of the logarithm to be $\log z = \log|z| + \imath \arg z$, with $z \in \mathbb{C}\backslash\Psi_0$ and $\log|z| \in \mathbb{R}$. Since, by (3.7), $\lambda_n(s) \in \sigma(\mathbf{H}(s))$ for all $s \in U_n$, it is easy to see that (3.16) is a sufficient condition for $\mathbf{G}_{\varepsilon_n}(s) = \mathbf{H}(s)(I + e^{-\varepsilon_n s}\mathbf{H}(s))^{-1}$ to have a pole at p_n.

For each $n \in \mathbb{N}$ and each $s \in [z_n', z_n'']$ the ray

$$R_n(s) := \{\log \lambda_n(s) - es \mid e \in [0, \infty)\}$$

intersects the horizontal line

$$L := \{s \in \mathbb{C} \mid \operatorname{Im} s = -3\pi\}.$$

Indeed, for $e = 0$ the corresponding point of $R_n(s)$ is above L, while for large $e > 0$ the corresponding point is below L.

Thus, for each $n \in \mathbb{N}$, we can define real-valued functions $w_n(s)$ and $e_n(s) > 0$ for $s \in [z_n', z_n'']$ such that

$$\log \lambda_n(s) - e_n(s)s = w_n(s) - 3\imath\pi. \tag{3.17}$$

Taking real and imaginary parts in (3.17) it follows that, for all $n \in \mathbb{N}$ and for all $s \in [z_n', z_n'']$,

$$e_n(s) = \frac{\arg \lambda_n(s) + 3\pi}{\operatorname{Im} s}, \tag{3.18}$$

$$w_n(s) = \log|\lambda_n(s)| - \frac{(\arg \lambda_n(s) + 3\pi) \operatorname{Re} s}{\operatorname{Im} s}. \tag{3.19}$$

Using (3.14) and (3.15) we see that, for all $n \in \mathbb{N}$,

$$w_n(z_n') < 0, \qquad w_n(z_n'') > 0.$$

[3] This is only true if $U_n \cap \mathfrak{C}_\mathbf{H} = \varnothing$, where $\mathfrak{C}_\mathbf{H}$ denotes the set of critical points of $\mathbf{H}$ as defined in [16]. However, since $\mathfrak{C}_\mathbf{H}$ is a discrete set, we can choose the x_n such that $[x_n, x_n + \imath\beta x_n] \cap \mathfrak{C}_\mathbf{H} = \varnothing$. Consequently, $U_n \cap \mathfrak{C}_\mathbf{H} = \varnothing$ for sufficiently small sets U_n.

Hence, since w_n is continuous on $[z_n', z_n'']$, there exists $p_n \in [z_n', z_n'']$ such that $w_n(p_n) = 0$ for all $n \in \mathbb{N}$. Setting $\varepsilon_n := e_n(p_n) > 0$ it follows from (3.17) that p_n and ε_n satisfy (3.16). Finally, Re $p_n = x_n \to \infty$, and, moreover,

$$\operatorname{Im} p_n \geq \alpha_n x_n = \frac{\mu x_n}{2v \log(\sqrt{1 + \beta^2 x_n})} \to \infty. \tag{3.20}$$

Combining (3.18) and (3.20) shows that $\varepsilon_n \to 0$. ∎

The next result gives a sufficient condition for assumption (2) in Theorem 3.7 to be satisfied.

Proposition 3.8. *Let* **H** *be a transfer function and suppose that assumption* (1) *in Theorem* 3.7 *is satisfied. Moreover, assume that there exist a number* $\alpha \in \mathbb{R}$ *and a transfer function* $\tilde{\mathbf{H}}$ *such that*

$$\mathbf{H}(s) = (a + bs)^v \tilde{\mathbf{H}}(s), \qquad \forall s \in \mathbb{C}_\alpha,$$

where $a, b \in \mathbb{C}$, $b \neq 0$, *and* $v > 0$. *If there exists* $\delta > 0$ *such that*

$$\lim_{|s| \to \infty, s \in \mathscr{S}(\delta)} \tilde{\mathbf{H}}(s) = D, \qquad D \in \mathbb{C}^{m \times m},$$

with det $D \neq 0$, *then* $\mathbf{H}(s)$ *satisfies assumption* (2) *in Theorem* 3.7.

The proof of Proposition 3.8 is straightforward and is left to the reader.

Finally, we give a simple instability result which applies to regular, well-posed as well as to ill-posed transfer functions.

Proposition 3.9. *Let* **H** *be a transfer function and assume that* $\mathbf{H} \in \mathscr{M}_-^{m \times m}$. *If*

$$\liminf_{\omega \to \infty, \imath\omega \notin \mathfrak{P}_\mathbf{H}} \sigma_{\max}(\mathbf{H}(\imath\omega)) < 1 \quad \textit{and} \quad \limsup_{\omega \to \infty, \imath\omega \notin \mathfrak{P}_\mathbf{H}} \sigma_{\min}(\mathbf{H}(\imath\omega)) > 1, \tag{3.21}$$

where $\sigma_{\max}$ *and* $\sigma_{\min}$ *denote the maximal*[4] *and the minimal singular value, respectively, then there exist sequences* (ε_n) *and* (ω_n) *in* $(0, \infty)$ *with*

$$\lim_{n \to \infty} \varepsilon_n = 0, \qquad \lim_{n \to \infty} \omega_n = \infty,$$

and such that, for any $n \in \mathbb{N}$, $\imath\omega_n$ *is a pole of* $\mathbf{G}_{\varepsilon_n}$.

We obtain a similar result if in (3.21) ∞ is replaced by $-\infty$. For the proof of Proposition 3.9 we need the following lemma.

Lemma 3.10. *Suppose that* **H** *is a transfer function and suppose there exists* $\varepsilon \geq 0$ *such that* $\mathbf{G}_\varepsilon \in \mathscr{M}_-^{m \times m}$. *If* $s_0 \in \mathbb{C}_0^{cl}$ *is a pole of* **H**, *but not a pole of* $\mathbf{G}_\varepsilon$, *then* $\lim_{s \to s_0} r(\mathbf{H}(s)) = \infty$.

[4] Recall that the maximal singular value of a matrix is equal to the operator norm induced by the Euclidean norm.

The above lemma is a slight generalization of Lemma 6.3 in [16] whose proof is extended easily to the present situation.

Proof of Proposition 3.9. First notice that

$$\sigma_{\min}(\mathbf{H}(s)) \le r(\mathbf{H}(s)) \le \sigma_{\max}(\mathbf{H}(s)), \qquad \forall s \in \mathbb{C}_0^{cl}, \quad s \notin \mathfrak{P}_{\mathbf{H}}. \tag{3.22}$$

By assumption there exist sequences (ω_n') and (ω_n'') in $(0, \infty)$ with

$$\omega_n' < \omega_n'', \qquad \lim_{n\to\infty} \omega_n' = \lim_{n\to\infty} \omega_n'' = \infty,$$

and such that

$$\sigma_{\max}(\mathbf{H}(\imath\omega_n')) < 1, \qquad \sigma_{\min}(\mathbf{H}(\imath\omega_n'')) > 1. \tag{3.23}$$

There are two cases: either the transfer function $\mathbf{H}$ is continuous on $[\imath\omega_n', \imath\omega_n'']$ or it is not. In the latter case the segment $[\imath\omega_n', \imath\omega_n'']$ contains poles of $\mathbf{H}$.

Case 1. If $\mathbf{H}(s)$ is continuous on $[\imath\omega_n', \imath\omega_n'']$, then so is $r(\mathbf{H}(s))$. Therefore, it follows from (3.22) and (3.23) that there exists $\omega_n \in (\omega_n', \omega_n'')$ such that $r(\mathbf{H}(\imath\omega_n)) = 1$. This shows that there exist numbers $\psi_n \in [0, 2\pi)$ such that $e^{\imath\psi_n} \in \sigma(\mathbf{H}(\imath\omega_n))$. Defining $\varepsilon_n := (\psi_n + \pi)/\omega_n$, we obtain

$$-1 = e^{\imath\psi_n} e^{-\imath\varepsilon_n\omega_n} \in \sigma(e^{-\imath\varepsilon_n\omega_n}\mathbf{H}(\imath\omega_n)),$$

which in turn implies that $\mathbf{G}_{\varepsilon_n}$ has a pole at $\imath\omega_n$. By construction, $\lim_{n\to\infty} \omega_n = \infty$, and therefore $\lim_{n\to\infty} \varepsilon_n = 0$.

Case 2. If $\mathbf{H}$ is not continuous on $[\imath\omega_n', \imath\omega_n'']$, then there exists $\tilde{\omega}_n \in (\omega_n', \omega_n'')$ such that $\imath\tilde{\omega}_n$ is a pole of $\mathbf{H}$. Moreover, we may choose $\tilde{\omega}_n$ such that $\mathbf{H}$ is continuous on $[\imath\omega_n', \imath\tilde{\omega}_n)$. If $\lim_{\omega\to\tilde{\omega}_n} r(\mathbf{H}(\imath\omega)) = \infty$, then, by (3.22) and (3.23), there exists $\omega_n \in (\omega_n', \tilde{\omega}_n)$ such that $r(\mathbf{H}(\imath\omega_n)) = 1$ and we can argue as in Case 1. If $\lim_{\omega\to\tilde{\omega}_n} r(\mathbf{H}(\imath\omega)) \neq \infty$ or if this limit does not exist, then an application of Lemma 3.10 shows that $\imath\tilde{\omega}_n$ is a pole of $\mathbf{G}_\varepsilon$ for any $\varepsilon \ge 0$. ■

Robustness Results

The following result complements Theorem 3.3.

Theorem 3.11. *Let* $\mathbf{H}$ *be a transfer function and suppose that* $\mathbf{G}_0$ *is* L^2*-stable (spectrally stable). If* $\gamma(\mathbf{H}) < 1$, *then there exists* $\varepsilon^* > 0$ *such that* $\mathbf{G}_\varepsilon$ is L^2-stable *(spectrally stable) for all* $\varepsilon \in (0, \varepsilon^*)$.

The part of the theorem which relates to L^2-stability is proved in Theorem 6.1 of [16]. Robustness of spectral stability can be shown in a similar way and is therefore left to the reader.

Theorem 3.11 deals with perturbations of the form $e^{-\varepsilon s}$. In a multivariable setting it is natural to consider the more general class of *multidelay perturbations* of the form $\mathbf{E}_{\bar{\varepsilon}}$. There are simple examples which show that Theorem 3.11 does not remain true for multidelay perturbations, cf. Example 6.4 of [16]. A robustness result for multidelay perturbations can be derived by using structured singular values (see, e.g., [19]). To this end set

$$\mathbf{\Delta} := \{\mathrm{diag}_{1\le j\le m}(s_j) \mid s_j \in \mathbb{C}\} \subset \mathbb{C}^{m\times m}.$$

The structured singular value $\mu_{\mathbf{\Delta}}(M)$ of $M \in \mathbb{C}^{m\times m}$ with respect to $\mathbf{\Delta}$ is defined by

$$\mu_{\mathbf{\Delta}}(M) := \frac{1}{\min\{\|\Delta\| \mid \Delta \in \mathbf{\Delta},\, \det(I - M\Delta) = 0\}},$$

unless no $\Delta \in \mathbf{\Delta}$ makes $I - M\Delta$ singular, in which case $\mu_{\mathbf{\Delta}}(M) := 0$. For any $M \in \mathbb{C}^{m\times m}$ we have that $r(M) \le \mu_{\mathbf{\Delta}}(M) \le \|M\|$, see [19].

Theorem 3.12. *Let* $\mathbf{H}$ *be a transfer function and suppose that* $\mathbf{G}_0$ *is spectrally stable* (L^2*-stable*). *If*

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0\setminus\mathfrak{P}_{\mathbf{H}}} \mu_{\mathbf{\Delta}}(\mathbf{H}(s)) < 1, \tag{3.24}$$

then there exists $\varepsilon^* > 0$ *such that the transfer function* $\mathbf{G}_{\vec{\varepsilon}}$ *is spectrally stable* (L^2*-stable*) *for all* $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m) \in [0, \infty)^m$ *satisfying* $\|\vec{\varepsilon}\| < \varepsilon^*$.

Proof. In the following sections we do not use the L^2-stability part of the theorem. Therefore we prove only the part which relates to spectral stability. So we assume that (3.24) holds and that $\mathbf{G}_0$ is spectrally stable, i.e., $\mathbf{G}_0 \in \mathscr{H}_-^{m\times m}$. As already mentioned, the latter implies that $\mathbf{H} \in \mathscr{M}_-^{m\times m}$. We proceed in three steps.

Step 1. By (3.24) there exist numbers $R > 0$ and $\rho \in (0, 1)$ such that

$$r(\mathbf{H}(s)) \le \mu_{\mathbf{\Delta}}(\mathbf{H}(s)) \le \rho, \qquad \forall s \in E_R\setminus\mathfrak{P}_{\mathbf{H}}, \tag{3.25}$$

where $E_R := \{s \in \mathbb{C}_0 \mid |s| > R\}$. Since $\mathbf{G}_0$ is spectrally stable, it follows from Lemma 3.10 that $\mathbf{H}$ has no poles in E_R^{cl}.

Step 2. By Step 1, $E_R^{cl} \cap \mathfrak{P}_{\mathbf{H}} = \varnothing$, and hence we obtain from (3.25) using the continuity of $\mu_{\mathbf{\Delta}}(\cdot)$ (cf. [19])

$$\mu_{\mathbf{\Delta}}(\mathbf{H}(s)) \le \rho < 1, \qquad \forall s \in E_R^{cl}. \tag{3.26}$$

Clearly, $\|\mathbf{E}_{\vec{\varepsilon}}(s)\| \le 1$ for all $\vec{\varepsilon} \in [0, \infty)^m$ and all $s \in E_R^{cl}$, and hence

$$\det(I + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s)) \ne 0, \qquad \forall \vec{\varepsilon} \in [0, \infty)^m, \qquad \forall s \in E_R^{cl}.$$

Combining this with the result of Step 1, we see that, for any $\vec{\varepsilon} \in [0, \infty)^m$, $\mathbf{G}_{\vec{\varepsilon}}$ has no poles in E_R^{cl}.

Step 3. It remains to show that there exists $\varepsilon^* > 0$ such that, for all $\vec{\varepsilon} \in \mathbb{R}_+^m$ with $\|\vec{\varepsilon}\| < \varepsilon^*$, $\mathbf{G}_{\vec{\varepsilon}}$ has no poles in the set $D_R := \{s \in \mathbb{C}_0^{cl} \mid |s| \le R\}$. To this end let $(\mathbf{N}, \mathbf{D})$ be a right-coprime factorization of $\mathbf{H}$ over $\mathscr{H}_-$. Then $\mathbf{G}_{\vec{\varepsilon}} = \mathbf{N}(\mathbf{D} + \mathbf{E}_{\vec{\varepsilon}}\mathbf{N})^{-1}$, and it is easy to show that $(\mathbf{N}, \mathbf{D} + \mathbf{E}_{\vec{\varepsilon}}\mathbf{N})$ is a right-coprime factorization of $\mathbf{G}_{\vec{\varepsilon}}$ over $\mathscr{H}_-$.

Since $\mathbf{G}_0 \in \mathscr{H}_-^{m \times m}$, it follows via Lemma 2.1(i) that

$$\inf_{s \in D_R} |\det(\mathbf{D}(s) + \mathbf{N}(s))| > 0. \tag{3.27}$$

By the compactness of D_R it is clear that $\mathbf{E}_{\vec{\varepsilon}}(s)$ converges uniformly to I on D_R as $\|\vec{\varepsilon}\| \to 0$. Consequently, (3.27) yields that there exists $\varepsilon^* > 0$ such that, for all $\vec{\varepsilon} \in \mathbb{R}_+^m$ with $\|\vec{\varepsilon}\| < \varepsilon^*$,

$$\inf_{s \in D_R} |\det(\mathbf{D}(s) + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{N}(s))| > 0,$$

which in turn implies that for all $\vec{\varepsilon}$ as above $\mathbf{G}_{\vec{\varepsilon}}$ has no poles in D_R. ∎

It seems to be a difficult open problem whether the condition

$$\limsup_{|s| \to \infty, s \in \mathbb{C}_0 \setminus \mathfrak{P}_{\mathrm{H}}} \mu_\Delta(\mathbf{H}(s)) > 1$$

implies a lack of robustness with respect to small multidelay perturbations.

4. Robustness and Nonrobustness Results for PDEs with Boundary and In-Span Control

In the following we introduce a class of multiple-input multiple-output controlled and observed PDEs which are linear in time and of spatial dimension 1, with coefficients which may depend upon the spatial variable. Our choice of class is motivated by our desire to include any such system that arises in the control literature, including dynamic control. We wish to include coupled systems, so we break up the space interval so that on different parts of the interval different PDEs are satisfied, and we allow natural in-span coupling conditions. The control is allowed to be applied, through general linear boundary operators, on the boundary or at the in-span coupling points. These boundary operators are sufficiently general to allow higher-order differential equations at the boundary and in-span points. The observation operators are even more general, since we allow distributed observation as well. Distributed control requires a different analysis, which can be found in [14].

The class of systems in this section includes all of the systems considered by Datko and collaborators [4]–[8] which have spatial dimension 1. For instance, we can use this approach to explain fully the modal destabilization results in [6] and [7]. On the other hand, the destabilization described in [8], Example 2 of [4], or Section 3 of [4] cannot be analyzed by the methods in this paper, since the control space in these systems is infinite dimensional.

We suppose that the space variable x belongs to some closed interval $[a, b]$. Without loss of generality we assume that $[a, b] = [0, 1]$. Let $\lambda \in \mathbb{N}$, and $\{x_l\}_{l=1}^{\lambda} \subset (0, 1)$, where $x_1 < x_2 < \cdots < x_\lambda$. These numbers determine a decomposition of $(0, 1)$ into $\lambda + 1$ open intervals $\{I_k\}_{k=0}^{\lambda}$. Let $\imath \in \mathbb{N}$. For $j = 0, \ldots, \imath$ and $k = 0, \ldots, \lambda$, let p_j^k be polynomials and let a_j^k be continuous functions on I_k^{cl}. The class of PDEs we consider is of the form

$$\sum_{j=0}^{\iota} a_j^k(x)p_j^k(\mathscr{D})\frac{\partial^j w}{\partial t^j}(x,t)=0, \qquad x\in I_k, \quad t>0, \tag{4.1}$$

where $\mathscr{D}$ denotes differentiation with respect to x. Set $n_k := \max_{0\le j\le \iota} \deg p_j^k$ and let $p_k(x, s)$ be the coefficient of $\mathscr{D}^{n_k}$ in the expression

$$\sum_{j=0}^{\iota} a_j^k(x)s^j p_j^k(\mathscr{D}),$$

where s is a complex variable. We introduce the following assumption:

(A1) There exists an open set $\Omega \supset \mathbb{C}_0^{cl}$ such that, for any $k = 0, \ldots, \lambda$,

$$p_k(x,s)\neq 0, \qquad \forall x\in I_k^{cl}, \quad \forall s\in\Omega.$$

This condition guarantees that when the Laplace transforms of the PDE in I_k are taken, the resulting ordinary differential equation is not degenerate.

To define boundary operators for the PDE (4.1), we note that, by **(A1)**, the PDE has spatial order n_k in I_k, so we need

$$n := \sum_{k=0}^{\lambda} n_k$$

boundary conditions. While the boundary for $\bigcup_{k=0}^{\lambda} I_k$ is $\{x_l\}_{l=1}^{\lambda} \cup \{0, 1\}$, for the purpose of defining boundary conditions each x_j should be represented by x_j^+ and x_j^-. This allows coupling conditions (for example, $\mathscr{D}w(0.5^-, t) = \mathscr{D}w(0.5^+, t)$) and in-span control (see, for instance, [2], [13], and [21]). Therefore, we consider the boundary set to be $\{x_l^-\}_{l=1}^{\lambda} \cup \{x_l^+\}_{l=1}^{\lambda} \cup \{0, 1\}$, which we rename as $\{z_l\}_{l=1}^{\mu}$, where $\mu = 2(\lambda + 1)$. For any piecewise continuous function $f\colon [0, 1] \to \mathbb{C}$ and for $l = 1, \ldots, \lambda$ we define

$$f(x_l^-) := \lim_{x\nearrow x_l} f(x), \qquad f(x_l^+) := \lim_{x\searrow x_l} f(x),$$

so that $f(z_l)$ is a well-defined complex number for all $l = 1, \ldots, \mu$.

Let $q_{l,j}^i$ be polynomials for $i = 1, \ldots, n$, $j = 0, \ldots, \iota$, $l = 1, \ldots, \mu$. We define boundary operators B_i on solutions $w(x, t)$ of (4.1) by

$$(B_i w)(t) = \sum_{l=1}^{\mu}\sum_{j=0}^{\iota} q_{l,j}^i(\mathscr{D})\frac{\partial^j}{\partial t^j} w(z_l, t). \tag{4.2}$$

We need to impose bounds on the order of the spatial derivatives in (4.2). In particular, we do not wish to take spatial derivatives at the boundary of I_k which are of order larger than $n_k - 1$. To this end it is useful to introduce the function $\kappa\colon \{1, \ldots, \mu\} \to \{0, \ldots, \lambda\}$ given by

$$\kappa(l) := \begin{cases} 0 & \text{if} \quad z_l = 0, \\ \lambda & \text{if} \quad z_l = 1, \\ l_0 & \text{if} \quad z_l = x_{l_0}^+, \\ l_0 - 1 & \text{if} \quad z_l = x_{l_0}^-. \end{cases} \tag{4.3}$$

We assume that, for any $i = 1, \ldots, n, j = 0, \ldots, \iota, l = 1, \ldots, \mu$,

$$\deg q^i_{l,j} \leq n_{\kappa(l)} - 1. \tag{4.4}$$

The boundary operators (4.2) are sufficiently general to allow higher-order differential equations at the boundary, as in the "hybrid systems" in [12].

Let m be a positive integer with $m \leq n$. We consider the following boundary conditions for the PDE (4.1):

$$(B_i w)(t) = u_i(t), \qquad i = 1, \ldots, m, \tag{4.5a}$$

$$(B_i w)(t) = 0, \qquad i = m + 1, \ldots, n, \tag{4.5b}$$

where $u(t) := [u_1(t), \ldots, u_m(t)]^T \in \mathbb{C}^m$ is the control input. We need to impose a suitable independence condition on the boundary operators. This is done later in the section, when its relevance can be made clear.

The observation operators for the system are more general than the boundary operators B_i, since we allow a distributed component. For $i = 1, \ldots, m, j = 0, \ldots, \iota$, $l = 1, \ldots, \mu$, let $r^i_{l,j}$ be polynomials, let $f^i_j \in L^2(0, 1; \mathbb{C})$, and define the observation operators by

$$(C_i w)(t) = \sum_{l=1}^{\mu} \sum_{j=0}^{\iota} r^i_{l,j}(\mathscr{D}) \frac{\partial^j}{\partial t^j} w(z_l, t) + \sum_{j=0}^{\iota} \int_0^1 f^i_j(x) \frac{\partial^j}{\partial t^j} w(x, t)\, dx. \tag{4.6}$$

We assume that, for any $i = 1, \ldots, m, j = 0, \ldots, \iota, l = 1, \ldots, \mu$,

$$\deg r^i_{l,j} \leq n_{k(l)} - 1. \tag{4.7}$$

The observation for the system given by the PDEs (4.1) and the boundary conditions (4.5) is $y(t) = [y_1(t), \ldots, y_m(t)]^T \in \mathbb{C}^m$, where

$$y_i(t) = (C_i w)(t), \qquad i = 1, \ldots, m. \tag{4.8}$$

We refer to the observed boundary control system given by (4.1), (4.5), and (4.8) as the *open-loop system.* In the following it is denoted by (**OBC**).

Application of output feedback of the form $u_i(t) = v_i(t) - y_i(t - \varepsilon_i)$, $i = 1, \ldots, m$, leads to

$$(B_i w)(t) + (C_i w)(t - \varepsilon_i) = v_i(t), \qquad i = 1, \ldots, m, \tag{4.9}$$

where the $\varepsilon_i \geq 0$ are time-delays and $v(t) = [v_1(t), \ldots v_m(t)]^T$ denotes the input of the feedback system. We set $\vec{\varepsilon} := (\varepsilon_1, \ldots \varepsilon_m)$ and refer to the system given by (4.1), (4.5b), (4.9), and (4.8) as the *closed-loop system with delay* $\vec{\varepsilon}$. In the following it is denoted by ($\mathbf{CBC}_{\vec{\varepsilon}}$). If $\vec{\varepsilon} = 0$, then we call ($\mathbf{CBC}_0$) the *undelayed closed-loop system.* If $u(t) \equiv 0$ (resp. $v(t) \equiv 0$), then we refer to (**OBC**) (resp. ($\mathbf{CBC}_{\vec{\varepsilon}}$)) as the *uncontrolled* open-loop system (resp. *uncontrolled* closed-loop system with delay $\vec{\varepsilon}$).

Existence of Exponential Solutions

Let $s \in \mathbb{C}$. We are looking for exponential solutions of the form

$$w(x, t) = e^{st}\varphi(x), \qquad \text{where} \quad \varphi \in L^2(0, 1; \mathbb{C}), \quad \varphi \not\equiv 0, \tag{4.10}$$

of the uncontrolled open and closed-loop systems. A solution of the form (4.10) is called a *mode* of (**OBC**) (resp. (**CBC**$_{\bar{\varepsilon}}$)). The complex number s is called the *exponent* of the mode. We say that a mode is *stable* if $\operatorname{Re} s < 0$, otherwise we say it is *unstable*.

In the following we first derive a necessary and sufficient condition for the existence of modes for (**OBC**). To this end suppose that (4.10) is a solution of the uncontrolled system (**OBC**). Since φ in general depends on s, we write $\varphi(x) = \varphi(x, s)$. Now $w(x, t)$ solves the underlying PDE (4.1), and hence it follows that $\varphi(x, s)$ satisfies

$$\left(\sum_{j=0}^{l} a_j^k(x)s^j p_j^k(\mathscr{D})\right)\varphi(x, s) = 0, \qquad x \in I_k. \tag{4.11}$$

Using assumption (**A1**), we see that, for every $s \in \Omega \supset \mathbb{C}_0^{cl}$ and every $k = 0, \ldots, \lambda$, (4.11) is an ordinary differential equation of order n_k on I_k^{cl}. Let $\{e_j^k\}_{j=1}^{n_k}$ be a basis of $\mathbb{C}^{n_k}$ and let $\{\varphi_j^k(\cdot, s)\}_{j=1}^{n_k}$ be solutions of (4.11) on I_k^{cl} satisfying

$$(\varphi_j^k(x_k, s), \mathscr{D}\varphi_j^k(x_k, s), \ldots, \mathscr{D}^{n_k-1}\varphi_j^k(x_k, s)) = e_j^k, \qquad j = 1, \ldots, n_k, \quad s \in \Omega,$$

where $x_0 := 0$. Clearly, for any $k = 0, \ldots, \lambda$ and any $s \in \Omega$, the functions $\varphi_j^k(\cdot, s)$ are linearly independent, and hence span the solution space of (4.11) on I_k^{cl}. In particular, for every $s \in \Omega$, $\varphi_j^k(\cdot, s) \in L^2(I_k^{cl}, \mathbb{C})$, and every solution of (4.11) in I_k can be written in the form

$$\sum_{j=1}^{n_k} A_j^k(s)\varphi_j^k(x, s), \qquad x \in I_k,$$

for some coefficients $A_j^k(s)$. For $k = 0, \ldots, \lambda$ and $j = 1, \ldots, n_k$, we define

$$\Phi_j^k(x, s) := \begin{cases} \varphi_j^k(x, s) & \text{for } x \in I_k^{cl}, \\ 0 & \text{for } x \in [0, 1] \backslash I_k^{cl}. \end{cases}$$

It is convenient to rename these n vectors as $\{\Phi_p(x, s)\}_{p=1}^n$. It is clear that any solution of (4.11) can be written as

$$\varphi(x, s) = \sum_{p=1}^{n} A_p(s)\Phi_p(x, s), \qquad x \in \bigcup_{k=0}^{\lambda} I_k, \tag{4.12a}$$

$$\varphi(z_l, s) = \sum_{p=1}^{n} A_p(s)\Phi_p(z_l, s), \qquad l = 1, \ldots, \mu, \tag{4.12b}$$

for some coefficients $A_p(s)$.

Lemma 4.1. *The following statements hold true:*

(i) *For any $p = 1, \ldots, n$ and any $l = 1, \ldots, \mu$, the function*

$$\Omega \to \mathbb{C}, \qquad s \mapsto \mathscr{D}^i\Phi_p(z_l, s)$$

is holomorphic, provided that $i \le n_{\kappa(l)} - 1$, where κ is given by (4.3).

(ii) *For any* $p = 1, \dots, n$ *and any function* $f \in L^2(0, 1; \mathbb{C})$, *the function*

$$\Omega \to \mathbb{C}, \qquad s \mapsto \int_0^1 f(x)\Phi_p(x, s)\,dx$$

is holomorphic.

Proof. There exists $k \in \{0, \dots, \lambda\}$ and $p_0 \in \{1, \dots, n_k\}$ such that $\Phi_p = \Phi_{p_0}^k$. It is convenient to define $x_0^+ := 0$ and $x_{\lambda+1}^- := 1$.

In order to prove statement (i), assume first that $z_l \in \{x_k^+, x_{k+1}^-\} = \partial I_k$. Then $\kappa(l) = k$, and therefore $i \le n_k - 1$ by hypothesis. Consequently, the claim follows from a well-known result on parameter-dependent ordinary differential equations —see Theorem 8.4 in Chapter 1 of [3]. If $z_l \notin \{x_k^+, x_{k+1}^-\}$, then there exists $x^* \in \{x_1, \dots, x_\lambda\} \cup \{0, 1\}$ such that, for all $i \in \mathbb{N}$ and all $s \in \Omega$,

$$\mathscr{D}^i\Phi_p(z_l, s) = \mathscr{D}^i\Phi_{p_0}^k(z_l, s) = \lim_{x \to x^*, x \notin I_k^{cl}} \mathscr{D}^i\Phi_{p_0}^k(x, s) = 0,$$

showing that the claim is true in this case also.

To prove statement (ii), recall from Chapter 1, Theorem 8.4, of [3] that $\Phi_{p_0}^k$ is continuous in (x, s) for $x \in I_k^{cl}$ and $s \in \Omega$ and holomorphic in s for each fixed $x \in I_k^{cl}$. The result then follows from a standard argument using theorems of Morera and Fubini. ∎

Making use of (4.12b), the boundary conditions $(B_i w)(t) = 0$, $i = 1, \dots, n$, of the uncontrolled open-loop system can be expressed as follows:

$$\sum_{p=1}^{n} A_p(s)(\hat{B}_i\Phi_p)(s) = 0, \qquad i = 1, \dots, n, \tag{4.13}$$

where

$$(\hat{B}_i\Phi_p)(s) = \sum_{l=1}^{\mu} \sum_{j=0}^{l} s^j q_{l,j}^i(\mathscr{D})\Phi_p(z_l, s). \tag{4.14}$$

Let $\mathbf{D}(s)$ be the $n \times n$ matrix

$$\mathbf{D}(s) := ((\hat{B}_i\Phi_p)(s)), \qquad i, p = 1, \dots, n. \tag{4.15}$$

Notice that, by (4.4) and Lemma 4.1, $\mathbf{D}(s)$ is holomorphic on Ω. Setting $A(s) := (A_1(s), \dots, A_n(s))^T$, (4.13) can be written as $\mathbf{D}(s)A(s) = 0$, and we see that if the system (**OBC**) has a mode with exponent $s_0 \in \Omega$, then

$$\det \mathbf{D}(s_0) = 0. \tag{4.16}$$

It is easy to show that the above argument can be reversed, i.e., (4.16) guarantees that (**OBC**) has a mode with exponent s_0.

Summarizing our discussion we obtain the following result.

Proposition 4.2. *Suppose that* (**A1**) *is satisfied and let* $s_0 \in \Omega$. *Then* (**OBC**) *has a mode with exponent* s_0 *if and only if* (4.16) *is satisfied.*

In order to prove a similar result for the closed-loop system ($\mathbf{CBC}_{\vec{\varepsilon}}$), suppose that (4.10) is a solution of the uncontrolled system ($\mathbf{CBC}_{\vec{\varepsilon}}$). Again we indicate the dependence of φ on s explicitly by writing $\varphi(x) = \varphi(x, s)$. Since the underlying PDE (see (4.1)) is the same as for the open-loop system (**OBC**), it follows that $\varphi(x, s)$ is of the form (4.12). Consequently, the boundary conditions (4.5b) and the closed-loop boundary conditions (4.9) with $v_i \equiv 0$ lead to

$$\sum_{p=1}^{n} A_p(s)((\hat{B}_i\Phi_p)(s) + e^{-\varepsilon_i s}(\hat{C}_i\Phi_p)(s)) = 0, \qquad i = 1, \dots, m, \tag{4.17}$$

$$\sum_{p=1}^{n} A_p(s)(\hat{B}_i\Phi_p)(s) = 0, \qquad i = m+1, \dots, n, \tag{4.18}$$

where

$$(\hat{C}_i\Phi_p)(s) := \sum_{l=1}^{\mu} \sum_{j=0}^{t} s^j r_{l,j}^i(\mathscr{D})\Phi_p(z_l, s) + \sum_{j=0}^{t} s^j \int_0^1 f_j^i(x)\Phi_p(x, s)\, dx. \tag{4.19}$$

Let $\mathbf{N}(s)$ be the $m \times n$ matrix

$$\mathbf{N}(s) := ((\hat{C}_i\Phi_p)(s)), \qquad i = 1, \dots, m, \quad p = 1, \dots, n, \tag{4.20}$$

and set

$$\tilde{\mathbf{N}}(s) :\equiv \begin{pmatrix} I_{m \times m} \\ 0_{(n-m) \times m} \end{pmatrix}. \tag{4.21}$$

It follows from (4.7) and Lemma 4.1 that $\mathbf{N}(s)$ is holomorphic on Ω. Equations (4.17) and (4.18) can be written in the more compact form

$$(\mathbf{D}(s) + \tilde{\mathbf{N}}(s)\mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{N}(s))A(s) = 0, \tag{4.22}$$

where $\mathbf{E}_{\vec{\varepsilon}}(s) = \operatorname{diag}_{1 \le i \le m}(e^{-\varepsilon_i s})$. Setting

$$\mathbf{D}_{\vec{\varepsilon}}(s) := \mathbf{D}(s) + \tilde{\mathbf{N}}(s)\mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{N}(s), \tag{4.23}$$

we obtain the following closed-loop counterpart of Proposition 4.2.

Proposition 4.3. *Suppose that* (**A1**) *is satisfied and let* $s_0 \in \Omega$. *Then* ($\mathbf{CBC}_{\vec{\varepsilon}}$) *has a mode with exponent* s_0 *if and only if* $\det \mathbf{D}_{\vec{\varepsilon}}(s_0) = 0$.

While by construction $\mathbf{D}$ and $\mathbf{D}_{\vec{\varepsilon}}$ depend on the choice of the basis for the solution space of (4.11) on I_k^{cl}, it is a routine exercise to show that the zeros of $\det \mathbf{D}$ and $\det \mathbf{D}_{\vec{\varepsilon}}$ do not. Since the basis $\{\varphi_j^k(\cdot, s)\}_{j=1}^{n_k}$ is not always easy to compute, we can use a different, more convenient, choice of basis in order to compute the zeros of $\det \mathbf{D}$ and $\det \mathbf{D}_{\vec{\varepsilon}}$.

Transfer Functions

We shall need the following assumption:

(**A2**) $\det \mathbf{D}(s) \not\equiv 0.$

For a given $s_0 \in \Omega$, let $\mathscr{E}_{s_0}$ denote the vector space of all functions $\psi: (\bigcup_{k=0}^{\lambda} I_k) \times [0, \infty) \to \mathbb{C}$ of the form $\psi(x, t) = e^{s_0 t}\tilde{\psi}(x)$, where $\tilde{\psi}: \bigcup_{k=0}^{\lambda} I_k \to \mathbb{C}$ solves the ordinary differential equation (4.11) for $s = s_0$. It is easy to see that assumption **(A2)** is equivalent to

(A2′) There exists $s_0 \in \Omega$ such that the restricted boundary operators $B_i|_{\mathscr{E}_{s_0}}$ are linearly independent, i.e., if $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are such that $\sum_{i=1}^{n} \alpha_i B_i \psi = 0$ for all $\psi \in \mathscr{E}_{s_0}$, then $\alpha_i = 0$ for all $i = 1, \ldots, n$.

Remark 4.4. We illustrate **(A2)** and **(A2′)** in a simple situation. Suppose that there are no in-span points (i.e., $I_0 = [0, 1]$) and that $n = n_0 = 2$. Let the boundary operators B_1 and B_2 be given by $(B_1 w)(t) = w(0, t)$ and $(B_2 w)(t) = w_t(a, t)$, where $a \in \{0, 1\}$. Then it is easy to show that **(A2′)**, or equivalently **(A2)**, is satisfied if and only if $a \neq 0$. Notice, however, that B_1 and B_2 are linearly independent on the solution space of (4.1) even if $a = 0$.

If **(A2)** is satisfied, then we can define

$$\mathbf{H}(s) := \mathbf{N}(s)\mathbf{D}^{-1}(s)\tilde{\mathbf{N}}(s), \tag{4.24}$$

where $\mathbf{D}$, $\mathbf{N}$, and $\tilde{\mathbf{N}}$ are given by (4.15), (4.20), and (4.21), respectively. While $\mathbf{N}$ and $\mathbf{D}$ depend on the choice of the basis for the solution space of (4.11) on I_k^{cl}, it is a routine exercise to prove that the product $\mathbf{N}(s)\mathbf{D}^{-1}(s)$ does not (the corresponding transformation matrices cancel). Consequently, $\mathbf{H}$ is independent of the choice of $\{\varphi_j^k\}_{j=1}^{n_k}$, $k = 0, \ldots, \lambda$. Since $\mathbf{D}$, $\mathbf{N}$, and $\tilde{\mathbf{N}}$ have all their entries in $\mathscr{H}_-$, it follows from **(A2)** that $\mathbf{H} \in \mathscr{M}_-^{m \times m}$. In particular, $\mathbf{H}$ is a transfer function in the sense of Section 3. We say that $\mathbf{H}$ is the transfer function of **(OBC)**.

In order to show that $\mathbf{H}$ admits the usual dynamical interpretation, let the Laplace transform be denoted by the superscript "$\hat{\ }$" and let $u(\cdot)$ be a Laplace transformable input function. Moreover, let $w(x, t; u)$ denote the solution of **(OBC)** with initial conditions given by

$$\frac{\partial^j}{\partial t^j} w(\cdot, 0; u) = 0, \qquad j = 0, \ldots, \iota - 1.$$

Then $\hat{w}(x, s; u)$ satisfies the ordinary differential equation (4.11), and, consequently, the function $\hat{w}(x, s; u)$ is of the form (4.12). Hence, the boundary conditions (4.5) then imply

$$\sum_{p=1}^{n} A_p(s)(\hat{B}_i\Phi_p)(s) = \hat{u}_i(s), \qquad i = 1, \ldots, m,$$

$$\sum_{p=1}^{n} A_p(s)(\hat{B}_i\Phi_p)(s) = 0, \qquad i = m + 1, \ldots, n.$$

This is equivalent to

$$\mathbf{D}(s)A(s) = \tilde{\mathbf{N}}(s)\hat{u}(s),$$

so, by assumption (**A2**)

$$A(s) = \mathbf{D}^{-1}(s)\tilde{\mathbf{N}}(s)\hat{u}(s) \tag{4.25}$$

for all $s \in \Omega$ such that $\det \mathbf{D}(s) \neq 0$. Since $\hat{y}_i(s) = \sum_{p=1}^{n} A_p(s)(\hat{C}_i\Phi_p)(s)$, we can write $\hat{y}(s) = \mathbf{N}(s)A(s)$. Therefore, by (4.25), we see that

$$\hat{y}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)\tilde{\mathbf{N}}(s)\hat{u}(s) = \mathbf{H}(s)\hat{u}(s).$$

As in Section 3 we define

$$\mathbf{G}_{\vec{\varepsilon}} = \mathbf{H}(I + \mathbf{E}_{\vec{\varepsilon}}\mathbf{H})^{-1}. \tag{4.26}$$

Using Lemma 2.4(i), $\mathbf{G}_{\vec{\varepsilon}}$ can be written in the form

$$\mathbf{G}_{\vec{\varepsilon}}(s) = \mathbf{N}(s)(\mathbf{D}(s) + \tilde{\mathbf{N}}(s)\mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{N}(s))^{-1}\tilde{\mathbf{N}}(s) = \mathbf{N}(s)\mathbf{D}_{\vec{\varepsilon}}^{-1}(s)\tilde{\mathbf{N}}(s). \tag{4.27}$$

Of course, (4.27) only makes sense if $\det \mathbf{D}_{\vec{\varepsilon}}(s) \not\equiv 0$, or, equivalently, if $\det(I + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s)) \not\equiv 0$, see Lemma 2.4(i). We call $\vec{\varepsilon} \in [0, \infty)^m$ an *admissible* delay for (**OBC**) if $\det \mathbf{D}_{\vec{\varepsilon}}(s) \not\equiv 0$. If $\vec{\varepsilon}$ is an admissible delay for (**OBC**), then $\mathbf{G}_{\vec{\varepsilon}} \in \mathscr{M}_{-}^{m \times m}$, and we say that $\mathbf{G}_{\vec{\varepsilon}}$ is the transfer function of ($\mathbf{CBC}_{\vec{\varepsilon}}$). Going through the above steps with $\mathbf{D}$ replaced by $\mathbf{D}_{\vec{\varepsilon}}$, we see that the response $y(\cdot)$ of ($\mathbf{CBC}_{\vec{\varepsilon}}$) to the input function $v(\cdot)$ under zero initial conditions is given by $\hat{y}(s) = \mathbf{G}_{\vec{\varepsilon}}(s)\hat{v}(s)$.

Modal Stability and Small Delays

We say that ($\mathbf{CBC}_{\vec{\varepsilon}}$) is *modally stable* if ($\mathbf{CBC}_{\vec{\varepsilon}}$) has no unstable modes. This is the kind of internal stability which is considered in the literature on robust stabilization of PDEs, see [4]–[9] and [12]. The following corollary shows that spectral stability combined with bi-coprimeness is equivalent to modal stability.

Corollary 4.5. *Suppose that* (**A1**) *and* (**A2**) *are satisfied and let* $\vec{\varepsilon} \in [0, \infty)^m$. *Then the following statements are equivalent*:

(i) ($\mathbf{CBC}_{\vec{\varepsilon}}$) *is modally stable.*
(ii) $\vec{\varepsilon}$ *is an admissible delay for* (**OBC**), $\mathbf{G}_{\vec{\varepsilon}}$ *given by* (4.26) *is spectrally stable, and the triple* ($\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}}$) *is bi-coprime.*

The proof of Corollary 4.5 follows immediately from Proposition 2.3, Lemma 2.4, Proposition 4.3, and (4.27). In the following we write ($\mathbf{CBC}_\varepsilon$) for ($\mathbf{CBC}_{\vec{\varepsilon}}$) and $\mathbf{D}_\varepsilon$ for $\mathbf{D}_{\vec{\varepsilon}}$ if the components ε_i of $\vec{\varepsilon}$ satisfy $\varepsilon_i = \varepsilon$ for all $i = 1, \ldots, m$.

It follows from (4.27) and the analyticity of $\mathbf{N}$, $\tilde{\mathbf{N}}$, and $\mathbf{D}_{\vec{\varepsilon}}$ that if $\mathbf{G}_{\vec{\varepsilon}}(s)$ has a pole in $\mathbb{C}_0^{cl}$, then the closed-loop system ($\mathbf{CBC}_{\vec{\varepsilon}}$) has an unstable mode. Therefore, by an application of Theorem 3.3, Theorem 3.7, and Proposition 3.9, we immediately obtain conditions which guarantee the existence of unstable closed-loop modes. We give a precise formulation of these conditions in the following three corollaries.

Corollary 4.6. *Assume that* (**A1**) *and* (**A2**) *hold and suppose that* $\mathbf{H}$ *given by* (4.24) *satisfies one of the following two conditions*:

(i) $\mathbf{H}$ *is regular and the feedthrough matrix* D *of* $\mathbf{H}$ *satisfies* $r(D) > 1$.
(ii) $\mathbf{H}$ *is not well-posed and satisfies conditions* (1) *and* (2) *in Theorem* 3.7.

Then there exist sequences (ε_n) *and* (s_n) *with*

$$\varepsilon_n > 0, \qquad \lim_{n\to\infty} \varepsilon_n = 0, \qquad s_n \in \mathbb{C}_0, \qquad \lim_{n\to\infty} |\mathrm{Im}\, s_n| = \infty, \qquad \lim_{n\to\infty} \mathrm{Re}\, s_n = \infty, \tag{4.28}$$

and such that, for any $n \in \mathbb{N}$, $\det(\mathbf{D}_{\varepsilon_n}(s_n)) = 0$, *i.e., for any* $n \in \mathbb{N}$ *the closed-loop system* $(\mathbf{CBC}_{\varepsilon_n})$ *has a mode with exponent* s_n.

Corollary 4.7. *Assume that* **(A1)** *and* **(A2)** *hold and suppose that* $\mathbf{H}$ *given by* (4.24) *is regular. If* $\mathbf{G}_0$ *given by* (4.26) *is in* $(\mathbf{H}^\infty)^{m\times m}$ *and if* $\gamma(\mathbf{H}) > 1$ *(where* $\gamma(\mathbf{H})$ *is given by* (3.2)*), then there exist sequences* (ε_n) *and* (s_n) *with*

$$\varepsilon_n > 0, \qquad \lim_{n\to\infty} \varepsilon_n = 0, \qquad s_n \in \mathbb{C}_0, \qquad \lim_{n\to\infty} |\mathrm{Im}\, s_n| = \infty,$$

and such that, for any $n \in \mathbb{N}$, $\det(\mathbf{D}_{\varepsilon_n}(s_n)) = 0$, *i.e., for any* $n \in \mathbb{N}$ *the closed-loop system* $(\mathbf{CBC}_{\varepsilon_n})$ *has a mode with exponent* s_n.

Corollary 4.8. *Assume that* **(A1)** *and* **(A2)** *hold and suppose that* $\mathbf{H}$ *given by* (4.24) *satisfies* (3.21). *Then there exist sequences* (ε_n) *and* (ω_n) *in* $(0, \infty)$ *with*

$$\lim_{n\to\infty} \varepsilon_n = 0, \qquad \lim_{n\to\infty} \omega_n = \infty,$$

and such that, for any $n \in \mathbb{N}$, $\det(\mathbf{D}_{\varepsilon_n}(\imath\omega_n)) = 0$, *i.e., for any* $n \in \mathbb{N}$ *the closed-loop system* $(\mathbf{CBC}_{\varepsilon_n})$ *has a mode with exponent* $\imath\omega_n$.

On the other hand, when there exists $\varepsilon^* > 0$ such that $\mathbf{G}_{\vec{\varepsilon}}$ is spectrally stable for all $\vec{\varepsilon} \in [0, \infty)^m$ with $\|\vec{\varepsilon}\| < \varepsilon^*$, we cannot immediately conclude modal stability of $(\mathbf{CBC}_{\vec{\varepsilon}})$ for all such $\vec{\varepsilon}$. However, the following result shows that modal stability can be obtained by using Lemma 2.4 and Corollary 4.5.

Corollary 4.9. *Assume that* **(A1)** *and* **(A2)** *hold and suppose that* $(\mathbf{CBC}_0)$ *is modally stable. If* $\mathbf{H}$ *given by* (4.24) *satisfies* (3.24), *then there exists* $\varepsilon^* > 0$ *such that* $(\mathbf{CBC}_{\vec{\varepsilon}})$ *is modally stable for all* $\vec{\varepsilon} \in [0, \infty)^m$ *satisfying* $\|\vec{\varepsilon}\| < \varepsilon^*$.

Proof. Since $(\mathbf{CBC}_0)$ is modally stable, it follows from Proposition 4.3 that

$$\det(\mathbf{D}(s) + \tilde{\mathbf{N}}(s)\mathbf{N}(s)) \neq 0, \qquad \forall s \in \mathbb{C}_0^{cl}, \tag{4.29}$$

and it follows from (4.27) that $\mathbf{G}_0$ is spectrally stable. It follows from (3.24) that $I + \mathbf{E}_{\vec{\varepsilon}}\mathbf{H}$, and hence, by Lemma 2.4, also $\mathbf{D} + \tilde{\mathbf{N}}\mathbf{E}_{\vec{\varepsilon}}\mathbf{N}$, is invertible for all $\vec{\varepsilon} \in [0, \infty)^m$. Thus all $\vec{\varepsilon} \in [0, \infty)^m$ are admissible delays. Moreover, by Theorem 3.12 there exists $\varepsilon^* > 0$ such that $\mathbf{G}_{\vec{\varepsilon}}$ is spectrally stable for all $\vec{\varepsilon} \in [0, \infty)^m$ with $\|\vec{\varepsilon}\| < \varepsilon^*$. From Lemma 2.4 and (4.29) we obtain that $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ is bi-coprime. An application of Corollary 4.5 shows that $(\mathbf{CBC}_{\vec{\varepsilon}})$ is modally stable for all $\vec{\varepsilon} \in [0, \infty)^m$ with $\|\vec{\varepsilon}\| < \varepsilon^*$ and all $s \in \mathbb{C}_0^{cl}$. ■

Remark 4.10. In [17] the robustness of modal stability with respect to small delays is proved for a class of boundary controlled systems with arbitrary spatial dimension; the class of systems dealt with in [17] requires an analytic semigroup, and is hence less general (in all regards except spatial dimension) than the systems considered here.

5. Examples

In this section we illustrate Theorem 3.7, Proposition 3.9, and the corresponding Corollaries 4.6(ii) and 4.8 with some simple examples. Theorem 3.3 and Corollaries 4.6(i) and 4.7 are illustrated by the examples in [15].

Example 5.1. In the first example we consider an equation for two coupled vibrating strings, with two observation and two controls. This example is similar to Example 9.3 in [15], except instead of incorporating viscous damping into the model we use Kelvin–Voigt damping, and we draw very different conclusions. We assume that each string satisfies the damped wave equation

$$w_{tt}(x, t) - w_{xx}(x, t) - aw_{xxt}(x, t) = 0, \qquad x \in (0, 1) \cup (1, 2), \quad t > 0, \tag{5.1}$$

where $a > 0$. As in [15] we consider the following boundary conditions:

$$w(1^-, t) = w(1^+, t), \qquad w(2, t) = 0, \tag{5.2}$$

with boundary controls

$$w_x(1^-, t) - w_x(1^+, t) = u_1(t), \qquad w_x(0, t) = u_2(t), \tag{5.3}$$

and boundary observations

$$y_1(t) = k_1 w_t(1, t), \qquad y_2(t) = -k_2 w_t(0, t), \tag{5.4}$$

where $k_1, k_2 \geq 0$. Since $a \geq 0$, it is easily checked that assumption **(A1)** is satisfied.

Let $u(t) := [u_1(t), u_2(t)]^T$ and $y(t) := [y_1(t), y_2(t)]^T$ and set

$$r(s) := \frac{s}{\sqrt{1 + as}},$$

where $\sqrt{s}$ denotes the principle branch of the square root. Setting

$$A(s) := \frac{e^{4r(s)} - 1}{2(e^{4r(s)} + 1)}, \qquad B(s) := \frac{e^{r(s)} - e^{3r(s)}}{e^{4r(s)} + 1},$$

a routine calculation shows that assumption **(A2)** is satisfied and that the transfer function for (5.1)–(5.4) is given by

$$\mathbf{H}(s) = \sqrt{1 + as}\tilde{\mathbf{H}}(s),$$

where

$$\tilde{\mathbf{H}}(s) := \begin{pmatrix} k_1 A(s) & k_1 B(s) \\ k_2 B(s) & 2k_2 A(s) \end{pmatrix}.$$

Note that, for any $\delta \in (0, \pi/2)$, $\operatorname{Re} r(s) \to \infty$ as $|s| \to \infty$ in $\mathscr{S}(\delta)$. This implies that

$$\lim_{|s| \to \infty, s \in \mathscr{S}(\delta)} A(s) = \tfrac{1}{2}, \qquad \lim_{|s| \to \infty, s \in \mathscr{S}(\delta)} B(s) = 0,$$

so

$$\lim_{|s|\to\infty, s\in\mathscr{S}(\delta)} \tilde{\mathbf{H}}(s) = \begin{pmatrix} k_1/2 & 0 \\ 0 & k_2 \end{pmatrix}.$$

Consequently, **H** satisfies the conditions in Proposition 3.8, and hence conditions (1) and (2) in Theorem 3.7, provided that $k_1 k_2 \neq 0$. Therefore, the conclusions of Corollary 4.6(ii) hold for this example. In particular, we find that there exist sequences (ε_n) and (s_n) with $\varepsilon_n \downarrow 0$ and $\operatorname{Re} s_n \to \infty$ as $n \to \infty$ and such that the system (5.1)–(5.4) with feedback $u(t) = -y(t - \varepsilon_n)$ has a mode with exponent s_n.

We can contrast this to the situation in [15], where the damping term $aw_{xxt}(x, t)$ is replaced by the viscous damping term $2aw_t(x, t) + a^2 w(x, t)$. In [15] it is found that for every $a > 0$ there are values of k_1 and k_2 such that the input–output system is robust with respect to delays. In the viscous damping case all of the modes oscillate and decay with the same exponential rate, whilst in the Kelvin–Voigt case only finitely many modes oscillate and the rest have exponents which are negative real (pure exponential decay). So, even though, in this sense, Kelvin–Voigt damping is "stronger" than viscous damping, the present example shows that it also causes the system to be ill-posed, destroying robustness of the closed-loop system.

Example 5.2. In this example we consider a one-dimensional beam equation with structural damping (also known as $A^{1/2}$ damping). This type of damping causes the underlying semigroup to be analytic. The partial differential equation for the displacement $w(x, t)$ is

$$w_{tt}(x, t) + 2aw_{xxt}(x, t) + w_{xxxx}(x, t) = 0, \qquad x \in (0, 1), \quad t > 0, \tag{5.5}$$

with the damping parameter $a \in (0, 1)$. We consider the following boundary conditions:

$$w(0, t) = 0, \qquad w(1, t) = 0, \qquad w_x(0, t) = 0, \tag{5.6}$$

with boundary control

$$w_{xx}(1, t) = u(t) \tag{5.7}$$

and boundary observation

$$y(t) = cw_{xt}(1, t) + dw_t(1, t), \tag{5.8}$$

where we assume that $c, d \in \mathbb{R}$ and $c \neq 0$. Trivially, assumption (**A1**) is satisfied.

If there is no damping, i.e., $a = 0$, it is well known that this system is ill-posed in the sense that its transfer function is ill-posed. We will find that in spite of a positive damping term $a \in (0, 1)$, the system is still ill-posed, and in fact satisfies the conditions in Proposition 3.8. Let $\sqrt{s}$ denote the principle branch of the square root, define $\theta \in (\pi/2, \pi)$ by $e^{\imath\theta} = -a + \imath\sqrt{1 - a^2}$, and set $\eta := e^{\imath\theta/2}$. Computing the transfer function **H** of (5.5)–(5.8), we find that (**A2**) holds and we obtain

$$\mathbf{H}(s) = \sqrt{s}\tilde{\mathbf{H}}(s),$$

where

$$\tilde{\mathbf{H}}(s) = \frac{(c\eta + d/\sqrt{s})e^{\eta\sqrt{s}} + (c\bar{\eta} + d/\sqrt{s})e^{\bar{\eta}\sqrt{s}}g_1(s)}{[\eta^2 e^{\eta\sqrt{s}} + \bar{\eta}^2 e^{\bar{\eta}\sqrt{s}}g_1(s) + \eta^2 e^{-\eta\sqrt{s}}g_2(s) + \bar{\eta}^2 e^{-\bar{\eta}\sqrt{s}}g_3(s)]} + \frac{(-c\eta + d/\sqrt{s})e^{-\eta\sqrt{s}}g_2(s) + (-c\bar{\eta} + d/\sqrt{s})e^{-\bar{\eta}\sqrt{s}}g_3(s)}{[\eta^2 e^{\eta\sqrt{s}} + \bar{\eta}^2 e^{\bar{\eta}\sqrt{s}}g_1(s) + \eta^2 e^{-\eta\sqrt{s}}g_2(s) + \bar{\eta}^2 e^{-\bar{\eta}\sqrt{s}}g_3(s)]},$$

and the functions g_1, g_2, and g_3 are given by

$$g_1(s) = \frac{(\eta - \bar{\eta})e^{\eta\sqrt{s}} + (\eta + \bar{\eta})e^{-\eta\sqrt{s}} - 2\eta e^{-\bar{\eta}\sqrt{s}}}{(-\eta + \bar{\eta})e^{\bar{\eta}\sqrt{s}} + (\eta + \bar{\eta})e^{-\bar{\eta}\sqrt{s}} - 2\bar{\eta}e^{-\eta\sqrt{s}}},$$

$$g_2(s) = \frac{\eta + \bar{\eta} + 2\bar{\eta}g_1(s)}{\eta - \bar{\eta}},$$

$$g_3(s) = \frac{2\eta + (\eta - \bar{\eta})g_1(s)}{\bar{\eta} - \eta}.$$

We first note that $\pi/4 < \theta/2 < \pi/2$, so in particular $\operatorname{Re}\eta > 0$. Since $\operatorname{Re}(\eta - 2\bar{\eta}) < 0$, there exists $\delta > 0$ such that $e^{-\eta\sqrt{s}}$, $e^{-\bar{\eta}\sqrt{s}}$, and $e^{(\eta-2\bar{\eta})\sqrt{s}}$ all go to zero as s goes to infinity in $\mathscr{S}(\delta)$. Setting

$$\varepsilon(s) := g_1(s) + e^{(\eta-\bar{\eta})\sqrt{s}},$$

it is easy to see that

$$\lim_{|s|\to\infty, s\in\mathscr{S}(\delta)} \varepsilon(s) = 0, \qquad \lim_{|s|\to\infty, s\in\mathscr{S}(\delta)} \varepsilon(s)e^{(\bar{\eta}-\eta)\sqrt{s}} = 0. \tag{5.9}$$

Therefore

$$\lim_{|s|\to\infty, s\in\mathscr{S}(\delta)} g_2(s)e^{-\eta\sqrt{s}} = \lim_{|s|\to\infty, s\in\mathscr{S}(\delta)} \left(\frac{\eta + \bar{\eta}}{\eta - \bar{\eta}}e^{-\eta\sqrt{s}} - \frac{2\bar{\eta}}{\eta - \bar{\eta}}e^{-\bar{\eta}\sqrt{s}} + \varepsilon(s)\frac{2\bar{\eta}}{\eta - \bar{\eta}}e^{-\eta\sqrt{s}}\right) = 0,$$

and, similarly,

$$\lim_{|s|\to\infty, s\in\mathscr{S}(\delta)} g_3(s)e^{-\bar{\eta}\sqrt{s}} = 0.$$

Thus, by (5.9)

$$\lim_{|s|\to\infty, s\in\mathscr{S}(\delta)} \tilde{\mathbf{H}}(s) = \lim_{|s|\to\infty, s\in\mathscr{S}(\delta)} \frac{(c\eta + d/\sqrt{s})e^{\eta\sqrt{s}} - (c\bar{\eta} + d/\sqrt{s})(e^{\eta\sqrt{s}} - \varepsilon(s)e^{\bar{\eta}\sqrt{s}})}{\eta^2 e^{\eta\sqrt{s}} - \bar{\eta}^2(e^{\eta\sqrt{s}} - \varepsilon(s)e^{\bar{\eta}\sqrt{s}})} = \frac{c}{\eta + \bar{\eta}}.$$

Since $\operatorname{Re}\eta \neq 0$ and $c \neq 0$, this limit is finite and nonzero. Consequently, $\mathbf{H}$ satisfies the conditions in Proposition 3.8, and hence conditions (1) and (2) in Theorem 3.7. Therefore, the conclusions of Corollary 4.6(ii) hold, so there exist sequences (ε_n)

and (s_n) with $\varepsilon_n \downarrow 0$ and $\operatorname{Re} s_n \to \infty$ as $n \to \infty$ and such that the system (5.5)–(5.8) with feedback $u(t) = -y(t - \varepsilon_n)$ has a mode with exponent s_n.

Example 5.3. In this example we illustrate Proposition 3.9 by considering robustness with respect to delays for the coupled beam example from [21]. Let $x_1 \in (0, 1)$ and consider the following system:

$$w_{tt}(x, t) - w_{xxxx}(x, t) = 0, \qquad x \in (0, x_1) \cup (x_1, 1), \quad t > 0, \tag{5.10}$$

$$w(0, t) = w_{xx}(0, t) = w(1, t) = w_{xx}(1, t) = 0, \tag{5.11}$$

$$w(x_1^-, t) = w(x_1^+, t), \qquad w_x(x_1^-, t) = w_x(x_1^+, t), \qquad w_{xxx}(x_1^-, t) = w_{xxx}(x_1^+, t), \tag{5.12}$$

$$w_{xx}(x_1^-, t) - w_{xx}(x_1^+, t) = u(t), \qquad y(t) = kw_{xt}(x_1, t), \tag{5.13}$$

where $k > 0$. Clearly, **(A1)** is satisfied and it is shown in [21] that the feedback $u(t) = -y(t)$ renders the system exponentially stable. We show that this stability is not robust with respect to delays.

Let $s = \imath\omega^2$, where we choose $\omega \in \{re^{\imath\theta} \mid r \geq 0, \theta \in [-\pi/2, 0]\}$ for $\operatorname{Re} s \geq 0$, and let

$$h(x_1, \omega) = \frac{\sinh(\omega x_1) - \cosh(\omega x_1) \tanh \omega(x_1 - 1)}{\cos(\omega x_1) \tan \omega(x_1 - 1) - \sin(\omega x_1)}.$$

Computing the transfer function for (5.10)–(5.13) we find that **(A2)** holds and we obtain

$$\mathbf{H}(s) = \frac{\omega[\cosh(\omega x_1) + h(x_1, \omega) \cos(\omega x_1)]}{(-2\imath)[\sinh(\omega x_1) - \cosh(\omega x_1) \tanh \omega(x_1 - 1)]}.$$

From Section 4, we see that the poles of $\mathbf{H}(s)$ are contained in the set of exponents of the modes of (5.10)–(5.13) with $u(t) = 0$. This set of exponents is the set of eigenvalues of the operator

$$A := \begin{pmatrix} 0 & I \\ -\mathscr{D}^4 & 0 \end{pmatrix}, \tag{5.14}$$

where, as in Section 4, $\mathscr{D}$ denotes spatial differentiation. Here the state space is

$$X = \{(w_1, w_2)^T \in H^2[0, 1] \oplus L^2[0, 1] \mid w_1(0) = w_1(1) = 0\},$$

and the domain of A is given by

$$\operatorname{dom}(A) = \{(w_1, w_2)^T \in H^4[0, 1] \oplus H^2[0, 1] \mid w_1(0) = w_1(1) = \mathscr{D}^2 w_1(0) = \mathscr{D}^2 w_1(1) = w_2(0) = w_2(1) = 0\}.$$

It is well known that A is skew-adjoint and has compact resolvent, hence the spectrum of A consists of purely imaginary eigenvalues. Moreover, there are infinitely many eigenvalues, with ∞ being the only accumulation point. As already mentioned, an application of the feedback law $u(t) = -y(t)$ to (5.10)–(5.13) results in a closed-loop system which is exponentially stable, and hence modally stable. A combination of Lemma 2.4 and Proposition 2.3 then shows that each eigenvalue of A is a pole of $\mathbf{H}$, and hence $\mathbf{H}$ has infinitely many poles on the imaginary axis.

The zeros of $\mathbf{H}(s)$ are contained in the set of exponents of the modes of the zero dynamics of (5.10)–(5.13). The zero dynamics are given by the uncontrolled inverse system of (5.10)–(5.13), i.e., the system which is obtained by interchanging the roles of u and y. Clearly, the transfer function of the inverse system is given by $1/\mathbf{H}$. The exponents of the modes of the zero dynamics are identical to the eigenvalues of the operator A^{inv}, where A^{inv} is given by the right-hand side of (5.14) with

$$\begin{aligned}\operatorname{dom}(A^{\text{inv}}) = \{(w_1, w_2)^T \in X \,|\, w_1|_{[0,x_1)} \in H^4[0, x_1),\ w_1|_{(x_1,1]} \in H^4(x_1, 1],\\ w_2 \in H^2[0, 1],\ \mathscr{D}^2 w_1(0) = \mathscr{D}^2 w_1(1) = w_2(0) = w_2(1) = 0,\\ \mathscr{D} w_2(x_1) = 0,\ \mathscr{D}^3 w_1(x_1^-) = \mathscr{D}^3 w_1(x_1^+)\}.\end{aligned}$$

It is shown in [21] that this operator is skew-adjoint and has compact resolvent, and so the spectrum of A^{inv} consists of purely imaginary eigenvalues. Moreover, there are infinitely many eigenvalues, with ∞ being the only accumulation point. Since the feedback $y(\cdot) = -u(\cdot)$ applied to the inverse system leads to an exponentially stable closed-loop system, we can argue as above to show that $\mathbf{H}$ has infinitely many zeros on the imaginary axis.

Combining our findings, it follows that $\mathbf{H}$ satisfies (3.21). So there exist sequences (ε_n) and (ω_n) in $(0, \infty)$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and $\lim_{n\to\infty} \omega_n = \infty$ and such that the system (5.10)–(5.13) with the feedback $u(t) = -y(t - \varepsilon_n)$ has a mode with exponent $\imath\omega_n$.

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