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Extending hysteresis operators to spaces of piecewise continuous functions ☆

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Abstract

We consider continuous-time hysteresis operators, defined to be causal and rate independent operators mapping input signals $u:\mathbb{R}_+ \to \mathbb{R}$ to output signals $v:\mathbb{R}_+ \to \mathbb{R}$. We show how a hysteresis operator defined on the set of continuous piecewise monotone functions can be naturally extended to piecewise continuous piecewise monotone functions. We prove that the extension is also a hysteresis operator and that a number of important properties of the original operator are inherited by the extension. Moreover, we define the concept of a discrete-time hysteresis operator and we show that discretizing continuous-time hysteresis operators using standard sampling and hold operations leads to discrete-time hysteresis operators. We apply the concepts and results described above in the context of sampled-data feedback control of linear systems with input hysteresis.

Keywords: Causality; Hysteresis operators; Piecewise continuous functions; Piecewise monotone functions; Rate independence; Sampled-data systems

1. Introduction

Generally speaking, hysteresis is a special type of memory-based relation between a scalar input signal $u(\cdot)$ and a scalar output signal $v(\cdot)$ that cannot be expressed in terms of a single-valued function but takes the form of "hysteresis" loops. The memory effects ex-

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hibited by hysteresis phenomena are rate independent in contrast to rate dependent memory which is typically fading and hence scale dependent. The concept of a hysteresis operator has been introduced in Brokate [3], Brokate and Sprekels [4], and Visintin [9], to be an operator which is both causal and rate independent. For continuous piecewise monotone input signals u this means that at time $t \in \mathbb{R}_+$, the value $v(t)$ of the output signal v is dependent only on the local extrema of u restricted to the time interval $[0,t]$. Hysteresis operators encompass nonlinearities important in applications such as relay (or passive), backlash (or play) and elastic-plastic hysteresis. This type of behaviour arises in mechanical plays, thermostats, elastoplasticity, ferromagnetism, and in smart material structures such as piezoelectric elements and magnetostrictive transducers (see Banks et al. [2] for hysteresis phenomena in smart materials). There exists a substantial literature on mathematical modelling and mathematical theory of hysteresis phenomena, see, for example, Brokate [3], Brokate and Sprekels [4], Krasnosel'skiĭ and Pokrovskiĭ [5], Macki et al. [8], and Visintin [9].

Hysteresis operators are defined on sets of piecewise monotone functions and usually these functions are assumed to be continuous. Whilst the domains of some of the hysteresis operators considered in [4] contain certain discontinuous functions, the framework in [4] excludes many functions which change their monotonicity behaviour at a point of discontinuity (see Remark 3.5). For certain applications such as sampled-data control for systems with hysteresis effects (see Section 5), it is desirable to extend hysteresis operators defined on the set of continuous piecewise monotone functions to the whole set of (normalized) piecewise continuous piecewise monotone functions. We show in this paper that this can be done in a natural way in great generality and that the extension is again a hysteresis operator inheriting a number of important properties of the original operator. Furthermore, we introduce a concept of discrete-time hysteresis and show that discretizing continuous-time hysteresis operators using standard sampling and hold operations yields discrete-time hysteresis operators.

In Section 2 of the paper we first introduce continuous-time hysteresis operators defined on a general domain and then restrict our attention to hysteresis operators on domains of continuous piecewise monotone functions. In contrast to most of the literature (see [4,9]), where hysteresis operators act on function spaces with a finite time horizon, we consider hysteresis operators acting on functions defined on the infinite time interval $[0,\infty)$. This is motivated by our interest in asymptotic properties of feedback control systems with hysteresis nonlinearities, see [6,7]. We show that for every hysteresis operator there exists a representing rate independent functional and derive a number of properties enjoyed by hysteresis operators. In Section 3 we consider hysteresis operators defined on the set of continuous piecewise monotone functions, extend the representing functional of such a hysteresis operator to a domain of piecewise continuous piecewise monotone functions and then use the extended functional to define an extension of the hysteresis operator. We show that the extension itself is a hysteresis operator and that other important properties of the original operator are inherited by the extension. In Section 4 we introduce the concept of a discrete-time hysteresis operator and give a discrete-time analogue of some of the continuous-time hysteresis results of Section 2. We show that the discretization of a continuous-time hysteresis operator obtained by standard sampling and hold operations is a discrete-time hysteresis operator. In Section 5 we apply some of the concepts and results developed in Sections 3 and 4 in the context of sampled-data feedback control of linear systems with input hysteresis.

Notation and terminology. We define

$$\mathbb{Z}_+ := \{x \in \mathbb{Z} \mid x \geqslant 0\}, \qquad \mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geqslant 0\}.$$

For sets M and N we denote the set of all functions $f : M \to N$ by $F(M, N)$. We say that a function $f \in F(\mathbb{R}_+, \mathbb{R})$ is *piecewise monotone* if there exists a sequence $0 = t_0 < t_1 < t_2 < \cdots$ such that $\lim_{i\to\infty} t_i = \infty$ and f is monotone on each of the open intervals (t_i, t_{i+1}). We say that a function $f \in F(\mathbb{R}_+, \mathbb{R})$ is *piecewise continuous* if there exists a sequence $0 = t_0 < t_1 < t_2 < \cdots$ such that $\lim_{i\to\infty} t_i = \infty$, f is continuous on each of the intervals (t_i, t_{i+1}), and the right and left limits of f exist and are finite at each t_i. The space of all piecewise continuous functions $f \in F(\mathbb{R}_+, \mathbb{R})$ is denoted by $\mathrm{PC}(\mathbb{R}_+, \mathbb{R})$. As usual, for $f \in \mathrm{PC}(\mathbb{R}_+, \mathbb{R})$, we define

$$f(t+) = \lim_{\tau \downarrow t} f(\tau) \quad (\text{for } t \geqslant 0) \quad \text{and} \quad f(t-) = \lim_{\tau \uparrow t} f(\tau) \quad (\text{for } t > 0).$$

Let $\mathbb{T} = \mathbb{R}_+, \mathbb{Z}_+$; a function $f \in F(\mathbb{T}, \mathbb{R})$ is called *ultimately constant* if there exists $T \in \mathbb{T}$ such that f is constant on $[T, \infty) \cap \mathbb{T}$.

Remark 1.1. Note that our concept of piecewise monotonicity is less restrictive than that in [4], where a piecewise monotone function is required to be monotone on the closed intervals $[t_i, t_{i+1}]$. Whilst the two definitions coincide for continuous functions, the definition in [4] seems to be somewhat unnatural in the context of discontinuous functions. For example, the functions f and g defined on $\mathbb{R}_+$ and given by

$$f(t) = \begin{cases} t & \text{if } t \in [0, 1), \\ 1 - t & \text{if } t \in [1, \infty) \end{cases} \quad \text{and} \quad g(t) = \begin{cases} t & \text{if } t \in [0, 1], \\ 3 - t & \text{if } t \in (1, \infty), \end{cases}$$

respectively, are piecewise monotone in our sense, but not in the sense of [4].

2. Continuous-time hysteresis operators

In this section we present basic background material on hysteresis operators which is needed for the subsequent developments in Sections 3 and 4. Our treatment of hysteresis operators is strongly influenced by Chapter 2 in book [4] by Brokate and Sprekels. Most of the results in this section can be found in a somewhat different and less general form in Chapter 2 of [4].

We call a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ a *time transformation* if f is continuous, nondecreasing and satisfies $f(0) = 0$ and $\lim_{t\to\infty} f(t) = \infty$, in other words $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a time transformation if and only if f is continuous, nondecreasing and surjective. We denote the set of all time transformations $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $\mathcal{T}$. For each $\tau \in \mathbb{R}_+$, we define a projection operator $Q_\tau : F(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ by

$$(Q_\tau u)(t) = \begin{cases} u(t) & \text{for } 0 \leqslant t \leqslant \tau, \\ u(\tau) & \text{for } t > \tau. \end{cases}$$

In the following let $\mathcal{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathcal{F} \neq \emptyset$. We introduce the following two assumptions on $\mathcal{F}$:

(**F1**) $u \circ f \in \mathcal{F}$ for all $u \in \mathcal{F}$ and all $f \in \mathcal{T}$;
(**F2**) $Q_t(\mathcal{F}) \subset \mathcal{F}$ for all $t \in \mathbb{R}_+$.

We call an operator $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ *causal* if for all $u, v \in \mathcal{F}$ and all $\tau \in \mathbb{R}_+$ with $u(t) = v(t)$ for all $t \in [0, \tau]$ it follows that $(\Phi(u))(t) = (\Phi(v))(t)$ for all $t \in [0, \tau]$. An operator $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ is called *rate independent* if $\mathcal{F}$ satisfies (F1) and

$$\big(\Phi(u \circ f)\big)(t) = \big(\Phi(u)\big)\big(f(t)\big), \quad \forall u \in \mathcal{F},\ \forall f \in \mathcal{T},\ \forall t \in \mathbb{R}_+.$$

A functional $\varphi : \mathcal{F} \to \mathbb{R}$ is called *rate independent* if $\mathcal{F}$ satisfies (F1) and

$$\varphi(u \circ f) = \varphi(u), \quad \forall u \in \mathcal{F},\ \forall f \in \mathcal{T}.$$

Definition 2.1. Let $\mathcal{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathcal{F} \neq \emptyset$. An operator $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ is called a *hysteresis operator* if $\mathcal{F}$ satisfies (F1) and Φ is causal and rate independent.

For $\mathcal{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathcal{F} \neq \emptyset$, let $\mathcal{F}^{\mathrm{uc}}$ denote the set of all ultimately constant $u \in \mathcal{F}$, i.e.,

$$\mathcal{F}^{\mathrm{uc}} = \{u \in \mathcal{F} \mid u \text{ is ultimately constant}\}.$$

Clearly, if $\mathcal{F}$ satisfies (F2), then $\mathcal{F}^{\mathrm{uc}} \neq \emptyset$. Moreover, if $\mathcal{F}$ satisfies (F1), then so does $\mathcal{F}^{\mathrm{uc}}$.

Theorem 2.2. *Let $\mathcal{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathcal{F} \neq \emptyset$, and assume that* (F1) *and* (F2) *hold. If $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ is a hysteresis operator, then the following statements hold*:

(1) *$Q_\tau \Phi = \Phi Q_\tau$ for all $\tau \in \mathbb{R}_+$;*
(2) *the functional*

$$\varphi : \mathcal{F}^{\mathrm{uc}} \to \mathbb{R}, \quad u \mapsto \lim_{t \to \infty} \big(\Phi(u)\big)(t), \tag{2.1}$$

is rate independent and satisfies

$$\big(\Phi(u)\big)(t) = \varphi(Q_t u), \quad \forall u \in \mathcal{F},\ \forall t \in \mathbb{R}_+. \tag{2.2}$$

Conversely, if $\varphi : \mathcal{F}^{\mathrm{uc}} \to \mathbb{R}$ is a rate independent functional, then $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ given by (2.2) *is a hysteresis operator and satisfies*

$$\lim_{t \to \infty} \big(\Phi(u)\big)(t) = \varphi(u), \quad \forall u \in \mathcal{F}^{\mathrm{uc}}. \tag{2.3}$$

For a hysteresis operator $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$, we call the rate independent functional $\varphi : \mathcal{F}^{\mathrm{uc}} \to \mathbb{R}$ defined by (2.1) the *representing functional* of Φ.

Remark 2.3. There exist causal operators Φ satisfying the commutativity property $Q_\tau \Phi = \Phi Q_\tau$ for all $\tau \in \mathbb{R}_+$, but which are not hysteresis operators. For example, consider the (linear) operator $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ given by

$$\big(\Phi(u)\big)(t) = \big(1 + a(t)\big)u(t) - \int_0^t (da/ds)(s)u(s)\,ds,$$

where $a:\mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable. Clearly, Φ is causal and a routine calculation shows that $Q_\tau \Phi = \Phi Q_\tau$ for all $\tau \in \mathbb{R}_+$. However, unless a is constant, Φ is not, in general, rate independent, and hence not a hysteresis operator.

Proof of Theorem 2.2. Assume that $\Phi:\mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ is a hysteresis operator. To prove statement (1), let $u \in \mathcal{F}$ and $\tau \in \mathbb{R}_+$. Trivially, since Φ is causal,

$$\big(\Phi(Q_\tau u)\big)(t) = \big(\Phi(u)\big)(t), \quad \forall t \in [0, \tau]. \tag{2.4}$$

Let $s > \tau$ and define a time transformation $f \in \mathcal{T}$ by

$$f(t) = \begin{cases} t & \text{for } 0 \leqslant t \leqslant \tau, \\ \tau & \text{for } \tau < t \leqslant s, \\ t + \tau - s & \text{for } t > s. \end{cases}$$

Then, using the causality and rate independence of Φ, we have for $t \in [\tau, s]$,

$$\big(\Phi(Q_\tau u)\big)(t) = \big(\Phi(u \circ f)\big)(t) = \big(\Phi(u)\big)\big(f(t)\big) = \big(\Phi(u)\big)(\tau). \tag{2.5}$$

Since $s > \tau$ was arbitrary, (2.5) holds for all $t \geqslant \tau$. Together with (2.4) this yields statement (1). To prove statement (2), we first note that the limit in (2.1) exists since for ultimately constant u, $\Phi(u)$ is ultimately constant by statement (1). Using the rate independence of Φ, we see that for all $u \in \mathcal{F}^{\mathrm{uc}}$ and all $f \in \mathcal{T}$,

$$\varphi(u \circ f) = \lim_{t\to\infty} \big(\Phi(u \circ f)\big)(t) = \lim_{t\to\infty} \big(\Phi(u)\big)\big(f(t)\big) = \lim_{t\to\infty} \big(\Phi(u)\big)(t) = \varphi(u),$$

showing that φ is rate independent. Using statement (1), we obtain for all $u \in \mathcal{F}$ and all $t \in \mathbb{R}_+$,

$$\big(\Phi(u)\big)(t) = \big(\Phi(Q_t u)\big)(t) = \lim_{s\to\infty} \big(\Phi(Q_t u)\big)(s) = \varphi(Q_t u),$$

which is (2.2).

Conversely, assume that $\varphi:\mathcal{F}^{\mathrm{uc}} \to \mathbb{R}$ is rate independent and define $\Phi:\mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ by (2.2), i.e., $(\Phi(u))(t) = \varphi(Q_t u)$. Then, trivially, Φ is causal. Moreover, for all $u \in \mathcal{F}$, $f \in \mathcal{T}$, and $t \in \mathbb{R}_+$,

$$\big(\Phi(u \circ f)\big)(t) = \varphi\big(Q_t(u \circ f)\big) = \varphi\big((Q_{f(t)}u) \circ f\big) = \varphi(Q_{f(t)}u) = \big(\Phi(u)\big)\big(f(t)\big),$$

thus Φ is rate independent. Finally, let $u \in \mathcal{F}^{\mathrm{uc}}$; then

$$\lim_{t\to\infty} \big(\Phi(u)\big)(t) = \lim_{t\to\infty} \varphi(Q_t u) = \varphi(u),$$

which is (2.3). □

Let $\mathcal{S}^r$ denote the set of all right-continuous step functions $u:\mathbb{R}_+ \to \mathbb{R}$, that is there exists a sequence $0 = t_0 < t_1 < t_2 < \cdots$ such that $\lim_{i\to\infty} t_i = \infty$ and u is constant on each of the intervals $[t_i, t_{i+1})$. For $\tau > 0$ define $\mathcal{S}^r_\tau \subset \mathcal{S}^r$ to be the set of all right-continuous step functions $u:\mathbb{R}_+ \to \mathbb{R}$ of step length τ, i.e., u is constant on each interval $[i\tau, (i+1)\tau)$. We note that whilst $\mathcal{S}^r$ satisfies (F1) and (F2), $\mathcal{S}^r_\tau$ satisfies (F2), but not (F1). The following corollary is an immediate consequence of Theorem 2.2(1).

Corollary 2.4. *Let $\mathcal{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathcal{F} \neq \emptyset$ and assume that* (F1) *and* (F2) *hold. Let $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator. Then*

$$\Phi(\mathcal{F}^{\mathrm{uc}}) \subset \mathcal{F}^{\mathrm{uc}}, \qquad \Phi(\mathcal{F} \cap \mathcal{S}^r) \subset \mathcal{S}^r, \qquad \Phi\big(\mathcal{F} \cap \mathcal{S}^r_\tau\big) \subset \mathcal{S}^r_\tau.$$

For any $u \in F(\mathbb{R}_+, \mathbb{R})$ and any $t \in \mathbb{R}_+$, we define

$$M(u,t) := \big\{\tau \in (t, \infty) \mid u \text{ is monotone on } (t, \tau)\big\}.$$

If u is piecewise monotone, then $M(u,t) \neq \emptyset$ for all $t \in \mathbb{R}_+$, and the *standard monotonicity partition* $t_0 < t_1 < t_2 < \cdots$ of u is defined recursively by setting $t_0 = 0$ and $t_{i+1} = \sup M(u, t_i)$ for all $i \in \mathbb{Z}_+$ such that $M(u, t_i)$ is bounded. If u is piecewise monotone and ultimately constant, then the standard monotonicity partition of u is finite. The set of all piecewise monotone $u \in C(\mathbb{R}_+, \mathbb{R})$ is denoted by $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. It can be shown that the sum of two continuous piecewise monotone functions is not necessarily piecewise monotone, implying that $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ is not a linear space. We define $C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ to be the set of all ultimately constant $u \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. We note that $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and $C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ both satisfy (F1) and (F2). Let $F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R})$ denote the space of ultimately constant $u : \mathbb{Z}_+ \to \mathbb{R}$. We define the *restriction operator* $R : C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R})$ by

$$\big(R(u)\big)(k) = \begin{cases} u(t_k) & \text{for } k \in [0, m] \cap \mathbb{Z}_+, \\ \lim_{t \to \infty} u(t) & \text{for } k \in \mathbb{Z}_+ \setminus [0, m], \end{cases}$$

where $0 = t_0 < t_1 < \cdots < t_m$ is the standard monotonicity partition of u.

The following lemma will be an important tool in the next section.

Lemma 2.5. *Let $u, v \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Then $R(u) = R(v)$ if and only if there exist $f, g \in \mathcal{T}$ such that $u \circ f = v \circ g$.*

The above lemma can be found in Brokate and Sprekels [4] (see [4, Lemma 2.2.4]).[2] As an immediate consequence of Lemma 2.5 and Theorem 2.2(2), we obtain the following corollary.

Corollary 2.6. *Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator and let $u, v \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$. Then*

$$R(Q_t u) = R(Q_t v) \quad \Rightarrow \quad \big(\Phi(u)\big)(t) = \big(\Phi(v)\big)(t).$$

The above corollary says that the output $(\Phi(u))(t)$ at time $t \in \mathbb{R}_+$ of a hysteresis operator Φ corresponding to a continuous piecewise monotone input u is determined completely by the local extrema of u restricted to the time interval $[0, t]$.

[2] Only a sketch of the proof is given in [4]. A complete proof can be found in the appendix of an extended version of the present paper contained in Mathematics Preprint 00/14 (University of Bath, 2000), which is available at http://www.maths.bath.ac.uk/mathematics/preprints.html.

3. Extending hysteresis operators defined on $C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ to piecewise continuous functions

Let $\mathrm{NPC}(\mathbb{R}_+,\mathbb{R}) \subset \mathrm{PC}(\mathbb{R}_+,\mathbb{R})$ denote the space of all *normalized piecewise continuous* functions $u:\mathbb{R}_+ \to \mathbb{R}$, that is u is piecewise continuous and is right-continuous or left-continuous at each point $t \in \mathbb{R}_+$. In particular, if $u \in \mathrm{NPC}(\mathbb{R}_+,\mathbb{R})$, then u is right-continuous at $t = 0$. The set of all piecewise monotone functions $u \in \mathrm{NPC}(\mathbb{R}_+,\mathbb{R})$ is denoted by $\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$, whilst $\mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ denotes the set of all ultimately constant $u \in \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$. We note that $\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ and $\mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ both satisfy (F1) and (F2). For $u \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$, we define $u(\infty) := \lim_{\tau\to\infty} u(\tau)$.

Lemma 3.1. *Let $u \in \mathrm{NPC}(\mathbb{R}_+,\mathbb{R})$, $f \in \mathcal{T}$, and $t > 0$. Define*

$$\tau = \begin{cases} \max f^{-1}(\{t\}) & \text{if } u(t-) = u(t), \\ \min f^{-1}(\{t\}) & \text{if } u(t-) \neq u(t). \end{cases}$$

Then $(u \circ f)(\tau+) = u(t+)$ and $(u \circ f)(\tau-) = u(t-)$.

Proof. Since f is continuous, nondecreasing and surjective, for all $t \in \mathbb{R}_+$, $f^{-1}(\{t\})$ is a compact interval and therefore τ is well defined. We consider two cases.

Case 1. Suppose that $u(t-) = u(t)$. Then, $f(\tau + h) > t$ for all $h > 0$ and so $(u \circ f)(\tau+) = u(f(\tau)+) = u(t+)$. Moreover, if $f^{-1}(\{t\})$ is a singleton, we have $f(\tau - h) < t$ for all $h \in (0, \tau]$ and so $(u \circ f)(\tau-) = u(f(\tau)-) = u(t-)$. If $f^{-1}(\{t\})$ is not a singleton, we have $f(\tau - h) = t$ for all sufficiently small $h > 0$ and so $(u \circ f)(\tau-) = u(t) = u(t-)$.

Case 2. Suppose that $u(t-) \neq u(t)$. Then, since $u \in \mathrm{NPC}(\mathbb{R}_+,\mathbb{R})$, it follows that $u(t+) = u(t)$. Adopting an argument similar to that in Case 1 yields the claim. □

Let $u \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ and let $0 = t_0 < t_1 < \cdots < t_m$ be the standard monotonicity partition of u. We define the map $\rho : \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}) \to F^{\mathrm{uc}}(\mathbb{Z}_+,\mathbb{R})$ by

$$\rho(u) = \bigl(u(t_0), u(t_1-), u(t_1+), u(t_2-), u(t_2+), \ldots, \\ u(t_m-), u(t_m+), u(\infty), u(\infty), \ldots\bigr).$$

Let $\tau > 0$. We define the *prolongation operator* $P_\tau : F(\mathbb{Z}_+,\mathbb{R}) \to C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ by letting $P_\tau u$ be the piecewise affine-linear function satisfying $(P_\tau u)(i\tau) = u(i)$ and having constant slope on $(i\tau, (i+1)\tau)$ for all $i \in \mathbb{Z}_+$. Moreover, we introduce the operator

$$\tilde{R} : \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}) \to F^{\mathrm{uc}}(\mathbb{Z}_+,\mathbb{R}), \quad u \mapsto R\bigl((P_\tau \circ \rho)(u)\bigr).$$

Clearly, for any $\tau > 0$,

$$R \circ P_\tau \circ R = R, \tag{3.1}$$

and using (3.1) it is easy to show that $R \circ P_\tau \circ \tilde{R} = \tilde{R}$.

For illustration, consider the function u shown in Fig. 1. Clearly, $u \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ with standard monotonicity partition $0 = t_0 < t_1 < t_2 < t_3 < t_4$. The sequences $\rho(u)$ and $\tilde{R}(u)$ are given by

$$\rho(u) = (u_0, u_7, u_6, u_4, u_4, u_5, u_3, u_7, u_2, u_1, u_1, u_1, \ldots)$$

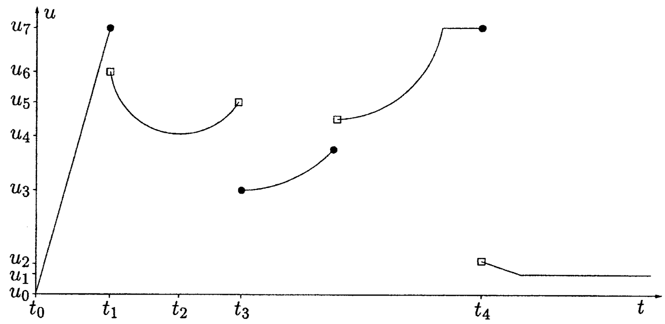

Fig. 1. Example of a function in $\mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$.

and

$$\tilde{R}(u) = (u_0, u_7, u_4, u_5, u_3, u_7, u_1, u_1, u_1, \ldots).$$

Lemma 3.2. *$\tilde{R}$ is an extension of R, the definition of $\tilde{R}$ does not depend upon the choice of $\tau > 0$ and*

$$\tilde{R}(u \circ f) = \tilde{R}(u), \quad \forall u \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}),\ \forall f \in \mathcal{T}. \tag{3.2}$$

Proof. Let $u \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$, then, since $u(t+) = u(t-)$ for all $t > 0$,

$$\tilde{R}(u) = R\big((P_\tau \circ \rho)(u)\big) = R(u),$$

showing that $\tilde{R}$ is an extension of R. Let $\tau_1, \tau_2 > 0$ and $u \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$. Clearly, $R((P_{\tau_1} \circ \rho)(u)) = R((P_{\tau_2} \circ \rho)(u))$, from which it follows that the definition of $\tilde{R}$ does not depend upon the choice of $\tau > 0$.

Finally, let $u \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$, $f \in \mathcal{T}$, and let $0 = t_0 < t_1 < \cdots < t_m$ be the standard monotonicity partition of u. Define $\tau_0 := 0$ and for $i = 1, \ldots, m$,

$$\tau_i := \begin{cases} \max f^{-1}(\{t_i\}) & \text{if } u(t_i-) = u(t_i), \\ \min f^{-1}(\{t_i\}) & \text{if } u(t_i-) \neq u(t_i). \end{cases}$$

Then $0 = \tau_0 < \tau_1 < \cdots < \tau_m$ is the standard monotonicity partition of $u \circ f$ and by Lemma 3.1, $(u \circ f)(\tau_i \pm) = u(t_i \pm)$ for $i = 0, 1, \ldots, m$. Hence, $\rho(u) = \rho(u \circ f)$, and therefore, $\tilde{R}(u) = \tilde{R}(u \circ f)$, showing that (3.2) holds. $\Box$

For any rate independent $\varphi : C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}) \to \mathbb{R}$ we define

$$\tilde{\varphi} : \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}) \to \mathbb{R}, \quad u \mapsto \varphi\big((P_\tau \circ \tilde{R})(u)\big), \tag{3.3}$$

where $\tau > 0$. We show that the definition of $\tilde{\varphi}$ does not depend on τ. To this end, let $\tau_1, \tau_2 > 0$ and $u \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$. Then, clearly there exists $f \in \mathcal{T}$ such that $(P_{\tau_1} \circ \tilde{R})(u) = (P_{\tau_2} \circ \tilde{R})(u) \circ f$ and therefore, by the rate independence of φ,

$$\varphi\big((P_{\tau_1} \circ \tilde{R})(u)\big) = \varphi\big((P_{\tau_2} \circ \tilde{R})(u)\big).$$

Lemma 3.3. *Let $\varphi : C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}$ be rate independent and define $\tilde{\varphi}$ by* (3.3). *Then*

(1) *$\tilde{\varphi}$ is an extension of φ, i.e.,*

$$\tilde{\varphi}(u) = \varphi(u), \quad \forall u \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R});$$

(2) *for any $\tau > 0$,*

$$\tilde{\varphi}(u) = \varphi\big((P_\tau \circ \rho)(u)\big), \quad \forall u \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R});$$

(3) *for $u, v \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$,*

$$\tilde{R}(u) = \tilde{R}(v) \quad \Rightarrow \quad \tilde{\varphi}(u) = \tilde{\varphi}(v);$$

(4) *$\tilde{\varphi}$ is rate independent, i.e.,*

$$\tilde{\varphi}(u \circ f) = \tilde{\varphi}(u), \quad \forall u \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}),\ \forall f \in \mathcal{T}.$$

Proof. Let $\tau > 0$ and $u \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Clearly, $R(u) = R((P_\tau \circ R)(u))$, and so using Lemma 2.5, there exist $f, g \in \mathcal{T}$ such that $u \circ f = (P_\tau \circ R)(u) \circ g$. Thus the rate independence of φ in combination with Lemma 3.2 gives

$$\tilde{\varphi}(u) = \varphi\big((P_\tau \circ \tilde{R})(u)\big) = \varphi\big((P_\tau \circ R)(u) \circ g\big) = \varphi(u \circ f) = \varphi(u),$$

which is statement (1). To prove statement (2), let $u \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. By definition $\tilde{R}(u) = R((P_\tau \circ \rho)(u))$ and therefore,

$$(P_\tau \circ \tilde{R})(u) = P_\tau\big(R\big((P_\tau \circ \rho)(u)\big)\big).$$

Thus, invoking (3.1),

$$R\big((P_\tau \circ \tilde{R})(u)\big) = R\big(P_\tau\big(R\big((P_\tau \circ \rho)(u)\big)\big)\big) = R\big((P_\tau \circ \rho)(u)\big),$$

and so using the rate independence of φ and Lemma 2.5,

$$\tilde{\varphi}(u) = \varphi\big((P_\tau \circ \tilde{R})(u)\big) = \varphi\big((P_\tau \circ \rho)(u)\big).$$

For statement (3), let $u, v \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Suppose that $\tilde{R}(u) = \tilde{R}(v)$; then by the definition of $\tilde{R}$, $R((P_\tau \circ \rho)(u)) = R((P_\tau \circ \rho)(v))$. Since $(P_\tau \circ \rho)(u), (P_\tau \circ \rho)(v) \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, it follows from an application of Lemma 2.5, the rate independence of φ, and statement (2) that

$$\tilde{\varphi}(u) = \varphi\big((P_\tau \circ \rho)(u)\big) = \varphi\big((P_\tau \circ \rho)(v)\big) = \tilde{\varphi}(v).$$

Statement (4) follows immediately from (3.2) and statement (3). □

Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator and define

$$\tilde{\Phi} : \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$$

by setting

$$\big(\tilde{\Phi}(u)\big)(t) = \tilde{\varphi}(Q_t u), \quad \forall t \in \mathbb{R}_+, \tag{3.4}$$

where φ is the representing functional of Φ and $\tilde{\varphi}$ is the extension of φ to $\mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ given by (3.3).

Theorem 3.4. *Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator and define the operator $\tilde{\Phi} : \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ by* (3.4). *Then*

(1) *$\tilde{\Phi}$ is an extension of Φ;*
(2) *$\tilde{\Phi}$ is a hysteresis operator with representing functional $\tilde{\varphi}$;*
(3) *for $u, v \in \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$,*

$$\tilde{R}(Q_t u) = \tilde{R}(Q_t v) \quad \Rightarrow \quad \big(\tilde{\Phi}(u)\big)(t) = \big(\tilde{\Phi}(v)\big)(t);$$

(4) *$\tilde{\Phi}(\mathcal{S}^r) \subset \mathcal{S}^r$ and $\tilde{\Phi}(\mathcal{S}^r_\tau) \subset \mathcal{S}^r_\tau$.*

Proof. Statement (1) is clear since $\tilde{\varphi}$ is an extension of φ and thus $\tilde{\Phi}$ is an extension of Φ. By Lemma 3.3(4), $\tilde{\varphi}$ is rate independent. Therefore, by Theorem 2.2, $\tilde{\Phi}$ is a hysteresis operator with representing functional $\tilde{\varphi}$, showing that statement (2) holds. Statement (3) follows from the definition of $\tilde{\Phi}$ and Lemma 3.3(3). Since $\mathcal{S}^r_\tau \subset \mathcal{S}^r \subset \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, statement (4) follows from statement (2) combined with Corollary 2.4. □

Remark 3.5. We mention that whilst the domains of the hysteresis operators defined in [4] include certain discontinuous functions, they do not contain the set $\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. This is due to the more restrictive concept of piecewise monotonicity adopted in [4] (see Remark 1.1), excluding many functions which change their monotonicity behaviour at a point of discontinuity (two examples of such functions are given in Remark 1.1).

In the following we define continuous, piecewise monotone "approximations" $u_1, u_2, u_3, \ldots$ of a given function $u \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ such that $\Phi(u_k)$ "approximates" $\tilde{\Phi}(u)$ as $k \to \infty$. Let $0 < \tau_1 < \tau_2 < \cdots < \tau_n$ denote the points of discontinuity of u and set $\tau_0 := 0$. For each $k \in \mathbb{Z}_+$, define

$$\varepsilon_k := \frac{1}{k+2} \min_{1 \leqslant i \leqslant n} (\tau_i - \tau_{i-1}). \tag{3.5}$$

For each $k \in \mathbb{Z}_+$, define an operator $C_k : \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ by setting

(1) if $t \in [\tau_j - \varepsilon_k, \tau_j)$ and u is right-continuous at τ_j, then

$$\big(C_k(u)\big)(t) = \begin{cases} \lambda[\tau_j - \varepsilon_k, \tau_j - \varepsilon_k/2; u(\tau_j - \varepsilon_k), u(\tau_j -)](t), \\ \quad t \in [\tau_j - \varepsilon_k, \tau_j - \varepsilon_k/2], \\ \lambda[\tau_j - \varepsilon_k/2, \tau_j; u(\tau_j -), u(\tau_j)](t), \\ \quad t \in [\tau_j - \varepsilon_k/2, \tau_j), \end{cases}$$

(2) if $t \in (\tau_j, \tau_j + \varepsilon_k]$ and u is left-continuous at τ_j, then

$$\big(C_k(u)\big)(t) = \begin{cases} \lambda[\tau_j, \tau_j + \varepsilon_k/2; u(\tau_j), u(\tau_j +)](t), \\ \quad t \in (\tau_j, \tau_j + \varepsilon_k/2], \\ \lambda[\tau_j + \varepsilon_k/2, \tau_j + \varepsilon_k; u(\tau_j +), u(\tau_j + \varepsilon_k)](t), \\ \quad t \in [\tau_j + \varepsilon_k/2, \tau_j + \varepsilon_k], \end{cases}$$

(3) $(C_k(u))(t) = u(t)$ otherwise,

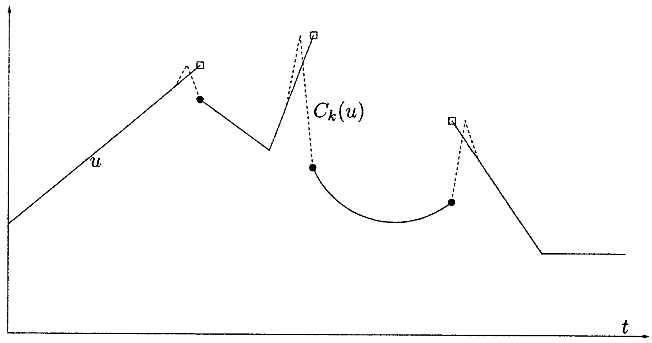

Fig. 2. Example of a function $u \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and its approximation $C_k(u)$.

where, in (1) and (2), $\lambda[t_1, t_2; a_1, a_2]$ (with $t_1, t_2, a_1, a_2 \in \mathbb{R}$, $t_1 \neq t_2$) denotes the affine-linear function defined on $\mathbb{R}$ and satisfying $\lambda[t_1, t_2; a_1, a_2](t_i) = a_i$, $i = 1, 2$. See Fig. 2 for an illustration of the operator C_k.

Lemma 3.6. *Let* $u \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. *Then*

(1) *for any* $t \in \mathbb{R}_+$, *there exists* $l > 0$ *such that*

$$\tilde{R}(Q_t u) = R\big(Q_t C_k(u)\big), \quad \forall k \geqslant l;$$

(2) *for any* $t_2 > t_1 \geqslant 0$, *if* u *is continuous on* $[t_1, t_2]$, *there exists* $l > 0$ *such that*

$$\tilde{R}(Q_t u) = R\big(Q_t C_k(u)\big), \quad \forall t \in [t_1, t_2],\ \forall k \geqslant l.$$

Proof. Let $u \in \mathrm{NPC}^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, let $0 < \tau_1 < \tau_2 < \cdots < \tau_n$ denote the points of discontinuity of u and let $0 = t_0 < t_1 < \cdots < t_m$ be the standard monotonicity partition of u. Define ε_k by (3.5).

To prove statement (1), let $t \in \mathbb{R}_+$ and choose $l > 0$ such that

$$\varepsilon_l < \min\big\{|t_i - \tau_j| \mid 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n,\ t_i \neq \tau_j\big\}$$

and

$$\varepsilon_l < \min\big\{|t - \tau_j| \mid 1 \leqslant j \leqslant n,\ t \neq \tau_j\big\}.$$

Then $\tilde{R}(Q_t u) = R(Q_t C_k(u))$ for all $k \geqslant l$.

To prove statement (2), let $t_2 > t_1 \geqslant 0$ be such that u is continuous on $[t_1, t_2]$. Hence, there exists $l_1 > 0$ such that

$$C_k(u)|_{[t_1,t_2]} = u|_{[t_1,t_2]}, \quad \forall k \geqslant l_1. \tag{3.6}$$

Moreover, by statement (1), there exists $l_2 > 0$ such that

$$\tilde{R}(Q_{t_1} u) = R\big(Q_{t_1} C_k(u)\big), \quad \forall k \geqslant l_2. \tag{3.7}$$

Hence, by (3.6) and (3.7), statement (2) holds for $l := \max(l_1, l_2)$. $\square$

For $u \in \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, $t > 0$, and $\varepsilon > 0$, we define

$$J_\varepsilon(u,t) := \bigcup_{i=1}^{n} (\tau_i - \varepsilon, \tau_i + \varepsilon) \quad \text{and} \quad d(u,t) := \min_{1 \leqslant i \leqslant n-1} (\tau_{i+1} - \tau_i)/2,$$

where $0 < \tau_1 < \tau_2 < \cdots < \tau_n$ denote the points of discontinuity of $Q_t u$.

Proposition 3.7. *Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator. Then, for an arbitrary $u \in \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, the following statements hold*:

(1) *for all $t \in \mathbb{R}_+$, there exists $l > 0$ such that*

$$\big(\tilde{\Phi}(u)\big)(t) = \big(\Phi\big(C_k(Q_t u)\big)\big)(t), \quad \forall k \geqslant l;$$

(2) *if for $t_3 \geqslant t_2 > t_1 \geqslant 0$, u is continuous on $[t_1, t_2]$, then there exists $l > 0$ such that*

$$\big(\tilde{\Phi}(u)\big)(s) = \big(\Phi\big(C_k(Q_{t_3} u)\big)\big)(s), \quad \forall s \in [t_1, t_2],\ \forall k \geqslant l;$$

(3) *for all $t \in \mathbb{R}_+$ and all $\varepsilon \in (0, d(u,t))$, there exists $l > 0$ such that*

$$\big(\tilde{\Phi}(u)\big)(s) = \big(\Phi\big(C_k(Q_t u)\big)\big)(s), \quad \forall s \in [0,t) \setminus J_\varepsilon(u,t),\ \forall k \geqslant l.$$

Proof. Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator and $u \in \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Statement (1) follows from Theorem 3.4 and Lemma 3.6(1), and statement (2) follows from Theorem 3.4 and Lemma 3.6(2). For statement (3), let $t \in \mathbb{R}_+$, $\varepsilon \in (0, d(u,t))$, and let $0 < \tau_1 < \tau_2 < \cdots < \tau_n$ denote the points of discontinuity of $Q_t u$. Clearly u is continuous on $[\tau_i + \varepsilon, \tau_{i+1} - \varepsilon]$ for $1 \leqslant i \leqslant n-1$. Therefore by statement (2) and the causality of Φ, there exists $l_i > 0$ such that for $1 \leqslant i \leqslant n-1$,

$$\big(\tilde{\Phi}(u)\big)(s) = \big(\Phi\big(C_k(Q_t u)\big)\big)(s), \quad \forall s \in [\tau_i + \varepsilon, \tau_{i+1} - \varepsilon],\ \forall k \geqslant l_i.$$

To conclude the proof, we distinguish between two cases: $\tau_n + \varepsilon < t$ and $\tau_n + \varepsilon \geqslant t$.

If $\tau_n + \varepsilon < t$, then u is continuous on $[\tau_n + \varepsilon, t]$ and therefore again by statement (2), there exists $l_n > 0$ such that

$$\big(\tilde{\Phi}(u)\big)(s) = \big(\Phi\big(C_k(Q_t u)\big)\big)(s), \quad \forall s \in [\tau_n + \varepsilon, t],\ \forall k \geqslant l_n.$$

If $\tau_n + \varepsilon \geqslant t$, then set $l_n := 0$.

In both cases define $l := \max_{1 \leqslant i \leqslant n} l_i$ and statement (3) then follows. □

The operator $\tilde{\Phi}$ defined by (3.4), extending a given hysteresis operator $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ to $\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, is not unique. There are other hysteretic extensions of Φ to $\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ as the following example shows.

Example 3.8. Define $Z_{\mathrm{e}} : \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ by

$$\big(Z_{\mathrm{e}}(u)\big)(t) = \begin{cases} 0 & \text{if } t = 0, \\ \sum_{0 < \tau \leqslant t} (u(\tau) - u(\tau-)) & \text{if } t > 0. \end{cases}$$

Clearly, Z_{e} is a causal extension of the trivial operator

$$Z : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R}), \quad u \mapsto 0.$$

We show that Z_{e} is rate independent. Let $u \in \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, $f \in \mathcal{T}$, $t > 0$, and let $0 < t_1 < \cdots < t_m \leqslant f(t)$ be the points at which $Q_{f(t)}u$ is not left-continuous. Define, for $i = 1, \ldots, m$, $\tau_i := \min f^{-1}(\{t_i\})$. Then $0 < \tau_1 < \cdots < \tau_m \leqslant t$ are the points at which $Q_t(u \circ f)$ is not left-continuous. By Lemma 3.1, $(u \circ f)(\tau_i -) = u(t_i -)$ for $i = 1, \ldots, m$, and thus

$$\begin{aligned}\big(Z_{\mathrm{e}}(u)\big)\big(f(t)\big) &= \sum_{0<\tau\leqslant f(t)} \big(u(\tau) - u(\tau-)\big) = \sum_{i=1}^{m} \big(u(t_i) - u(t_i-)\big) \\ &= \sum_{i=1}^{m} \big((u \circ f)(\tau_i) - (u \circ f)(\tau_i-)\big) \\ &= \sum_{0<\tau\leqslant t} \big((u \circ f)(\tau) - (u \circ f)(\tau-)\big) = \big(Z_{\mathrm{e}}(u \circ f)\big)(t),\end{aligned}$$

showing that Z_{e} is rate independent. Therefore Z_{e} is a hysteresis operator, but $Z_{\mathrm{e}} \neq \tilde{Z} = 0$. It follows that if $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ is a hysteresis operator, then $\tilde{\Phi} + Z_{\mathrm{e}}$, as well as $\tilde{\Phi}$, are hysteresis operators which extend Φ to $\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$.

The following corollary says that, given a hysteresis operator Φ on $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, $\tilde{\Phi}$ is the unique operator extending Φ to $\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and satisfying statement (3) of Theorem 3.4.

Corollary 3.9. *Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator. Suppose that Φ_{e} is an extension of Φ to* $\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. *If for all $u, v \in \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and all $t \in \mathbb{R}_+$,*

$$\tilde{R}(Q_t u) = \tilde{R}(Q_t v) \quad \Rightarrow \quad \big(\Phi_{\mathrm{e}}(u)\big)(t) = \big(\Phi_{\mathrm{e}}(v)\big)(t), \tag{3.8}$$

then $\Phi_{\mathrm{e}} = \tilde{\Phi}$.

Proof. Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator, let Φ_{e} be an extension of Φ to $\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ satisfying (3.8), let $u \in \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$. By Lemma 3.6(1), for all sufficiently large k, we have

$$\tilde{R}(Q_t u) = \tilde{R}\big(Q_t C_k(Q_t u)\big).$$

Hence, by (3.8), for all sufficiently large k,

$$\big(\Phi_{\mathrm{e}}(u)\big)(t) = \big(\Phi_{\mathrm{e}}\big(C_k(Q_t u)\big)\big)(t) = \big(\Phi\big(C_k(Q_t u)\big)\big)(t).$$

It follows from Proposition 3.7(1), that $(\Phi_{\mathrm{e}}(u))(t) = (\tilde{\Phi}(u))(t)$. □

Corollary 3.10. *Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator. Assume that $\Phi(C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$. Then*

$$\tilde{\Phi}\big(\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})\big) \subset \mathrm{NPC}(\mathbb{R}_+, \mathbb{R}),$$

and for $u \in \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, right-continuity of u at $t \in \mathbb{R}_+$ (respectively, left-continuity at $t > 0$) implies right-continuity of $\tilde{\Phi}(u)$ at $t \in \mathbb{R}_+$ (respectively, left-continuity at $t > 0$).

Proof. Let $\Phi : C_{\rm pm}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator and assume that the inclusion $\Phi(C_{\rm pm}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$ holds. Let $u \in {\rm NPC}_{\rm pm}(\mathbb{R}_+, \mathbb{R})$. We proceed in four steps.

Step 1. Let us suppose that u is right-continuous at $t \in \mathbb{R}_+$; then there exists $\tau > t$ such that u is continuous on $[t, \tau]$. So by Proposition 3.7(2), there exist $l > 0$ such that

$$\big(\tilde{\Phi}(u)\big)(s) = \big(\Phi\big(C_k(Q_\tau u)\big)\big)(s), \quad \forall s \in [t, \tau],\ \forall k \geqslant l.$$

Since by assumption $\Phi(C_{\rm pm}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$, $\Phi(C_k(Q_\tau u))$ is a continuous function and so $\tilde{\Phi}(u)$ is right-continuous at t.

Step 2. Similarly, if u is left-continuous at $t > 0$, then it is easy to show that $\tilde{\Phi}(u)$ is left-continuous at t.

Step 3. Assume that u is left-continuous at $t > 0$. We show that the right limit $\lim_{s\downarrow t}(\tilde{\Phi}(u))(s)$ exists and is finite. To this end, define $w = u$ on $\mathbb{R}_+ \setminus \{t\}$ and $w(t) = \lim_{s\downarrow t} u(s)$. Thus w is right-continuous at t. Now $\tilde{R}(Q_\tau u) = \tilde{R}(Q_\tau w)$ for all $\tau \in \mathbb{R}_+ \setminus \{t\}$ and therefore by Theorem 3.4(3), $\tilde{\Phi}(u) = \tilde{\Phi}(w)$ on $\mathbb{R}_+ \setminus \{t\}$. Thus

$$\lim_{s\downarrow t}\big(\tilde{\Phi}(u)\big)(s) = \lim_{s\downarrow t}\big(\tilde{\Phi}(w)\big)(s) = \big(\tilde{\Phi}(w)\big)(t),$$

since $\tilde{\Phi}(w)$ is right-continuous at t by Step 1.

Step 4. Similarly, if u is right-continuous at $t > 0$, then it is easy to show that the left limit $\lim_{s\uparrow t}(\tilde{\Phi}(u))(s)$ exists and is finite. □

We end this section by considering the extension of the backlash (or play) operator.

Example 3.11. Let $h \in \mathbb{R}_+$ and $\xi \in \mathbb{R}$. Defining the function $b_h : \mathbb{R}^2 \to \mathbb{R}$ by

$$b_h(v, w) = \max\big\{v - h, \min\{v + h, w\}\big\}, \tag{3.9}$$

we introduce the backlash (or play) operator $\mathcal{B}_{h,\xi} : C_{\rm pm}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ by setting

$$\big(\mathcal{B}_{h,\xi}(u)\big)(t) = \begin{cases} b_h(u(0), \xi) & \text{for } t = 0, \\ b_h(u(t), (\mathcal{B}_{h,\xi}(u))(t_i)) & \text{for } t_i < t \leqslant t_{i+1},\ i \in \mathbb{Z}_+, \end{cases}$$

where $0 = t_0 < t_1 < t_2 < \cdots$ is such that $\lim_{n\to\infty} t_n = \infty$ and u is monotone on each interval (t_i, t_{i+1}). We remark that ξ plays the role of an “initial state.” It is not difficult to show that the definition is independent of the choice of the partition (t_i); see [6]. It is well known that $\mathcal{B}_{h,\xi}$ is a hysteresis operator; see, for example, [4]. The backlash operator $\mathcal{B}_{h,\xi}$ is illustrated in Fig. 3.

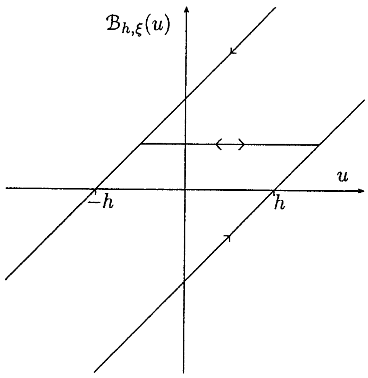

Fig. 3. Backlash hysteresis.

By Theorem 3.4, the extension $\tilde{\mathcal{B}}_{h,\xi}$ of $\mathcal{B}_{h,\xi}$ to $\mathrm{NPC_{pm}}(\mathbb{R}_+, \mathbb{R})$ given by (3.4) is a hysteresis operator. By Corollary 3.10, $\tilde{\mathcal{B}}_{h,\xi}(\mathrm{NPC_{pm}}(\mathbb{R}_+, \mathbb{R})) \subset \mathrm{NPC}(\mathbb{R}_+, \mathbb{R})$. A lengthy, but straightforward argument[3] shows that $\tilde{\mathcal{B}}_{h,\xi}$ can be written recursively as

$$\big(\tilde{\mathcal{B}}_{h,\xi}(u)\big)(t) = \begin{cases} b_h(u(0), \xi) & \text{for } t = 0, \\ b_h(u(t), (\tilde{\mathcal{B}}_{h,\xi}(u))(0)) & \text{for } 0 < t < t_1, \\ b_h(u(t_i), (\tilde{\mathcal{B}}_{h,\xi}(u))(t_i-)) & \text{for } t = t_i,\ i \in \mathbb{Z}_+ \setminus \{0\}, \\ b_h(u(t), b_h(u(t_i+), (\tilde{\mathcal{B}}_{h,\xi}(u))(t_i-))) & \\ \quad \text{for } t_i < t < t_{i+1},\ i \in \mathbb{Z}_+ \setminus \{0\}, & \end{cases} \tag{3.10}$$

where $0 = t_0 < t_1 < t_2 < \cdots$ is such that $\lim_{n\to\infty} t_n = \infty$ and u is monotone on each interval (t_i, t_{i+1}).

4. Discrete-time hysteresis operators

We call a function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ a (*discrete-time*) *time transformation* if f is surjective and nondecreasing. We denote the set of all discrete-time transformations $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ by $\mathcal{T}^d$. For each $k \in \mathbb{Z}_+$, we define a (discrete-time) projection operator $Q_k^d : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ by

$$\big(Q_k^d u\big)(n) = \begin{cases} u(n) & \text{for } n \in [0, k] \cap \mathbb{Z}_+, \\ u(k) & \text{for } m \in \mathbb{Z}_+ \setminus [0, k]. \end{cases}$$

[3] This argument is spelt out in detail in the appendix of an extended version of the present paper contained in Mathematics Preprint 00/14 (University of Bath, 2000), which is available at http://www.maths.bath.ac.uk/mathematics/preprints.html.

We call an operator $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ *causal* if for all $u, v \in F(\mathbb{Z}_+, \mathbb{R})$ and all $k \in \mathbb{Z}_+$ with $u(n) = v(n)$ for all $n \in [0, k] \cap \mathbb{Z}_+$ it follows that $(\Phi(u))(n) = (\Phi(v))(n)$ for all $n \in [0, k] \cap \mathbb{Z}_+$. An operator $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ is called *rate independent* if

$$\big(\Phi(u \circ f)\big)(n) = \big(\Phi(u)\big)\big(f(n)\big), \quad \forall u \in F(\mathbb{Z}_+, \mathbb{R}),\ \forall f \in \mathcal{T}^d,\ \forall n \in \mathbb{Z}_+.$$

Definition 4.1. An operator $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ is called a (*discrete-time*) *hysteresis operator* if Φ is causal and rate independent.

Recall that $F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R})$ denotes the set of all ultimately constant $u \in F(\mathbb{Z}_+, \mathbb{R})$. A functional $\varphi : F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R}) \to \mathbb{R}$ is called *rate independent* if

$$\varphi(u \circ f) = \varphi(u), \quad \forall u \in F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R}),\ \forall f \in \mathcal{T}^d.$$

The proof of the following theorem is analogous to the proof of Theorem 2.2 and is therefore omitted.

Theorem 4.2. *If $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ is a hysteresis operator, then*

(1) $Q_k^d \Phi = \Phi Q_k^d$ *for all* $k \in \mathbb{Z}_+$;
(2) *The functional*

$$\varphi : F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R}) \to \mathbb{R}, \quad u \mapsto \lim_{n\to\infty} \big(\Phi(u)\big)(n), \tag{4.1}$$

is rate independent and satisfies

$$\big(\Phi(u)\big)(n) = \varphi\big(Q_n^d u\big), \quad \forall u \in F(\mathbb{Z}_+, \mathbb{R}),\ \forall n \in \mathbb{Z}_+. \tag{4.2}$$

Conversely, if $\varphi : F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R}) \to \mathbb{R}$ is a rate independent functional, then $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ given by (4.2) *is a hysteresis operator and satisfies*

$$\lim_{n\to\infty} \big(\Phi(u)\big)(n) = \varphi(u), \quad \forall u \in F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R}).$$

For a hysteresis operator $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$, we call the rate independent functional $\varphi : F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R}) \to \mathbb{R}$ defined by (4.1) the *representing functional* of Φ.

Let $\tau > 0$. The τ-*hold* operator $H_\tau : F(\mathbb{Z}_+, \mathbb{R}) \to \mathcal{S}_\tau^r$ is defined by

$$(H_\tau u)(n\tau + t) = u(n), \quad \forall n \in \mathbb{Z}_+,\ \forall t \in [0, \tau), \tag{4.3}$$

and the τ-*sampling* operator $S_\tau : F(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ by

$$(S_\tau u)(n) = u(n\tau), \quad \forall n \in \mathbb{Z}_+. \tag{4.4}$$

The above hold and sampling operations are standard in the context of sampled-data control where continuous-time systems are controlled by discrete-time controllers via hold and sampling mechanisms.

Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a continuous-time hysteresis operator and define $\Phi^d : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ by

$$\Phi^d := S_\tau \tilde{\Phi} H_\tau, \tag{4.5}$$

where $\tilde{\Phi}$ is the extension of Φ to $\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ defined by (3.4). The definition of Φ^d is independent of the choice of τ due to the rate independence of $\tilde{\Phi}$.

Proposition 4.3. *Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}) \to F(\mathbb{R}_+,\mathbb{R})$ be a continuous-time hysteresis operator. Then $\Phi^d : F(\mathbb{Z}_+,\mathbb{R}) \to F(\mathbb{Z}_+,\mathbb{R})$ defined by* (4.5) *is a discrete-time hysteresis operator.*

Proof. It is clear that Φ^d is causal. It remains to show that Φ^d is rate independent. Let $u \in F(\mathbb{Z}_+,\mathbb{R})$ and $f \in \mathcal{T}^d$; then $f^c := \tau P_\tau(f) \in \mathcal{T}$ and $(H_\tau u) \circ f^c = H_\tau(u \circ f)$. Hence, using the rate independence of $\tilde{\Phi}$,

$$\begin{aligned}\big(\Phi^d(u \circ f)\big)(n) &= \big(\tilde{\Phi}\big(H_\tau(u \circ f)\big)\big)(n\tau) = \big(\tilde{\Phi}\big((H_\tau u) \circ f^c\big)\big)(n\tau)\\ &= \big(\tilde{\Phi}(H_\tau u)\big)\big(f^c(n\tau)\big) = \big(\tilde{\Phi}(H_\tau u)\big)\big(f(n)\tau\big) = \big(\Phi^d(u)\big)\big(f(n)\big),\end{aligned}$$

showing that Φ^d is rate independent. $\square$

Let $\mathbb{T} = \mathbb{Z}_+, \mathbb{R}_+$ and $\mathcal{F} \subset F(\mathbb{T},\mathbb{R})$, $\mathcal{F} \neq \emptyset$; then the *numerical value set* $\mathrm{NVS}\,\Psi$ of an operator $\Psi : \mathcal{F} \to F(\mathbb{T},\mathbb{R})$ is defined by

$$\mathrm{NVS}\,\Psi := \big\{\big(\Psi(u)\big)(t) \mid u \in \mathcal{F},\, t \in \mathbb{T}\big\}.$$

The following proposition shows that for a continuous-time hysteresis operator Φ defined on $C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$, the numerical value sets of Φ and Φ^d coincide. This result is important in the context of sampled-data low-gain control of systems subject to input hysteresis (see [7]), but is also of some interest in its own right.

Proposition 4.4. *Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}) \to F(\mathbb{R}_+,\mathbb{R})$ be a continuous-time hysteresis operator and define the operator $\Phi^d : F(\mathbb{Z}_+,\mathbb{R}) \to F(\mathbb{Z}_+,\mathbb{R})$ by* (4.5)*. Then*

$$\big(\Phi^d(u)\big)(n) = \big(\Phi(P_\tau u)\big)(n\tau), \quad \forall u \in F(\mathbb{Z}_+,\mathbb{R}),\ \forall n \in \mathbb{Z}_+, \tag{4.6}$$

and $\mathrm{NVS}\,\Phi^d = \mathrm{NVS}\,\Phi$.

Proof. Let $u \in F(\mathbb{Z}_+,\mathbb{R})$ and $n \in \mathbb{Z}_+$. We note that $\tilde{R}(Q_{n\tau} H_\tau u) = R(Q_{n\tau} P_\tau u)$ and so by Theorem 3.4(1) and Theorem 3.4(3),

$$\big(\Phi^d(u)\big)(n) = \big(\tilde{\Phi}(H_\tau u)\big)(n\tau) = \big(\Phi(P_\tau u)\big)(n\tau).$$

To prove that $\mathrm{NVS}\,\Phi^d = \mathrm{NVS}\,\Phi$, note first, that by (4.6), $\mathrm{NVS}\,\Phi^d \subset \mathrm{NVS}\,\Phi$. To show the reverse inclusion, let $a \in \mathrm{NVS}\,\Phi$. Then there exist $v \in C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ and $t \in \mathbb{R}_+$ such that $a = (\Phi(v))(t)$. Set $w := Q_t v \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$. Clearly

$$Q_{k\tau} w = w, \quad \forall k \geqslant t/\tau.$$

Moreover, $(P_\tau \circ R)(w) \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ and so there exists $k_0 > 0$ such that

$$Q_{k\tau}\big((P_\tau \circ R)(w)\big) = (P_\tau \circ R)(w), \quad \forall k \geqslant k_0.$$

For $k \geqslant \max(k_0, t/\tau) =: k_1$ we then have

$$(P_\tau \circ R)(Q_{k\tau} w) = (P_\tau \circ R)(w) = Q_{k\tau}\big((P_\tau \circ R)(w)\big). \tag{4.7}$$

Let φ be the representing functional of Φ, then for $k \geqslant k_1$,

$$\begin{aligned} a &= \big(\Phi(w)\big)(t) = \big(\Phi(w)\big)(k\tau) = \varphi(Q_{k\tau}w) = \tilde{\varphi}(Q_{k\tau}w) \\ &= \varphi\big((P_\tau \circ R)(Q_{k\tau}w)\big), \end{aligned} \tag{4.8}$$

where we have used Theorem 2.2(1) and Theorem 2.2(2) and the fact that $\tilde{\varphi}$ is an extension of φ. Combining Theorem 2.2(2) and (4.6)–(4.8), we obtain for any $k \geqslant k_1$,

$$\begin{aligned} a &= \varphi\big(Q_{k\tau}(P_\tau \circ R)(w)\big) = \big(\Phi\big((P_\tau \circ R)(w)\big)\big)(k\tau) \\ &= \big(\Phi^d(Rw)\big)(k) \in \mathrm{NVS}\,\Phi^d. \qquad \square \end{aligned}$$

We finally look at the discretization of the backlash operator.

Example 4.5. Let $h \in \mathbb{R}_+$ and $\xi \in \mathbb{R}$. We define the discrete-time backlash operator $\mathcal{B}^d_{h,\xi}: F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ by setting

$$\mathcal{B}^d_{h,\xi} := S_\tau \tilde{\mathcal{B}}_{h,\xi} H_\tau.$$

Using (4.6), we see that for all $u \in F(\mathbb{Z}_+, \mathbb{R})$, $\mathcal{B}^d_{h,\xi}(u)$ can be expressed recursively as

$$\big(\mathcal{B}^d_{h,\xi}(u)\big)(n) = \begin{cases} b_h(u(0), \xi) & \text{for } n = 0, \\ b_h(u(n), (\mathcal{B}^d_{h,\xi}(u))(n-1)) & \text{for } n \in \mathbb{Z}_+ \setminus \{0\}, \end{cases}$$

where $b_h : \mathbb{R}^2 \to \mathbb{R}$ is given by (3.9).

5. Applications to sampled-data control of linear systems with input hysteresis

Let $\tau > 0$. A *generalized τ-hold* operator $\hat{H}_\tau : F(\mathbb{Z}_+, \mathbb{R}) \to \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ is an operator of the form

$$(\hat{H}_\tau u)(n\tau + t) = h(t)u(n), \quad \forall n \in \mathbb{Z}_+,\ \forall t \in [0, \tau),$$

where the so-called hold function $h : [0, \tau] \to \mathbb{R}$ is normalized piecewise continuous and piecewise monotone. Trivially, if $h(t) \equiv 1$, then $\hat{H}_\tau = H_\tau$, where H_τ is the τ-hold operator given by (4.3). A *generalized τ-sampling* operator $\hat{S}_\tau : C(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ is an operator of the form

$$(\hat{S}_\tau u)(n) = \begin{cases} u(0) \int_{-\tau}^{0} dw(t), & n = 0, \\ \int_{-\tau}^{0} u(n\tau + t)\, dw(t), & n \in \mathbb{Z}_+ \setminus \{0\}, \end{cases}$$

where the weighting function $w : [-\tau, 0] \to \mathbb{R}$ is of bounded variation. If $w(t) = 0$ for all $t \in [-\tau, 0)$ and $w(0) = 1$, then $\hat{S}_\tau = S_\tau$, where S_τ is the sampling operator given by (4.4). For an overview on generalized hold and generalized sampling techniques we refer the reader to [1].

Consider the system shown in Fig. 4, where Φ is a hysteresis operator defined on $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and Σ is a linear state-space system of the form

$$\dot{x} = Ax + bv, \quad x(0) = \xi \in \mathbb{R}^n, \qquad y = c^T x,$$

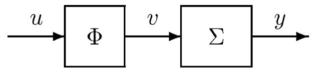

Fig. 4. Linear system with input hysteresis.

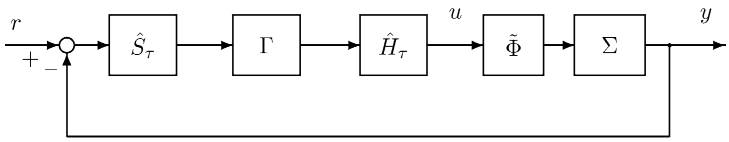

Fig. 5. Sampled-data feedback system.

where $A \in \mathbb{R}^{n\times n}$ and $b, c \in \mathbb{R}^n$. In Fig. 4, $v = \Phi(u)$, and hence the system shown in Fig. 4 is mathematically described by

$$\dot{x} = Ax + b\Phi(u), \quad x(0) = \xi \in \mathbb{R}^n, \qquad y = c^T x. \tag{5.1}$$

Let $r \in C(\mathbb{R}_+, \mathbb{R})$ and let $\Gamma : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ be a causal discrete-time operator. A (generalized) sampled-data feedback control is a control law of the form

$$u = (\hat{H}_\tau \Gamma \hat{S}_\tau)(r - y). \tag{5.2}$$

The number $\tau > 0$ is called the sampling period and the causal discrete-time operator Γ is usually called the (discrete-time) controller. The function r models an external input signal (which in some applications is also called the reference signal). Of course, the feedback-control law (5.2) produces in general a discontinuous control $u \in \mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Consequently, the feedback interconnection of (5.1) and (5.2) only makes sense if we replace the hysteresis operator Φ (which is defined $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$) by its “canonical” extension $\tilde{\Phi}$ (which is defined on $\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and is given by (3.4)). We see that the need for extensions of standard hysteresis operators to sets of piecewise continuous functions arises naturally in the context of sampled-data control in the presence of input hysteresis.

The sampled-data feedback system given by (5.1) and (5.2) is shown in Fig. 5 and is mathematically described by

$$\dot{x} = Ax + b(\tilde{\Phi}\hat{H}_\tau \Gamma \hat{S}_\tau)(r - c^T x), \quad x(0) = \xi \in \mathbb{R}^n. \tag{5.3}$$

Consider the integral form of (5.3), namely

$$x(t) = e^{At}\xi + \int_0^t e^{A(t-s)} b\big[(\tilde{\Phi}\hat{H}_\tau \Gamma \hat{S}_\tau)(r - c^T x)\big](s)\, ds, \quad \xi \in \mathbb{R}^n. \tag{5.4}$$

In the following we assume that

$$\tilde{\Phi}\big(\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})\big) \subset L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}), \tag{5.5}$$

which is not restrictive in so far as we believe that hysteresis operators not satisfying (5.5) are of limited (or even no) physical relevance. If $\Phi(C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$, then it follows from Corollary 3.10 that $\tilde{\Phi}(\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})) \subset \mathrm{NPC}(\mathbb{R}_+, \mathbb{R})$, implying that (5.5) holds. Using (5.5) it is not difficult to show that, for every $\xi \in \mathbb{R}^n$, (5.4) admits a unique absolutely continuous solution $x_\xi : \mathbb{R}_+ \to \mathbb{R}$ which satisfies (5.3) almost everywhere.[4] For the rest of this section we assume that $\hat{H}_\tau = H_\tau$ and $\hat{S}_\tau = S_\tau$. We want to relate (5.4) to a discrete-time problem. To this end define $x_\xi^d, r^d \in F(\mathbb{Z}_+, \mathbb{R})$ by $x_\xi^d(n) := x_\xi(n\tau)$ and $r^d(n) := r(n\tau)$, respectively. It follows from (5.4) that

$$x_\xi^d(n+1) = e^{A\tau} x_\xi^d(n) + \int_{n\tau}^{(n+1)\tau} e^{A((n+1)\tau - s)} b\big[(\tilde{\Phi} H_\tau \Gamma)(f)\big](s)\, ds, \tag{5.6}$$

where $f \in F(\mathbb{Z}_+, \mathbb{R})$ is defined by

$$f(n) = \big(S_\tau(r - c^T x)\big)(n) = r^d(n) - c^T x_\xi^d(n).$$

Since $(H_\tau \Gamma)(f) \in \mathcal{S}_\tau^r$, it follows from Theorem 3.4(4) that $(\tilde{\Phi} H_\tau \Gamma)(f) \in \mathcal{S}_\tau^r$, and so,

$$\big[(\tilde{\Phi} H_\tau \Gamma)(f)\big](s) = \big[(\tilde{\Phi} H_\tau \Gamma)(f)\big](n\tau), \quad \forall s \in \big[n\tau, (n+1)\tau\big),\ \forall n \in \mathbb{Z}_+. \tag{5.7}$$

Moreover, obviously, $H_\tau S_\tau v = v$ for all $v \in \mathcal{S}_\tau^r$, and so it follows from Theorem 3.4(4) that $\tilde{\Phi} H_\tau = H_\tau \Phi^d$, where the discrete-time hysteresis operator Φ^d is given by (4.5). Combining this with (5.7) yields

$$\big[(\tilde{\Phi} H_\tau \Gamma)(f)\big](s) = \big[(\Phi^d \Gamma)(f)\big](n), \quad \forall s \in \big[n\tau, (n+1)\tau\big),\ \forall n \in \mathbb{Z}_+. \tag{5.8}$$

Inserting this into (5.6), we obtain

$$\begin{aligned}
x_\xi^d(n+1) &= e^{A\tau} x_\xi^d(n) + \left(\int_{n\tau}^{(n+1)\tau} e^{A((n+1)\tau - s)}\, ds\right) b\big[(\Phi^d \Gamma)(f)\big](n),\\
&= e^{A\tau} x_\xi^d(n) + \left(\int_0^\tau e^{As}\, ds\right) b\big[(\Phi^d \Gamma)\big(r^d - c^T x_\xi^d\big)\big](n),\\
&= A^d x_\xi^d(n) + b^d\big[(\Phi^d \Gamma)\big(r^d - c^T x_\xi^d\big)\big](n),
\end{aligned}$$

where $A^d := e^{A\tau}$ and $b^d := (\int_0^\tau e^{As}\, ds) b$. This shows that $n \mapsto x_\xi^d(n) = x_\xi(n\tau)$ is the solution of the discrete-time initial-value problem

$$z(n+1) = A^d z(n) + b^d\big[(\Phi^d \Gamma)(r^d - c^T z)\big](n), \quad z(0) = \xi.$$

We see that the discrete-time hysteresis operator Φ^d introduced in Section 4 (see (4.5)) arises naturally in the context of sampled-data control of continuous-time systems with input hysteresis.

Finally, the intersampling behaviour of x_ξ (that is the behaviour of $x_\xi|_{(n\tau,(n+1)\tau)}$) can be bounded in terms of x_ξ^d and r^d. To show this, we use (5.4) (with $\hat{H}_\tau = H_\tau$) and (5.8) to obtain

[4] If we assume that $\Phi(C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$ and if we denote the points of discontinuity of the hold-function h by $t_i \in (0, \tau)$ $(i = 1, 2, \ldots, m)$, then it follows from Corollary 3.10 that x_ξ is continuously differentiable on the intervals $(n\tau, n\tau + t_1)$, $(n\tau + t_1, n\tau + t_2), \ldots, (n\tau + t_{m-1}, n\tau + t_m)$, $(n\tau + t_m, (n+1)\tau)$ (where $n \in \mathbb{Z}_+$) and thus, x_ξ satisfies the differential equation in (5.3) on these intervals.

$$x_\xi(n\tau + t) = e^{At} x_\xi^d(n) + \left(\int_0^t e^{As}\, ds \right) b\big[(\Phi^d \Gamma)\big(r^d - c^T x_\xi^d\big)\big](n),$$
$$\forall t \in [0, \tau),\ \forall n \in \mathbb{Z}_+.$$

Consequently, we may conclude that there exist constants $\alpha, \beta \geqslant 0$ (not depending on t or n) such that

$$\big\|x_\xi(n\tau + t)\big\| \leqslant \alpha \big\|x_\xi^d(n)\big\| + \beta \big|\big[(\Phi^d \Gamma)\big(r^d - c^T x_\xi^d\big)\big](n)\big|, \quad \forall t \in [0, \tau),\ \forall n \in \mathbb{Z}_+.$$

For applications of the results in Sections 3 and 4 to sampled-data low-gain integral control of infinite-dimensional linear systems in the presence of input hysteresis we refer the reader to [7].

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