Chapter 14

Low-Gain Integral Control of Infinite-Dimensional Regular Linear Systems Subject to Input Hysteresis

Hartmut Logemann
Adam D. Mawby

ABSTRACT In the present chapter we introduce a general class of causal dynamic nonlinearities with certain monotonicity and Lipschitz continuity properties. It is shown that closing the loop around an exponentially stable, single-input, single-output, infinite-dimensional, regular, linear system, subject to an input nonlinearity from this class and compensated by an integral controller, guarantees asymptotic tracking of constant reference signals, provided that (i) the steady-state gain of the linear part of the plant is positive, (ii) the positive integrator gain is smaller than a certain constant given by a positive-real condition in terms of the linear part of the plant, and (iii) the reference value is feasible in a very natural sense. The class of nonlinearities under consideration contains in particular relay hysteresis, backlash, and hysteresis operators of Prandtl and Preisach types.

14.1 Introduction

This chapter extends a sequence [12, 13] of recent results pertaining to integral control of infinite-dimensional systems subject to static input nonlinearities. Underpinning these results are generalizations of the well-known principle (see, e.g., [6, 15, 17]) that closing the loop around a stable, linear, finite-dimensional, continuous-time, single-input, single-output plant, with transfer function $\mathbf{G}$ compensated by a pure integral controller k/s, will result in a stable closed-loop system that achieves asymptotic tracking of arbitrary constant reference signals, provided that $|k|$ is sufficiently small and $\mathbf{G}(0)k > 0$.[1] This result has been extended to various classes of infinite-dimensional systems; see [11] and the references therein. In par-

[1]Therefore, under the above assumptions on the plant, the problem of tracking constant reference signals reduces to that of tuning the gain parameter k. This so-called "tuning regulator theory" [6] has been successfully applied in process control (see [5, 14]).

F. Colonius et al. (eds.), *Advances in Mathematical Systems Theory*
© Birkhäuser Boston 2001

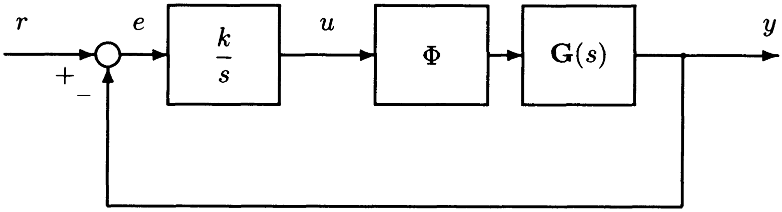

FIGURE 14.1. Low-gain control with input nonlinearity.

ticular, Logemann et al. [13] have shown that the above principle remains true for a single-input, single-output, linear, regular, infinite-dimensional system subject to a static, nondecreasing, globally Lipschitz input nonlinearity Φ (such as, e.g., saturation), provided the reference value r is feasible in the sense that $r/\mathbf{G}(0)$ is in the image of the nonlinearity; see Figure 14.1. Here we consider the same problem as in [13], but for a wider class of causal dynamic nonlinearities that satisfy a certain Lipschitz condition. This class encompasses, in particular, a large number of hysteresis nonlinearities important in applications such as relay (or passive), backlash, and plastic-elastic hysteresis. Generally speaking, hysteresis is a special type of memory-based relation between a scalar input signal $u(\cdot)$ and a scalar output signal $v(\cdot)$ that cannot be expressed in terms of a single-valued function, but takes the form of "hysteresis" loops; in particular, the operator $u(\cdot) \mapsto v(\cdot)$ is causal and rate independent. This type of behavior arises in mechanical plays, thermostats, elastoplasticity, ferromagnetism, and in smart material structures such as piezoelectric elements and magnetostrictive transducers (see Banks et al. [1] for hysteresis phenomena in smart materials). There exists a substantial literature on mathematical modeling and mathematical theory of hysteresis phenomena; see, for example, Brokate [3], Brokate and Sprekels [4], Krasnosel'skiĭ and Pokrovskiĭ [9], Macki et al. [16], and Visintin [19]. Of particular importance in a systems and control context is the pioneering work [9].

As in [13] we assume that the linear part of the system to be controlled (described in Figure 14.1 by the transfer function $\mathbf{G}(s)$) is an exponentially stable, single-output, single-input, regular, infinite-dimensional system. This class, introduced by Weiss (see [23] through [22]), is rather general and allows for highly unbounded control and observation operators. It includes most distributed parameter systems and time-delay systems of interest in control engineeering.

The main result in this chapter shows that for the class of dynamic nonlinearities under consideration, the output $y(t)$ of the closed-loop system, shown in Figure 14.1, converges to r as $t \to \infty$, provided that $\mathbf{G}(0) > 0$, r is feasible in some natural sense, and $k \in (0, K/\lambda)$, where $\lambda > 0$ is a Lipschitz constant for the nonlinearity and K is the supremum of the set of all numbers $k > 0$ such that the function

$$1 + k \operatorname{Re} \frac{\mathbf{G}(s)}{s}$$

is positive real.

The chapter is organized as follows. In Section 14.2 we briefly discuss regular linear infinite-dimensional systems. In Section 14.3 we define a class of dynamic nonlinear operators for which, in Section 14.4, we show that the output $y(t)$ of the closed-loop system (shown in Figure 14.1) converges to the reference r as $t \to \infty$. In Section 14.5 we introduce various hysteresis operators, such as relay, backlash, and elastic-plastic as well as hysteresis operators of Prandtl and Preisach types; we show that under a few natural assumptions they are contained within the class of nonlinearities introduced in Section 14.3. Section 14.6 contains simulations of two controlled diffusion process examples with input hysteresis nonlinearities. Finally, a number of technicalities have been relegated to the Appendix (Section 14.7), which, in particular, contains an existence and uniqueness result for the solutions of the nonlinear abstract Cauchy problem describing the closed-loop system shown in Figure 14.1.

Notation: If $I \subset \mathbb{R}$ is a compact interval, then $AC(I, \mathbb{R})$ denotes the space of absolutely continuous real-valued functions defined on I; $AC(\mathbb{R}_+, \mathbb{R})$ denotes the space of real-valued functions defined on $\mathbb{R}_+$ that are absolutely continuous on any compact interval $I \subset \mathbb{R}_+$; that is, a function $f : \mathbb{R}_+ \to \mathbb{R}$ is in $AC(\mathbb{R}_+, \mathbb{R})$ if and only if there exists a function $g \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ such that

$$f(t) = f(0) + \int_0^t g(\tau)\, d\tau\,, \quad \forall\, t \geq 0\,.$$

We call a function $f : [a, b] \to \mathbb{R}$ piecewise monotone if there exist numbers $a = t_0 < t_1 < \ldots < t_n = b$ such that f is monotone on $[t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$. A function $f : \mathbb{R}_+ \to \mathbb{R}$ is called piecewise monotone if f is piecewise monotone on each compact interval $I \subset \mathbb{R}_+$. We denote the space of piecewise monotone continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ by $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. It is straightforward to show that $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ is dense in $C(\mathbb{R}_+, \mathbb{R})$ in the sense that for all $f \in C(\mathbb{R}_+, \mathbb{R})$ and all $\varepsilon > 0$, there exists $g \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ such that

$$|f(t) - g(t)| \leq \varepsilon\,, \quad \forall\, t \in \mathbb{R}_+\,.$$

For $\alpha \in \mathbb{R}$, we define the exponentially weighted L^p-space

$$L^p_\alpha(\mathbb{R}_+, \mathbb{R}) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}) |\ f(\cdot) \exp(-\alpha\, \cdot) \in L^p(\mathbb{R}_+, \mathbb{R})\}\,.$$

$L(X, Y)$ denotes the space of bounded linear operators from a Banach space X to a Banach space Y. Let $\mathbb{N}$ denote the nonnegative integers. For $\alpha \in \mathbb{R}$, we define $\mathbb{C}_\alpha := \{s \in \mathbb{C} \,|\, \operatorname{Re} s > \alpha\}$. The Laplace transform is denoted by $\mathfrak{L}$.

14.2 Preliminaries on Regular Linear Systems

In Figure 14.1 the underlying linear system (i.e., the system with transfer function $\mathbf{G}(s)$) is assumed to be a single-input, single-output, continuous-time, regular system Σ with state space X (a Hilbert space) and with generating operators (A, B, C, D). This means in particular that A generates a strongly continuous semigroup $\mathbf{T} = (\mathbf{T}_t)_{t \geq 0}$, $C \in L(X_1, \mathbb{R})$ is an admissible observation operator for $\mathbf{T}$, $B \in L(\mathbb{R}, X_{-1})$ is an admissible control operator for $\mathbf{T}$, and $D \in \mathbb{R}$ is the feedthrough of the system. Here X_1 denotes the space $\mathrm{dom}(A)$ (the domain of A) endowed with the graph norm and X_{-1} denotes the completion of X with respect to the norm $\|x\|_{-1} = \|(s_0 I - A)^{-1}x\|$, where s_0 is any fixed element in the resolvent set of A. The norm on X is denoted by $\|\cdot\|$, while $\|\cdot\|_1$ and $\|\cdot\|_{-1}$ denote the norms on X_1 and X_{-1}, respectively. Then $X_1 \hookrightarrow X \hookrightarrow X_{-1}$ and $\mathbf{T}$ restricts (resp., extends) to a strongly continuous semigroup on X_1 (resp., X_{-1}). The exponential growth constant

$$\omega(\mathbf{T}) := \lim_{t \to \infty} \frac{1}{t} \ln \|\mathbf{T}_t\|$$

is the same on all three spaces. The generator of $\mathbf{T}$ on X_{-1} is an extension of A to X (which is bounded as an operator from X to X_{-1}). We use the same symbol $\mathbf{T}$ (resp., A) for the original semigroup (resp., its generator) and the associated restrictions and extensions. With this convention, we may write $A \in L(X, X_{-1})$. Considered as a generator on X_{-1}, the domain of A is X.

We regard a regular system Σ as synonymous with its generating operators and simply write $\Sigma = (A, B, C, D)$. The regular system is said to be *exponentially stable* if the semigroup $\mathbf{T}$ is exponentially stable; that is, $\omega(\mathbf{T}) < 0$. The control operator B (resp., observation operator C) is said to be *bounded* if $B \in L(\mathbb{R}, X)$ (resp., $C \in L(X, \mathbb{R})$); otherwise, B (resp., C) is said to be *unbounded.* In terms of the generating operators (A, B, C, D), the transfer function $\mathbf{G}(s)$ can be expressed as

$$\mathbf{G}(s) = C_L(sI - A)^{-1}B + D\,,$$

where C_L denotes the so-called Lebesgue extension of C. The transfer function $\mathbf{G}(s)$ is bounded and holomorphic in any half-plane $\mathrm{Re}\, s > \alpha$ with $\alpha > \omega(\mathbf{T})$. Moreover,

$$\lim_{s \to \infty,\, s \in \mathbb{R}} \mathbf{G}(s) = D\,.$$

For any $x_0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$, the state and output functions $x(\cdot)$ and $y(\cdot)$, respectively, satisfy the equations

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \qquad x(0) = x_0\,, \tag{14.1a}$$

$$y(t) = C_L x(t) + Du(t)\,, \tag{14.1b}$$

for almost all $t \geq 0$ (in particular, $x(t) \in \mathrm{dom}(C_L)$ for almost all $t \geq 0$). The derivative on the left-hand side of (14.1a) has, of course, to be understood in X_{-1}. In other words, if we consider the initial-value problem (14.1a) in the space X_{-1}, then for any $x_0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$, (14.1a) has a unique strong solution (in the sense of Pazy [18, p. 109]) given by the variation of parameters formula

$$x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau\,. \tag{14.2}$$

For any $x_0 \in X$ and any $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$, (14.2) defines a continuous X-valued function.

Denoting the input-output operator of (14.1) by $\mathbf{F}$, we have that $\mathbf{F}$ is a shift-invariant (and thus causal) operator from $L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ into $L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$. For $\alpha \in \mathbb{R}$ and $u \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$,

$$(\mathfrak{L}(\mathbf{F}u))(s) = \mathbf{G}(s)(\mathfrak{L}(u))(s)\,, \quad \mathrm{Re}\, s > \max(\omega(\mathbf{T}), \alpha)\,.$$

Finally, we introduce the state-to-output map $\boldsymbol{\Psi} : X \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ defined by

$$(\boldsymbol{\Psi} x_0)(t) = C_L \mathbf{T}_t x_0\,, \quad \text{a.e. } t \in \mathbb{R}_+\,.$$

For more details on regular systems see Weiss [23] through [22]. For details on regular systems in the context of low-gain control the reader is referred to [11] and [13].

For future reference we state the following lemma, the proof of which can be found in [13].

Lemma 14.1. *Assume that $\mathbf{T}$ is exponentially stable and that $B \in L(\mathbb{R}, X_{-1})$ is an admissible control operator for $\mathbf{T}$.*

If $u \in L^\infty(\mathbb{R}_+, \mathbb{R})$ is such that $\lim_{t\to\infty} u(t) = u_\infty$ exists, then, for all $x_0 \in X$, the state $x(\cdot)$ given by (14.2) satisfies

$$\lim_{t\to\infty} \|x(t) + A^{-1} B u_\infty\| = 0\,.$$

14.3 A Class of Causal Monotone Nonlinear Operators

Let $a \in (0, \infty]$ and let $J \subset \mathbb{R}_+$ be an interval of the form $[0, a)$ or $[0, a]$. For $\tau \in J$, we define the operator $\mathbf{Q}_\tau : C(J, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ by

$$(\mathbf{Q}_\tau u)(t) = \begin{cases} u(t) & \text{for } 0 \leq t \leq \tau\,, \\ u(\tau) & \text{for } t > \tau\,. \end{cases}$$

If the domain space of $\mathbf{Q}_\tau$ is $C(\mathbb{R}_+, \mathbb{R})$ (i.e., $J = [0, \infty)$), then $\mathbf{Q}_\tau$ is a projection operator. Let $\mathcal{C} \subset C(\mathbb{R}_+, \mathbb{R})$, $\mathcal{C} \neq \emptyset$. Recall that an operator

$\Phi : \mathcal{C} \to C(\mathbb{R}_+, \mathbb{R})$ is called *causal* if for all $u, v \in \mathcal{C}$ and all $\tau \geq 0$ with $u(t) = v(t)$ for all $t \in [0, \tau]$ it follows that $(\Phi(u))(t) = (\Phi(v))(t)$ for all $t \in [0, \tau]$. If $\mathcal{C}$ is invariant under $\mathbf{Q}_t$ for all $t \in \mathbb{R}_+$ (i.e., $\mathbf{Q}_t(\mathcal{C}) \subset \mathcal{C}$ for all $t \in \mathbb{R}_+$) then it is easy to show that Φ is causal if and only if for all $u \in \mathcal{C}$,

$$(\Phi(\mathbf{Q}_t u))(t) = (\Phi(u))(t), \quad \forall t \in \mathbb{R}_+ .$$

Given an operator $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ and a number $a > 0$ and denoting the space of all functions $f : [0, a) \to \mathbb{R}$ by $\mathcal{F}([0, a), \mathbb{R})$, we define an operator $\tilde{\Phi} : C([0, a), \mathbb{R}) \to \mathcal{F}([0, a), \mathbb{R})$ by setting

$$(\tilde{\Phi}(u))(t) = (\Phi(\mathbf{Q}_t u))(t), \quad \forall t \in [0, a) .$$

If Φ is causal, then for each $\tau \in [0, a)$ we have

$$(\tilde{\Phi}(u))(t) = (\Phi(\mathbf{Q}_\tau u))(t), \quad \forall t \in [0, \tau] ,$$

implying in particular that $\tilde{\Phi}(C([0, a), \mathbb{R})) \subset C([0, a), \mathbb{R})$. In the following, we use the same symbol Φ to denote the original operator acting on $C(\mathbb{R}_+, \mathbb{R})$ and the associated operator $\tilde{\Phi}$ acting on $C([0, a), \mathbb{R})$.

Let $u \in C(\mathbb{R}_+, \mathbb{R})$. The function u is called *ultimately nondecreasing* if there exists $T \in \mathbb{R}_+$ such that u is nondecreasing on $[T, \infty)$; u is said to be *approximately ultimately nondecreasing*, if for all $\varepsilon > 0$, there exists an ultimately nondecreasing function $v \in C(\mathbb{R}_+, \mathbb{R})$ such that

$$|u(t) - v(t)| \leq \varepsilon, \quad \forall t \in \mathbb{R}_+ .$$

The *numerical value set* $\mathrm{NVS}\,\Phi$ of an operator $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ is defined by

$$\mathrm{NVS}\,\Phi := \{(\Phi(u))(t) \mid u \in C(\mathbb{R}_+, \mathbb{R}),\ t \in \mathbb{R}_+\} .$$

For $\alpha \geq 0$, $w \in C([0, \alpha], \mathbb{R})$, and $\delta_1, \delta_2 > 0$, we define $\mathcal{C}(w; \delta_1, \delta_2)$ to be the set of all $u \in C(\mathbb{R}_+, \mathbb{R})$ such that

$$u(t) = w(t), \ \forall t \in [0, \alpha] \qquad \text{and} \qquad |u(t) - w(\alpha)| \leq \delta_1, \ \forall t \in [\alpha, \alpha + \delta_2] .$$

We introduce the following assumptions on the nonlinear operator $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$.

(N1) Φ is causal.

(N2) For all $u \in C(\mathbb{R}_+, \mathbb{R})$ and all $\tau \in \mathbb{R}_+$, $(\Phi(\mathbf{Q}_\tau u))(t) = (\Phi(\mathbf{Q}_\tau u))(\tau)$ for all $t \geq \tau$.

(N3) $\Phi(AC(\mathbb{R}_+, \mathbb{R})) \subset AC(\mathbb{R}_+, \mathbb{R})$.

(N4) Φ is monotone in the sense that for all $u \in AC(\mathbb{R}_+, \mathbb{R})$ with $\Phi(u) \in AC(\mathbb{R}_+, \mathbb{R})$,

$$\frac{d}{dt}(\Phi(u))(t)\,\dot{u}(t) \geq 0\,, \quad \text{a.e. } t \in \mathbb{R}_+\,.$$

(N5) There exists $\lambda > 0$ such that for all $\alpha \in \mathbb{R}_+$, $w \in C([0,\alpha], \mathbb{R})$, there exist numbers $\delta_1, \delta_2 > 0$ such that for all $u, v \in \mathcal{C}(w; \delta_1, \delta_2)$,

$$\sup_{t\in[\alpha,\alpha+\delta_2]} |(\Phi(u))(t) - (\Phi(v))(t)| \leq \lambda \sup_{t\in[\alpha,\alpha+\delta_2]} |u(t) - v(t)|\,.$$

(N6) If $u \in C(\mathbb{R}_+, \mathbb{R})$ is approximately ultimately nondecreasing and furthermore $\lim_{t\to\infty} u(t) = \infty$, then $\Phi(u)(t)$ and $\Phi(-u)(t)$ converge to $\sup \mathrm{NVS}\,\Phi$ and $\inf \mathrm{NVS}\,\Phi$, respectively, as $t \to \infty$.

(N7) If $u \in C(\mathbb{R}_+, \mathbb{R})$ is such that $\lim_{t\to\infty}(\Phi(u))(t) \in \mathrm{int}\,\mathrm{NVS}\,\Phi$, then u is bounded.

(N8) For all $a > 0$ and all $u \in C([0,a), \mathbb{R})$, there exist $\alpha, \beta > 0$ such that

$$\sup_{t\in[0,\tau]} |(\Phi(u))(t)| \leq \alpha + \beta \sup_{t\in[0,\tau]} |u(t)|\,, \quad \forall\, \tau \in [0,a)\,.$$

Remark 14.1. (i) Assumption (N2) says that if the input u of the nonlinearity Φ is constant on $[\tau,\infty)$, then the output $\Phi(u)$ is constant and equal to $(\Phi(u))(\tau)$ on $[\tau,\infty)$.

(ii) Hysteresis operators as defined in [4] are simply causal and rate independent operators defined on $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ (see [4, Definition 2.2.8 and Proposition 2.2.9]). Most hysteresis operators admit causal extensions to $C(\mathbb{R}_+, \mathbb{R})$ satisfying (N2). In this sense, a large class of hysteresis operators satisfies the assumptions (N1) and (N2). This is in particular true for hysteresis operators that are Lipschitz continuous in the sense of Definition 14.2. We mention that none of the assumptions (N1) through (N8) imply rate independence. However, most of the operators satisfying (N1) and (N2) and which are of interest in the modeling of dynamic actuator nonlinearities will be hysteresis operators in the sense of [4].

(iii) Assumptions (N1) and (N5) ensure local existence and uniqueness of solutions to the nonlinear closed-loop system shown in Figure 14.1; if (N1), (N5), and (N8) hold, then the solution of this closed-loop system exists and is unique on the time interval $\mathbb{R}_+$ (see Section 14.4).

(iv) If (N1) and (N6) hold, then $\mathrm{NVS}\,\Phi$ is an interval.

We show in Section 14.5 that the assumptions (N1) through (N8) are satisfied by a large class of hysteresis operators. Some of the implications of the assumptions (N1) through (N5) are described in the following lemma.

Lemma 14.2. *For an operator $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ the following statements hold.*

(i) If Φ satisfies (N1), (N2), and (N5), then for all $u \in C(\mathbb{R}_+, \mathbb{R})$ and all $\alpha \in \mathbb{R}_+$, there exists $\delta > 0$ such that for all $t \in [\alpha, \alpha + \delta]$,

$$|(\Phi(u))(t) - (\Phi(u))(\alpha)| \leq \lambda \sup_{\tau \in [\alpha, t]} |u(\tau) - u(\alpha)| . \tag{14.3}$$

(ii) If Φ satisfies (N1) through (N3) and (N5), then for all $u \in AC(\mathbb{R}_+, \mathbb{R})$,

$$\left| \frac{d}{dt}(\Phi(u))(t) \right| \leq \lambda |\dot{u}(t)| , \quad \text{a.e. } t \in \mathbb{R}_+ . \tag{14.4}$$

(iii) If Φ satisfies (N1) through (N5), then for every $u \in AC(\mathbb{R}_+, \mathbb{R})$, there exists a measurable function $d_u : \mathbb{R}_+ \to [0, \lambda]$ such that

$$\frac{d}{dt}(\Phi(u))(t) = d_u(t)\dot{u}(t) , \quad \text{a.e. } t \in \mathbb{R}_+ . \tag{14.5}$$

Proof: To prove statement (i), let $u \in C(\mathbb{R}_+, \mathbb{R})$ and $\alpha \in \mathbb{R}_+$ and define $w \in C([0, \alpha], \mathbb{R})$ by $w(t) = u(t)$ for all $t \in [0, \alpha]$. By (N5), there exist numbers $\delta_1, \delta_2 > 0$ such that for all $v_1, v_2 \in \mathcal{C}(w; \delta_1, \delta_2)$,

$$\sup_{t \in [\alpha, \alpha + \delta_2]} |(\Phi(v_1))(t) - (\Phi(v_2))(t)| \leq \lambda \sup_{t \in [\alpha, \alpha + \delta_2]} |v_1(t) - v_2(t)| .$$

By continuity of u, there exists $\delta \in (0, \delta_2)$ such that $\mathbf{Q}_t u \in \mathcal{C}(w; \delta_1, \delta_2)$ for all $t \in [\alpha, \alpha + \delta]$. Thus, using (N1) and (N2), we may conclude that for $t \in [\alpha, \alpha + \delta]$,

$$\begin{aligned} |(\Phi(u))(t) - (\Phi(u))(\alpha)| &\leq \sup_{\tau \in [\alpha, t]} |(\Phi(u))(\tau) - (\Phi(u))(\alpha)| \\ &= \sup_{\tau \in [\alpha, \alpha + \delta_2]} |(\Phi(\mathbf{Q}_t u))(\tau) - (\Phi(\mathbf{Q}_\alpha u))(\tau)| \\ &\leq \lambda \sup_{\tau \in [\alpha, \alpha + \delta_2]} |(\mathbf{Q}_t u)(\tau) - (\mathbf{Q}_\alpha u)(\tau)| \\ &= \lambda \sup_{\tau \in [\alpha, t]} |u(\tau) - u(\alpha)| , \end{aligned}$$

which is (14.3).

To prove statements (ii) and (iii), let $u \in AC(\mathbb{R}_+, \mathbb{R})$. Let E be the set of all $t \in \mathbb{R}_+$ such that u or $\Phi(u)$ is not differentiable at t. By (N3), E is of measure zero. Using statement (i), we obtain for all $t \in \mathbb{R}_+ \setminus E$,

$$\begin{aligned} \left| \frac{d}{dt}(\Phi(u))(t) \right| &= \lim_{\varepsilon \downarrow 0} \frac{|(\Phi(u))(t + \varepsilon) - (\Phi(u))(t)|}{\varepsilon} \\ &\leq \lambda \lim_{\varepsilon \downarrow 0} \frac{\sup_{\tau \in [t, t + \varepsilon]} |u(\tau) - u(t)|}{\varepsilon} \\ &\leq \lambda \lim_{\varepsilon \downarrow 0} \left(\sup_{\tau \in (t, t + \varepsilon]} \left| \frac{u(\tau) - u(t)}{\tau - t} \right| \right) = \lambda |\dot{u}(t)| , \end{aligned}$$

which is (14.4). Finally, to prove statement (iii), let $E' \subset \mathbb{R}_+$ be of measure zero and such that $E \subset E'$ and

$$\left|\frac{d}{dt}(\Phi(u))(t)\right| \leq \lambda|\dot{u}(t)|\,, \quad \frac{d}{dt}(\Phi(u))(t)\,\dot{u}(t) \geq 0\,; \qquad \forall\, t \in \mathbb{R}_+ \setminus E'\,.$$

The existence of such a set E' is guaranteed by statement (ii) and (N4). Set $F = \{t \in \mathbb{R}_+ \setminus E \mid \dot{u}(t) = 0\}$ and define

$$d_u(t) = \begin{cases} \frac{d}{dt}(\Phi(u))(t)/\dot{u}(t) & \text{if } t \in \mathbb{R}_+ \setminus (E' \cup F), \\ 0 & \text{if } t \in E' \cup F. \end{cases}$$

By construction the function d_u is measurable, $d_u(t) \in [0, \lambda]$ for all $t \in \mathbb{R}_+$, and (14.5) holds. □

The following remark proves useful in Section 14.5.

Remark 14.2. Consider the following assumption which is slightly stronger than Assumption (N4).

(N4′) Φ is monotone in the sense that for all $u \in AC(\mathbb{R}_+, \mathbb{R})$ with $\Phi(u) \in AC(\mathbb{R}_+, \mathbb{R})$,

$$\frac{d}{dt}(\Phi(u))(t)\,\dot{u}(t) \geq 0\,, \quad \forall\, t \in \mathbb{R}_+ \setminus E_u\,,$$

where E_u is the set of all $t \in \mathbb{R}_+$ such that u or $\Phi(u)$ is not differentiable at t.

If in statement (iii) of Lemma 14.2, (N4) is replaced by (N4′), then for every $u \in AC(\mathbb{R}_+, \mathbb{R})$, there exists a measurable function $d_u : \mathbb{R}_+ \to [0, \lambda]$ such that

$$\frac{d}{dt}(\Phi(u))(t) = d_u(t)\dot{u}(t)\,, \quad \forall\, t \in \mathbb{R}_+ \setminus E_u\,.$$

This follows from the observation that $E' = E$ in the proof of Lemma 14.2.

We are now in the position to define the class of nonlinear operators we consider in the context of the low-gain integral control problem in Section 14.4. If $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ satisfies (N5), then any number $l > 0$ such that (N5) holds for $\lambda = l$, is called a *Lipschitz constant* of Φ.

Definition 14.1. *Let $\lambda > 0$. The set of all operators $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ satisfying (N1) through (N8) and having Lipschitz constant λ is denoted by $\mathcal{N}(\lambda)$.*

We next introduce a concept of Lipschitz continuity for operators from $C(\mathbb{R}_+, \mathbb{R})$ to $C(\mathbb{R}_+, \mathbb{R})$ and show that if $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ is Lipschitz continuous and satisfies (N1) and (N2), then Φ also satisfies (N3), (N5), and (N8).

Definition 14.2. *Let* $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R})$ *or* $\mathcal{C} = C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. *An operator* $\Phi : \mathcal{C} \to C(\mathbb{R}_+, \mathbb{R})$ *is called* Lipschitz continuous with Lipschitz continuity constant $l > 0$ *if*

$$\sup_{t\in\mathbb{R}_+} |(\Phi(u))(t) - (\Phi(v))(t)| \leq l \sup_{t\in\mathbb{R}_+} |u(t) - v(t)|, \quad \forall\, u, v \in \mathcal{C}.$$

For later convenience we define for every $\tau \geq 0$ a seminorm

$$\sigma_\tau : C(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}_+, \quad u \mapsto \sup_{t\in[0,\tau]} |u(t)|.$$

Remark 14.3. Let $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R})$ or $\mathcal{C} = C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Then $\Phi : \mathcal{C} \to C(\mathbb{R}_+, \mathbb{R})$ is causal and Lipschitz continuous with Lipschitz continuity constant $l > 0$ if and only if

$$\sigma_\tau(\Phi(u) - \Phi(v)) \leq l\sigma_\tau(u - v), \quad \forall\, u, v \in \mathcal{C}, \ \forall\, \tau \in \mathbb{R}_+.$$

Lemma 14.3. *If* $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ *is Lipschitz continuous with Lipschitz continuity constant* $l > 0$ *and satisfies (N1) and (N2), then assumptions (N3), (N5) (with Lipschitz constant* $\lambda = l$*), and (N8) hold.*

Proof: Let $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ be Lipschitz continuous with Lipschitz continuity constant $l > 0$ and let Φ satisfy assumptions (N1) and (N2). To show that (N3) holds, let $u \in AC(\mathbb{R}_+, \mathbb{R})$, $\varepsilon > 0$, and $b > a \geq 0$. Then there exists $\delta > 0$ such that

$$\sum_{k=1}^{n} |u(b_k) - u(a_k)| \leq \frac{\varepsilon}{l},$$

for every finite family of pairwise disjoint subintervals $(a_k, b_k) \subset [a, b]$ of total length

$$\sum_{k=1}^{n} (b_k - a_k) \leq \delta. \tag{14.6}$$

Since u is continuous, there exists $c_k \in [a_k, b_k]$ such that

$$|u(c_k) - u(a_k)| = \max_{t\in[a_k,b_k]} |u(t) - u(a_k)|.$$

Using (N1), (N2), Lipschitz continuity, and Remark 14.3, we obtain for any $\tau_1, \tau_2 \in \mathbb{R}_+$ with $\tau_2 \geq \tau_1$

$$\begin{aligned}
|(\Phi(u))(\tau_2) - (\Phi(u))(\tau_1)| &= |(\Phi(\mathbf{Q}_{\tau_2}u))(\tau_2) - (\Phi(\mathbf{Q}_{\tau_1}u))(\tau_2)| \\
&\leq l\sigma_{\tau_2}(\mathbf{Q}_{\tau_2}u - \mathbf{Q}_{\tau_1}u) \\
&= l \max_{t\in[\tau_1,\tau_2]} |u(t) - u(\tau_1)|.
\end{aligned} \tag{14.7}$$

Now suppose that the family of intervals (a_k, b_k) satisfies (14.6). Then

$$\sum_{k=1}^{n}(c_k - a_k) \le \delta\,,$$

and so

$$\sum_{k=1}^{n} \max_{t\in[a_k,b_k]} |u(t) - u(a_k)| = \sum_{k=1}^{n} |u(c_k) - u(a_k)| \le \frac{\varepsilon}{l}\,. \tag{14.8}$$

Using (14.7) and (14.8), we may conclude

$$\sum_{k=1}^{n} |(\Phi(u))(b_k) - (\Phi(u))(a_k)| \le l \sum_{k=1}^{n} \max_{t\in[a_k,b_k]} |u(t) - u(a_k)| \le \varepsilon\,,$$

showing that $\Phi(u) \in AC(\mathbb{R}_+, \mathbb{R})$.

By (N1) and Lipschitz continuity (with Lipschitz continuity constant l), it is clear that (N5) holds with Lipschitz constant $\lambda = l$. Finally, we show that (N8) is satisfied. To this end let $a > 0$ and $u \in C([0,a), \mathbb{R})$, then by (N1), Lipschitz continuity, and Remark 14.3,

$$\sigma_\tau(\Phi(\mathbf{Q}_\tau u) - \Phi(0)) \le l\sigma_\tau(\mathbf{Q}_\tau u)\,, \quad \forall \tau \in [0,a)\,.$$

Therefore, by (N1) and (N2),

$$\sup_{t\in[0,\tau]} |(\Phi(u))(t)| \le l \sup_{t\in[0,\tau]} |u(t)| + |(\Phi(0))(0)|\,, \quad \forall \tau \in [0,a)\,,$$

showing that assumption (N8) is satisfied with $\alpha = (\Phi(0))(0)$ and $\beta = l$. □

For future reference we state the following lemma.

Lemma 14.4. *Let $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ be Lipschitz continuous. If for any ultimately nondecreasing $u \in C(\mathbb{R}_+, \mathbb{R})$ with $\lim_{t\to\infty} u(t) = \infty$,*

$$\lim_{t\to\infty} (\Phi(u))(t) = \sup \mathrm{NVS}\,\Phi \quad \textit{and} \quad \lim_{t\to\infty} (\Phi(-u))(t) = \inf \mathrm{NVS}\,\Phi\,,$$

then Φ satisfies (N6).

The proof of Lemma 14.4 is straightforward and is therefore omitted.

14.4 Integral Control in the Presence of Input Nonlinearities Satisfying (N1) to (N8)

In the following, let (A, B, C, D) be the generating operators of a linear single-input single-output regular system Σ with state space X and transfer function $\mathbf{G}$, and let $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ be a dynamic input nonlinearity. Denoting the constant reference signal by r, an application of the integrator

$$u(t) = u_0 + k \int_0^t [r - C_L x(\tau) - D(\Phi(u))(\tau)]\, d\tau\,,$$

where k is a real parameter (see Figure 14.1), leads to the following nonlinear system of differential equations

$$\begin{aligned} \dot{x} &= Ax + B\Phi(u)\,, \qquad x(0) = x_0 \in X\,, && (14.9a)\\ \dot{u} &= k[r - C_L x - D\Phi(u)]\,, \qquad u(0) = u_0 \in \mathbb{R}\,. && (14.9b) \end{aligned}$$

A continuous function

$$[0,\tau) \to X \times \mathbb{R}\,, \quad t \mapsto (x(t), u(t))$$

is called a *solution* of (14.9) if $(x(\cdot), u(\cdot))$ is absolutely continuous as a $(X_{-1} \times \mathbb{R})$-valued function, $x(t) \in \operatorname{dom}(C_L)$ for a.e. $t \in [0,\tau)$, $(x(0), u(0)) = (x_0, u_0)$, and the differential equations in (14.9) are satisfied a.e. on $[0,\tau)$. Of course, the derivative on the left-hand side of (14.9a) has to be understood in X_{-1}.[2]

An application of a well-known result on abstract Cauchy problems (see Pazy [18, Theorem 2.4, p. 107]) shows that a continuous $(X \times \mathbb{R})$-valued function $(x(\cdot), u(\cdot))$ is a solution of (14.9) if and only if it satisfies the following integrated version of (14.9),

$$\begin{aligned} x(t) &= \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B(\Phi(u))(\tau)\, d\tau\,,\\ u(t) &= u_0 + k \int_0^t [r - C_L x(\tau) - D(\Phi(u))(\tau)]\, d\tau\,. \end{aligned}$$

The next result shows that (14.9) has a unique solution.

Proposition 14.1. *For any $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ satisfying (N1), (N5), and (N8) and any pair $(x_0, u_0) \in X \times \mathbb{R}$ of initial conditions, there exists a unique solution $(x(\cdot), u(\cdot))$ of (14.9) defined on $\mathbb{R}_+$.*

For the proof of the above result it is useful to consider the following initial-value problem for u,

$$\dot{u} = k[r - \Psi x_0 - \mathbf{F}(\Phi(u))]\,, \qquad u(0) = u_0\,, \qquad (14.10)$$

[2] Being a Hilbert space, $X_{-1} \times \mathbb{R}$ is reflexive. Hence any absolutely continuous $(X_{-1} \times \mathbb{R})$-valued function is a.e. differentiable and can be recovered from its derivative by integration; see [2, Theorem 3.1, p. 10].

where $\mathbf{\Psi}$ and $\mathbf{F}$ are the state-to-output and input-to-output operators of Σ, respectively (see Section 14.2). Clearly, (14.10) is obtained from (14.9b) on noting that $C_L x(t) + D(\Phi(u))(t) = (\mathbf{\Psi} x_0)(t) + [\mathbf{F}(\Phi(u))](t)$. An absolutely continuous function $u : [0, \tau) \to \mathbb{R}$ is a *solution* of (14.10) if $u(0) = u_0$ and the differential equation in (14.10) is satisfied a.e. on $[0, \tau)$.

Lemma 14.5. *Let $x_0 \in X$. For any $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ satisfying (N1), (N5) and (N8) and any initial condition $u_0 \in \mathbb{R}$, there exists a unique solution $u(\cdot)$ of (14.10) defined on $\mathbb{R}_+$.*

The proof of this lemma is relegated to the Appendix (Section 14.7).

Proof: (of Proposition 14.1) Let $u : \mathbb{R}_+ \to \mathbb{R}$ be the unique solution of (14.10) (the existence of such a solution is guaranteed by Lemma 14.5) and *define* $x(\cdot)$ to be the unique solution of

$$\dot{x} = Ax + B\Phi(u)\,, \qquad x(0) = x_0\,.$$

Then $(x(\cdot), u(\cdot))$ is the unique solution of (14.9) defined on $\mathbb{R}_+$. □

If $\mathbf{G}$ is holomorphic and bounded on $\mathbb{C}_\alpha$ for some $\alpha < 0$ (which is the case if $\mathbf{T}_t$ is exponentially stable) and $\mathbf{G}(0) > 0$, then it is easy to show that

$$1 + k \operatorname{Re} \frac{\mathbf{G}(s)}{s} \geq 0\,, \quad \forall\, s \in \mathbb{C}_0\,, \tag{14.11}$$

for all sufficiently small $k > 0$; see [11, Lemma 3.10]. We define

$$K := \sup\{k > 0 \,|\, (14.11) \text{ holds}\}\,. \tag{14.12}$$

Henceforth, let $\mathcal{M}_f(\mathbb{R}_+)$ denote the space of all finite signed Borel measures on $\mathbb{R}_+$. Recall that a signed measure μ on $\mathbb{R}_+$ is called finite if $|\mu|(\mathbb{R}_+) < \infty$, where $|\mu|$ denotes the total variation of μ.

The main result of this section is the following theorem.

Theorem 14.1. *Let $\lambda > 0$. Assume that $\Phi \in \mathcal{N}(\lambda)$, $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_f(\mathbb{R}_+)$, $\mathbf{T}_t$ is exponentially stable, $\mathbf{G}(0) > 0$, $k \in (0, K/\lambda)$, and $r \in \mathbb{R}$ is such that*

$$\Phi_r := r/\mathbf{G}(0) \in \operatorname{clos}(\mathrm{NVS}\,\Phi)\,. \tag{14.13}$$

Then, for all $(x_0, u_0) \in X \times \mathbb{R}$, a unique solution $(x(\cdot), u(\cdot))$ of (14.9) exists on $\mathbb{R}_+$ and satisfies

(i) $\lim_{t\to\infty} (\Phi(u))(t) = \Phi_r\,,$

(ii) $\lim_{t\to\infty} \|x(t) + A^{-1} B \Phi_r\| = 0\,,$

(iii) $\lim_{t\to\infty}[r - y(t) + (\Psi x_0)(t)] = 0$, *where* $y(t) = C_L x(t) + D(\Phi(u))(t)$,

(iv) *if* $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$, *then* $u(\cdot)$ *is bounded.*

Remark 14.4. (i) Since $(\Psi x_0)(t)$ converges exponentially to 0 as $t \to \infty$ for all $x_0 \in X_1 = \mathrm{dom}(A)$, it follows from (iii) that the error $e(t) = r - y(t)$ converges to 0 for all $x_0 \in \mathrm{dom}(A)$. If C is bounded, then this statement is true for all $x_0 \in X$. If C is unbounded and $x_0 \notin \mathrm{dom}(A)$, then $e(t)$ does not necessarily converge to 0 as $t \to \infty$. However, the proof of Theorem 14.1 shows that $e(t)$ is small for large t in the sense that $e(t) = e_1(t) + e_2(t)$, where the function e_1 is bounded with $\lim_{t\to\infty} e_1(t) = 0$ and $e_2 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$.

(ii) The assumption that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_f(\mathbb{R}_+)$ is not very restrictive and seems to be satisfied in all practical examples of exponentially stable systems. In particular, this assumption is satisfied if B or C is bounded (see [10, Lemma 2.3]).

(iii) In applying Theorem 14.1 it is important to know the constant K or at least a lower bound for K. In principle, K can be obtained from frequency/step response experiments performed on the linear part of the plant; see [12] for details.

Proof: **(of Theorem 14.1)** By Proposition 14.1, there exists a unique solution of (14.9) on $\mathbb{R}_+$. We denote this solution by $(x(\cdot), u(\cdot))$ and introduce new variables by defining

$$z(t) := x(t) + A^{-1}B(\Phi(u))(t)\,, \qquad v(t) := (\Phi(u))(t) - \Phi_r\,; \qquad \forall\, t \geq 0\,.$$

By regularity it follows that $z(t) \in \mathrm{dom}\,(C_L)$ for a.e. $t \in \mathbb{R}_+$. Moreover, by Lemma 14.2 (iii), there exists a measurable function $d_u : \mathbb{R}_+ \to [0, \lambda]$ such that $\dot v(t) = d_u(t)\dot u(t)$ for a.e. $t \in \mathbb{R}_+$. Therefore an easy calculation yields that for a.e. $t \in \mathbb{R}_+$,

$$\begin{aligned} \dot z(t) &= Az(t) - kd_u(t)A^{-1}B(C_L z(t) + \mathbf{G}(0)v(t))\,, \qquad z(0) = z_0\,, && (14.14a)\\ \dot v(t) &= -kd_u(t)(C_L z(t) + \mathbf{G}(0)v(t))\,, \qquad v(0) = v_0\,, && (14.14b) \end{aligned}$$

where

$$z_0 := x_0 + A^{-1}B(\Phi(u))(0)\,, \qquad v_0 := (\Phi(u))(0) - \Phi_r\,.$$

The derivative on the left-hand side of (14.14a) has to be understood in X_{-1}. We observe that, while in these new variables we still have an unbounded operator $A^{-1}BC_L$, the operator $A^{-1}B$ is in $L(\mathbb{R}, X)$. Since d_u is a measurable function satisfying $d_u(t) \in [0, \lambda]$ for all $t \in \mathbb{R}_+$ and $k \in (0, K/\lambda)$, it follows from the Lyapunov argument developed in [13] (see the proof of Theorem 3.3 in [13]) that the limit of $v(t)$ as $t \to \infty$ exists and is finite, and hence there exists a number $L \in \mathbb{R}$ such that

$$\lim_{t\to\infty} (\Phi(u))(t) = L\,.$$

The essence of the proof is to show that $L = \Phi_r$. Setting

$$y_0(t) = (\Psi x_0)(t)\,, \qquad y_1(t) = [\mathfrak{L}^{-1}(\mathbf{G}) \star (\Phi(u))](t)\,,$$

where $\star$ denotes convolution, we have

$$\dot{u}(t) = k[r - y_0(t) - y_1(t)], \quad \text{a.e. } t \in \mathbb{R}. \tag{14.15}$$

Since $\lim_{t\to\infty}(\Phi(u))(t) = L$ and $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_f(\mathbb{R}_+)$, it follows that

$$\lim_{t\to\infty} y_1(t) = \mathbf{G}(0)L; \tag{14.16}$$

see [8, Theorem 6.1 (ii), p. 96]. Define a function $\tilde{y}_1 : \mathbb{R}_+ \to \mathbb{R}$ by setting

$$\tilde{y}_1(t) = r - y_1(t) = \mathbf{G}(0)\Phi_r - y_1(t).$$

Seeking a contradiction, suppose that $L \neq \Phi_r$. Then, either $\Phi_r > L$ or $\Phi_r < L$. If $\Phi_r > L$, then by (14.16), there exists a number $\tau_0 \geq 0$ such that

$$\tilde{y}_1(t) \geq \frac{1}{2}\mathbf{G}(0)(\Phi_r - L) > 0, \quad \forall\, t \geq \tau_0. \tag{14.17}$$

Hence, integrating (14.15) yields

$$u(t) = u(\tau) + k\left(\int_\tau^t \tilde{y}_1(s)\,ds - \int_\tau^t y_0(s)\,ds\right), \quad t \geq \tau \geq \tau_0. \tag{14.18}$$

By exponential stability, $y_0 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$, and thus $y_0 \in L^1(\mathbb{R}_+, \mathbb{R})$. Therefore, for given $\varepsilon > 0$, there exists $\tau_\varepsilon \geq \tau_0$ such that

$$\int_{\tau_\varepsilon}^\infty |y_0(s)|\,ds \leq \frac{\varepsilon}{k}. \tag{14.19}$$

Defining $u_\varepsilon \in C(\mathbb{R}_+, \mathbb{R})$ by

$$u_\varepsilon(t) = \begin{cases} u(t) & \text{for } 0 \leq t \leq \tau_\varepsilon, \\ u(\tau_\varepsilon) + k\int_{\tau_\varepsilon}^t \tilde{y}_1(s)\,ds & \text{for } t > \tau_\varepsilon, \end{cases}$$

it follows from (14.17) that u_ε is ultimately nondecreasing, and moreover, by (14.18) and (14.19),

$$|u(t) - u_\varepsilon(t)| \leq \varepsilon, \quad \forall\, t \in \mathbb{R}_+,$$

showing that u is approximately ultimately nondecreasing. Since $u(t) \to \infty$ as $t \to \infty$, we may invoke (N6) to conclude that

$$\Phi_r > L = \lim_{t\to\infty}(\Phi(u))(t) = \sup \mathrm{NVS}\,\Phi,$$

which is in contradiction to (14.13). If $\Phi_r < L$, then a very similar argument shows that $-u$ is approximately ultimately nondecreasing and $\lim_{t\to\infty}(-u)(t) = \infty$. Invoking (N6) gives

$$\Phi_r < L = \lim_{t\to\infty}(\Phi(u))(t) = \inf \mathrm{NVS}\,\Phi,$$

which again is in contradiction to (14.13). Therefore, we may conclude that $L = \Phi_r$ and thus $\lim_{t\to\infty}(\Phi(u))(t) = \Phi_r$, which is statement (i). Statement (ii) follows from statement (i) and Lemma 14.1. Statement (iii) is a consequence of statement (i), together with the identity

$$r - y(t) + (\Psi x_0)(t) = \mathbf{G}(0)\Phi_r - [\mathfrak{L}^{-1}(\mathbf{G}) \star (\Phi(u))](t)\,,$$

and the fact that $\lim_{t\to\infty}[\mathfrak{L}^{-1}(\mathbf{G}) \star (\Phi(u))](t) = \mathbf{G}(0)\Phi_r$. Finally, to prove statement (iv), let $\Phi_r \in \operatorname{int}\mathrm{NVS}\,\Phi$. Then, boundedness of u follows immediately from statement (i) and (N7). □

Remark 14.5. (i) For $\lambda, \tilde\lambda > 0$, define $\mathcal{N}(\lambda, \tilde\lambda)$ to be the set of all $\Phi \in \mathcal{N}(\lambda)$ such that for all $u \in AC(\mathbb{R}_+, \mathbb{R})$, there exists a measurable function $d_u : \mathbb{R}_+ \to [0, \tilde\lambda]$ such that (14.5) holds. Of course, by Lemma 14.2 (iii), $\mathcal{N}(\lambda, \tilde\lambda) = \mathcal{N}(\lambda)$ if $\tilde\lambda \geq \lambda$. The proof of Theorem 14.1 shows that if $\Phi \in \mathcal{N}(\lambda, \tilde\lambda)$, then statements (i) through (iv) of Theorem 14.1 are true for all $k \in (0, K/\tilde\lambda)$. If $\tilde\lambda < \lambda$, then $K/\tilde\lambda > K/\lambda$, which means that in this case a wider range of gain parameters k may be used. This observation is relevant for Section 14.5, where we show that there exist $\lambda, \tilde\lambda > 0$ with $\tilde\lambda < \lambda$ and hysteresis operators $\Phi \in \mathcal{N}(\lambda, \tilde\lambda)$ such that $\Phi \notin \mathcal{N}(l)$ for all $l \in (0, \lambda)$.

(ii) We see from the proof of Theorem 14.1 that (N7) is only needed for statement (iv).

14.5 Hysteresis Nonlinearities Satisfying (N1) to (N8)

In this section we consider various classes of hysteresis operators and we show that under certain conditions these operators satisfy (N1) to (N8).

Static nonlinearities

Although static nonlinearities do not describe hysteresis phenomena, we include them here because (i) they form a special subclass of hysteresis operators as defined in [4] and (ii) we would like to recover the main result in [13] whose content is essentially Theorem 14.1 for *static* nondecreasing globally Lipschitz nonlinearities.

For a continuous function $\phi : \mathbb{R} \to \mathbb{R}$, define the corresponding static nonlinearity by

$$\mathcal{S}_\phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})\,, \quad u \mapsto \phi \circ u\,.$$

The proof of the following proposition is straightforward and is therefore left to the reader.

Proposition 14.2. *If $\phi : \mathbb{R} \to \mathbb{R}$ is nondecreasing and globally Lipschitz with Lipschitz constant $\lambda > 0$, then $\mathcal{S}_\phi \in \mathcal{N}(\lambda)$.*

By a combination of Theorem 14.1 and Proposition 14.2 we recover the main result of [13]. Note that saturation and deadzone nonlinearities satisfy the assumptions of Proposition 14.2.

Relay hysteresis

In *relay* (also called *passive* or *positive*) hysteresis, the relationship between input and output is determined by two threshold values $a_1 < a_2$ for the input. The output $v(t) = (\mathcal{R}_\xi(u))(t)$ moves, for a given continuous input $u(t)$, on one of two fixed curves $\rho_1 : [a_1, \infty) \to \mathbb{R}$ and $\rho_2 : (-\infty, a_2] \to \mathbb{R}$ (see Figure 14.2), depending on which threshold, a_1 or a_2, was last attained. In the following we restrict our attention to "continuous" relay hysteresis nonlinearities; that is, the two curves ρ_1 and ρ_2 join at a_1 and a_2.

More formally, let $a_1, a_2 \in \mathbb{R}$ with $a_1 < a_2$ and let $\rho_1 : [a_1, \infty) \to \mathbb{R}$ and $\rho_2 : (-\infty, a_2] \to \mathbb{R}$ be continuous and such that $\rho_1(a_1) = \rho_2(a_1)$ and $\rho_1(a_2) = \rho_2(a_2)$. For $u \in C(\mathbb{R}_+, \mathbb{R})$ and $t \geq 0$ define

$$S(u,t) := u^{-1}(\{a_1, a_2\}) \cap [0,t]\,, \quad \tau(u,t) := \begin{cases} \max S(u,t) & \text{if } S(u,t) \neq \emptyset\,, \\ -1 & \text{if } S(u,t) = \emptyset\,. \end{cases}$$

Following Macki et al. [16], for each $\xi \in \mathbb{R}$, we define an operator $\mathcal{R}_\xi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ by

$$(\mathcal{R}_\xi(u))(t) = \begin{cases} \rho_2(u(t)) & \text{if } u(t) \leq a_1, \\ \rho_1(u(t)) & \text{if } u(t) \geq a_2, \\ \rho_2(u(t)) & \text{if } u(t) \in (a_1, a_2),\ \tau(u,t) \neq -1,\ u(\tau(u,t)) = a_1, \\ \rho_1(u(t)) & \text{if } u(t) \in (a_1, a_2),\ \tau(u,t) \neq -1,\ u(\tau(u,t)) = a_2, \\ \rho_1(u(t)) & \text{if } u(t) \in (a_1, a_2),\ \tau(u,t) = -1,\ \xi > 0, \\ \rho_2(u(t)) & \text{if } u(t) \in (a_1, a_2),\ \tau(u,t) = -1,\ \xi \leq 0. \end{cases} \tag{14.20}$$

The number ξ plays the role of an "initial state" that determines the output value $(\mathcal{R}_\xi(u))(t)$ if $u(s) \in (a_1, a_2)$ for all $s \in [0,t]$. The operator $\mathcal{R}_\xi$ is called a *relay hysteresis* operator and is illustrated in Figure 14.2.

Proposition 14.3. *If ρ_1 and ρ_2 are both nondecreasing and globally Lipschitz with Lipschitz constant $\lambda > 0$, then for each $\xi \in \mathbb{R}$, the operator $\mathcal{R}_\xi$ defined by* (14.20) *is in $\mathcal{N}(\lambda)$.*

Proof: A straightforward consequence of the definition of the relay hysteresis operator is that $\mathcal{R}_\xi$ satisfies conditions (N1), (N2), (N5), (N6), and (N8). To show that (N3) and (N4) hold, let $u \in AC(\mathbb{R}_+, \mathbb{R})$. For any compact interval $J \subset \mathbb{R}_+$,

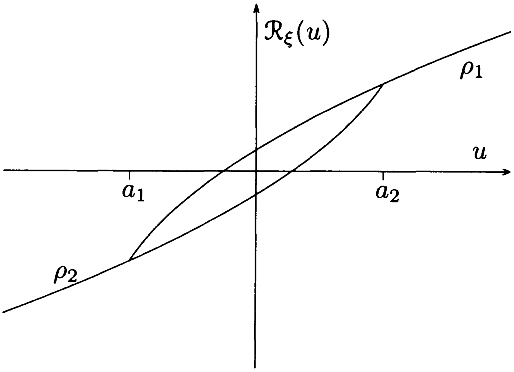

FIGURE 14.2. Relay hysteresis.

u is uniformly continuous on J, and therefore, using that $a_1 \neq a_2$, there exists $\delta > 0$, such that for all $t_1, t_2 \in J$,

$$u(t_1) = a_1\,, \quad u(t_2) = a_2 \quad \Longrightarrow \quad |t_2 - t_1| \geq \delta\,.$$

As a consequence, there exist numbers $t_i \uparrow \infty$ as $i \to \infty$, with $t_0 = 0$ and a map $j : \mathbb{N} \to \{1, 2\}$ such that for all $i \in \mathbb{N}$,

$$(\mathcal{R}_\xi(u))(t) = \rho_{j(i)}(u(t))\,, \quad \forall\, t \in [t_i, t_{i+1}]\,. \tag{14.21}$$

It follows that $\mathcal{R}_\xi(u)$ is absolutely continuous on $[t_i, t_{i+1}]$ for each $i \in \mathbb{N}$. Hence, by continuity of $\mathcal{R}_\xi(u)$, we may conclude that $\mathcal{R}_\xi(u) \in AC(\mathbb{R}_+, \mathbb{R})$, showing that (N3) holds. Furthermore, since ρ_1 and ρ_2 are nondecreasing and Lipschitz, (14.21) yields that for all $i \in \mathbb{N}$,

$$\frac{d}{dt}(\mathcal{R}_\xi(u))(t)\dot{u}(t) \geq 0\,, \quad \text{a.e. } t \in [t_i, t_{i+1}]\,,$$

which implies that (N4) holds.

Finally, to show that (N7) is satisfied, note first that $\mathrm{NVS}\,\mathcal{R}_\xi = \operatorname{im}\rho_1 \cup \operatorname{im}\rho_2$. Let $u \in C(\mathbb{R}_+, \mathbb{R})$ and suppose $\lim_{t\to\infty}(\mathcal{R}_\xi(u))(t) = l \in \operatorname{int}\mathrm{NVS}\,\mathcal{R}_\xi$. Then there exist $\varepsilon > 0$ and $T \geq 0$ such that $I_\varepsilon := (l - \varepsilon, l + \varepsilon) \subset \operatorname{int}\mathrm{NVS}\,\mathcal{R}_\xi$ and $(\mathcal{R}_\xi(u))(t) \in I_\varepsilon$ for all $t \geq T$, which implies

$$u(t) \in \rho_1^{-1}(I_\varepsilon) \cup \rho_2^{-1}(I_\varepsilon) =: U\,, \quad \forall\, t \geq T\,. \tag{14.22}$$

But the set U is bounded since $\sup\rho_1, \inf\rho_2 \notin I_\varepsilon$, and ρ_1 and ρ_2 are nondecreasing. Combining this with (14.22) shows that u is bounded. □

We remark that although the relay hysteresis operator $\mathcal{R}_\xi$ as defined in (14.20) satisfies the Lipschitz condition (N5), $\mathcal{R}_\xi$ is not Lipschitz continuous in the sense of Definition 14.2. In fact, it is easy to show that $\mathcal{R}_\xi$ is

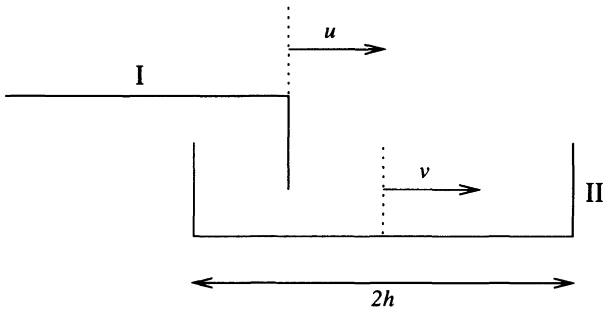

FIGURE 14.3. Schematic representation of backlash.

not even continuous with respect to the topology on $C(\mathbb{R}_+, \mathbb{R})$ given by the family of seminorms $\{\sigma_n \mid n \in \mathbb{N}\}$. In particular, when we talk about "continuous" relay hysteresis, we simply mean that the output corresponding to a continuous input is continuous, but not that the relay hysteresis operator is continuous with respect to any natural topology on $C(\mathbb{R}_+, \mathbb{R})$.

Backlash hysteresis

The *backlash* operator (also called *play* operator) has been discussed in a mathematically rigorous context in a number of references; see, for example [3, 4, 9, 19]. Intuitively, the backlash operator describes the input-output behavior of a simple mechanical play between two mechanical elements I and II shown in Figure 14.3. The position of element I at time t is denoted by $u(t)$. The position $v(t)$ of the middle point of element II at time t will remain constant as long as $u(t)$ moves in the interior and it will change at the rate $\dot{v}=\dot{u}$ as long as $u(t)$ hits the boundary of element II with an outward directed velocity.

To give a formal definition of backlash, define for each $h \in \mathbb{R}_+$ the function $b_h : \mathbb{R}^2 \to \mathbb{R}$ by

$$b_h(v, w) = \max\{v - h, \min\{v + h, w\}\}.$$

The proof of the following semigroup property can be found in the Appendix (Section 14.7).

Lemma 14.6. *Let $t_1 < t_2$, $u : [t_1, t_2] \to \mathbb{R}$ be monotone and $w \in [u(t_1) - h, u(t_1) + h]$. Then, for all $t, \tau \in [t_1, t_2]$ with $t \geq \tau$,*

$$b_h(u(t), w) = b_h(u(t), b_h(u(\tau), w)).$$

For all $h \in \mathbb{R}_+$ and all $\xi \in \mathbb{R}$ we introduce an operator $\mathcal{B}_{h,\xi}$ on $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$

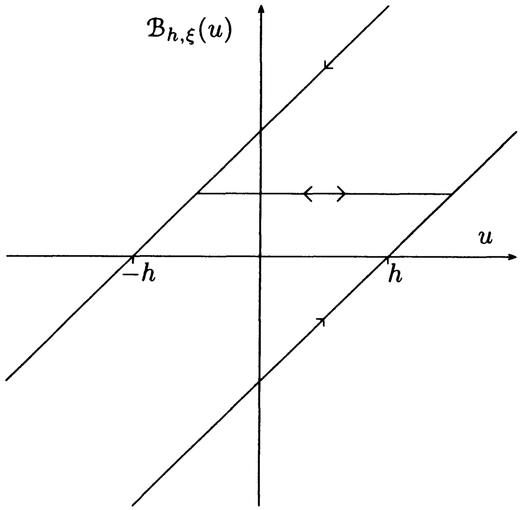

FIGURE 14.4. Backlash hysteresis.

by defining recursively for every $u \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$,

$$(\mathcal{B}_{h,\xi}(u))(t) = \begin{cases} b_h(u(0), \xi) & \text{for } t = 0\,, \\ b_h(u(t), (\mathcal{B}_{h,\xi}(u))(t_i)) & \text{for } t_i < t \leq t_{i+1},\ i \in \mathbb{N}\,, \end{cases} \tag{14.23}$$

where $0 = t_0 < t_1 < t_2 < \dots$ is a partition of $\mathbb{R}_+$, such that u is monotone on each of the intervals $[t_i, t_{i+1}]$. Again, ξ plays the role of an "initial state." Using Lemma 14.6 it is not difficult to show that the definition of $\mathcal{B}_{h,\xi}(u)$ is independent of the choice of partition. Clearly, $\mathcal{B}_{h,\xi}(u)$ is continuous at each $t \in \mathbb{R}$, $t \neq t_i$ for all $i \in \mathbb{N}$ and is left-continuous at t_i for all $i \in \mathbb{N}\backslash\{0\}$. Moreover, an application of Lemma 14.6 shows that $\mathcal{B}_{h,\xi}(u)$ is right-continuous at t_i for all $i \in \mathbb{N}$. Consequently, $\mathcal{B}_{h,\xi}(u)$ is continuous for all $u \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. The backlash operator $\mathcal{B}_{h,\xi}$ is illustrated in Figure 14.4.

Proposition 14.4. *Let $(h, \xi) \in \mathbb{R}_+ \times \mathbb{R}$. The backlash operator $\mathcal{B}_{h,\xi}$ has the following properties.*

(i) $\mathcal{B}_{h,\xi} : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ *is Lipschitz continuous with Lipschitz continuity constant $l = 1$ and uniquely extends to a Lipschitz continuous operator* $\mathcal{B}_{h,\xi} : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ *with Lipschitz continuity constant $l = 1$.*

(ii) $\mathcal{B}_{h,\xi} : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ *satisfies (N1) to (N3), (N4′), and (N5) to (N8). In particular,* $\mathcal{B}_{h,\xi} \in \mathcal{N}(1)$.

Proof: Statement (i) follows from [4, p. 42]. For statement (ii), we first note that as an immediate consequence of the definition of $\mathcal{B}_{h,\xi}$ and statement (i), (N1) and (N2) are satisfied. Combining statement (i) and Lemma 14.3, $\mathcal{B}_{h,\xi}$ also satisfies conditions (N3), (N5), and (N8). To show that (N4′) holds, let $u \in AC(\mathbb{R}_+, \mathbb{R})$ and let E be the set of all $t \in \mathbb{R}_+$ such that u or $\mathcal{B}_{h,\xi}(u)$ is not differentiable at t. Clearly, E has zero measure. We need to show that

$$\frac{d}{dt}(\mathcal{B}_{h,\xi}(u))(t)\dot{u}(t) \geq 0\,, \quad \forall t \in \mathbb{R}_+ \setminus E\,. \tag{14.24}$$

Let $t \in \mathbb{R}_+ \setminus E$. If $\dot{u}(t) = 0$, then (14.24) holds trivially. If $\dot{u}(t) > 0$, then there exist $t_1 > t$ and $u_n \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ such that $\sigma_{t_1}(u_n - u) \to 0$ as $n \to \infty$ and $u_n(\tau) \geq u_n(t)$ for all $\tau \in (t, t_1)$ and all $n \in \mathbb{N}$. It follows that $(\mathcal{B}_{h,\xi}(u_n))(\tau) \geq (\mathcal{B}_{h,\xi}(u_n))(t)$ for all $\tau \in (t, t_1)$, which in turn implies $(\mathcal{B}_{h,\xi}(u))(\tau) \geq (\mathcal{B}_{h,\xi}(u))(t)$ for all $\tau \in (t, t_1)$. Therefore

$$\frac{d}{dt}(\mathcal{B}_{h,\xi}(u))(t) = \lim_{\varepsilon \downarrow 0} \frac{(\mathcal{B}_{h,\xi}(u))(t+\varepsilon) - (\mathcal{B}_{h,\xi}(u))(t)}{\varepsilon} \geq 0\,,$$

and so (14.24) holds. If $\dot{u}(t) < 0$, then (14.24) can be obtained by a very similar argument.

To show that (N6) is satisfied, note first that NVS $\mathcal{B}_{h,\xi} = \mathbb{R}$. Let $u \in C(\mathbb{R}_+, \mathbb{R})$ be ultimately nondecreasing with $\lim_{t\to\infty} u(t) = \infty$. Then there exists $T \in \mathbb{R}_+$ such that $(\mathcal{B}_{h,\xi}(u))(t) = u(t) - h$ for all $t \geq T$. Thus, $\lim_{t\to\infty}(\mathcal{B}_{h,\xi}(u))(t) = \infty$. Similarly, $\lim_{t\to\infty}(\mathcal{B}_{h,\xi}(-u))(t) = -\infty$. It follows from statement (i) and Lemma 14.4 that (N6) holds.

For (N7), let $u \in C(\mathbb{R}_+, \mathbb{R})$ and suppose $\lim_{t\to\infty}(\mathcal{B}_{h,\xi}(u))(t) = l \in \mathbb{R}$. Then there exist $\varepsilon > 0$, $T \in \mathbb{R}_+$ such that $(\mathcal{B}_{h,\xi}(u))(t) \in (l - \varepsilon, l + \varepsilon)$ for all $t \geq T$. Consequently, $u(t) \in (l - \varepsilon - h, l + \varepsilon + h)$ for all $t \geq T$, and hence, u is bounded. □

Remark 14.6. Let $\beta_1, \beta_2 : \mathbb{R} \to \mathbb{R}$ be continuous and such that $\operatorname{im} \beta_1 = \operatorname{im} \beta_2$ and $\beta_1(v) \leq \beta_2(v)$ for all $v \in \mathbb{R}$. Setting

$$b(v, w) = \max\{\beta_1(v), \min\{\beta_2(v), w\}\}\,,$$

we can define for each $\xi \in \mathbb{R}$ a *generalized backlash* operator $\mathcal{B}_\xi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ by (14.23) with b_h replaced by b. Generalized backlash (also called *generalized play*) was introduced in [9]. If β_1 and β_2 are both nondecreasing and globally Lipschitz with Lipschitz constant $\lambda > 0$, then a suitably modified version of Proposition 14.23 holds for generalized backlash, in particular $\mathcal{B}_\xi \in \mathcal{N}(\lambda)$. An example of generalized backlash is illustrated in Figure 14.5.

Elastic-plastic hysteresis

The *elastic-plastic* operator (also called *stop* operator) models the stress-strain relationship in a one-dimensional elastic-plastic element. As long as the modulus of the stress v is smaller than the yield stress h, the strain u is related to v through the linear Hooke's Law. Once the stress exceeds

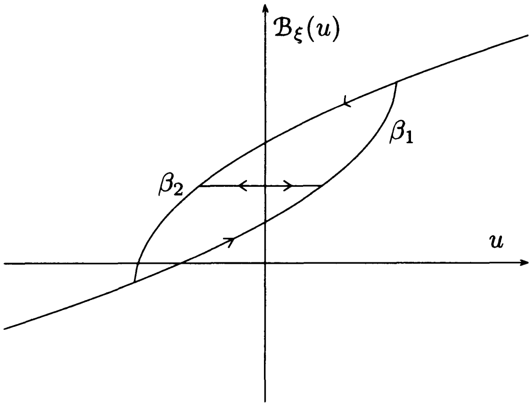

FIGURE 14.5. Generalized backlash hysteresis.

the yield value it remains constant under further increasing of the strain; however, the elastic behavior is instantly recovered when the strain is again decreased. As we show, elastic-plastic hysteresis is closely related to backlash hysteresis.

To give a formal definition of the elastic-plastic operator, define for each $h \in \mathbb{R}_+$ the function $e_h : \mathbb{R} \to \mathbb{R}$ by

$$e_h(u) = \min\{h, \max\{-h, u\}\}\,.$$

Following [4], for all $h \in \mathbb{R}_+$ and all $\xi \in \mathbb{R}$, we introduce an operator $\mathcal{E}_{h,\xi}$ on $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ by defining recursively for every $u \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$,

$$(\mathcal{E}_{h,\xi}(u))(t) = \begin{cases} e_h(u(0) - \xi) & \text{for } t = 0, \\ e_h(u(t) - u(t_i) + (\mathcal{E}_{h,\xi}(u))(t_i)) & \text{for } t_i < t \le t_{i+1},\ i \in \mathbb{N}, \end{cases} \tag{14.25}$$

where $0 = t_0 < t_1 < t_2 < \dots$ is a partition of $\mathbb{R}_+$, such that u is monotone on each of the intervals $[t_i, t_{i+1}]$. As with backlash we note that the definition is independent of the choice of partition and $\mathcal{E}_{h,\xi}(C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$. The elastic-plastic operator $\mathcal{E}_{h,\xi}$ is illustrated in Figure 14.6.

To show that elastic-plastic hysteresis and the Preisach operator (introduced later) satisfy (N1) through (N8), we need the following lemma. The proof is routine and is therefore omitted.

Lemma 14.7. *Let $u \in C(\mathbb{R}_+, \mathbb{R})$ be unbounded. Then there exists an increasing sequence $(t_n) \subset \mathbb{R}_+$ with $\lim_{n\to\infty} t_n = \infty$ such that either*

$$u(t_n) = \sup_{t\in[0,t_n]} |u(t)|\,, \quad \forall n \in \mathbb{N} \quad \textit{or} \quad u(t_n) = -\sup_{t\in[0,t_n]} |u(t)|\,, \quad \forall n \in \mathbb{N}\,.$$

We recall the definition of $\mathcal{N}(\lambda, \tilde{\lambda})$ from Remark 14.5.

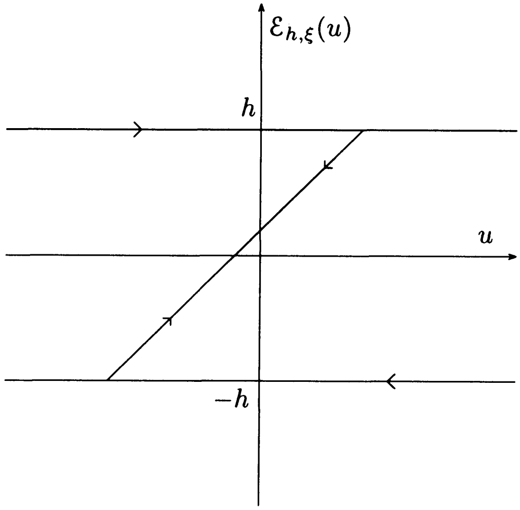

FIGURE 14.6. Elastic-plastic hysteresis.

Proposition 14.5. *Let $(h,\xi) \in \mathbb{R}_+ \times \mathbb{R}$. The elastic-plastic operator $\mathcal{E}_{h,\xi}$ has the following properties.*

(i) $\mathcal{E}_{h,\xi} : C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ *is Lipschitz continuous with Lipschitz continuity constant $l = 2$ and uniquely extends to a Lipschitz continuous operator $\mathcal{E}_{h,\xi} : C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ with Lipschitz continuity constant $l = 2$.*

(ii) for $H \in \mathbb{R}_+$, globally Lipschitz $\zeta : \mathbb{R}_+ \to \mathbb{R}$ with Lipschitz constant 1, $u \in C(\mathbb{R}_+,\mathbb{R})$, and $t \in \mathbb{R}_+$,

$$(\mathcal{E}_{H,\zeta(H)}(u))(t) = H \quad \Longrightarrow \quad (\mathcal{E}_{h,\zeta(h)}(u))(t) = h\,, \quad \forall\, h \in [0,H]\,,$$

and

$$(\mathcal{E}_{H,\zeta(H)}(u))(t) = -H \quad \Longrightarrow \quad (\mathcal{E}_{h,\zeta(h)}(u))(t) = -h\,, \quad \forall\, h \in [0,H]\,.$$

(iii) $\mathcal{E}_{h,\xi} : C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ *satisfies (N1) to (N3), (N4′), and (N5) to (N8). Furthermore, $\mathcal{E}_{h,\xi} \in \mathcal{N}(2,1) \subset \mathcal{N}(2)$.*

Remark 14.7. In statement (i), $l = 2$ is the smallest possible Lipschitz continuity constant for $\mathcal{E}_{h,\xi}$. To illustrate this, consider $u, v \in C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ defined by

$$u(t) = \begin{cases} t+\xi & \text{for } t \in [0,h], \\ h+\xi & \text{for } t > h, \end{cases}$$

$$v(t) = \begin{cases} t+\xi & \text{for } t \in [0,3h/2], \\ 3h-t+\xi & \text{for } t \in (3h/2,5h/2], \\ h/2+\xi & \text{for } t > 5h/2. \end{cases}$$

Then $\sigma_\tau(u-v) = h/2$ and $\sigma_\tau(\mathcal{E}_{h,\xi}(u) - \mathcal{E}_{h,\xi}(v)) = h$ for all $\tau \geq 5h/2$.

Proof: **(of Proposition 14.5)** Statement (i) follows from [4, p. 44]. To prove statement (ii), note that by [4, p. 42] and the Lipschitz continuity of $\mathcal{B}_{h,\xi}$, we have for every $u \in C(\mathbb{R}_+, \mathbb{R})$, $\xi_1, \xi_2 \in \mathbb{R}$, and $t, h_1, h_2 \in \mathbb{R}_+$,

$$|(\mathcal{B}_{h_1,\xi_1}(u))(t) - (\mathcal{B}_{h_2,\xi_2}(u))(t)| \leq \max(|h_1 - h_2|, |\xi_1 - \xi_2|)\,. \tag{14.26}$$

Also from [4, p. 44],

$$\mathcal{E}_{h,\xi}(u) + \mathcal{B}_{h,\xi}(u) = u\,, \quad \forall\, u \in C(\mathbb{R}_+, \mathbb{R})\,. \tag{14.27}$$

Now let $H \in \mathbb{R}_+$, $\zeta : \mathbb{R}_+ \to \mathbb{R}$ be globally Lipschitz with Lipschitz constant 1, $u \in C(\mathbb{R}_+, \mathbb{R})$, $t \in \mathbb{R}_+$ and suppose $(\mathcal{E}_{H,\zeta(H)}(u))(t) = H$. Then using (14.26) and (14.27), we have for all $h \in [0, H]$,

$$H - (\mathcal{E}_{h,\zeta(h)}(u))(t) = (\mathcal{E}_{H,\zeta(H)}(u))(t) - (\mathcal{E}_{h,\zeta(h)}(u))(t) \leq H - h\,,$$

and so since $(\mathcal{E}_{h,\zeta(h)}(u))(t) \leq h$, we obtain $(\mathcal{E}_{h,\zeta(h)}(u))(t) = h$ for all $h \in [0, H]$. The second implication in statement (ii) can be proved in a similar way.

To prove statement (iii), we first note that as an immediate consequence of the definition of $\mathcal{E}_{h,\xi}$ and statement (i), (N1) and (N2) are satisfied. Therefore, combining statement (i) and Lemma 14.3, $\mathcal{E}_{h,\xi}$ also satisfies conditions (N3), (N5), and (N8). To show (N4′) holds, let $u \in AC(\mathbb{R}_+, \mathbb{R})$ and E be the set of all $t \in \mathbb{R}_+$ such that u or $\mathcal{E}_{h,\xi}(u)$ is not differentiable at t. By (N3), E has zero measure. We need to show that

$$\frac{d}{dt}(\mathcal{E}_{h,\xi}(u))(t)\dot{u}(t) \geq 0\,, \quad \forall\, t \in \mathbb{R}_+ \setminus E\,. \tag{14.28}$$

Let $t \in \mathbb{R}_+ \setminus E$, then by (14.27), u, $\mathcal{E}_{h,\xi}(u)$ and $\mathcal{B}_{h,\xi}(u)$ are all differentiable at t and

$$\frac{d}{dt}(\mathcal{E}_{h,\xi}(u))(t) = -\frac{d}{dt}(\mathcal{B}_{h,\xi}(u))(t) + \dot{u}(t)\,.$$

Therefore, since $\mathcal{B}_{h,\xi} \in \mathcal{N}(1)$ and $\mathcal{B}_{h,\xi}$ satisfies (N4′), it follows from Lemma 14.2 (i), and Remark 14.2 that there exists a measurable function $d_u : \mathbb{R}_+ \to [0,1]$ such that

$$\frac{d}{dt}(\mathcal{E}_{h,\xi}(u))(t) = (1 - d_u(t))\dot{u}(t)\,, \quad \forall\, t \in \mathbb{R}_+ \setminus E\,,$$

and thus (14.28) holds. We note that $1 - d_u(t) \in [0,1]$ for all $t \in \mathbb{R}_+$ (although the smallest possible Lipschitz constant of $\mathcal{E}_{h,\xi}$ is $\lambda = 2$; see Remark 14.7). It follows that $\mathcal{E}_{h,\xi} \in \mathcal{N}(2,1) \subset \mathcal{N}(2)$ (cf. Remark 14.5 (i)) once we have shown that (N6) and (N7) hold.

To show that (N6) is satisfied, let $u \in C(\mathbb{R}_+, \mathbb{R})$ be ultimately non-decreasing with $\lim_{t\to\infty} u(t) = \infty$; then

$$\lim_{t\to\infty} (\mathcal{E}_{h,\xi}(u))(t) = h = \sup \mathrm{NVS}\, \mathcal{E}_{h,\xi}$$

and, similarly, $\lim_{t\to\infty}(\mathcal{E}_{h,\xi}(-u))(t) = -h = \inf \mathrm{NVS}\, \mathcal{E}_{h,\xi}$. It follows from statement (i) and Lemma 14.4 that (N6) holds.

For (N7), let $u \in C(\mathbb{R}_+, \mathbb{R})$ and suppose

$$\lim_{t\to\infty}(\mathcal{E}_{h,\xi}(u))(t) \in \operatorname{int} \mathrm{NVS}\, \mathcal{E}_{h,\xi} = (-h, h)\,.$$

Seeking a contradiction, assume that u is unbounded. Then, by Lemma 14.7, without loss of generality, we may assume that there exists an increasing sequence $(t_n) \subset \mathbb{R}_+$ such that $\lim_{n\to\infty} t_n = \infty$ and $u(t_n) - \xi = \sup_{t\in[0,t_n]} |u(t) - \xi|$. Moreover, again without loss of generality, we may assume that $u(t_n) > h + \xi$ for all $n \in \mathbb{N}$. Define for each $n \in \mathbb{N}$, $H_n := u(t_n) - \xi > h$; then $(\mathcal{E}_{H_n,\xi}(u))(t_n) = H_n$ for all $n \in \mathbb{N}$. By statement (ii), $(\mathcal{E}_{h,\xi}(u))(t_n) = h$ for all $n \in \mathbb{N}$, which is in contradiction to the assumption that $\lim_{t\to\infty}(\mathcal{E}_{h,\xi}(u))(t) \in (-h, h)$. □

Preisach Operators

All the hysteresis operators considered so far model relatively simple hysteresis loops. The Preisach operator, introduced below, represents a far more general type of hysteresis which for certain input functions exhibits nested loops in the corresponding input-output graphs.

In the following, let $\mathcal{M}_c(\mathbb{R}_+)$ denote the set of all signed Borel measures μ on $\mathbb{R}_+$ such that $|\mu|(S) < \infty$ for all compact sets $S \subset \mathbb{R}_+$.[3] Clearly, $\mathcal{M}_f(\mathbb{R}_+) \subset \mathcal{M}_c(\mathbb{R}_+)$. We denote the Lebesgue measure on $\mathbb{R}$ by μ_L.

Let $C_0(\mathbb{R}_+, \mathbb{R})$ be the set of all continuous functions $\zeta : \mathbb{R}_+ \to \mathbb{R}$ with compact support. We define the set of Preisach memory curves

$$P := \{\zeta \in C_0(\mathbb{R}_+, \mathbb{R}) \,|\, |\zeta(h_1) - \zeta(h_2)| \le |h_1 - h_2| \ \forall\, h_1, h_2 \in \mathbb{R}_+\}\,.$$

For given $\zeta \in P$, the *Preisach operator*, $\mathcal{P}_\zeta : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$, is defined by

$$(\mathcal{P}_\zeta(u))(t) = \int_0^\infty \int_0^{(\mathcal{B}_{h,\zeta(h)}(u))(t)} w(h,s)\, ds\, d\mu(h) + w_0\,, \tag{14.29}$$

where $\mu \in \mathcal{M}_c(\mathbb{R}_+)$, $w \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}; \mu \otimes \mu_L)$, and $w_0 \in \mathbb{R}$. It is clear that for fixed $\zeta \in P$, $u \in C(\mathbb{R}_+, \mathbb{R})$, and $t \in \mathbb{R}_+$, the map

$$\psi : \mathbb{R}_+ \to \mathbb{R}\,, \quad h \mapsto (\mathcal{B}_{h,\zeta(h)}(u))(t)\,,$$

is in P: by (14.26), ψ is globally Lipschitz with Lipschitz constant 1, and as a direct consequence of the definition of the backlash operator, ψ also has compact support. Consequently, the right-hand side of (14.29) is finite for all $u \in C(\mathbb{R}_+, \mathbb{R})$ and all $t \in \mathbb{R}_+$.

[3] If $\mu \in \mathcal{M}_c(\mathbb{R}_+)$, then it follows that the measure $|\mu|$ is regular, and hence that μ is a signed Radon measure; see [7, pp. 205–216].

The following two lemmas are useful for the verification of (N1) through (N8) for a large class of Preisach operators.

Lemma 14.8. *Suppose that $\mu \in \mathcal{M}_c(\mathbb{R}_+)$, $w \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}; \mu \otimes \mu_L)$, $w_0 \in \mathbb{R}$, $\zeta \in P$, and that $\mathcal{P}_\zeta$ is defined by* (14.29). *Let $u \in C(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$. If $u(t) = \sup_{\tau \in [0,t]} |u(\tau)|$ and $\zeta = 0$ on $[u(t), \infty)$, then*

$$(\mathcal{P}_\zeta(u))(t) = \int_0^{u(t)} \int_0^{u(t)-h} w(h,s)\, ds\, d\mu(h) + w_0\,.$$

Proof: Let $u \in C(\mathbb{R}_+, \mathbb{R})$, $t \in \mathbb{R}_+$, and suppose that $u(t) = \sup_{\tau \in [0,t]} |u(\tau)|$ and $\zeta = 0$ on $[u(t), \infty)$. Setting $H := u(t)$, we have $(\mathcal{B}_{H,\zeta(H)}(u))(t) = 0 = u(t) - H$ and $(\mathcal{B}_{h,\zeta(h)}(u))(t) = 0$ for all $h > H$. Combining Proposition 14.5 (ii) and (14.27) shows that $(\mathcal{B}_{h,\zeta(h)}(u))(t) = u(t) - h$ for all $h \in [0, H]$ and therefore by (14.29),

$$(\mathcal{P}_\zeta(u))(t) = \int_0^{u(t)} \int_0^{u(t)-h} w(h,s)\, ds\, d\mu(h) + w_0\,.$$

□

The proof of the following lemma follows immediately from [4, pp. 58–60].

Lemma 14.9. *Let $\mu \in \mathcal{M}_c(\mathbb{R}_+)$, $w \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}; \mu \otimes \mu_L)$, and $w_0 \in \mathbb{R}$. Suppose that $\lambda := \int_0^\infty \sup_{s \in \mathbb{R}} |w(h,s)|\, d|\mu|(h) < \infty$. Then for all $\zeta \in P$, the Preisach operator $\mathcal{P}_\zeta : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$, defined by* (14.29), *is Lipschitz continuous with Lipschitz continuity constant λ and for $u \in AC(\mathbb{R}_+, \mathbb{R})$,*

$$(\mathcal{P}_\zeta(u))'(t) = \int_0^\infty w(h, (\mathcal{B}_{h,\zeta(h)}(u))(t))(\mathcal{B}_{h,\zeta(h)}(u))'(t)\, d\mu(h)\,, \quad \text{a.e. } t \in \mathbb{R}_+\,,$$

where $'$ denotes differentiation with respect to t.

Remark 14.8. Let $\zeta \in \mathcal{P}$ and $u \in AC(\mathbb{R}_+, \mathbb{R})$. It is implicit in Lemma 14.9 that for μ_L-almost every $t \in \mathbb{R}_+$, $(\mathcal{B}_{h,\zeta(h)}(u))'(t)$ exists for $|\mu|$-almost every $h \in \mathbb{R}_+$. This result is proved in [4, Lemma 2.4.8].

Proposition 14.6. *Let $\mu \in \mathcal{M}_c(\mathbb{R}_+)$ be positive, let $w \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}; \mu \otimes \mu_L)$ be nonnegative, and let $w_0 \in \mathbb{R}$. Suppose that $\lambda := \int_0^\infty \sup_{s \in \mathbb{R}} w(h,s)\, d\mu(h) < \infty$. Then, for all $\zeta \in P$, the Preisach operator $\mathcal{P}_\zeta$, defined by* (14.29), *is in $\mathcal{N}(\lambda)$.*

Proof: By Lemma 14.9, $\mathcal{P}_\zeta : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ is Lipschitz continuous with Lipschitz continuity constant λ and, by its definition, satisfies conditions (N1) and (N2). Therefore, by Lemma 14.3, $\mathcal{P}_\zeta$ also satisfies conditions (N3), (N5), and (N8).

To show that (N4) holds, let $u \in AC(\mathbb{R}_+, \mathbb{R})$. By (N3) and Lemma 14.9 (see also Remark 14.8) there exists $E \subset \mathbb{R}_+$ with $\mu_L(E) = 0$ and such that for all

$t \in \mathbb{R}_+ \setminus E$, $u'(t)$ and $(\mathcal{P}_\zeta(u))'(t)$ exist, $(\mathcal{B}_{h,\zeta(h)}(u))'(t)$ exists for μ-almost every $h \in \mathbb{R}_+$, and

$$(\mathcal{P}_\zeta(u))'(t) = \int_0^\infty w(h, (\mathcal{B}_{h,\zeta(h)}(u))(t))(\mathcal{B}_{h,\zeta(h)}(u))'(t)\, d\mu(h)\,. \tag{14.30}$$

Let $t \in \mathbb{R}_+ \setminus E$. If $u'(t) = 0$, (N4) immediately follows. If $u'(t) > 0$, then, since (N4′) holds for $\mathcal{B}_{h,\zeta(h)}$, we have $(\mathcal{B}_{h,\zeta(h)}(u))'(t) \geq 0$, whenever this derivative exists (which is the case for μ-almost every $h \in \mathbb{R}_+$). Since w and μ are nonnegative, we obtain from (14.30) that $(\mathcal{P}_\zeta(u))'(t) \geq 0$. If $u'(t) < 0$, then (N4) can be shown to hold by a similar argument.

To show that (N6) is satisfied, let $u \in C(\mathbb{R}_+, \mathbb{R})$ be ultimately non-decreasing with $\lim_{t\to\infty} u(t) = \infty$. Then there exists $T \in \mathbb{R}_+$ such that for all $t \geq T$, $\sup_{\tau\in[0,t]} |u(\tau)| = u(t)$ and $\zeta = 0$ on $[u(t), \infty)$. So by Lemma 14.8,

$$(\mathcal{P}_\zeta(u))(t) = \int_0^{u(t)} \int_0^{u(t)-h} w(h,s)\, ds\, d\mu(h) + w_0\,, \quad \forall t \geq T\,,$$

and since $\lim_{t\to\infty} u(t) = \infty$,

$$\lim_{t\to\infty} (\mathcal{P}_\zeta(u))(t) = \int_0^\infty \int_0^\infty w(h,s)\, ds\, d\mu(h) + w_0 \in [w_0, \infty]\,. \tag{14.31}$$

We note that because μ and w are nonnegative

$$\sup \mathrm{NVS}\, \mathcal{P}_\zeta \leq \int_0^\infty \int_0^\infty w(h,s)\, ds\, d\mu(h) + w_0\,,$$

and therefore, by (14.31),

$$\lim_{t\to\infty} (\mathcal{P}_\zeta(u))(t) = \sup \mathrm{NVS}\, \mathcal{P}_\zeta = \int_0^\infty \int_0^\infty w(h,s)\, ds\, d\mu(h) + w_0\,. \tag{14.32}$$

Similarly, $\lim_{t\to\infty}(\mathcal{P}_\zeta(-u))(t) = \inf \mathrm{NVS}\, \mathcal{P}_\zeta$. It follows from Lipschitz continuity and Lemma 14.4 that (N6) holds.

For (N7), let $u \in C(\mathbb{R}_+, \mathbb{R})$ and suppose that

$$\lim_{t\to\infty} (\mathcal{P}_\zeta(u))(t) \in \mathrm{int}\, \mathrm{NVS}\, \mathcal{P}_\zeta\,.$$

Let $H \in \mathbb{R}_+$ be such that $\zeta = 0$ on $[H, \infty)$. Seeking a contradiction, suppose that u is unbounded. Then, by Lemma 14.7, without loss of generality, we may assume that there exists an increasing sequence $(t_n) \subset \mathbb{R}_+$ such that $\lim_{n\to\infty} t_n = \infty$ and $u(t_n) = \sup_{t\in[0,t_n]} |u(t)|$. Moreover, again without loss of generality, we may assume that $u(t_n) \geq H$ for all $n \in \mathbb{N}$. By Lemma 14.8

$$(\mathcal{P}_\zeta(u))(t_n) = \int_0^{u(t_n)} \int_0^{u(t_n)-h} w(h,s)\, ds\, d\mu(h) + w_0\,, \quad \forall n \in \mathbb{N}\,.$$

Since $\lim_{n\to\infty} u(t_n) = \infty$, it follows from the second equation in (14.32) that

$$\lim_{t\to\infty} (\mathcal{P}_\zeta(u))(t) = \lim_{n\to\infty} (\mathcal{P}_\zeta(u))(t_n) = \sup \mathrm{NVS}\, \mathcal{P}_\zeta\,,$$

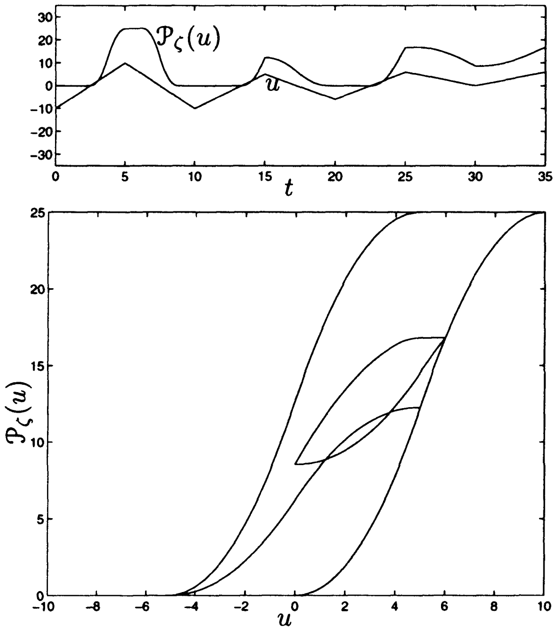

FIGURE 14.7. Example of Preisach hysteresis.

which is in contradiction to $\lim_{t\to\infty}(\mathcal{P}_\zeta(u))(t) \in \operatorname{int} \mathrm{NVS}\, \mathcal{P}_\zeta$. □

As an example, we consider the operator $\mathcal{P}_\zeta$ obtained by setting $\zeta \equiv 0$, $\mu = \mu_L$, $w_0 = 0$, and $w \equiv 2 \cdot \chi_{[0,5]\times[0,5]}$, where χ_S denotes the indicator function of the set S. This operator is illustrated in Figure 14.7.

If we set $w \equiv 1$ and $w_0 = 0$ in (14.29), we obtain the *Prandtl operator*

$$(\mathcal{P}_\zeta(u))(t) = \int_0^\infty (\mathcal{B}_{h,\zeta(h)}(u))(t)\, d\mu(h)\,, \quad \forall\, u \in C(\mathbb{R}_+, \mathbb{R})\,, \ \forall\, t \in \mathbb{R}_+\,, \tag{14.33}$$

where $\zeta \in P$ and $\mu \in \mathcal{M}_c(\mathbb{R}_+)$ (cf. [4, pp. 54]). The following corollary is a special case of Proposition 14.6.

Corollary 14.1. *Let μ be a finite positive Borel measure on $\mathbb{R}_+$. Then for all $\zeta \in P$, the Prandtl operator $\mathcal{P}_\zeta$, defined by* (14.33)*, is in $\mathcal{N}(\lambda)$, where $\lambda := \mu(\mathbb{R}_+)$.*

For example, defining the measure μ by $\mu(E) = \int_E (\sin(\pi h) + 1)\chi_{[0,10]} dh$ and setting $\zeta \equiv 0$ yields the operator illustrated in Figure 14.8.

Another example covered by Corollary 14.1 is backlash hysteresis. Indeed, the backlash operator $\mathcal{B}_{h,\xi}$ can be obtained from (14.33) by setting $\mu = \delta_h$ (where δ_h is the unit point mass at h) and by letting $\zeta : \mathbb{R}_+ \to \mathbb{R}$ be any continuous function with compact support and such that $\zeta(h) = \xi$.

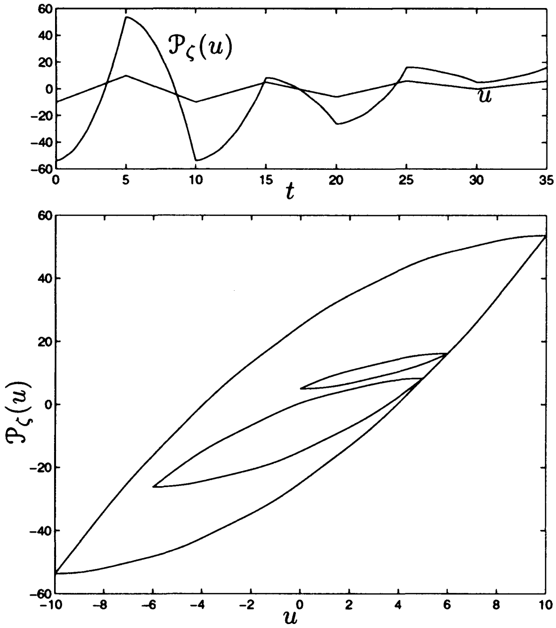

FIGURE 14.8. Example of Prandtl hysteresis.

Let $p \in L^1(\mathbb{R}_+, \mathbb{R})$; then setting $\mu = \left(\int_0^\infty p(h)\, dh\right) \delta_0 - p\mu_L$ in (14.33), we obtain, for $\zeta \in P$,

$$(\mathcal{P}_\zeta(u))(t) = \int_0^\infty p(h)(\mathcal{E}_{h,\zeta(h)}(u))(t)\, dh\,, \quad \forall\, u \in C(\mathbb{R}_+, \mathbb{R})\,, \ \ \forall\, t \in \mathbb{R}_+\,, \tag{14.34}$$

where we have used (14.27) and the fact that for all $\xi \in \mathbb{R}$ and $u \in C(\mathbb{R}_+, \mathbb{R})$, $\mathcal{B}_{0,\xi}(u) = u$.

Proposition 14.7. *Let $p \in L^1(\mathbb{R}_+, \mathbb{R})$ be nonnegative. Then for all $\zeta \in P$, the Prandtl operator $\mathcal{P}_\zeta : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$, given by* (14.34)*, is in $\mathcal{N}(\lambda)$, where $\lambda := 2\int_0^\infty p(h)\, dh$.*

Proof: By Lemma 14.9, $\mathcal{P}_\zeta : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ is Lipschitz continuous with Lipschitz continuity constant $\lambda = 2\int_0^\infty p(h)\, dh$ and clearly satisfies conditions (N1) and (N2) and therefore by Lemma 14.3, also satisfies conditions (N3), (N5), and (N8).

To show that (N4) holds, fix $u \in AC(\mathbb{R}_+, \mathbb{R})$. By (N3), (14.27), and Lemma 14.9 (see also Remark 14.8) there exists $E \subset \mathbb{R}_+$ with $\mu_L(E) = 0$ and such that for all $t \in \mathbb{R}_+ \setminus E$, $u'(t)$ and $(\mathcal{P}_\zeta(u))'(t)$ exist, $(\mathcal{E}_{h,\zeta(h)}(u))'(t)$ exists for $|\mu|$-almost every $h \in \mathbb{R}_+$, and

$$(\mathcal{P}_\zeta(u))'(t) = \int_0^\infty p(h)(\mathcal{E}_{h,\zeta(h)}(u))'(t)\,dh\,.$$

Let $t \in \mathbb{R}_+ \setminus E$. If $u'(t) = 0$, (N4) immediately follows. If $u'(t) > 0$, then, since (N4′) holds for $\mathcal{E}_{h,\zeta(h)}$, we have $(\mathcal{E}_{h,\zeta(h)}(u))'(t) \geq 0$ whenever this derivative exists (which is the case for $|\mu|$-almost every $h \in \mathbb{R}_+$). Since p is nonnegative, we may conclude that $(\mathcal{P}_\zeta(u))'(t) \geq 0$. If $u'(t) < 0$, then (N4) can be shown to hold by a similar argument.

To prove that (N6) is satisfied, let $u \in C(\mathbb{R}_+, \mathbb{R})$ be ultimately non-decreasing with $\lim_{t\to\infty} u(t) = \infty$. Then there exists $T \in \mathbb{R}_+$ such that for all $t \geq T$, $\sup_{\tau\in[0,t]} |u(\tau)| = u(t)$ and $\zeta = 0$ on $[u(t), \infty)$. So, by Lemma 14.8, with $w_0 = 0$, $w \equiv 1$, and $\mu = \left(\int_0^\infty p(h)\,dh\right)\delta_0 - p\mu_L$, we obtain

$$(\mathcal{P}_\zeta(u))(t) = \int_0^{u(t)} p(h)h\,dh + u(t)\int_{u(t)}^\infty p(h)\,dh\,, \quad \forall t \geq T\,. \tag{14.35}$$

We note that because p is nonnegative, $\sup \mathrm{NVS}\,\mathcal{P}_\zeta \leq \int_0^\infty p(h)h\,dh \in [0, \infty]$. Now using (14.35) and the fact that p is nonnegative

$$\int_0^\infty p(h)h\,dh \geq \sup \mathrm{NVS}\,\mathcal{P}_\zeta \geq (\mathcal{P}_\zeta(u))(t) \geq \int_0^{u(t)} p(h)h\,dh\,, \quad \forall t \geq T\,.$$

Since $\lim_{t\to\infty} u(t) = \infty$, it follows that

$$\lim_{t\to\infty} (\mathcal{P}_\zeta(u))(t) = \int_0^\infty p(h)h\,dh = \sup \mathrm{NVS}\,\mathcal{P}_\zeta\,.$$

Similarly, $\lim_{t\to\infty}(\mathcal{P}_\zeta(-u))(t) = \inf \mathrm{NVS}\,\mathcal{P}_\zeta$. Consequently, (N6) follows from the Lipschitz continuity of $\mathcal{P}_\zeta$ and an application of Lemma 14.4.

For (N7), let $u \in C(\mathbb{R}_+, \mathbb{R})$ and suppose $\lim_{t\to\infty}(\mathcal{P}_\zeta(u))(t) \in \operatorname{int} \mathrm{NVS}\,\mathcal{P}_\zeta$. Let $H \in \mathbb{R}_+$ be such that $\zeta = 0$ on $[H, \infty)$. Seeking a contradiction, suppose that u is unbounded. Then, by Lemma 14.7, without loss of generality, we may assume that there exists an increasing sequence $(t_n) \subset \mathbb{R}_+$ such that $\lim_{n\to\infty} t_n = \infty$ and $u(t_n) = \sup_{t\in[0,t_n]} |u(t)|$. Moreover, again without loss of generality, we may assume that $u(t_n) \geq H$ for all $n \in \mathbb{N}$. Then, by Lemma 14.8,

$$(\mathcal{P}_\zeta(u))(t_n) = \int_0^{u(t_n)} p(h)h\,dh + u(t_n)\int_{u(t_n)}^\infty p(h)\,dh\,, \quad \forall n \in \mathbb{N}\,.$$

Combining this with $\lim_{n\to\infty} u(t_n) = \infty$, we may conclude as in the proof of (N6) that $\lim_{t\to\infty}(\mathcal{P}_\zeta(u))(t) = \lim_{n\to\infty}(\mathcal{P}_\zeta(u))(t_n) = \sup \mathrm{NVS}\,\mathcal{P}_\zeta$, which is in contradiction to $\lim_{t\to\infty}(\mathcal{P}_\zeta(u))(t) \in \operatorname{int} \mathrm{NVS}\,\mathcal{P}_\zeta$. □

14.6 Example: Controlled Diffusion Process with Output Delay

Consider a diffusion process (with diffusion coefficient $\kappa > 0$ and with Dirichlet boundary conditions) on the one-dimensional spatial domain $[0, 1]$, with scalar nonlinear pointwise control action (applied at point $x_b \in (0,1)$, via an operator $\Phi : C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$, as defined below) and delayed (delay $T \geq 0$) pointwise scalar observation (output at point $x_c \in (0,1)$, $x_c \geq x_b$).

We formally write this single-input, single-output system as

$$\begin{aligned} z_t(t,x) &= \kappa z_{xx}(t,x) + \delta(x - x_b)(\Phi(u))(t)\,, \\ y(t) &= z(t-T, x_c)\,, \end{aligned}$$

with boundary conditions

$$z(t,0) = 0 = z(t,1)\,, \qquad \forall\, t > 0\,.$$

For simplicity, we assume zero initial conditions

$$z(t,x) = 0\,, \qquad \forall\,(t,x) \in [-T,0] \times [0,1]\,.$$

With input $(\Phi(u))(\cdot)$ and output $y(\cdot)$, this example qualifies as a regular linear system with transfer function given by

$$\mathbf{G}(s) = \frac{e^{-sT} \sinh\left(x_b\sqrt{(s/\kappa)}\right) \sinh\left((1-x_c)\sqrt{(s/\kappa)}\right)}{\kappa\sqrt{(s/\kappa)}\sinh\sqrt{(s/\kappa)}}\,.$$

In this case, a detailed analysis (see [12] for related investigations) shows that K defined by (14.12) satisfies

$$K = \frac{1}{|\mathbf{G}'(0)|} = \frac{6\kappa^2}{x_b(1-x_c)(6T\kappa + 1 - x_b^2 - (1-x_c)^2)}\,.$$

Therefore, by Theorem 14.1, if $\Phi \in \mathcal{N}(\lambda)$ for some $\lambda > 0$ and $k \in (0, K/\lambda)$, the integral control, $\dot{u}(t) = k[r - y(t)]$, with $u(0) = 0$, guarantees asymptotic tracking of all feasible constant reference signals r. For purposes of illustration, we adopt the following values

$$\kappa = 0.1, \qquad x_b = \frac{1}{3}, \qquad x_c = \frac{2}{3}, \qquad T = 1\,,$$

and so $K = 243/620 \approx 0.3919$.

We consider relay and Prandtl hysteresis operators.

(i) Let $\Phi = \mathcal{R}_\xi$ be a relay hysteresis operator defined by (14.20), where $\xi = 0$, $a_1 = -1$, $a_2 = 1$, $\rho_1(u) = \sqrt{u + 1.1}$, and $\rho_2(u) = \sqrt{0.1} + \sqrt{2.1} - \sqrt{1.1 - u}$. Then $\Phi \in \mathcal{N}(\lambda)$, where $\lambda = 1.6$ and $\mathrm{NVS}\,\Phi = \mathrm{im}\,\rho_1 \cup \mathrm{im}\,\rho_2 = \mathbb{R}$. Hence $K/\lambda \approx 0.245$ and taking $r = 1.42$ gives

$$\Phi_r = \frac{r}{\mathbf{G}(0)} = \frac{r\kappa}{x_b(1-x_c)} = 1.278 \in \mathrm{int}\,(\mathrm{NVS}\,\Phi)\,.$$

In each of the following cases of admissible controller gains

$$\text{(i) } k = 0.24, \qquad \text{(ii) } k = 0.17, \qquad \text{(iii) } k = 0.1,$$

Figure 14.9 depicts the output behavior of the system under integral control, Figure 14.10 depicts the corresponding control input, and Figure 14.11 shows the input to the hysteresis nonlinearity. We see from Figure 14.11 that for (i), $\lim_{t\to\infty} u(t) = \rho_1^{-1}(\Phi_r)$ and for (ii) and (iii), $\lim_{t\to\infty} u(t) = \rho_2^{-1}(\Phi_r)$.

(ii) Let $\Phi = \mathcal{P}_\zeta$ be a Prandtl operator, as defined in (14.33), where $\mu = (1/8)\chi_{[1,9]}\mu_L$ and $\zeta \equiv 0$. Then $\Phi \in \mathcal{N}(\lambda)$, where $\lambda = 1$, and $\mathrm{NVS}\,\Phi = \mathbb{R}$. Then $K/\lambda = K \approx 0.3919$. For $r = 1$, we have

$$\Phi_r = \frac{r}{\mathbf{G}(0)} = \frac{r\kappa}{x_b(1-x_c)} = 0.9 \in \mathrm{int}\,(\mathrm{NVS}\,\Phi)\,.$$

In each of the following cases of admissible controller gains

$$\text{(i) } k = 0.39, \qquad \text{(ii) } k = 0.2, \qquad \text{(iii) } k = 0.1,$$

Figure 14.12 depicts the output behavior of the system under integral control, Figure 14.13 depicts the corresponding control input, and Figure 14.14 shows the input to the Prandtl operator. We see from Figure 14.12 that for (i) (the largest gain) the output exhibits a small overshoot, which does not occur in (ii) and (iii). The overshoot leads to the formation of a hysteresis loop. Hence in Figure 14.14, the hysteresis input converges to a different value as compared to the other two cases.

Figures 14.9 through 14.14 were generated using SIMULINK Simulation Software within MATLAB wherein a truncated eigenfunction expansion, of order 10, was adopted to model the diffusion process.

14.7 Appendix

Proof of Lemma 14.5

Let $\alpha \geq 0$ and $w \in C([0,\alpha],\mathbb{R})$ and consider the following initial-value problem

$$\begin{aligned} \dot{u}(t) &= k[r - (\mathbf{\Psi}x_0)(t) - (\mathbf{F}\Phi(u))(t)]\,, \quad t > \alpha\,, && (14.36a)\\ u(t) &= w(t)\,, \quad t \in [0,\alpha]\,, && (14.36b) \end{aligned}$$

where $\Phi : C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$.

Lemma 14.10. *Assume that Φ satisfies (N1) and (N5) and let $x_0 \in X$. For any initial function $w \in C([0,\alpha],\mathbb{R})$, there exists $\varepsilon > 0$ and a unique function $u \in C([0,\alpha+\varepsilon],\mathbb{R})$ with $u(t) = w(t)$ for all $t \in [0,\alpha]$ and such that $u|_{[\alpha,\alpha+\varepsilon]}$ is absolutely continuous and* (14.36a) *is satisfied for a.e. $t \in [\alpha,\alpha+\varepsilon]$.*

Remark 14.9. For $\alpha = 0$, the above initial-value problem is identical to the initial-value problem (14.10). Note that Steps 2 and 3 in the proof of Lemma 14.5

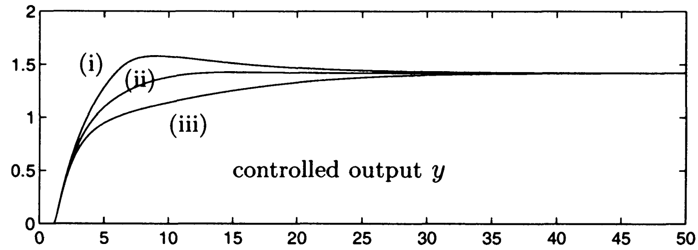

FIGURE 14.9. Controlled output.

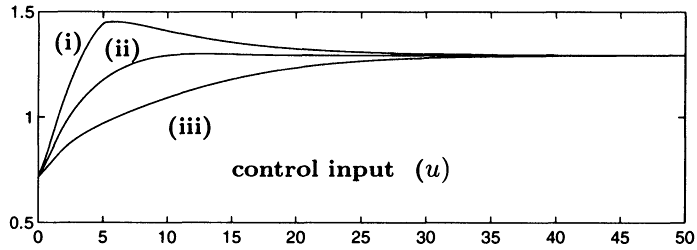

FIGURE 14.10. Control input.

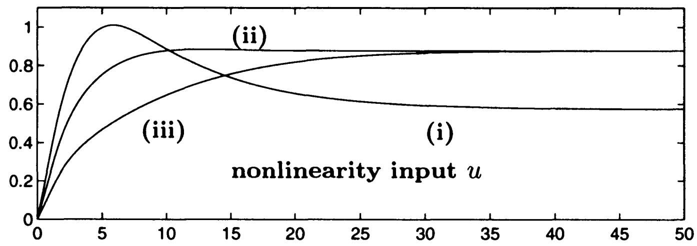

FIGURE 14.11. Input to relay nonlinearity.

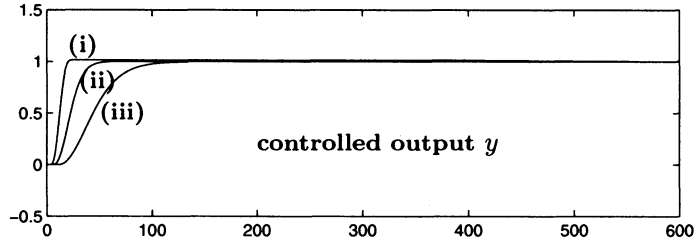

FIGURE 14.12. Controlled output.

(see below) require the existence and uniqueness result given in Lemma 14.10 in the more general context of $\alpha \geq 0$.

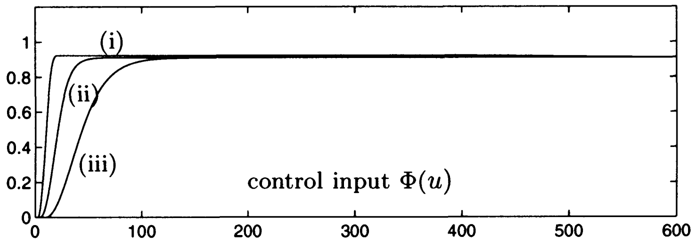

FIGURE 14.13. Control input.

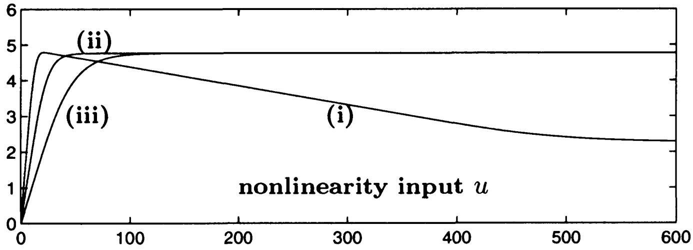

FIGURE 14.14. Input to Prandtl operator.

For the proof of Lemma 14.10 it is convenient to introduce a "truncation" of the operator $\mathbf{F}$ acting on functions defined on a finite time interval. To this end let $\tau > 0$ and define $\pi_\tau : L^2([0,\tau],\mathbb{R}) \to L^2(\mathbb{R}_+,\mathbb{R})$ by

$$(\pi_\tau v)(t) = \begin{cases} v(t) & \text{for } 0 \le t \le \tau\,, \\ 0 & \text{for } t > \tau\,. \end{cases}$$

Defining the linear operator $\mathbf{F}_\tau : L^2([0,\tau],\mathbb{R}) \to L^2([0,\tau],\mathbb{R})$ by

$$(\mathbf{F}_\tau v)(t) = (\mathbf{F}(\pi_\tau v))(t)\,, \quad \forall\, t \in [0,\tau]\,,$$

we have that $\mathbf{F}_\tau$ is bounded and $\|\mathbf{F}_{\tau_1}\| \le \|\mathbf{F}_{\tau_2}\|$ for all $\tau_2 \ge \tau_1 > 0$.

Proof: **(of Lemma 14.10)** Let $w \in C([0,\alpha],\mathbb{R})$. Then by (N5), there exist numbers $\delta_1, \delta_2 > 0$ such that for all $u, v \in \mathcal{C}(w;\delta_1,\delta_2)$,

$$\sup_{t\in[\alpha,\alpha+\delta_2]} |(\Phi(u))(t) - (\Phi(v))(t)| \le \lambda \sup_{t\in[\alpha,\alpha+\delta_2]} |u(t) - v(t)|\,. \tag{14.37}$$

For $\varepsilon, \eta > 0$ set

$$\mathcal{C}_{\eta,\varepsilon} := \{u \in C([0,\alpha+\varepsilon],\mathbb{R}) \mid u(t) = w(t) \text{ if } t \in [0,\alpha]\,; \\ |u(t) - w(\alpha)| \le \eta \text{ if } t \in [\alpha,\alpha+\varepsilon]\}\,. \tag{14.38}$$

Let $\eta \in (0, \delta_1)$, $\varepsilon > 0$, and $u, v \in \mathcal{C}_{\eta,\varepsilon}$. Then

$$\mathbf{Q}_{\alpha+\varepsilon} u,\ \mathbf{Q}_{\alpha+\varepsilon} v \in \mathcal{C}(w; \delta_1, \delta_2)\,,$$

and hence, by (14.37), we obtain for every $\eta \in (0, \delta_1)$ and every $\varepsilon \in (0, \delta_2)$,

$$\begin{aligned}
&\sup_{t\in[\alpha,\alpha+\varepsilon]} |(\Phi(u))(t) - (\Phi(v))(t)| \\
&\qquad \le \sup_{t\in[\alpha,\alpha+\delta_2]} |(\Phi(\mathbf{Q}_{\alpha+\varepsilon} u))(t) - (\Phi(\mathbf{Q}_{\alpha+\varepsilon} v))(t)| \\
&\qquad \le \lambda \sup_{t\in[\alpha,\alpha+\delta_2]} |(\mathbf{Q}_{\alpha+\varepsilon} u)(t) - (\mathbf{Q}_{\alpha+\varepsilon} v)(t)| \\
&\qquad = \lambda \sup_{t\in[\alpha,\alpha+\varepsilon]} |u(t) - v(t)|\,, \quad \forall\, u, v \in \mathcal{C}_{\eta,\varepsilon}\,.
\end{aligned}$$

Using the causality of $\mathbf{F}$ and Φ (Φ is causal by (N1)), the boundedness of $\mathbf{F}_\tau$ for every $\tau \ge 0$, Hölder's inequality, and (14.7), we conclude that there exists $L > 0$ such that, for every $\varepsilon \in (0, \delta_2)$,

$$\int_\alpha^{\alpha+\varepsilon} |\mathbf{F}\Phi(u) - \mathbf{F}\Phi(v)| \le \varepsilon L \sup_{t\in[\alpha,\alpha+\varepsilon]} |u(t) - v(t)|\,, \quad \forall\, u, v \in \mathcal{C}_{\eta,\varepsilon}\,. \tag{14.39}$$

Moreover, an application of (14.39) for $v = \mathbf{Q}_\alpha u$ shows that, for every $\varepsilon \in (0, \delta_2)$, we have for all $u \in \mathcal{C}_{\eta,\varepsilon}$,

$$\int_\alpha^{\alpha+\varepsilon} |\mathbf{F}\Phi(u)| \le \int_\alpha^{\alpha+\varepsilon} |(\mathbf{F}\Phi(\mathbf{Q}_\alpha u))(\tau)|\, d\tau + \varepsilon L \sup_{t\in[\alpha,\alpha+\varepsilon]} |u - w(\alpha)|\,. \tag{14.40}$$

Set $f(t) = r - (\mathbf{\Psi} x_0)(t)$ and choose $\rho > 0$ such that

$$\int_\alpha^{\alpha+\rho} (|f(\tau)| + |(\mathbf{F}\Phi(\mathbf{Q}_\alpha u))(\tau)|)\; d\tau \le \frac{\eta}{2|k|}\,. \tag{14.41}$$

In the following let $\eta \in (0, \delta_1)$ and choose $\varepsilon > 0$ such that

$$\varepsilon < \delta_2\,, \quad \varepsilon < \rho\,, \quad \varepsilon < \frac{1}{2|k|L}\,. \tag{14.42}$$

Define the operator Γ by

$$\begin{aligned}
(\Gamma u)(t) &= w(\alpha) + k\left(\int_\alpha^t f(\tau)\, d\tau - \int_\alpha^t (\mathbf{F}\Phi(u))(\tau)\, d\tau\right), \quad t \ge \alpha\,,\\
(\Gamma u)(t) &= w(t)\,, \quad 0 \le t \le \alpha\,.
\end{aligned}$$

Clearly, $\mathcal{C}_{\eta,\varepsilon}$ is a complete metric space, and the claim follows if we can show that Γ is a contraction on $\mathcal{C}_{\eta,\varepsilon}$.

We first show that $\Gamma(\mathcal{C}_{\eta,\varepsilon}) \subset \mathcal{C}_{\eta,\varepsilon}$. Using (14.40) through (14.42) we obtain, for all $u \in \mathcal{C}_{\eta,\varepsilon}$ and all $t \in [\alpha, \alpha+\varepsilon]$,

$$|(\Gamma u)(t) - w(\alpha)| \le \frac{\eta}{2} + \varepsilon|k|L \sup_{t\in[\alpha,\alpha+\varepsilon]} |u(t) - w(\alpha)| \le \frac{\eta}{2} + \varepsilon|k|L\eta \le \eta\,,$$

which shows that $\Gamma(\mathcal{C}_{\eta,\varepsilon}) \subset \mathcal{C}_{\eta,\varepsilon}$. It remains to show that Γ is a contraction on $\mathcal{C}_{\eta,\varepsilon}$. To this end, let $u, v \in \mathcal{C}_{\eta,\varepsilon}$. Using (14.39), we obtain

$$\sup_{\tau\in[\alpha,\alpha+\varepsilon]} |(\Gamma u)(\tau) - (\Gamma v)(\tau)| \leq \varepsilon|k|L \sup_{\tau\in[\alpha,\alpha+\varepsilon]} |u(\tau) - v(\tau)| \,.$$

By (14.42) we have that $\varepsilon|k|L < 1$, showing that Γ is a contraction on $\mathcal{C}_{\varepsilon,\eta}$. $\square$

Proof: (of Lemma 14.5) We proceed in several steps.
STEP 1. Existence and uniqueness on a small interval.

An application of Lemma 14.10 with $\alpha = 0$ shows that there exists an $\varepsilon > 0$ such that (14.10) has a unique solution on the interval $[0, \varepsilon)$.

STEP 2. Extended uniqueness.

Let u_i be a solution of (14.10) on the interval $[0, a_i)$, $i = 1, 2$. We claim that $u_1(t) = u_2(t)$ for all $t \in [0, a)$, where $a = \min(a_1, a_2)$. Seeking a contradiction, assume that there exists $t \in (0, a)$ such that $u_1(t) \neq u_2(t)$. Defining

$$a^* = \inf\{t \in (0, a) \,|\, u_1(t) \neq u_2(t)\}$$

it follows that $a^* > 0$ (by STEP 1), $a^* < a$ (by assumption), and $u_1(a^*) = u_2(a^*)$ (by continuity of u_1 and u_2). Clearly, the initial-value problem

$$\begin{aligned} \dot{u}(t) &= k[r - (\mathbf{\Psi} x_0)(t) - (\mathbf{F}\Phi(u))(t)]\,, \quad t \geq a^*\,, \\ u(t) &= u_1(t)\,, \quad t \in [0, a^*]\,, \end{aligned}$$

is solved by u_1 and u_2 on [0,a). This implies (by Lemma 14.10) that there exists an $\varepsilon > 0$ such that $u_1(t) = u_2(t)$ for all $t \in [0, a^* + \varepsilon)$, which contradicts the definition of a^*.

STEP 3. Global existence.

Let $\mathcal{T} \subset \mathbb{R}_+$ be the set of all $\tau > 0$ such that there exists a solution u^τ of (14.10) on the interval $[0, \tau)$. Set $t^* := \sup \mathcal{T}$ and define a function $u : [0, t^*) \to \mathbb{R}$ by setting

$$u(t) = u^\tau(t)\,, \quad \text{for } t \in [0, \tau)\,, \text{ where } \tau \in \mathcal{T}\,.$$

By STEP 2 the function u is well defined (i.e., the definition of $u(t)$ for a particular value $t \in [0, t^*)$ does not depend on the choice of $\tau \in \mathcal{T} \cap (t, \infty)$) and u is the unique solution of (14.10) on the interval $[0, t^*)$.

We claim that $t^* = \infty$. Seeking a contradiction, assume that $t^* < \infty$. Multiplying $\dot{u}$ by u, we obtain using (14.10),

$$u(t)\dot{u}(t) = ku(t)\,[f(t) - (\mathbf{F}\Phi(u))(t)]\,, \quad \forall\, t \in [0, t^*)\,,$$

where again $f(t) = r - (\mathbf{\Psi} x_0)(t)$. Integration yields

$$\frac{1}{2}u^2(t) = \frac{1}{2}u^2(0) + k\left(\int_0^t fu - \int_0^t \mathbf{F}\Phi(u)u\right)\,, \quad \forall\, t \in [0, t^*)\,. \qquad (14.43)$$

For $v \in C([0,t^*),\mathbb{R})$ and $t \in [0,t^*)$, we define

$$\sigma_t(v) = \sup_{\tau\in[0,t]} |v(\tau)|\,.$$

Using (14.43), the boundedness of $\mathbf{F}_{t^*}$, and applying Hölder's inequality, shows that there exists $\gamma_1 > 0$ such that for all $t \in [0,t^*)$,

$$\frac{1}{2}\sigma_t(u^2) \le \frac{1}{2}u^2(0) + |k|\left(\int_0^{t^*} f^2\right)^{1/2}\left(\int_0^t u^2\right)^{1/2} + |k|\gamma_1\left(\int_0^t [\Phi(u)]^2\right)^{1/2}\left(\int_0^t u^2\right)^{1/2}.$$

Denoting the map

$$C([0,t^*),\mathbb{R}) \to \mathbb{R}_+\,,\quad v \mapsto [\sigma_t(v)]^2$$

by σ_t^2, we see that there exist suitable constants $\gamma_2,\gamma_3,\gamma_4 > 0$ such that for all $t \in [0,t^*)$

$$\sigma_t^2(u) \le \gamma_2 + \gamma_3\left(\int_0^t \sigma_\tau^2(u)\,d\tau\right)^{1/2} + \gamma_4\left(\int_0^t \sigma_\tau^2(\Phi(u))\,d\tau\right)^{1/2}\left(\int_0^t \sigma_\tau^2(u)\,d\tau\right)^{1/2}.$$

Using Assumption (N8), we may conclude that there exist numbers $\alpha,\beta > 0$ such that for all $t \in [0,t^*)$,

$$\sigma_t^2(u) \le \gamma_2 + \gamma_3\left(\int_0^t \sigma_\tau^2(u)\,d\tau\right)^{1/2} + \gamma_4\left(\int_0^t [\alpha + \beta\sigma_\tau(u)]^2\,d\tau\right)^{1/2}\left(\int_0^t \sigma_\tau^2(u)\,d\tau\right)^{1/2}.$$

From this we obtain that there exist numbers $\gamma_5,\gamma_6 > 0$ such that

$$\sigma_t^2(u) \le \gamma_5 + \gamma_6\int_0^t \sigma_\tau^2(u)\,d\tau\,,\quad \forall\, t \in [0,t^*)\,,$$

and an application of Gronwall's lemma then shows that

$$\sigma_t^2(u) \le \gamma_5 e^{\gamma_6 t}\,,\quad \forall\, t \in [0,t^*)\,.$$

Since, by assumption, $t^* < \infty$, it follows that u is bounded on $[0,t^*)$. Consequently, the right-hand side of (14.43) converges to a finite limit as $t \uparrow t^*$, and so $\lim_{t\uparrow t^*} u^2(t)$ exists and is finite. By continuity of u, this in turn implies that there exists $\gamma \in \mathbb{R}$ such that $\lim_{t\uparrow t^*} u(t) = \gamma$, and hence setting $u(t^*) = \gamma$ makes u into a continuous function on $[0,t^*]$. Finally, Lemma 14.10 shows that the initial value problem

$$\begin{aligned} \dot{v} &= k[r - \Psi x_0 - \mathbf{F}\Phi(v)]\,,\quad t \ge a\,,\\ v(t) &= u(t)\,,\quad t \in [0,t^*]\,, \end{aligned}$$

has a unique solution $\tilde{u}$ on $[0, t^* + \varepsilon)$ for some $\varepsilon > 0$. By the causality of the map $\mathbf{F}\Phi$, the function $\tilde{u}$ is a solution of (14.10) on $[0, t^* + \varepsilon)$; that is, $\tilde{u}$ is a continuation of u. But this means that $t^* + \varepsilon \in \mathcal{T}$, which is in contradiction to the definition of t^*. □

Proof: (of Lemma 14.6) Let $t_1 < t_2$, $u : [t_1, t_2] \to \mathbb{R}$ be monotone and $w \in [u(t_1) - h, u(t_1) + h]$. Fix $t, \tau \in [t_1, t_2]$ with $t \geq \tau$. We first note that $w = b_h(u(t_1), w)$ since $w \in [u(t_1) - h, u(t_1) + h]$. Without loss of generality we may assume that u is nondecreasing and so $w = b_h(u(t_1), w) \leq b_h(u(\tau), w)$. If $w = b_h(u(\tau), w)$, then, trivially, $b_h(u(t), w) = b_h(u(t), b_h(u(\tau), w))$. If $w < b_h(u(\tau), w)$, then $b_h(u(\tau), w) = u(\tau) - h$ and thus $w < b_h(u(\tau), w) \leq u(t) - h$. Consequently,

$$b_h(u(t), w) = u(t) - h = b_h(u(t), b_h(u(\tau), w)) .$$

□

Acknowledgment: This work was supported by the UK EPSRC Council (Grant GR/L78086).

References

[1] H.T. Banks, R.C. Smith and Y. Wang, *Smart Material Structures: Modeling, Estimation and Control*, Masson, Paris, 1996.

[2] V. Barbu, *Analysis and Control of Nonlinear Infinite-Dimensional Systems*, Academic Press, Boston, 1993.

[3] M. Brokate, Hysteresis operators, in *Phase Transitions and Hysteresis*, A. Visintin ed., Springer-Verlag, Berlin, 1–38, 1994.

[4] M. Brokate and J. Sprekels, *Hysteresis and Phase Transitions*, Springer-Verlag, New York, 1996.

[5] G.W.M. Coppus, S.L. Sha, and R.K. Wood, Robust multivariable control of a binary distillation column, *IEE Proc., D* 130:201–208, 1983.

[6] E.J. Davison, Multivariable tuning regulators: The feedforward and robust control of a general servomechanism problem, *IEEE Trans. Auto. Control* 21:35–47, 1976.

[7] G.B. Folland, *Real Analysis*, Wiley, New York, 1984.

[8] G. Gripenberg, S.-O. Londen, and O. Staffans, *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge, 1990.

[9] M.A. Krasnosel'skiĭ and A.V. Pokrovskiĭ, *Systems with Hysteresis*, Springer-Verlag, Berlin, 1989.

[10] H. Logemann and E.P. Ryan, Time-varying and adaptive integral control of infinite-dimensional regular linear systems with input nonlinearities, *SIAM J. Control Optimization* 38:1120–1144, 2000.

[11] H. Logemann and S. Townley, Low-gain control of uncertain regular linear systems, *SIAM J. Control & Optimization* 35:78–116, 1997.

[12] H. Logemann, E.P. Ryan, and S. Townley, Integral control of linear systems with actuator nonlinearities: lower bounds for the maximal regulating gain, *IEEE Trans. Auto. Control* 44:1315–1319, 1999.

[13] H. Logemann, E.P. Ryan, and S. Townley, Integral control of infinite-dimensional linear systems subject to input saturation, *SIAM J. Control & Optimization* 36:1940–1961, 1998.

[14] J. Lunze, Experimentelle Erprobung einer Einstellregel für PI-Mehrgrößenregler bei der Herstellung von Ammoniumnitrat-Harnstoff-Lösung, *Messen Steuern Regeln* 30:2–6, 1987.

[15] J. Lunze, *Robust Multivariable Feedback Control*, Prentice-Hall, London, 1988.

[16] J.W. Macki, P. Nistri, and P. Zecca, Mathematical models for hysteresis, *SIAM Review* 35:94–123, 1993.

[17] M. Morari, Robust stability of systems with integral control, *IEEE Trans. Auto. Control* 30:574–577, 1985.

[18] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[19] A. Visintin, *Differential Models of Hysteresis*, Springer-Verlag, Berlin, 1994.

[20] G. Weiss, Admissibility of unbounded control operators, *SIAM J. Control & Optimization* 27:527–545, 1989.

[21] G. Weiss, Admissible observation operators for linear semigroups, *Israel J. Math.* 65:17–43, 1989.

[22] G. Weiss, The representation of regular linear systems on Hilbert spaces, in *Distributed Parameter System*, F. Kappel, K. Kunisch, and W. Schappacher eds., Birkhäuser Verlag, Basel, 401–416, 1989.

[23] G. Weiss, Transfer functions of regular linear systems, part I: Characterization of regularity, *Trans. Amer. Math. Soc.* 342:827–854, 1994.