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INPUT–OUTPUT THEORY OF HIGH-GAIN ADAPTIVE STABILIZATION OF INFINITE-DIMENSIONAL SYSTEMS WITH NON-LINEARITIES

HARTMUT LOGEMANN AND DAVID H. OWENS

Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, U.K.

SUMMARY

This paper develops an input–output theory of high-gain adaptive stabilization of certain infinite-dimensional processes with actuator and sensor non-linearities. It is shown that there is a wide range of gain adaption rules achieving stability for the class of processes under consideration if proportional output feedback is used. The abstract input–output results are applied to retarded systems and Volterra integrodifferential systems. The paper shows that the scope of applicability of universal adaptive stabilization ideas extends far beyond finite-dimensional linear systems.

KEY WORDS Global adaptive stabilization Infinite-dimensional systems Non-linearities Nussbaum gains Input–output methods

NOMENCLATURE

$\mathbb{R}_+ :=$ set of non-negative real numbers.

$\mathbb{C}_+ :=$ open right-half complex plane.

Let $J \subset \mathbb{R}$ be an interval, then $C(J, \mathbb{R}^n) :=$ vector space of $\mathbb{R}^n$-valued continuous functions on J.

$C(J) := C(J, \mathbb{R})$.

$L^p(J) :=$ vector space of real-valued p-integrable functions on J.

$LL^p(J) :=$ vector space of real-valued locally p-integrable functions on J.

$L^p_+ := L^p(\mathbb{R}_+)$, $LL^p_+ := LL^p(\mathbb{R}_+)$, $LL^p := LL^p(\mathbb{R})$.

$BV(J, \mathbb{R}^{n\times n}) :=$ vector space of $\mathbb{R}^{n\times n}$-valued functions of bounded variation on J.

$H^\infty :=$ algebra of bounded holomorphic functions on $\mathbb{C}_+$.

$\mathrm{id}_{\mathbb{R}} :=$ the map $\mathbb{R} \to \mathbb{R}$, $x \mapsto x$.

Let $f \in LL^p_+$, then for all $t \geqslant 0$

$$(P_t f)(\tau) := \begin{cases} f(\tau) & 0 \leqslant \tau \leqslant t \\ 0 & \tau > t \end{cases}$$

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If $f \in C([0,\alpha))$, then for all $0 \leqslant t < \alpha$

$$(P_t f)(\tau) := \begin{cases} f(\tau) & 0 \leqslant \tau \leqslant t \\ 0 & \tau > t \end{cases}$$

$S_+(\alpha,\beta)(\beta \geqslant \alpha > 0)$ denotes the set of all Borel functions
$f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that $f(t,0) = 0$ $\forall t \in \mathbb{R}_+$ and $\beta x^2 \geqslant x f(t,x) \geqslant \alpha x^2$ $\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

$S_-(\alpha,\beta)(\beta \geqslant \alpha > 0)$ denotes the set of all Borel functions
$f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that $f(t,0) = 0$ $\forall t \in \mathbb{R}_+$ and $(-\alpha)x^2 \geqslant x f(t,x) \geqslant (-\beta)x^2$ $\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

$S(\alpha,\beta) := S_+(\alpha,\beta) \cup S_-(\alpha,\beta)$.

$S := \bigcup_{\beta \geqslant \alpha > 0} S(\alpha,\beta)$.

Given $f \in S$, there exist numbers $\beta \geqslant \alpha > 0$ such that $f \in S(\alpha,\beta)$,
$\mathrm{sign}(f) := +1$ if $f \in S_+(\alpha,\beta)$ and $\mathrm{sign}(f) := -1$ if $f \in S_-(\alpha,\beta)$ (notice that $\mathrm{sign}(f) = \mathrm{sign}(f(t,x))$ $\forall (t,x) \in \mathbb{R}_+ \times (0,\infty)$).

For $f \in S$ and $x: \mathbb{R}_+ \to \mathbb{R}$ the symbol $f(.,x(.))$ denotes the function $t \mapsto f(t,x(t))$.

1. INTRODUCTION

The initial steps taken in the area of 'high-gain' adaptive stabilization included the result by Willems and Byrnes[1] that linear, single-input/single-output, minimum-phase systems with finite-dimensional state-space realizations (A,B,C) are stabilized by the proportional output feedback law

$$u(t) = -\mathrm{sign}(CB)k(t)y(t)$$

where the gain $k(t)$ is obtained from the differential equation

$$\dot{k}(t) = (y(t))^2$$

These basic results appeared to depend upon the assumption of linearity, finite-dimensionality, the specific form of the control law and the gain evolution as a quadratic in 'real' output. In particular, a state-space representation seemed to be necessary and exact implementation of $u(t)$ based upon exact measurements of $y(t)$ appeared to be essential. For practical applications and robustness studies an input–output theory would have advantages and could enable the removal of the finite-dimensionality assumption. Also, the inclusion of imperfect measurement and actuation in the form of non-linear sensor and actuator characteristics would be of great benefit. These ideas and their implications are addressed in this paper. The analysis is necessarily technical but, in essence, the results of previous studies (see the references below) carry over to these more complex situations with little change other than the need for a new 'scaling-invariance' property in the controller gain characterization. However, the proofs of the results in the linear finite-dimensional case do not carry over to the non-linear infinite-dimensional situation and therefore the generalizations are far from being trivial.

Robust (non-adaptive) high-gain control of infinite-dimensional systems has previously been studied in some detail by the authors.[2] In the present paper we shall develop an input–output approach to adaptive high-gain stabilization of certain single-input/single-output non-linear infinite-dimensional processes. The class of processes Π to be considered is shown in Figure 1.

Π

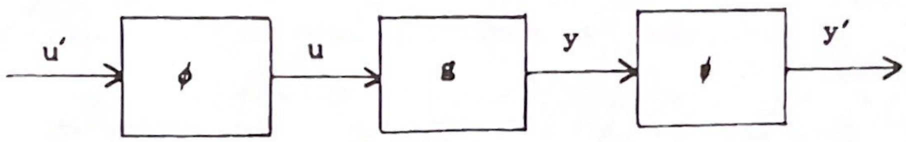

Figure 1

We suppose that ϕ and ψ are memoryless actuator and sensor non-linearities and that g is the transfer function of a linear time-invariant (not necessarily finite-dimensional) system. Moreover, we assume the following:

1. The functions ϕ and ψ are unbiased time-varying non-linearities lying either in a positive or negative sector (i.e. ϕ and ψ are elements of S as defined in the list of notation).
2. The transfer function g is minimum-phase (i.e. g has no zeros in the closed right-half plane) and there exists a real number $a \neq 0$ such that

$$sg(s) - a = O(s^{-1}) \quad \text{as } |s| \to \infty \text{ in } \mathbb{C}_+$$

The above equation generalizes the 'relative-degree one' condition for finite-dimensional systems when a is just CB in a state-space realization (A, B, C) (see Section 2).

Our adaptive stabilization results do not require parameter identification algorithms and can hence be regarded as being in the same spirit as the papers by Nussbaum,[3] Willems and Byrnes,[1] Dahleh and Hopkins,[4] Mårtensson,[5] Owens *et al.*[6] and Kobayashi.[7] The above references have in common that they are based on state-space methods. The articles by Nussbaum[3] and Willems and Byrnes[1] and the thesis by Mårtensson[5] deal with linear finite-dimensional systems, while Owens *et al.*[6] consider finite-dimensional systems with certain non-linearities in the state. Dahleh and Hopkins[4] extend the main result of Willems and Byrnes[1] to a class of linear differential-delay systems. Generalizations to linear distributed parameter systems described by holomorphic semigroups are given by Kobayashi.[7] The theory developed in the present paper contains the results of Dahleh and Hopkins[4] and Kobayashi[7] but includes a considerably larger class of systems, feedback laws and gain adaption rules. In particular, our results apply to retarded systems and Volterra integrodifferential systems. It is emphasized that the new theory can deal with systems subject to unbiased time-varying sensor and actuator non-linearities lying either in a positive or negative sector — an important problem which has not been considered in the above references. Although input–output descriptions are used throughout the paper, non-zero initial conditions are taken into account by using 'initial-condition terms' (see Section 2).

In Section 2 we establish a number of preliminary results and define precisely the problem to be considered. Section 3 investigates the stabilization problem in the case that the sign of the process Π defined by

$$\text{sign}(\Pi) = \text{sign}(\phi)\text{sign}(a)\text{sign}(\psi)$$

is known. We prove a lemma describing the dynamics of the process Π subject to the control law

$$u(t) = -\text{sign}(\Pi)\phi[t, k(t)\psi(t, y(t))]$$

under the assumption that the time-varying gain $k(t)$ is non-decreasing and $\lim_{t\to\infty}k(t)=\infty$. The analysis of 'gain divergence' is then used to prove that for a large class of causal gain adaption rules $y \mapsto k$ the above control law stabilizes the process Π in the sense that $y \in L^2_+ \cap L^\infty_+$, $\lim_{t\to\infty}y(t)=0$ and $\lim_{t\to\infty}k(t)$ exists and is finite. In Section 4 we consider the adaptive stabilization problem in the absence of information on sign(Π). In this case it is useful to employ so-called 'switching functions', a concept which has its origin in the papers by Nussbaum[3] and Willems and Byrnes.[1] In particular, it will turn out that the notion of a *scaling-invariant* switching function introduced in Section 4 plays an essential role in the problem of dealing with the non-linearities ϕ and ψ. A switching function is a locally (essentially) bounded function (taking on both positive and negative values) which enables the controller to 'learn' the sign of the process; in other words, it provides the controller gain with the 'right' sign. A scaling-invariant switching function has this property despite the presence of arbitrary unbiased time-varying actuator and sensor non-linearities lying either in a positive or negative sector. A precise definition of these concepts is given in Section 4. As Owens *et al.*[6] did, we allow switching as a function of both current and past gain and input data. This leads to a wide class of stabilizing adaptive high-gain controllers with convergence of the switching mechanism being independent of the gain adaption rules for generating $k(t)$. More precisely, we investigate the dynamics of the process Π under the control law

$$u(t)=\phi[t,N(\xi(t))k(t)\psi(t,y(t))]$$
$$\dot{\xi}(t)=k(t)(\psi(t,y(t)))^2 \qquad \xi(0)=\xi_0\in\mathbb{R}$$

where $k(t)$ is a strictly positive time-varying gain and N is a scaling-invariant switching function. It turns out that the above controller stabilizes the process Π in the sense that $y\in L^2_+\cap L^\infty_+$ and $\lim_{t\to\infty}\xi(t)$ exists and is finite. Moreover, if k is bounded, it follows that $\lim_{t\to\infty}y(t)=0$. In view of this result it is natural to consider the gain k as the image of a causal map operating on $\xi(.)$ and $\psi(.,y(.))$. A large class of gain adaption rules is given, ensuring closed-loop stability in the sense that $y\in L^2_+\cap L^\infty_+$, $\lim_{t\to\infty}y(t)=0$, the limits $\lim_{t\to\infty}\xi(t)$ and $\lim_{t\to\infty}k(t)$ exist and are finite. These gain adaption laws include $\dot{k}(t)=\psi^2(t,y(t))$ and therefore fully generalize previous work. Section 5 is devoted to the application of the input–output results of Sections 3 and 4 to retarded systems and Volterra integrodifferential systems. It is shown that for these two classes of systems the adaptive control laws of the previous sections achieve '*internal*' stability. The proofs of some of the results in Sections 2–4 are given in Appendices I–III.

2. PRELIMINARIES

In the following sections a certain functional differential equation will play an important role. It is therefore useful to give a precise statement of existence and uniqueness of solutions. Consider the initial-value problem

$$\dot{\mathbf{x}}(t)=(\mathbf{T}\mathbf{x})(t)+\mathbf{f}_1(t,\mathbf{x}(t))+\mathbf{f}_2(t) \quad t\geqslant\alpha \tag{1a}$$

$$\mathbf{x}\,|_{[0,\alpha]}=\mathbf{x}_0\in C([0,\alpha],\mathbb{R}^n) \quad \alpha\geqslant 0 \tag{1b}$$

where:

(i) $\mathbf{T}$: $(LL^2_+)^n\to(LL^2_+)^n$. We assume that $\mathbf{T}(\mathbf{0})=\mathbf{0}$ and that there exists $\varkappa>0$ such that $\|P_t(\mathbf{T}\mathbf{x}-\mathbf{T}\mathbf{x}')\|\leqslant\varkappa\|P_t(\mathbf{x}-\mathbf{x}')\|$ $\forall\mathbf{x},\mathbf{x}'\in(LL^2_+)^n$, $\forall t\geqslant 0$, i.e. $\mathbf{T}$ is unbiased, causal and of finite incremental gain.

(ii) $\mathbf{f}_1$: $\mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous and locally Lipschitzian function.
(iii) $\mathbf{f}_2$ is a function in $(LL^1_+)^n$.

Of course, if $\alpha = 0$ in (1b), then $C([0, \alpha], \mathbb{R}^n) = \mathbb{R}^n$. In order to define what we mean by a solution of (1) on $[0, \beta)$ $(\alpha < \beta \leqslant \infty)$, we have to give a meaning to $\mathbf{Tx}$ if $\mathbf{x} \in C([0, \beta), \mathbb{R}^n)$ (remember that $\mathbf{T}$ operates on functions which are defined on $\mathbb{R}_+$ and are elements in $(LL^2_+)^n$). We set $(\mathbf{Tx})(t) = (\mathbf{TP}_\tau \mathbf{x})(t)$ for $0 \leqslant t \leqslant \tau < \beta$. Since $\mathbf{T}$ is causal, this definition does not depend on the choice of τ.

2.1. *Definition*

A solution of (1) on $[0, \beta)$ is a function $\mathbf{x} \in C([0, \beta), \mathbb{R}^n)$ $(\alpha < \beta \leqslant \infty)$ such that:

(i) $\mathbf{x}$ is absolutely continuous on $[\alpha, \beta)$ and satisfies (1a) a.e. on $[\alpha, \beta)$.
(ii) (1b) is satisfied by $\mathbf{x}$.

2.2. *Theorem*

The initial-value problem (1) has a unique solution $\mathbf{x}$ on $[0, \beta)$ where $\alpha < \beta \leqslant \infty$. If $\beta < \infty$ and β cannot be increased, then there exists a sequence $\alpha < t_1 < t_2 < \cdots \to \beta$ such that $\lim_{i \to \infty} |\mathbf{x}(t_i)| = \infty$.

Although we hardly believe the above theorem to be unknown, we were unable to find it in the literature (e.g. the 'classical' paper by Driver[8] and the book by Hale[9]). We provide a proof in Appendix I.

Let us consider the linear part of the non-linear process Π shown in Figure 1. It is not assumed that the transfer function g is rational. Suppose that g is meromorphic on $\mathbb{C}_+$ and that

$$g^{-1}(s) = a^{-1}s + h(s) \tag{2}$$

where $a \in \mathbb{R}\backslash\{0\}$ and $h \in H^\infty$. Of course (2) is equivalent to

$$g(s) = \left(1 + \frac{a}{s} h(s)\right)^{-1} \frac{a}{s} \tag{3}$$

i.e. g is the feedback interconnection of the integrator a/s and the transfer function h (see Figure 2).

2.3. *Remark*

(i) Suppose that g is meromorphic in some open set containing $\bar{\mathbb{C}}_+$. Then (2) holds iff g has no zeros in $\bar{\mathbb{C}}_+$ and $sg(s) - a = O(s^{-1})$ if $|s| \to \infty$ in $\mathbb{C}_+$.
(ii) If g is the transfer function of a finite-dimensional system Σ: $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{b}u$, $y = \mathbf{c}^{\mathrm{T}}\mathbf{x}$, then (2) is satisfied if Σ is minimum-phase and $\mathbf{c}^{\mathrm{T}}\mathbf{b} \neq \mathbf{0}$.

In the following we shall assign an operator H: $L^2_+ \to L^2_+$ to the H^∞-function h. The operator H is given by $H = \mathbf{L}^{-1}M_h\mathbf{L}$, where $\mathbf{L}$ denotes the Laplace transform and M_h denotes the multiplication by h on the Hardy space H^2 (of the right-half plane). The operator H is linear, bounded and shift-invariant (in the sense of Vidyasagar[10]). As a consequence H is causal[10] and therefore has a unique causal extension to LL^2_+. This extension will also be denoted by H. It should be mentioned that the converse is also true; i.e. given a linear bounded shift-invariant g

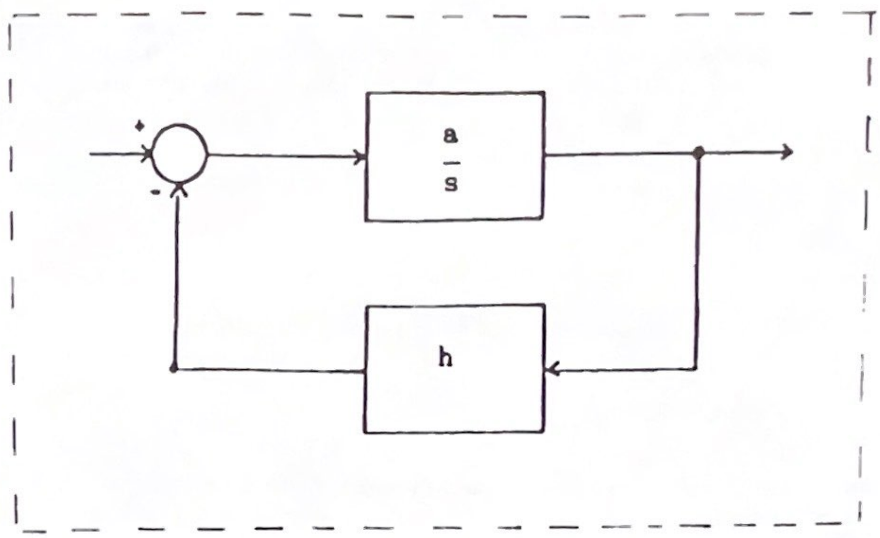

Figure 2

operator H: $L^2_+ \to L^2_+$ there exists a function $h \in H^\infty$ such that $H = \mathbf{L}^{-1} M_h \mathbf{L}$ (see Harris and Valenca[11] or Logemann[12]).

The function g can be thought of as being the transfer function of

$$\dot{y} = a(u - (Hy + w)) \qquad y(0) = y_0 \tag{4}$$

where $w \in L^2_+$ is due to a non-zero initial condition in the system with the transfer function h. The initial-value problem (4) is of the form (1) and therefore has a unique solution for each $u \in LL^1_+$.

The *problem* is to find an adaptive control law $u'(t) = F(y'(\tau), 0 \leqslant \tau \leqslant t)$ (i.e. $u(t) = \phi[t, F(\psi(\tau, y(\tau)), 0 \leqslant \tau \leqslant t)]$) which 'stabilizes' the process Π shown in Figure 1 $\forall y_0 \in \mathbb{R}$ and $\forall w \in L^2_+$ under the assumptions that $a \neq 0$, H: $L^2_+ \to L^2_+$ is a linear bounded shift-invariant operator and $\phi, \psi \in S$. By 'stability' we mean that $y, y' \in L^2_+ \cap L^\infty_+$ and $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0$. In Section 2 we study this problem under the assumption that sign(Π) is known. This assumption is dropped in Section 4.

3. A GENERAL CLASS OF ADAPTIVE CONTROLLERS IF SIGN(Π) IS KNOWN

The following result will be useful in the sequel. The proof is left to the reader.

3.1. Proposition

(i) Suppose that $\theta \in S$ and that f: $\mathbb{R}_+ \to \mathbb{R}$ is measurable. Then $\theta(., f(.)) \in L^q_+$ iff $f \in L^q_+$ $(q = 1, 2, \ldots, \infty)$.

(ii) Suppose that $\theta_i \in S(\delta_i, \Delta_i)(i = 1, 2)$ and $\text{sign}(\theta_1)\text{sign}(\theta_2) = +1$. Then

$$|\theta_1(t, r\theta_2(t, x))| \leqslant |r| \Delta_1 \Delta_2 |x| \quad \forall t \in \mathbb{R}_+ \quad \forall r, x \in \mathbb{R}$$

$$x\theta_1(t, -r\theta_2(t, x)) \leqslant -r\,\delta_1\,\delta_2 x^2 \quad \forall t, r \in \mathbb{R}_+ \quad \forall x \in \mathbb{R}$$

and

$$x\theta_1(t, r\theta_2(t, x)) \geqslant r\,\delta_1\,\delta_2 x^2 \quad \forall t, r \in \mathbb{R}_+ \quad \forall x \in \mathbb{R}$$

Let us apply proportional output feedback multiplied by $-\text{sign}(\Pi)$ to the process shown in Figure 1, i.e.

$$u(t) = \phi[t, -\text{sign}(\Pi)k(t)\psi(t, y(t))] \tag{5}$$

where k: $[0, \beta) \to \mathbb{R}$ is a time-varying gain $(0 < \beta \leqslant \infty)$.

3.2. *Lemma (unbounded gain lemma)*

Suppose that ϕ and ψ are in S and that the gain function k in (5) is non-decreasing and unbounded. Furthermore, we assume that the feedback system given by (4) and (5) has a unique absolutely continuous solution which can be continued uniquely to the right as long as it remains bounded. Under these conditions it follows that

$$y, y' = \psi(., y(.)) \in L^2(0, \beta) \cap L^\infty(0, \beta)$$

3.3. *Remark*

The closed-loop system (4) and (5) is of the form (1). Suppose that k, ϕ and ψ are continuous and that ϕ and ψ have continuous partial derivatives $D_2\phi$ and $D_2\psi$. Then it follows from Theorem 2.2 that there exists a unique solution of (4) and (5) which can be continued uniquely to the right as long as it remains bounded.

Proof of Lemma 3.2

Step 1. We show first that the closed-loop system given by (4) and (5) does not have an escape time smaller than β, i.e. that the solution exists on the entire interval $[0, \beta)$. We shall need the estimate

$$\left| \int_0^t f(\tau)(Hf)(\tau)\, \mathrm{d}\tau \right| \leqslant \|H\| \int_0^t f^2(\tau)\, \mathrm{d}\tau \quad \forall f \in LL_+^2 \quad \forall t \geqslant 0 \tag{6}$$

which follows from the causality and boundedness of H and Hölder's inequality. It will be useful to consider the two different cases $\text{sign}(\phi)\text{sign}(\psi) = +1$ and $\text{sign}(\phi)\text{sign}(\psi) = -1$.

(a) sign(ϕ)sign(ψ) = +1. Using (4) and (5), we obtain

$$(\dot{y}y)(t) = a\{y(t)\phi[t, -\text{sign}(\Pi)k(t)\psi(t, y(t))] - y(t)(Hy)(t) - y(t)w(t)\} \tag{7}$$

Let $0 < \gamma < \beta$ be any number such that y exists on $[0, \gamma)$. Moreover, let δ_ϕ, Δ_ϕ, δ_ψ and Δ_ψ be positive numbers such that $\phi \in S(\delta_\phi, \Delta_\phi)$ and $\psi \in S(\delta_\psi, \Delta_\psi)$. Setting $k^* := \max(|k(0)|, |k(\gamma)|)$

and integrating (7) from 0 to t, it follows by Proposition 3.1(ii) and (6) that

$$y^2(t) \leqslant y_0^2 + 2|a| \left[(k^* \Delta_\phi \Delta_\psi + \| H \|) \int_0^t y^2(\tau)\, \mathrm{d}\tau + \| w \|_2 \left(\int_0^t y^2(\tau)\, \mathrm{d}\tau \right)^{1/2} \right] \quad \forall 0 \leqslant t < \gamma \tag{8}$$

It is now sufficient to show that $y(t)$ is bounded on $[0, \gamma)$. In this case the solution $y(t)$ cannot have an escape time smaller than β and hence it exists on $[0, \beta)$.

We shall consider two cases:

(i)

$$\int_0^t y^2(\tau)\, \mathrm{d}\tau \leqslant 1 \quad \forall 0 \leqslant t < \gamma$$

It follows immediately from (8) that $y(t)$ is bounded on $[0, \gamma)$.

(ii)

$$\exists 0 < t_0 < \gamma : \int_0^{t_0} y^2(\tau)\, \mathrm{d}\tau > 1$$

Setting

$$M := 2|a| (k^* \Delta_\phi \Delta_\psi + \| H \| + \| w \|_2) \qquad N := y_0^2 + M \int_0^{t_0} y^2(\tau)\, \mathrm{d}\tau$$

we obtain from (8)

$$y^2(t) \leqslant N + M \int_{t_0}^t y^2(\tau)\, \mathrm{d}\tau \quad \forall t \in [t_0, \gamma)$$

and, moreover, by Gronwall's inequality

$$y^2(t) \leqslant N \exp(M(\gamma - t_0)) \quad \forall t \in [t_0, \gamma)$$

Hence y is bounded on $[t_0, \gamma)$. Since y is continuous on $[0, t_0]$, it follows that y is bounded on $[0, \gamma)$.

(b) sign(ϕ)sign(ψ) = -1. Realize that the diagrams in Figures 1 and 3 are equivalent. It then follows from (a) that the closed-loop system given by (4) and (5) does not have a finite escape time.

Step 2. It remains to show that $y \in L^2(0, \beta) \cap L^\infty(0, \beta)$. It then follows from Proposition

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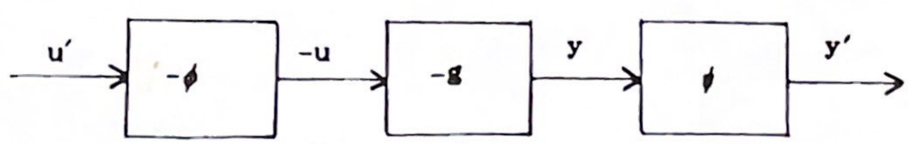

Figure 3

(3.1)(i) that y' is in $L^2(0,\beta)\cap L^\infty(0,\beta)$ as well. Let us first consider the case when $a>0$ and $\operatorname{sign}(\phi)\operatorname{sign}(\psi)=+1$.

(a) $a>0$, $\operatorname{sign}(\phi)\operatorname{sign}(\psi)=+1$. Choose $t_0\in[0,\beta)$ such that

$$k_0 := k(t_0) > \frac{1}{\delta_\phi\ \delta_\psi}\ \|H\|$$

Integrating (7) from t_0 to t and using Proposition 3.1(ii) and (6) gives

$$y^2(t)+2a(\delta_\phi\ \delta_\psi k_0-\|H\|)\int_{t_0}^{t} y^2(\tau)\,\mathrm{d}\tau-2a\|w\|_2\left(\int_{t_0}^{t} y^2(\tau)\,\mathrm{d}\tau\right)^{1/2}\leqslant L<\infty\quad t\in[t_0,\beta)$$

where

$$L := y^2(t_0)+2a\|H\|\int_0^{t_0} y^2(\tau)\,\mathrm{d}\tau$$

As a consequence $y\in L^2(t_0,\beta)\cap L^\infty(t_0,\beta)$ and hence by continuity of y we have $y\in L^2(0,\beta)\cap L^\infty(0,\beta)$.

(b) $a<0$, $\operatorname{sign}(\phi)\operatorname{sign}(\psi)=+1$. The proof of the lemma in this case is very similar to the one in case (a). The details are therefore omitted.

(c) $a\gtrless 0$, $\operatorname{sign}(\phi)\operatorname{sign}(\psi)=-1$. Using the fact that the diagrams in Figures 1 and 3 are equivalent, it follows from (a) and (b) that the claim is true in this particular case.

Consider the following gain adaption rule:

$$k = Z[\psi(.,y(.))] \tag{9}$$

where

$$Z\colon D_z\supset\bigcup_{\psi\in S}\{\psi(.,f(.))\mid f\in LL_+^\infty\}\to LL_+^1$$

is a causal map satisfying

(A1) $Z(D_z\cap L_+^2\cap L_+^\infty)\subset L_+^\infty$
(A2) $Z(f)\in L_+^\infty\Rightarrow f\in L_+^2$
(A3) $(Z(f)(0)>-\infty$ and $Z(f)$ is non-decreasing $\forall f\in D_z$.

3.4. Remark

(i) The space LL_+^∞ is an example of a function space containing $\cup_{\psi\in S}\{\psi(.,f(.))\mid f\in LL_+^\infty\}$.
(ii) It is necessary to give a meaning to $Z[\psi(.,f(.))]$ if $f\in C([0,\beta))$ $(0<\beta<\infty)$ (Z operates on functions which are defined on $\mathbb{R}_+$). We set

$$Z[\psi(.,f(.))](t) := Z[\psi(.,(P_\tau f)(.))](t)\quad \forall t\leqslant\tau<\beta$$

Since Z is causal, this definition does not depend on the choice of τ.

An example satisfying the assumptions (A1)–(A3) is given by

$$\dot{k}=\psi^2(.,y(.))\qquad k(0)=k_0 \tag{10}$$

i.e.

$$Z: LL_+^2 \to LL_+^1 \qquad f \mapsto k_0 + \int_0^{\cdot} f^2(\tau)\,\mathrm{d}\tau \tag{11}$$

The following theorem shows that the controller given by (5) and (9) stabilizes the system (4).

3.5. *Theorem*

Let ϕ and ψ be in S and suppose that (A1)–(A3) are satisfied. Furthermore, we assume that the closed-loop system (4), (5) and (9) has a unique absolutely continuous solution y which can be continued uniquely to the right as long as it remains bounded. Under this condition it follows that:

(i) $\lim_{t\to\infty} k(t)$ exists and is finite.
(ii) $y, y' = \psi(.,y(.)) \in L_+^2 \cap L_+^\infty$.
(iii) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0$.

3.6. *Remark*

Consider the special case that Z is given by (11) and realize that the closed-loop system (4), (5) and (10) is of the form (1). Suppose that ϕ and ψ are continuous with continuous partial derivatives $D_2\phi$ and $D_2\psi$. Then it follows from Theorem 2.2 that there exists a unique solution (y,k) of (4), (5) and (10) which can be continued uniquely to the right as long as y remains bounded.

Proof of Theorem 3.5

Step 1. Choose $0 < \beta \leqslant \infty$ such that the closed-loop system has unique absolutely continuous solution on $[0,\beta)$. We claim that $k(t)$ given by (9) is bounded on $[0,\beta)$. Assume the contrary, i.e. k is unbounded on $[0,\beta)$. Since we know by (A3) that k is non-decreasing and $k(0) > -\infty$, we obtain that $\lim_{t\to\beta} k(t) = +\infty$ and hence by Lemma 3.2 $y \in L^2(0,\beta) \cap L^\infty(0,\beta)$. Define the function $\tilde{y} \in L_+^2 \cap L_+^\infty$ by setting $\tilde{y}(t) = y(t)$ for $t \in [0,\beta)$ and $\tilde{y}(t) = 0$ for $t \geqslant \beta$. It follows from (A1) that $Z[\psi(.,\tilde{y}(.))] \in L_+^\infty$ and, using the causality of Z, we obtain $k \in L^\infty(0,\beta)$, which is a contradiction.

Step 2. We claim that the solution y of the closed loop exists on $[0,\infty)$. Let $0 < \beta < \infty$ be any number such that the solution y exists on $[0,\beta)$. It is sufficient to show that y is bounded on $[0,\beta)$. In this case y cannot have a finite escape time. Step 1 implies that k is bounded on $[0,\beta)$. Therefore we can show, using exactly the same arguments as in step 1 of the proof of Lemma 3.2, that y is bounded on $[0,\beta)$, which proves the claim.

Step 3. It follows from steps 1 and 2 that $k \in L_+^\infty$ and by (A3) we obtain (i). Moreover, by (A2) we have $y \in L_+^2$. Using (i) and Proposition 3.1(i), we obtain that $\dot{y} \in L_+^2$. As a consequence we have $\lim_{t\to\infty} y(t) = 0$ and hence $\lim_{t\to\infty} y'(t) = 0$, which proves (iii). Since y is continuous, it follows that $y \in L_+^\infty$. Therefore $y \in L_+^2 \cap L_+^\infty$ and by Proposition 3.1(i) $y' \in L_+^2 \cap L_+^\infty$, which shows that (ii) holds true.

3.7. Remark

Suppose that $\phi(t, x) = x \ \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}$ and apply an L^2_+-input signal v to the closed-loop system (4), (5) and (9), i.e. (5) becomes

$$u = v - \operatorname{sign}(\Pi) k\psi(., y(.))$$

An inspection of the proof of Theorem 3.5 shows that the conclusions of Theorem 3.5 remain true in this case. In particular we have that the closed-loop operators $v \mapsto y$ and $v \mapsto y'$ are L^2-stable.

4. ADAPTIVE HIGH–GAIN STABILIZATION WITH GAIN SWITCHING

An underlying requirement of the previous section is that the sign of the process Π shown in Figure 1 is known. If this knowledge is not available, then the control problem becomes more complex. It will turn out that it is useful to introduce the concept of a *switching function* which has its origin in the paper by Nussbaum.[3]

4.1. Definition

A function $N \in LL^\infty$ is called:

(i) a *switching function* if for some $x_0 \in \mathbb{R}$

$$\sup_{x > x_0} \frac{1}{x - x_0} \int_{x_0}^{x} N(\lambda) \, d\lambda = +\infty \tag{12a}$$

and

$$\inf_{x > x_0} \frac{1}{x - x_0} \int_{x_0}^{x} N(\lambda) \, d\lambda = -\infty \tag{12b}$$

(ii) a *Nussbaum gain* if the function $\mathrm{id}_{\mathbb{R}} N$ is a switching function.

A switching function N is called *scaling-invariant* if the function $(\Gamma_\alpha^\beta \circ N)N$ is a switching function for all $\alpha, \beta > 0$, where the function Γ_α^β is given by

$$\Gamma_\alpha^\beta(\lambda) = \begin{cases} \alpha & \lambda > 0 \\ 0 & \lambda = 0 \\ \beta & \lambda < 0 \end{cases}$$

A Nussbaum gain N is called *scaling-invariant* if $(\Gamma_\alpha^\beta \circ N)N$ is a Nussbaum gain for all $\alpha, \beta > 0$.

It is easily seen that if conditions (12) are satisfied for some $x_0 \in \mathbb{R}$, then they are satisfied for all $x_0 \in \mathbb{R}$. Moreover, it is clear that a Nussbaum gain N is scaling-invariant iff the switching function $\mathrm{id}_{\mathbb{R}} N$ is.

We given an example of a scaling-invariant switching function and a scaling-invariant Nussbaum gain.

4.2. Example

Consider the function $N(\lambda) = \cos(\frac{1}{2}\pi\lambda)\exp(\lambda^2)$. In the paper by Nussbaum[3] it is shown that for all $x_0 \in \mathbb{R}$

$$\sup_{x > x_0} \int_{x_0}^{x} N(\lambda) \, d\lambda = +\infty$$

and

$$\inf_{x > x_0} \int_{x_0}^{x} N(\lambda)\, d\lambda = -\infty$$

Using the same ideas it is possible to show that N is a scaling-invariant switching function and a scaling-invariant Nussbaum gain. For completeness a proof is provided in Appendix II.

It is easy to find Nussbaum gains which are not scaling-invariant. The Nussbaum gain

$$N(\lambda = \begin{cases} 1 & n^2 \leqslant |\lambda| < (n+1)^2 \quad n \text{ even} \\ -1 & n^2 \leqslant |\lambda| < (n+1)^2 \quad n \text{ odd} \end{cases}$$

has been considered by Willems and Byrnes.[1] It can be shown that $(\Gamma_\alpha^\beta \circ N)N$ is not a Nussbaum gain if $\alpha \neq \beta (\alpha, \beta > 0)$. However, it is not difficult to show that there exist *bounded* scaling-invariant Nussbaum gains. An example is given by

$$N(\lambda) = \begin{cases} 1 & 0 \leqslant |\lambda| < \lambda_0 \\ 1 & \lambda_n \leqslant |\lambda| < \lambda_{n+1} \quad n \text{ even} \\ -1 & \lambda_n \leqslant |\lambda| < \lambda_{n+1} \quad n \text{ odd} \end{cases}$$

where $\lambda_{n+1} = (\lambda_n)^2$ and $\lambda_0 > 1$.

Let us consider the same process as in Sections 2 and 3 (see Figure 1). We shall study the behaviour of the process II if the following control law is applied:

$$u(t) = \phi[t, N(\xi(t))k(t)\psi(t, y(t))] \tag{13a}$$

$$\dot{\xi}(t) = k(t)(\psi(t, y(t)))^2 \qquad \xi(0) = \xi_0 \in \mathbb{R} \tag{13b}$$

where N is a scaling-invariant switching function and k: $\mathbb{R}_+ \to \mathbb{R}$ is a strictly positive function, i.e. $k(t) \geqslant \varepsilon > 0 \forall t \in \mathbb{R}_+$.

4.3. *Lemma*

Let ϕ and ψ be in S and suppose that the feedback system given by (4) and (13) has a unique absolutely continuous solution which can be continued uniquely to the right as long as it remains bounded. Under these conditions the following is true:

(i) $y, y' \in L_+^2 \cap L_+^\infty$.
(ii) $\lim_{t\to\infty} \xi(t)$ exists and is finite.

Moreover, if k is bounded, we have:

(iii) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0$.

4.4. *Remark*

Suppose that k is continuous, N is continuously differentiable and ϕ and ψ are continuous with continuous partial derivatives $D_2\phi$ and $D_2\psi$. Then it follows from Theorem 2.2 that the feedback system given by (4) and (13) has a unique solution which can be continued uniquely to the right as long as it remains bounded.

For the proof of Lemma 4.3 we need the following.

4.5. Proposition

Let $\phi \in S(\delta_\phi, \Delta_\phi)$ and $\psi \in S(\delta_\psi, \Delta_\psi)$ and suppose that $\text{sign}(\phi)\text{sign}(\psi) = +1$. Then the inequalities

$$x\phi(t, \lambda r\psi(t, x)) \leqslant \Gamma(\lambda)\lambda r(\psi(t, x))^2$$

and

$$x\phi(t, \lambda r\psi(t, x)) \geqslant \Gamma'(\lambda)\lambda r(\psi(t, x))^2$$

hold $\forall (t, r, \lambda, x) \in \mathbb{R}_+^2 \times \mathbb{R}^2$, where

$$\Gamma(\lambda) := \begin{cases} \Delta_\phi/\delta_\psi & \lambda > 0 \\ 0 & \lambda = 0 \\ \delta_\phi/\Delta_\psi & \lambda < 0 \end{cases} \tag{14a}$$

and

$$\Gamma'(\lambda) := \Gamma(-\lambda) \tag{14b}$$

See Appendix III for a proof of Proposition 4.5.

Proof of Lemma 4.3

It follows from the equivalence of the diagrams in Figures 1 and 3 that we can assume without loss of generality that $\text{sign}(\phi)\text{sign}(\psi) = +1$. Furthermore, let δ_ϕ, Δ_ϕ, δ_ψ and Δ_ψ be positive numbers such that $\phi \in S(\delta_\phi, \Delta_\phi)$ and $\psi \in S(\delta_\psi, \Delta_\psi)$. Consider the equation

$$y\dot{y} = -a(yHy + yw) + ay\phi[\,.\,, (N \circ \xi)k\psi(\,.\,, y(\,.\,))] \tag{15}$$

Integrating (15) from 0 to t, using (6) and (14) and applying Proposition 4.5 yields

$$\tfrac{1}{2}(y^2(t) - y_0^2) \leqslant |a| \left[\|H\| \int_0^t y^2(\tau)\,\mathrm{d}\tau + \|w\|_2 \left(\int_0^t y^2(\tau)\,\mathrm{d}\tau \right)^{1/2} \right] + |a| \begin{cases} \displaystyle\int_0^t (\Gamma \circ N)(\xi(\tau)) N(\xi(\tau)) k(\tau) \psi^2(\tau, y(\tau))\,\mathrm{d}\tau & \text{if } a > 0 \\ \displaystyle -\int_0^t (\Gamma' \circ N)(\xi(\tau)) N(\xi(\tau)) k(\tau) \psi^2(\tau, y(\tau))\,\mathrm{d}\tau & \text{if } a < 0 \end{cases} \tag{16}$$

Using (13b) and the change of variables formula for Lebesgue integrable functions (see e.g. Reference 13, p.195), it follows that

$$\tfrac{1}{2}(y^2(t) - y_0^2) \leqslant |a| \left[\|H\| \int_0^t y^2(\tau)\,\mathrm{d}\tau + \|w\|_2 \left(\int_0^t y^2(\tau)\,\mathrm{d}\tau \right)^{1/2} \right] + |a| \begin{cases} \displaystyle\int_{\xi_0}^{\xi(t)} (\Gamma \circ N)(\lambda) N(\lambda)\,\mathrm{d}\lambda & \text{if } a > 0 \\ \displaystyle -\int_{\xi_0}^{\xi(t)} (\Gamma' \circ N)(\lambda) N(\lambda)\,\mathrm{d}\lambda & \text{if } a < 0 \end{cases} \tag{17}$$

Now realize that by (13b) and the properties of k

$$\begin{aligned}\int_0^t y^2(\tau)\,d\tau &\leqslant \frac{1}{\varepsilon}\int_0^t k(\tau)y^2(\tau)\,d\tau\\ &\leqslant \frac{1}{\varepsilon}\frac{1}{(\delta_\psi)^2}\int_0^t k(\tau)\psi^2(\tau, y(\tau))\,d\tau\\ &= \frac{1}{\varepsilon}\frac{1}{(\delta_\psi)^2}(\xi(t)-\xi_0) \qquad (18)\end{aligned}$$

Setting

$$K_1 := |a|\frac{1}{\varepsilon}\frac{1}{(\delta_\psi)^2}\|H\| \qquad K_2 := |a|\,\|w\|_2\frac{1}{\sqrt{\varepsilon}}\frac{1}{\delta_\psi}$$

and using (17) and (18), we obtain

$$\tfrac{1}{2}y^2(t) \leqslant \tfrac{1}{2}y_0^2 + (\xi(t)-\xi_0)\left[K_1 + \frac{K_2}{\sqrt{(\xi(t)-\xi_0)}} + |a|\frac{1}{\xi(t)-\xi_0}\begin{cases}\displaystyle\int_{\xi_0}^{\xi(t)}(\Gamma\circ N)(\lambda)N(\lambda)\,d\lambda & \text{if } a>0\\ \displaystyle -\int_{\xi_0}^{\xi(t)}(\Gamma'\circ N(\lambda)N(\lambda)\,d\lambda & \text{if } a<0\end{cases}\right. \qquad (19)$$

Equations (18) and (19) hold on each interval of the form $[0,\alpha)$ where the solution $(y(t), \xi(t))$ of the feedback system given by (4) and (13) exists. Since the RHS of (19) has to be non-negative, it follows from the properties of the function N that $\xi(t)$ and hence (by (19)) $y(t)$ remain bounded. As a consequence the solution $(y(t), \xi(t))$ exists on $[0,\infty)$ and it follows that $\lim_{t\to\infty}\xi(t)$ exists and is finite, which is (ii). Moreover, by (18) and (19) $y\in L_+^2\cap L_+^\infty$ and hence by Proposition 3.1(i) $y'\in L_+^2\cap L_+^\infty$, which is (i). Finally, if k is bounded, it follows from (4) and (13) via (i) and (ii) that $\dot{y}\in L_+^2$. Therefore we have $\lim_{t\to\infty}y(t)=0$ and, since $\psi\in S$, we obtain that $\lim_{t\to\infty}y'(t)=0$, which proves (iii)

4.6. Remark

(i) Suppose that we replace (13b) by $\dot{\xi} = k(\psi(\cdot, y(\cdot)))^{2p}$ where $p\geqslant 2$ is an integer. Then the conclusions of Lemma 4.3 remain true if the operator H in (4) is a linear bounded shift-invariant operator from L_+^{2p} into L_+^{2p} and from L_+^∞ into L_+^∞ and if the function w in (4) is in L_+^{2p}.

(ii) In the linear case, i.e. $\phi(t,x)=\psi(t,x)=x$ for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, Lemma 4.3 remains true for all (i.e. not necessarily scaling-invariant) switching functions N. It should be regarded as a generalization of the result of Willems and Byrnes,[1] which is just the special case given by:

(a) g is a *rational* transfer function.
(b) $\phi(t,x)=\psi(t,x)=x\ \forall (t,x)\in\mathbb{R}_+\times\mathbb{R}$.
(c) $k(t)=1\ (t\geqslant 0)$.
(d) $N=N_0\mathrm{id}_{\mathbb{R}}$ where N_0 is a Nussbaum gain.

It is natural to regard the gain k as the image of a causal map Z operating on ξ and $\psi(\cdot, y(\cdot))$. Let us consider the following gain adaption rule:

$$k = Z(\xi, \psi(\cdot, y(\cdot))) \qquad (20)$$

where $Z\colon D_z \to LL^1_+$ is a causal map whose domain D_z contains the set

$$C(\mathbb{R}_+) \times \bigcup_{\psi \in S} \{\psi(.,f(.)) \mid f \in LL^\infty_+\}.$$

Moreover, we assume that Z satisfies:

(A4) $Z[D_z \cap (L^\infty_+ \times (L^\infty_+ \cap L^2_+))] \subset L^\infty_+$.
(A5) $Z(f)$ is non-decreasing for all $f \in D_z$.
(A6) There exists $\varepsilon > 0$ such that $\inf_{t \geq 0}\{Z(f))(t)\} \geq \varepsilon \ \forall f \in D_z$.

Consider the following example:

$$Z\colon C(\mathbb{R}_+) \times LL^\infty_+ \to LL^1_+$$

$$(f_1, f_2) \mapsto k_0 + \int_0^{\cdot} f_2^2(\tau) P(|f_1(\tau)|, |f_2(\tau)|)\, \mathrm{d}\tau \tag{21}$$

where $k_0 > 0$ and P is a polynomial in two variables with positive coefficients. It is trivial that (A4)–(A6) are satisfied. The gain adaption induced by (21) can be written in the form of an ordinary differential equation

$$\dot{k}(t) = \psi^2(t, y(t)) P(|\xi(t)|, |\psi(t, y(t))|) \qquad k(0) = k_0 \tag{22}$$

The following theorem is a simple consequence of Lemma 4.3.

4.7. Theorem

Let ϕ and ψ be in S and suppose that (A4)–(A6) are satisfied. Moreover, assume that the feedback system given by (4), (13) and (20) has a unique absolutely continuous solution (y, ξ) which can be continued uniquely to the right as long as it remains bounded. Under these conditions the following is true:

(i) $y, y' \in L^2_+ \cap L^\infty_+$.
(ii) $\lim_{t\to\infty} \xi(t)$ exists and is finite.
(iii) $\lim_{t\to\infty} k(t)$ exists and is finite.
(iv) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0$.

4.8. Remark

(i) Consider the special case that Z is given by (21). Suppose that N is continuously differentiable and that ϕ and ψ are continuous and have continuous partial derivatives $D_2\phi$ and $D_2\psi$. Then it follows from Theorem 2.2 that there exists a unique solution (y, ξ, k) of the feedback system given by (4), (13) and (22) which can be continued to the right as long as (y, ξ) remains bounded.

(ii) Under the condition of Remark 4.6(i) the above theorem remains true if we replace (13b) by $\dot{\xi} = k(\psi(.,y(.)))^{2p}$ where $p \geq 2$ is an integer.

(iii) Suppose that $\phi(t, x) = x \ \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}$ and apply an L^2_+-input signal v to the closed-loop system given by (4), (13) and (20), i.e. (13a) becomes

$$u(t) = v(t) + N(\xi(t)) k(t) \psi(t, y(t))$$

It is easy to show that the conclusions of Theorem 4.7 remain true in this case. In particular it follows that the closed-loop operators $v \mapsto y$ and $v \mapsto y'$ are L^2-stable.

5. APPLICATIONS TO RETARDED SYSTEMS AND VOLTERRA INTEGRODIFFERENTIAL SYSTEMS

In this section we show that retarded systems and Volterra integrodifferential systems fit into the abstract framework developed in Sections 2–4.

5.1. Retarded systems

In the following we extend any function $\mathbf{F} \in BV([\alpha, \beta], \mathbb{R}^{n \times n})$ to the whole real axis by setting $\mathbf{F}(t) = \mathbf{F}(\alpha)\ \forall t < \alpha$ and $\mathbf{F}(t) = \mathbf{F}(\beta)\ \forall t > \beta$. Any measurable function $\mathbf{f}: \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}$ will be extended to the whole real axis by defining $\mathbf{f}(t) = \mathbf{0}\quad \forall t \notin \Omega$. For $\mathbf{F} = (F_{ij}) \in BV([0, r], \mathbb{R}^{n \times n})$ and $\mathbf{f} = (f_1, \ldots, f_n)^{\mathrm{T}}$, $f_i \in LL^1 (1 \leqslant i \leqslant n)$, we define

$$\mathbf{dF} * \mathbf{f} := \begin{bmatrix} \sum_{j=1}^{n} \mathrm{d}F_{1j} * f_j \\ \\ \sum_{j=1}^{n} \mathrm{d}F_{nj} * f_j \end{bmatrix}$$

where $\mathrm{d}F_{ij}$ denotes the measure on $\mathbb{R}$ induced by F_{ij} and $\mathrm{d}F_{ij} * f_j$ denotes the convolution of the measure $\mathrm{d}F_{ij}$ and the function f_j. If $\mathbf{f}$ is continuous on $[-r, \infty)$, then of course

$$(\mathbf{dF} * \mathbf{f})(t) = \int_0^r \mathrm{d}\mathbf{F}(\tau)\mathbf{f}(t - \tau) \quad \forall t \geqslant 0$$

Consider the single-input/single-output retarded system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathrm{d}\mathbf{A} * \mathbf{x} + \mathbf{b}u \\ y &= \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \mathbf{x}\,|_{[-r,0]} &= \mathbf{x}_0 \in C([-r, 0], \mathbb{R}^n) \end{aligned} \tag{23}$$

where $\mathbf{A} \in BV([0, r], \mathbb{R}^{n \times n})$. We assume that

$$\mathbf{c}^{\mathrm{T}}\mathbf{b} \neq 0 \tag{24}$$

and

$$\det \begin{bmatrix} s\mathbf{I} - \hat{\mathbf{A}}(s) & -\mathbf{b} \\ \mathbf{c}^{\mathrm{T}} & 0 \end{bmatrix} \neq 0 \quad \forall s \in \overline{\mathbb{C}}_+ \tag{25}$$

where $\hat{\mathbf{A}}(s) := \int_0^r \exp(-s\tau)\, \mathrm{d}\mathbf{A}(\tau)$. The transfer function $g(s)$ of (23) is given by $g(s) = \mathbf{c}^{\mathrm{T}}(s\mathbf{I} - \hat{\mathbf{A}}(s))^{-1}\mathbf{b}$.

5.2. Remark

As in the finite-dimensional case, we shall call (25) the minimum-phase condition. It can be shown that (25) holds iff $\forall s \in \overline{\mathbb{C}}_+$

$$\begin{aligned} g(s) &\neq 0 \\ rk(s\mathbf{I} - \hat{\mathbf{A}}(s), \mathbf{b}) &= n \end{aligned}$$

and

$$rk \begin{bmatrix} s\mathbf{I} - \hat{\mathbf{A}}(s) \\ \mathbf{c}^{\mathrm{T}} \end{bmatrix} = n$$

It is easy to show[14] that there exists a non-singular real transformation $\mathbf{Q} \in \mathbb{R}^{n\times n}$ such that

$$\mathbf{Q}^{-1}\mathbf{b} = \begin{bmatrix} \mathbf{c}^{\mathrm{T}}\mathbf{b} \\ \mathbf{0} \end{bmatrix} \qquad \mathbf{c}^{\mathrm{T}}\mathbf{Q} = (1, \mathbf{0})$$

Partition the matrix $\mathbf{Q}^{-1}\mathbf{A}(.)\mathbf{Q}$ as follows:

$$\mathbf{Q}^{-1}\mathbf{A}(.)\mathbf{Q} = \begin{bmatrix} A_{11}(.) & \mathbf{A}_{12}(.) \\ \mathbf{A}_{21}(.) & \mathbf{A}_{22}(.) \end{bmatrix}$$

where $\mathbf{A}_{11}(.)$, $\mathbf{A}_{12}(.)$, $\mathbf{A}_{21}(.)$ and $\mathbf{A}_{22}(.)$ are matrices with entries in $BV([0,r], \mathbb{R})$ of size 1×1, $1\times(n-1)$, $(n-1)\times 1$ and $(n-1)\times(n-1)$ respectively. Setting $\bar{\boldsymbol{\eta}}(t) = \mathbf{Q}^{-1}\mathbf{x}(t)$, it follows from (23) that

$$\begin{aligned} \dot{\bar{\boldsymbol{\eta}}} &= \mathrm{d}(\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}) * \bar{\boldsymbol{\eta}} + (\mathbf{Q}^{-1}\mathbf{b})u \\ y &= (\mathbf{c}^{\mathrm{T}}\mathbf{Q})\bar{\boldsymbol{\eta}} \\ \bar{\boldsymbol{\eta}}\,|_{[-r,0]} &= \mathbf{Q}^{-1}\mathbf{x}_0 \end{aligned} \tag{26}$$

If we realize that $\bar{\boldsymbol{\eta}}$ can be written as $\bar{\boldsymbol{\eta}} = (y, \boldsymbol{\eta})^{\mathrm{T}}$, then it follows that (26) can be expressed as

$$\dot{y} = (\mathbf{c}^{\mathrm{T}}\mathbf{b})u_1 \tag{27a}$$

$$\dot{\boldsymbol{\eta}} = \mathrm{d}\mathbf{A}_{22} * \boldsymbol{\eta} + \mathrm{d}\mathbf{A}_{21} * u_2 \tag{27b}$$

$$z = -\frac{1}{\mathbf{c}^{\mathrm{T}}\mathbf{b}}(\mathrm{d}\mathbf{A}_{12} * \boldsymbol{\eta} + \mathrm{d}A_{11} * u_2) \tag{27c}$$

$$u_1 = u - z \qquad u_2 = y \tag{27d}$$

$$y\,|_{[-r,0]} = \eta_1 \qquad \boldsymbol{\eta}\,|_{[-r,0]} = \boldsymbol{\eta}_2 \tag{27e}$$

where $(\eta_1, \boldsymbol{\eta}_2) = \bar{\boldsymbol{\eta}}\,|_{[-r,0]}$ and in particular $\eta_1 = \mathbf{c}^{\mathrm{T}}\mathbf{x}_0$. Let $\boldsymbol{\eta}(\boldsymbol{\eta}_2, \eta_1, v)$ denote the solution of the retarded functional differential equation (27b) corresponding to the initial conditions $\boldsymbol{\eta}\,|_{[-r,0]} = \boldsymbol{\eta}_2$, $u_2\,|_{[-r,0]} = \eta_1$ and the input $u_2\,|_{[0,\infty)} = v \in LL^1_+$. The corresponding output $z(\boldsymbol{\eta}_2, \eta_1, v)$ given by (27c) can be written in the form

$$z(\boldsymbol{\eta}_2, \eta_1, v) = Hv + w$$

where

$$Hv = -\frac{1}{\mathbf{c}^{\mathrm{T}}\mathbf{b}}(\mathrm{d}\mathbf{A}_{12} * \boldsymbol{\eta}(\mathbf{0}, 0, v) + \mathrm{d}A_{11} * v)$$

and

$$w = -\frac{1}{\mathbf{c}^{\mathrm{T}}\mathbf{b}}[\mathrm{d}\mathbf{A}_{12} * (\boldsymbol{\eta}_2, 0, 0) + \boldsymbol{\eta}(\mathbf{0}, \eta_1, 0)) + \mathrm{d}A_{11} * \eta_1]$$

Now define $\hat{\mathbf{A}}_{22}(s) = \int_0^r \exp(-s\tau)\,\mathrm{d}\mathbf{A}_{22}(\tau)$ and realize that by (25) we have $\det(s\mathbf{I} - \hat{\mathbf{A}}_{22}(s)) \neq 0\,\forall s \in \overline{\mathbb{C}}_+$. (This can be shown on the basis of the same reasoning as in the finite-dimensional case.) Hence it follows that the retarded system given by (27b) and (27c) is exponentially stable,[9] and as a consequence the mapping $v \mapsto Hv$ is linear and bounded from L^2_+ into L^2_+ and the function w is in L^2_+.[15] Moreover, it is clear that the operator H is shift-invariant. Finally, notice that the system (27) can be written as

$$\begin{aligned} \dot{y}(t) &= \mathbf{c}^{\mathrm{T}}\mathbf{b}(u(t) - (Hy)(t) - w(t)) \quad t \geqslant 0 \\ y(0) &= \mathbf{c}^{\mathrm{T}}\mathbf{x}_0(0) \end{aligned}$$

The above equation is of the form (4), which shows that the abstract theory developed in the previous sections applies if the linear part of the process Π (see Figure 1) is given by the retarded system (23) satisfying (24) and (25). In particular, we mention that the adaptive control laws of Sections 3 and 4 achieve 'stability' of the 'internal' variable $\mathbf{x}(t)$ of (23). We formulate a precise result for the case that sign(Π) is not known.

5.3. *Corollary*

Suppose that the retarded system (23) satisfies (24) and (25) and assume that the conditions of Theorem 4.7 hold. Then, for all initial conditions $\mathbf{x}_0 \in \mathbf{C}([-r, 0], \mathbb{R}^n)$ and $\xi_0 \in \mathbb{R}$, the closed-loop system given by (23), (13) and (20) has the following properties:

(i) $\lim_{t\to\infty} \xi(t)$ exists and is finite.
(ii) $\lim_{t\to\infty} k(t)$ exists and is finite.
(iii) $y, y' \in L^2_+ \cap L^\infty_+$
(iv) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0$.
(v) $\mathbf{x} \in (L^2_+)^n \cap (L^\infty_+)^n$.
(vi) $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

Proof. The statements (i)–(iv) follow from Theorem 4.7. Since $y \in L^2_+ \cap L^\infty_+$ and by the exponential stability of the retarded system given by (27b) and (27c), we obtain $\boldsymbol{\eta} \in (L^2_+)^{n-1} \cap (L^2_+)^{n-1}$ and hence $\bar{\boldsymbol{\eta}} \in (L^2_+)^n \cap (L^\infty_+)^n$, which implies (v). Finally, it is easy to see that $\dot{\mathbf{x}} \in (L^2_+)^n$ as well and therefore $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$, which is (vi).

5.4. *Volterra integrodifferential systems*

Consider the single-input/single-output Volterra integrodifferential system

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{A}_0\delta + \mathbf{A}_1)*\mathbf{x} + \mathbf{b}u \\ y &= \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0 \in \mathbb{R}^n \end{aligned} \tag{28}$$

where $\mathbf{A}_0 \in \mathbb{R}^{n\times n}$, $\mathbf{A}_1 \in (L^1_+)^{n\times n}$, δ denotes the Dirac distribution (with support in 0) and $*$ denotes convolution. We assume that

$$\mathbf{c}^{\mathrm{T}}\mathbf{b} \neq 0 \tag{29}$$

and

$$\det\begin{bmatrix} s\mathbf{I} - \mathbf{A}_0 - \hat{\mathbf{A}}_1(s) & -\mathbf{b} \\ \mathbf{c}^{\mathrm{T}} & 0 \end{bmatrix} \neq 0 \quad \forall s \in \bar{\mathbb{C}}_+ \tag{30}$$

where $\hat{}$ now denotes the Laplace transform. As in Section 5.1, let $\mathbf{Q} \in \mathbb{R}^{n\times n}$ be a non-singular transformation such that

$$\mathbf{Q}^{-1}\mathbf{b} = \begin{bmatrix} \mathbf{c}^{\mathrm{T}}\mathbf{b} \\ \mathbf{0} \end{bmatrix} \qquad \mathbf{c}^{\mathrm{T}}\mathbf{Q} = (1, \mathbf{0})$$

Set $\mathbf{A}(.) = \mathbf{A}_0\,\delta + \mathbf{A}_1(.)$ and partition the matrix $\mathbf{Q}^{-1}\mathbf{A}(.)\mathbf{Q}$ as follows:

$$\mathbf{Q}^{-1}\mathbf{A}(.)\mathbf{Q} = \begin{bmatrix} A_{11}(.) & \mathbf{A}_{12}(.) \\ \mathbf{A}_{21}(.) & \mathbf{A}_{22}(.) \end{bmatrix}$$

where $\mathbf{A}_{11}(.)$, $\mathbf{A}_{12}(.)$, $\mathbf{A}_{21}(.)$ and $\mathbf{A}_{22}(.)$ are matrices with entries in $\delta\mathbb{R} + L^1_+$ of size 1×1, $1 \times (n-1)$, $(n-1) \times 1$ and $(n-1) \times (n-1)$ respectively. If we realize that $(1, \mathbf{0})\,\mathbf{Q}^{-1}\mathbf{x} = \mathbf{y}$, then after a co-ordinate transformation $\mathbf{x} \mapsto \mathbf{Q}^{-1}\mathbf{x} = (y, \boldsymbol{\eta})^{\mathrm{T}}$ the system (28) can be written in the form

$$\dot{y} = \mathbf{c}^{\mathrm{T}}\mathbf{b}u_1 \tag{31a}$$

$$\dot{\boldsymbol{\eta}} = \mathbf{A}_{22}*\boldsymbol{\eta} + \mathbf{A}_{21}*u_2 \tag{31b}$$

$$z = -\frac{1}{\mathbf{c}^{\mathrm{T}}\mathbf{b}}(\mathbf{A}_{12}*\boldsymbol{\eta} + A_{11}*u_2) \tag{31c}$$

$$u_1 = u - z \qquad u_2 = y \tag{31d}$$

$$(y(0), \boldsymbol{\eta}(0))^{\mathrm{T}} = \mathbf{Q}^{-1}\mathbf{x}_0 \tag{31e}$$

Let $\mathbf{R}$ denote the resolvent of the homogeneous part of (31b), i.e. $\mathbf{R}$ is the unique solution of

$$\dot{\mathbf{R}} = \mathbf{A}_{22}*\mathbf{R} \qquad \mathbf{R}(0) = \mathbf{I}$$

The solution of (31b) is then given by[16]

$$\boldsymbol{\eta}(t) = \mathbf{R}(t)\boldsymbol{\eta}(0) + (\mathbf{R}*\mathbf{A}_{21}*u_2)(t)$$

and the corresponding output z can be written in the form

$$z = Hu_2 + w$$

where

$$Hu_2 = -\frac{1}{\mathbf{c}^{\mathrm{T}}\mathbf{b}}(\mathbf{A}_{12}*\mathbf{R}*\mathbf{A}_{21} + A_{11})*u_2 \tag{32}$$

and

$$w(t) = -\frac{1}{\mathbf{c}^{\mathrm{T}}\mathbf{b}}(\mathbf{A}_{12}*\mathbf{R})\boldsymbol{\eta}(0) \tag{33}$$

Now it follows from (30) that $\det(s\mathbf{I} - \hat{\mathbf{A}}_{22}(s)) \neq 0\ \forall s \in \overline{\mathbb{C}}_+$, which is equivalent to $\mathbf{R}$ being integrable.[17] As a consequence the operator H defined in (32) is linear and bounded from L^2_+ into L^2_+ and it is trivial that H is shift-invariant. Moreover, since the matrix $\mathbf{R}$ is integrable, we obtain that the entries of $\mathbf{R}$ are square-integrable.[18] Therefore the function w defined in (33) is in L^2_+. It follows that (31) can be written in the form (4). This shows that the abstract results of the previous sections apply if the linear part of the process $\boldsymbol{\Pi}$ (see Fig. 1) is given by the Volterra integrodifferential system (28) satisfying (29) and (30). As for retarded systems (see Corollary 5.3), it is easy to show that the adaptive control laws of Sections 3 and 4 achieve 'stability' of the 'internal' variable $\mathbf{x}(t)$ of (28), i.e. we have for the closed-loop system that $\mathbf{x} \in (L^2_+)^n \cap (L^\infty_+)^n$ and $\lim_{t\to\infty}\mathbf{x}(t) = \mathbf{0}$.

6. CONCLUSIONS

An input–output theory of high-gain adaptive stabilization of infinite-dimensional processes with actuator and sensor non-linearities has been developed. In the absence of information on the sign of the process the concept of a *scaling-invariant* switching function turned out to be useful. This notion may also be of independent interest. It was shown that retarded systems and Volterra integrodifferential systems fit into the abstract theory of Sections 2–4 and that the adaptive controllers of Sections 3 and 4 achieve 'internal' stability if they are applied to these particular systems. It is not difficult to show that the same is true for the class of distributed systems considered by Kobayashi.[7] The results of the present paper are restricted to single-input/single-output systems. However, if the sign of the process is known, it is fairly obvious how the results extend to the multivariable case.

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APPENDIX I: PROOF OF THEOREM 2.2

Step 1. Existence and uniqueness on a 'small' interval

There exist positive numbers L, r and δ such that

$$|\mathbf{f}_1(t,\mathbf{x}) - \mathbf{f}_1(t,\mathbf{x}')| \leqslant L|\mathbf{x}-\mathbf{x}'|$$

for all $t \in J := [\alpha, \alpha+\delta]$ and for all $\mathbf{x}, \mathbf{x}' \in \{\boldsymbol{\eta} \in \mathbb{R}^n \mid |\boldsymbol{\eta} - \mathbf{x}_0(\alpha)| \leqslant r\}$. Moreover, let $\rho > 0$ be a number satisfying

$$\int_\alpha^{\alpha+\rho} |\mathbf{f}_2(t)|\,\mathrm{d}t \leqslant r/3$$

and pick $\varepsilon > 0$ such that

$$\left.\begin{array}{l} \varepsilon \leqslant \delta \qquad \varepsilon \leqslant \rho \qquad \varepsilon < 1/(L+\varkappa) \\ \varepsilon \leqslant \dfrac{1}{3}\dfrac{r}{\max_{t\in J}|\mathbf{f}_1(t,\mathbf{x}_0(\alpha))| + Lr} \\ \varepsilon \leqslant \left(\dfrac{1}{3}\dfrac{r}{\varkappa\sqrt{(\alpha+\rho)}\max(|\mathbf{x}_0(\alpha)|+r, \|\mathbf{x}_0\|_\infty)}\right)^2 \end{array}\right\} \qquad (34)$$

Define the operator $\mathbf{B}$ by

$$(\mathbf{Bx})(t) = \mathbf{x}_0(t) \quad \forall 0 \leqslant t \leqslant \alpha$$

$$(\mathbf{Bx})(t) = \int_\alpha^t (\mathbf{Tx})(\tau)\,\mathrm{d}\tau + \int_\alpha^t \mathbf{f}_1(\tau,\mathbf{x}(\tau))\,\mathrm{d}\tau + \int_\alpha^t \mathbf{f}_2(\tau)\,\mathrm{d}\tau + \mathbf{x}_0(\alpha) \quad \forall t \geqslant \alpha$$

It is clear that $\mathbf{x}$ is a solution of (1) on $[0, \alpha+\varepsilon)$ iff $\mathbf{x}$ is a solution of

$$\mathbf{x}(t) = (\mathbf{Bx})(t) \quad t \in [0, \alpha+\varepsilon) \qquad (35)$$

Using Banach's fixed point theorem, we shall show that (35) has a unique solution. Let us introduce the following closed subset of $C([0,\alpha+\varepsilon], \mathbb{R}^n)$:

$$C^* := \{\mathbf{x} \in C([0,\alpha+\varepsilon],\mathbb{R}^n) \mid \mathbf{x}(t) = \mathbf{x}_0(t) \quad \forall 0 \leqslant t \leqslant \alpha$$
$$|\mathbf{x}(t) - \mathbf{x}_0(\alpha)| \leqslant r \quad \forall \alpha \leqslant t \leqslant \alpha+\varepsilon\}$$

In order to apply the fixed point theorem, we shall need the estimate

$$\int_{\alpha}^{\alpha+\varepsilon} |(\mathbf{Tx})(\tau) - (\mathbf{Tx}')(\tau)|\, \mathrm{d}\tau \leqslant \sqrt{(\varepsilon)}\varkappa \left(\int_0^{\alpha+\varepsilon} |\mathbf{x}(\tau) - \mathbf{x}'(\tau)|^2\, \mathrm{d}\tau \right)^{1/2} \qquad (36)$$

$$(\mathbf{x}, \mathbf{x}' \in C([0, \alpha+\varepsilon], \mathbb{R}^n))$$

which follows from the properties of $\mathbf{T}$ (causality, finite incremental gain) and Hölder's inequality.

We claim that $\mathbf{B}(C^*) \subset C^*$ and that $\mathbf{B}|_{C^*}$ is a contraction, i.e. the conditions of Banach's fixed point theorem are satisfied for the (metric) space C^* and the operator $\mathbf{B}|_{C^*}$. Using (36) and (34), we obtain for $\mathbf{x} \in C^*$ and $t \in [\alpha, \alpha+\varepsilon]$

$$\begin{aligned} |(\mathbf{Bx})(t) - \mathbf{x}_0(\alpha)| &\leqslant \sqrt{(\varepsilon)}\varkappa \left(\int_0^{\alpha+\varepsilon} |\mathbf{x}(\tau)|^2\, \mathrm{d}\tau \right)^{1/2} + r/3 + \varepsilon(Lr + \max_{\tau \in J} |\mathbf{f}_1(\tau, \mathbf{x}_0(\alpha))|) \\ &\leqslant \tfrac{2}{3} r + \sqrt{(\varepsilon)}\varkappa\sqrt{(\alpha+\varepsilon)} \max(\|\mathbf{x}_0\|_\infty, |\mathbf{x}_0(\alpha)| + r) \\ &\leqslant \tfrac{2}{3} r + \sqrt{(\varepsilon)}\varkappa\sqrt{(\alpha+\rho)} \max(\|\mathbf{x}_0\|_\infty, |\mathbf{x}_0(\alpha)| + r) \\ &\leqslant r \end{aligned}$$

which shows that $\mathbf{B}(C^*) \subset C^*$. It remains to show that $\mathbf{B}|_{C^*}$ is a contraction. Let $\mathbf{x}$ and $\mathbf{x}'$ be in C^*, then it follows from (34) and (36) that

$$\max_{\tau \in [0, \alpha+\varepsilon]} |(\mathbf{Bx})(\tau) - (\mathbf{Bx}')(\tau)| = \max_{\tau \in [\alpha, \alpha+\varepsilon]} |(\mathbf{Bx}(\tau) - (\mathbf{Bx}')(\tau)|$$

$$\leqslant \sqrt{(\varepsilon)}\varkappa \left(\int_0^{\alpha+\varepsilon} |\mathbf{x}(\tau) - \mathbf{x}'(\tau)|^2\, \mathrm{d}\tau \right)^{1/2} + L \int_{\alpha}^{\alpha+\varepsilon} |\mathbf{x}(\tau) - \mathbf{x}'(\tau)|\, \mathrm{d}\tau$$

$$\leqslant \varepsilon(\varkappa + L) \max_{\tau \in [0, \alpha+\varepsilon]} |\mathbf{x}(\tau) - \mathbf{x}'(\tau)|$$

By (34) we have $\varepsilon(\varkappa + L) < 1$, which shows that $\mathbf{B}|_{C^*}$ is a contraction.

Step 2. Extended uniqueness

Let $\mathbf{x}_i$ be a solution of (1) on $[0, \gamma_i)$ $(i = 1, 2)$. We claim that on $[0, \gamma)$ $\mathbf{x}_1 \equiv \mathbf{x}_2$, where $\gamma := \min(\gamma_1, \gamma_2)$. Let us assume the contrary, i.e. there exists $t \in (\alpha, \gamma)$ for which $\mathbf{x}_1(t) \neq \mathbf{x}_2(t)$. Defining

$$t^* := \inf\{t \in (\alpha, \gamma) \mid \mathbf{x}_1(t) \neq \mathbf{x}_2(t)\}$$

it follows that $t^* > \alpha$ (by step 1) and $\mathbf{x}_1(t^*) = \mathbf{x}_2(t^*)$ (by the continuity of $\mathbf{x}_1$ and $\mathbf{x}_2$). Now realize that the initial-value problem

$$\dot{\mathbf{x}}(t) = (\mathbf{Tx})(t) + \mathbf{f}_1(t, \mathbf{x}(t)) + \mathbf{f}_2(t) \quad t \geqslant t^*$$

$$\mathbf{x}|_{[0,t^*]} = \mathbf{x}_1|_{[0,t^*]}$$

is solved by $\mathbf{x}_1$ and $\mathbf{x}_2$. This implies (by step 1) that there is an $\varepsilon > 0$ such that $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ on $[0, t^* + \varepsilon)$, which contradicts the definition of t^*.

Step 3. Continuation of solutions

Let $\mathbf{x}$ be a solution of (1) on $[0, \beta)$, $\beta < \infty$. It is sufficient to show that $\mathbf{x}$ can be continued to the right (beyond β) if $\mathbf{x}$ is bounded on $[0, \beta)$. Define

$$\mathbf{x}'(t) := \begin{cases} \mathbf{x}(t) & 0 \leqslant t < \beta \\ \mathbf{0} & t \geqslant \beta \end{cases}$$

and set $\mathbf{x}(\beta) := (\mathbf{B}\mathbf{x}')(\beta)$. Since $\mathbf{x}(t) = (\mathbf{B}\mathbf{x})(t) = (\mathbf{B}\mathbf{x}')(t)$ on $[0, \beta)$, it follows that $\lim_{t \to \beta} \mathbf{x}(t) = (\mathbf{B}\mathbf{x}')(\beta)$ and we see that the above definition of $\mathbf{x}(\beta)$ makes $\mathbf{x}$ into a continuous function on $[0, \beta]$. Step 1 shows that the initial-value problem

$$\dot{\mathbf{z}}(t) = (\mathbf{T}\mathbf{z})(t) + \mathbf{f}_1(t, \mathbf{z}(t)) + \mathbf{f}_2(t) \quad t \geqslant \beta$$
$$\mathbf{z}(t) = \mathbf{x}(t) \quad 0 \leqslant t \leqslant \beta$$

has a unique solution $\mathbf{x}^*$ on $[0, \beta + \varepsilon)$ for some $\varepsilon > 0$. Finally, realize that (by the causality of $\mathbf{T}$) $\mathbf{x}^*$ is a solution of (1) on $[0, \beta + \varepsilon)$, i.e. $\mathbf{x}^*$ is a continuation of $\mathbf{x}$.

APPENDIX II: PROOF THAT $\lambda \mapsto \cos(\frac{1}{2}\pi\lambda)\exp(\lambda^2)$ IS A SCALING-INVARIANT SWITCHING FUNCTION

For $N(\lambda) = \cos(\frac{1}{2}\pi\lambda)\exp(\lambda^2)$ and fixed but arbitrary $\alpha, \beta > 0$ define

$$I(x) := \int_0^x (\Gamma_\alpha^\beta \circ N)(\lambda) N(\lambda) \, d\lambda$$

We claim

$$\sup_{x > 0} \frac{1}{x} I(x) = +\infty \tag{37}$$

$$\inf_{x > 0} \frac{1}{x} I(x) = -\infty \tag{38}$$

Proof of (37)

$N(\lambda)$ is positive on $(4n - 1, 4n + 1)$ $\forall n \in \mathbb{N}$. It is sufficient to show that

$$\frac{1}{4n + 1} I(4n + 1) \to +\infty \quad \text{as } n \to \infty$$

Set $\gamma := \max(\alpha, \beta)$ and realize that

$$I(4n + 1) = I(4n - 1) + \alpha \int_{4n-1}^{4n+1} N(\lambda) \, d\lambda$$

$$|I(4n - 1)| \leqslant \gamma(4n - 1)\exp[(4n - 1)^2]$$

$$\int_{4n-1}^{4n+1} N(\lambda) \, d\lambda \geqslant \int_{4n-1/2}^{4n+1/2} N(\lambda) \, d\lambda \geqslant \cos(\pi/4)\exp[(4n - \tfrac{1}{2})^2]$$

Hence

$$I(4n + 1) \geqslant (\alpha/\sqrt{2})\exp[(4n - \tfrac{1}{2})^2] - \gamma(4n - 1)\exp[(4n - 1)^2]$$
$$= \exp[(4n - 1)^2][(\alpha/\sqrt{2})\exp(4n - \tfrac{3}{4}) - \gamma(4n - 1)]$$

and as a consequence

$$\frac{1}{4n+1}\, I(4n+1) \to +\infty \quad \text{as } n \to +\infty$$

Proof of (38)

$N(\lambda)$ is negative on the interval $(4n+1, 4n+3)$ $\forall n \in \mathbb{N}$. Since

$$I(4n+3) = I(4n+1) + \beta \int_{4n+1}^{4n+3} N(\lambda)\, \mathrm{d}\lambda$$

$$|\, I(4n+1)\,| \leqslant \gamma(4n+1)\exp[(4n+1)^2]$$

$$\int_{4n+1}^{4n+3} N(\lambda)\, \mathrm{d}\lambda \leqslant \int_{4n+3/2}^{4n+5/2} N(\lambda)\, \mathrm{d}\lambda \leqslant \cos(\tfrac{3}{4}\pi)\exp[(4n+\tfrac{3}{2})^2]$$

it follows that

$$\begin{aligned} I(4n+3) &\leqslant \gamma(4n+1)\exp[(4n+1)^2] - (\beta/\sqrt{2})\exp[(4n+\tfrac{3}{2})^2] \\ &= \exp[(4n+1)^2]\,[\gamma(4n+1) - (\beta/\sqrt{2})\exp(4n+\tfrac{5}{4})] \end{aligned}$$

Therefore

$$\frac{1}{4n+3}\, I(4n+3) \to -\infty \quad \text{as } n \to \infty$$

We have proved that N is a scaling-invariant switching function. An inspection of the above proof shows that N is a scaling-invariant Nussbaum gain as well.

APPENDIX III: PROOF OF PROPOSITION 4.5

Without loss of generality we may assume that $\operatorname{sign}(\phi) = \operatorname{sign}(\psi) = +1$. Indeed, if the claim is true in this case, then it follows easily that the claim is true in the case that $\operatorname{sign}(\phi) = \operatorname{sign}(\psi) = -1$. Moreover, we restrict ourselves to the proof of the inequality

$$x\phi(t, \lambda r\psi(t, x)) \leqslant \Gamma(\lambda)\lambda r(\psi(t, x))^2 \quad ((t, r, \lambda, x) \in \mathbb{R}_+^2 \times \mathbb{R}^2) \tag{39}$$

The proof of the second inequality in Proposition 4.5 is very similar and is therefore omitted. In order to see why (39) holds, realize that

$$x\phi(t, \lambda r\psi(t, x)) \leqslant \Delta_\phi \lambda r\psi(t, x)x \quad ((t, r, \lambda, x) \in \mathbb{R}_+^2 \times \mathbb{R}_+ \times \mathbb{R}) \tag{40}$$

$$x\phi(t, \lambda r\psi(t, x)) \leqslant \delta_\phi \lambda r\psi(t, x)x \quad ((t, r, \lambda, x) \in \mathbb{R}_+^2 \times (-\infty, 0] \times \mathbb{R}) \tag{41}$$

Using the inequalities

$$0 \leqslant x \leqslant (1/\delta_\psi)\psi(t, x) \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+$$

$$0 \geqslant x \geqslant (1/\delta_\psi)\psi(t, x) \quad \forall (t, x) \in \mathbb{R}_+ \times (-\infty, 0]$$

it follows from (40) that

$$x\phi(t, \lambda r\psi(t, x)) \leqslant (\Delta_\phi/\delta_\psi)\lambda r\psi^2(t, x) \quad ((t, r, \lambda, x) \in \mathbb{R}_+^2 \times \mathbb{R}_+ \times \mathbb{R}) \tag{42}$$

The inequalities

$$x \geqslant (1/\Delta_\psi)\psi(t,x) \geqslant 0 \quad \forall(t,x)\in \mathbb{R}_+ \times \mathbb{R}_+$$
$$x \leqslant (1/\Delta_\psi)\psi(t,x) \leqslant 0 \quad \forall(t,x)\in \mathbb{R}_+ \times (-\infty,0]$$

together with (41) imply

$$x\phi(t,\lambda r\psi(t,x)) \leqslant (\delta_\phi/\Delta_\psi)\lambda r\psi^2(t,x) \quad ((t,r,\lambda,x)\in \mathbb{R}_+^2 \times (-\infty,0] \times \mathbb{R}) \tag{43}$$

Combining (42) and (43) gives (39).

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