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Robust High-gain Feedback Control of Infinite-Dimensional Minimum-Phase Systems

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This paper deals with a certain class of infinite-dimensional minimum-phase systems. We show that these systems can be stabilized by multivariable PI-controllers with sufficiently high gains; it turns out that the PI-controllers under consideration achieve almost decoupling and almost perfect tracking at high gain. Further, we prove several sufficient conditions for robust stability. Two results on pole–zero cancellations and closed-loop stability, which are used throughout the paper, may be of independent interest.

1. Introduction

THE PROBLEMS of finite-dimensional stabilization and finite-dimensional regulation of infinite-dimensional systems by output feedback has received considerable attention in recent years; see e.g. Schumacher (1981), Curtain & Salamon (1984), Kamen *et al.* (1985), Balas (1986), Logemann (1986a) and Jacobson & Nett (1987). Unfortunately, the controllers derived by the above authors may be of high order in certain cases. Moreover, if approximation techniques are used (cf. e.g. Kamen *et al.*, 1985; Balas, 1986; Logemann, 1986a), the relationship between the particular approximation method and the order of the stabilizing controller is not yet understood. Intuitively, it is clear that restrictions on the plant such as being minimum-phase or having stability should lead to simple low-order controllers. Pojohlainen (1982) has shown that a large class of *stable* infinite-dimensional plants can be stabilized and regulated by a *low-gain* PI-controller. In the present paper we shall study the 'dual' situation. More precisely: we investigate the problem of *high-gain* control of certain infinite-dimensional *minimum-phase* (possibly unstable) systems. As far as the authors are aware, this problem has not been considered in the literature before.

The present investigation is based on frequency-domain methods in contrast to the above papers (with the exception of Kamen *et al.* (1985), Logemann (1986a) and Jacobson & Nett (1987)) where the analysis is done using state-space methods. In particular, we use the framework of Callier & Desoer (1978, 1980a, 1980b) and a slight generalization of their set-up, which is sometimes more suitable. Moreover the notion of a 'multivariable first-order lag' (cf. Owens, 1978; Owens & Chotai, 1982; Owens *et al.*, 1984) plays an important role in our analysis.

We prove, in Section 4, that a large class of infinite-dimensional minimum-phase systems can be stabilized by high-gain PI-controllers. Further, we show that the proposed PI-controller achieves 'almost' decoupling and 'almost' perfect tracking at high gain.

In Section 5, we apply our frequency-domain results to certain minimum-phase retarded systems and Volterra integrodifferential systems. It turns out that the proposed PI-controller achieves *internal* stability if the gains are sufficiently large. In Section 6, we investigate robustness properties of the PI-controller under consideration. We study perturbations in the parameters of the inverse transfer matrix of the plant and perturbations that are induced by measurement nonlinearities. In both cases, conditions are developed which guarantee stability at high gain. It should be mentioned that some of the results in Sections 4–6 extend results of Owens & Chotai (1982) and Owens *et al.* (1984) to an infinite-dimensional setting. Section 2 is devoted to preliminaries. In Section 3 we present two results on multivariable pole–zero cancellations and closed-loop stability which we need in Section 4 and Section 6. These results, which are proved in Section 7 (=Appendix), may also be of independent interest.

2. Preliminaries

Let $\mathbf{R}_+$ denote the closed positive half-axis. For $\sigma \in \mathbb{R}$, we define

$$\mathbb{C}_\sigma := \{s \in \mathbb{C} : \operatorname{Re} s > \sigma\}.$$

Let $\Omega \subset \mathbb{C}$ be a region. The ring of holomorphic functions on Ω is denoted by $\mathrm{H}(\Omega)$. The symbol $\mathrm{M}(\Omega)$ denotes the quotient field of $\mathrm{H}(\Omega)$, i.e. the field of meromorphic functions on Ω. For $f \in \mathrm{M}(\Omega)$ and $a \in \Omega$ we define

$$\operatorname{ord}_a f := \min \left\{ n \geq 0 : \frac{\mathrm{d}^n f}{\mathrm{d}z^n}(a) \neq 0 \right\}.$$

Further, we denote the Hardy space of all bounded holomorphic functions in the right half-plane by $\mathbf{H}^\infty := \{f \in \mathrm{H}(\mathbb{C}_0) : f \text{ is bounded}\}$. Suppose f is a distribution with support in the interval $[0, \infty)$ of the form

$$f = \sum_{i=0}^{\infty} f_i \delta_{t_i} + f_a, \tag{2.1}$$

where $t_0 := 0$, $t_i > 0 \; \forall \, i \geq 1$, δ_{t_i} denotes the Dirac distribution at t_i, $f_i \in \mathbb{C}$, and f_a is a $\mathbb{C}$-valued Lebesgue-measurable function. The set A consists of all distributions f of the form (2.1) such that

$$\|f\|_{\mathrm{A}} = \sum_{i=0}^{\infty} |f_i| + \int_0^\infty |f_a(t)| \, \mathrm{d}t$$

is finite. It can be shown that A is a Banach algebra (Hille & Phillips 1957: p. 141). If $f \in \mathrm{A}$, then the Laplace transformation of f

$$\hat{f}(s) := \sum_{i=0}^{\infty} f_i \mathrm{e}^{-st_i} + \int_0^\infty f_a(t) \mathrm{e}^{-st} \, \mathrm{d}t$$

is well-defined for all $s \in \bar{\mathbb{C}}_0$. In particular, we have

$$\|\hat{f}\|_\infty = \sup_{s \in \bar{\mathbb{C}}_0} |\hat{f}(s)| \leq \|f\|_{\mathrm{A}}$$

and therefore $\hat{f} \in \mathbf{H}^\infty$. It is useful to introduce the following sets:

$$\mathrm{A}_- := \{f \in \mathrm{A} : \exists\ \varepsilon > 0 \text{ s.t. } f e^{\varepsilon \cdot} \in \mathrm{A}\},$$

$$\hat{\mathrm{A}} := \{\hat{f} : f \in \mathrm{A}\}, \qquad \hat{\mathrm{A}}_- := \{\hat{f} : f \in \mathrm{A}_-\},$$

$$\hat{\mathrm{A}}_-^\infty := \left\{\hat{f} \in \hat{\mathrm{A}}_- : \exists\ \rho \text{ s.t. } \inf_{s \in \mathbb{C}_0, |s| \geq \rho} |\hat{f}(s)| > 0\right\}.$$

Let $\hat{\mathrm{B}} = \hat{\mathrm{A}}_-(\hat{\mathrm{A}}_-^\infty)^{-1}$ denote the quotient ring of $\hat{\mathrm{A}}_-$ with respect to $\hat{\mathrm{A}}_-^\infty$, i.e.

$$\hat{\mathrm{B}} = \{n/d : n \in \hat{\mathrm{A}}_-,\ d \in \hat{\mathrm{A}}_-^\infty\}$$

(cf. Callier & Desoer, 1978, 1980a, 1980b). Moreover, we define

$$\mathbf{H}_-^\infty := \left\{f \in \mathbf{H}^\infty : \exists\ \sigma < 0 \text{ s.t. } f \in \mathrm{H}(\mathbb{C}_\sigma) \quad \text{and} \quad \sup_{s \in \mathbb{C}_\sigma} |f(s)| < \infty\right\},$$

$$\mathrm{D}_- := \left\{f \in \mathbf{H}_-^\infty : \exists\ \rho > 0 \text{ s.t. } \inf_{s \in \mathbb{C}_0, |s| > \rho} |f(s)| > 0\right\}, \qquad \mathrm{T} := \mathbf{H}_-^\infty(\mathrm{D}_-)^{-1}.$$

The following inclusions hold: $\hat{\mathrm{A}} \subset \mathbf{H}^\infty$, $\hat{\mathrm{A}}_- \subset \mathbf{H}_-^\infty$, $\hat{\mathrm{A}}_-^\infty \subset \mathrm{D}_-$, and $\hat{\mathrm{B}} \subset \mathrm{T}$.

Remark 2.1 (i) Let $f \in \mathrm{D}_-$. Then there exists $\sigma < 0$ such that f is holomorphic and bounded on $\mathbb{C}_\sigma$. Therefore f is uniformly continuous on every strip $\{s : \alpha \leq \operatorname{Re} s \leq \beta\}$, with $\sigma < \alpha < \beta$ (Corduneanu, 1968: p. 72), and there exist real numbers $\rho > 0$ and γ, with $\sigma < \gamma < 0$, such that

$$\inf_{s \in \mathbb{C}_\gamma, |s| \geq \rho} |f(s)| > 0.$$

(ii) Let $f \in \mathrm{T}$. Then it follows from (i) that there exists a number $\sigma < 0$ such that f has at most finitely many poles in $\mathbb{C}_\sigma$.

(iii) It is trivial to show that $f \in \mathbf{H}^\infty$ is a unit iff $\inf_{s \in \mathbb{C}_0} |f(s)| > 0$. It is by no means trivial that

$$f \in \hat{\mathrm{A}} \text{ is a unit (in } \hat{\mathrm{A}}) \text{ iff } \inf_{s \in \mathbb{C}_0} |f(s)| > 0.$$

(cf. Hille & Phillips 1957: pp. 141). It follows from (i) that $f \in \mathbf{H}_-^\infty$ (resp. $\hat{\mathrm{A}}_-$) is a unit in $\mathbf{H}_-^\infty$ (resp. $\hat{\mathrm{A}}_-$) iff $\inf_{s \in \mathbb{C}_0} |f(s)| > 0$.

(iv) $f \in \mathrm{T}$ (resp. $f \in \hat{\mathrm{B}}$) is a unit in T (resp. $\hat{\mathrm{B}}$) iff there exists $\rho > 0$ such that

$$\inf_{s \in \mathbb{C}_0, |s| > \rho} |f(s)| > 0.$$

It is an important property of T (resp. $\hat{\mathrm{B}}$) that every matrix in $\mathrm{T}^{m \times n}$ (resp. $\hat{\mathrm{B}}^{m \times n}$) admits right and left Bezout factorizations. This can be stated more precisely as follows.

LEMMA 2.2 *For $G \in \mathrm{T}^{m \times n}$ (resp. $\hat{\mathrm{B}}^{m \times n}$) there exists a* right Bezout factorization over $\mathbf{H}_-^\infty$ *(resp. $\hat{\mathrm{A}}_-$), i.e. there exist $N \in \mathbf{H}_-^{\infty\, m \times n}$ (resp. $\hat{\mathrm{A}}_-^{m \times n}$) and $D \in \mathbf{H}_-^{\infty\, n \times n}$ (resp. $\hat{\mathrm{A}}_-^{n \times n}$) such that*:

(i) $\det D \in \mathrm{D}_-$ *(resp. $\hat{\mathrm{A}}_-^\infty$)*; (ii) $G = ND^{-1}$;

(iii) *N and D are* right Bezout-coprime, *i.e. there exist matrices $U \in \mathbf{H}_-^{\infty\, n \times m}$ (resp. $\hat{\mathrm{A}}_-^{n \times m}$) and $V \in \mathbf{H}_-^{\infty\, n \times m}$ (resp. $\hat{\mathrm{A}}_-^{n \times m}$) such that*

$$UN + VD = I.$$

Moreover it is possible to choose D rational. A similar statement is valid for left Bezout factorizations.

Proof. See Callier & Desoer (1980b) if $G \in \hat{\mathrm{B}}^{m \times n}$. In case $G \in \mathrm{T}^{m \times n}$, the proof is similar (cf. Logemann, 1986c). □

Remark 2.3 (i) Two right Bezout factorizations over $\mathbf{H}_-^\infty$ (resp. $\hat{\mathrm{A}}_-$) are unique up to a unimodular matrix over $\mathbf{H}_-^\infty$ (resp. $\hat{\mathrm{A}}_-$). Of course, the same is true for left Bezout factorizations.

(ii) Let $G \in \mathrm{T}^{m \times n}$ (resp. $\hat{\mathrm{B}}^{m \times n}$) and let $G = N_r D_r^{-1}$ and $G = D_\ell^{-1} N_\ell$ be a right and a left Bezout factorization (respectively) over $\mathbf{H}_-^\infty$ (resp. $\hat{\mathrm{A}}_-$). Then there exists $\sigma < 0$ such that $\det D_r$ and $\det D_\ell$ have the same zeros in $\mathbb{C}_\sigma$ (counting multiplicities). Moreover, if $n = m$, then $\det N_r$ and $\det N_\ell$ have the same zeros in $\mathbb{C}_\sigma$ (counting multiplicities).

Because of Remark 2.3 the following definitions make sense.

DEFINITION 2.4 Let $G \in \mathrm{T}^{m \times n}$ (resp. $\hat{\mathrm{B}}^{m \times n}$) and let $G = ND^{-1}$ be a right Bezout factorization over $\mathbf{H}_-^\infty$ (resp. $\hat{\mathrm{A}}_-$).

(i) A complex number $s_0 \in \bar{\mathbb{C}}_0$ is called a *pole* of G if $\det D(s_0) = 0$. Moreover we define $\mathrm{p}_{s_0}(G) := \mathrm{ord}_{s_0} \det D$ (*multiplicity* of the pole s_0).

(ii) The matrix G is called $\mathbf{H}_-^\infty$-*stable* (resp. $\hat{\mathrm{A}}_-$-*stable*) if $\mathrm{p}_s(G) = 0$ for all $s \in \bar{\mathbb{C}}_0$, i.e. $G \in \mathbf{H}_-^{\infty\, m \times n}$ (resp. $\hat{\mathrm{A}}_-^{m \times n}$).

(iii) G is called *minimum-phase* if $\mathrm{rk}\, N(s) = \min\{m, n\}$, for all $s \in \bar{\mathbb{C}}_0$.

3. Pole-zero cancellations and closed-loop stability

LEMMA 3.1 *Let* $F \in \mathrm{T}^{m \times n}$ *and* $G \in \mathrm{T}^{n \times p}$. *Then*

$$\mathrm{p}_s(FG) \leq \mathrm{p}_s(F) + \mathrm{p}_s(G) \quad \text{for all } s \in \bar{\mathbb{C}}_0.$$

Proof. See Appendix.

We now define the concept of pole-zero cancellation.

DEFINITION 3.2 Let $F \in \mathrm{T}^{m \times n}$, $G \in \mathrm{T}^{n \times p}$, and $s_0 \in \bar{\mathbb{C}}_0$. We say that FG contains a *pole-zero cancellation* at s_0 if $\mathrm{p}_{s_0}(FG) < \mathrm{p}_{s_0}(F) + \mathrm{p}_{s_0}(G)$. Otherwise (i.e. $\mathrm{p}_{s_0}(FG) = \mathrm{p}_{s_0}(F) + \mathrm{p}_{s_0}(G)$, by Lemma 3.1) we say that FG contains no pole-zero cancellation at s_0.

The following lemma gives a sufficient condition for the absence of pole-zero cancellations in case of square plants.

THEOREM 3.3 *Let* $F, G \in \mathrm{T}^{n \times n}$ (*resp.* $\hat{\mathrm{B}}^{n \times n}$) *and suppose that* $\det F \not\equiv 0$ *and* $\det G \not\equiv 0$. *Moreover let* $F = N_F D_F^{-1}$ *and* $G = N_G D_G^{-1}$ *be right Bezout factorizations over* $\mathbf{H}_-^\infty$ (*resp.* $\hat{\mathrm{A}}_-$). *FG contains no pole-zero cancellation at* $s_0 \in \bar{\mathbb{C}}_0$ *if*

$$|\det N_F(s_0)| + |\det D_G(s_0)| > 0, \qquad |\det N_G(s_0)| + |\det D_F(s_0)| > 0. \tag{3.1}$$

Proof. See Appendix.

Remark 3.4 The condition in Theorem 3.3 is not necessary for the absence of

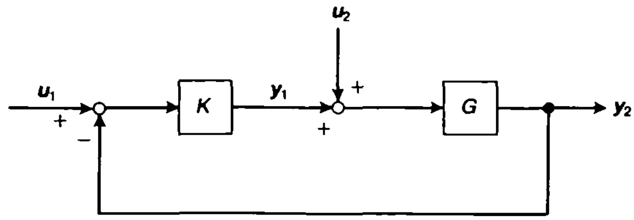

FIG. 1.

pole–zero cancellations, as the following example shows:

$$F(s) = G(s) = \begin{bmatrix} 1/(s-1) & 0 \\ 0 & (s-1)/(s+2) \end{bmatrix}.$$

Then

$$F(s)G(s) = \begin{bmatrix} 1/(s-1)^2 & 0 \\ 0 & (s-1)^2/(s+2)^2 \end{bmatrix}.$$

It follows from the right Bezout factorizations

$$F(s) = G(s) = \begin{bmatrix} 1/(s+1) & 0 \\ 0 & (s-1)/(s+1) \end{bmatrix} \begin{bmatrix} (s-1)/(s+1) & 0 \\ 0 & (s+2)/(s+1) \end{bmatrix}^{-1},$$

$$F(s)G(s) = \begin{bmatrix} 1/(s+1)^2 & 0 \\ 0 & (s-1)^2/(s+1)^2 \end{bmatrix} \begin{bmatrix} (s-1)^2/(s+1)^2 & 0 \\ 0 & (s+2)^2/(s+1)^2 \end{bmatrix}^{-1}$$

that $p_1(FG) = p_1(F) + p_1(G) = 2$, i.e. FG contains no pole–zero cancellation at 1. Of course, the condition of Theorem 3.3 is not satisfied.

Consider the feedback system in Fig. 1. We call the feedback system stable if every transfer function $u_i \to y_j$ that occurs around the loop is stable. More precisely:

DEFINITION 3.4 Let $G \in \mathbb{T}^{m\times n}$ (resp. $\hat{\mathbb{B}}^{m\times n}$) and $K \in \mathbb{T}^{n\times m}$ (resp. $\hat{\mathbb{B}}^{n\times m}$) be such that $\det(I + GK)$ is a unit in $\mathbb{T}$ (resp. $\hat{\mathbb{B}}$). The feedback system in Fig. 1 is called $\mathbf{H}^\infty_-$-*stable* (resp. $\hat{\mathbb{A}}_-$-*stable*) if the matrix

$$\mathrm{H}(G, K) := \begin{bmatrix} (I+KG)^{-1}K & -(I+KG)^{-1}KG \\ (I+GK)^{-1}GK & (I+GK)^{-1}G \end{bmatrix}$$

is in $\mathbf{H}^{\infty(m+n)\times(m+n)}_-$ (resp. $\hat{\mathbb{A}}^{(m+n)\times(m+n)}_-$).

LEMMA 3.5 *Let $G \in \mathbb{T}^{m\times n}$ (resp. $\hat{\mathbb{B}}^{m\times n}$) and $K \in \mathbb{T}^{n\times m}$ (resp. $\hat{\mathbb{B}}^{n\times m}$) be such that $\det(I + GK)$ is a unit in $\mathbb{T}$ (resp. $\hat{\mathbb{B}}$). Moreover let $G = D_G^{-1}N_G$ and $K = N_K D_K^{-1}$ be a left Bezout factorization over $\mathbf{H}^\infty_-$ (resp. $\hat{\mathbb{A}}_-$) and a right Bezout factorization over $\mathbf{H}^\infty_-$ (resp. $\hat{\mathbb{A}}_-$), respectively. The feedback system in Fig. 1 is $\mathbf{H}^\infty_-$-stable (resp. $\hat{\mathbb{A}}_-$-stable) iff*

$$\inf_{s\in\mathbb{C}_0} |\det[D_G(s)D_K(s) + N_G(s)N_K(s)]| > 0.$$

Proof. The proof is by straightforward application of the theory of fractional representation (cf. Vidyasagar *et al.* 1982). □

Remark 3.6 In case $G \in \hat{\mathbb{B}}^{m\times n}$ and $K \in \hat{\mathbb{B}}^{n\times m}$, the content of Lemma 3.5 is well known (Callier & Desoer, 1980b).

The following lemma gives another necessary and sufficient condition for closed-loop stability.

Lemma 3.7 *Let* $G \in T^{m\times n}$ *(resp.* $\hat{\mathbb{B}}^{m\times n}$*) and* $K \in \mathbb{T}^{n\times m}$ *(resp.* $\hat{\mathbb{B}}^{n\times m}$*) be such that* $\det(I+GK)$ *is a unit in* $\mathbb{T}$ *(resp.* $\hat{\mathbb{B}}$*). The feedback system in Fig.* 1 *is* $\mathbf{H}_-^\infty$*-stable (resp.* $\hat{\mathbf{A}}_-$*-stable) iff*

(i) $(I+GK)^{-1}GK \in \mathbf{H}_-^{\infty\, m\times m}$ (resp. $\hat{\mathbf{A}}_-^{m\times m}$).

(ii) *GK contains no pole–zero cancellations in* $\bar{\mathbb{C}}_0$.

Remark 3.8 A similar result is proved in Anderson & Gevers (1981) for finite-dimensional discrete systems. We present a simpler proof.

Proof. Let $G = D_G^{-1}N_G$ and $K = N_K D_K^{-1}$ be a left Bezout factorization over $\mathbf{H}_-^\infty$ (resp. $\hat{\mathbf{A}}_-$) and a right Bezout factorization over $\mathbf{H}_-^\infty$ (resp. $\hat{\mathbf{A}}_-$), respectively. Moreover, let $GK = D^{-1}N$ be a left Bezout factorization over $\mathbf{H}_-^\infty$ (resp. $\hat{\mathbf{A}}_-$). Consider the matrix

$$\mathrm{H}(GK, I) = \begin{bmatrix} (I+GK)^{-1} & -(I+GK)^{-1}GK \\ (I+GK)^{-1}GK & (I+GK)^{-1}GK \end{bmatrix}$$

and realize that

$$\begin{aligned} \det(D_G D_K + N_G N_K) &= (\det D_G)(\det D_K)\det(I+GK) \\ &= \frac{(\det D_G)(\det D_K)}{\det D}\det(D+N). \end{aligned} \tag{3.2}$$

It follows from (3.2) and Lemma 3.5 that the matrix $\mathrm{H}(G, K)$ is $\mathbf{H}_-^\infty$-stable (resp. $\hat{\mathbf{A}}_-$-stable) iff GK contains no pole–zero cancellations in $\bar{\mathbb{C}}_0$ and the matrix $\mathrm{H}(GK, I)$ is $\mathbf{H}_-^\infty$-stable (resp. $\hat{\mathbf{A}}_-$-stable). It follows from the identity

$$(I+GK)^{-1} = I - (I+GK)^{-1}GK$$

that $\mathrm{H}(GK, I)$ is $\mathbf{H}_-^\infty$-stable (resp. $\hat{\mathbf{A}}_-$-stable) iff $(I+GK)^{-1}GK$ is $\mathbf{H}_-^\infty$-stable (resp. $\hat{\mathbf{A}}_-$-stable).

4. High-gain feedback control using first-order models

Let $G \in \mathbb{T}^{n\times n}$ (resp. $\hat{\mathbb{B}}^{n\times n}$) have inverse of the form

$$G^{-1}(s) = sA_0 + A_1 + H(s), \tag{4.1}$$

where $A_0, A_1 \in \mathbb{C}^{n\times n}$, $\det(A_0) \neq 0$, and $H \in \mathbf{H}_-^{\infty\, n\times n}$ (resp. $\hat{\mathbf{A}}_-^{n\times n}$). As an approximate model of G we choose a proper rational transfer matrix G_A defined by

$$G_A^{-1}(s) = sA_0 + A_1. \tag{4.2}$$

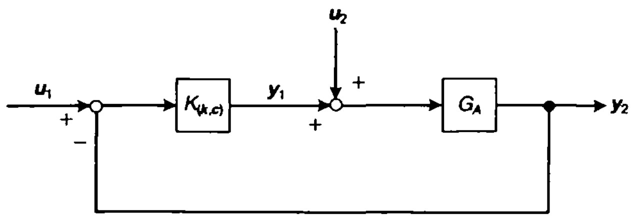

FIG. 2.

G_A is a so-called *multivariable first-order lag.* (cf. Owens 1978, Owens & Chotai 1982, Owens *et al.*, 1984). The controller for G_A suggested by Owens & Chotai (1982) has the form

$$K_{(\boldsymbol{k},\boldsymbol{c})}(s) = A_0 \operatorname*{diag}_{1 \leqslant j \leqslant n} (k_j + c_j + k_j c_j / s) - A_1, \tag{4.3}$$

where

$$\boldsymbol{k} := [k_1, \ldots, k_n]^{\mathrm{T}}, \qquad k_j > 0 \quad (j = 1, \ldots, n), \qquad \boldsymbol{c} := [c_1, \ldots, c_n]^{\mathrm{T}} \in \mathbb{R}^n_+.$$

This controller has the nice properties that (Owens & Chotai, 1982):

(a) it produces time constants in loop j of the order k_j^{-1} with reset times of the order of c_j^{-1};

(b) the steady-state errors are zero in any loop j where $c_j \neq 0$;

(c) the resulting feedback scheme (cf. Fig. 2) is stable in the range $k_j > 0$ and $c_j \geqslant 0$ $(j = 1, \ldots, n)$;

(d) interaction effects in all loops become arbitrarily small as $\boldsymbol{k} \to \infty$ (i.e. $\min_j k_j \to \infty$) with the c_j fixed.

Clearly $K_{(\boldsymbol{k},\boldsymbol{c})}$ is an 'ideal' controller for the approximate model, with improving theoretical response characteristics as the k_j increase. We shall show that the application of $K_{(\boldsymbol{k},\boldsymbol{c})}$ to the original infinite-dimensional plant yields a stable feedback system for all sufficiently large k_j, and that the closed-loop response characteristics of the two feedback schemes (see Fig. 2 and Fig. 3) become arbitrarily close (in a well-defined sense) as $\boldsymbol{k}$ tends to ∞. We need the following lemma.

LEMMA 4.1 (i) *If* $G \in \mathrm{T}^{n \times n}$ (*resp.* $\hat{\mathrm{B}}^{n \times n}$) *is of the form* (4.1), *then* G *is minimum-phase.*

(ii) *The transfer matrix* $K_{(\boldsymbol{k},\boldsymbol{c})}$ *defined by* (4.3) *is minimum-phase if* $\min_j k_j > \bar{\sigma}(A_0^{-1} A_1)$, *where* $\bar{\sigma}(M)$ *denotes the largest singular value of* M.

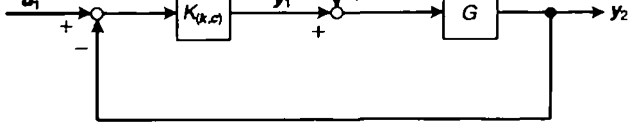

FIG. 3.

Proof. (i) Define

$$D(s):=I+\frac{1}{s+1}(A_0^{-1}A_1-I),\qquad N(s):=\frac{1}{s+1}A_0^{-1},\qquad P(s):=A_0-A_1,\quad (4.4)$$

and note that

$$G_A=D^{-1}N,\qquad NP+D\equiv I.\qquad (4.5a,b)$$

Since $G=(I+G_AH)^{-1}G_A$, it follows from (4.5a) that $G=(D+NH)^{-1}N$. Then (4.5b) yields $N(P-H)+D+NH\equiv I$.

(ii) Define

$$f_j(s):=\begin{Bmatrix}\dfrac{s}{s+1} & (c_j\neq 0)\\ 1 & (c_j=0)\end{Bmatrix}\quad\text{and}\quad g_j:=\begin{Bmatrix}\dfrac{1}{k_jc_j} & (c_j\neq 0)\\ 1 & (c_j=0)\end{Bmatrix}\quad (j=1,\ldots,n,),$$

$$D(s):=\Big(\operatorname{diag}_j f_j(s)\Big)A_0^{-1},$$

$$N(s):=\operatorname{diag}_j\frac{k_jc_j}{s+1}+\Big(\operatorname{diag}_j f_j(s)\Big)\Big(\operatorname{diag}_j(k_j+c_j)-A_0^{-1}A_1\Big).$$

Then

$$K_{(k,c)}(s)=D^{-1}(s)N(s),\qquad (4.6)$$

$$D(s)\Big[A_0-A_0\Big(\operatorname{diag}_j(k_j+c_j)-A_0^{-1}A_1\Big)\operatorname{diag}_j g_j\Big]+N(s)\operatorname{diag}_j g_j\equiv I.$$

Hence (4.6) is a left Bezout factorization of $K_{(k,c)}$. Now, $N(s)$ can be written in the form

$$N(s)=\Big(\operatorname{diag}_j h_j(s)\Big)\Big[I-\Big(\operatorname{diag}_j\frac{f_j(s)}{h_j(s)}\Big)A_0^{-1}A_1\Big],\qquad (4.7)$$

where

$$h_j(s):=\begin{cases}\dfrac{s(k_j+c_j)+k_jc_j}{s+1} & (c_j\neq 0),\\ k_j & (c_j=0).\end{cases}$$

An easy computation shows that $|f_j(s)/h_j(s)|\leq 1/k_j$ $(s\in\bar{\mathbb{C}}_0)$, and therefore we obtain that $\det N(s)\neq 0$, for all $s\in\bar{\mathbb{C}}_0$, if $\min_j k_j>\bar{\sigma}(A_0^{-1}A_1)$. □

THEOREM 4.2 *Let $G\in\mathbf{T}^{n\times n}$ (resp. $\hat{\mathbf{B}}^{n\times n}$) be such that G^{-1} is of the form* (4.1). *Then the feedback system in Fig.* 3 *is H_-^∞-stable (resp. $\hat{\mathrm{A}}_-$-stable) if*

$$\min_j k_j>\max\{\bar{\sigma}(A_0^{-1}A_1),\ \bar{\sigma}(A_0^{-1})\ \|H\|_\infty\},$$

where, for $M\in\mathbf{H}^{\infty\, n\times n}$, we define $\|M\|_\infty:=\sup_{s\in\mathbb{C}_0}\bar{\sigma}(M(s))$.

Proof. Define:

$$L_{(k,c)}:=(I+GK_{(k,c)})^{-1}GK_{(k,c)},\qquad L_{A,(k,c)}:=(I+G_AK_{(k,c)})^{-1}G_AK_{(k,c)}.\qquad (4.8a,b)$$

An elementary computation shows that

$$L_{(\boldsymbol{k},\boldsymbol{c})} = \left[I + \left(\operatorname{diag}_j f_j\right)A_0^{-1}H\right]^{-1} L_{A,(\boldsymbol{k},\boldsymbol{c})}, \tag{4.9}$$

where $f_j(s) := s/(s+k_j)(s+c_j)$ $(j = 1, \ldots, n)$.

We claim that $L_{(\boldsymbol{k},\boldsymbol{c})} \in \mathbf{H}_-^{\infty\, n\times n}$ (resp. $\hat{\mathbf{A}}_-^{n\times n}$) in the range $k_j > \bar{\sigma}(A_0^{-1}) \|H\|_\infty$ with $c_j \geq 0$ $(j = 1, \ldots, n)$. First note that $L_{A,(\boldsymbol{k},\boldsymbol{c})} \in \hat{\mathbf{A}}_-^{n\times n}$ for $k_j > 0$ and $c_j \geq 0$ $(j = 1, \ldots, n)$. Since $(\mathbf{H}^{\infty\, n\times n}, \|\bullet\|_\infty)$ is a Banach algebra, it follows that

$$\left[I + \left(\operatorname{diag}_j f_j\right)A_0^{-1}H\right]^{-1} \in \mathbf{H}^{\infty\, n\times n}$$

if

$$\left\|\left(\operatorname{diag}_j f_j\right)A_0^{-1}H\right\|_\infty < 1 \tag{4.10}$$

(cf. Rudin 1974: p. 388). Because $\|f_j\|_\infty \leq 1/k_j$ $(j = 1, \ldots, n)$, equation (4.10) will be satisfied if $\min_j k_j > \bar{\sigma}(A_0^{-1}) \|H\|_\infty$. So far, we have shown that

$$L_{(\boldsymbol{k},\boldsymbol{c})} \in \mathbf{H}^{\infty\, n\times n}$$

if $\min_j k_j > \bar{\sigma}(A_0^{-1}) \|H\|_\infty$ and $c_j \geq 0$ $(j = 1, \ldots, n)$. This implies that

$$L_{(\boldsymbol{k},\boldsymbol{c})} \in \mathbf{H}_-^{\infty\, n\times n} \quad (\text{resp. } \hat{\mathbf{A}}_-^{n\times n})$$

if $\min_j k_j > \bar{\sigma}(A_0^{-1}) \|H\|_\infty$ and $c_j \geq 0$ $(j = 1, \ldots, n)$, because we know that $L_{(\boldsymbol{k},\boldsymbol{c})} \in \mathbf{T}^{n\times n}$ (resp. $\mathbf{B}^{n\times n}$) for all values of k_j and c_j. Moreover, it follows from Lemma 4.1 and Theorem 3.3 that $GK_{(\boldsymbol{k},\boldsymbol{c})}$ contains no pole–zero cancellations in $\bar{\mathbb{C}}_0$ if

$$\min_j k_j > \bar{\sigma}(A_0^{-1}A_1).$$

Now we can use Lemma 3.7 in order to establish the theorem. □

In the following, we shall study the closed-loop response characteristics of the feedback system in Fig. 3 as $\boldsymbol{k}$ tends to ∞. In particular, we shall compare the performance of the feedback schemes in Fig. 2 and Fig. 3 if $\boldsymbol{k} \to \infty$. In order to do this, we need some more notation.

We equip the space $\mathrm{L}^q(\mathbb{R}_+)^n$ $(:= [\mathrm{L}^q(\mathbb{R}_+)]^n)$ with the norm $\|\boldsymbol{f}\|_q = \max_j \|f_j\|_q$, for $\boldsymbol{f} = [f_1, \ldots, f_n]^{\mathrm{T}} \in \mathrm{L}^q(\mathbb{R}_+)^n$. Let L_n denote the Laplace–Plancherel transformation (cf. Hoffman 1962: p. 131; or Doetsch 1971: p. 419):

$$\mathsf{L}_n : \mathrm{L}^2(\mathbb{R}_+)^n \to (\mathbf{H}^2)^n : \boldsymbol{f} \mapsto (2\pi)^{-\frac{1}{2}} \int_0^\infty \boldsymbol{f}(t) e^{-st}\, dt,$$

where $\mathbf{H}^2$ is the usual Hardy space in the right half-plane (cf. Hoffman, 1962: p. 121; or Doetsch, 1971: p. 419). For $M \in \mathbf{H}^{\infty\, n\times n}$, we define

$$\bar{M} : \mathrm{L}^2(\mathbb{R}_+)^n \to \mathrm{L}^2(\mathbb{R}_+)^n : \boldsymbol{f} \mapsto \mathsf{L}_n^{-1}(M \mathsf{L}_n \boldsymbol{f}).$$

Since L_n is a bijective linear bounded operator (see Hoffman 1962: p. 131; or Doetsch 1971: p. 419) the same is true for L_n^{-1}; also, $\tilde{M}$ is a bounded translation-invariant linear operator.

If M is a matrix in $\hat{\mathrm{A}}^{n\times n}$, we define

$$\tilde{M} : \mathrm{L}^q(\mathbb{R}_+)^n \to \mathrm{L}^q(\mathbb{R}_+)^n : f \mapsto \check{M} \star f,$$

for $1 \leqslant q \leqslant \infty$, where $\check{M} \in \mathrm{A}^{n\times n}$ is the inverse Laplace transform of M, and $\star$ denotes convolution on $\mathbb{R}_+$. The bounded translation-invariant linear operator $\tilde{M}$ maps $\mathrm{L}^q(\mathbb{R}_+)^n$ into $\mathrm{L}^q(\mathbb{R}_+)^n$ $(1 \leqslant q \leqslant \infty)$ (cf. Vidyasagar 1978: p. 250). Finally let $\mathscr{X}$ be a $\mathbb{C}$-vector-space and let $T : \mathscr{X}^n \to \mathscr{X}^n$ be a linear operator. Let $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ denote the canonical basis of $\mathbb{C}^n$. We define

$$T_{ij} : \mathscr{X} \to \mathscr{X} : x \mapsto \big(T(x\boldsymbol{e}_j)\big)_i \quad (i,j = 1, \ldots, n).$$

If $M = [m_{ij}] \in \mathbf{H}^{\infty\, n\times n}$ then, for all $f \in \mathrm{L}^2(\mathbb{R}_+)$, we have

$$\tilde{M}_{ij} f = \mathsf{L}_1^{-1}(m_{ij}\mathsf{L}_1 f). \tag{4.11}$$

If $M = [m_{ij}] \in \hat{\mathrm{A}}^{n\times n}$ then for all $f \in \mathrm{L}^q(\mathbb{R}_+)$ (with $1 \leqslant q \leqslant \infty$),

$$\tilde{M}_{ij} f = \check{m}_{ij} \star f. \tag{4.12}$$

The following remark is useful in order to prove Theorem 4.4.

Remark 4.3 (i) Note that the closed-loop transfer matrix of the approximate feedback system $L_{A,(\boldsymbol{k},\boldsymbol{c})}$ (see (4.8b)) is given by

$$L_{A,(\boldsymbol{k},\boldsymbol{c})}(s) = \Big(\operatorname*{diag}_j \frac{1}{(s+k_j)(s+c_j)}\Big)\Big(\operatorname*{diag}_j [(k_j + c_j)s + k_j c_j] - sA_0^{-1}A_1\Big). \tag{4.13}$$

(ii) It follows from (4.13) that the off-diagonal elements of the matrix $L_{A,(\boldsymbol{k},\boldsymbol{c})}$ tend to zero (in the sup norm) if $\boldsymbol{k} \to \infty$ ('almost decoupling').

(iii) Moreover, it follows from (4.13) that

$$\limsup_{\boldsymbol{k}\to\infty} \|L_{A,(\boldsymbol{k},\boldsymbol{c})}\|_\infty \leqslant 2. \tag{4.14}$$

The next theorem shows that the closed-loop operators $u_1 \mapsto y_2$ of the real and approximate feedback system (see Fig. 2 and Fig. 3) become arbitrarily close in the L^2-induced norm if $\boldsymbol{k} \to \infty$. Moreover, the controller $K_{(\boldsymbol{k},\boldsymbol{c})}$ achieves almost decoupling for sufficiently large k_j.

THEOREM 4.4 *Let $G \in \mathrm{T}^{n\times n}$ be such that G^{-1} is of the form* (4.1). *Then*

$$\text{(i)} \quad \lim_{\boldsymbol{k}\to\infty} \|\tilde{L}_{(\boldsymbol{k},\boldsymbol{c})} - \tilde{L}_{A,(\boldsymbol{k},\boldsymbol{c})}\|_{i2} = 0, \qquad \text{(ii)} \quad \lim_{\boldsymbol{k}\to\infty} \|\tilde{L}_{(\boldsymbol{k},\boldsymbol{c})ij}\|_{i2} = 0 \quad (i \neq j),$$

for fixed $\boldsymbol{c} \in \mathbb{R}_+^n$, where $\|\bullet\|_{i2}$ denotes the L^2*-induced norm.*

Proof. (i) It follows from (4.9) that

$$L_{(\boldsymbol{k},\boldsymbol{c})} - L_{A,(\boldsymbol{k},\boldsymbol{c})} = \Big\{\Big[I + \Big(\operatorname*{diag}_j f_j\Big)A_0^{-1}H\Big]^{-1} - I\Big\}L_{A,(\boldsymbol{k},\boldsymbol{c})},$$

where $f_j(s) = s/(s+k_j)(s+c_j)$ $(j = 1, \ldots, n)$. Now, because

$$\left\|\left(\operatorname{diag}_j f_j\right)A_0^{-1}H\right\|_\infty \to 0 \quad \text{if } \boldsymbol{k} \to \infty,$$

it follows from Remark 4.3(iii) that

$$\lim_{\boldsymbol{k}\to\infty} \|L_{(\boldsymbol{k},\boldsymbol{c})} - L_{A,(\boldsymbol{k},\boldsymbol{c})}\|_\infty = 0.$$

The inequality $\|\tilde{L}_{(\boldsymbol{k},\boldsymbol{c})} - \tilde{L}_{A,(\boldsymbol{k},\boldsymbol{c})}\|_{i2} \leq n^{\frac{1}{2}} \|L_{(\boldsymbol{k},\boldsymbol{c})} - L_{A,(\boldsymbol{k},\boldsymbol{c})}\|_\infty$ (cf. Logemann, 1986c: p. 68) yields the claim.

(ii) Set $[l_{ij}^{(\boldsymbol{k},\boldsymbol{c})}] = L_{(\boldsymbol{k},\boldsymbol{c})}$. Then we can conclude from (4.9) and from Remark 4.3(ii) that $\|l_{ij}^{(\boldsymbol{k},\boldsymbol{c})}\|_\infty \to 0$ $(\boldsymbol{k}\to\infty)$ for all i and j such that $i \neq j$. Now the claim follows from (4.11) and from the fact that $\|l_{ij}^{(\boldsymbol{k},\boldsymbol{c})}\|_\infty = \|\tilde{L}_{(\boldsymbol{k},\boldsymbol{c})ij}\|_{i2}$ (cf. Harris & Valenca, 1983: p. 83; or Logemann, 1986c: p. 64). □

COROLLARY 4.5 *Let the assumptions of Theorem* 4.4 *be satisfied. Then*

$$\lim_{\boldsymbol{k}\to\infty} \|\tilde{L}_{(\boldsymbol{k},\boldsymbol{c})}\boldsymbol{r} - \boldsymbol{r}\|_2 = 0,$$

for all $\boldsymbol{r} \in \mathrm{L}^2(\mathbb{R}_+)^n$.

Proof. For the sake of brevity, we assume that $k_j = \kappa$ and $c_j = \gamma$ $(j = 1, \ldots, n)$. Denote $L_{A,(\boldsymbol{k},\boldsymbol{c})}$ by $L_{A,(\kappa,\gamma)}$. Equip the space $(\mathbf{H}^2)^n$ with the norm

$$\|\boldsymbol{f}\|_{\mathbf{H}^2} := \max_j \|f_j\|_{\mathbf{H}^2} \quad \text{for } f = [f_1, \ldots, f_n]^{\mathrm{T}} \in (\mathbf{H}^2)^n.$$

Then the Laplace–Plancheral transform $\mathsf{L}_n : \mathrm{L}^2(\mathbb{R}_+)^n \to (\mathbf{H}^2)^n$ is an isometry (see Hoffman, 1962: p. 131; or Doetsch, 1971: p. 419). By Theorem 4.4, it is sufficient to show that, for all $\boldsymbol{f} \in (\mathbf{H}^2)^n$, we have

$$\lim_{\kappa\to\infty} \|L_{A,(\kappa,\gamma)}\boldsymbol{f} - \boldsymbol{f}\|_{\mathbf{H}^2} = 0. \tag{4.15}$$

Now an easy calculation yields

$$L_{A,(\kappa,\gamma)}(s) = \frac{(\kappa+\gamma)s + \kappa\gamma}{(s+\kappa)(s+\gamma)} I - \frac{s}{(s+\kappa)(s+\gamma)} A_0^{-1}A_1.$$

In order to establish (4.15), it is sufficient to show that

$$\lim_{\kappa\to\infty} \int_{-\infty}^{\infty} \left|\frac{(\mathrm{i}\omega)^2}{(\mathrm{i}\omega+\kappa)(\mathrm{i}\omega+\gamma)} f(\mathrm{i}\omega)\right|^2 \mathrm{d}\omega = 0 \quad \forall f \in \mathbf{H}^2. \tag{4.16}$$

Equation (4.16) is true by Lebesgue's bounded convergence theorem. □

Corollary 4.5 says that the controller $K_{(\boldsymbol{k},\boldsymbol{c})}$ achieves almost perfect tracking of L^2-signals for sufficiently large k_j $(j = 1, \ldots, n)$. Theorem 4.4 can be improved in case $G \in \hat{\mathcal{B}}^{n\times n}$ and $H \in \hat{\mathrm{A}}_-^{n\times n}$.

Remark 4.6 (i) It follows from (4.13) that the inverse Laplace transform of

$L_{A,(k,c)}$ is given by

$$\check{L}_{A,(k,c)}(t) = \operatorname{diag}_j \left(\frac{c_j^2}{c_j - k_j} e^{-c_j t} + \frac{k_j^2}{k_j - c_j} e^{-k_j t} \right) - \left[\operatorname{diag}_j \left(\frac{c_j}{c_j - k_j} e^{-c_j t} + \frac{k_j t}{k_j - c_j} e^{-k_j t} \right) \right] A_0^{-1} A_1 \qquad (4.17)$$

if $k_j \neq c_j$ $(j = 1, \ldots, n)$.

(ii) Inspection of (4.17) yields that the off-diagonal elements of $\check{L}_{A,(k,c)}$ tend to zero (in the A-norm) if $k \to \infty$.

(iii) Moreover, it follows from (4.17) that the A-norm of the diagonal elements of $\check{L}_{A,(k,c)}$ tends to 1 if $k \to \infty$. Therefore (by (ii))

$$\lim_{k \to \infty} \|\check{L}_{A,(k,c)}\|_{A^{n\times n}} = 1, \qquad (4.18)$$

where $\|\bullet\|_{A^{n\times n}}$ is defined by

$$\|[f_{ij}]\|_{A^{n\times n}} := \max_{1 \leq i \leq n} \sum_{j=1}^{n} \|f_{ij}\|_A,$$

for $[f_{ij}] \in A^{n\times n}$. Equipped with the norm $\|\bullet\|_{A^{n\times n}}$, the space $A^{n\times n}$ becomes a Banach algebra.

THEOREM 4.7 *Let $G \in \hat{B}^{n\times n}$ be such that G^{-1} is of the form* (4.1) *with $H \in \hat{A}_{-}^{n\times n}$. Then, for $1 \leq q \leq \infty$ and $c \in \mathbb{R}_+^n$ fixed,*

$$\text{(i)} \quad \lim_{k \to \infty} \|\check{L}_{(k,c)} - \check{L}_{A,(k,c)}\|_{lq} = 0, \qquad \text{(ii)} \quad \lim_{k \to \infty} \|\check{L}_{(k,c)ij}\|_{lq} = 0 \quad (i \neq j).$$

Proof. (i) It follows from (4.9) that

$$\check{L}_{(k,c)} - \check{L}_{A,(k,c)} = \left\{ \left[\delta I + \left(\operatorname{diag}_j \check{f}_j \right) \star (A_0^{-1} \check{H}) \right]^{-1} - \delta I \right\} \star \check{L}_{A,(k,c)}$$

where

$$\check{f}_j(t) = \frac{c_j}{c_j - k_j} e^{-c_j t} + \frac{k_j}{k_j - c_j} e^{-k_j t} \quad (j = 1, \ldots, n).$$

and δ is the Dirac distribution on $\mathbb{R}_+$ with support at 0. Now it is easy to show that $\|(\operatorname{diag}_j \check{f}_j) \star (A_0^{-1} \check{H})\|_{A^{n\times n}} \to 0$ if $k \to \infty$ and therefore, by Remark 4.6(iii), we have

$$\lim_{k \to \infty} \|\check{L}_{(k,c)} - \check{L}_{A,(k,c)}\|_{A^{n\times n}} = 0. \qquad (4.19)$$

Finally, note that

$$\|\check{L}_{(k,c)} - \check{L}_{A,(k,c)}\|_{lq} \leq \|\check{L}_{(k,c)} - \check{L}_{A,(k,c)}\|_{A^{n\times n}}$$

for all $1 \leq q \leq \infty$ (cf. Vidyasagar 1978: p. 250).

(ii) Set $[\check{l}_{ij}^{(k,c)}] := \check{L}_{(k,c)}$ and $[\check{l}_{Aij}^{(k,c)}] := \check{L}_{A,(k,c)}$. It follows from (4.19) that $\lim_{k \to \infty} \left| \|\check{l}_{ij}^{(k,c)}\|_A - \|\check{l}_{Aij}^{(k,c)}\|_A \right| = 0$. Therefore (by Remark 4.6(ii)) $\lim_{k \to \infty} \|\check{l}_{ij}^{(k,c)}\|_A =$

0, if $i \neq j$. Now the claim follows from (4.12) and from the inequality

$$\|\bar{L}_{(k,c)ij}\|_{1q} \leqslant \|l_{ij}^{(k,c)}\|_{A} \quad (1 \leqslant q \leqslant \infty)$$

(cf. Vidasagar 1978, p. 250). □

5. Examples

In this section, we shall consider examples of infinite-dimensional systems whose inverse transfer matrices are of the form (4.1).

5.1 *Volterra Integrodifferential Systems*

Consider the Volterra integrodifferential system

$$\begin{aligned} \dot{x}(t) &= [(E_0\delta + E_1) \star x](t) + [(B_0\delta + B_1) \star u](t), \\ y(t) &= [(C_0\delta + C_1) \star x](t), \end{aligned} \tag{5.1}$$

where $x(t) \in \mathbb{R}^m$, $u(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^n$, $E_0 \in \mathbb{R}^{m\times m}$, $B_0 \in \mathbb{R}^{m\times n}$, and $C_0 \in \mathbb{R}^{n\times m}$. The entries of the matrices E_1, B_1, and C_1 are all functions in

$$\mathrm{L}^1_- = \{f \in \mathrm{L}^1(\mathbb{R}_+) \ : \ \exists \varepsilon > 0 \text{ s.t. } fe^{\varepsilon\cdot} \in \mathrm{L}^1(\mathbb{R}_+)\}.$$

The sizes of E_1, B_1, and C_1 are given by $m \times m$, $m \times n$, and $n \times m$, respectively. For the transfer matrix G of the system (5.1) we obtain the following expression

$$G(s) = [C_0 + \hat{C}_1(s)][sI - E_0 - \hat{E}_1(s)]^{-1}[B_0 + \hat{B}_1(s)], \tag{5.2}$$

where $\hat{}$ denotes the Laplace transformation.

LEMMA 5.1 *The transfer matrix* (5.2) *of the Volterra integrodifferential system is in* $\hat{\mathrm{B}}^{n\times n}$.

Proof. The entries of $E_0 + \hat{E}_1(s)$, $B_0 + \hat{B}_1(s)$, and $C_0 + \hat{C}_1(s)$ are elements of $\hat{\mathrm{A}}_-$. It remains to show that $[sI - E_0 - \hat{E}_1(s)]^{-1} \in \hat{\mathrm{B}}^{m\times m}$. Let $\operatorname{Adj} X(s)$ denote the adjoint matrix of $X(s) = sI - E_0 - \hat{E}_1(s)$; then, by Cramer's rule, we have

$$X^{-1}(s) = \frac{\operatorname{Adj} X(s)}{(s+1)^m}\left(\frac{\det X(s)}{(s+1)^m}\right)^{-1}. \tag{5.3}$$

$\det X(s)$ can be written in the form

$$\det X(s) = s^m + e_{m-1}(s)s^{m-1} + \cdots + e_0(s),$$

where the e_i belong to the subalgebra of $\hat{\mathrm{A}}_-$ generated by the entries of $E_0 + \hat{E}_1(s)$. Therefore we have

$$\frac{\det X(s)}{(s+1)^m} \in \hat{\mathrm{A}}^\infty_-.$$

Finally we note that

$$\frac{\operatorname{Adj} X(s)}{(s+1)^m} \in \hat{\mathrm{A}}_-^{m\times m}.$$

It follows from (5.3) that $X^{-1} \in \hat{\mathrm{B}}^{m\times m}$. □

We need the following assumptions on the Volterra integrodifferential system (5.1).

(V1) The transfer matrix given by (5.2) is minimum-phase.

(V2) $\det(C_0B_0) \neq 0$.

(V3) The entries of $s\hat{B}_1(s)$ and $s\hat{C}_1(s)$ belong to $\mathbf{H}^\infty_-$.

(V4) $\operatorname{rk}[sI - E_0 - \hat{E}_1(s),\, B_0 + \hat{B}_1(s)] = \operatorname{rk}\begin{bmatrix} sI - E_0 - \hat{E}_1(s) \\ C_0 + \hat{C}_1(s) \end{bmatrix} = m \quad \forall s \in \bar{\mathbb{C}}_0$.

Remark 5.2 (i) (V3) is satisfied if either (a) the entries of B_1 and C_1 are elements of the space

$$\mathrm{W}^1_{1-} := \{f \in \mathrm{W}^1_1(\mathbb{R}_+) : \exists\, \varepsilon > 0 \text{ s.t. } fe^{\varepsilon\cdot} \in \mathrm{W}^1_1(\mathbb{R}_+)\},$$

where $\mathrm{W}^1_1(\mathbb{R}_+)$ denotes the Sololev space of all functions $f \in \mathrm{L}^1(\mathbb{R}_+)$ such that the distributional derivative of f is in $\mathrm{L}^1(\mathbb{R}_+)$, or (b) the entries of B_1 and C_1 satisfy the conditions of Satz 1, p. 477, or Satz 4, p. 480, in Doetsch (1971).

(ii) (V4) is the generalized Hautus condition (cf. Hautus, 1970), which is necessary in order to establish internal stability.

THEOREM 5.3 (i) *If* (V1)–(V3) *are satisfied, then the inverse of the transfer matrix* (5.2) *is of the form* (4.1) *with* $H \in \mathbf{H}^{\infty\, n\times n}_-$.

(ii) *If* (V1) *and* (V2) *are satisfied,* $B_1 \equiv 0$, *and* $C_1 \equiv 0$, *then the inverse of the transfer matrix* (5.2) *is of the form* (4.1) *with* $H \in \hat{\mathbf{A}}^{n\times n}_-$.

Proof. (i) Define $F(s) := (s+1)G(s) \in \mathrm{T}^{n\times n}$, where G is given by (5.2). It is sufficient to show that

$$F(s) - C_0B_0 = O(s^{-1}) \quad \text{if } |s| \to \infty \text{ in } \mathbb{C}_\alpha, \text{ for some } \alpha < 0. \tag{5.4}$$

Indeed, if (5.4) is true, then it follows in particular that

$$\lim_{|s|\to\infty,\, s\in\mathbb{C}_\alpha} F(s) = C_0B_0.$$

Hence $F^{-1} \in \mathrm{T}^{n\times n}$ by (V2) (cf. Remark 2.1 (iv)). Further, it follows from (V1) and Remark 2.1 (ii) that there exists $\beta \in (\alpha, 0)$ such that $F^{-1}(s)$ has no poles in $\mathbb{C}_\beta$, which means that $F^{-1} \in \mathbf{H}^{\infty\, n\times n}_-$. Therefore, using (5.4), we obtain

$$H(s) := (s+1)[F^{-1}(s) - (C_0B_0)^{-1}] \in \mathbf{H}^{\infty\, n\times n}_-.$$

Now note that $G^{-1}(s) = (s+1)F^{-1}(s) = (s+1)(C_0B_0)^{-1} + H(s)$, which is (4.1).

It remains to show that (5.4) holds. Let $\alpha < 0$ be such that

$$\hat{E}(s) = E_0 + \hat{E}_1(s)$$

and $s\hat{B}_1(s)$ and $s\hat{C}_1(s)$ are bounded and holomorphic on $\mathbb{C}_\alpha$ (this is possible by (V3)). For all $s \in \mathbb{C}_\alpha$ such that $|s| > \sup_{z\in\mathbb{C}_\alpha} \bar{\sigma}(\hat{E}(z))$, the following equation holds:

$$\begin{aligned} s[F(s)-C_0B_0] &= s[C_0+\hat{C}_1(s)][I-s^{-1}\hat{E}(s)]^{-1}[B_0+\hat{B}_1(s)](I+s^{-1}I)-sC_0B_0 \\ &= C_0\Big(\sum_{i=1}^{\infty}\frac{1}{s^{i-1}}\hat{E}^i(s)\Big)B_0+C_0\Big(\sum_{i=0}^{\infty}\frac{1}{s^i}\hat{E}^i(s)\Big)s\hat{B}_1(s) \\ &\quad + s\hat{C}_1(s)\Big(\sum_{i=0}^{\infty}\frac{1}{s^i}\hat{E}^i(s)\Big)B_0+s\hat{C}_1(s)\Big(\sum_{i=0}^{\infty}\frac{1}{s^i}\hat{E}^i(s)\Big)\hat{B}_1(s) \\ &\quad + [C_0+\hat{C}_1(s)]\Big(\sum_{i=0}^{\infty}\frac{1}{s^i}\hat{E}^i(s)\Big)[B_0+\hat{B}_1(s)]. \end{aligned} \tag{5.5}$$

Since the five terms on the r.h.s. of (5.5) are bounded in the region

$$\{s\in\mathbb{C}_\alpha : |s|>\rho\},$$

for some $\rho>\sup_{z\in\mathbb{C}_\alpha}\bar{\sigma}(\hat{E}(z))$, then (5.5) proves (5.4).

(ii) It follows from Owens *et al.* (1984) that there exists a nonsingular $T\in\mathbb{R}^{m\times m}$ such that

$$T^{-1}B_0=\begin{bmatrix}C_0B_0\\0\end{bmatrix},\qquad CT=[\mathrm{I}_n\quad 0].$$

Define

$$\begin{bmatrix}E_{11} & E_{12}\\ E_{21} & E_{22}\end{bmatrix}:=T^{-1}E_0T\delta+T^{-1}E_1(\bullet)T,$$

where E_{11}, E_{12}, E_{21}, and E_{22} are matrices with entries in $\delta\mathbb{R}+\mathrm{L}^1(\mathbb{R}_+)$, of size $n\times n$, $n\times(m-n)$, $(m-n)\times n$, and $(m-n)\times(m-n)$, respectively. Setting

$$z(t):=T^{-1}x(t),$$

we get

$$\dot{z}(t)=[(T^{-1}E_0T\delta+T^{-1}E_1T)\star z](t)+T^{-1}B_0u(t),\qquad y(t)=C_0Tz(t). \tag{5.6}$$

Note that (5.6) can be written in the form

$$\dot{z}_1(t)=C_0B_0u_1(t),\qquad y_1(t)=z_1(t), \tag{5.7}$$

$$\left.\begin{aligned}\dot{z}_2(t)&=(E_{22}\star z_2)(t)+(E_{21}\star u_2)(t),\\ y_2(t)&=-(C_0B_0)^{-1}[(E_{12}\star z_2)(t)+(E_{11}\star u_2)(t)],\end{aligned}\right\} \tag{5.8}$$

$$u_1(t)=u(t)-y_2(t),\qquad u_2(t)=y_1(t); \tag{5.9}$$

i.e. the system (5.6) is the feedback interconnection of the integrator (5.7) and the Volterra integrodifferential system (5.8). Let H denote the transfer function of (5.8). Then, by Lemma 5.1, $H\in\hat{\mathcal{B}}^{n\times n}$. It follows from (5.6) – (5.9) that

$$G(s)=C_0[sI-E_0-\hat{E}_1(s)]^{-1}B_0=s^{-1}C_0B_0[I+H(s)s^{-1}C_0B_0]^{-1}.$$

Therefore, $G^{-1}(s)=s(C_0B_0)^{-1}+H$. Moreover, $H\in\hat{\mathcal{A}}^{n\times n}$ by (V1). □

As a consequence of Theorem 5.3 we have the following.

COROLLARY 5.4 *Suppose that the conditions* (V1) – (V4) *are satisfied and, for a fixed* $c \in \mathbb{R}^n_+$, *choose a realization (parametrized by* k) *of* $K_{(k,c)}$ *that is stabilizable and detectable in the range* $k_j > 0$ $(j = 1, \ldots, n)$. *Under these conditions, the feedback interconnection of the Volterra integrodifferential system* (5.1) *and the* PI-*controller* $K_{(k,c)}$ *is uniformly asymptotically stable (in the sense of Miller* (1971, 1972)) *for all sufficiently large* k_j $(j = 1, \ldots, n)$.

Proof. Combine Theorem 5.3(i) and Theorem 4.2, using Coroll. 3.6 of Logemann (1986b). □

5.2 *Retarded Systems*

Consider the retarded system

$$\dot{x}(t) = \int_0^h [\mathrm{d}A(\tau)\, x(t-\tau)] + B_0 u(t) + \int_0^h B_1(\tau) u(t-\tau)\, \mathrm{d}\tau, \qquad (5.10)$$
$$y(t) = C_0 x(t) + \int_0^h C_1(\tau)\, x(t-\tau)\, \mathrm{d}\tau;$$

here, $x(t) \in \mathbb{R}^m$, $u(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^n$, $B_0 \in \mathbb{R}^{m \times n}$, $C_0 \in \mathbb{R}^{n \times m}$, and $h > 0$; A is a function of bounded variation on the interval $[0, h]$ with values in $\mathbb{R}^{m \times m}$; and the entries of the matrices B_1 and C_1 are all functions in $L^1(0, h)$.

For the transfer matrix G of the system (5.10) we obtain

$$G(s) = [C_0 + \hat{C}_1(s)][sI - \hat{A}(s)]^{-1}[B_0 + \hat{B}_1(s)], \qquad (5.11)$$

where

$$\hat{A}(s) = \int_0^h \mathrm{e}^{-s\tau}\, \mathrm{d}A(\tau), \qquad \hat{B}_1(s) = \int_0^h B_1(\tau)\mathrm{e}^{-s\tau}\, \mathrm{d}\tau, \qquad \hat{C}_1(s) = \int_0^h C_1(\tau)\mathrm{e}^{-s\tau}\, \mathrm{d}\tau.$$

The entries of $\hat{A}$, $\hat{B}_1$, and $\hat{C}_1$ are entire functions. If we extend A, B_1, and C_1 to the complete positive real axis by defining $A(\tau) = A(h)$, $B_1(\tau) = 0$, and $C_1(\tau) = 0$, for $\tau > h$, then the function $\hat{A}$ is the Laplace–Stieltjes transform (Widder, 1972: p. 27) of A and, of course, $\hat{B}_1$ and $\hat{C}_1$ are the Laplace transforms of B_1 and C_1, respectively.

We need the following assumptions on the retarded system (5.10).

(R1) The transfer matrix given by (5.11) is minimum-phase.

(R2) $\det(C_0 B_0) \neq 0$.

(R3) The entries of $s\hat{B}_1(s)$ and $s\hat{C}_1(s)$ belong to $\mathbf{H}^\infty_-$.

(R4) The function A of bounded variation contains no singular part (see e.g. Kolmogorov & Fomin (1975: p. 341)).

(R5) $\operatorname{rk}[sI - \hat{A}(s), B_0 + \hat{B}_1(s)] = \operatorname{rk}\begin{bmatrix} sI - \hat{A}(s) \\ C_0 + \hat{C}_1(s) \end{bmatrix} = m \quad \forall s \in \bar{\mathbb{C}}_0.$

See Remark 5.2 for comments on the conditions (R3) and (R5).

LEMMA 5.5 (i) *The transfer matrix* (5.11) *of the retarded system is in* $\mathrm{T}^{n \times n}$.

(ii) *Suppose that* (R4) *is satisfied. Then the transfer matrix* (5.11) *belongs to* $\hat{\mathcal{B}}^{n \times n}$.

Proof. (i) The proof is very similar to that of Lemma 5.1.
(ii) See Logemann (1986a). □

THEOREM 5.6 (i) *If* (R1)–(R3) *are satisfied then the inverse of the transfer matrix* (5.11) *is of the form* (4.1) *with* $H \in \mathbf{H}_-^{\infty n\times n}$.

(ii) *If* (R1), (R2), *and* (R4) *are satisfied, and if* $B_1 \equiv 0$ *and* $C_1 \equiv 0$, *then the inverse of the transfer matrix* (5.11) *is of the form* (4.1) *with* $H \in \hat{\mathbf{A}}_-^{n\times n}$.

Proof. The proof is similar to that of Theorem 5.3. □

As a consequence of Theorem 5.6 we have the following.

COROLLARY 5.7 *Suppose that the conditions* (R1)–(R3) *and* (R5) *are satisfied. For a fixed* $\boldsymbol{c} \in \mathbb{R}_+^n$, *choose a realization (parametrized by* $\boldsymbol{k}$*) of* $K_{(\boldsymbol{k},\boldsymbol{c})}$ *that is stabilizable and detectable in the range* $k_j > 0$ $(j = 1, \ldots, n)$. *Under these conditions, the feedback interconnection of the retarded system* (5.10) *and the* PI-*controller* $K_{(\boldsymbol{k},\boldsymbol{c})}$ *is exponentially stable (i.e. the strongly continuous solution semigroup of the closed-loop system is exponentially stable) for all sufficiently large* k_j $(j = 1, \ldots, n)$.

Proof. Combine Theorems 5.6 (i) and 4.2, using Coroll. 3.4 of Logemann (1986b). □

Remark 5.8 Using ideas of Hale (1974), the results of Subsection 5.2 can be extended to certain functional-differential equations with infinite delays.

6. Robustness properties of the PI-controller $K_{(\boldsymbol{k},\boldsymbol{c})}$ at high gain

6.1 *Robustness with respect to perturbations in the parameters of* A_0 *and* A_1

Consider a transfer matrix $G \in \mathbf{T}^{n\times n}$ (resp. $\hat{\mathbf{B}}^{n\times n}$) with inverse of the form (4.1). Let A_0^* and A_1^* be numerical estimates of A_0 and A_1 obtained from a complex model or from open-loop step response data (if available). We assume that $\det A_0^* \neq 0$. This numerical information can be used to construct a PI-controller of the form

$$K^*_{(\boldsymbol{k},\boldsymbol{c})}(s) = A_0^* \operatorname*{diag}_j (k_j + c_j + k_j c_j / s) - A_1^*, \qquad (6.1)$$

where $\boldsymbol{k} := [k_1, \ldots, k_n]^{\mathrm{T}}$, $k_j > 0$, and $\boldsymbol{c} := [c_1, \ldots, c_n]^{\mathrm{T}} \in \mathbb{R}_+^n$. Suppose that we now apply the controller to the real system given by (4.1); we will use the notation $\sigma(X)$ and $\mathrm{r}(X)$ respectively for the spectrum and the spectral radius of a matrix X.

THEOREM 6.1 *Let* $G \in \mathbf{T}^{n\times n}$ *(resp.* $\hat{\mathbf{B}}^{n\times n}$*) be such that* G^{-1} *is of the form* (4.1). *Suppose that* $k_j = \kappa$ *and* $c_j = \gamma$ $(j = 1, \ldots, n)$, *and denote* $K^*_{(\boldsymbol{k},\boldsymbol{c})}$ *by* $K^*_{(\kappa,\gamma)}$. *The feedback scheme in Fig.* 4 *is* $\mathbf{H}_-^{\infty}$*-stable (resp.* $\hat{\mathbf{A}}_-$*-stable) for all sufficiently large* κ *if*

(i) $\|A_0^{*-1}(A_0^* - A_0)\| < 1$, *where* $\|\bullet\|$ *is any matrix norm on* $\mathbb{C}^{n\times n}$,
or (ii) $\mathrm{r}\big(A_0^{*-1}(A_0^* - A_0)\big) < 1$,
or (iii) $\sigma(A_0^{*-1}A_0) \subset \mathbb{C}_0$.

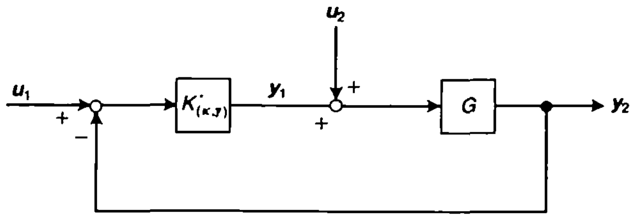

FIG. 4.

Proof. Since (i)$\Rightarrow$(ii)$\Rightarrow$(iii), it is sufficient to prove the claim in case that (iii) is satisfied.

Define $G_A^* \in \mathbb{C}(s)^{n\times n}$ by $G_A^{*-1}(s) = sA_0^* + A_1^*$. Further, define

$$L^*_{A,(\kappa,\gamma)} := (I + G_A^* K^*_{(\kappa,\gamma)})^{-1} G_A^* K^*_{(\kappa,\gamma)}, \qquad L^*_{(\kappa,\gamma)} := (I + GK^*_{(\kappa,\gamma)})^{-1} GK^*_{(\kappa,\gamma)}.$$

Of course, $L^*_{A,(\kappa,\gamma)}$ is $\hat{\mathbf{A}}_-$-stable for all $\kappa > 0$ and $\gamma \geq 0$. It is sufficient to show that $L^*_{(\kappa,\gamma)}$ is $\mathbf{H}^\infty_-$-stable (resp. $\hat{\mathbf{A}}_-$-stable) for all sufficiently large κ (cf. the proof of Theorem 4.2). An elementary computation shows that

$$L^*_{(\kappa,\gamma)} = (P_\kappa + Q_{(\kappa,\gamma)})^{-1} L^*_{A,(\kappa,\gamma)}, \tag{6.2}$$

where

$$P_\kappa(s) := I + \frac{s}{s+\kappa} A_0^{*-1}(A_0 - A_0^*), \tag{6.3}$$

$$Q_{(\kappa,\gamma)}(s) := \frac{s}{(s+\kappa)(s+\gamma)} \{A_0^{*-1}[A_1 - A_1^* + H(s)] - \gamma A_0^{*-1}(A_0 - A_0^*)\}.$$

Note that

$$\|Q_{(\kappa,\gamma)}\|_\infty \leq \frac{1}{\kappa}\bar{\sigma}(A_0^{*-1})[\bar{\sigma}(A_1 - A_1^*) + \|H\|_\infty] + \frac{\gamma}{\kappa}\bar{\sigma}(A_0^{*-1})\bar{\sigma}(A_0 - A_0').$$

Let $\gamma \in \mathbb{R}_+$ be fixed. If we show that $\inf_{s\in\mathbb{C}_0} |\det P_\kappa(s)| = \varepsilon > 0$ (independent of κ), then it follows that $(P_\kappa + Q_{(\kappa,\gamma)})^{-1}$ is $\mathbf{H}^\infty_-$-stable (resp. $\hat{\mathbf{A}}_-$-stable) for all sufficiently large κ, and hence that $L^*_{(\kappa,\gamma)}$ is $\mathbf{H}^\infty_-$-stable (resp. $\hat{\mathbf{A}}_-$-stable) (by (4.2)) for all sufficiently large κ. Now choose a fixed $\kappa_0 > 0$ and realize that

$$P_{\kappa_0}(s) = \frac{\kappa_0}{s+\kappa_0}\left(\frac{s}{\kappa_0} A_0^{*-1} A_0 + I\right).$$

Then condition (iii) gives $\inf_{s\in\mathbb{C}_0} |\det P_{\kappa_0}(s)| = \varepsilon > 0$. Finally note that, for $\kappa > 0$ and $\lambda > 0$, we have

$$\{s/(s+\kappa) : s \in \mathbb{C}_0\} = \{s/(s+\lambda) : s \in \mathbb{C}_0\}$$

and therefore, by (4.3), $\inf_{s\in\mathbb{C}_0} |\det P_\kappa(s)| = \varepsilon$, independent of κ.

Remark 6.2 In the general case, i.e. the k_j and c_j are different, the condition (iii) is not sufficient for stability at high gain. We present a counterexample: Let

$$H \equiv 0, \qquad A_1 = A_1^* = 0, \qquad A_0 = \begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}, \qquad A_0^* = I_2.$$

Of course $\sigma(A_0^{*-1}A_0) = \sigma(A_0) \subset \mathbb{C}_0$. Further, choose the controller to be

$$K_k^* = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \qquad k = [k_1, k_2]^{\mathrm{T}},$$

i.e. $c_1 = c_2 = 0$ in (6.1). Then an elementary calculation shows

$$[I + G(s)K_k^*]^{-1}G(s)K_k^* = \frac{1}{2s^2 + (2k_1 - k_2)s + k_1k_2} \begin{bmatrix} 2s + k_2 & 2s \\ 2s & k_1 - s \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

So, if we choose $k_1 = \kappa$ and $k_2 = \kappa^2$, the closed-loop system will be unstable for $\kappa \geq 2$.

In the following,

$$\mathrm{W}(X) := \{x^{\mathrm{H}}Xx : x \in \mathbb{C}^n, x^{\mathrm{H}}x = 1\}$$

denotes the *numerical range* of an $n \times n$ matrix X; see e.g. Halmos (1982: p. 112).

THEOREM 6.3 *Let $G \in \mathrm{T}^{n\times n}$ (resp. $\hat{\mathrm{B}}^{n\times n}$) and assume that G^{-1} is of the form* (4.1). *Then, for $c \in \mathbb{R}^n$ fixed, the following holds.*

(i) *The feedback scheme in Fig. 5 is $\mathbf{H}_-^\infty$-stable (resp. $\hat{\mathrm{A}}_-$-stable) for all sufficiently large k_j $(j = 1, \dots, n)$ if $\|A_0^{*-1}(A_0^* - A_0)\| < 1$, where $\|\bullet\|$ is any submultiplicative norm on $\mathbb{C}^{n\times n}$ with the additional property that $\|\mathrm{diag}_j\, a_j\| \leq \max_j |a_j|$ for arbitrary $a_1, \dots, a_n \in \mathbb{C}$.*

(ii) *Under the additional assumption that $k_j = \nu_j\kappa$ with $\nu_j > 0$ fixed $(j = 1, \dots, n)$, the feedback system in Fig. 5 is $\mathbf{H}_-^\infty$-stable (resp. $\hat{\mathrm{A}}_-$-stable) for all sufficiently large κ if*

$$\text{(a)}\ \sigma\Big(\Big(\operatorname*{diag}_j \nu_j\Big)A_0^{*-1}A_0\Big) \subset \mathbb{C}_0 \quad \text{or} \quad \text{(b)}\ \mathrm{W}(A_0^{*-1}A_0) \subset \mathbb{C}_0.$$

Proof. (i) Define $G_A^* \in \mathbb{C}(s)^{n\times n}$ by $G_A^{*-1}(s) = sA_0^* + A_1^*$. Further, define

$$L_{A,(k,c)}^* := (I + G_A^*K_{(k,c)}^*)^{-1}G_A^*K_{(k,c)}^*, \qquad L_{(k,c)}^* := (I + GK_{(k,c)}^*)^{-1}GK_{(k,c)}^*.$$

$L_{A,(k,c)}^*$ is $\hat{\mathrm{A}}_-$-stable for all $k_j > 0$ and $c_j \geq 0$ $(j = 1, \dots, n)$. It is sufficient to show that $L_{(k,c)}^*$ is $\mathbf{H}_-^\infty$-stable (resp. $\hat{\mathrm{A}}_-$-stable) for all sufficiently large k_j $(j = 1, \dots, n)$ (cf. the proof of Theorem 4.2). An elementary computation shows that

$$L_{(k,c)}^*(s) = \left[I + \left(\operatorname*{diag}_j \frac{s}{s + k_j}\right)A_0^{*-1}(A_0 - A_0^*) + Q_{(k,c)}(s)\right]^{-1} L_{A,(k,c)}^*(s), \quad (6.4)$$

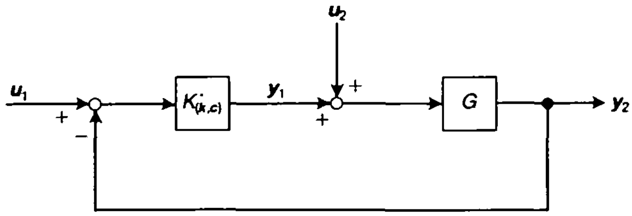

FIG. 5.

where

$$Q_{(k,c)}(s) := \left(\operatorname{diag}_j \frac{s}{(s+k_j)(s+c_j)}\right) A_0^{*-1}[A_1 - A_1^* + H(s)] - \left(\operatorname{diag}_j \frac{c_j s}{(s+k_j)(s+c_j)}\right) A_0^{*-1}(A_0 - A_0^*).$$

For fixed $c \in \mathbb{R}_+^n$, we have

$$\lim_{k\to\infty} \sup_{s\in\mathbb{C}_0} \|Q_{(k,c)}(s)\| = 0. \tag{6.5}$$

The claim follows from (6.4) and (6.5) if we note that $\mathbf{H}^{\infty\, n\times n}$ equipped with the norm $\sup_{s\in\mathbb{C}_0} \|M(s)\|$ $(M \in \mathbf{H}^{\infty\, n\times n})$ is a Banach algebra (cf. the proof of Theorem 4.2).

(ii) The proof of part (a) is similar to that of Theorem 6.1 and is therefore omitted. We shall prove part (b). Define

$$P_\kappa(s) := I + \left(\operatorname{diag}_j \frac{s}{s+\nu_j\kappa}\right) A_0^{*-1}(A_0 - A_0^*). \tag{6.6}$$

By (6.4) and (6.5) it is sufficient to show that

$$\inf_{s\in\mathbb{C}_0} |\det P_\kappa(s)| = \varepsilon > 0, \tag{6.7}$$

independent of κ (cf. the proof of Theorem 6.1). It follows from (6.6) that

$$P_\kappa(s) = \left(\operatorname{diag} \frac{s}{s+\nu_j\kappa}\right)\left(\operatorname{diag} \frac{\nu_j\kappa}{s} + A_0^{*-1}A_0\right).$$

Moreover, the following inclusions hold for all $s \in \bar{\mathbb{C}}_0\backslash\{0\}$:

$$\sigma\left(\operatorname{diag}_j \frac{\nu_j\kappa}{s} + A_0^{*-1}A_0\right) \subset \mathrm{W}\left(\operatorname{diag}_j \frac{\nu_j\kappa}{s} + A_0^{*-1}A_0\right) \subset \mathrm{W}\left(\operatorname{diag}_j \frac{\nu_j\kappa}{s}\right) + \mathrm{W}(A_0^{*-1}A_0) \subset \mathbb{C}_0. \tag{6.8}$$

The matrices $P_\kappa(0)$ and $P_\kappa(\infty)$ are nonsingular. Therefore it follows from (6.8) that

$$\inf_{s\in\mathbb{C}_0} |\det P_\kappa(s)| = \varepsilon_\kappa > 0. \tag{6.9}$$

Finally note that

$$\left\{\operatorname{diag}_j \frac{s}{s+\nu_j\kappa} : s \in \mathbb{C}_0\right\} = \left\{\operatorname{diag}_j \frac{s}{s+\nu_j\lambda} : s \in \mathbb{C}_0\right\} \quad (\kappa > 0,\ \lambda > 0).$$

It follows now from (6.6) that ε_κ in (6.9) is independent of κ.

6.2 *Robustness with respect to Certain Measurement Nonlinearities*

We now study the effect of certain measurement nonlinearities on the feedback schemes in Figs 1–2. We consider nonlinear functions $\phi : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the

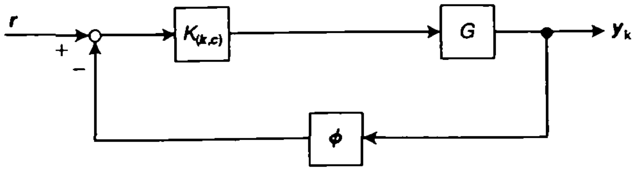

FIG. 6.

conditions

(N1) $\exists\, \alpha > 0\ \forall x \in \mathbb{R}^n$: $|x - \phi(x)|_\infty \leq \alpha,$

(N2) $\exists\, \beta > 0\ \forall x, y \in \mathbb{R}^n$: $|\phi(x) - \phi(y)|_\infty \leq \beta\, |x - y|_\infty$

($|\bullet|_\infty$ denotes the maximum-norm on $\mathbb{R}^n$); we give two examples:

(a) (Dead-zone) Define $\phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\phi_i(x) = \begin{cases} x_i + \gamma_i & (x_i < -\gamma_i), \\ 0 & (-\gamma_i \leq x_i \leq \gamma_i), \\ x_i - \gamma_i & (x_i > \gamma_i), \end{cases}$$

where $\gamma_i \geq 0$ $(i = 1, \ldots, n)$; then ϕ satisfies (N1) and (N2).

(b) Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a bounded differentiable function such that the differential $(D\psi)(x)$ is bounded. Then $\phi := \mathrm{id}_{\mathbb{R}^n} + \psi$ satisfies (N1) and (N2).

Let $\hat{\mathcal{B}}_r$ denote the $\mathbb{R}$-algebra of all transfer functions in $\hat{\mathcal{B}}$ with 'real coefficients', i.e. any Laurent expansion about a real point has real coefficients. Note that

$$\hat{\mathcal{B}}_r = \{f \in \hat{\mathcal{B}} : \overline{f(s)} = f(\bar{s})\ \forall s \in \bar{\mathbb{C}}_0\}.$$

We shall consider transfer matrices $G \in \hat{\mathcal{B}}_r^{n \times n}$ of the form

$$G^{-1}(s) = sA_0 + A_1 + H, \tag{6.10}$$

where $A_0, A_1 \in \mathbb{R}^{n \times n}$, $\det A_0 \neq 0$, and $H \in \hat{A}_-^{n \times n}$. As usual, we define the rational matrix function G_A by

$$G_A^{-1}(s) = sA_0 + A_1. \tag{6.11}$$

The time-domain equations of the feedback systems in Figs 6–7 are

$$y_k = \check{L}_{(k,c)} \star r - \check{L}_{(k,c)} \star (\phi \circ y_k - y_k), \tag{6.12}$$

$$y_{A,k} = \check{L}_{A,(k,c)} \star r - \check{L}_{A,(k,c)} \star (\phi \circ y_{A,k} - y_{A,k}), \tag{6.13}$$

where $L_{(k,c)}$ and $L_{A,(k,c)}$ are given by (4.8), and $\check{}$ denotes the inverse Laplace transform.

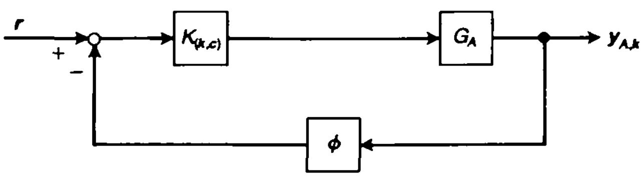

FIG. 7.

The following lemma shows that the feedback systems (6.12) and (6.13) are well-posed. We denote by LL_+^q the space of all measurable functions $f:\mathbb{R}_+\to\mathbb{R}$ such that $f|_I\in \mathrm{L}^q(I)$ for all compact intervals $I\subset\mathbb{R}_+$ $(1\leq q\leq\infty)$.

LEMMA 6.4 *Let* $A\in(\mathrm{LL}_+^1)^{n\times n}$ *and* $f\in(\mathrm{LL}_+^\infty)^n$, *and assume that* $\phi:\mathbb{R}^n\to\mathbb{R}^n$ *satisfies condition* (N2). *Then the equation*

$$y=f+A\star(\phi\circ y-y)$$

has a unique solution in $(\mathrm{LL}_+^\infty)^n$.

Proof. Apply Theorem 2 in Desoer & Vidyasagar (1975: p. 48). □

THEOREM 6.5 *Let* $G\in\hat{\mathrm{B}}_{\mathrm{r}}^{n\times n}$ *be a transfer matrix of the form* (6.10) *and let the transfer matrix* G_A *be given by* (6.11). *Further, we assume that the function* $\phi:\mathbb{R}^n\to\mathbb{R}^n$ *satisfies the conditions* (N1) *and* (N2). *Then*

(i) *The feedback system*

$$y=\check{G}_A\star e,\qquad e=\check{K}_{(k,c)}\star(r-\phi\circ y)\tag{6.14}$$

is L^∞-*stable for all* $k_j>0$ *and* $c_j\geq 0$ $(j=1,\ldots,n)$.

(ii) *The feedback system*

$$y=\check{G}\star e,\qquad e=\check{K}_{(k,c)}\star(r-\phi\circ y)\tag{6.15}$$

is L^∞-*stable for all* $c\in\mathbb{R}_+^n$ *and all* $k=[k_1,\ldots,k_n]^{\mathrm{T}}$ *satisfying* $\min_j k_j>\bar{\sigma}(A_0^{-1})\,\|H\|_\infty$.

Proof. (i) We have that $\check{L}_{A,(k,c)}\in\mathrm{L}^1(\mathbb{R}_+)^{n\times n}$ for all $k_j>0$ and $c_j\geq 0$ $(j=1,\ldots,n)$. Therefore it follows from Lemma 6.4 that (6.13) has a unique solution in $(\mathrm{LL}_+^\infty)^n$ for all $r\in(\mathrm{LL}_+^\infty)^n$. Realize that

$$\|\phi\circ f-f\|_\infty\leq\alpha\quad\forall f\in(\mathrm{LL}_+^\infty)^n.$$

The L^∞-stability of the feedback system (6.14) is now implied by (6.13).

(ii) Define $f_j(s):=s/(s+k_j)(s+c_j)$ $(j=1,\ldots,n)$. It is clear that

$$\Big[I+\Big(\operatorname*{diag}_j f_j\Big)A_0^{-1}H\Big]^{-1}\in\hat{\mathrm{A}}^{n\times n}$$

for all k such that $\min_j k_j>\bar{\sigma}(A_0^{-1})\,\|H\|_\infty$ and for all $c\in\mathbb{R}_+^n$. It follows from (4.9) that $\check{L}_{(k,c)}\in\mathrm{L}^1(\mathbb{R}_+)^{n\times n}$ for all k such that $\min_j k_j>\bar{\sigma}(A_0^{-1})\,\|H\|_\infty$ and for all $c\in\mathbb{R}_+^n$. We now conclude from (6.12) that the feedback system (6.15) is L^∞-stable for all k such that $\min_j k_j>\bar{\sigma}(A_0^{-1})\,\|H\|_\infty$ and for all $c\in\mathbb{R}_+^n$ (cf. the proof of part (i)).

Remark 6.6 Under the assumptions of Theorem 6.5, we have, for all $k\in\mathbb{R}_+^n$ satisfying $\min_j k_j>\bar{\sigma}(A_0^{-1})\,\|H\|_\infty$, and for all $c\in\mathbb{R}_+^n$ and $r\in\mathrm{L}^\infty(\mathbb{R}_+)^n$:

$$\|y_k-\check{L}_{(k,c)}\star r\|_\infty\leq\alpha\,\|\bar{L}_{(k,c)}\|_{i\infty},\tag{6.16}$$

$$\|y_k-\check{L}_{A,(k,c)}\star r\|_\infty\leq\|(\check{L}_{(k,c)}-\check{L}_{A,(k,c)})\star r\|_\infty+\alpha\,\|\bar{L}_{(k,c)}\|_{i\infty}.\tag{6.17}$$

(The inequalities (6.16) and (6.17) are easily derived from (6.12) and (6.13)).

Moreover, it follows from Theorem 4.7 and Remark 4.6(iii) that

$$\left.\begin{aligned}\limsup_{k\to\infty} \|y_k - \check{L}_{(k,c)} \star r\|_\infty \leq \alpha \quad (6.18)\\ \limsup_{k\to\infty} \|y_k - \check{L}_{A,(k,c)} \star r\|_\infty \leq \alpha \quad (6.19)\end{aligned}\right\} \quad \forall c \in \mathbb{R}^n_+ \quad \forall r \in L^\infty(\mathbb{R}_+)^n.$$

The inequality (6.16) provides an upper bound on the peak transient error induced by the nonlinearity in the infinite-dimensional feedback system in Fig. 6. The inequality (6.17) gives an upper bound on the peak transient error owing to the nonlinearity and to the modelling error $G - G_A$ if we replace the nonlinear infinite-dimensional feedback system in Fig. 6 by the linear finite-dimensional feedback system in Fig. 2. The inequalities (6.18) and (6.19) show that both L^∞-errors are asymptotically bounded by α as $k \to \infty$. In particular, we see that the peak transient effects will not be amplified by system dynamics if the k_j are sufficiently large.

Remark 6.7 It should be mentioned that the results in Subsection 6.2 are of similar nature to the results for the discrete-time finite-dimensional case in Boland & Owens (1980).

Acknowledgement

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7. Appendix

In this appendix we prove Lemma 3.1 and Theorem 3.3. In order to do this, we recall some facts concerning the algebraic structure of the ring $H(\Omega)$. The units of the integral domain $H(\Omega)$ are exactly those functions in $H(\Omega)$ which have no zeros in Ω. It is well known that $H(\Omega)$ is a Bezout domain, i.e. every finitely generated ideal in $H(\Omega)$ is principal (Narasimhan, 1985: p. 136). Thus, any finite set of functions in $H(\Omega)$ has a greatest common divisor. Moreover it is known that $H(\Omega)$

forms a so-called elementary divisor ring, i.e. every matrix of holomorphic functions admits a Smith normal form (Narasimhan, 1985: p. 141). Therefore every meromorphic matrix is unimodular-equivalent to its Smith–McMillan form. More precisely, if $M \in \mathrm{M}(\Omega)^{n\times m}$, then there exist unimodular matrices $U \in \mathrm{H}(\Omega)^{n\times n}$ and $V \in \mathrm{H}(\Omega)^{m\times m}$ such that

$$UMV = \begin{bmatrix} \mathrm{diag}(\varepsilon_1/\psi_1, \ldots, \varepsilon_r/\psi_r) & 0_{r\times(m-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{bmatrix},$$

where $r := \mathrm{rk}\, M \leq \min\{m, n\}$ and

$$\left.\begin{matrix} \varepsilon_i, \psi_i \in \mathrm{H}(\Omega) \\ \gcd(\varepsilon_i, \psi_i) = 1 \end{matrix}\right\} \quad (i = 1, \ldots, r), \qquad \left.\begin{matrix} \varepsilon_i \mid \varepsilon_{i+1} \\ \psi_{i+1} \mid \psi_i \end{matrix}\right\} \quad (i = 1, \ldots, r-1).$$

The holomorphic functions ε_i and ψ_i are unique up to units in $\mathrm{H}(\Omega)$. Therefore, it makes sense to write $\varepsilon_i = \varepsilon_i(M)$ and $\psi_i = \psi_i(M)$. We define (up to units in $\mathrm{H}(\Omega)$) the *pole function* and the *zero function* of M to be

$$\psi(M) := \prod_{i=1}^{r} \psi_i(M), \qquad \varepsilon(M) := \prod_{i=1}^{r} \varepsilon_i(M),$$

respectively. In case $r < \min\{m, n\}$ we set $\varepsilon_i(M) :\equiv 0$ and $\psi_i(M) :\equiv 1$ $(r+1 \leq i \leq \min\{m, n\})$.

Remark 7.1 (i) Let $G \in \mathrm{T}^{m\times n}$. Then it is not difficult to show that

$$\mathrm{ord}_s\, \psi(G) = \mathrm{p}_s(G) \quad \forall s \in \bar{\mathbb{C}}_0.$$

(ii) Let $G \in \mathrm{T}^{n\times n}$ and let $G = ND^{-1}$ be a right Bezout factorization. Then

$$\mathrm{ord}_s\, \varepsilon(G) = \mathrm{ord}_s \det N \quad \forall s \in \bar{\mathbb{C}}_0.$$

Proof of Lemma 3.1. There exists $\sigma < 0$ such that $\psi(F)$, $\psi(G)$, and $\psi(FG)$ all belong to $\mathrm{H}(\mathbb{C}_\sigma)$. It is sufficient to show that

$$\psi(FG) \mid \psi(F)\psi(G). \tag{7.1}$$

Coppel (1974) has proved (7.1) for rational matrices. The generalization to meromorphic matrices is straightforward and it is therefore omitted. □

Proof of Theorem 3.3. First note that the inequalities (3.1) can be written

$$|\varepsilon(F)(s_0)| + |\psi(G)(s_0)| > 0, \qquad |\varepsilon(G)(s_0)| + |\psi(F)(s_0)| > 0. \tag{7.2}$$

By Lemma 3.1, it is sufficient to show that

$$\mathrm{ord}_{s_0}[\psi(F)\psi(G)] \leq \mathrm{ord}_{s_0} \psi(FG). \tag{7.3}$$

We split the proof into three steps. The idea behind the first step is due to Coppel (1974).

Step 1. Realize that $\varepsilon(G) = \psi(G^{-1})$ and write $F = (FG)G^{-1}$. Then it follows from (7.1) that $\psi(F) \mid \psi(FG)\varepsilon(G)$, and hence (by (7.2))

$$\mathrm{ord}_{s_0} \psi(F) \leq \mathrm{ord}_{s_0} \psi(FG). \tag{7.4}$$

In exactly the same way, we can show

$$\operatorname{ord}_{s_0} \psi(G) \leq \operatorname{ord}_{s_0} \psi(FG). \tag{7.5}$$

Step 2. Note that $\varepsilon(F)$, $\psi(F)$, $\varepsilon(G)$, $\psi(G)$ $\varepsilon(FG)$, and $\psi(FG)$ all belong to $\mathrm{H}(\mathbb{C}_\sigma)$ for some $\sigma < 0$. In Step 2, '=' means equality up to units in $\mathrm{H}(\mathbb{C}_\sigma)$. Define

$$f := \gcd[\varepsilon(F), \psi(F)], \qquad g := \gcd[\varepsilon(G), \psi(G)].$$

The equation $\varepsilon(FG)/\psi(FG) = \varepsilon(F)\varepsilon(G)/\psi(F)\psi(G)$ yields

$$\varepsilon(FG) = \frac{[\varepsilon(F)/f][\varepsilon(G)/g]}{[\psi(F)/f][\psi(G)/g]}\psi(FG)$$

It follows from (7.2) that $d := \gcd[\varepsilon(F)\varepsilon(G)/fg,\ \psi(F)\psi(G)/fg]$ has no zero in s_0, i.e. $\operatorname{ord}_{s_0} d = 0$. Hence

$$\operatorname{ord}_{s_0}[\psi(F)\psi(G)/fg] \leq \operatorname{ord}_{s_0} \psi(FG). \tag{7.6}$$

Step 3. We show that (7.3) is true. We have to deal with three cases.

(i) $\operatorname{ord}_{s_0} f = \operatorname{ord}_{s_0} g = 0$: then

$\operatorname{ord}_{s_0}[\psi(F)\psi(G)] = \operatorname{ord}_{s_0}[\psi(F)\psi(G)/fg] \leq \operatorname{ord}_{s_0} \psi(FG)$, by (7.6).

(ii) $\operatorname{ord}_{s_0} f > 0$: then it follows from (7.2) that $\operatorname{ord}_{s_0} \psi(G) = 0$, and hence, by (7.4),

$$\operatorname{ord}_{s_0}[\psi(F)\psi(G)] = \operatorname{ord}_{s_0} \psi(F) \leq \operatorname{ord}_{s_0} \psi(FG).$$

(iii) $\operatorname{ord}_{s_0} g > 0$: using (7.5) we can show in exactly the same way as in (ii) that (7.3) holds true.