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A Note on Stability and Stabilizability of Neutral Systems

Hartmut Logemann and Luciano Pandolfi

***Abstract*— This note presents frequency-domain characterizations of exponential stability and stabilizability of neutral systems based on transfer-function matrices and the existence of 'nice' solutions of certain Bezout equations. It turns out that the existence of H^∞-solutions is not sufficient for exponential stabilizability, but that they have to satisfy an additional growth assumption as well. Whilst the proofs of our results are based on an abstract infinite-dimensional representation of the neutral system, we emphasize that the results are expressed in terms of the original parameters of the neutral equation and do not require a reformulation of the system in an abstract state-space form. The sufficiency parts of the results hold even when the delay operator acting on the derivative contains a singular part.**

I. Introduction

Stabilizability and the relationship between internal and external stability for infinite-dimensional systems have been investigated by many researchers in the last 20 years, see e.g., [15], [10]–[12], [9], [1], [14], [8], [4] for stabilizability and [6], [2], [4], and [13] for the relationship between internal and external stability. For an abstract infinite-dimensional semigroup system of the form

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{1.1}$$

defined on a Hilbert space with finite-dimensional input and output spaces it has been shown [4] that (1.1) is exponentially stable if and only if its transfer function belongs to H^∞ (i.e., it is holomorphic and bounded in the open right-half plane) provided that the system is exponentially stabilizable and exponentially detectable. Moreover, it is proved in [4] that a well-known sufficient condition for exponential stabilizability (cf. [15]) is also necessary. In particular, it became clear that exponential stabilization by bounded state-feedback with finite-dimensional range is only possible if there exists a constant $\epsilon > 0$ such that the spectrum of A in the half-plane $\mathrm{Re}\,(s) > -\epsilon$ consists of at most finitely many eigenvalues with finite multiplicities.

Manuscript received March 15, 1991; revised December 6, 1991 and November 17, 1992. This work was supported in part by the Italian MRST within the program of GNAFA-CNR.

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IEEE Log Number 9212843.

In this note, we shall investigate stability and stabilizability properties of neutral systems of the form

$$\frac{d}{dt}\left(\xi(t) - \int_{-h}^{0} dM(\tau)\xi(t+\tau)\right) = \int_{-h}^{0} dL(\tau)\xi(t+\tau) + B_0\omega(t)$$

$$\gamma(t) = C_0\xi(t), \tag{1.2}$$

where $h > 0$ is the length of the delay, M and L are functions of bounded variation on $[-h, 0]$ with values in $\mathbb{R}^{n\times n}$, M is right-continuous at 0, $B_0 \in \mathbb{R}^{n\times m}$ and $C_0 \in \mathbb{R}^{p\times n}$. We give a characterization of exponential stability of (1.2) in terms of its transfer function and the solvability in H^∞ of certain Bezout equations and derive a necessary and sufficient condition for exponential stabilizability of (1.2) by bounded state-feedback. These results, presented in Section III, are expressed in terms of the coefficients $M(\cdot)$, $L(\cdot)$, B_0, C_0 of (1.2) and are obtained by applying a number of abstract results proved in Section II. More precisely, in Section II we show that internal and external stability of an abstract state-space system (A, B, C) coincide provided that the conditions C1) and C2) hold:

Condition C1): The Bezout equation

$$(sI - A)X(s) + BU(s) = I \tag{1.3}$$

admits solutions in H^∞.

Condition C2): There exist H^∞ solutions for the Bezout equation

$$\tilde{X}(s)(sI - A) + \tilde{Y}(s)C = I. \tag{1.4}$$

We emphasize that this result is quite general in the sense that the input and output spaces are not assumed to be finite-dimensional. Moreover, we prove that if $C(sI - A)^{-1}B \in H^\infty$ and if C1) and C2) hold then $(sI - A)^{-1} \in H^\infty$, provided that A is closed and densely defined, even if A does not generate a C_0 – semigroup. Since it is well known that the condition $(sI - A)^{-1} \in H^\infty$ implies exponential stability if A generates a C_0 – semigroup on a Hilbert space (see e.g., [16]), our result contains the one in [4] as a special case. Furthermore, we give necessary and sufficient conditions for the stabilizability of an abstract pair (A, B) in terms of condition C1) and the function $(sI - A)^{-1}B$.

Notation:

- $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \mathrm{Re}\,(s) > 0\}$.
- For Y a Banach space and $\alpha \in \mathbb{R}$ let $H^p(\alpha; Y)$ denote the usual Hardy space of functions on $\mathrm{Re}\,(s) > \alpha$. If $\alpha = 0$ we write simply $H^p(Y)$. In this note $p = \infty$ or $p = 2$.
- Z, Z_i, Z_o are (complex) Banach spaces, denoting the 'state space', the input space and the output space, respectively, and $A: D(A) \subseteq Z \to Z$ is a linear closed densely defined operator. The spectrum and the resolvent set of A are denoted by $\sigma(A)$ and $\rho(A)$, respectively. We shall write H, H_i, and H_o instead of Z, Z_i, and Z_o if these spaces are assumed to be Hilbert spaces.

II. External Stability, Internal Stability, andStabilizability of Abstract Systems

In this section, we shall investigate the relationship between external and internal stability of an abstract triple (A, B, C), where

A is not necessarily supposed to generate a C_0 – semigroup. Furthermore, we shall give a necessary and sufficient condition for stabilizability. By 'external stability' we mean that the 'transfer function' $C(sI-A)^{-1}B$ of the triple (A, B, C) belongs to H^∞, while 'internal stability' means that the resolvent operator $(sI-A)^{-1}$ has this property. It is obvious that internal stability implies external stability. If A generates a C_0-semigroup $E(t)$ on a Hilbert space, exponential stability of $E(t)$ is equivalent to internal stability (see [16]). In our abstract setting, the pair (A, B) is 'stabilizable' if there exists a bounded operator F such that $(sI-A-BF)^{-1}$ belongs to H^∞. If A is a generator on a Hilbert space this implies that the semigroup generated by $A+BF$ is exponentially stable. Similarly, we say that the pair (C, A) is 'detectable' if there exists a bounded operator H such that $(sI-A-HC)^{-1}$ is in H^∞.

Theorem 2.1: Suppose that $\rho(A)\cap\mathbb{C}_+\neq\emptyset$, $B\in\mathcal{L}(Z_i, Z)$ and $C\in\mathcal{L}(Z, Z_o)$. Then the following statements are equivalent

1. The resolvent $(sI-A)^{-1}$ is in $H^\infty(\mathcal{L}(Z))$.
2. The transfer function $G(s) := C(sI-A)^{-1}B$ is in $H^\infty(\mathcal{L}(Z_i, Z_o))$ and conditions C1) and C2) hold.

Proof:

1. a) $\Rightarrow$ b): Trivial, set $X(s)=\tilde{X}(s)=(sI-A)^{-1}$, $U(s)\equiv 0$ and $Y(s)\equiv 0$.
2. b) $\Rightarrow$ a): For $s\in\rho(A)\cap\mathbb{C}_+$ we have

$$CX(s)+G(s)U(s)=C(sI-A)^{-1} \tag{2.1}$$

and

$$\tilde{X}(s)+\tilde{Y}C(sI-A)^{-1}=(sI-A)^{-1}. \tag{2.2}$$

By (2.1) $C(sI-A)^{-1}$ is bounded on $\rho(A)\cap\mathbb{C}_+$ and hence it follows from (2.2) that $(sI-A)^{-1}$ is bounded on $\rho(A)\cap\mathbb{C}_+$. It remains to show that $\rho(A)\cap\mathbb{C}_+=\mathbb{C}_+$ or equivalently that $\sigma(A)\cap\mathbb{C}_+=\emptyset$. Let us assume the contrary, i.e., $\sigma(A)\cap\mathbb{C}_+\neq\emptyset$. Then there exist $s_0\in\sigma(A)\cap\mathbb{C}_+$ and $s_n\in\rho(A)\cap\mathbb{C}_+$ such that $\lim_{n\to\infty}s_n=s_0$. As a consequence the sequence $\|(s_nI-A)^{-1}\|$ is unbounded which leads to a contradiction. □

Corollary 2.2: Suppose that A generates a C_0-semigroup $E(t)$ on H, $B\in\mathcal{L}(H_i, H)$ and $C\in\mathcal{L}(H, H_o)$. Under these conditions the exponential stability of $E(t)$ is equivalent to statement b) in Theorem 2.1.

The above corollary follows immediately from Theorem 2.1 since we mentioned already that the exponential stability of $E(t)$ is equivalent to statement a) in Theorem 2.1 (see [16]).

The next result shows how condition C1) is related to stabilizability.

Proposition 2.3: i) Let $B\in\mathcal{L}(Z_i, Z)$ and suppose that there exists $F\in\mathcal{L}(Z, Z_i)$ such that $(sI-A-BF)^{-1}\in H^\infty(\mathcal{L}(Z))$, then C1) is satisfied.

ii) There exist Banach spaces Z and Z_i, an operator $A: D(A)\subseteq Z\to Z$ generating a C_0-semigroup and $B\in\mathcal{L}(Z_i, Z)$ such that

- C1) is satisfied on $\mathbb{C}_{-\alpha} := \{s\in\mathbb{C}\,|\,Re(s)>-\alpha\}$ for any $\alpha\in(0,1)$
- There does not exist an operator $F\in\mathcal{L}(Z, Z_i)$ such that the C_0-semigroup generated by $A=BF$ is exponentially stable.

Proof: i) Simply set $X(s) := (sI-A-BF)^{-1}$ and $U(s) := -F(sI-A-BF)^{-1}$.

ii) We consider the example presented in [7, p. 61]. Let Z be the space of all complex-valued functions defined on $\mathbb{R}_+$ which vanish at ∞ and are integrable with respect to $e^x\,dx$, i.e.,

$$Z=C_0(\mathbb{R}_+, \mathbb{C})\cap L^1(\mathbb{R}_+, \mathbb{C}; e^x\,dx).$$

Endowed with the norm

$$\|f\|=\|f\|_\infty+\|f\|_1=\sup\{|f(x)|: x\in\mathbb{R}_+\}+\int_0^\infty |f(x)|\,e^x\,dx,$$

Z becomes a Banach space. The translation semigroup

$$(E(t)f)(x)=f(x+t)$$

is strongly continuous on Z. Since $\|E(t)\|=1$ for all $t\geq 0$ it follows that $E(t)$ is not exponentially stable. For $s\in\mathbb{C}_{-1}$ and $f\in Z$ define

$$\Lambda(s)f=\int_0^\infty e^{-st}\,E(t)f\,dt. \tag{2.3}$$

It is routine to show that the integral in (2.3) exists in Z for all $s\in\mathbb{C}_{-1}$ and that $(\Lambda(s)f)(x)=\int_0^\infty e^{-st}f(t+x)\,dt$ for all $f\in Z$ and $x\in\mathbb{R}_+$. Moreover, two simple calculations lead to

$$\|\Lambda(s)f\|_1\leq\frac{1}{1+\mathrm{Re}(s)}\|f\|_1 \quad\text{and}\quad \|\Lambda(s)f\|_\infty\leq\|f\|_1.$$

Hence,

$$\|\Lambda(s)f\|\leq\left(\frac{1}{1+\mathrm{Re}(s)}+1\right)\|f\| \quad \text{for all } s\in\mathbb{C}_{-1}. \tag{2.4}$$

Let A be the infinitesimal generator of $E(t)$. Since the integral in (2.3) exists for all $f\in Z$ and all $s\in\mathbb{C}_{-1}$ it follows that $\rho(A)\supseteq\mathbb{C}_{-1}$ and $\Lambda(s)=(sI-A)^{-1}$ for $s\in\mathbb{C}_{-1}$. The estimate (2.4) shows that $(sI-A)^{-1}$ is bounded in s on any half-plane $\mathbb{C}_{-\alpha}$, where $\alpha\in(0,1)$. Hence, we see that for the pair $(A, 0)$ condition C1) is satisfied on $\mathbb{C}_{-\alpha}$ [for any $\alpha\in(0,1)$]. However, $(A, 0)$ is not exponentially stabilizable, since $E(t)$ is not exponentially stable. □

Remark 2.4:

i) A similar result holds for the relationship between condition C2) and detectability.

ii) It follows from Proposition 2.3 i) that if A generates a C_0-semigroup, $B\in\mathcal{L}(Z_i, Z)$ and there exists $F\in\mathcal{L}(Z, Z_i)$ such that $A+BF$ generates an exponentially stable C_0-semigroup, then C1) is satisfied.

iii) In the light of Corollary 2.2 and statement ii) of this remark, it becomes clear that Theorem 19 in [4] is a special case of Theorem 2.1. We emphasize that the proof of Theorem 2.1 is based on algebraic manipulations and does not make use of a spectrum decomposition of the operator A.

Next we shall derive a sufficient condition for stabilizability which is based on C1).

Theorem 2.5: Suppose that $\sigma(A)\cap\overline{\mathbb{C}}_+$ consists of isolated eigenvalues with finite multiplicity and $B\in\mathcal{L}(Z_i, Z)$. Under these conditions there exists a feedback operator $F\in\mathcal{F}(Z, Z_i)$ such that $(sI-A-BF)^{-1}\in H^\infty(\mathcal{L}(Z))$ if:

a) Condition C1) holds.

and

b)

$$\lim_{\substack{|s|\to\infty\\ s\in\mathbb{C}_+}}\|(sI-A)^{-1}B\|=0. \tag{2.5}$$

Remark 2.6: By applying again Weiss' result in [16] we see that Theorem 2.5 is a sufficient condition for exponential stabilizability provided Z is a Hilbert space and A is the generator of a C_0-semigroup on Z.

Proof of Theorem 2.5: By condition C1) we have on $\mathbb{C}_+$

$$(sI-A)^{-1}=X(s)+(sI-A)^{-1}BU(s). \tag{2.6}$$

Combining (2.6) and (2.5) it follows that there exist constants $\rho>0$ and $R>0$ such that

$$\|(sI-A)^{-1}\|\leq R \quad \text{for all } |s|\geq\rho,\ s\in\mathbb{C}_+. \tag{2.7}$$

Using the assumption on $\sigma(A) \cap \overline{\mathbb{C}}_+$ we see that $(sI - A)^{-1}$ is holomorphic on $\overline{\mathbb{C}}_+$ apart from at most finitely many points where $(sI - A)^{-1}$ has poles. An application of Theorem 6.17 in [5] shows that it is possible to decompose Z as a direct sum $Z = Z_u \oplus Z_s$, where $\dim Z_u < \infty$. This decomposition reduces the operator A. Using obvious notation we write $A = \operatorname{diag}(A_u, A_s)$ and $B = \operatorname{col}(B_u, B_s)$. It is easily seen that condition a) implies that the finite-dimensional system (A_u, B_u) satisfies the rank condition rk $(sI - A_u, B_u) = \dim Z_u$ for all $s \in \overline{\mathbb{C}}_+$. It follows from the Hautus test that (A_u, B_u) is exponentially stabilizable, so that there exists $F = (F_u, 0)$ such that $\sigma(A + BF) \cap \overline{\mathbb{C}}_+ = \varnothing$ (cf. [15]). Since

$$(sI - A - BF)^{-1} = (I - (sI - A)^{-1}BF)^{-1}(sI - A)^{-1}$$

we see that there exist constants $\rho' > 0$ and $R' > 0$ such that

$$\|((sI - A)BF)^{-1}\| \leq R' \quad \text{for all} |s| \geq \rho', s \in \mathbb{C}_+$$

where we have used (2.7) and (2.5). Now $(sI - A - BF)^{-1}$ is bounded on $\{s \in \overline{\mathbb{C}}_+ | \, |s| \leq \rho'\}$ and we conclude that $(sI - A - BF)^{-1} \in H^\infty(\mathcal{L}(Z))$. Under more restrictive assumptions on the input operator B the converse of Theorem 2.5 holds as well. □

Theorem 2.7: Suppose that $\mathbb{C}_+ \cap \rho(A)$ is unbounded and that $B \in \mathcal{L}(Z_i, Z)$ is the limit of a sequence of finite rank operators $B_n \in \mathcal{L}(Z_i, Z)$ in the uniform topology of $\mathcal{L}(Z_i, Z)$. If there exists $F \in \mathcal{L}(Z, Z_i)$ such that $(sI - A - BF)^{-1} \in H^\infty(\mathcal{L}(Z))$, then C1) and (2.5) hold.

Proof: It follows from Proposition 2.3 i) that C1) is satisfied. In order to show that (2.5) holds we proceed in three steps.

Step 1: We show that

$$\lim_{\substack{|s| \to \infty \\ s \in \mathbb{C}_+}} \|(sI - A - BF)^{-1}x\| = 0 \quad \text{for all} x \in Z. \tag{2.8}$$

First let x be in $D(A)$. Pick $z \in \mathbb{C}_+$ and let $y \in Z$ be such that $x = (zI - A - BF)^{-1}y$. Setting $\beta := \|(sI - A - BF)^{-1}\|_\infty$ it follows from the first resolvent equation that

$$\|(sI - A - BF)^{-1}x\| \leq \frac{1}{|z - s|}\{\beta + \|(zI - A - BF)^{-1}\|\}\|y\| \quad \text{for all } s \in \mathbb{C}_+.$$

Hence (2.8) is true for all $x \in D(A)$. Now, for $x \in Z$ choose a sequence $x_n \in D(A)$ such that $x = \lim_{n \to \infty} x_n$. For given $\epsilon > 0$ there exists $N \in \mathbb{N}$ and $\rho > 0$ satisfying

$$\|x - x_N\| \leq \frac{\epsilon}{2\beta}$$

and

$$\|(sI - A - BF)^{-1}\| \leq \frac{\epsilon}{2} \quad \text{for all } |s| \geq \rho, s \in \mathbb{C}_+.$$

Hence, we have for all $|s| \geq \rho$, $s \in \mathbb{C}_+$

$$\begin{aligned}
&\|(sI - A - BF)^{-1}x\| \\
&= \|(sI - A - BF)^{-1}(x - x_N) + (sI - A - BF)^{-1}x_N\| \\
&\leq \beta\|x - x_N\| + \|(sI - A - BF)^{-1}x_N\| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Step 2: Since the B_n are operators of finite rank it follows from Step 1 that

$$\lim_{\substack{|s| \to \infty \\ s \in \mathbb{C}_+}} \|(sI - A - BF)^{-1}B_n\| = 0.$$

Using the boundedness of $(sI - A - BF)^{-1}$ on $\mathbb{C}_+$ and the fact that $B_n \to B$ in the uniform topology of $\mathcal{L}(Z_i, Z)$ it follows that

$$\lim_{\substack{|s| \to \infty \\ s \in \mathbb{C}_+}} \|(sI - A - BF)^{-1}B\| = 0. \tag{2.9}$$

Step 3: Realizing that for all $s \in \mathbb{C}_+ \cap \rho(A)$ of sufficiently large modulus

$$(sI - A)^{-1}B = (I + (sI - A - BF)^{-1}BF)^{-1} \cdot (sI - A - BF)^{-1}B$$

and using (2.9) we obtain

$$\lim_{\substack{|s| \to \infty \\ s \in \mathbb{C}_+}} \|(sI - A)^{-1}B\| = 0,$$

which is (2.5). □

Remark 2.8: The assumption in Theorem 2.7 that B is the uniform limit of a sequence of finite rank operators is for example satisfied if one of the following conditions holds

i) $B \in \mathcal{L}(Z_i, Z)$ is nuclear, i.e., of the form

$$Bu = \sum_{\lambda=1}^{\infty} \alpha_\lambda f_\lambda(u) z_\lambda \quad \text{for all } u \in Z_i,$$

where $z_\lambda \in Z$, $\|z_\lambda\| \leq 1$, $f_\lambda \in Z_i^*$, $\|f_\lambda\| \leq 1$ and (α_λ) is a summable sequence in $\mathbb{C}$.

ii) Z has a Schauder basis (that is in particular true if Z is a Hilbert space) and B is compact.

III. External Stability, Internal Stability, and Stabilizability of Neutral Systems

In this section, we apply the abstract results of Section II to the neutral system given by (1.2). However, the resulting criteria for stability and stabilizability will be in terms of the original 'parameters' $M(\cdot)$, $L(\cdot)$, B_0 and C_0 and not in terms of an abstract state-space description of the neutral system (1.2). We introduce some notations which will turn out to be useful: For $\varphi \in C(-h, 0; \mathbb{C}^n)$ we define

$$\mathcal{M}(\varphi) := \int_{-h}^{0} dM(\tau)\varphi(\tau) \quad \text{and} \quad \mathcal{L}(\varphi) := \int_{-h}^{0} dL(\tau)\varphi(\tau).$$

Moreover, using the notation $\epsilon_s(\tau) = e^{s\tau}$, we set for $s \in \mathbb{C}$

$$\hat{M}(s) := \mathcal{M}(\epsilon_s), \ \hat{L}(s) := \mathcal{L}(\epsilon_s) \quad \text{and}$$
$$\Delta(s) := s(I - \hat{M}(s)) - \hat{L}(s).$$

It is natural to associate the following Bezout equations with the neutral system

$$\Delta(s)\Xi(s) + B_0\Omega(s) = I \tag{3.1}$$

and

$$\tilde{\Xi}(s)\Delta(s) + \tilde{\Gamma}(s)C_0 = I. \tag{3.2}$$

We shall show that if we know solutions of (3.1) and (3.2) then it is possible to construct solutions of the 'abstract' Bezout equations (1.3) and (1.4), where now $Z = M^2 = \mathbb{C}^n \times L^2(-h, 0; \mathbb{C}^n)$ is a Hilbert space, $Z_i = \mathbb{C}^m$, $Z_0 = \mathbb{C}^p$, the operators $B: \mathbb{C}^m \to M^2$ and $C: M^2 \to \mathbb{C}^p$ are given by $B := \operatorname{col}(B_0, 0)$ and $C := (C_0, 0)$ and the operator A is defined as follows:

$$D(A) := \left\{ \begin{pmatrix} \varphi_0 \\ \varphi(\cdot) \end{pmatrix} \in M^2 \,|\, \varphi(\cdot) \in W^{1,2}, \varphi_0 = \varphi(0) - \mathcal{M}(\varphi(\cdot)) \right\} \tag{3.3}$$

$$A\begin{pmatrix} \varphi_0 \\ \varphi(\cdot) \end{pmatrix} := \begin{pmatrix} \mathcal{L}(\varphi(\cdot)) \\ \frac{d\varphi}{d\Theta} \end{pmatrix}. \tag{3.4}$$

It is well known (see e.g., [14]) that with this choice for the state-space and the operators, system (1.1) becomes an abstract state-space model for the neutral system (1.2). In particular A is the generator of a C_0-semigroup on the Hilbert space M^2. Moreover, the transfer functions $C(sI - A)^{-1}B$ and $C_0\Delta^{-1}(s)B_0$ coincide.

The term 'stability' (exponential or external) as referred to (1.2) coincides with the same property for the corresponding abstract system.

Theorem 3.1: The neutral system (1.2) is exponentially stable if the following two conditions are satisfied:

1. The transfer function $G(s) := C_0\Delta^{-1}(s)B_0$ is in $H^\infty(\mathbb{C}^{p\times m})$.
2. The Bezout equations (3.1) and (3.2) admit solutions $\Xi, \hat{\Xi} \in H^\infty(\mathbb{C}^{n\times n})$, $\Omega \in H^\infty(\mathbb{C}^{m\times n})$ and $\hat{\Gamma} \in H^\infty(\mathbb{C}^{n\times p})$ which are $O(1/s)$ as $|s| \to \infty$ in $\mathbb{C}_+$.

Conversely under the extra assumption that M contains no singular part, the conditions a) and b) are necessary for exponential stability.

The reader should notice that in the above theorem the solutions of the Bezout equations (3.1) and (3.2) are required to converge to 0 at least like $1/s$ as $|s| \to \infty$ in $\mathbb{C}_+$. This condition is used in order to ensure that the solutions of the corresponding abstract Bezout equations are bounded (see formulas below).

Proof of Theorem 3.1:

Suppose that a) and b) hold. Using Theorem 2.1 and Corollary 2.3 it is sufficient to show that the abstract Bezout equations (1.3) and (1.4) admit solutions which are holomorphic and bounded on $\mathbb{C}_+$. We shall construct solutions of (1.3) and leave the solution of (1.4) to the reader. Let us write $X(s)$ and $U(s)$ in the block form

$$X(s) = \begin{pmatrix} X'_{0s} & X''_{0s} \\ X'_s & X''_s \end{pmatrix}, \qquad U(s) = (U'_s, U''_s).$$

The meaning of the symbols X'_{0s}, X''_{0s}, etc. is clear since $X(s) \in \mathcal{L}(M^2)$, $U(s) \in \mathcal{L}(M^2, \mathbb{C}^m)$ for each $s \in \mathbb{C}_+$ and M^2 is the product space $\mathbb{C}^n \times L^2(-h, 0; \mathbb{C}^n)$.

Equation (1.3) makes sense on the whole space M^2 only if the condition $X(s)M^2 \subseteq D(A)$ is satisfied for all $s \in \mathbb{C}_+$. This will be the case if and only if for all $\Phi = \mathrm{col}(\varphi_0, \varphi) \in M^2$

$$X'_{0s}(\varphi_0) = X'_s(\varphi_0)(0) - \mathcal{M}(X'_s(\varphi_0)(\cdot)) \tag{3.5}$$
$$X''_{0s}(\varphi) = X''_s(\varphi)(0) - \mathcal{M}(X''_s(\varphi)(\cdot)). \tag{3.6}$$

Moreover, the first column of (1.3) gives

$$sX'_{0s}(\varphi_0) - \mathcal{L}(X'_s(\varphi_0)(\cdot)) + B_0U'_s(\varphi_0) = \varphi_0 \tag{3.7}$$
$$sX'_s(\varphi_0)(\Theta) - \frac{d}{d\Theta}(X'_s(\varphi_0)(\Theta)) = 0.$$

It follows from the last equation that

$$X'_s(\varphi_0)(\Theta) = e^{s\Theta}X'_s(\varphi_0)(0). \tag{3.8}$$

For simplicity set $v_s := X'_s(\varphi_0)(0)$. Then, by (3.5), (3.7), and (3.8), we obtain

$$\Delta(s)v_s + B_0U'_s(\varphi_0) = \varphi_0. \tag{3.9}$$

Equation (3.9) is satisfied by

$$v_s = \Xi(s)\varphi_0, \qquad U'_s(\varphi_0) = \Omega(s)\varphi_0$$

so that

$$X'_s(\varphi_0)(\Theta) = e^{s\Theta}\Xi(s)\varphi_0$$

and

$$X'_{0s}(\varphi_0) = (I - \hat{M}(s))\Xi(s)\varphi_0.$$

It follows from the assumption that the operator-valued functions X'_{0s}, X'_s, and U'_s just derived are holomorphic and bounded on $\mathbb{C}_+$.

In order to calculate the remaining blocks of $X(s)$ and $U(s)$ let $\varphi \in L^2(-h, 0; \mathbb{C}^n)$ and apply (1.3) to $\Phi = \mathrm{col}(0, \varphi)$. Then we obtain

$$sX''_{0s}(\varphi) - \mathcal{L}(X''_s(\varphi)(\cdot)) + B_0U''_s(\varphi) = 0 \tag{3.10}$$

and

$$s\eta_s(\Theta) - \frac{d}{d\Theta}\eta_s(\Theta) = \varphi(\Theta),$$

where $\eta_s(\Theta) := X''_s(\varphi)(\Theta)$. Consequently,

$$\eta_s(\Theta) = e^{s\Theta}\eta_s(0) - \int_0^\Theta e^{s(\Theta-\tau)}\varphi(\tau)\,d\tau. \tag{3.11}$$

Combining (3.6), (3.10), and (3.11) gives

$$\Delta(s)\eta_s(0) + B_0U''_s(\varphi) = s\mathcal{M}(\varphi_s^*) + \mathcal{L}(\varphi_s^*). \tag{3.12}$$

where $\varphi_s^*(\Theta) := -\int_0^\Theta e^{s(\Theta-\tau)}\varphi(\tau)\,d\tau$. A solution of (3.12) is given by

$$\eta_s(0) = \Xi(s)(s\mathcal{M}(\varphi_s^*) + \mathcal{L}(\varphi_s^*))$$
$$U''_s(\varphi) = \Omega(s)(s\mathcal{M}(\varphi_s^*) + \mathcal{L}(\varphi_s^*))$$

so that

$$X''_s(\varphi)(\Theta) = e^{s\Theta}\Xi(s)(s\mathcal{M}(\varphi_s^*) + \mathcal{L}(\varphi_s^*)) + \varphi_s^*(\Theta)$$

and, from (3.6)

$$X''_{0s}(\varphi) = (I - \hat{M}(s))\Xi(s)(s\mathcal{M}(\varphi_s^*) + \mathcal{L}(\varphi_s^*)) - \mathcal{M}(\varphi_s^*).$$

The operator-valued functions X''_{0s}, X''_s, and U''_s just derived are holomorphic and bounded on $\mathbb{C}_+$ by the assumption on $\Xi(s)$ and $\Omega(s)$. Doing the same calculations backwards we see that $(sI - A)X(s)\binom{\varphi_0}{0} + BU(s)\binom{\varphi_0}{0} = \binom{\varphi_0}{0}$ for all $\varphi_0 \in \mathbb{C}^n$ and $(sI - A)X(s)\binom{0}{\varphi(\cdot)} + BU(s)\binom{0}{\varphi(\cdot)} = \binom{0}{\varphi(\cdot)}$ for all $\varphi(\cdot) \in L^2(-h, 0; \mathbb{C}^n)$. Hence, $X(s)$ and $U(s)$ are H^∞-solutions of the Bezout equation (1.3).

Now conversely assume that the system (1.2) is internally stable. It is trivial that the transfer function $G(s)$ is in $H^\infty(\mathbb{C}^{p\times m})$, i.e., condition a) holds. In order to show that condition b) is satisfied assume that M contains no singular part. This means that $\hat{M}(s)$ can be written in the form

$$\hat{M}(s) = \sum_{j=1}^\infty M_j e^{-h_j s} + \int_{-h}^0 M_\infty(\tau)e^{s\tau}\,d\tau$$

where $0 < h_j \le h$ for $j \in \mathbb{N}$ and

$$\sum_{j=1}^\infty \|M_j\| + \int_{-h}^0 \|M_\infty(\tau)\|\,d\tau < \infty.$$

It is convenient to define

$$\Delta_0(s) := I - \sum_{j=1}^\infty M_j e^{-h_j s}$$

and

$$\hat{M}_c(s) := \int_{-h}^0 M_\infty(\tau)e^{s\tau}\,d\tau.$$

Since the system is internally stable we have that $\Delta^{-1}(s)$ is holomorphic on a half plane $\mathrm{Re}(s) > \alpha$ for some $\alpha < 0$ and moreover $\Delta_0^{-1}(s) \in H^\infty(\mathbb{C}^{n\times n})$ (cf. [14, pp. 160–162]). Now realize that

$$\Delta^{-1}(s) = \frac{1}{s}\left[I - \Delta_0^{-1}(s)\left(\hat{M}_c(s) + \frac{1}{s}\hat{L}(s)\right)\right]^{-1}\Delta_0^{-1}(s).$$

Since $\hat{L}(s)$ is bounded on $\mathbb{C}_+$ and $\hat{M}_c(s)$ tends to zero as $|s| \to \infty$ in $\mathbb{C}_+$ by well-known results on Laplace transforms and by the Riemann–Lebesgue lemma, it follows that $\|\Delta_0^{-1}(s)(\hat{M}_c(s) + (1/s)\hat{L}(s))\| \leq (1/2)$ for all $s \in \mathbb{C}_+$ of sufficiently large modulus. Hence, we obtain that $\Delta^{-1}(s) \in H^\infty(\mathbb{C}^{n\times n})$ and moreover $\Delta^{-1}(s) = O(1/s)$ as $|s| \to \infty$ in $\mathbb{C}_+$. Setting $\Xi(s) = \tilde{\Xi}(s) = \Delta^{-1}(s)$, $\Omega(s) \equiv 0$ and $\tilde{\Gamma}(s) \equiv 0$ we obtain solutions of the Bezout equations (3.1) and (3.2) which have the desired properties. □

Finally, we consider the stabilizability problem for neutral systems.

Theorem 3.2: The neutral system (1.2) is exponentially stabilizable by bounded feedback if the following two conditions hold:

a) The Bezout equation (3.1) admits solutions $\Xi \in H^\infty(\mathbb{C}^{n\times n})$ and $\Omega \in H^\infty(\mathbb{C}^{m\times n})$ which are $O(1/s)$ as $|s| \to \infty$ in $\mathbb{C}_+$.

b) $\lim_{|s|\to\infty, s\in\mathbb{C}_+} \Delta^{-1}(s)B_0 = 0$.

Conversely, under the extra assumption that M contains no singular part, the conditions a) and b) are necessary for exponential stabilizability.

We stress that Theorem 3.2 concerns stabilization with a feedback of the form

$$\omega(t) = \int_{-h}^{0} F(\tau)\xi(t+\tau)\,d\tau + F_0\xi(t) \tag{3.13}$$

where $F \in L^2(-h, 0; \mathbb{R}^{m\times n})$ and $F_0 \in \mathbb{R}^{m\times n}$, i.e., a feedback which is bounded on M^2. Stabilization of neutral systems by bounded state-feedback has been studied by several researchers, cf. e.g., [11], [14], and [1]. It has been proved [11] that if

$$\mathrm{rk}\,(\Delta(s), B_0) = n \quad \text{for all } s \in \overline{\mathbb{C}}_+ \tag{3.14}$$

then it is possible to construct a bounded feedback of the form (3.13) which arbitrarily relocates any finite number of poles. We mention that the statement in [11] on stabilizability is not correct, see [9]. It has been shown in [14] that (3.14) is necessary and sufficient for stabilizability provided that M contains no singular part and $\det(\Delta_0(s)) \neq 0$ for all $s \in \overline{\mathbb{C}}_+$. The result in [14] contains the one in [1] as a special case. It is clear that (3.14) is equivalent to the existence of solutions $\Xi(s)$ and $\Omega(s)$ of (3.1) which are holomorphic on $\overline{\mathbb{C}}_+$.

Proof of Theorem 3.2: First suppose that conditions a) and b) hold. A straightforward calculation shows that

$$(sI - A)^{-1}\begin{pmatrix} B_0 \\ 0 \end{pmatrix} = \begin{pmatrix} I - \hat{M}(s) \\ e^{s\cdot} \end{pmatrix}\Delta^{-1}(s)B_0$$

where the operator A is given by (3.3) and (3.4). By condition b) we see that

$$\lim_{\substack{|s|\to\infty \\ s\in\mathbb{C}_+}} (sI - A)^{-1}\begin{pmatrix} B_0 \\ 0 \end{pmatrix} = 0. \tag{3.15}$$

As in the proof of Theorem 3.1 it follows from condition a) that there exist functions $X \in H^\infty(\mathcal{L}(M^2))$ and $U \in H^\infty(\mathcal{L}(M^2, \mathbb{R}^m))$ satisfying

$$(sI - A)X(s) + \begin{pmatrix} B_0 \\ 0 \end{pmatrix}U(s) = I \quad \text{for all } s \in \mathbb{C}_+, \tag{3.16}$$

i.e., condition C1) for the abstract version of system (1.2). Using Remark 2.5 it follows from (3.15) and (3.16) that the system (1.2) is exponentially stabilizable by bounded feedback.

Conversely assume that M contains no singular part and suppose that there exist an exponentially stabilizing feedback of the form (3.13). Setting

$$\hat{F}(s) := F_0 + \int_{-h}^{0} F(\tau)\,e^{s\tau}\,d\tau$$

it follows that $(\Delta(s) - B_0\hat{F}(s))^{-1}$ is holomorphic on a half-plane $\mathrm{Re}\,(s) > \alpha$ for some $\alpha < 0$. Moreover, since the difference equation associated with (1.2) is invariant under bounded feedback we have that $\Delta_0^{-1}(s) \in H^\infty(\mathbb{C}^{n\times n})$ (see [14, pp. 160–162]). Since

$$(\Delta(s) - B_0\hat{F}(s))^{-1} = \frac{1}{s}\left[I - \Delta_0^{-1}(s)\left(\hat{M}_c(s) + \frac{1}{s}\hat{L}(s) + \frac{1}{s}B_0\hat{F}(s)\right)\right]^{-1}\Delta_0^{-1}(s),$$

we obtain that $(\Delta(s) - B_0\hat{F}(s))^{-1} \in H^\infty(\mathbb{C}^{n\times n})$ and $(\Delta(s) - B_0\hat{F}(s))^{-1} = O(1/s)$ as $|s| \to \infty$ in $\mathbb{C}_+$. Setting $\Xi(s) := (\Delta(s) - B_0\hat{F}(s))^{-1}$ and $\Omega(s) := -\hat{F}(s)(\Delta(s) - B_0\hat{F}(s))^{-1}$ we obtain solutions of the Bezout equation (3.1) which satisfy condition a). Finally, since

$$\Delta^{-1}(s)B_0 = (I + \Xi(s)B_0\hat{F}(s))^{-1}\Xi(s)B_0,$$

we see that conditions b) holds as well. □

Remark 3.3:

i) It is known (see [3], [14]) that if $\det\Delta_0(s_0) = 0$ for some $s_0 \in \mathbb{C}$ then system (1.2) has a vertical root chain with real parts tending to $\mathrm{Re}\,(s_0)$. So, if $s_0 \in \overline{\mathbb{C}}_+$, the system is not exponentially stable and moreover not exponentially stabilizable by a feedback of the form (3.13) (see [14]).

ii) The growth condition for the solutions of the Bezout equations (3.1) and (3.2) is used in order to absorb the unboundedness of the term $s\mathcal{M}(\varphi_s^*)$ (see the proof of Theorem 3.1). This is not needed if $M \equiv 0$, i.e., for retarded systems. Moreover, condition b) of Theorem 3.2 is always satisfied in the retarded case.

iii) An example given in [6] shows that boundedness of the transfer function and the existence of entire solutions of the Bezout equations (3.1) and (3.2) are not sufficient for exponential stability.

(iv) Finally, we mention that the sufficient parts of Theorem 3.1 and 3.2 hold even when M contains a singular part.

Acknowledgment

The authors would like to thank R. Rebarber for his painstaking care in refereeing this note and for providing a number of helpful suggestions.

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Linear-Quadratic Zero-Sum Differential Games for Generalized State Space Systems

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***Abstract*— In this note, we consider linear-quadratic zero-sum differential games for generalized state space systems. It is well known that a unique linear feedback saddle-point solution can exist in the game of state space systems. However, for the generalized state space system, we show that the game admits uncountably many linear feedback saddle-point solutions. Sufficient conditions for the existence of linear feedback saddle-point solutions are found. A constructive method is given to find these linear feedback saddle-point solutions. A simple example is included to illustrate the nonuniqueness of the linear feedback saddle-point solutions.**

I. Introduction

We consider the zero-sum linear quadratic differential game with valuefunctional

$$J = \frac{1}{2}x^T(t_f)E^TQ_fEx(t_f)+\frac{1}{2}\int_{t_0}^{t_f}\{x^TQx+u^TR_1u-v^TR_2v\}\,dt, \tag{1}$$

subject to the linear time-invariant continuous generalized state space system

$$E\dot{x} = Ax + Bu + Cv, \quad Ex(0^-) = Ex_0, \tag{2}$$

where $x(t) \in R^n$, $u(t) \in R^m$ and $v(t) \in R^l$. E is a square matrix of rank $r \le n$. The pencil $(sE - A)$ is assumed to be regular (i.e., $|(sE - A)| \not\equiv 0$). $R_1 > 0$, $R_2 > 0$ and all other weighting matrices are nonnegative definite. The time interval $[t_0, t_f]$ is fixed. The superscript ^{T} denotes the transpose of the matrix. A minimizing player controls u and a maximizing player controls v. The initial condition for the part of $x(t)$ in the orthogonal complement of the kernel of E is known by both players. We also assume that each player has access to closed-loop no memory information on x. The strategies of two players are denoted by γ_1 and γ_2, which belong to strategy spaces Γ_1 and Γ_2, respectively. In this note, the restriction that Γ_1 and Γ_2 are composed of linear feedback strategies of the form

$$\gamma_1(x, t) = -K_1(t)x, \quad \gamma_2(x, t) = -K_2(t)x, \tag{3}$$

is made.

Manuscript received December 20, 1991; revised April 24, 1992 and August 11, 1992.

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IEEE Log Number 9212844.

Definition 1: A linear feedback strategy pair$(\gamma_1, \gamma_2) \in \Gamma_1^a \times \Gamma_2^a \subset \Gamma_1 \times \Gamma_2$ is called an admissible strategy pair if the closed-loop system obtained has no impulsive solution. Correspondingly, $\Gamma_1^a \times \Gamma_2^a \subset \Gamma_1 \times \Gamma_2$ is called the admissible strategy space.

Definition 2: An admissible linear feedback strategy pair$(\gamma_1^*, \gamma_2^*) \in \Gamma_1^a \times \Gamma_2^a$ constitutes a saddle-point equilibrium pair if

$$J(\gamma_1^*, \gamma_2) \le J(\gamma_1^*, \gamma_2^*) \le J(\gamma_1, \gamma_2^*) \tag{4}$$

for all $(\gamma_1, \gamma_2) \in \Gamma_1^a \times \Gamma_2^a$.

According to [8], [9], there exist nonsingular matrices M and H such that the system (2) can take the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}v, \tag{5}$$

where

$$H^{-1}x = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad MEH = \operatorname{diag}(I_r\, 0), \quad MAH = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad MC = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \tag{6}$$

and the value functional (1) becomes

$$J = \frac{1}{2}z_1^T(t_f)Q_{11f}z_1(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left\{\begin{bmatrix} z_1^T & z_2^T \end{bmatrix}\begin{bmatrix} Q_{11} & Q_{22} \\ Q_{12}^T & Q_{22} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + u^TR_1u - v^TR_2v\right\}dt. \tag{7}$$

where

$$H^TQH = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \quad M^{-T}Q_fM^{-1} = \begin{bmatrix} Q_{11f} & Q_{12f} \\ Q_{12f}^T & Q_{22f} \end{bmatrix}. \tag{8}$$

Without any loss of generality, we use (5) and (7) instead of (2) and (1) in the following discussion. Suppose that the system output of (5) is $y = C_{11}z_1 + C_{22}z_2$, where $Q_{11} = C_{11}^TC_{11}$, $Q_{12} = C_{11}^TC_{22}$ and $Q_{22} = C_{22}^TC_{22}$. Then, a basic assumption is made.

Assumption 1: The system (5) is impulse controllable and impulsereconstructible.

To the end of this section, some well-known results are summarized which givea necessary and sufficient condition for the closed-loop system not to have impulsive solution as well as some verifiable conditions under which Assumption 1 holds. The reader is referred to [5], [8] for details.

Lemma 1: For the system (5), there exist linear feedback strategy pair$(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$ such that the closed-loop system has no impulsive solution if and only if it is impulse controllable.

Lemma 2: The system (5) is impulse controllable if and only if the rows ofthe matrix $[A_{22}\ B_2\ C_2]$ are independent.

Lemma 3: The system (5) is impulse reconstructible if and only if the rowsof the matrix $[A_{22}^T\ C_{22}^T]$ are independent.

II. Preliminary Results

The following lemma on the zero-sum dynamic game is useful implicitly intreating the problem of this note.

Lemma 4 [1]: If the zero-sum dynamic game admits a unique pure-feedbacksaddle-point solution (γ_1^*, γ_2^*), and if (u^*, v^*) is any open-loop saddle-point solution, then

1. $\hat{\gamma}_1^* = u^*$,
2. $\hat{\gamma}_2^* = v^*$,
3. (u^*, v^*) is the unique open-loop saddle-point solution,