Adaptive Stabilization of Infinite-Dimensional Systems

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Abstract—**In [1], it was shown that in order to stabilize an unknown linear, time invariant, finite dimensional system, it is sufficient to know the order of any stabilizing controller. The main result of the present paper is to generalize this result to a large class of infinite-dimensional systems. For high-gain stabilizable infinite-dimensional systems, an algorithm is presented which takes this additional *a priori* knowledge into account. Simulation results are presented.**

I. Introduction

During the eighties, some success was made in establishing theoretical foundations for adaptive control: New controllers were proposed, solving adaptive control problems for larger classes of plants than was previously thought possible. For a recent general overview of adaptive control, see [2]. In particular, the advent of controllers with a low number (typically one) of adjustable parameters, capable of controlling a class containing plants with arbitrary (but finite) McMillan degree was an important step, in particular in comparison with traditional approaches, where it had been demonstrated that under some circumstances, an arbitrarily small violation of the assumptions on the high-frequency behavior could make traditional model reference systems—mathematically proven to be globally stable under the stated assumptions—unstable, [3].

From the positive results, e.g., [4] and [1], one would guess that under some reasonable extra conditions, these results on stabilization of systems with arbitrarily high—but finite—McMillan degree would carry over to infinite dimensional systems. The present contribution, together with some contributions quoted below, show that this intuition is correct. The main result, Theorem 8, generalizes the result by Mårtensson [1]—namely, that in order to stabilize a linear, time invariant, finite dimensional plant, it is sufficient to know the order of *any* stabilizing controller—to a large class of infinite-dimensional systems. Therefore, the present contribution can also be interpreted as a sort of robustness results; showing that certain finite dimensional adaptive controllers are in fact robust against small, infinite dimensional perturbations. This is an important point, since it is often claimed that "real-world systems," in particular technical or biological, always contain delays.

Some attempts have been made to generalize "traditional" adaptive algorithms to classes of infinite dimensional systems: References [5], [6], and [7] deal with model reference adaptive control for semigroup systems on Hilbert spaces. In [8], Fernández *et al.* show that some standard adaptive schemes designed for first-order systems remain (locally) stable in the presence of "small" input delays. Ortega *et al.* [9] presents a globally stable adaptive controller for scalar plants with one exactly known delay in the input. In [10], a $2D$-system approach is taken: a class of delay-differential systems, with the delays consisting of a finite number of point delays, all having a rational relationship, is considered. By utilizing algebraic methods, a standard model reference adaptive controller is proposed.

In adaptive stabilization of finite-dimensional linear systems, a high-gain approach has had some success. To our knowledge, the first contribution in this tradition was [4], later followed by approaches based on a generalization of Tychonov's theorem [11] to time-varying systems, first proved by Byrnes and Mårtensson [12] (with incorrect proof), [13]. Kobayashi [14] generalizes this to a class of infinite dimensional systems described by semigroups on a Hilbert space. The paper considers adaptive stabilization of multivariable systems of a somewhat limited structure: It is assumed that the eigenvalues of the instantaneous gain all reside *either* in the (open) left or in the (open) right-half plane. Furthermore, the system is assumed to satisfy fairly restrictive smoothness assumptions. Kobayashi's results have been considerably improved by Logemann and Zwart in [15], which considers systems described by strongly continuous semigroups on Banach spaces. Dahleh and Hopkins have presented similar algorithms [16], [17], stabilizing single-input, single-output delay systems. Byrnes [18] considers systems with *bounded* infinitesimal generator, thereby excluding "all" interesting examples. Further, the paper contains two gaps in that global existence of the nonlinear, infinite-dimensional equation are not established, and it only follows that the state goes to zero in the weak topology. Logemann and Owens [19] use an input–output approach, allowing for certain cone-bounded, memoryless nonlinearities in the inputs and outputs, and present results for retarded systems and Volterra integro-differential systems. A modification of that scheme is presented in [20], presenting an

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algorithm stabilizing a class of nonlinear retarded processes with a prescribed rate of exponential decay α. For classes of finite-dimensional systems, similar α-stabilizing algorithms are given in [21], [22], and [23]. It should be noted that in all these references, exponential stabilization is achieved at the price of unbounded adaptation gain (as $t \to \infty$). Even in the finite-dimensional case it is still an open problem how to achieve—without external excitation—adaptive stabilization with prescribed decay rate, with all involved quantities bounded.

There also have been approaches to "universal adaptive stabilization" which are not based on high-gain concepts. In [1], Mårtensson proved that in order to adaptively stabilize an unknown linear, finite dimensional, time-invariant system, knowledge of the order of *any* asymptotically stabilizing controller is sufficient. The algorithm was based on a dense "search" through controller space. Similar algorithms for finite-dimensional systems have later been presented e.g., by Miller and Davison [24], [25], Mårtensson [13], and by Fu and Barmish [26]. An attempt to generalize the result to a class of infinite-dimensional systems consisting of a set of delay systems was made by Dahleh [27]. Unfortunately, this paper has many shortcomings: Most importantly, the author formulates a theorem for a set of systems satisfying a certain condition (continuous initial observability). He does not elaborate on whether this is a restrictive condition or not. In Section III, we show that this assumption is not satisfied in almost all interesting cases. Furthermore, the proof of the main theorem fails to establish both the global existence of solutions and that the state goes to zero in a suitable sense.

In Section II, we introduce the set of infinite-dimensional systems we shall be dealing with, the so-called Pritchard–Salamon class of systems with unbounded control and observation. As an example on how more concrete systems fit into the abstract framework, retarded systems with output delays are presented in some detail. In Section III, we analyze a set of infinite-dimensional systems with respect to the property of continuous initial observability. From this point on, we shall restrict our attention to the subset of the Pritchard–Salamon class which consists of the exponentially stabilizable and exponentially detectable systems. Section IV presents background results on stabilization of this subclass by finite-dimensional linear, time-invariant controllers. Although we do not claim that any result in the section is genuinely new, we were not able to find them suitably formulated for our purposes anywhere else. Section V introduces switching function adaptive controllers, which will be our vehicle for the positive results to follow. The machinery differs from the finite-dimensional case in e.g., [13] not only by being more technical, but also in that some problems are genuinely approached differently. Therefore, we believe that this is also of interest for adaptive stabilization of finite-dimensional systems. We prove that the knowledge of the order of any stabilizing compensator is sufficient *a priori* information for adaptive stabilization of the set of systems under consideration. Section VI presents an adaptive algorithm which stabilizes a class of high-gain stabilizable systems. The algorithm is based on a combination of results on high-gain stabilization of infinite-dimensional systems with the switching function controller machinery developed in Section V. In Section VII, some examples and simulations are given, illustrating the ideas in Sections V and VI. In the last section, some conclusions are drawn. Finally, a lemma of a more technical nature is proved in the appendix.

II. Infinite Dimensional State-Space Systems with Unbounded Control and Observation

In this section, we introduce a set of infinite dimensional state-space systems in an abstract setting, which will be referred to as the Pritchard–Salamon class. The presentation will be very brief. For a fuller presentation with proofs and motivation, the reader is referred to the references given.

In a formal sense, our basic model is

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

where $u \in L_2^{loc}(0, \infty; \mathbb{R}^m)$, A is the infinitesimal generator of a C_0-semigroup $S(t)$ on a real Hilbert-space W, $C \in \mathfrak{L}(W, \mathbb{R}^p)$, and $B \in \mathfrak{L}(\mathbb{R}^m, V)$ where V is a real Hilbert-space satisfying $V \supset W$. We are interested in the *mild solution of (1a)*, i.e., in the trajectory given by the variation-of-constants formula

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)Bu(\tau)\,d\tau. \tag{2}$$

In order to make the expression under the integral in (2) meaningful, we assume that A also generates a C_0-semigroup on V, which we will also denote by $S(t)$. We introduce the following assumptions:

A1) The inclusion map $W \to V$ is bounded and W is dense in V.

A2) There exist $\alpha, t_1 > 0$ such that $\|CS(\cdot)x\|_{L_2(0,t_1)} \le \alpha\|x\|_V$ for all $x \in W$.

A3) There exist $\beta, t_2 > 0$ such that $\int_0^{t_2} S(t_2 - \tau)Bu(\tau)\,d\tau \in W$ and

$$\left\| \int_0^{t_2} S(t_2 - \tau)Bu(\tau)\,d\tau \right\|_W \le \beta\|u\|_{L_2(0,t_2;\mathbb{R}^m)}$$

for all $u \in L_2(0, t_2; \mathbb{R}^m)$.

A4) There exists $\epsilon > 0$ such that

$$\int_0^t CS(\tau)Bu\,d\tau = C\int_0^t S(\tau)Bu\,d\tau \tag{3}$$

for all $u \in \mathbb{R}^m$ and $t \in [0, \epsilon]$.

Note that in (3), the left-hand side is to be interpreted via A2) [cf. Remark 1, i) and iii)], while the right-hand side makes sense because of A3) and Remark 1, iv) below.

If $V = W$, we will call (1) a system with *bounded control and observation.*

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Remark 1:

i) Suppose that A1) and A2) are satisfied. Then the bounded linear operator $\mathscr{O}_W: W \to L_2(0, t_1; \mathbb{R}^p)$, $x \mapsto CS(\cdot)x$ can be uniquely extended to a bounded linear operator $\mathscr{O}_V: V \to L_2(0, t_1; \mathbb{R}^p)$. For every $x \in V$ we define $CS(\cdot)x := \mathscr{O}_V x$.

ii) Assumption A3) implies that for every $x_0 \in W$ and every $u \in L_2(0, t_2; \mathbb{R}^m)$, (2) defines a continuous function $x(\cdot)$ on $[0, t_2]$ with values in W. Of course, we define the output by

$$y(t) = CS(t)x_0 + C\int_0^t S(t-\tau)Bu(\tau)\,d\tau \tag{4}$$

for $t \in [0, t_2]$.

iii) If A2) holds for one particular $t_1 > 0$, then it can be shown that it is satisfied for all $t_1 > 0$, where α will depend on t_1. If $S(t)$ is exponentially stable on V then we can choose α independent of t_1.

iv) If A3) holds for one particular $t_2 > 0$, then it can be shown that it is satisfied for all $t_2 > 0$, where β will depend on t_2. If $S(t)$ is exponentially stable on W then β can be chosen independently of t_2.

v) Let A_V denote the infinitesimal generator of $S(t)$ on V. Assume that A1)–A3) holds, together with

(A5) $D(A_V) \subset W$ with continuous dense injection, where $D(A_V)$ is endowed with the graph norm of A_V.

Then it can be shown that A4) holds (cf. [28]).

vi) For $t \geq 0$ define $\mathfrak{C}_t: L_2(0, t; \mathbb{R}^m) \to V$, $u \mapsto \int_0^t S(t-\tau)Bu(\tau)\,d\tau$. Assumption A3) means that there exists a $t_2 > 0$ such that

$$\operatorname{Im} \mathfrak{C}_{t_2} \subset W \tag{5}$$

and

$$\mathfrak{C}_{t_2} \in \mathfrak{L}(L_2(0, t_2; \mathbb{R}^p), W). \tag{6}$$

If A1) and A5) are satisfied, it has been shown in [29] that (6) is implied by (5).

vii) It is easy to show that $\int_t^T CS(\tau)Bu\,d\tau = C\int_t^T S(\tau)Bu\,d\tau$ for all $T \geq t \geq 0$ and $u \in \mathbb{R}^m$ provided that A1)–A4) hold. ■

The above setup and various modifications thereof have been introduced and investigated in [28], and [30]–[34]. Related work has been done in [29] and [35].

For examples of systems satisfying A1)–A4), we refer the reader to [30], [28], [32], [33]. It is known that for a large class of neutral systems with delays in the input or the output A1)–A4) hold (cf. [30], [28]). Furthermore, it has been shown in [36] that A1)–A4) are satisfied for retarded systems with delays both in the control and the observation. For parabolic, hyperbolic, and spectral systems sufficient conditions for A1)–A4) were given in [28] and [32]. They were applied to partial differential equation models of flexible structures in [37].

The next result shows that we obtain a well posed closed-loop system if state feedback or output injection is applied to the plant (1) (see [30], [32], [34]).

Lemma 1:

i) Suppose A1) and A3) are satisfied. Then for $F \in \mathfrak{L}(W, \mathbb{R}^m)$ there exists a C_0-semigroup $S_F(t)$ on W which is the unique solution of

$$S_F(t)x = S(t)x + \int_0^t S(t-\tau)BFS_F(\tau)x\,d\tau \tag{7}$$

for all $x \in W$ and $t \geq 0$.

ii) If A1) and A2) hold then for $H \in \mathfrak{L}(\mathbb{R}^p, W)$ there exists a C_0-semigroup $S_H(t)$ on W and V which is the unique solution of

$$S_H(t)x = \int_0^t S(t-\tau)HCS_H(\tau)x\,d\tau \tag{8}$$

for all $x \in W$ and $t \geq 0$. Further, $S_H(t)$ and C satisfy A2). Under the extra assumption that A3) and A4) hold $S_H(t)$, B, and C satisfy A3) and A4).

Remark 2: Suppose A1)–A4) hold. Since C and $S_H(t)$ satisfy A2) the expression $CS_H(\cdot)x$ makes sense as a function in $L_2^{loc}(0, \infty; \mathbb{R}^p)$ for all $x \in V$ [see Remark 1, i) and iii)]. It follows that (8) holds for all $x \in V$. ■

We will next show how retarded systems fit into the abstract framework:

Example 3: Consider the system with delays in the output

$$\begin{aligned} \dot{x}(t) &= Ax_t + Bu(t) \\ y(t) &= Cx_t \end{aligned} \tag{9}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and x_t denotes the function segment given by $x_t(\tau) = x(t+\tau)$ for $\tau \in [-h, 0]$, where h is the length of the delay. Further, $B \in \mathbb{R}^{n\times m}$, $A \in \mathfrak{L}(\mathfrak{C}, \mathbb{R}^n)$ and $C \in \mathfrak{L}(\mathfrak{C}, \mathbb{R}^p)$, where $\mathfrak{C}$ denotes $\mathfrak{C}(-h, 0; \mathbb{R}^n)$. From elementary functional analysis, it follows that A and C can be represented as

$$A\varphi = \int_{-h}^0 d\alpha(\tau)\varphi(\tau), \quad C\varphi = \int_{-h}^0 d\gamma(\tau)\varphi(\tau), \quad \varphi \in \mathscr{C}$$

for some functions $\alpha \in BV(-h, 0; \mathbb{R}^{n\times n})$ and $\gamma \in BV(-h, 0; \mathbb{R}^{p\times n})$. In the sequel, let $W_{1,2}$ denote the Sobolev space $W_{1,2}(-h, 0; \mathbb{R}^n)$ (cf. [38]). Define

$$W := \{(\varphi(0), \varphi): \varphi \in W_{1,2}\}$$

$$V := M_2 = \mathbb{R}^n \times L_2(-h, 0; \mathbb{R}^n)$$

$$\mathbf{x}(t) := (x(t), x_t) =: (x^0, x^1) \in M_2.$$

It is clear that W and V satisfy A1). In the following we shall identify W with $W_{1,2}$. We define the operators $\mathscr{A}, \mathscr{B}, \mathscr{C}$, corresponding to A, B, and C in (1), according to

$$D(\mathscr{A}) = \{\varphi \in M_2: \varphi^1 \in W_{1,2}, \varphi^0 = \varphi^1(0)\} = W$$

$$\mathscr{A}\varphi = (A\varphi^1, \dot{\varphi}^1)$$

and $\mathscr{B} \in \mathfrak{L}(\mathbb{R}^m, M_2)$, $\mathscr{B}u = (Bu, 0)$, while $\mathscr{C} \in \mathfrak{L}(W, \mathbb{R}^p)$ is given by

$$\mathscr{C}\varphi = \int_{-h}^0 d\gamma(\tau)\varphi^1(\tau).$$

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It follows from e.g., [30] that A2) and A3) are satisfied. Moreover, A4) holds by Remark 1, v).

In particular, note that we are faced with two difficulties here, which to some extent motivates the technical framework in Section II:

- The operator $\mathscr{B}$ is unbounded in the sense that $\mathscr{B} \in \mathfrak{L}(\mathbb{R}^m, V)$ but $\mathscr{B} \notin \mathfrak{L}(\mathbb{R}^m, W)$.
- The operator $\mathscr{C} \in \mathfrak{L}(W, \mathbb{R}^p)$ cannot (in general) be extended to an operator in $\mathfrak{L}(V, \mathbb{R}^p)$. ■

Example 4: Retarded systems with delays in the input can also be modeled within the Pritchard–Salamon class. This is technically more delicate than the output delay case and we refer the reader to [30] and [31]. Moreover, retarded systems with delays in the control *and* observation variables fit into the Pritchard–Salamon class as well, see [36]. ■

Example 5: Single-input, single-output linear systems with time delay in the input plays an important role in control engineering. In transfer function form they are given as

$$g(s) = g_1(s)e^{-sh}$$

where $g_1(s)$ is a rational function and $h > 0$ is the time delay. Alternatively, the system may be given as

$$\dot{x}(t) = Ax(t) + bu(t-h)$$
$$y(t) = cx(t)$$

or

$$\dot{x}(t) = Ax(t) + bu(t)$$
$$y(t) = cx(t-h)$$

where $g_1(s) = c(sI - A)^{-1}b$. Industrial examples include time delays due to conveyor belts, flows in tubes, and computational delays in digital control systems. The sampled implementation of a continuous-time control law is often modeled as a time delay equal to half the sampling period ([39]). In many cases the time delay is unpredictable (e.g., computational delays in digital computers), changing, or operation point dependent (flows in tubes: the time delay being inversely proportional to the flow). Therefore, considering the problem of control of uncertain systems with uncertain time delays is highly industrially relevant. ■

III. Continuously Initially Observable Systems

In [27], Dahleh deals with the adaptive stabilization of a class of delay systems which are "continuously initially observable." Unfortunately, he does not elaborate upon the restrictivity of the assumption. This will be done in the following.

It would be possible to carry out the analysis for a slightly larger class than the Pritchard–Salamon class introduced in Section II. To save space and notations, we shall not do so.

Definition: Let $u \equiv 0$ and $T > 0$, and assume that A1) and A2) are satisfied. Let X be a normed space satisfying $X \subset V$ (set theoretically). We call (1) *initially observable on X on the interval* $[0, T]$ if the output map $x_0 \mapsto CS(\cdot)x_0$ (as a mapping from X to $L_2(0, T; \mathbb{R}^p)$) is one-to-one. We say that (1) is *continuously initially observable on X on the interval* $[0, T]$ if there is a constant γ (in general depending on T) such that $\|x_0\|_X \le \gamma \|CS(\cdot)x_0\|_{L_2(0,T;\mathbb{R}^p)}$ for all $x_0 \in X$. If there exists a $T > 0$ such that (1) is (continuously) initially observable on X on $[0, T]$, we will say that (1) *is (continuously) initially observable on X.*

By Remark 1, i) the definition makes sense also in the case $X \not\subset W$. Initial observability on X on $[0, T]$ is the property that the mapping from state to output is left invertible, while continuous initial observability on X on $[0, T]$ says that it is *continuously* left invertible.

To get some intuition, we will first consider the possibly simplest nontrivial delay system.

Example 6: Consider the delay system on $\mathbb{R}$

$$\dot{x}(t) = 0$$
$$y(t) = x(t-1)$$
$$x(0) = \varphi^0, \quad x(\tau) = \varphi^1(\tau), \quad \tau \in [-1, 0]$$
$$\varphi = (\varphi^0, \varphi^1) \in M_2(-1, 0). \tag{10}$$

By Example 3, (10) can be reformulated as a system of the form (1), where $W = \{(\varphi(0), \varphi) \colon \varphi \in W_{1,2}(-1,0)\}$ and $V = M_2(-1,0)$. The output of (10) is given by

$$y(t) = \begin{cases} \varphi^1(t-1), & t \in [0,1) \\ \varphi^0, & t \ge 1. \end{cases} \tag{11}$$

It is clear that for any subspace $X \subset M_2(-1,0)$ and $T > 1$ (10) is initially observable on X on $[0, T]$. On $W(V)$ the system is initially observable on $[0, T]$ if and only if $T \ge 1$ ($T > 1$). Using (11) it is not difficult to show that (10) is continuously initially observable on $V = M_2(-1,0)$ on the interval $[0, T]$ if and only if $T > 1$. However, as we will show next, *there is no T such that the system is continuously initially observable on W on* $[0, T]$.

Let ψ be a real-valued function and $\mathbb{R}$ be a function satisfying supp $\psi \subset [-1, 0]$ and $\int_{-\infty}^{\infty} \psi^2(\tau)\, d\tau = 1$, and define $\psi_i \in W_{1,2}$ by $\psi_i(\tau) = \psi(i\tau)$ for $\tau \in [-1, 0]$. Further, let φ_i be a sequence of initial conditions in W defined by $\varphi_i = (\psi_i(0), \psi_i) = (0, \psi_i)$. Denote the output of (10) with initial condition $\varphi = \varphi_i$ by y_i. Clearly, for $T \ge 1$ it holds that $\|y_i\|_{L_2[0,T]} = 1/i \to 0$ as $i \to \infty$. The corresponding initial conditions φ_i go to 0 only in coarse topologies such as the M_2 topology, not in finer topologies as the W topology. ■

An immediate observation is that a systems with delay h in the output operator cannot be initially observable on intervals of length less than h. More importantly, even for $T > h$, the next proposition shows that the assumption of continuous initial observability prohibits most delay systems.

We now return the focus to general systems of the form (1). The above example has demonstrated an initially observable system which is not continuously initially observable on the space W, where the observation operator is defined and bounded. This is not a coincidence.

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Proposition 2: Let $T > 0$. Then the system (1) is continuously initially observable on W on $[0, T]$ if and only if it is initially observable on W on $[0, T]$ and $\dim W < \infty$.

The proposition follows via duality arguments from a result in [40] on exact controllability. However, a direct proof can also be constructed along the following lines, which, basically, is a dual version of Proposition 3.2 in [40]: Assume that the system is initially observable on $[0, T]$, otherwise there is nothing to prove. To say that the system is continuously initially observable is to say that the output operator has a bounded left inverse. Next, show that the state-to-output operator, called $\mathscr{O}_W$ in Remark 1 i), can be approximated (in the strong operator topology) by a sequence of finite rank operators, i.e., it is compact. For this, use the semigroup property and that $\lim_{\tau \to 0} \|CS(\tau) - C\| = 0$, where the latter follows from the fact that $S^*(t)$ is a strongly continuous semigroup on W^*. A compact, one-to-one, operator has a bounded inverse if and only if the underlying space is finite dimensional.

In particular, it follows from the proposition that systems with bounded control and observation operators—such as retarded systems with no input- or output delays—are not continuously initially observable on $V = W$.

Remark 7: Proposition 2 shows that (except for the case $\dim W < \infty$) the system (1) is not continuously initially observable on W. If A1) and A2) are satisfied, by Remark 1, i), the input of (1) can be defined as an element in $L_2^{loc}(0, \infty; \mathbb{R}^p)$ (not necessarily pointwise—compare Example 6!) for all initial conditions $x_0 \in V$. It is an interesting problem to find necessary and/or sufficient conditions for continuous initial observability of (1) on V. One such characterization (in terms of duality) follows from [41, theorem 3.2]: (1) is continuously initially observable on V on the interval $[0, t_1]$ if and only if $\operatorname{Im} \mathscr{O}_W^* \supset V^*$ [with $\mathscr{O}_W$ as in Remark 1, i)][1], i.e., the controlled system $(S^*(t), C^*)$ in the state space W^* is *approximately* controllable (on $[0,t_1]$) and the range of its controllability map contains V^*. ∎

IV. Finite-Dimensional Stabilization

We now define exponential stabilizability and detectability on W.

Definition: i) Suppose that A1) and A3) hold. We say that the system (1) is *unbounded exponentially stabilizable on W* if there exists an operator $F \in \mathfrak{L}(W, \mathbb{R}^m)$ such that the perturbed semigroup $S_F(t)$ defined by (7) is exponentially stable on W.

ii) Suppose that A1) and A2) are satisfied. The system (1) is called *exponentially detectable on W* if there exists an operator $H \in \mathfrak{L}(\mathbb{R}^p, W)$ such that the perturbed semigroup $S_H(t)$ defined by (8) is exponentially stable on W.

We shall now turn our attention towards finite-dimensional dynamic output feedback of (1). Introduce the finite-dimensional compensator of order $l \geq 0$

$$\dot{z} = Fz + Gy \qquad z \in \mathbb{R}^l$$
$$u = Hz + Ky. \tag{12}$$

For $l = 0$ (12) should be understood as $u = Ky$.

We will consider dynamic feedback as static feedback applied to an augmented plant, by conceptually adding the dynamics of (12) to the plant. Define

$$\tilde{K} := \begin{pmatrix} K & H \\ G & F \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}, \quad \tilde{C} := \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}$$
$$\tilde{S}(t) := \begin{pmatrix} S(t) & 0 \\ 0 & I \end{pmatrix}, \quad \tilde{W} := W \oplus \mathbb{R}^l, \quad \tilde{V} := V \oplus \mathbb{R}^l. \tag{13}$$

It is clear that $\tilde{W}, \tilde{V}, \tilde{S}(\cdot), \tilde{B}, \tilde{C}$ will satisfy A1), A2), A3), and/or A4) if and only if $W, V, S(\cdot), B, C$ do. Moreover, it is easily seen that unbounded exponential stabilizability on W and exponential detectability on W are preserved under augmentation. If A1) and A3) are satisfied by (1) it follows from Lemma 1, i) that there exists a unique C_0-semigroup $S_{cl}(t)$ on $\tilde{W}$ solving

$$S_{cl}(t)x = \tilde{S}(t)x + \int_0^t \tilde{S}(t-\tau)\tilde{B}\tilde{K}\tilde{C}S_{cl}(\tau)x\,d\tau \tag{14}$$

for all $x \in \tilde{W}$, and all $t \geq 0$.

Definition: Suppose A1) and A3) hold. We say that *(12) exponentially stabilizes (1) on W* if the C_0-semigroup $S_{cl}(t)$ on $\tilde{W}$ given by (14) is exponentially stable.

Remark 8: Suppose that A1) and A3) are satisfied and (1) is exponentially stabilizable on W by a compensator of order l. Then it is easy to show that for all $l' \geq l$ there exists a compensator of order l' which stabilizes (1) exponentially on W. ∎

In the following let $\mathfrak{S}$ denote the set of all systems of the form (1) which satisfy A1)–A4) and which are unbounded exponentially stabilizable and exponentially detectable on W. The set of all systems in $\mathfrak{S}$ which can be exponentially stabilized on W by some compensator of order l will be denoted by $\mathfrak{S}_l$.

Proposition 3: Any system in $\mathfrak{S}$ can be exponentially stabilized on W by some finite-dimensional compensator, i.e.,

$$\mathfrak{S} = \bigcup_{l=1}^{\infty} \mathfrak{S}_l.$$

Proof: It follows from [32] and [34] that any system in $\mathfrak{S}$ has a well-defined transfer matrix whose entries belong to the so-called Callier–Desoer class (cf. [42], [43]). As a consequence, any system in $\mathfrak{S}$ can be stabilized in the input–output sense by a rational compensator (cf. [44]–[46]). An application of the result in [32] on the equivalence of input-output stability and exponential stability for infinite-dimensional systems proves the claim. □

Related results on finite-dimensional stabilization of infinite-dimensional systems can be found in [47]–[49], [31], and [50].

[1] Note that the map $V^* \to W^*$, $\varphi \mapsto \varphi|_W$ is bounded, one-to-one, and has dense range.

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Just as in the finite dimensional case, it suffices to consider stabilization by controllers with e.g., rational coefficients:

Lemma 4: Let $\mathfrak{K}$ be a dense subset of the set of all controllers of the form (12). Then any plant in $\mathfrak{I}_l$ can be stabilized by a controller in $\mathfrak{K}$.

"Dense" should of course be interpreted with respect to the topology induced by identifying the controller (12) with a point in $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$.

Proof: By of the construction above, it is enough to show the lemma for static feedback, i.e., $l = 0$. Let G denote the transfer function matrix of (1) and suppose that $K \in \mathbb{R}^{m\times p}$ exponentially stabilizes (1) on W. We shall show that $K + \Delta$ has the same property for all sufficiently small $\Delta \in \mathbb{R}^{m\times p}$. It is known (see [32] and [34]) that the entries of G belong to the Callier–Desoer class. It follows from the equivalence of exponential stability and input–output stability for systems in $\mathfrak{I}$ ([32]) that it is sufficient to show that it is sufficient to show that

$$(I + G(K+\Delta))^{-1}G \in H_\infty^{p\times m} \quad \text{for all sufficiently small} \quad \Delta \in \mathbb{R}^{m\times p} \tag{15}$$

where H_∞ denotes the functions which are analytic and bounded on the open right-half plane. Since $(I + GK)^{-1}G \in H_\infty^{p\times m}$ and

$$(I + G(K+\Delta))^{-1}G = \left(I + (I+GK)^{-1}G\Delta\right)^{-1} \cdot (I+GK)^{-1}G$$

we may consider $(I + G(K+\Delta))^{-1}G$ as the feedback interconnection of two stable subsystems: the system represented by the transfer function $(I + GK)^{-1}G$, and the system represented by Δ. Thus (15) follows from the small gain theorem, [51]. □

V. Stabilization by Switching Control

We next introduce some concepts and definitions for adaptive stabilization. The type of switching function controllers we will consider first appeared in [13]. It was the main tool of [27]. Similar approaches can be found in [24] and [26]. It should be noted that the concept of adaptive stabilization by switching was introduced in the papers by Byrnes and Willems [4] and [52].

Let $\mathfrak{K} = \{K_i\}_{i\in\mathbb{N}}$ be a countable set of controllers of the form (12). By $u = Ky$ we mean the operator relationship between u and y, for some initial condition $z(0)$, which is to be considered as a part of the operator K. Further, let $\{\tau_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers, increasing towards infinity. We call a function $\sigma:\mathbb{R}\to\mathbb{N}$ a *switching function with switching points* $\{\tau_i\}$ if for all $a \in \mathbb{R}$ it holds that $\sigma([a,\infty)) = \mathbb{N}$ and its discontinuity points are $\{\tau_i\}$. We also require σ to be right continuous.

The switching function controller associated with $\mathfrak{K}$ and σ is now defined to be

$$u = K_{\sigma(k)}y$$
$$\dot{k} = \|y\|^2 + \|u\|^2, \qquad k(0) = k_0. \tag{16}$$

The structure is illustrated in Fig. 1.

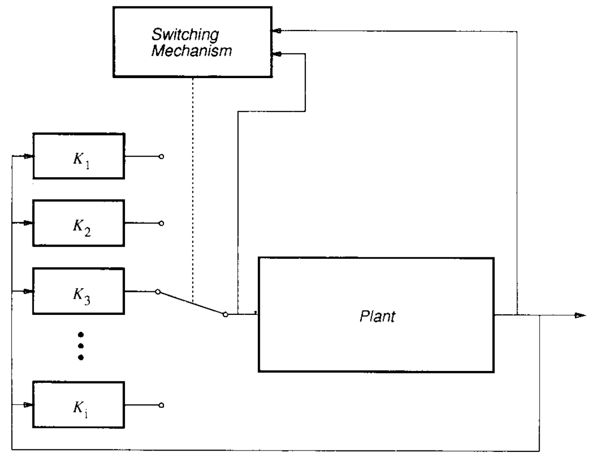

Fig. 1. Switching function controller.

In this paper, we will only consider the switching point sequence given by

$$\tau_i = \tau_{i-1}^2 \qquad i = 1, 2, \cdots,$$
$$\tau_0 = \alpha \tag{17}$$

for some $\alpha > 1$.

Remark 9: The way (16) is written (compare Fig. 1) it says that all controllers K_i are processing the plant output for all $t \geq 0$. Thus, unless all but a finite number of K_i are memoryless, (16) is an infinite-dimensional controller. If all K_i have a realization on a common state space, say $\mathbb{R}^l$, with a common initial condition $z(0)$, this difficulty can be avoided by considering static feedback applied to an augmented plant, as in (13). We will write this controller

$$\tilde{u} = \tilde{K}_{\sigma(k)}\tilde{y}$$
$$\dot{k} = \|\tilde{y}\|^2 + \|\tilde{u}\|^2, \qquad k(0) = k_0 \tag{18}$$

with notation as in (13). ■

Consider the augmented system (1), i.e.,

$$\tilde{x}(t) = \tilde{S}(t)\tilde{x}_0 + \int_0^t \tilde{S}(t-\tau)\tilde{B}\tilde{u}(\tau)\,d\tau, \quad \tilde{x}_0 \in \tilde{W}$$
$$\tilde{y}(t) = \tilde{C}\tilde{x}(t) \tag{19}$$

where $\tilde{S}(t)$, $\tilde{B}$, $\tilde{C}$, and $\tilde{W}$ are defined as in (13). For $T \in \mathbb{R}_+ \cup \{\infty\}$ we shall call a function $(\tilde{x}, k): [0,T) \to \tilde{W} \times \mathbb{R}$ a solution of the closed-loop equations (19) and (18) on $[0,T)$ if $\tilde{x}:[0,T) \to \tilde{W}$ is continuous, $k:[0,T)\to\mathbb{R}$ is absolutely continuous, and $(\tilde{x}, k)$ satisfies (19) and (18) for almost all $t \in [0,T)$.

Definition: By saying that (18) stabilizes (19) we shall mean that for all initial conditions $\tilde{x}(0) \in \tilde{W}$ and $k(0) \in \mathbb{R}$ it holds that $(\|\tilde{x}(t)\|_{\tilde{W}}, k(t)) \to (0, k_\infty)$ (where $k_\infty < \infty$) as $t \to \infty$. For $\mathfrak{I}'$ a subset of $\mathfrak{I}$ we say that (18) stabilizes $\mathfrak{I}'$ if it stabilizes every member in $\mathfrak{I}'$. If $\mathfrak{I}' \subset \mathfrak{I}$ is such that there exists a controller of the type (18) stabilizing it, we say that $\mathfrak{I}'$ can be adaptively stabilized.

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In [13], a general theorem on adaptive stabilization of finite-dimensional linear systems by switching function controllers was proved. It was extended to a class of nonlinear systems in [53]. Here we shall prove a generalization which covers the class $\mathfrak{S}$ of infinite-dimensional systems.

In order not to burden the paper with generality we will not take advantage of, we will not attempt to formulate the theorem in the greatest possible generality. However, some comments will be given in a remark.

Theorem 5: Let $\mathfrak{S}' \subset \mathfrak{S}$. Assume that $\mathfrak{K} = \{K_i\}_{i \in \mathbb{N}}$ is a set of controllers of the type (12) (with a bound on the l's), with the property that for any system in $\mathfrak{S}'$ there is a controller K (depending on the system) in $\mathfrak{K}$ exponentially stabilizing it on W. Then the controller (18) [with τ_i according to (17)] will stabilize any system in $\mathfrak{S}'$.

Without loss of generality, we may assume that $\mathfrak{S}' \subset \mathfrak{S}_{l^*}$, for some l^*.

We would like to stress the "modular" character of the theorem. By the "selection" of $\mathfrak{K}$ it is possible to take advantage of available *a priori* knowledge.

At the end of this section, the theorem will be used to show that, just as in the finite dimensional case [1], the order of a stabilizing controller is sufficient *a priori* information for adaptive stabilization. In the next section, the theorem will be used to construct an adaptive controller for a class of high-gain stabilizable systems.

In previous work on adaptive stabilization by searching dense sets of controllers, a crucial step was an estimate of $\|x\|$ in terms of the L_2-norm of y and u; see [1], [13], [24], and [25]. This estimate expresses the fact that the initial state of a finite-dimensional observable system can be continuously reconstructed from the observation. As Proposition 2 showed, generalization to infinite-dimensional systems is not entirely straightforward. Somewhat surprisingly, exponential detectability, instead of observability, turns out to be a fruitful approach.

Proposition 6: Suppose A1)–A4) are satisfied and (1) is exponentially detectable on W. Then we have the following:

i) For all $x_0 \in W$ there exist constants $c_0 = c_0(x_0)$ and c_1 (not depending on x_0), such that for all $t \geq 0$ and $u \in L_2^{loc}(0, \infty; \mathbb{R}^m)$ it holds that

$$\|x(t)\|_W^2 \leq c_0 + c_1\left(\int_0^t \|y(\tau)\|^2\, d\tau + \int_0^t \|u(\tau)\|^2\, d\tau\right). \quad (20)$$

i. Let $x_0 \in W$. If $u \in L_2(0, \infty; \mathbb{R}^m)$ produces an output $y \in L_2(0, \infty; \mathbb{R}^p)$ then

$$\lim_{t \to \infty} \|x(t)\|_V = \lim_{t \to \infty} \|x(t)\|_W = 0. \quad (21)$$

For the proof we need the following technical result, which is proven in the appendix. The lemma makes sense of the (purely formal) equation

$$\dot{x} = Ax + Bu = (A + HC)x - HCx + Bu.$$

Lemma 7: Suppose A1)–A4) hold. For $H \in \mathfrak{L}(\mathbb{R}^p, W)$, let $S_H(\cdot)$ be as in Lemma 1, ii). Then, for all $x_0 \in W$ and $u \in L_2^{loc}(0, \infty; \mathbb{R}^m)$ the integral equation

$$w(t) = S_H(t)x_0 + \int_0^t S_H(t - \tau)Bu(\tau)\, d\tau - \int_0^t S_H(t - \tau)HCw(\tau)\, d\tau \quad (22)$$

has a unique solution $x \in \mathfrak{C}(0, \infty; W)$ which is given by

$$x(t) = S(t)x_0 + \int_0^t S(t - \tau)Bu(\tau)\, d\tau,$$

i.e., the mild solution of (1a).

Proof of Proposition 6: By the definition of exponential detectability, there exists an operator $H \in \mathfrak{L}(\mathbb{R}^p, W)$ such that the C_0-semigroup $S_H(t)$ defined by (8) is exponentially stable on W, i.e., there exist $\Lambda, \lambda > 0$ such that

$$\|S_H(t)\|_{\mathfrak{L}(W)} \leq \Lambda e^{-\lambda t}$$

for all $t \geq 0$. It follows from Lemma 7 that for $x_0 \in W$ the mild solution of (1a) satisfies

$$x(t) = S_H(t)x_0 - \int_0^t S_H(t - \tau)Hy(\tau)\, d\tau + \int_0^t S_H(t - \tau)Bu(\tau)\, d\tau. \quad (23)$$

By Lemma 1, ii), the exponential stability of $S_H(\cdot)$ on W and Remark 1, iv) there exists a constant $\gamma > 0$ (not depending on t or u) such that

$$\left\|\int_0^t S_H(t - \tau)Bu(\tau)\, d\tau\right\|_W \leq \gamma\|u\|_{L_2(0,t)} \quad (24)$$

for all $t \geq 0$ and $u \in L_2^{loc}(0, \infty; \mathbb{R}^p)$. From (23) we get

$$\|x(t)\|_W \leq \Lambda\|x_0\|_W + \gamma\left(\int_0^t \|u(\tau)\|^2\, d\tau\right)^{1/2} + \Lambda\|H\|\left(\int_0^t e^{-2\lambda\tau}\, d\tau\right)^{1/2}\left(\int_0^t \|y(\tau)\|^2\, d\tau\right)^{1/2}.$$

With $\Gamma = \max(\gamma, \Lambda\|H\|(\int_0^\infty e^{-2\lambda t}\, d\tau)^{1/2})$ it follows that

$$\|x(t)\|_W^2 \leq 2\Lambda^2\|x_0\|_W^2 + 2\Gamma^2\left[\left(\int_0^t \|y(\tau)\|^2\, d\tau\right)^{1/2} + \left(\int_0^t \|u(\tau)\|^2\, d\tau\right)^{1/2}\right]^2 \leq 2\Lambda^2\|x_0\|_W^2 + 4\Gamma^2\left(\int_0^t \|y(\tau)\|^2\, d\tau + \int_0^t \|u(\tau)\|^2\, d\tau\right).$$

With $c_0 = 2\Lambda^2\|x_0\|_W^2$ and $c_1 = 4\Gamma^2$, this is the first statement.

By assumption A1), it is enough to show the second equality of statement ii). For this, note that $\lim_{t \to \infty} \|S_H(t)x_0\|_W = 0$, and, moreover,

$$\lim_{t \to \infty} \left\|\int_0^t S_H(t - \tau)Hy(\tau)\, d\tau\right\|_W = 0$$

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since $S_H(t)$ is exponentially stable on W and $Hy \in L_2(0,\infty; W)$. Setting

$$z(t) := \int_0^t S_H(t-\tau)Bu(\tau)\,d\tau$$

it follows from (23) that it only remains to show that $\lim_{t\to\infty} \|z(t)\|_W = 0$. (This does not follow immediately from $u \in L_2$, since $Bu(\cdot)$ may take on values outside of W.) To this end, let $\epsilon > 0$ be given. Since $u \in L_2(0,\infty)$ there is a T such that

$$\int_T^\infty \|u(\tau)\|^2\,d\tau \le \frac{\epsilon}{2\gamma}.$$

For $t > T$ we have

$$z(t) = \int_0^T S_H(t-\tau)Bu(\tau)\,d\tau + \int_T^t S_H(t-\tau)Bu(\tau)\,d\tau$$
$$= S_H(t-T)\int_0^T S_H(T-\tau)Bu(\tau)\,d\tau + \int_T^t S_H(t-\tau)Bu(\tau)\,d\tau.$$

Therefore, by (24)

$$\|z(t)\|_W \le \Lambda e^{-\lambda(t-T)}\gamma\|u\|_{L_2(0,\infty)} + \frac{\epsilon}{2}.$$

Clearly, $\|z(t)\|_W \le \epsilon$ for large t. Since $\epsilon > 0$ was arbitrary, it follows that $\lim_{t\to\infty}\|z(t)\|_W = 0$. □

Remark 10: By replacing Lemma 3.1 in [13] and the lemma in [1] by Proposition 6, the results on stabilization of observable plants immediately become valid for exponentially detectable plants, with no other changes in the proofs. Precisely, this applies to the theorem in [1], and to Theorem 3.18, Theorem 4.1, and Theorem 5.1 in [13]. ■

Proof of Theorem 5: Using routing arguments, it can be shown that the closed-loop equations (19) and (18) locally admits a unique solution. Further, if it exists and is bounded (in $\tilde{W} \times \mathbb{R}$) on an interval $[0, T)$, it can also be continued beyond T.

We claim that it is enough to prove that $k(t)$ is bounded, or equivalently, that $\sigma(k(t))$ only switches a finite number of times when $t \to t^\dagger$, where $[0, t^\dagger)$ is the maximal interval of existence of the solution to the closed-loop equations (19) and (18). Under this assumption, it follows from Proposition 6, i) (with x replaced by $\tilde{x}$ etc.,) that $\|\tilde{x}(t)\|_{\tilde{W}}$ is bounded on $[0, t^\dagger)$. Therefore, by the above, the global existence of the solution is established. Moreover, if k stays bounded, it converges, since it is monotonically increasing. However, this is exactly the statement that $\tilde{y} \in L_2(0,\infty;\mathbb{R}^{p+l})$ and $\tilde{u} \in L_2(0,\infty;\mathbb{R}^{m+l})$. It follows from Proposition 6, ii) that $\lim_{t\to\infty}\|\tilde{x}(t)\|_{\tilde{W}} = 0$, which proves the claim.

Next, assume that $\sigma(k(\cdot))$ switches an infinite number of times. We consider a fixed system in $\mathfrak{S}'$. Let K_q denote a controller which exponentially stabilizes on W. By the assumption, there is an infinite sequence $\{t_j\}$ and a subsequence $\{\tau_{i_j}\} \subset \{\tau_i\}$, such that $k(t_j) = \tau_{i_j}$ and $\sigma(\tau_{i_j}) = q$ for all i. ("A 'good' controller K_q exists, and will be used infinitely often.") Define $t_j' = \min\{t: k(t) = \tau_{i_j+1}\}$ (which exists under the assumption of $k(t)$ being unbounded). ("The 'good' controller will always be disconnected after a finite time.") But this is to say

$$\int_{t_j}^{t_j'} \|\tilde{y}(t)\|^2 + \|\tilde{u}(t)\|^2\,dt = \int_{t_j}^{t_j'} \|\tilde{y}(t)\|^2 + \|\tilde{K}_q\tilde{y}(t)\|^2\,dt \ge \tau_{i_j+1} - \tau_{i_j}. \tag{25}$$

Since $\tilde{K}_q$ by assumption is exponentially stabilizing on $\tilde{W}$, there exists a constant c [not depending on t_j or $x(t_j)$] such that

$$\int_{t_j}^{t_j'} \|\tilde{y}(t)\|^2 + \|\tilde{u}(t)\|^2\,dt \le c\|\tilde{x}(t_j)\|_{\tilde{W}}^2. \tag{26}$$

By Proposition 6, i), for some c_0, c_1, we have $\|\tilde{x}(t_j)\|_{\tilde{W}}^2 \le c_0 + c_1(k(t_j) - k_0)$ for all j. Noting that $k(t_j) = \tau_{i_j}$ and combining these estimates, we arrive at the inequality

$$\tau_{i_j+1} - \tau_{i_j} \le cc_0 + cc_1(\tau_{i_j} - k_0). \tag{27}$$

But, with $\{\tau_i\}$ chosen as in (17), it is easy to see that for any c, c_0, and c_1, (27) will be violated for all sufficiently large j. □

Remark 11: We list a number of simple generalizations of Theorem 5. They all follow from an inspection of the above proof. The details are omitted.

i) The parameter updating law $\dot{k} = \|\tilde{y}\|^2 + \|\tilde{u}\|^2$ can be replaced by a continuous, causal, functional $\Phi(\tilde{y}(\cdot), \tilde{u}(\cdot))$ satisfying $\Phi(\tilde{y}(\cdot), \tilde{u}(\cdot))(t_1) \le \Phi(\tilde{y}(\cdot), \tilde{u}(\cdot))(t_2)$ for $t_1 \le t_2$, and $\Phi(\tilde{y}(\cdot), \tilde{u}(\cdot))(t_1) \ge c\int_0^{t_1}(\|\tilde{y}(t)\|^2 + \|\tilde{u}(t)\|^2)\,dt$ for some $c > 0$. It is enough that the last condition holds in a "weak" sense, for example for large t_1.

ii) If $\mathfrak{K}$ is bounded (as a subset of $\mathbb{R}^{(m+l)\times(p+l)}$), the parameter updating law $k(t) = \|\tilde{y}(t)\|^2$ can be used (by the previous remark).

iii) The set of switching points given by (17) is of course not magic. Required is a sequence violating (27) for large indexes. If (27) is to be violated for *any* c, c_0, and c_1, we need a sequence growing faster than exponential. With some *a priori* knowledge about $\mathfrak{S}'$, i.e., some knowledge about c, c_0, and c_1, a less violently growing sequence may be possible. For $\mathfrak{S}'$ such that the possible c, c_0, and c_1 lie in a bounded set (if $\mathfrak{S}'$ is compact in some sense), an exponentially growing switching sequence is possible.

iv) Let $\alpha > 0$ be given. Under the additional assumption that $\mathfrak{K}$ for any system in $\mathfrak{S}'$ contains a controller stabilizing it with exponential decay rate α, the controller

$$\tilde{u}(t) = \tilde{K}_{\sigma(k(t))}\tilde{y}(t)$$
$$\dot{k}(t) = e^{\alpha t}\left(\|\tilde{y}(t)\|^2 + \|\tilde{u}(t)\|^2\right)$$

will stabilize any system in $\mathfrak{S}'$ with decay rate α. Compare [23], [22], and [20]. ■

Finally, we generalize the result by Mårtensson [1] to the class $\mathfrak{S}$ of infinite-dimensional systems. In order to emphasize the existence proof character of the controller, we formulate the result both in a nonconstructive and in a constructive theorem. Of course, the latter will be proved.

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Theorem 8 (Nonconstructive Version): $\mathfrak{S}_l$ can be adaptively stabilized. That is, sufficient *a priori* information for adaptive stabilization of the infinite dimensional system (1) is the order of a stabilizing controller.

Theorem 8 (Constructive Version): Let $\mathfrak{K} = \{K_i\}_{i\in\mathbb{N}}$ be an enumeration of all lth order controllers of the type (12) with rational coefficients. Assume that system (1) is in $\mathfrak{S}_l$, i.e., exponentially stabilizable on W by some controller (12) of order l. Under these assumptions, the controller (18) will stabilize (19) in the sense that for all $\tilde{x}(0) \in \tilde{W}$ and $k(0) \in \mathbb{R}$ it holds that $(\|\tilde{x}(t)\|_{\tilde{W}}, k(t)) \to (0, k_\infty)$ (where $k_\infty < \infty$) as $t \to \infty$.

Proof: By Lemma 4 a system in $\mathfrak{S}_l$ is exponentially stabilizable on W by a controller in $\mathfrak{K}$. The theorem now follows from Theorem 5. □

Remark 12: Let $\alpha > 0$ be given and let $\mathfrak{S}_{l,\alpha}$ denote the subset of systems in $\mathfrak{S}_l$ for which there exists $F \in \mathfrak{L}(W, \mathbb{R}^m)$ and $H \in \mathfrak{L}(\mathbb{R}^p, W)$ such that the semigroups $S_F(t)$ and $S_H(t)$ defined by (7) and (8) have a decay rate of at least α on W. Then there exists a countable set $\mathfrak{K}$ which for any system in $\mathfrak{S}_{l,\alpha}$ contains a controller stabilizing it with decay rate α. Hence, the control law in Remark 11, iv) will stabilize any system in $\mathfrak{S}_{l,\alpha}$ with decay rate α. ■

VI. High-Gain Stabilization

In this section, we turn our attention to a class of infinite-dimensional systems which are stabilizable by high-gain output feedback. We will only consider square systems, i.e., with $p = m$. By $\mathfrak{H}$ we will denote the set of all systems of the form (1) in $\mathfrak{S}$ satisfying

(HG) The inverse of the transfer function matrix G of the system (1) satisfies

$$G^{-1}(s) = sD + H(s) \tag{28}$$

where

$$D \in GL(m) \quad \text{and} \quad H \in H_\infty^{m\times m}.$$

We remark that (HG) makes sense since it is known (see [32], [34]) that each system in $\mathfrak{S}$ has a well-defined transfer function matrix whose entries belong to the Callier–Desoer class ([42], [43]). Moreover, D and H are uniquely determined by G.

We next introduce two subclasses $\mathfrak{B}$ and $\mathfrak{R}$ of $\mathfrak{S}$ and show that they are contained in $\mathfrak{H}$.

By $\mathfrak{B}$ we will denote the subset of $\mathfrak{S}$ with *bounded control and observation* (i.e., $V = W$) satisfying the following additional properties:

- The system (1) is *exponentially minimum-phase*, i.e., there exists a $\gamma > 0$ such that the kernel of the operator

$$\begin{pmatrix} sI - A & B \\ C & 0 \end{pmatrix}$$

is trivial for all complex s with $\operatorname{Re} s \geq -\gamma$.

- $\det CB \neq 0$.
- (At least) one of the following properties is satisfied:
  1) $\operatorname{Im} B \subset D(A)$ and $\operatorname{Im} C^* \subset D(A^*)$,
  2) $\operatorname{Im} B \subset D(A^2)$,
  3) $\operatorname{Im} C^* \subset D(A^{*2})$,
  4) $S(t)$ is analytic and $\operatorname{Im} B \subset D(A)$,
  5) $S(t)$ is analytic and $\operatorname{Im} C^* \subset D(A^*)$.

It has been proven in [54] that $\mathfrak{B} \subset \mathfrak{H}$. Further, D is given by $(CB)^{-1}$.

Retarded systems (in general) do not satisfy any of the smoothness conditions 1)–5). However, a direct analysis of the transfer function matrix of a minimum-phase retarded system with invertible instantaneous gain will show that (HG) holds. We turn our attention to a set of retarded systems of the form

$$\begin{aligned} \dot{x}(t) &= \int_{-h}^{0} dA(\tau)x(t+\tau) + Bu(t) \\ y(t) &= Cx(t) + \int_{-h}^{0} \Gamma(\tau)x(t+\tau)\,d\tau \\ x(0) &= x^0 \quad x(\tau) = x^1(\tau), \quad \tau \in [-h, 0) \\ (x^0, x^1) &\in M_2 = \mathbb{R}^n \times L_2(-h, 0; \mathbb{R}^n) \end{aligned} \tag{29}$$

where $A \in BV(-h, 0; \mathbb{R}^{n\times n})$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $\Gamma \in AC(-h, 0; \mathbb{R}^{m\times n})$. By $\mathfrak{R}$ we will denote the set of all systems of the form (29) having the properties

- $$\det \begin{pmatrix} sI - \hat{A}(s) & B \\ C + \hat{\Gamma}(s) & 0 \end{pmatrix} \neq 0 \quad \text{for all } s \in \overline{\mathbb{C}_+} \tag{30}$$

where $\hat{A}(s) = \int_{-h}^{0} dA(\tau)e^{s\tau}$, $\hat{\Gamma}(s) = \int_{-h}^{0}\Gamma(\tau)e^{s\tau}\,d\tau$, and $\mathbb{C}_+$ denotes the open right-half plane

- $$\det CB \neq 0. \tag{31}$$

Proposition 9: The set $\mathfrak{R}$ is contained in $\mathfrak{H}$. Further, $D = (CB)^{-1}$.

Proof: Let the operators $\mathscr{A}$ and $\mathscr{B}$ be as in Example 3, and define the output operator $\mathscr{C}$ by

$$\mathscr{C}: M_2 \to \mathbb{R}^m$$

$$(\varphi^0, \varphi^1) \mapsto C\varphi^0 + \int_{-h}^{0} \Gamma(\tau)\varphi^1(\tau)\,d\tau.$$

Since $\Gamma \in L_2(-h, 0; \mathbb{R}^{n\times m})$ it follows that $\mathscr{C} \in \mathscr{L}(M_2, \mathbb{R}^m)$ and hence (29) can be represented as the infinite-dimensional system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathscr{A}\mathbf{x} + \mathscr{B}u \\ y &= \mathscr{C}\mathbf{x} \end{aligned} \tag{32}$$

with state space $M_2(-h, 0; \mathbb{R}^n)$. The assumptions A1)–A4) are trivially satisfied by (32) because $\mathscr{B} \in \mathscr{L}(\mathbb{R}^m, M_2)$ and $\mathscr{C} \in \mathscr{L}(M_2, \mathbb{R}^m)$. Moreover, it follows from (30) that

$$\operatorname{rk}\begin{pmatrix} sI - \hat{A}(s) & B \end{pmatrix} = n \quad \text{for all } s \in \overline{\mathbb{C}_+}$$

and

$$\operatorname{rk}\begin{pmatrix} sI - \hat{A}(s) \\ C + \hat{\Gamma}(s) \end{pmatrix} = n \quad \text{for all } s \in \overline{\mathbb{C}_+}$$

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which implies that (32) is exponentially stabilizable and exponentially detectable (cf. e.g., [30]).

We have shown that $\mathfrak{R} \subset \mathfrak{S}$, and it remains to verify that the transfer matrix of (29)

$$G(s) = \left(C + \hat{\Gamma}(s)\right)\left(sI + \hat{A}(s)\right)^{-1} B$$

satisfies assumption (HG), with $D = (CB)^{-1}$. This will be done in several steps.

Steps 1: Denote the ring of entire functions and the field of meromorphic functions on $\mathbb{C}$ by $\mathscr{H}$ and $\mathscr{M}$, respectively. Realize that $G \in \mathscr{M}^{m \times m}$. It is well known that $\mathscr{M}$ is the quotient field of $\mathscr{H}$ and further that $\mathscr{H}$ is a Bezout ring (i.e., every finitely generated ideal is principal), see e.g., [55]. Therefore, it follows ([56]) that G admits a right-coprime factorization over $\mathscr{H}$, i.e., there exist $P, Q, X, Y \in \mathscr{H}^{m \times m}$ satisfying $G = PQ^{-1}$ and $XP + YQ = I$. Now

$$\det \begin{pmatrix} sI - \hat{A}(s) & B \\ C + \hat{\Gamma}(s) & 0 \end{pmatrix} = \frac{\det\left(sI - \hat{A}(s)\right)}{\det Q(s)} \det P(s)$$

and since $\det Q$ divides $\det(sI - \hat{A}(s))$ in the ring $\mathscr{H}$ (cf. [46]) it follows from (30) that

$$\det P(s) \neq 0 \qquad \text{for all } s \in \overline{\mathbb{C}_+}. \tag{33}$$

Step 2: Setting $F(s) := (s+1)G(s)$ we claim that it is sufficient to show that

$$F(s) - CB = O(s^{-1}) \quad \text{as} \quad |s| \to \infty \quad \text{in} \quad \overline{\mathbb{C}_+}. \tag{34}$$

If (34) is true we have in particular

$$\lim_{\substack{|s| \to \infty \\ s \in \mathbb{C}_+}} F(s) = CB. \tag{35}$$

Combining (31), (33), and (35) shows that $F^{-1}(s)$ is continuous on $\overline{\mathbb{C}_+}$ and belongs to $H_\infty^{m \times m}$. Therefore, by (34)

$$K(s) := (s+1)\left(F^{-1}(s) - (CB)^{-1}\right) \in H_\infty^{m \times m}.$$

It follows that $G^{-1}(s) = (s+1)F^{-1}(s) = (s+1)(CB)^{-1} + K(s)$, which is (28).

Step 3: It remains to show that (34) holds. Realize that by the assumptions on A and Γ the functions $\hat{A}(s)$, $\hat{\Gamma}(s)$, and $s\hat{\Gamma}(s)$ are bounded on $\overline{\mathbb{C}_+}$. Hence, we obtain from

$$\begin{aligned} s(F(s) - CB) &= s\left(C + \hat{\Gamma}(s)\right)\left(I - s^{-1}\hat{A}(s)\right)^{-1} \\ &\quad \cdot B(I + s^{-1}I) - sCB \\ &= C\left(\sum_{i=1}^{\infty} s^{-(i-1)}\hat{A}^i(s)\right)B \\ &\quad + s\hat{\Gamma}(s)\left(\sum_{i=0}^{\infty} s^{-i}\hat{A}^i(s)\right)B \\ &\quad + \left(C + \hat{\Gamma}(s)\right)\left(\sum_{i=0}^{\infty} s^{-i}\hat{A}^i(s)\right)B \end{aligned}$$

that $s(F(s) - CB)$ is bounded on the set $\{s \in \overline{\mathbb{C}_+} : |s| > \varrho\}$, for ϱ some large constant. □

The next result shows that all systems in $\mathfrak{H}$ can be exponentially stabilized by static high-gain output feedback. In the finite-dimensional case, it is well known, e.g., from multivariable root-locus (e.g., [57]), that if spec $CB \subset \mathbb{C}_+$, all such systems will be stabilized by the feedback $u = -ky$ for all sufficiently large k. Under the additional technical assumptions introduced above, this also holds in infinite dimensions:

Proposition 10: Suppose that (1) belongs to $\mathfrak{H}$, and that spec $D \subset \mathbb{C}_+$. Then there exists a $k_0 > 0$ such that the control law $u = -ky$ will stabilize (1) for all $k > k_0$.

Proof: For positive integers q, r, for H_1 (H_2) a $q \times r$ ($r \times q$) matrix of functions from $\mathbb{C}$ to $\mathbb{C}$, we write $\mathfrak{F}(H_1, H_2) := (I + H_1H_2)^{-1}H_1$, the "transfer function" of the "plant" H_1 "controlled" by $u = -H_2y + v$. By a general result on the equivalence of input–output stability and exponentially stability for infinite-dimensional systems (cf. [58] or [32]) it is sufficient to show that $G_k := \mathfrak{F}(G, kI) \in H_\infty^{m \times m}$ for all sufficiently large k. From (28) and a short computation (or, possibly easier, a block diagram argument) we have]

$$G_k(s) = \mathfrak{F}\left(\mathfrak{F}\left(\frac{1}{s}D^{-1}, kI\right), H(s)\right).$$

Since, by definition, $H \in H_\infty^{m \times m}$ it will follow from the small gain theorem that $G_k \in H_\infty^{m \times m}$ if we can show that $\|\mathfrak{F}((1/s)D^{-1}, kI)\|_{H_\infty^{m \times m}}$ can be made arbitrarily small by selecting k large enough.

By an elementary computation, $\mathfrak{F}((1/s)D^{-1}, kI) = (sI + kD^{-1})^{-1}D^{-1}$. Therefore, since spec $D^{-1} = (\text{spec } D)^{-1} \subset \mathbb{C}_+$, it holds that $\mathfrak{F}((1/s)D^{-1}, kI) \in H_\infty^{m \times m}$ for all $k > 0$, and furthermore, $\|\mathfrak{F}((1/s)D^{-1}, kI)\|_{H_\infty^{m \times m}} \to 0$ as $k \to \infty$. □

We note the following simple corollary.

Corollary 11: Suppose that (1) belongs to $\mathfrak{H}$, and that, for some $Q \in \mathbb{R}^{m \times m}$, spec $DQ \subset \mathbb{C}_+$. Then there exists a $k_0 > 0$ such that the control law $u = -kQy$ will stabilize (1) for all $k > k_0$.

Proof: The condition spec $DQ \subset \mathbb{C}_+ \not\ni 0$ means in particular that Q is nonsingular. A system in $\mathfrak{S}$ given by the triple (A, B, C) clearly belongs to $\mathfrak{H}$ if and only if $(A, B, Q^{-1}C) \in \mathfrak{H}$. An application of Proposition 10 to be latter system proves the corollary. □

In the light of this corollary, the problem of finding a (constant coefficient) controller for a system in $\mathfrak{H}$ reduces to the problem of finding $Q \in GL(m)$ such that spec $DQ \in \mathbb{C}_+$, and to find k large enough. We will than invoke the switching function machinery developed in Section V to switch among these to arrive at an adaptive controller stabilizing any member of $\mathfrak{H}$.

The following terminology was originally suggested by C. I. Byrnes: For $m \geq 1$ we call a set $\mathscr{Q} \subset GL(m)$ *unmixing* if it holds that for any $A \in GL(m)$ there is a $Q \in \mathscr{Q}$ such that spec AQ = spec $QA \subset \mathbb{C}_+$.

An extensive treatment of the unmixing problem can be found in [59] or [13, section VI-B]. The following proposition collects some relevant facts therein.

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Proposition 12: i) For all $m \geq 1$ there exist finite unmixing sets.

ii) It is enough to unmix $SO(m)$. More precisely: If $\mathcal{Q} \subset SO(m)$ is such that for any $A \in SO(m)$ there is a $Q \in \mathcal{Q}$ such that spec AQ = spec $QA \subset \mathbb{C}_+$ then $\mathcal{Q} \cup J\mathcal{Q}$ (where $J = \text{diag}(1, \ldots, 1, -1) \in O(n)$) is unmixing. iii) A necessary, but not sufficient, condition for a set $\mathcal{Q} \subset GL(m)$ to be unmixing is that there is no proper subspace of $\mathbb{R}^m$ which is invariant for all $\mathcal{Q} \in \mathcal{Q}$ (For $\mathcal{Q}$ a group: The natural representation of $\mathcal{Q}$ on $\mathbb{R}^m$ is irreducible.)

For the proof, as well as more comments, the reader is referred to [59] or [13].

For $m = 1$, clearly $\{1, -1\}$ is an unmixing set. For $m = 2$ it is easy to show ([59] or [13]) that the three element set consisting of the identity and the rotations $\pm 2\pi/3$ unmixes $SO(2)$. It can be shown that $O(3, \mathbb{Z})$ (the orthogonal matrices with integer elements) is unmixing for $GL(3)$. However, it is not true that $O(m, \mathbb{Z})$ is unmixing for $GL(m)$ for large m ([59] or [13]). Exactly for what m the statement holds is unknown. In [13], it was made plausible that the subgroup of index 2 in $O(3, \mathbb{Z})$ consisting of the matrices of the group of rotations of the unit cube in 3-space, is unmixing. Recently, [60] showed that this statement is true, except for a finite number of matrices. A 16 element solution for $SO(3)$ is also given there.

For any m a finite unmixing set is explicitly given in [59] and [13]. The construction can be described as follows: The Euler angles on $SO(m)$ can be considered as a mapping from a finite interval $I \subset \mathbb{R}^M$ (where $M = \dim SO(m) = m(m-1)/2$) to $SO(m)$. This mapping is onto, and almost one-to-one (one-to-one except for some of the edges). Putting a sufficiently fine lattice on I, the image of the lattice points under the Euler angle mapping will be an unmixing set.

We now tie the pieces together in an adaptive algorithm, capable of stabilizing any system in $\mathfrak{H}$. The result should be compared with e.g., the results in [14], which only considers multivariable systems having an instantaneous gain with unmixed spectrum, i.e., either spec $D^{-1} \subset \mathbb{C}_+$ or spec $D^{-1} \subset \mathbb{C}_-$. Here a "mixed" spectrum is allowed, with eigenvalues in both half planes. Moreover, the set $\mathfrak{H}$ is much larger than the class of systems considered in [14].

Theorem 13: Assume that (1) describes a system in $\mathfrak{H}$. Let $\mathcal{Q}$ be a finite or countable unmixing set for $GL(m)$, and let $\mathfrak{K} = \{K_i\}_{i=1}^{\infty}$ be an enumeration of $\{kQ: k \in \mathbb{N}, Q \in \mathcal{Q}\}$. Under these assumptions, the controller (16) will stabilize (1) in the sense that for all $x(0) \in W$ and $k(0) \in \mathbb{R}$ it holds that $(\|x(t)\|_W, k(t)) \to (0, k_\infty)$ (where $k_\infty < \infty$) as $t \to \infty$.

By Proposition 12, the theorem is not void, since such sets $\mathcal{Q}$ do exist. The proof is similar to the proof of Theorem 8.

Proof: Apply Theorem 5 and Corollary 11. □

VII. Examples and Simulations

Example 13: Many industrial processes can be described as a first-order linear system with time delay in the input. One example would be the concentration of a substance in a tank with ideal mixing and a transport delay in the inlet tube. Let the plant be given as

$$\dot{y}(t) = ay(t) + bu(t - h). \tag{36}$$

An elementary computation using the Nyquist criterion (or applying Theorem 13.8 in [61]) shows that (36) is stabilizable by static feedback if and only if $b \neq 0$ and $ha < 1$. Assume that (36) is known to satisfy this assumption. More precisely: let Ξ' be the set of all systems of the form (36) for which $ha < 1$ and $b \neq 0$. Let $\{K_i\}$ be an enumeration of $\mathbb{Q}$ and let σ be a switching function. It follows from Theorem 5 that the controller

$$\begin{aligned} u &= K_{\sigma(k)} y \\ \dot{k} &= y^2 + u^2 \end{aligned} \tag{37}$$

will stabilize all plants in Ξ'.

A pseudocode implementation of the involved functions will now be presented: First, we compute the "index" $\iota(k)$ of k with respect to the switching sequence (τ_i), defined as the smallest $i \geq 1$ such that $\iota(k) < \tau_{i-1}$. This is a well-defined function from the reals to the integers, and is implemented as follows:

```
ι(k)
begin
  τ ← τ0
  for i ← 1 to ∞
    if k < τ
      return i
    else
      τ ← τ^2
end
```

(The **return** statement has the same meaning as in the C programming language. The code computes how many times τ_0 has to be squared in order for $k < \tau$.) Secondly, a map $e: \mathbb{N} \to \mathbb{Q}$ is implemented in the following pseudocode:

```
e(ι)
begin
  j ← 1
  for i ← 1 to ∞
    for d ← 1 to i
      for n ← −i to i
        if ι = x
          return n/d
        else
          j ← j + 1
end
```

Note that e has the property that for all $a \in \mathbb{N}$, $e(\{n \geq a\}) = \mathbb{Q}$. Therefore $e \circ \iota: \mathbb{R} \to \mathbb{Q}$ has the properties required of $K_{\sigma(k)}$. As $K_{\sigma(k)}$ we thus take $e(\iota(k))$.

Figs. 2 and 3 show the outcome of the simulation of the closed-loop system (36) and (37) for two different values of the parameter b. The other parameters were: $a = 0.1$,

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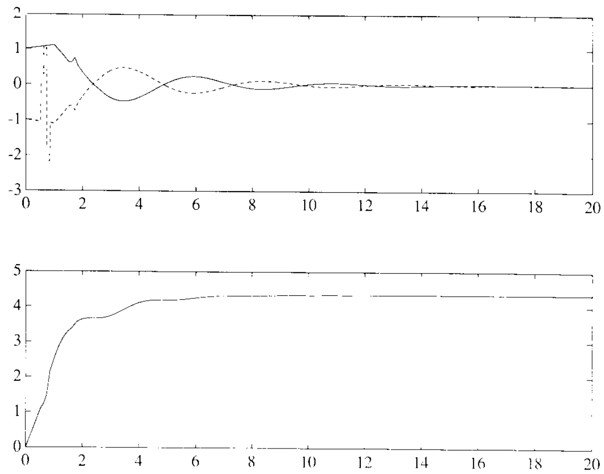

Fig. 2. The systems (36) and (37) with $b = 1$. The upper plot shows y, u, and K, while the lower one shows k.

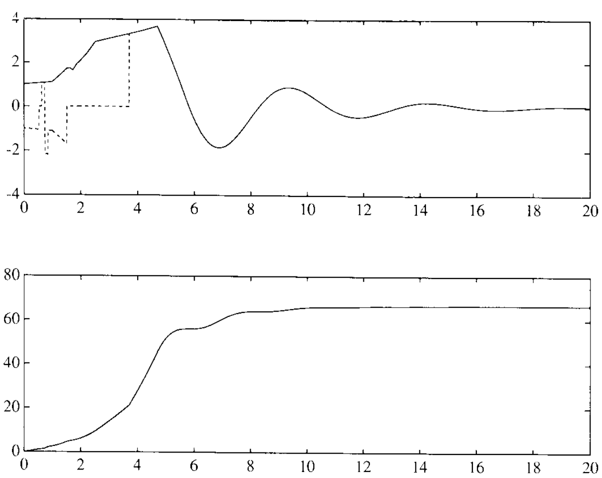

Fig. 3. The systems (36) and (37) with $b = -1$. The upper plot shows y, u, and K, while the lower one shows k.

$h = 1$, $\tau_0 = 1.1$. The initial conditions were given by $x(0) = 1$ and $u(s) = 0$, $s < 0$. It should be pointed out that for some parameter values and some initial conditions, the performance was very bad—a fact that at least to some extent has to be considered as a price for using an algorithm utilizing only an absolute minimum of *a priori* knowledge. ∎

Remark 14:

i) The above code can be speeded up considerably. Since our emphasis is on ideas, not on efficient implementations—and since we do not claim that the algorithm in its present form is very practical anyhow—we have refrained to do so.

ii) Note that it is straightforward to modify the above given pseudocode to generate a mapping onto $\mathbb{Q}^n$ instead of $\mathbb{Q}$. ∎

To the best of our knowledge, there is no algorithm available in the literature which is capable of stabilizing the class of systems considered in Example 13. Note that it cannot be treated within the approaches given in [9], [10], [16], [17], [19], and [20], because in [9] and [10] it is assumed that h is known exactly, while [16], [17], [19], and [20] deal with a high-gain situation requiring the plant to be of "generalized" relative degree one (in [16], [19] and [20]) or two (in [17]), which is clearly not satisfied for systems with input delays. Finally, in [8], local asymptotic stability of a standard adaptive scheme is ensured only if the parameters a and h satisfy conditions which are more restrictive than those in Example 13, e.g., an upper bound on a is known and $3ha < 1$.

Example 15: Consider the plant given by

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t)$$
$$y(t) = Cx(t) \tag{38}$$

where

$$A_0 = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 2 & -3 \\ 1 & 3 & 2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{pmatrix},$$
$$B = \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Except for the delay term $A_1 x(t-h)$, this is the same as [13, example 3.12]. A simple computation shows that

$$\det \begin{pmatrix} sI - A_0 - A_1 e^{-sh} & B \\ C & 0 \end{pmatrix} = 5(s + 1 - a_{11} e^{-sh})$$

By inspection, it is seen that (30) is satisfied if $|a_{11}| < 1$. Moreover, spec $CB = \{1 \pm 2i\} \subset \mathbb{C}_+$. Therefore, by Proposition 9, the system is in $\mathfrak{H}$. (With a similar analysis, we may also allow several delay terms of the form $A_i x(t - h)$. For simplicity, we shall not do so.)

For the case that it is known that $\text{spec}(CB) \subset \mathbb{C}_+$, in Theorem 13 we may replace $\{kQ: k \in \mathbb{N}, Q \in \mathscr{O}\}$ by $\{kI: k \in \mathbb{N}\}$ and the theorem will still hold, applied to the smaller class. Let ι be as in the previous example, and define $e(\iota)$ by the algorithm

```
e(ι)
begin
  j ← 1
  for i ← 1 to ∞
    for n ← 1 to i
      if ι = j
        return n
      else
        j ← j + 1
end
```

For all a, it holds that $e(\{x > a\}) = \mathbb{N}$. Therefore, the controller (16), with $K_{\sigma(k)} = e(\iota(k))$, will stabilize system (38) provided $|a_{11}| < 1$.

In Fig. 4, we show the outcome of a simulation. The initial conditions were $x_i(0) = 1$, $x(s) = 0$, $s < 0$, $i =$

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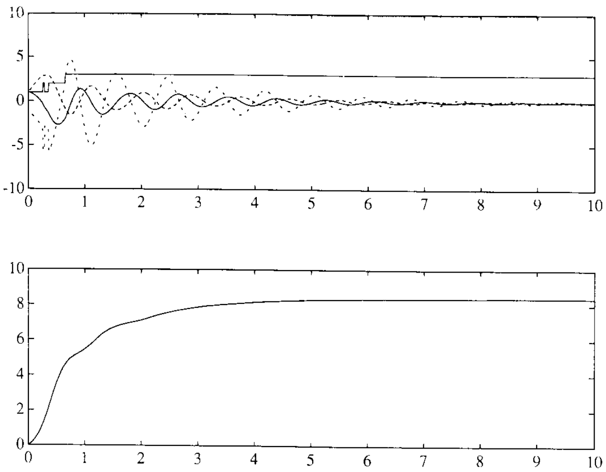

Fig. 4. Simulation of the system in Example 15. The upper plot shows y, u, and K, while the lower one shows k.

1, 2, 3. Parameters were $\tau_0 = 1.1$, $h = 1$, $A_1 = \operatorname{diag}(0, 1, 1)$. ■

VIII. Conclusions

We have shown that the subclass of the Pritchard–Salamon class consisting of the systems which are exponentially stabilizable and detectable has the "order-is-enough" property (namely, that sufficient *a priori* information for adaptive stabilization is the knowledge of *any* stabilizing, fixed coefficient controller). For a class of high-gain stabilizable systems, a generalization of infinite-dimensional scalar relative-degree-one systems, an algorithm was presented which takes advantage of this *a priori* knowledge. It should be remarked that in both cases, the controllers look exactly as in the finite dimensional case. Therefore, the results of the present paper can be interpreted as robustness results: they show that the adaptive switching algorithms introduced in [13] are robust against certain infinite-dimensional perturbations of the plant.

The question almost asks itself: To what classes of (nonlinear) systems can the results of this paper, in particular Theorem 8, be generalized? What reasonable set of (nonlinear) systems have the order-is-enough property? This property should not be taken for granted: In [62] a class of systems was presented, which does not have the order-is-enough property. Explicitly, it was shown that it is not possible to adaptively stabilize all systems of the form $\dot{x} = f(x) + u$, $x(0) = x_0 > 0$ where $x \in \mathbb{R}$ and f is any smooth function with $f(0) = 0$.

For the finite-dimensional case, it should be noted that a weak converse of the result in [1] was proved in [63] namely that the existence of an adaptively stabilizing controller for a finite-dimensional linear time-invariant plant implies the existence of a linear time-invariant controller, with at most the same dynamic order, placing the poles of the closed-loop system in the *closed* left-half plane. Under certain conditions, the result generalizes to the adaptive stabilization of infinite-dimensional systems controlled by finite-dimensional controllers. Details will be presented in a future paper. A completely satisfying characterization of necessary and sufficient conditions for adaptive stabilization of linear, time invariant, finite-dimensional systems is still missing.

In this paper, we have exclusively considered stabilization by finite-dimensional controllers. We would like to remark that due to the progress of the VLSI technology, and, to a lesser extent, computer technology in general, a future exclusive emphasis on finite-dimensional stabilization seems unnatural.

Appendix
Proof of Lemma 7

Using Remark 1, i) and iii) the following definition makes sense.

Definition: Assume that A1) and A2) are satisfied and let $e_1, \dots, e_m$ denote the canonical basis of $\mathbb{R}^m$. We can give a meaning to $CS(\cdot)B$ as an element in $L_2^{loc}(0, \infty; \mathbb{R}^{p \times m})$ by defining $CS(\cdot)B := (CS(\cdot)Be_1, \dots, CS(\cdot)Be_m)$. The proof of the next result can be found in [34].

Lemma 14: Let A1)–A4) be satisfied. Then

$$\int_0^t CS(t-\tau)Bu(\tau)\,d\tau = C\int_0^t S(t-\tau)Bu(\tau)\,d\tau$$

for all $u \in L_2^{loc}(0, \infty; \mathbb{R}^m)$ and $t \geq 0$.

Proof of Lemma 7:

Step 1: It is well known from a semigroup theory that for all $x \in W$

$$\int_0^t S(t-\tau)HCS_H(\tau)x\,d\tau = \int_0^t S_H(t-\tau)HCS(\tau)x\,d\tau. \tag{39}$$

For $x \in V$ pick a sequence $x_n \in W$ such that $x = \lim_{n\to\infty} x_n$. By Remark 1, i), iii), and Lemma 1, ii) the following definitions make sense:

$$CS(\cdot)x := \lim_{n\to\infty} CS(\cdot)x_n$$

and

$$CS_H(\cdot)x := \lim_{n\to\infty} CS_H(\cdot)x_n$$

where the limits have to be understood in $L_2(0, t; \mathbb{R}^m)$. It follows that (39) holds for all $x \in V$.

Step 2: We show that

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)Bu(\tau)\,d\tau \tag{40}$$

is a solution of (22). Plugging in x into the right-hand side of (22) gives

$$\text{RHS} = S_H(t)x_0 + \int_0^t S_H(t-\tau)Bu(\tau)\,d\tau$$
$$-\int_0^t S_H(t-\tau)HC\left(S(\tau)x_0 + \int_0^\tau S(\tau-s)Bu(s)\,ds\right)d\tau.$$

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Using (8) and Remark 2 we obtain

$$\begin{aligned}\mathrm{RHS} = S(t)x_0 &+ \int_0^t S(t-\tau)HCS_H(\tau)x_0\,d\tau \\ &+ \int_0^t S(t-\tau)Bu(\tau)\,d\tau \\ &+ \int_0^t\int_0^{t-\tau} S(t-\tau-s)HCS_H(s)Bu(\tau)\,ds\,d\tau \\ &- \int_0^t S_H(t-\tau)HCS(\tau)x_0\,d\tau \\ &- \int_0^t S_H(t-\tau)HC\left(\int_0^\tau S(\tau-s)Bu(s)\,ds\right)d\tau.\end{aligned}$$

Hence, by (39)

$$\mathrm{RHS} = x(t) + X(t) - Y(t)$$

where

$$X(t) := \int_0^t\int_0^{t-\tau} S(t-\tau-s)HCS_H(s)Bu(\tau)\,ds\,d\tau$$

$$Y(t) := \int_0^t S_H(t-\tau)HC\left(\int_0^\tau S(\tau-s)Bu(s)\,ds\right)d\tau.$$

It remains to show that $X(t) = Y(t)$. Defining all integrands to be zero for negative arguments, using Lemma 14, interchanging the order of integration, and using Step 1 we obtain

$$\begin{aligned}Y(t) &= \int_0^t\int_0^t S_H(t-\tau)HCS(\tau-s)Bu(s)\,d\tau\,ds \\ &= \int_0^t\int_{-s}^{t-s} S_H(t-s-\sigma)HCS(\sigma)Bu(s)\,d\sigma\,ds \\ &= \int_0^t\int_0^{t-s} S_H(t-s-\sigma)HCS(\sigma)Bu(s)\,d\sigma\,ds \\ &= \int_0^t\int_0^{t-s} S(t-s-\sigma)HCS_H(\sigma)Bu(s)\,d\sigma\,ds \\ &= X(t).\end{aligned}$$

Step 3: We show that x given by (40) is the unique solution of (22). Suppose $\tilde{x} \in \mathfrak{C}(0,\infty;W)$ is another solution of (22). Then both Cx and $C\tilde{x}$ solve the equation

$$y(t) = CS_H(t)x_0 + C\int_0^t S_H(t-\tau)Bu(\tau)\,d\tau - \int_0^t CS_H(t-\tau)Hy(\tau)\,d\tau \quad (41)$$

which is obtained by applying C to both sides of (22). The above equation is a linear Volterra integral equation in finite dimensions. In particular, this means that (41) possesses a unique, continuous solution. It follows that $Cx \equiv C\tilde{x}$, which implies, via (22), that $x \equiv \tilde{x}$. □

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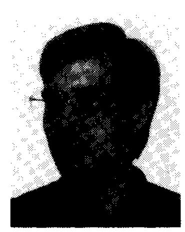

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