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# Destabilizing Effects of Small Time Delays on Feedback-Controlled Descriptor Systems*

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ABSTRACT

In the last 15 years the problem of stabilizability and stabilization of descriptor systems have received considerable attention. In this paper it is shown that if a descriptor system $E\dot{x} = Ax + Bu$ exhibits impulsive behavior, then the stability of the closed-loop system is extremely sensitive to small delays. More precisely, if $F$ is the feedback which leads to a stable and impulsive-free closed-loop system, then there exist numbers $\varepsilon_j > 0$ and $s_j \in \mathbb{C}$ with $\lim_{j\to\infty} \varepsilon_j = 0$ and $\lim_{j\to\infty} \operatorname{Re} s_j = +\infty$ and such that the delayed closed-loop system obtained by applying the feedback $u(t) = Fx(t - \varepsilon_j)$ has a pole at $s_j$. Moreover, if the open-loop system does not have impulsive behavior, the same phenomenon occurs, provided that the spectral radius of the matrix $\lim_{|s|\to\infty} F(sE - A)^{-1}B$ is greater than 1. If this spectral radius is smaller than 1, it is shown that the closed-loop stability is robust with respect to small delays. © 1998 Elsevier Science Inc.

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## 1. INTRODUCTION

Stabilizability and pole-placement problems for linear descriptor systems given by

$$E\dot{x}(t) = Ax(t) + Bu(t)$$

have received a great deal of attention in the last 15 years; see for example Bunse-Gerstner et al. [2, 3], Dai [4], Kautsky et al. [11], Mehrmann [17], Shayman [20], and Shayman and Zhou [21], to mention just a few references. Stabilization (pole assignment) by feedback of the form $u(t) = Fx(t)$ requires not only the closed-loop system to be stable (to have prescribed poles), but also that it be robust, in the sense that closed-loop stability (the configuration of the closed-loop poles) is insensitive to perturbations in the plant and controller data. The problem of robust pole placement for descriptor systems has for example been addressed in [11], where numerical procedures for generating robust feedback systems with prescribed poles are given. The perturbations considered in [11] are of the form

$$E \rightsquigarrow E + \delta E, \qquad A \rightsquigarrow A + \delta A, \qquad B \rightsquigarrow B + \delta B, \qquad F \rightsquigarrow F + \delta F.$$

In this paper we consider perturbations which are induced by "small" time delays in the feedback loop, symbolically

$$u(t) = Fx(t) \rightsquigarrow u(t) = Fx(t - \varepsilon).$$

It will be shown that if the descriptor system to be controlled exhibits impulsive behavior, then the stability of the feedback system is extremely sensitive to such perturbations. More precisely, if $u(t) = Fx(t)$ is a feedback control which leads to an impulsive-free closed-loop system, then there exist numbers $\varepsilon_j > 0$ and $s_j \in \mathbb{C}$ with $\lim_{j\to\infty} \varepsilon_j = 0$ and $\lim_{j\to\infty} \operatorname{Re} s_j = +\infty$ and such that the delayed closed-loop system obtained by applying the feedback $u(t) = Fx(t - \varepsilon_j)$ has a pole at $s_j$. Of course, if $F$ is strongly stabilizing in the sense that it simultaneously stabilizes the system and eliminates its impulsive behavior, then the above result shows that arbitrarily small delays destabilize the feedback system. Moreover, if the system to be controlled does not have impulsive behavior, then the above conclusion remains true, provided that the spectral radius of the matrix $\lim_{|s|\to\infty} F(sE - A)^{-1}B$ is greater than 1. If the spectral radius is smaller than 1, it is shown that closed-loop stability is robust with respect to small delays.

The phenomenon of destabilization of feedback systems by arbitrarily small delays in the loop is not new and has been thoroughly studied in the context of infinite-dimensional systems, but to the best of our knowledge has not been investigated for descriptor systems. It seems that the paper [1] by Barman et al. from 1973 is the first one devoted to this topic. More recently, researchers working in control of partial differential equations, rediscovered the destabilizing effect of small delays in various examples involving vibrating systems; see for example Datko [5, 6], Datko et al. [7], and Desch and Wheeler [8]. Whilst these papers are based on partial-differential-equation and related techniques, a frequency-domain point of view is taken in [1]. The approach developed by Logemann and Rebarber [13, 14] and Logemann et al. [15] is similar in spirit to that in [1], but is not tied to the restrictions imposed in [1] such as the assumptions that the open-loop transfer function has at most finitely many poles in the closed right half plane or that the transfer function is bounded in some right half plane. Using results from [15], Logemann and Townley [16], have shown that small delays in the feedback loop can also have a destabilizing effect on certain neutral functional differential equations.

The paper is organized as follows. In Section 2 we present some preliminaries on descriptor systems which are needed later in the paper. Three simple examples are discussed in Section 3 to motivate the investigations in the following sections. In Section 4 we state two destabilization results from [13] and [15]. One of them applies to systems described by ill-posed (not necessarily rational) transfer function matrices. By "ill-posed" we mean that the transfer-function matrix is unbounded in any right half plane of the complex plane. We then show that the crucial assumption in this result is always satisfied for improper rational matrices. In Section 5 we apply the results of Section 4 to descriptor systems with and without impulsive behavior. The feedback $x(t) = Fu(t)$ is assumed to render the closed-loop system impulsive-free. An important subclass of such controls are the so-called strongly stabilizing feedbacks which stabilize the system and simultaneously eliminate its impulsive behavior. The key idea here is to reformulate the closed-loop system as a system which has been obtained by applying output feedback to a controlled and observed descriptor system with transfer function matrix $-F(sE - A)^{-1}B$. We derive destabilization as well as robustness results.

A result on the zeros of a certain quasipolynomial needed in Example 3.1 is proved in the Appendix.

NOTATION AND TERMINOLOGY. In the following let $\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\}$ (where $\alpha \in \mathbb{R}$), and let $\mathbb{D}_\rho := \{s \in \mathbb{C} \mid |s| < \rho\}$ (where $\rho > 0$). For a set $U \subset \mathbb{C}$, let $U^{cl}$ denote the closure of $U$. The field of all meromorphic functions on $\mathbb{C}_\alpha$ is denoted by $M_\alpha$, while $H_\alpha^\infty$ denotes the algebra of all bounded holomorphic functions defined on $\mathbb{C}_\alpha$. If $T$ is a matrix in $\mathbb{C}^{n\times n}$, then $\sigma(T)$ and $r(T)$ denote the spectrum and the spectral radius of $T$, respectively. Finally, a real-rational $m \times n$ matrix $R(s)$ is called *proper* if the limit

$$\lim_{|s|\to\infty} R(s) =: R(\infty) \in \mathbb{C}^{m\times n}$$

exists. If $R(\infty) = 0$, then $R(s)$ is called *strictly proper*.

## 2. PRELIMINARIES ON DESCRIPTOR SYSTEMS

Consider a controlled descriptor system of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{2.1}$$

where $E, A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. For the rest of the paper we shall assume that (2.1) is *regular*, i.e.,

$$\det(sE - A) \not\equiv 0. \tag{A1}$$

As is well known and as the following example shows, the system $E\dot{x} = Ax$ may have impulsive behavior.

EXAMPLE 2.1. Let us consider the following single-input systems

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\dot{x}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u(t), \qquad t > 0. \tag{2.2}$$

If $u(t) \equiv 0$, then we obtain for $t > 0$

$$\dot{x}_2(t) = x_1(t) \quad \text{and} \quad x_2(t) = 0.$$

Hence $x_1(t) = -x_2(0-)\delta(t)$, and we see that the system (2.2) shows impulsive behavior.

Since (2.1) is regular, there exists nonsingular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that

$$QEP = \begin{pmatrix} L_{n_1} & 0 \\ 0 & N \end{pmatrix}, \quad QAP = \begin{pmatrix} J & 0 \\ 0 & I_{n_2} \end{pmatrix}, \tag{2.3}$$

where $n_1 + n_2 = n$, $N \in \mathbb{R}^{n_2\times n_2}$ and $J \in \mathbb{R}^{n_1 \times n_1}$ are in Jordan form and $N$ is nilpotent; see Dai [4, p.7] or Gantmacher [9, p. 28]. This classical result is due to Weierstrass, and therefore (2.3) is called the *Weierstrass canonical form* of the pencil $sE - A$. The *index* of the pencil, denoted by ind($E$, $A$), is defined to be the nilpotency degree $\nu$ of $N$ (i.e., $N^{\nu} = 0$ and $N^{\nu-1} \neq 0$). If $n_2 = 0$, then we set ind($E$, $A$) $= 0$. Of course, ind($E$, $A$) $= 0$ if and only if the matrix $E$ is nonsingular.

Premultiplying (2.1) by $Q$ and setting $z = P^{-1}x$, we may represent (2.1) as

$$\dot{z}_1(t) = J z_1(t) + B_1 u(t), \tag{2.4a}$$

$$N\dot{z}_2(t) = z_2(t) + B_2 u(t), \tag{2.4b}$$

where

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = z(t), \qquad \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = QB.$$

The solution of (2.4) is given by

$$z_1(t) = e^{Jt} z_1(0) + \int_0^t e^{J(t-\tau)} B_1 u(\tau)\, d\tau, \tag{2.5a}$$

$$z_2(t) = \begin{cases} -B_2 u(t) & \text{if} \quad \nu = 1, \\ -\sum\limits_{i=1}^{\nu-1} \delta^{(i-1)}(t) N^i z_2(0-) - \sum\limits_{i=0}^{\nu-1} N^i B_2 u^{(i)}(t) & \text{if} \quad \nu > 1, \end{cases} \tag{2.5b}$$

with superscript $(i)$ indicating the $i$th distribution derivative. Whilst (2.5a) is the standard variation-of-parameters formula for ordinary differential equations, (2.5b) can be obtained by taking the Laplace transform of (2.4b); see [4,

p. 17]. It follows from (2.5) that the system (2.1) admits impulsive solutions if and only if $\operatorname{ind}(E, A) > 1$. The following simple result gives necessary and sufficient frequency-domain conditions for the existence of impulsive behavior.

PROPOSITION 2.2. *If* (A1) *is satisfied, then the following statements are equivalent*:

(i) $\operatorname{ind}(E, A) > 1$;
(ii) *the rational matrix* $(sE - A)^{-1}$ *is not proper*;
(iii) *the rational matrix* $(sE - A)^{-1}E$ *is not strictly proper.*

*Proof.* (i) ⇒ (ii): Suppose that $\nu = \operatorname{ind}(E, A) > 1$, and consider the Weierstrass canonical form (2.3). Since

$$s(sN - I)^{-1}N - I = (sN - I)^{-1} \tag{2.6}$$

and since $N$ is nilpotent with nilpotency degree $\nu$, it follows that

$$(sN - I)^{-1}N^{\nu-1} = -N^{\nu-1} \neq 0.$$

Therefore we may conclude that the rational matrix $(sN - I)^{-1}N$ is not strictly proper, which, by (2.6), implies that $(sN - I)^{-1}$ is not proper. Thus, using (2.3), the rational matrix $(sE - A)^{-1}$ is not proper.

(ii) ⇒ (iii): Assume that (ii) holds. Let $s_0 \in \mathbb{C}$ be such that

$$\det(s_0 E - A) \neq 0. \tag{2.7}$$

Setting $\mathbf{H}(s) := [(s + s_0)E - A]^{-1}(s_0 E - A)$, it follows from the identity

$$s[(s + s_0)E - A]^{-1}E + [(s + s_0)E - A]^{-1}(s_0 E - A) = I$$

that

$$[(s + s_0)E - A]^{-1}E = \frac{1}{s}[I - \mathbf{H}(s)]. \tag{2.8}$$

Using (2.7) and the hypothesis, we see that $\mathbf{H}(s)$ is not proper. Consequently the right-hand side of (2.8) is not strictly proper and so $(sE - A)^{-1}E$ is not strictly proper.

(iii) ⇒ (i): Suppose that $(sE - A)^{-1}E$ is not strictly proper. Now

$$(sE - A)^{-1}A = s(sE - A)^{-1}E - I,$$

and so $(sE - A)^{-1}A$ is not proper, implying that $(sE - A)^{-1}$ is not proper either. Hence $(sN - I)^{-1}$ is not proper by (2.3). This yields that $N \neq 0$, which in turn gives $\operatorname{ind}(E, A) > 1$. ■

If the system (2.1) is stable and impulsive-free [i.e., $\det(sE - A) \neq 0$, for all $s \in \mathbb{C}_0^{\mathrm{cl}}$, and $\operatorname{ind}(E, A) \leqslant 1$], we say that it is *strongly stable*. The system (2.1) is called *strongly stabilizable* if there exists a feedback matrix $F \in \mathbb{R}^{m \times n}$ such that the closed-loop system

$$E\dot{x}(t) = (A + BF)x(t)$$

obtained from (2.1) by applying the feedback control $u(t) = Fx(t)$ is stable and impulsive-free, i.e., $\det(sE - A - BF) \neq 0$ for all $s \in \mathbb{C}_0^{\mathrm{cl}}$ and $\operatorname{ind}(E, A + BF) \leqslant 1$.

It is easy to see that the system in Example 2.1 is strongly stabilizable and moreover the feedback $F = (f_1, f_2)$ is strongly stabilizing if and only if $f_1(1 + f_2) > 0$. Although we shall not use it, we state the following theorem which gives a necessary and sufficient algebraic condition for strong stabilizability. For the proof of this result, which is reminiscent of the Hautus criterion for state-space systems, see Bunse-Gerstner et al. [2].

THEOREM 2.3. *The system* (2.1) *is strongly stabilizable if and only if*

$$\operatorname{rank}(sE - A, B) = n \quad \textit{for all } s \in \mathbb{C}_0^{\mathrm{cl}} \quad \textit{and} \quad \operatorname{rank}(E, AS, B) = n,$$

*where the columns of the real matrix* $S$ *span* $\ker E \cap \ker B^T$ *and where* $S^TS = I$.

## 3. THREE SIMPLE EXAMPLES

Consider the descriptor system (2.1). An application of the feedback control

$$u(t) = Fx(t - \varepsilon), \qquad \text{where} \quad \varepsilon \geqslant 0,$$

leads to the closed-loop system

$$E\dot{x}(t) = Ax(t) + BFx(t - \varepsilon). \tag{3.1}$$

A solution $x(\cdot)$ of this equation is called a *mode* of (3.1) if

$$x(t) = e^{s_0 t} x_0 \qquad \text{for some} \quad s_0 \in \mathbb{C}, \quad x_0 \in \mathbb{C}^n \setminus \{0\}.$$

The number $s_0$ is called the *exponent* of the mode. We say that a mode is *stable* if Re $s_0 < 0$, and that it is *unstable* if Re $s_0 \geqslant 0$. Setting

$$\mathbf{\Delta}_\varepsilon(s) := sE - A - e^{-\varepsilon s} BF,$$

it is trivial to prove that (3.1) has a mode with exponent $s_0$ if and only if $\det \mathbf{\Delta}_\varepsilon(s_0) = 0$.

EXAMPLE 3.1. Consider the controlled descriptor system (2.2) in Example 2.1. This system is not strongly stable, since it is not impulsive-free. However, the feedback $u(t) = Fx(t)$ with $F = (1, 0)$ is strongly stabilizing. In particular we have

$$\det \mathbf{\Delta}_0(s) = \det(sE - A - BF) = s + 1.$$

A delayed feedback of the form $u(t) = Fx(t - \varepsilon)$ leads to the following closed-loop system:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x(t - \varepsilon). \tag{3.2}$$

For $\det \mathbf{\Delta}_\varepsilon$ we obtain

$$\det \mathbf{\Delta}_\varepsilon(s) = e^{-\varepsilon s} s + 1. \tag{3.3}$$

The perturbation induced by the time delay affects the leading term $s$ in $\det \mathbf{\Delta}_0(s)$ in a drastic way, namely $s \rightsquigarrow se^{-\varepsilon s}$. Even for small $\varepsilon$, this perturbation is "large" for $|s| \gg 0$. Therefore one might expect the stability of the feedback system to be sensitive to small delays. Indeed, it can be shown that for any $\varepsilon > 0$, the quasipolynomial (3.3) has (infinitely many) zeros in $\mathbb{C}_0$ (see the Appendix). This means that for any $\varepsilon > 0$ the delayed closed-loop system (3.2) has unstable modes.

EXAMPLE 3.2. Consider the descriptor system

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t). \tag{3.4}$$

Using Proposition 2.2, it is easily checked that this system is impulsive-free. It is unstable, since it has a pole at 1. It is also readily seen that the feedback $F = (-1, 2)$ is strongly stabilizing. The quasipolynomial corresponding to the delayed closed-loop system obtained by applying the feedback $u(t) = F(t - \varepsilon)$ is given by

$$\det \mathbf{\Delta}_{\varepsilon}(s) = se^{-\varepsilon s} - \tfrac{1}{2}s + \tfrac{1}{2}.$$

This is the characteristic quasipolynomial of the neutral differential delay equation

$$\dot{x}(t - \varepsilon) - \tfrac{1}{2}\dot{x}(t) = -\tfrac{1}{2}x(t),$$

and it follows from a standard result in the stability theory of neutral systems (see Salamon [18, p. 160]) that for any $\varepsilon > 0$, $\det \mathbf{\Delta}_{\varepsilon}(s)$ has (infinitely many) zeros in $\mathbb{C}_0$. Again we see that for any $\varepsilon > 0$ the delayed closed-loop system has unstable modes.

Next we give an example of a feedback-controlled descriptor system for which closed-loop stability is robust with respect to small delays.

EXAMPLE 3.3. Consider the descriptor system

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(t). \tag{3.5}$$

Using Proposition 2.2, we find that this system is impulsive-free. It is unstable, since it has a pole at $\frac{1}{2}$. It is also readily seen that the feedback $F = (1, -1)$ is strongly stabilizing.

The quasipolynomial corresponding to the delayed closed-loop system obtained by applying the feedback $u(t) = F(t - \varepsilon)$ is given by

$$\det \mathbf{\Delta}_{\varepsilon}(s) = 1 - 2s - 2e^{-\varepsilon s} \tag{3.6}$$

$$= (1 - 2s)[1 + \mathbf{H}(s)e^{-\varepsilon s}], \tag{3.7}$$

where

$$\mathbf{H}(s) = -F(sE - A)^{-1}B = \frac{2}{2s - 1}.$$

Now $\lim_{|s| \to \infty} \mathbf{H}(s) = 0 < 1$, and hence it follows that there exists $\tilde{\rho} > 0$ such that

$$1 + \mathbf{H}(s)e^{-\varepsilon s} \neq 0 \qquad \text{for all} \quad s \in \mathbb{C}_0^{\text{cl}} \setminus \mathbb{D}_{\tilde{\rho}}^{\text{cl}}, \quad \varepsilon > 0.$$

Choosing $\rho > \max(\frac{1}{2}, \tilde{\rho})$, we see from (3.7) that

$$\det \mathbf{\Delta}_{\varepsilon}(s) \neq 0 \qquad \text{for all} \quad s \in \mathbb{C}_0^{\text{cl}} \setminus \mathbb{D}_{\rho}^{\text{cl}}, \quad \varepsilon > 0. \tag{3.8}$$

On the other hand, we obtain using (3.6) that there exists $\varepsilon^* > 0$ such that

$$\det \mathbf{\Delta}_{\varepsilon}(s) \neq 0 \qquad \text{for all} \quad s \in \mathbb{C}_0^{\text{cl}} \cap \mathbb{D}_{\rho}^{\text{cl}}, \quad \varepsilon \in (0, \varepsilon^*). \tag{3.9}$$

Combining (3.8) and (3.9) shows that $\det \mathbf{\Delta}_{\varepsilon}(s)$ has no zeros in $\mathbb{C}_0^{\text{cl}}$ for all $\varepsilon \in (0, \varepsilon^*)$. As a consequence closed-loop stability is robust with respect to small delays.

For the robustness argument in Example 3.3 it was essential that

$$\lim_{|s| \to \infty} \left| F(sE - A)^{-1}B \right| < 1. \tag{3.10}$$

Notice that in Example 3.2 we have

$$F(sE - A)^{-1}B = \frac{2s}{s - 1},$$

so that (3.10) is not satisfied.

In general, for a system of the form (2.1) and for any feedback matrix $F \in \mathbb{R}^{m \times n}$, we define

$$\Gamma_F := \lim_{|s| \to \infty} F(sE - A)^{-1}B, \tag{3.11}$$

provided the limit exists, i.e. provided the rational matrix $F(sE - A)^{-1}B$ is proper. By Proposition 2.2, the latter will be the case if $\operatorname{ind}(E, A) \leqslant 1$.

The three examples indicate that sensitivity with respect to small delays is closely related to the high-frequency behavior of the rational matrices $(sE - A)^{-1}$ and $F(sE - A)^{-1}B$. In Section 5 we will show that this is indeed the case. More precisely, we will prove the following results:

(1) If (2.1) has impulsive behavior and if the feedback $F$ removes the impulsive behavior, then there exist numbers $\varepsilon_j > 0$ and $s_j \in \mathbb{C}$ with $\varepsilon_j \to 0$ and $\operatorname{Re} s_j \to \infty$ as $j \to \infty$ and such that for any $j \in \mathbb{N}$, the delayed closed-loop system (3.1) has a mode with exponent $s_j$;

(2) if neither (2.1) nor the closed-loop system (3.1) has impulsive behavior and if $r(\Gamma_F) > 1$, then there exist numbers $\varepsilon_j > 0$ and $s_j \in \mathbb{C}$ with $\varepsilon_j \to 0$ and $\operatorname{Re} s_j \to \infty$ as $j \to \infty$ and such that for any $j \in \mathbb{N}$, the delayed closed-loop system (3.1) has a mode with exponent $s_j$;

(3) if (2.1) does not have impulsive behavior, the feedback $F$ is strongly stabilizing, and $r(\Gamma_F) < 1$, then there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ the delayed feedback system is stable in the sense that every mode is stable.

## 4. DESTABILIZATION BY SMALL DELAYS IN THE FREQUENCY DOMAIN

Let $\Omega \subset \mathbb{C}$. A function $\mathbf{H} : \Omega \to \mathbb{C}^{m \times m}$ is called a ($\mathbb{C}^{m \times m}$-valued) *transfer function* (*matrix*) if there exists $\alpha \in \mathbb{R}$ such that $\mathbb{C}_\alpha \subset \Omega$ and $\mathbf{H}|_{\mathbb{C}_\alpha} \in M_\alpha^{m \times m}$.

Let $\mathbf{H}$ be a transfer function, and consider the feedback system shown in Figure 1, where $u$ is the input function, $y$ is the output function, and the block with transfer function $e^{-\varepsilon s}$ represents a delay of length $\varepsilon \geqslant 0$. Delayed state feedback for descriptor systems is captured by the configuration shown in Figure 1 (see Section 5).

If $\det[I + e^{-\varepsilon s}\mathbf{H}(s)] \not\equiv 0$, then the function $\mathbf{G}_\varepsilon$ defined by

$$\mathbf{G}_\varepsilon(s) := \mathbf{H}(s)\left[I + e^{-\varepsilon s}\mathbf{H}(s)\right]^{-1} \tag{4.1}$$

is a transfer function, the so-called closed-loop transfer function of the feedback system shown in Figure 1.

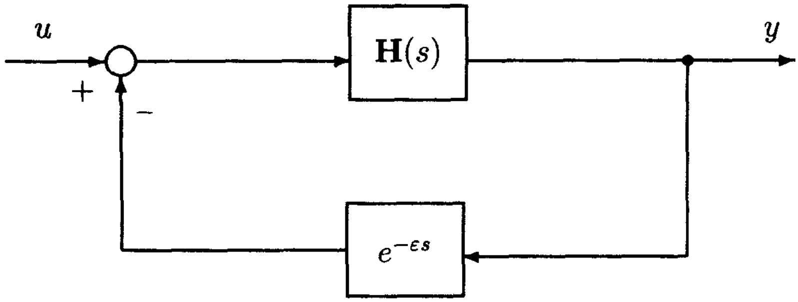


FIG. 4.1. Feedback system with delay.

DEFINITION 4.1. Let $\mathbf{H}$ be a transfer function. $\mathbf{H}$ is called *well posed* if $\mathbf{H} \in (H_\alpha^\infty)^{m \times m}$ for some $\alpha \in \mathbb{R}$. If $\mathbf{H}$ is not well posed, then it is called *ill posed*. $\mathbf{H}$ is called *regular* if it is well posed and the limit $\lim_{\xi \to \infty} \mathbf{H}(\xi) = D$ exists (where $\xi$ is real). The matrix $D$ is called the *feedthrough matrix*.

The terminology introduced in the above definition has its origin in the theory of abstract infinite-dimensional control systems. Roughly speaking, a well-posed transfer function can be realized by a "well-posed" state-space system and vice versa, see Salamon [19] and Weiss [22]. In many cases the transfer function of a descriptor system is an improper rational matrix, and hence ill posed in the above sense.

The following result can be found in [15].

THEOREM 4.2. *Let* $\mathbf{H}$ *be a regular transfer function with feedthrough matrix* $D$. *If* $r(D) > 1$, *then there exist sequences* $(\varepsilon_j)$ *and* $(s_j)$ *with*

$$\varepsilon_j > 0, \qquad \varepsilon_j \to 0, \qquad s_j \in \mathbb{C}_0, \qquad \operatorname{Im} s_j \to \infty, \qquad \operatorname{Re} s_j \to \infty$$

*and such that for any* $j \in \mathbb{N}$, $s_j$ *is a pole of* $\mathbf{G}_{\varepsilon_j}$.

The next result shows that for a large class of ill-posed transfer functions arbitrarily small delays lead to closed-loop poles with arbitrarily large real parts. In order to state the theorem we introduce some more notation. For $\delta \in (0, \pi]$ define the open sector $\mathscr{S}(\delta)$ by

$$\mathscr{S}(\delta) := \{\lambda e^{i\varphi} | \lambda \in (0, \infty),\ \varphi \in (-\delta, \delta)\},$$

and for $\psi \in [-\pi, \pi)$ set

$$\mathscr{S}(\psi, \delta) := e^{i\psi}\mathscr{S}(\delta).$$

THEOREM 4.3. *Let* **H** *be a transfer function, and assume that the following conditions hold*:

(i) *There exist* $\theta \in (0, \pi/2)$ *and* $\alpha > 0$ *such that* $\mathbf{H}(s)$ *is holomorphic in* $\mathscr{S}(\theta) \cap \mathbb{C}_\alpha$.

(ii) *There exist numbers* $\rho > \alpha$, $\mu > 0$, $\psi \in [-\pi, \pi)$, $\delta \in (0, \theta)$, *and* $\eta \in (0, \pi/2)$ *such that*

$$\lim_{|s| \to \infty,\ s \in \mathscr{S}(\delta)} r(\mathbf{H}(s)) = \infty, \tag{4.2}$$

$$r(\mathbf{H}(s)) \leqslant |s|^{\mu} \qquad \textit{for all} \quad s \in \mathscr{S}(\delta) \cap \mathbb{C}_\rho, \tag{4.3}$$

$$r(\mathbf{H}(s)) \subset \mathbb{C} \setminus \mathscr{S}(\psi, \eta) \qquad \textit{for all} \quad s \in \mathscr{S}(\delta) \cap \mathbb{C}_\rho. \tag{4.4}$$

*Then there exist sequences* $(\varepsilon_j)$ *and* $(s_j)$ *with*

$$\varepsilon_j > 0, \qquad \varepsilon_j \to 0, \qquad s_j \in \mathbb{C}_0, \qquad \operatorname{Im} s_j \to \infty, \qquad \operatorname{Re} s_j \to \infty$$

*and such that for any* $j \in \mathbb{N}$, $s_j$ *is a pole of* $\mathbf{G}_{\varepsilon_j}$.

The proof of Theorem 4.3 can be found in Logemann and Rebarber [13]. The condition (4.2) guarantees that **H** is ill posed, while (4.4) says that the spectrum of $\mathbf{H}(s)$ does not spiral around the origin as $s$ moves in a sector of sufficiently small angle.

In order to apply Theorem 4.3 to descriptor systems we need the following technical result.

PROPOSITION 4.4. *If* **H** *is a rational matrix of size* $m \times m$, *then there exist constants* $\delta, \rho > 0$, $\psi \in [-\pi, \pi)$, *and* $\eta \in (0, \pi/2)$ *such that the condition* (4.4) *is satisfied.*

*Proof.* Define $\tilde{\mathbf{H}}(s) := s^{\nu}\mathbf{H}(1/s)$, where we choose $\nu \in \mathbb{N}$ such that $\tilde{\mathbf{H}}(0) := \lim_{s \to 0} \tilde{\mathbf{H}}(s) = 0$. We want to study the behavior of $\sigma(\tilde{\mathbf{H}}(s))$ as $s \to 0$ in $\mathscr{S}(\delta)$ for some small $\delta > 0$. Consider

$$\mathbf{Q}(w, s) := \det\left[wI - \tilde{\mathbf{H}}(s)\right] = w^m + q_1(s)w^{m-1} + \cdots + q_m(s),$$

where the $q_1(s), \dots, q_m(s)$ are real-rational functions. Write $q_i(s) = n_i(s)/d_i(s)$, where $n_i(s)$ and $d_i(s)$ are coprime real polynomials, and let $p_0(s)$ denote the lowest common multiple of $d_1(s), \dots, d_m(s)$. Introducing the polynomials $p_i(s) = p_0(s)q_i(s)$, we define

$$\mathbf{P}(w, s) := p_0(s)\mathbf{Q}(s, w) = p_0(s)w^m + p_1(s)w^{m-1} + \cdots + p_m(s),$$

which is a real polynomial in the variables $s$ and $w$. Since $\tilde{\mathbf{H}}(s)$ is holomorphic at 0, it follows that $p_0(0) \neq 0$. Hence there exists an open neighborhood $U$ of 0 such that $p_0(s) \neq 0$ for all $s \in U$. As a consequence, if $s_0 \in U$, then a complex number $w_0$ is an eigenvalue of $\tilde{\mathbf{H}}(s_0)$ if and only if $w_0$ is a root of $\mathbf{P}(w, s_0) = 0$ and the multiplicities are the same.

*Case* 1. Suppose that $\mathbf{P}(w, s)$ is irreducible. It then follows from algebraic function theory (see e.g. Hille [10, p. 93] or Knopp [12, p. 110]) that there exists a multivalued analytic function $\mathbf{w}(s)$ satisfying $\mathbf{P}(\mathbf{w}(s), s) \equiv 0$. Notice that $\mathbf{w}(0) = \{0\}$, since $\sigma(\tilde{\mathbf{H}}(0)) = \sigma(0) = \{0\}$. Let $w_1(s), w_2(s), \dots, w_m(s)$ denote the branches of $\mathbf{w}(s)$. There exists a neighborhood $V_k \subset U$ of 0 such that for all $s \in V_k$, $w_k(s)$ can be represented by a Puiseux series of the form

$$w_k(s) = \sum_{j=1}^{\infty} \gamma_{kj} s^{j/p_k} \qquad \text{for some} \quad p_k \in \{1, 2, \dots, m\}.$$

For each $k \in \{1, 2, \dots, m\}$ let $l_k$ be the smallest integer such that $\gamma_{kl_k} \neq 0$. Then for any $\delta \in (0, \pi/2)$ we have that

$$\lim_{s \to 0,\, s \in \mathscr{S}(\delta)} \frac{w_k(s)}{\gamma_{kl_k} s^{l_k/p_k}} = 1.$$

Therefore we may write for all $s \in \mathscr{S}(\delta)$ of sufficiently small modulus

$$w_k(s) \approx \gamma_{kl_k} |s|^{l_k/p_k} e^{i(\arg s + 2\pi q_k)l_k/p_k} \qquad \text{for some} \quad q_k \in \{0, 1, \dots, p_k - 1\}.$$

Setting $\varphi_k := \arg \gamma_{kl_k} + 2\pi q_k l_k / p_k$, we see that for any $\varepsilon > 0$

$$w_k(s) \in \mathscr{S}(\varphi_k, l_k \delta / p_k + \varepsilon), \tag{4.5}$$

provided $s \in \mathscr{S}(\delta)$ and $|s|$ is sufficiently small. Consequently, for a sufficiently small $\delta_0 > 0$, there exist numbers $\psi \in [-\pi, \pi)$ and $\eta_0 \in (0, \pi/2)$ such that for all $s \in \mathscr{S}(\delta_0)$ of sufficiently small modulus, $\mathbf{w}(s) \subset \mathbb{C} \setminus \mathscr{S}(\psi, \eta_0)$, and hence

$$\sigma(\tilde{\mathbf{H}}(s)) \subset \mathbb{C} \setminus \mathscr{S}(\psi, \eta_0).$$

Therefore, there exist numbers $\delta \in (0, \delta_0)$ and $\eta \in (0, \eta_0)$ such that for all $s \in \mathscr{S}(\delta)$ of sufficiently small modulus

$$\sigma\left(\frac{1}{s^{\nu}} \tilde{\mathbf{H}}(s)\right) \subset \mathbb{C} \setminus \mathscr{S}(\psi, \eta).$$

Now recall that $\mathbf{H}(1/s) = (1/s^{\nu})\tilde{\mathbf{H}}(s)$, and so it follows that for all $s \in \mathscr{S}(\delta)$ of sufficiently large modulus

$$\sigma(\mathbf{H}(s)) \subset \mathbb{C} \setminus \mathscr{S}(\psi, \eta).$$

Hence, by choosing $\rho > 0$ sufficiently large, we have

$$\sigma(\mathbf{H}(s)) \subset \mathbb{C} \setminus \mathscr{S}(\psi, \eta) \qquad \text{for all} \quad s \in \mathscr{S}(\delta) \cap \mathbb{C}_\rho,$$

which is (4.4).

*Case* 2. If $\mathbf{P}(w, s)$ is not irreducible, we can write

$$\mathbf{P}(w, s) = \prod_{j=1}^{\iota} \mathbf{P}_j(w, s),$$

where the polynomials $\mathbf{P}_1(w, s), \ldots, \mathbf{P}_\iota(w, s)$ are irreducible. Then each of the equations $\mathbf{P}_i(w, s) = 0$ defines an algebraic function $\mathbf{w}_i(s)$ whose branches satisfy a condition similar to (4.5). Consequently, we can invoke the same argument as in case 1 to prove the claim. ■

## 5. DESCRIPTOR SYSTEMS WITH DELAYED FEEDBACK CONTROL

In addition to (A1), we need in this section an extra assumption on the controlled descriptor system (2.1) given by $(E, A, B)$:

$$\text{There exists } F \in \mathbb{R}^{m\times n} \text{ such that } \operatorname{ind}(E, A+BF) \leqslant 1. \tag{A2}$$

Assumption (A2) means that there exists a feedback $F$ which renders the closed-loop system impulsive-free. Clearly, (A2) is satisfied if there exists a strongly stabilizing $F$.

In order to apply Theorem 4.3 to state-feedback-controlled descriptor systems we need the following lemma.

LEMMA 5.1. *Suppose that* (A1) *and* (A2) *are satisfied, and let* $F \in \mathbb{R}^{m\times n}$ *be such that* $\operatorname{ind}(E, A+BF) \leqslant 1$. *If* $\operatorname{ind}(E, A) > 1$, *then the rational matrix* $F(sE-A)^{-1}B$ *is improper.*

*Proof.* Set $\mathbf{X}(s) := (sE - A - BF)^{-1}$. Then $\mathbf{X}(s)$ is a rational matrix, and since $\operatorname{ind}(E, A+BF) \leqslant 1$, it follows from Proposition 2.2 that

$$\mathbf{X}(s) \text{ is proper.} \tag{5.1}$$

Now $(sE-A)\mathbf{X}(s) - BF\mathbf{X}(s) = I$, and so

$$\mathbf{X}(s) - (sE-A)^{-1}BF\mathbf{X}(s) = (sE-A)^{-1}, \tag{5.2}$$

By hypothesis, $\operatorname{ind}(E, A) > 1$, and thus, using Proposition 2.2, we have that $(sE-A)^{-1}$ is improper. Combining (5.1) and (5.2), we see that

$$(sE-A)^{-1}B \text{ is improper.} \tag{5.3}$$

On the other hand, we obtain from $\mathbf{X}(s)(sE-A) - \mathbf{X}(s)BF = I$ that

$$\mathbf{X}(s)B - \mathbf{X}(s)BF(sE-A)^{-1}B = (sE-A)^{-1}B. \tag{5.4}$$

Combining (5.1), (5.3), and (5.4), we see that $F(sE-A)^{-1}B$ is improper. ∎

Consider the descriptor system (2.1). An application of the feedback control

$$u(t) = v(t) + Fx(t - \varepsilon), \qquad \text{where} \quad \varepsilon \geqslant 0,$$

leads to the closed-loop system

$$E\dot{x}(t) = Ax(t) + BFx(t - \varepsilon) + Bv(t). \tag{5.5}$$

Here $v(t)$ denotes the input into the feedback system. In order to apply the input-output results in Section 4, we introduce the controlled and observed descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = -Fx(t). \tag{5.6}$$

By an application of the output-feedback law

$$u(t) = v(t) - y(t - \varepsilon)$$

to the system (5.6) we obtain the same closed-loop system (5.5). Since the transfer function of (5.6) is given by

$$\mathbf{H}(s) = -F(sE - A)^{-1}B, \tag{5.7}$$

it follows that

$$\mathbf{G}_\varepsilon(s) = \mathbf{H}(s)[I + e^{-\varepsilon s}\mathbf{H}(s)]^{-1} = -F(sE - A - e^{-\varepsilon s}BF)^{-1}B$$

is the transfer function of the closed-loop system (5.5) with observation $y(t) = -Fx(t)$.

PROPOSITION 5.2. *Suppose that* (A1) *and* (A2) *are satisfied, and let* $F \in R^{m \times n}$ *be such that* $\operatorname{ind}(E, A + BF) \leqslant 1$. *If* $\operatorname{ind}(E, A) > 1$, *then* $\lim_{|s| \to \infty} r(F(sE - A)^{-1}B) = \infty$.

*Proof.* Defining $\mathbf{H}(s)$ by (5.7), it follows from Lemma 5.1 that

$$\lim_{|s| \to \infty} \|\mathbf{H}(s)\| = \infty, \tag{5.8}$$

where $\|\cdot\|$ denotes the operator norm induced by the Euclidean norm on $\mathbb{C}^m$. By hypothesis, $\operatorname{ind}(E, A + BF) \leqslant 1$, and so $(sE - A - BF)^{-1}$ is proper (by Proposition 2.2). Therefore, we obtain that the rational matrix

$$\mathbf{G}_0(s) = \mathbf{H}(s)[I + \mathbf{H}(s)]^{-1} = -F(sE - A - BF)^{-1}B \text{ is proper.} \quad (5.9)$$

It follows from the identity $\mathbf{H} = \mathbf{G}_0(I - \mathbf{G}_0)^{-1}$ via (5.8) and (5.9) that $\lim_{|s|\to\infty}\det[I - \mathbf{G}_0(s)] = 0$. Setting $D := \lim_{|s|\to\infty}\mathbf{G}_0(s)$, we obtain that $\det(I - D) = 0$, and thus

$$1 \in \sigma(D). \quad (5.10)$$

Let $(s_j)$ be a sequence of complex numbers such that $\lim_{j\to\infty}|s_j| = \infty$. For sufficiently large $j$, $s_j$ is not a pole of either $\mathbf{H}$ or $\mathbf{G}_0$. Thus $-1 \notin \sigma(\mathbf{H}(s_j))$, because otherwise $s_j$ would be a pole of $(I + \mathbf{H})^{-1}$, and hence of $\mathbf{G}_0 = \mathbf{H}(I + \mathbf{H})^{-1} = I - (I + \mathbf{H})^{-1}$. Consequently, if $\xi_j \in \sigma(\mathbf{H}(s_j))$, then $\lambda_j := \xi_j/(1 + \xi_j)$ is a well-defined complex number and $\lambda_j \in \mathbf{G}_0(s_j)$. Moreover, any $\lambda_j \in \sigma(\mathbf{G}_0(s_j))$ can be written as $\lambda_j = \xi_j/(1 + \xi_j)$ where $\xi_j \in \sigma(\mathbf{H}(s_j))$. It therefore follows from (5.10) that there exist numbers $\xi_j \in \sigma(\mathbf{H}(s_j))$ such that

$$\lim_{j\to\infty}\frac{\xi_j}{1 + \xi_j} = 1.$$

But this implies that $\lim_{j\to\infty}|\xi_j| = \infty$, and so $\lim_{j\to\infty} r(\mathbf{H}(s_j)) = \infty$. Since this is true for any sequence with $\lim_{j\to\infty}|s_j| = \infty$, it follows that $\lim_{|s|\to\infty} r(\mathbf{H}(s)) = \infty$. ■

We are now in the position to state and prove the main result of this paper. Let $\Gamma_F$ be defined as in (3.11).

THEOREM 5.3. *Suppose that* (A1) *and* (A2) *are satisfied, and let* $F \in \mathbb{R}^{m\times n}$ *be such that* $\operatorname{ind}(E, A + BF) \leqslant 1$. *If one of the conditions*

(i) $\operatorname{ind}(E, A) > 1$,
(ii) $\operatorname{ind}(E, A) \leqslant 1$ *and* $r(\Gamma_F) > 1$

*is satisfied, then there exist sequences* $(\varepsilon_j)$ *and* $(s_j)$ *with*

$$\varepsilon_j > 0, \qquad \varepsilon_j \to 0, \qquad s_j \in \mathbb{C}_0, \qquad \operatorname{Im} s_j \to \infty, \qquad \operatorname{Re} s_j \to \infty$$

*and such that for any $j \in \mathbb{N}$, the delayed closed-loop system*

$$E\dot{x}(t) = Ax(t) + BFx(t - \varepsilon_j)$$

*has a mode with exponent $s_j$.*

*Proof.* Define $\mathbf{H}(s)$ by (5.7). First suppose that $\operatorname{ind}(E, A) > 1$. Since $\mathbf{H}(s)$ is a rational matrix, there clearly exists a number $\mu > 0$ such that $r(\mathbf{H}(s)) \leqslant |s|^{\mu}$ for $|s|$ sufficiently large. Combining this with Propositions 4.4 and 5.2, it follows that $\mathbf{H}(s)$ satisfies the conditions (i) and (ii) in Theorem 4.3. Hence there there exist sequences $(\varepsilon_j)$ and $(s_j)$ with

$$\varepsilon_j > 0, \qquad \varepsilon_j \to 0, \qquad s_j \in \mathbb{C}_0, \qquad \operatorname{Im} s_j \to \infty, \qquad \operatorname{Re} s_j \to \infty$$

and such that $s_j$ is a pole of

$$\mathbf{G}_{\varepsilon_j}(s) = \mathbf{H}(s)\left[I + e^{-\varepsilon_j s}\mathbf{H}(s)\right]^{-1} = -F(sE - A - e^{-\varepsilon_j s}BF)^{-1}B. \tag{5.11}$$

Hence $s_j$ is a pole of $\mathbf{\Delta}_{\varepsilon_j}^{-1}(s) = (sE - A - e^{-\varepsilon_j s}BF)^{-1}$, and so $\det \mathbf{\Delta}_{\varepsilon_j}(s_j) = 0$. It follows that the closed-loop system $E\dot{x}(t) = Ax(t) + BFx(t - \varepsilon_j) + Bv(t)$ has a mode with exponent $s_j$.

If $\operatorname{ind}(E, A) \leqslant 1$ and $r(\Gamma_F) > 1$, then it follows from Theorem 4.2 that there exist sequences $(\varepsilon_j)$ and $(s_j)$ with the above properties and such that $s_j$ is a pole of $\mathbf{G}_{\varepsilon_j}(s)$. Hence, by (5.11), $\det \mathbf{\Delta}_{\varepsilon_j}(s_j) = 0$, and the claim follows. ∎

If in the above theorem $F$ is strongly stabilizing, then the result shows that closed-loop stability is not robust with respect to small delays.

For the system in Example 2.1 we have $\operatorname{ind}(E, A) > 1$. The feedback $F = (f_1, f_2)$ satisfies $\operatorname{ind}(E, A + BF) \leqslant 1$ if and only if $f_1 \neq 0$. Theorem 5.3 shows that for any such feedback there exist delays $\varepsilon_j > 0$ with $\lim_{j \to \infty} \varepsilon_j = 0$ and complex numbers $s_j$ with $\lim_{j \to \infty} \operatorname{Re} s_j = \infty$ and such that the closed-loop system

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} 1 & 0 \\ f_1 & 1 + f_2 \end{pmatrix} x(t - \varepsilon_j)$$

has a mode with exponent $s_j$. Of course, this applies in particular to the strongly stabilizing feedbacks which are characterized by the condition $f_1(1+f_2) > 0$.

We close this section with a simple robustness result.

PROPOSITION 5.4. *Suppose that* (A1) *is satisfied and that* (2.1) *is strongly stabilizable. Let* $F \in \mathbb{R}^{m\times n}$ *be a strongly stabilizing feedback. If* ind$(E, A) \leqslant 1$ *and if* $r(\Gamma_F) < 1$, *then there exists* $\varepsilon^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^*)$ *the delayed feedback system*

$$E\dot{x}(t) = Ax(t) + BFx(t - \varepsilon_\varepsilon)$$

*is stable in the sense that every mode is stable.*

*Proof.* The characteristic quasipolynomial

$$\det \boldsymbol{\Delta}_\varepsilon(s) = \det(sE - A - BFe^{-\varepsilon s}) \tag{5.12}$$

of the delayed closed-loop system can be written as

$$\det \boldsymbol{\Delta}_\varepsilon(s) = \det(sE - A)\det\left[I - (sE - A)^{-1}BFe^{-\varepsilon s}\right].$$

$$= \det(sE - A)\det\left[I - F(sE - A)^{-1}Be^{-\varepsilon s}\right].$$

Setting $\mathbf{H}(s) = -F(sE - A)^{-1}B$, we obtain that

$$\det \boldsymbol{\Delta}_\varepsilon(s) = e^{-m\varepsilon s}\det(sE - A)\det[e^{\varepsilon s}I + \mathbf{H}(s)]. \tag{5.13}$$

Since $\lim_{|s|\to\infty}\mathbf{H}(s) = -\Gamma_F$ and, by assumption, $r(\Gamma_F) < 1$, it follows that there exists $\rho_1 > 0$ such that

$$\det[e^{\varepsilon s}I + \mathbf{H}(s)] \neq 0 \qquad \text{for all} \quad s \in \mathbb{C}_0^{\mathrm{cl}}\setminus\mathbb{D}_{\rho_1}^{\mathrm{cl}}, \quad \varepsilon > 0.$$

Let $\rho_2 > 0$ be such that $\det(sE - A) \neq 0$ for all $|s| > \rho_2$. Choosing $\rho > \max(\rho_1, \rho_2)$, we see from (5.13) that

$$\det \boldsymbol{\Delta}_\varepsilon(s) \neq 0 \qquad \text{for all} \quad s \in \mathbb{C}_0^{\mathrm{cl}} \setminus \mathbb{D}_\rho^{\mathrm{cl}}, \qquad \varepsilon > 0. \tag{5.14}$$

On the other hand, we obtain from (5.12) and the fact that $F$ is stabilizing that

$$\det \mathbf{\Delta}_\varepsilon(s) \neq 0 \qquad \text{for all} \quad s \in \mathbb{C}_0^{\mathrm{cl}} \cap \mathbb{D}_\rho^{\mathrm{cl}}, \qquad \varepsilon \in (0, \varepsilon^*), \quad (5.15)$$

for some sufficiently small $\varepsilon^* > 0$. Combining (5.14) and (5.15) shows that $\det \mathbf{\Delta}_\varepsilon(s)$ has no zeros in $\mathbb{C}_0^{\mathrm{cl}}$ for all $\varepsilon \in (0, \varepsilon^*)$. As a consequence, every mode of the delayed closed-loop system is stable, provided that $\varepsilon \in (0, \varepsilon^*)$. ■

If $E$ is invertible, then $\mathrm{ind}(E, A) = 0$, any stabilizing feedback $F$ is strongly stabilizing, and $r(\Gamma_F) = 0$. Hence, by the above result, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ and the delayed feedback system is stable. Thus, Proposition 5.4 contains the (well-known) fact that the stability of a state-feedback-controlled state-space system is robust with respect to small delays.

## APPENDIX. ON THE ZEROS OF THE QUASIPOLYNOMIAL (3.3)

Here we show that for any $\varepsilon > 0$ the quasipolynomial $e^{-\varepsilon s}s + 1$ has infinitely many zeros in $\mathbb{C}_0$. To this end, set $y_1^n := (2n+1)\pi/\varepsilon$ and $y_2^n := (2n + \frac{3}{2})\pi/\varepsilon$, $n \in \mathbb{N}$, and define

$$f(y) := \exp\left(\varepsilon y \frac{\cos \varepsilon y}{\sin \varepsilon y}\right), \qquad g(y) := \frac{y}{-\sin \varepsilon y},$$

where $y \in (y_1^n, y_2^n]$. Then $f(y_2^n) = 1$ and $g(y_2^n) = y_2^n$. Hence there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$g(y_2^n) > f(y_2^n) \qquad \text{for all} \quad n \geqslant N. \qquad (\text{A}.1)$$

Setting $\tilde{y}_j^n := y_1^n + 1/j$, we have for sufficiently large $j$ that

$$\tilde{y}_j^n \in (y_1^n, y_2^n)$$

and

$$g(\tilde{y}_j^n) < f(\tilde{y}_j^n). \qquad (\text{A}.2)$$

Since $f$ and $g$ are continuous on $(y_1^n, y_2^n]$ it follows from (A.1) and (A.2) that there exists $y_n \in (y_1^n, y_2^n)$ such that

$$f(y_n) = g(y_n). \tag{A.3}$$

Defining

$$x_n := y_n \frac{\cos \varepsilon y_n}{\sin \varepsilon y_n} > 0, \qquad z_n := x_n + iy_n,$$

it is clear that $z_n \in \mathbb{C}_0$ and $z_i \neq z_j$ if $i \neq j$. We claim that $z_n$ is a zero of $e^{-\varepsilon s}s + 1$. Indeed, by the definition of $f$ and $x_n$ we have that $e^{\varepsilon x_n} = f(y_n)$, and therefore it follows from (A.3) that for all $n \geqslant N$

$$x_n + e^{\varepsilon x_n} \cos \varepsilon y_n = 0, \qquad y_n + e^{\varepsilon x_n} \sin \varepsilon y_n = 0.$$

Thus $z_n + e^{\varepsilon z_n} = 0$, yielding that

$$e^{-\varepsilon z_n} z_n + 1 = 0.$$

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