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Low-gain control of unknown infinite-dimensional systems: a frequency-domain approach

Hartmut Logemann
Institut für Dynamische Systeme, Universität Bremen, Postfach 330 440, 2800 *Bremen* 33, *West Germany*

and David H. Owens
Department of Mechanical and Process Engineering, Division of Dynamics and Control, University of Stratclyde, James Weir Building, 75 *Montrose Street, Glasgow* G1 1XJ

Abstract

This paper presents a frequency-domain based input–output theory of multivariable low-gain PI-control of infinite-dimensional stable systems. It is known that under very mild assumptions a large class of infinite-dimensional systems can be stabilized and regulated by multivariable PI-controllers with sufficiently low gain. The controller design can be accomplished using plant step data only. No other knowledge of the plant is required. The application of the input–output results to distributed parameter systems and neutral systems is discussed.

1. Introduction

The problem of finite-dimensional control of infinite-dimensional systems by output feedback has received a considerable amount of attention in recent years; see for example (Schumacher, 1983; Nett, 1984; Kamen *et al.*, 1985; Logemann, 1986a,c; Balas, 1986; Curtain and Salamon, 1986; Jacobson and Nett, 1987). Unfortunately the orders of the controllers derived by the above authors may be quite high in certain cases. Moreover, if approximation techniques are used (cf. (Nett, 1984; Kamen *et al.*, 1985; Logemann, 1986a,c; Balas, 1986)) the relationship between the particular approximation method and the order of the stabilizing controller is not yet understood. Intuitively it is clear that restrictions on the plant such as stability or minimum-phase should lead to simple low-order controllers.

In this paper the problem of *low-gain* PI-control of a certain class of infinite-dimensional *stable* systems is investigated. We do not assume that the plant is exactly known. However, in a similar manner to Owens and Chotai (1986) it is supposed that the designer has access to reliable plant step data. In finite dimensions a similar problem was considered by Davison (1976). He proved that under very mild assumptions a lumped stable plant can be stabilized and regulated by a multivariable PI-controller of the form $(1/s)kK_I + K_P$ for all values of the parameter k in some interval $0 < k < k^*$. Davison's result was generalized to certain distributed parameter systems by Pohjolainen (1982, 1985) and Logemann and Owens (1987c) and to a class of time-delay systems by Koivo and Pohjolainen (1985) and Jussila and Koivo (1986).

The present investigation is based on input–output and frequency-domain methods in contrast to the above papers by Davison (1976), Jussila and Koivo (1986), Koivo and Pohjolainen (1985) and Pohjolainen (1982, 1985), where the analysis is done using state-space methods. We develop a systemaic input–output theory of low-gain control of infinite-dimensional systems, which includes all plants, whose impulse response matrices have their entries in

$$\mathscr{A}_{-} := \{f \in \mathscr{A} \mid \exists \varepsilon > 0: \, f(\cdot)e^{\varepsilon \cdot} \in \mathscr{A}\}.$$

Here $\mathscr{A}$ denotes the algebra of distributions having support in $[0, \infty)$ and being the sum of a string of delayed impulses and a measurable function, with the additional property that the weights of the impulses form an absolutely summable sequence and the function is absolutely integrable (cf. Section 2). Although input–output descriptions are used throughout the paper, non-zero initial conditions are taken into account by introducing so-called 'initial-condition operators' (cf. Section 3). Using the frequency-domain framework provided by Callier and Desoer (1978, 1980) our results show that multivariable PI-controllers are capable of controlling a large class of infinite-dimensional systems to produce robust stability of the closed loop and tracking of step set-point changes. In particular the new methodology and results apply to a wider class of systems than those covered by Davison (1976), Jussila and Koivo (1986), Koivo and Pohjolainen (1985), Logemann and Owens (1987c) and Pohjolainen (1982, 1985).

The organization of the paper is as follows. Section 2 is devoted to preliminaries including the required notation, the definition of the class of systems to be considered and the proof of a preliminary lemma on final values. Section 3 provides the proof of a Davison-type result (cf. (Davison, 1976)) which applies to the class of infinite-dimensional systems under consideration. Section 4 is devoted to the application of the results of Section 3 to distributed parameter systems and neutral systems. It turns out that the PI-controller derived in Section 3 achieves *internal* stability in each case.

2. Preliminaries

Let $\mathbb{R}_+$ denote the interval $[0, \infty)$. The set $\mathbb{C}_\alpha$ $(\alpha \in \mathbb{R})$ is defined by

$$\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \operatorname{Re}(s) > \alpha\}.$$

Suppose that f is a distribution with support in $\mathbb{R}_+$ of the form

$$f = \sum_{i=0}^{\infty} f_i \delta_{t_i} + f_a, \tag{2.1}$$

where $t_0 := 0$, $t_i > 0$ for all $i \geq 1$, δ_{t_i} denotes the Dirac distribution at t_i, $f_i \in \mathbb{C}$ and f_a is a $\mathbb{C}$-valued Lebesgue-measurable function. The set $\mathscr{A}$ consists of all distributions f of the form (2.1) such that

$$\|f\|_{\mathscr{A}} = \sum_{i=0}^{\infty} |f_i| + \int_0^\infty |f_a(t)| \, dt$$

is finite. It can be shown that the convolution algebra $\mathscr{A}$ forms a Banach algebra (cf. (Hille and Phillips, 1957, p. 141)). If $f \in \mathscr{A}$, then the Laplace transform of f,

$$\hat{f}(s) := \sum_{i=0}^{\infty} f_i e^{-st_i} + \int_0^\infty f_a(t) e^{-st} \, dt,$$

is holomorphic on $\mathbb{C}_0$ and continuous on $\bar{\mathbb{C}}_0$. In particular we have

$$\|f\|_\infty := \sup_{s \in \mathbb{C}_0} |f(s)| \leq \|f\|_{\mathscr{A}}.$$

It is useful to define the following subalgebras of $\mathscr{A}$:

$$\mathscr{A}_- := \{f \in \mathscr{A} \mid \exists > 0\colon f(\cdot)e^{\varepsilon\cdot} \in \mathscr{A}\},$$

$$\mathscr{A}^0_- := \left\{f = \sum_{i=0}^{\infty} f_i \delta_{t_i} + f_a \in \mathscr{A}_- \mid \exists N > 0\colon f_i = 0 \; \forall i > N\right\},$$

$$\mathrm{L}^1_-(\mathbb{R}_+) := \{f \in \mathrm{L}^1(\mathbb{R}_+) \mid \exists \varepsilon > 0\colon f(\cdot)e^{\varepsilon\cdot} \in \mathrm{L}^1(\mathbb{R}_+)\}.$$

Obviously the following inclusions hold:

$$\mathrm{L}^1_-(\mathbb{R}_+) \subset \mathscr{A}^0_- \subset \mathscr{A}_- \subset \mathscr{A}.$$

Moreover, we define

$$\hat{\mathscr{A}} = \{\hat{f} \mid f \in \mathscr{A}\}.$$

It is now clear what is meant by $\hat{\mathscr{A}}_-$, $\hat{\mathscr{A}}^0_-$, etc.

2.1 Remark From a mathematical point of view the algebra $\mathscr{A}$ is studied, for example, by Hille and Phillips (1957, p. 141). As far as the authors are aware it was introduced into systems theory by Desoer and Wu (1968) (cf. also the book by Willems (1971)).

2.2 Remarks (i) Let

$$f = \sum_{i=0}^{\infty} f_i \delta_{t_i} + f_a \in \mathscr{A}.$$

Then

$$\lim_{\substack{|s| \to \infty \\ s \in \mathbb{C}_0}} \hat{f}(s) = 0$$

if and only if $\sum_{i=0}^{\infty} f_i \delta_{t_i} = 0$ (cf. (Callier and Desoer, 1978)).

(ii) $f \in \mathscr{A}_-$ is a unit of the algebra $\mathscr{A}_-$ if and only if

$$\inf_{s \in \mathbb{C}_0} |\hat{f}(s)| > 0$$

(cf. (Callier and Desoer, 1980)).

(iii) The final value theorem holds for transfer functions in $\hat{\mathscr{A}}_-$. More precisely, let

$$\theta(t) := \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases},$$

denote the unit step function and assume that $f \in \mathscr{A}_-$. Then there exist $M > 0$ and $\varepsilon > 0$ such that

$$|(f * r\theta)(t) - \hat{f}(0)r| \leq Me^{-\varepsilon t}|r|$$

for all $t \geq 0$ and all $r \in \mathbb{R}$; cf. (Callier and Winkin, 1986).

2.3 Lemma *Assume that* $f \in \mathscr{A}_-$ *and* $u \in L^\infty(\mathbb{R}_+)$.

(i) *If there exist* $M > 0$, $\varepsilon > 0$ *and* $u_\infty \in \mathbb{R}$ *such that* $|u(t) - u_\infty| \leq Me^{-\varepsilon t}$ *for all* $t \geq 0$, *then there exist* $N > 0$ *and* $\delta > 0$ *such that*

$$|(f * u)(t) - \hat{f}(0)u_\infty| \leq Ne^{-\delta t}.$$

(ii) *If* $\lim_{t\to\infty} u(t) := u_\infty$ *exists then* $\lim_{t\to\infty}(f * u)(t) = \hat{f}(0)u_\infty$.

Proof (i) For this, cf. (Callier and Winkin, 1986).

(ii) Define $v(t) := u(t) - u_\infty$. Then, if we write $f = \sum_{i=0}^\infty f_i\delta_{t_i} + f_a$,

$$(f * u)(t) = (f * u_\infty\theta)(t) + \int_0^t f_a(t-\tau)v(\tau)\,d\tau + \sum_{t_i \leq t} f_i v(t - t_i).$$

By Remark 2.2(iii), $\lim_{t\to\infty}(f * u_\infty\theta)(t) = \hat{f}(0)u_\infty$. Therefore it remains to show that

$$\lim_{t\to\infty} \int_0^t f_a(t-\tau)v(\tau)\,d\tau = 0 \tag{2.2}$$

and

$$\lim_{t\to\infty} \sum_{t_i \leq t} f_i v(t - t_i) = 0. \tag{2.3}$$

We show that (2.2) holds. Equation (2.3) can be established by analogous arguments. For given $\varepsilon > 0$ there exists a number $T > 0$ such that

$$|v(t)| \leq \varepsilon \quad \text{for all } t \geq T,$$

$$\int_t^{2t} |f_a(\tau)|\,d\tau \leq \varepsilon \quad \text{for all } t \geq T.$$

Hence

$$\left|\int_0^t f_a(t-\tau)v(\tau)\,d\tau\right| \leq \left(\int_0^{\frac{1}{2}t} + \int_{\frac{1}{2}t}^t\right) |f_a(t-\tau)|\,|v(\tau)|\,d\tau$$
$$\leq (\|v\|_\infty + \|f_a\|_1)\varepsilon \quad \text{for all } t \geq 2T.$$

In order to deal with unstable systems we shall be working with the algebra $\hat{\mathscr{B}} := \hat{\mathscr{B}}(0)$ of transfer functions introduced by Callier and Desoer (1978, 1980). An intuitive way of defining $\hat{\mathscr{B}}$ is to use (Callier and Desoer, 1978, Theorem 3.3). Then $\hat{g} \in \hat{\mathscr{B}}$ if and only if $\hat{g} = \hat{f} + \hat{r}$, where $f \in \mathscr{A}_-$ and $\hat{r}$ is a strictly proper rational function whose poles are in $\bar{\mathbb{C}}_0$.

2.4 Definition Let $\hat{G}$ and $\hat{K}$ be two transfer matrices of dimension $m \times q$ and $q \times m$, respectively. We assume that the entries of $\hat{G}$ and $\hat{K}$ are elements in $\hat{\mathscr{B}}$. The feedback system in Fig. 1 is called *i/o-stable* if the matrix

$$H(\hat{G}, \hat{K}) := \begin{pmatrix} (I + \hat{K}\hat{G})^{-1}\hat{K} & -(I + \hat{K}\hat{G})^{-1}\hat{K}\hat{G} \\ (I + \hat{G}\hat{K})^{-1}\hat{G}\hat{K} & (I + \hat{G}\hat{K})^{-1}\hat{G} \end{pmatrix}$$

is in $\hat{\mathscr{A}}_-^{(m+q)\times(m+q)}$. If the feedback configuration in Fig. 1 is i/o-stable we shall also say that $\hat{K}$ stabilizes $\hat{G}$.

2.5 Remarks (i) If the feedback system in Fig. 1 is i/o-stable then if follows in particular that $(I + \hat{G}\hat{K})^{-1} \in \hat{\mathscr{A}}_-^{m\times m}$ and $(I + \hat{K}\hat{G})^{-1} \in \hat{\mathscr{A}}_-^{q\times q}$.

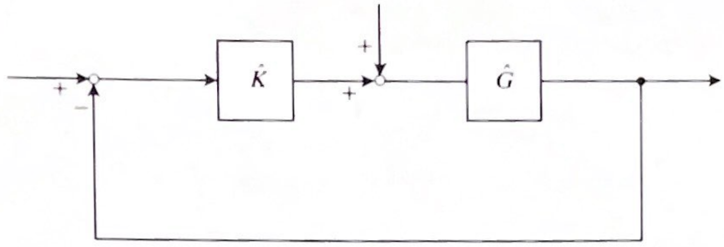

Fig. 1

(ii) Let $\hat{G} \in \hat{\mathscr{A}}_{-}^{m \times q}$ and suppose that $\hat{K} \in \hat{\mathscr{B}}^{q \times m}$. Under these conditions the feedback system in Fig. 1 is i/o-stable if and only if $(I + \hat{K}\hat{G})^{-1}\hat{K} \in \hat{\mathscr{A}}_{-}^{q \times m}$.

3. PI-controllers for infinite-dimensional systems

We shall first consider pure proportional controllers. The following simple result is easily obtained from Remarks 2.2(ii) and 2.5(ii).

3.1 Lemma *Let $G \in \mathscr{A}_{-}^{m \times q}$ and $K_p \in \mathbb{C}^{q \times m}$. Then K_P stabilizes $\hat{G}$ if*

$$\bar{\sigma}(K_P) < \frac{1}{\sup\limits_{\omega \in \mathbb{R}} \bar{\sigma}(\hat{G}(i\omega))},$$

where $\bar{\sigma}(\cdot)$ denotes the largest singular value of its argument.

In the sequel we denote the spectrum of a matrix $M \in \mathbb{C}^{n \times n}$ by $\sigma(M)$. The next result gives a condition under which stabilizing integral controllers exist for stable plants.

3.2 Theorem *Let $G \in \mathscr{A}_{-}^{m \times q}$ and assume that $\mathrm{rk}\,(\hat{G}(0)) = m$. Then there exists $K_I \in \mathbb{C}^{q \times m}$ such that*

$$\sigma(\hat{G}(0)K_I) \subset \mathbb{C}_0. \tag{3.1}$$

For each $K_I \in \mathbb{C}^{q \times m}$ satisfying (3.1) there exists $k^* > 0$ such that the controller $(1/s)kK_I$ stabilizes $\hat{G}$ for all $0 \leqslant k < k^*$.

Proof Note that $\hat{G}(0)\hat{G}(0)^T$ is non-singular ($\mathrm{rk}\,(\hat{G}(0)) = m$!). If we set

$$K_I := \hat{G}(0)^T(\hat{G}(0)\hat{G}(0)^T)^{-1}$$

then (3.1) is satisfied. Now consider an arbitrary matrix $K_I \in \mathbb{C}^{q \times m}$ satisfying the condition (3.1). Define

$$\hat{F}(s) := \hat{G}(s)K_I, \qquad \hat{C}_k(s) := (1/s)kI_m.$$

We have to show that there exists positive k^* such that $\hat{C}_k$ stabilizes $\hat{F}$ for all $0 \leqslant k < k^*$. Note that

$$\hat{C}_k(s) = \frac{k}{s+1}\left(\frac{s}{s+1} I_m\right)^{-1}$$

is a right-coprime factorization and it follows from fractional representation theory (cf. for example (Vidyasagar *et al.* 1982)) and Remark 2.2(ii) that $\hat{C}_k$ stabilizes $\hat{F}$ if and only if the function

$$f_k(s) := \det\left(\frac{s}{s+1} I_m + \frac{k}{s+1}\hat{F}(s)\right)$$

is bounded away from zero in $\mathbb{C}_0$; that is, $\inf_{s\in\mathbb{C}_0}|f_k(s)| > 0$. Let us rewrite f_k as follows:

$$f_k(s) = \left(\frac{s}{s+1}\right)^m \det\left(I_m + \frac{k}{s}\hat{F}(s)\right) \tag{3.2}$$

$$= \left(\frac{k}{s+1}\right)^m \det\left(\frac{s}{k} I_m + \hat{F}(s)\right). \tag{3.3}$$

We prove the claim by contraposition; that is, we suppose that there exists a sequence $(k_i)_{i\geq 0}$, $k_i \in (0, \infty)$, such that

$$\lim_{i\to\infty} k_i = 0 \quad \text{and} \quad \inf_{s\in\mathbb{C}_0} |f_{k_i}(s)| = 0.$$

Since $f_{k_i}(\infty) = 1$ for all $i \geq 0$ there exists a complex number $s_i \in \bar{\mathbb{C}}_0$ such that $f_{k_i}(s_i) = 0$. If follows from (3.2) that

$$|s_i| \leq \max_{i\geq 0}(k_i) \sup_{s\in\mathbb{C}_0} \bar{\sigma}(\hat{F}(s)).$$

So there exists a convergent subsequence $(z_j)_{j\geq 0}$ of $(s_i)_{i\geq 0}$. We conclude from (3.2) that $\lim_{j\to\infty} z_j = 0$. Finally it follows from (3.3) that $-z_j/k_j \in \sigma(\hat{F}(z_j))$, which means that $\sigma(\hat{F}(0)) \not\subset \mathbb{C}_0$.

The existence of stabilizing PI-controllers can now be established.

3.3 Theorem *Let $G \in \mathscr{A}_-^{m\times q}$ and choose a matrix $K_P \in \mathbb{C}^{q\times m}$ that stabilizes $\hat{G}$. Moreover, suppose that* rk $(\hat{G}(0)) = m$. *Then there are matrices $K_I \in \mathbb{C}^{q\times m}$ satisfying*

$$\sigma((I + \hat{G}(0)K_P)^{-1}\hat{G}(0)K_I \subset \mathbb{C}_0. \tag{3.4}$$

For each such K_I there exists $k^* > 0$ such that

$$\hat{K}_k(s) := \frac{1}{s} kK_I + K_P$$

stabilizes $\hat{G}$ for all $0 \leq k < k^*$.

Proof It follows from the assumptions that rk $(I + \hat{G}(0)K_P)^{-1}\hat{G}(0)) = m$. By Theorem 3.2 there exist matrices $K_I \in \mathbb{C}^{q\times m}$ that satisfy (3.4). Moreover, for each such matrix K_I there exists $k^* > 0$ such that the feedback system in Fig. 2 is i/o-stable for all $0 \leq k < k^*$. Application of the loop-transformation theorem (cf. (Desoer and Vidyasagar, 1975, p. 51)) shows that the feedback system in Fig. 3 is i/o-stable for all $0 \leq k < k^*$ as well.

In the sequel we shall study the tracking properties of the feedback system in Fig. 4. We assume that the impulse response of the plant is in $\mathscr{A}_-^{m\times q}$ and that the

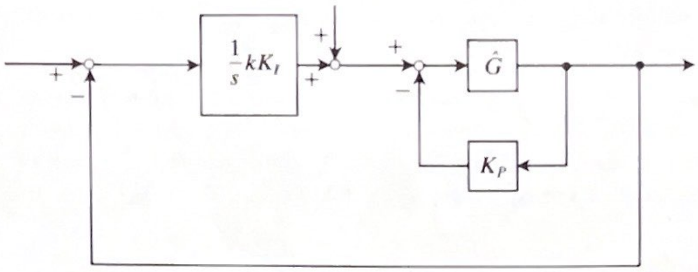

Fig. 2

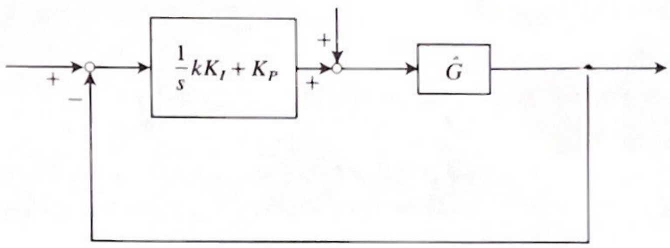

Fig. 3

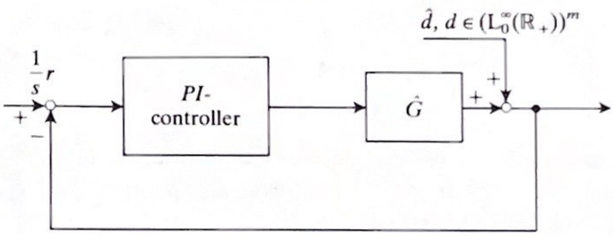

Fig. 4

disturbance d is a function in the space $(L_0^\infty(\mathbb{R}_+))^m$, where

$$L_0^\infty(\mathbb{R}_+) := \left\{ f \in L^\infty(\mathbb{R}_+) \mid \lim_{t\to\infty} f(t) \text{ exists} \right\}.$$

Notice that Fig. 5 (where $G \in \mathscr{A}_-^{m\times q}$ and $d_0 \in \mathbb{R}^q$) is a special case of Fig. 4 obtained by setting

$$d(t) = (G * (d_0\theta))(t).$$

Indeed it is obvious that $G * (d_0\theta) \in (L_0^\infty(\mathbb{R}_+))^m$.

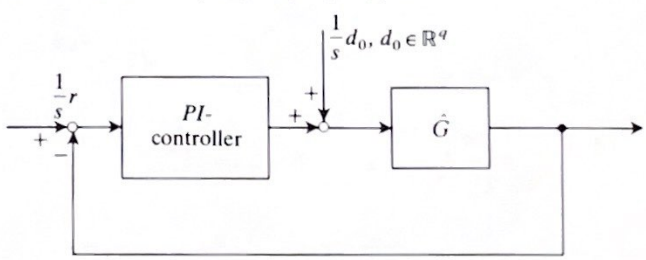

Fig. 5

In order to allow for non-zero initial conditions in the plant the following definition is useful

3.4 Definition A stable linear system with q inputs and m outputs is a triple (G, X, F), where $G \in \mathscr{A}_{-}^{m \times q}$, X denotes a vector space (the state space) and F is a linear operator mapping X into $(L_0^\infty(\mathbb{R}_+))^m$. The operator F is called the *initial condition operator* of the system (G, X, F). Let $u: \mathbb{R}_+ \to \mathbb{R}^q$ be a function such that

$$u\,|_{[0,T]} \in (L^1(0, T))^q \quad \text{for all } T > 0$$

(that is, u is locally integrable) and suppose that $x_0 \in X$. The output y of the system (G, X, F) corresponding to input u and the initial state x_0 is defined by $y = G * u + Fx_0$.

The following result shows that under certain conditions stabilization implies tracking.

3.5 Theorem *Let (G, X, F) be a stable linear system with q inputs and m outputs and let K_I, $K_P \in \mathbb{C}^{q \times m}$. It is assumed that* rk $(K_I) = m$. *We apply the* PI-*controller*

$$\left.\begin{aligned} \dot{z}(t) &= v(t), \qquad z(0) = z_0 \in \mathbb{R}^m, \\ w(t) &= K_I z(t) + K_P v(t) \end{aligned}\right\} \tag{3.5}$$

to the system (G, X, F). The resulting feedback system (cf. Fig. 6) *given by*

$$\left.\begin{aligned} y(t) &= (G * w)(t) + (Fx_0)(t) + d(t), \\ v(t) &= r\theta(t) - y(t) \end{aligned}\right\} \tag{3.6}$$

tracks the reference signal $r\theta(t)$, that is, $\lim_{t\to\infty} y(t) = r$ $(r \in \mathbb{R}^m)$, for arbitrary initial conditions $(x_0, z_0) \in X \times \mathbb{R}^m$ and arbitrary disturbances $d \in (L_0^\infty(\mathbb{R}_+))^m$ if $\hat{K}(s) := (1/s)K_I + K_P$ stabilizes $\hat{G}$.

Proof We obtain from equations (3.5) and (3.6)

$$\hat{y}(s) = \hat{G}(s)\left(K_I\left(\frac{1}{s}z_0\right) + \hat{K}(s)\left(\frac{1}{s} - \hat{y}(s)\right)\right) + \hat{y}_0(s) + \hat{d}(s),$$

where $y_0(t) := (Fx_0)(t)$. Hence

$$\begin{aligned} \hat{y}(s) = {} & (I + \hat{G}(s)\hat{K}(s))^{-1}\hat{G}(s)K_I\left(\frac{1}{s}z_0\right) + (I + \hat{G}(s)\hat{K}(s))^{-1}\hat{G}(s)\hat{K}(s)\left(\frac{1}{s}r\right) \\ & + (I + \hat{G}(s)\hat{K}(s))^{-1}(\hat{y}_0(s) + \hat{d}(s)). \end{aligned} \tag{3.7}$$

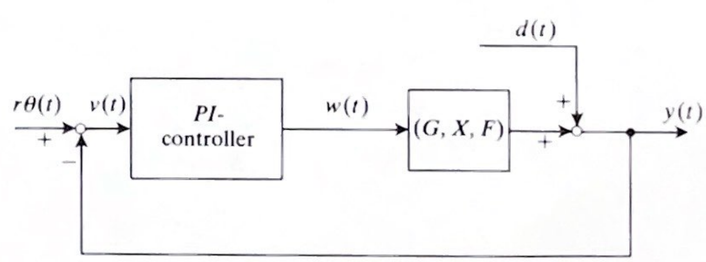

Fig. 6

Moreover, it is clear that

$$\hat{K}(s) = \Big(\frac{1}{s+1}K_I + \frac{s}{s+1}K_P\Big)\Big(\frac{s}{s+1}I_m\Big)^{-1}$$

is a right-coprime factorization, indeed we have

$$(K_I^TK_I)^{-1}K_I^T\Big(\frac{1}{s+1}K_I + \frac{s}{s+1}K_P\Big) + (I_m - (K_I^TK_I)^{-1}K_I^TK_P)\Big(\frac{s}{s+1}I_m\Big) \equiv I_m.$$

By assumption, $\hat{K}$ stabilizes $\hat{G}$ and therefore it follows from fractional representation theory (cf., for example, (Vidyasagar *et al.*, 1982)) that

$$\inf_{s\in\mathbb{C}_0}\Big|\det\Big(\frac{s}{s+1}I_m + \hat{G}(s)\Big(\frac{1}{s+1}K_I + \frac{s}{s+1}K_P\Big)\Big)\Big| > 0.$$

We obtain in particular that $\det(\hat{G}(0)K_I) = 0$. Furthermore, it follows from stability that the transfer matrices $T_1 := (I + \hat{G}\hat{K})^{-1}\hat{G}K_I$, $T_2 := (I + \hat{G}\hat{K})^{-1}\hat{G}\hat{K}$ and $T_3 := (I + \hat{G}\hat{K})^{-1}$ have all their entries in $\hat{\mathscr{A}}_-$. Now using the fact that $\hat{G}(0)K_I$ is non-singular it is easy to show that $T_1(0) = 0$, $T_2(0) = I$ and $T_3(0) = 0$. Hence it follows from Remark 2.2(iii), Lemma 2.3 and equation (3.7) that $\lim_{t\to\infty} y(t) = r$.

3.6 Corollary *Let (G, X, F) be a stable linear system with q inputs and m outputs. The following statements are equivalent.*

(i) *The rank of the matrix $\hat{G}(0)$ is equal to m.*

(ii) *There exist matrices $K_I, K_P \in \mathbb{C}^{q\times m}$ and a number $k^* > 0$ such that*

  (a) *for all $k \in [0, k^*)$ the controller $\hat{K}_k(s) = (1/s)kK_I + K_P$ stabilizes $\hat{G}$,*

  (b) *for all $k \in (0, k^*)$ the closed-loop system given by*

$$\dot{z}(t) = v(t), \qquad z(0) = z_0 \in \mathbb{R}^m, \tag{3.8}$$

$$w(t) = kK_Iz(t) + K_Pv(t), \tag{3.9}$$

$$y(t) = (G * w)(t) + (Fx_0)(t) + d(t), \tag{3.10}$$

$$v(t) = r\theta(t) - y(t), \qquad r \in \mathbb{R}^m \tag{3.11}$$

tracks constant reference signals (that is, $\lim_{t\to\infty} y(t) = r$) in the presence of arbitrary initial conditions $(x_0, z_0) \in X \times \mathbb{R}^m$ and arbitrary disturbances $d \in (L_0^\infty(\mathbb{R}_+))^m$.

Proof (i)⇒(ii) This follows from Lemma 3.1 and Theorems 3.3 and 3.5.

(ii)⇒(i) Let $k_0 \in (0, k^*)$ be fixed. By assumption it follows in particular that $(I + \hat{G}\hat{K}_{k_0})^{-1}\hat{G}\hat{K}_{k_0} \in \hat{\mathscr{A}}_-^{m\times m}$ and $(I + \hat{K}_{k_0}\hat{G})^{-1}\hat{K}_{k_0} \in \hat{\mathscr{A}}_-^{q\times m}$. Moreover, we have

$$\begin{aligned} I_m &= \lim_{s\to 0}\{(I + \hat{G}(s)\hat{K}_{k_0}(s))^{-1}\hat{G}(s)\hat{K}_{k_0}(s)\} \\ &= \hat{G}(0)\lim_{s\to 0}\{(I + \hat{K}_{k_0}(s)\hat{G}(s))^{-1}\hat{K}_{k_0}(s)\}. \end{aligned}$$

Hence it follows that $\operatorname{rk}(\hat{G}(0)) = m$.

3.7 Remark If a plant $G \in \mathscr{A}_-^{m\times q}$ does not satisfy the above rank condition, that $\operatorname{rk}(\hat{G}(0)) < m$, then it may be still possible to stabilize $\hat{G}$ by a PI-controller. A trivial example is given by

$$\hat{G}(s)=\begin{pmatrix}\dfrac{s}{s+1} & e^{-hs}\\ 0 & \dfrac{1}{s+2}\end{pmatrix}\quad (h>0),\qquad \hat{K}_k(s)=\frac{1}{s}k\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

Indeed a simple calculation shows that $\hat{K}_k$ stabilizes $\hat{G}$ for all $0\leq k<\infty$. Of course (by Corollary 3.6) the closed-loop system does not track arbitrary constant reference signals.

The following corollary follows immediately from Theorems 3.3 and 3.5.

3.8 Corollary *Suppose, for a plant $G\in\mathscr{A}_-^{m\times q}$, the rank condition* $\operatorname{rk}(\hat{G}(0))=m$ *is satisfied. If $(K_P, K_I)\in\mathbb{R}^{q\times m}\times\mathbb{R}^{q\times m}$ is a pair of matrices such that K_P stabilizes $\hat{G}$ and K_I satisfies the condition*

$$\sigma((I+\hat{G}(0)K_P)^{-1}\hat{G}(0)K_I)\subset\mathbb{C}_0,$$

then there exists $k^*>0$ such that the controller $\hat{K}_k(s)=(1/s)kK_I+K_P$ stabilizes $\hat{G}$ for all $0\leq k<k^*$. Moreover, suppose that $\hat{K}_k$ is applied to the linear system (G, X, F), where F is an initial-condition operator defined on the 'state-space' X. Then for all $k\in(0, k^*)$ the closed-loop system defined by (3.8) *to* (3.11) *tracks the reference signals in the presence of arbitrary initial conditions $(x_0, z_0)\in X\times\mathbb{R}^m$ and arbitrary disturbances $d\in(\mathrm{L}_0^\infty(\mathbb{R}_+))^m$.*

It is obvious from Corollary 3.8 that no exact knowledge of the system (G, X, F) is required in order to control it using a PI-controller of the form $\hat{K}_k(s)=(1/s)kK_I+K_P$. For pure integral control the knowledge of $\hat{G}(0)$ is sufficient. It follows from Remark 2.2(iii) that $\hat{G}(0)$ can be deduced from plant step response data. If proportional action is required then it is necessary to know an upper bound on $\sup_{\omega\in\mathbb{R}}\bar{\sigma}(\hat{G}(i\omega))$ (cf. Lemma 3.1). If we restrict ourselves to the case when $g\in\mathscr{A}_-^{0\,m\times q}$ then an upper bound on $\sup_{\omega\in\mathbb{R}}\bar{\sigma}(\hat{G}(i\omega))$ can be deduced from step response data as well. More precisely let h_{ij} denote the response from zero initial conditions of the ith output of the system (G, X, F) to a unit step in the jth input. Then it follows from Lemma 3.9 that

$$\sup_{\omega\in\mathbb{R}}\bar{\sigma}(\hat{G}(i\omega))\leq\sqrt{m}\max_{\substack{1\leq i\leq m\\ 1\leq j\leq q}}\left(\lim_{t\to\infty}\bigvee_0^t(h_{ij})\right),$$

where $\bigvee_0^t(\cdot)$ denotes the total variation of its argument on the interval $[0, t]$.

3.9 Lemma *Suppose that $g=\sum_{i=0}^n g_i\delta_{t_i}+g_a\in\mathscr{A}_-^0$ and let h denote the step response of g, that is, $h:=g*\theta$. Then we have*

$$\lim_{t\to\infty}\bigvee_0^t(h)=\|g\|_{\mathscr{A}}=\sum_{i=0}^n|g_i|+\int_0^\infty|g_a(\tau)|\,d\tau.$$

As a consequence

$$\|\hat{g}(s)\|_\infty=\sup_{s\in\mathbb{C}_0}|\hat{g}(s)|\leq\lim_{t\to\infty}\bigvee_0^t(h). \tag{3.12}$$

Equality holds in (3.12) *if h is monotonic and $g_i=0$, $i=0,\ldots,n$.*

The proof of this lemma can be found in the Appendix.

4. Examples

4.1 Semigroup systems

It has been shown by the authors (cf. (Logemann and Owens, 1987c)) that the results of Section 3 apply to a class of semigroup systems with unbounded control and observation operators introduced by Curtain and Pritchard (1978). In particular it is proved by Logemann and Owens (1987c) that the PI-controller of Section 3 achieves internal (that is, exponential) stability and tracking of step set-point changes in the presence of step disturbances in the state equations. Recent work by Curtain (1987) shows that this extends to a more general class of infinite-dimensional systems introduced by Salamon (cf. (Salamon, 1984; Curtain, 1987)).

4.2 Neutral systems

The results mentioned in subsection 4.1 do not cover neutral systems with general delays in the state-and-control variables. However, it will be shown that neutral systems fit into the framework developed in the previous sections and that the application of the PI-controller of Section 3 will result in *internal* stability of the closed-loop system. Consider the neutral system

$$\left.\begin{aligned} &\frac{d}{dt}\int_0^r dD(\tau)x(t-\tau) = \int_0^r dA(\tau)x(t-\tau) + \int_0^r dB(\tau)u(t-\tau) + d_0, \\ &x|_{[-r,0]} = \phi \in (C[-r,0]))^n, \\ &y(t) = \int_0^t dC(\tau)x(t-\tau), \end{aligned}\right\} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^q$ and $y(t) \in \mathbb{R}^m$. The disturbance d_0 is assumed to be a constant vector in $\mathbb{R}^n$. The functions A, B, C and D are of bounded variation on the interval $[-r, 0]$ with values in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times q}$, $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{n\times n}$, respectively. Moreover, we have to assume that

$$D = \theta I - E, \tag{4.2}$$

where E is a function of bounded variation on $[0, r]$ which is continuous at zero (cf. (Henry, 1974) or (Kappel, 1984)).

In order to introduce the transfer matrix $\hat{G}$ of the system (4.1) (with zero-initial condition) we define:

$$\hat{A}(s) := \int_0^t e^{-s\tau}\, dA(\tau), \qquad \hat{B}(s) := \int_0^r e^{-s\tau}\, dB(\tau),$$

$$\hat{C}(s) := \int_0^r e^{-s\tau}\, dC(\tau), \qquad \hat{D}(s) := \int_0^r e^{-s\tau}\, dD(\tau).$$

The entries of $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are entire functions. If we extend A, B, C and D to $\mathbb{R}_+$ by defining $A(t) = A(r)$, $B(t) = B(r)$, $C(t) = C(r)$ and $D(t) = D(r)$ for $t > r$, then the functions $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are the Laplace–Stieltjes transforms (cf. (Widder, 1972)) of A, B, C and D, respectively. We obtain the following expression for the transfer matrix G of (4.1):

$$\hat{G}(s) = \hat{C}(s)(s\hat{D}(s) - \hat{A}(s))^{-1}\hat{B}(s). \tag{4.3}$$

We shall need the following assumptions.

(N1) The functions of bounded variation A, B, C and D contain no singular part (see, for example, (Natanson, 1955, p. 263) or (Kolmogorov and Fomin, 1975, p. 341)).

(N2) The neutral system (4.1) is exponentially stable; that is, the strongly continuous solution semigroup on $(C([-r, 0]))^n$ of the homogeneous part of (4.1) is exponentially stable.

4.1 Remark Suppose that the function D contains no singular part. Then the condition (N2) is satisfied if and only if there exists a number $\alpha < 0$ such that

$$\det(s\hat{D}(s) - \hat{A}(s)) \neq 0 \quad \text{for all } s \in \mathbb{C}_\alpha$$

(cf. (Henry, 1974)).

The following lemma enables us to apply the results of Section 3 to a neutral system (4.1) satisfying (N1) and (N2).

4.2 Lemma *Suppose that* (N1) *and* (N2) *are satisfied. Then the inverse Laplace transform G of the transfer matrix $\hat{G}$ given by* (4.3) *is an element in* $(\mathrm{L}^1_-(\mathbb{R}_+))^{m\times q}$.

Proof By (4.2) and (N1) we can express $\hat{D}(s)$ as

$$\hat{D}(s) = I - \sum_{j=0}^{\infty} E_j e^{-r_j s} - \int_0^r E_\infty(\tau)e^{-s\tau}\, d\tau, \tag{4.4}$$

where $0 < r_j \leq r$ for all $j \geq 0$ and

$$\sum_{j=0}^{\infty} \|E_j\| + \int_0^r \|E_\infty(\tau)\|\, d\tau < \infty.$$

Define

$$\Delta_0(s) := I - \sum_{j=0}^{\infty} E_j e^{-r_j s}. \tag{4.5}$$

It follows from (N2) via Remark 4.1 and (Salamon, 1984, p. 160) that there exists $\alpha < 0$ such that

$$\det(\Delta_0(s)) \neq 0 \quad \text{for all } s \in \mathbb{C}_\alpha. \tag{4.6}$$

Inspection of (4.5) yields

$$\inf_{s\in\mathbb{C}_\beta} |\det(\Delta_0(s))| > 0 \tag{4.7}$$

for all large enough $\beta > 0$. Now observe that $\det(\Delta_0(s))$ is a holomorphic almost-periodic function (cf., for example (Corduneanu, 1968) or (Levin, 1964)) in every vertical strip of the complex plane. Hence by (4.6) and a result of Levin (1964, p. 268)

$$\inf_{0\leq \mathrm{Re}(s)\leq\beta} |\det(\Delta_0(s))| > 0. \tag{4.8}$$

Therefore we have

$$\inf_{s \in \mathbb{C}_0} |\det(\Delta_0(s))| > 0$$

(by (4.7) and (4.8)) and it follows from Logemann (1987) that $\hat{G} \in \hat{\mathscr{B}}^{m \times q}$. Moreover we conclude from (N2) via Remark 4.1 that $\hat{G}$ has no poles in $\mathbb{C}_\gamma$ for some $\gamma < 0$. As a consequence we obtain that $G \in \mathscr{A}_-^{m \times q}$. Finally it is not difficult to show that

$$\lim_{\substack{|s| \to \infty \\ s \in \mathbb{C}_0}} \hat{G}(s) = 0$$

which (by Remark 2.2 (i)) implies that $G \in (\mathrm{L}_-^1(\mathbb{R}_+))^{m \times q}$.

We shall apply the PI-controller

$$\left.\begin{aligned} \dot{z}(t) &= v(t), \qquad z(0) = z_0 \in \mathbb{R}^m, \\ w(t) &= kK_I z(t) + K_P v(t) \end{aligned}\right\} \tag{4.9}$$

to the neutral system (4.1); that is,

$$\left.\begin{aligned} v(t) &= r\theta(t) - y(t), \qquad r \in \mathbb{R}^m \\ u(t) &= w(t). \end{aligned}\right\} \tag{4.10}$$

4.3 Theorem *Suppose that the neutral system* (4.1) *satisfies* (N1) *and* (N2) and that rk $(\hat{G}(0)) = m$, *where* $\hat{G}$ *is defined by* (4.3). *Choose matrices* $K_P \in \mathbb{R}^{q \times m}$ *and* $K_I \in \mathbb{R}^{q \times m}$ *such that* K_P *stabilizes* G *and* K_I *satisfies the condition*

$$\sigma((I + \hat{G}(0)K_P^{-1})\hat{G}(0)K_I) \subset \mathbb{C}_0.$$

Then there exists a number $k^* > 0$ *such that for all* $0 < k < k^*$ *the neutral system defined by* (4.1), (4.9) *and* (4.10) *is exponentially stable and tracks constant reference signals in the presence of arbitrary initial values* $(\phi, z_0) \in (C([-r, 0]))^n \times \mathbb{R}^m$ *and arbitrary disturbances* $d_0 \in \mathbb{R}^n$.

Proof The exponential stability of the closed-loop system follows from the fact that the controller $\hat{K}_k(s) = (1/s)K_I + K_P$ stabilizes $\hat{G}$ for all $0 < k < k^*$ (cf. Corollary 3.8), and from (Logemann, 1986b, Corollary 3.2). Moreover, set

$$X := (C[-r, 0]))^n \quad \text{and} \quad (F\phi)(t) := \int_0^t dC(\tau)(S(t)\phi)(-\tau)$$

for $\phi \in X$, where $S(t)$ is the strongly continuous solution semigroup of the homogeneous part of (4.1). Using the exponential stability of $S(t)$ it is easy to show that $F\phi \in (\mathrm{L}_0^\infty(\mathbb{R}_+))^m$. Finally define

$$d(t) := \int_0^r dC(\tau)(Y * d_0\theta)(t - \tau), \tag{4.11}$$

where $Y(t)$ is the fundamental solution of the homogeneous part of (4.1) (cf. (Kappel, 1984)). If we realize that $Y(t) \in (\mathrm{L}^1(\mathbb{R}_+))^{n \times n}$ (by (N1) and (N2) via Remark 4.1 and by (Kappel, 1984, p. 18)) then if follows from (4.11) that $d \in (\mathrm{L}_0^\infty(\mathbb{R}_+))^m$. Now apply Corollary 3.8 to the system (G, X, F) and to disturbances of the form (4.11).

4.4 Remarks (i) The theoretical results of the papers by Koivo and Pohjolainen (1985) and Jussila and Koivo (1986) are contained in Theorem 4.3 as a special case.

(ii) Using ideas of Hale (1974) we can derive a result similar to Theorem 4.3 for certain retarded systems with infinite delays. Moreover, it has been shown by the authors (1987b) that the input–output results of Section 3 apply to Volterra integrodifferential systems and Volterra integral systems.

5. Conclusions

By considering infinite-dimensional systems having their impulse responses in $\mathscr{A}_{-}^{m\times q}$ it is possible to systematically design PI-controllers for the plant on the basis of open-loop step-response data. It has been shown that the resulting closed-loop system is stable, tracks step set-point changes and rejects disturbances belonging to $(L_0^\infty(\mathbb{R}_+))^m$. Furthermore, it has been proved that the PI-controller of Section 3 achieves *internal stability* if it is applied to neutral systems. It should be mentioned that all the results in Sections 3 and 4 remain true if we consider instead of step set-point changes reference signals in $(L_0^\infty(\mathbb{R}_+))^q$ (for example, ramps). The theory developed enables the generalization of recent results in low-gain control by Owens and Chotai (1986) to cover the infinite-dimensional case. Finally we note that there appears to be a form of 'duality' between the above '*low-gain* theory' of *stable* systems and '*high-gain* theory' of *minimum-phase* systems as illustrated in the finite-dimensional case, for example, by Owens and Chotai (1982) and in the infinite-dimensional case by Logemann and Owens (1987a).

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Appendix

In order to prove Lemma 3.9 the following lemma is useful.

A.1 Lemma *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation and let $c\in\mathbb{C}$. Define*

$$F(t):=\begin{cases} f(a), & t=a,\\ f(t)+c, & a<t\leq b.\end{cases}$$

If $\lim_{t\to a+} f(t)=f(a)$ then

$$\bigvee_a^b (F)=\bigvee_a^b (f)+|c|.$$

The proof is straightforward and is therefore omitted.

Proof of Lemma 3.9 Note that

$$h(x)=\sum_{t_i<x} g_i+\int_0^x g_a(\tau)\,d\tau$$

and choose $t>t_i$, $1\leq i\leq n$. Then

$$\bigvee_0^t (h)=\sum_{i=1}^n \bigvee_{t_{i-1}}^{t_i} (h)+\bigvee_{t_n}^t (h)$$

and it follows from Lemma A.1 and (Natanson, 1955, p. 259) that

$$\bigvee_0^t (h)=\sum_{i=1}^n \left(|g_{i-1}|+\int_{t_{i-1}}^{t_i} |g_a(\tau)|\,d\tau\right)+|g_n|+\int_{t_n}^t |g_a(\tau)|\,d\tau=\sum_{i=0}^n |g_n|+\int_0^t |g_a(\tau)|\,d\tau.$$

Hence

$$\lim_{t\to\infty} \bigvee_0^t (h) = \sum_{i=0}^{n} |g_i| + \int_0^{\infty} |g_a(\tau)|\, d\tau = \|g\|_{\mathscr{A}}$$

and it is now trivial that

$$\|\hat{g}\|_\infty \leq \lim_{t\to\infty} \bigvee_0^t (h).$$

Let us assume that $g_i = 0$ $(i = 0, \ldots, n)$ and that h is monotonic. This implies that h is absolutely continuous on $[0, t]$ for all $t > 0$ and

$$\int_0^t |\dot{h}(\tau)|\, d\tau = \left| \int_0^t \dot{h}(\tau)\, d\tau \right| \quad \text{for all } t > 0.$$

Therefore

$$\|\hat{g}\|_\infty \geq |\hat{g}(0)| = \lim_{t\to\infty} |h(t)| = \lim_{t\to\infty} \left| \int_0^t \dot{h}(\tau)\, d\tau \right| = \lim_{t\to\infty} \int_0^t |\dot{h}(\tau)|\, d\tau = \lim_{t\to\infty} \bigvee_0^t (h)$$

and by (3.12) it follows that $\|\hat{g}\|_\infty = \lim_{t\to\infty} \bigvee_0^t (h)$.