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Stabilization and Regulation of Infinite-Dimensional Systems Using Coprime Factorizations

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Abstract

This paper surveys some of the results on stabilization and regulation of infinite-dimensional systems which have been obtained within the so-called fractional representation approach to feedback system analysis and synthesis. The relationship with state-space concepts is carefully discussed. The following topics are addressed: Rings of transfer functions and coprime factorizations, Pritchard-Salamon systems, External and internal closed-loop stability, Closed-loop stability and pole-zero cancellations, The Nyquist stability criterion, Closed-loop stability and the existence of coprime factorizations, Parametrization of all stabilizing controllers for a given plant, Existence of finite-dimensional stabilizing compensators, Strong stabilization by finite-dimensional controllers, The internal model principle, PI-control of uncertain infinite-dimensional systems.

1 Introduction

Coprime *polynomial* factorizations of rational matrices have played a major role in feedback system analysis and synthesis for finite-dimensional plants since the work of Rosenbrock, [Rose70]. In case that the transfer function matrix is irrational it is difficult to identify a suitable class of holomorphic matrices containing the set of polynomial matrices and leading to a concept of coprime matrix factorizations which mimics the polynomial factorizations of the finite-dimensional theory. However, as early as 1972, it was recognized that in many situations it is possible to model an unstable infinite-dimensional plant as the coprime "ratio" of two *stable* transfer matrices, see [DeCa72] and [Vidy72]. Spezialized to the lumped case this means that a rational matrix is factorized as a "ratio" of two *stable rational* matrices. This simple idea gave rise to the so-called *fractional representation approach* to feedback system analysis and synthesis, an elegant methodology which leads in a simple natural way to the resolution of many control problems, see the key papers [DLMS80], [SaMu81], [ViSF82], and [FrVi83]. The starting point of this approach is the observation that in a wide variety of applications the set of all stable linear single-input single-output systems forms a ring $\mathcal{S}$; that is, parallel and cascade connections of stable linear systems are again stable linear systems. Moreover, in many cases (e.g. convolution operators or transfer functions) the ring $\mathcal{S}$ is commutative and is an integral domain (i.e. $\mathcal{S}$ has no divisors of zero). The set of all (stable and unstable) single-input single-output systems is denoted by $\mathcal{T}$ and is defined to be the quotient field $\mathcal{Q}(\mathcal{S})$ of $\mathcal{S}$ or the ring of fractions $\mathcal{S}\mathcal{D}^{-1}$ of $\mathcal{S}$ with respect to a multiplicative subset $\mathcal{D}$ of $\mathcal{S}$ with $1 \in \mathcal{D}$ and $0 \notin \mathcal{D}$. Multivariable plants are treated by considering matrices over $\mathcal{T}$. A central idea is that of expressing an unstable plant $G \in \mathcal{T}^{p\times m}$ as a ratio ND^{-1} of two stable transfer matrices N and D in such a way that N and D are coprime.

The advantage of an abstract fractional representation approach to feedback systems is that it embraces within a single framework, continuous-time as well as discrete-time and finite-dimensional as well as infinite-dimensional systems. The main features of this approach are:

- The stability of a feedback system can be characterized by simple algebraic criteria in terms of coprime factors.
- The set of all stabilizing compensators for a given plant can be parametrized via a linear-fractional transformation, provided the plant admits right and left coprime factorizations.
- In case that $\mathcal{S}$ is a normed ring or more generally a topological ring, the set of all unstable plants which admit a right-coprime and a left-coprime factorization can be endowed with a natural topology, the so-called graph topology, which is fundamental for robustness studies.
- The internal model principle for servomechanisms holds under some fairly weak assumptions.

The research in this area of control theory culminated in Vidyasagar's well known book [Vidy85], which deals mainly with finite-dimensional systems, but also contains a chapter on distributed parameter systems indicating which of the finite-dimensional results extend to an infinite-dimensional setting. One of the extra difficulties in the infinite-dimensional case is that a given plant might not admit a right or left coprime factorization, which is an essential requirement in fractional representation theory.

The fractional representation approach to feedback systems is a pure input-output theory; state-space concepts are hardly mentioned in [Vidy85]. For the finite-dimensional case this does not cause any problems because the relation between state-space and input-output notions is well understood for many years. This is certainly not the case in the area of infinite-dimensional systems theory and the wide gap between the frequency-domain approach, as presented e.g. in [CaDe78], [CaDe80b], [DeWa80], and [Zame81], and the semigroup approach, as documented in [CuPr78], has not really been bridged in [Bank83]. However, a synthesis of state-space and frequency-domain methods for distributed parameter systems has been of considerable interest to many researchers in the field during the last five years, and has lead to interesting and useful results on the relationship between exponential stability and input-output stability, see [Curt89] for an overview. In particular, we mention the paper [JaNe88] by Jacobson and Nett who recognized the need to link the so-called Callier-Desoer ring of transfer functions, [CaDe78], to a semigroup based state-space description. A crucial assumption in their paper is that the input and output operators are bounded, which is too restrictive for many applications. A more appropriate class of systems, which contains the class considered in [JaNe88], is the so-called Pritchard-Salamon class introduced in [PrSa87]. These are systems which evolve in an infinite-dimensional Hilbert space and which allow for a certain unboundedness in the control and observation operators. Whilst the Pritchard-Salamon class does include many examples of partial differential systems with boundary control and observation and of neutral systems with delayed control and sensing action, it is by no means the largest class of infinite-dimensional systems which has been treated in the literature. However, it has just the right properties for feedback system analysis and synthesis in both time and frequency domain.

It is the purpose of this paper to survey a number of results on stabilization and regulation of infinite-dimensional systems by output feedback which have been obtained within the fractional representation approach and to relate them to the Pritchard-Salamon class of state-space systems. The paper is organized as follows:

Section 2 collects a number of facts and results on various rings of irrational transfer functions which have been used in the literature, amongst them the Callier-Desoer ring. In particular, the important concepts of right and left coprime factorizations for irrational transfer function matrices are introduced. Moreover, we define the Pritchard-Salamon class of state-space systems, collect some of their properties, and relate it to the frequency-domain set-up presented in the first subsection of Section 2. In the third subsection we mention a few examples of different types of systems which occcur frequently in the applications, some of which fit into the frequency-domain and/or the state-space frameworks presented in the first two subsections of Section 2 and some of which do not. Section 3 is devoted to the stability of feedback systems. We introduce the concept of external closed-loop stability and show that it is equivalent to internal closed-loop stability under suitable stabilizability and detectability assumptions. Several characterrizations are given for external closed-loop stability, one of them in terms of a particular transfer function matrix and unstable pole-zero cancellations. Moreover, it is shown how the Nyquist stability criterion fits into the set-up of fractional representation theory and it is indicated that in many cases closed-loop stability implies the existence of coprime factorizations. The last subsection of Section 3 is devoted to a discussion of the so-called Youla-Bongiorno-Jabr parametrization of all stabilizing controllers of a given plant. Practical feedback control of infinite-dimensional systems must be accomplished with a finite (small) number of actuators and sensors and a control algorithm which can be implemented by an one-line digital computer. Therefore the controller should be finite-dimensional. Section 4 deals with the important problem of finite-dimensional stabilization of infinite-dimensional plants. In particular, it is shown that for a large class of transfer functions the existence of a strictly proper rational stabilizing compensator is equivalent to the fact that the entries of the transfer matrix of the plant belong to the Callier-Desoer ring. The servoproblem in infinite dimensions is the topic of Section 5, where the internal model principle and some of its applications to high and low gain PI-control of uncertain infinite-dimensional systems are dicussed. Finally, some conclusions are drawn in Section 6.

We mention that this paper does not address the topics of robustness analysis of closed-loop stability, robust controller synthesis[1], and H^∞-control, since these will be treated in the contributions of R. F. Curtain, M. C. Smith, and A. Tannenbaum.

Notation

- The superscript $\hat{}$ stands for Laplace transfomation.
- $\imath$:= imaginary unit.
- $\mathcal{L}(X,Y)$:= bounded linear operators from X to Y, where X and Y are normed spaces.
- $\mathbb{C}_\alpha := \{s \in \mathbb{C} : \mathrm{Re}(s) > \alpha\}$, where $\alpha \in \mathbb{R}$.
- $\mathbb{K}(s)$:= rational functions over $\mathbb{K} = \mathbb{R}, \mathbb{C}$.
- $\mathbb{K}_p(s)$:= proper rational functions over $\mathbb{K} = \mathbb{R}, \mathbb{C}$.
- $H^\infty(\mathbb{C}_\alpha)$:= bounded holomorphic functions on $\mathbb{C}_\alpha$.
- $H^\infty := H^\infty(\mathbb{C}_0)$. Endowed with the norm $\|f\|_\infty := \sup_{s\in\mathbb{C}_0} |f(s)|$ the space H^∞ becomes a Banach algebra.
- $H^\infty_- := \bigcup_{\alpha<0} H^\infty(\mathbb{C}_\alpha)$.
- $\mathcal{A} := \{f = f_a(\cdot) + \sum_{i=0}^\infty f_i\delta_{t_i} : f_a \in L^1(0,\infty;\mathbb{C}),\ (f_i)_{i\in\mathbb{N}} \in \ell^1\}$, where $t_0 = 0$, $t_i > 0$ for $i = 1,2,3,\ldots$, and δ_{t_i} denotes the Dirac distribution with support in $\{t_i\}$. $\mathcal{A}$ is a convolution algebra, and endowed with the norm $\|f\|_{\mathcal{A}} := \int_0^\infty |f_a(\tau)|\,d\tau + \sum_{i=0}^\infty |f_i|$ it becomes a Banach algebra.
- $\mathcal{A}_- := \{f \in \mathcal{A} : f(\cdot)\exp(\cdot\,\varepsilon) \in \mathcal{A}$ for some $\varepsilon = \varepsilon(f) > 0\}$.

[1]An exeption is the subsection on robust PI-controller design in Section 5.

- $\hat{\mathcal{A}} := \{\hat{f} : f \in \mathcal{A}\}$.
- $\hat{\mathcal{A}}_- := \{\hat{f} : f \in \mathcal{A}_-\}$.
- $C(\mathbb{C}_0^{cl}) :=$ complex-valued continuous functions on $\mathbb{C}_0^{cl}$.
- $BV(a,b;\mathbb{R}^{n\times n}) :=$ functions of bounded variation on $[a,b]$ with values in $\mathbb{R}^{n\times n}$.

2 Rings of transfer functions, coprime factorizations, and Pritchard-Salamon systems

In the following we introduce various rings of transfer functions and define the important concept of right and left coprime factorizations for irrational transfer function matrices. Moreover, we link the frequency-domain set-up to a class of state-space systems, the so-called Pritchard-Salamon class, which will be used throughout this paper.

Rings of transfer functions and coprime factorizations

It is convenient to use the abstract algebraic notion of a ring of fractions, see e.g. [Lang65] or [Vidy85]. Let $\mathcal{S}$ be an integral domain. The ring $\mathcal{S}$ should be interpreted as the ring of all "stable" transfer functions. Let $\mathcal{D} \subseteq \mathcal{S}$ be a multiplicative subset with $1 \in \mathcal{D}$ and $0 \notin \mathcal{D}$. Here *multiplicative* means that if $a, b \in \mathcal{D}$ then $ab \in \mathcal{D}$. Sometimes we shall make the extra assumption that $\mathcal{D}$ is *saturated*, i.e. if $a, b \in \mathcal{S}$ and $ab \in \mathcal{D}$ then it follows that a and b are in $\mathcal{D}$. The elements of $\mathcal{D}$ are the denominators of the "unstable" transfer functions. The ring of fractions $\mathcal{T} := \mathcal{S}\mathcal{D}^{-1}$ of $\mathcal{S}$ with respect to $\mathcal{D}$ is the set of all transfer functions of interest. The ring $\mathcal{T}$ is the smallest ring which contains $\mathcal{S}$ as a subring, and in which every element of $\mathcal{D}$ is invertible. If $\mathcal{D} = \mathcal{S} \setminus \{0\}$ then $\mathcal{T} = \mathcal{S}(\mathcal{S} \setminus \{0\})^{-1} =: \mathcal{Q}(\mathcal{S})$ is a field, the quotient field of $\mathcal{S}$.

Example 1 We give some examples which illustrate the above abstract concepts.

(i) *Rational functions:* If $\mathcal{S}_1 := \mathbb{C}(s) \cap H^\infty$ and $\mathcal{D}_1 := \mathcal{S}_1 \setminus \{0\}$, then $\mathcal{T}_1 := \mathcal{Q}(\mathcal{S}_1) = \mathbb{C}(s)$.

(ii) *Proper rational functions:* Set $\mathcal{S}_2 := \mathbb{C}(s) \cap H^\infty$ and $\mathcal{D}_2 := \{f \in \mathcal{S}_2 : f(\infty) \neq 0\}$. Then $\mathcal{T}_2 := \mathcal{S}_2\mathcal{D}_2^{-1} = \mathbb{C}_p(s)$.

(iii) *Callier-Desoer ring* (see [CaDe78], [CaDe80a], [CaDe80b]): If $\mathcal{S}_3 := \hat{\mathcal{A}}_-$ and $\mathcal{D}_3 := \hat{\mathcal{A}}_-^\infty = \{f \in \mathcal{S}_3 : f \text{ is bounded away from } 0 \text{ at } \infty \text{ in } \mathbb{C}_0\}$[2], then

$$\mathcal{T}_3 := \mathcal{S}_3\mathcal{D}_3^{-1} = \hat{\mathcal{A}}_-(\hat{\mathcal{A}}_-^\infty)^{-1} = \hat{\mathcal{A}}_-\mathcal{D}_2^{-1} = \hat{\mathcal{A}}_- + \mathcal{R}_{spu},$$

where $\mathcal{R}_{spu}$ denotes the ring of all strictly proper totally unstable rational functions, i.e. $\mathcal{R}_{spu} := \{f \in \mathbb{C}(s) : f(\infty) = 0 \text{ and } f(s) \neq \infty \text{ for all } s \in \mathbb{C} \setminus \mathbb{C}_0^{cl}\}$. The ring $\mathcal{T}_3$ will also be denoted by $\hat{\mathcal{B}}$, which is the usual notation in the literature.

[2]Recall that if f is a bounded holomorphic function on $\mathbb{C}_\alpha$ for some $\alpha < 0$, then f is uniformly continuous on any vertical strip $a \leq \mathrm{Re}(s) \leq b$, where $\alpha < a < b$ (see [Cord68], p.72). Hence if f is bounded away from 0 at ∞ in $\mathbb{C}_0$ then f is also bounded away from 0 at ∞ in $\mathbb{C}_\beta$ for some $\beta \in (\alpha, 0)$.

(iv) *A (slight) generalization of the Callier-Desoer set-up* (see [Loge86a], [LoOw87]): For $\mathcal{S}_4 := H^\infty_-$ and $\mathcal{D}_4 := \{f \in \mathcal{S}_4 : f$ is bounded away from 0 at ∞ in $\mathbb{C}_0\}$ we obtain

$$\mathcal{T}_4 := \mathcal{S}_4\mathcal{D}_4^{-1} = H^\infty_-\mathcal{D}_4^{-1} = H^\infty_-\mathcal{D}_2^{-1} = H^\infty_- + \mathcal{R}_{spu}$$

(v) *Quotient field of* $\hat{\mathcal{A}}$: Set $\mathcal{S}_5 := \hat{\mathcal{A}}$ and $\mathcal{D}_5 := \mathcal{S}_5 \setminus \{0\}$, then $\mathcal{T}_5 := \mathcal{S}_5\mathcal{D}_5^{-1} = \mathfrak{Q}(\hat{\mathcal{A}})$.

(vi) *Transfer functions of bounded type:* The elements of the quotient field of H^∞ are called functions of bounded type. If we set $\mathcal{S}_6 := H^\infty$ and $\mathcal{D}_6 := \mathcal{S}_6 \setminus \{0\}$, then the transfer functions of bounded type are given by $\mathcal{T}_6 := \mathcal{S}_6\mathcal{D}_6^{-1} = \mathfrak{Q}(H^\infty)$.

Note that the subsets $\mathcal{D}_i$ are saturated ($i = 1, ..., 6$) and that $\mathcal{T}_1 \subset \mathcal{T}_5 \subset \mathcal{T}_6$ and $\mathcal{T}_2 \subset \mathcal{T}_3 \subset \mathcal{T}_4 \subset \mathcal{T}_6$. The rings $\mathcal{T}_1$ and $\mathcal{T}_2$ contain only finite-dimensional systems. While $\mathcal{T}_3$ and $\mathcal{T}_4$ cover infinite-dimensional systems with finite-dimensional unstable part, the rings $\mathcal{T}_5$ and $\mathcal{T}_6$ contain also plants which have infinitely many unstable poles.

Many more rings of irrational transfer functions have been introduced in the literature, e.g. the ring of transfer functions of exponential order which is a subring of $\mathcal{T}_3 = \hat{\mathcal{B}}$ (see [CaWi86]) and the ring of pseudo-rational transfer functions (see [Yama88], [YaHa88], [Yama91], and [YaHa92]). For sake of simplicity we shall concentrate in this paper on the rings $\mathcal{T}_i$, $i = 3, 4, 5, 6$. It is not possible to say which one of these rings is the most suitable for control theory. This depends on the particular problem under consideration. Thus some comments in this direction are in order:

- If the plant under consideration has infinitely many unstable poles (this is for example the case for systems described by the wave equation, see Example 6 (v) and (vi) below), then $\mathcal{T}_5$ and $\mathcal{T}_6$ are the only possible candidates for a treatment of the system in the frequency-domain.

- If the problem is to show L^p-stability of a feedback system, then $\mathcal{T}_3$ and $\mathcal{T}_5$ are good candidates, since the input-output operator of a system with transfer function in $\hat{\mathcal{A}}$ is L^p-stable for $1 \leq p \leq \infty$. Of course, one would prefer to work with $\mathcal{T}_3$ unless the the number of unstable poles of the plant is infinite. If $p = 2$ then the rings $\mathcal{T}_4$ and $\mathcal{T}_6$ are appropriate as well, since L^2-stability is equivalent with the transfer function belonging to H^∞. Sometimes it is easier to verify that a transfer function belongs to $\mathcal{T}_4$ or $\mathcal{T}_6$, rather than to show that it is in $\mathcal{T}_3$ or $\mathcal{T}_5$.

- Under suitable stabilizability and detectability assumptions, most infinite-dimensional state-space systems will be exponentially stable if and only if the transfer function is in H^∞ (see for example Theorem 4). So, in order to establish internal stability via a frequency-domain analysis, the rings $\mathcal{T}_4$ and $\mathcal{T}_6$ are good choices.

- For regulation problems it is advantageous to use the rings $\mathcal{T}_3$ and $\mathcal{T}_5$. As in the finite-dimensional case, the stability requirements of the servoproblem imply asymptotic tracking and asymptotic disturbance rejection if the transfer function matrices of the plant and the compensator have all their entries in $\mathcal{T}_3$ or $\mathcal{T}_5$, see Section 5. This is not true for the rings $\mathcal{T}_4$ and $\mathcal{T}_6$.

It is also possible to introduce "real" versions $\mathcal{T}_{ir}$ of the rings $\mathcal{T}_i$ ($i = 1, ..., 6$) consisting of all f in $\mathcal{T}_i$ which have real "coefficients". More precisely, by $\mathcal{T}_{ir}$ we denote the subring of $\mathcal{T}_i$ which consists of all $f \in \mathcal{T}_i$ with the property that $\bar{f}(s) = f(\bar{s})$ for all $s \in \mathbb{C}_0$ (this means that coefficients of the Laurent expansion of f at a real point are real). For example: $\mathcal{T}_{1r} = \mathbb{R}(s)$, $\mathcal{T}_{2r} = \mathbb{R}_p(s)$, and $\mathcal{T}_{6r} = \mathcal{Q}(H^{\infty,r})$, where $H^{\infty,r} := \{f \in H^\infty : \bar{f}(s) = f(\bar{s})$ for all $s \in \mathbb{C}_0\}$. Real world systems usually have real coefficients. However, after partial fraction expansion or coordinate transformations, complex coefficients may creep in. Hence we consider $\mathcal{T}_i$ rather than $\mathcal{T}_{ir}$. All results in this paper remain true if $\mathcal{T}_i$ is replaced by $\mathcal{T}_{ir}$[3].

The above set-up models unstable systems as fractions of stable systems. In order to avoid cancellations the concept of right and left coprime factorizations of transfer function matrices is useful. It is convenient to introduce this notion within the abstract algebraic setting, which was introduced at the beginning of this section.

Definition 2 Suppose $G \in \mathcal{T}^{p\times m}$. A pair $(N, D) \in \mathcal{S}^{p\times m} \times \mathcal{S}^{m\times m}$ is called a right-coprime factorization (r.c.f.) of G (over $\mathcal{S}$ with respect to $\mathcal{D}$) if $\det D \in \mathcal{D}$, $G = ND^{-1}$, and N and D are right-coprime, i.e. there exist matrices $X \in \mathcal{S}^{m\times p}$ and $Y \in \mathcal{S}^{m\times m}$ such that $XN + YD = I_m$. A pair $(\tilde{D}, \tilde{N}) \in \mathcal{S}^{p\times p} \times \mathcal{S}^{p\times m}$ is called a left-coprime factorization (l.c.f) of G (over $\mathcal{S}$ with respect to $\mathcal{D}$) if $\det \tilde{D} \in \mathcal{D}$, $G = \tilde{D}^{-1}\tilde{N}$, and $\tilde{N}$ and $\tilde{D}$ are left-coprime, i.e. there exist matrices $\tilde{X} \in \mathcal{S}^{m\times p}$ and $\tilde{Y} \in \mathcal{S}^{p\times p}$ such that $\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I_p$.

A r.c.f. of G is unique up to multiplication from the right by a a unimodular factor, i.e. if (N_1, D_1) and (N_2, D_2) are right-coprime factorizations of G, then there exists a matrix U such that U is invertible in $\mathcal{S}^{m\times m}$ and $N_1 = N_2U$ and $D_1 = D_2U$. Moreover, if $\mathcal{D}$ is saturated and $G \in \mathcal{T}^{p\times m}$ admits a r.c.f. over $\mathcal{S}$ with respect to $\mathcal{D}$ then any r.c.f. of G over $\mathcal{S}$ with respect to $\mathcal{S} \setminus \{0\}$ is a r.c.f. over $\mathcal{S}$ with respect to $\mathcal{D}$. Similar statements hold for left-coprime factorizations.

It is well-known that any transfer function matrix with entries in $\mathcal{T}_i$, $i = 1, ..., 4$, (see Example 1) admits right-coprime and left-coprime factorizations, see e.g. [CaDe80b], [Vidy85], and [Loge86a]. This is not true (even in the single-input single-output case) for the rings $\mathcal{T}_5$ and $\mathcal{T}_6$. We remark that the positive result for $i = 3, 4$ follows easily from the result for $i = 2$ via the additive decomposition of a function in $\mathcal{T}_i$ ($i = 3, 4$) into a "stable" infinite-dimensional and an "unstable" finite-dimensional part, see (iii) and (iv) in Example 1. The negative result for $i = 5, 6$ follows from combining the fact that $\mathcal{S}_5 = \hat{\mathcal{A}}$ and $\mathcal{S}_6 = H^\infty$ are not Bezout rings[4] (cf. [Rent77], [ViSF82], and [Loge87a]) with the result that every matrix in $\mathcal{Q}(\mathcal{S})^{p\times m}$ has a r.c.f. if and only if $\mathcal{S}$ is a Bezout ring (see [Vidy85], corollary 8.1.8).

Pritchard-Salamon systems

Pritchard-Salamon systems are abstract infinite-dimensional control systems which evolve in an infinite-dimensional Hilbert space and which allow for a certain unboundedness in

[3] This is an important remark, since in sections 3 to 5 we are of course interested in "real" compensators if the plant is "real".

[4] An integral domain is called a *Bezout ring* if every finitely generated ideal is principal.

the control and observation operators. They were introduced in [Sala84], [PrSa85], and [PrSa87] and were further investigated in several publications, see [Curt89] and [CLTZ92] and the references therein (cf. also [Weis89a] and [Weis89b] for related work). Whilst the Pritchard-Salamon class does include many examples of partial differential systems with boundary control and observation and of neutral systems with delayed control and sensing action, it is by no means the largest class of infinite-dimensional systems which has been treated in the literature. However, it has just the right properties for control synthesis in both time and frequency domain.

Let W and V be complex Hilbert spaces satisfying $W \hookrightarrow V$, i.e. $W \subseteq V$ and the canonical injection $W \to V$, $x \mapsto x$ is bounded and has dense range. Let $S(t)$ be a C_0-semigroup on W and V (i.e. $S(t)$ is a C_0-semigroup on V which restricts to a C_0-semigroup on W). The infinitesimal generators of $S(t)$ on W and V will be denoted by A^W and A^V, respectively. Moreover, let ω_W and ω_V be the exponential growth constants of $S(t)$ on W and V. In general $\omega_W \neq \omega_V$, even if $\mathrm{dom}(A^V) \subseteq W$ [5], see [CLTZ92] for counterexamples. It is well known that the growth constants do coincide if $W = \mathrm{dom}(A^V)$ and $\langle x, x\rangle_W = \langle x, x\rangle_V + \langle A^V x, A^V x\rangle_V$.

We shall now introduce the concepts of admissible input and output operators for $S(t)$, which are fundamental for the following development. For this paper it is sufficient to concentrate on finite-dimensional input and output spaces, although most of the results of this section will extend to the case of infinite-dimensional input and output spaces, see [Sala84], [PrSa87], [Weis89a], [Weis89b], [Weis90a], and [CLTZ92].

Definition 3 (i) An operator $B \in \mathcal{L}(\mathbb{C}^m, V)$ is called admissible input operator for S(t) if there exist $t_1 > 0$ and $\alpha > 0$ such that for all $u \in L^2(0, t_1; \mathbb{C}^p)$ it holds that

$$\int_0^{t_1} S(t_1 - \tau)Bu(\tau)\, d\tau \in W \quad \text{and} \quad \| \int_0^{t_1} S(t_1 - \tau)Bu(\tau)\, d\tau \|_W \leq \alpha \|u\|_{L^2(0,t_1)} \tag{1}$$

(ii) An operator $C \in \mathcal{L}(W, \mathbb{C}^p)$ is called admissible output operator for $S(t)$ if there exist $t_2 > 0$ and $\beta > 0$ such that

$$\|CS(\cdot)x\|_{L^2(0,t_2)} \leq \beta \|x\|_V \text{ for all } x \in W. \tag{2}$$

Remark 4 (i) If (1) holds for one particular t_1, then it can be shown that it holds for all $t_1 > 0$, where α will depend on t_1. Moreover, if $S(t)$ is exponentially stable on W, then we can choose α independent of t_1 and (1) holds for $0 \leq t_1 \leq \infty$.
(ii) Statement (i) remains valid if we replace (1) by (2), t_1 by t_2, α by β and exponential stability on W by exponential stability on V.
(iii) If $B \in \mathcal{L}(\mathbb{C}^m, V)$ is an admissible input operator for $S(t)$ then the controllability operator at time $t > 0$

$$\mathfrak{C}_t : L^2(0, t; \mathbb{C}^m) \to V,\ u \mapsto \int_0^t S(t - \tau)Bu(\tau)\, d\tau$$

[5] Suppose that $\mathrm{dom}(A^V)$ is endowed with the graph norm of the operator A^V. Then an application of the closed graph theorem shows that $\mathrm{dom}(A^V) \hookrightarrow W$ if $\mathrm{dom}(A^V) \subseteq W$.

has the properties that $\operatorname{ran}(\mathfrak{C}_t) \subseteq W$ and $\mathfrak{C}_t \in \mathcal{L}(L^2(0,t;\mathbb{C}^m), W)$.
(iv) Suppose that $C \in \mathcal{L}(W, \mathbb{C}^p)$ is an admissible output operator for $S(t)$. Then the bounded linear operator $\mathfrak{O}_t^W : W \to L^2(0,t;\mathbb{C}^p)$, $x \mapsto CS(\cdot)x$, the observability operator on W at time t, can be extended uniquely to a bounded linear operator $\mathfrak{O}_t^V : V \to L^2(0,t;\mathbb{C}^p)$, the observability operator on V at time t. Moreover, we define the operator $\mathfrak{O}_\infty^V : V \to L^2_{loc}(0,\infty;\mathbb{C}^p)$ by $(P_t\mathfrak{O}_\infty^V x)(\tau) = (\mathfrak{O}_t^V x)(\tau)$ for all $\tau \in [0,t]$, where P_t is the usual truncation operator at time t.
(v) If $B \in \mathcal{L}(\mathbb{C}^m, V)$ is an admissible input operator for $S(t)$ and $\operatorname{Re}(s) > \max(\omega_W, \omega_V)$ then $(sI - A^V)^{-1}B \in \mathcal{L}(\mathbb{C}^m, W)$, see [Weis90a] and [Curt88].
(vi) If $C \in \mathcal{L}(W, \mathbb{C}^p)$ is an admissible output operator for $S(t)$ and $\operatorname{Re}(s) > \max(\omega_W, \omega_V)$ then ther exists a constant $M = M(s) > 0$ such that $\|C(sI - A^W)^{-1}x\|_{\mathbb{C}^p} \leq M\|x\|_V$ for all $x \in W$, see [CLTZ92]. Hence the operator $C(sI - A^W)^{-1} \in \mathcal{L}(W, \mathbb{C}^p)$ can be uniquely extended to an operator $\mathfrak{O}(s) \in \mathcal{L}(V, \mathbb{C}^p)$.

The control system

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)Bu(\tau)\,d\tau\,, \text{ where } x_0 \in V,\, t \geq 0 \tag{3a}$$

$$y(t) = Cx(t) + Du(t) \tag{3b}$$

is called a *Pritchard-Salamon system* if $B \in \mathcal{L}(\mathbb{C}^m, V)$ is an admissible input operator for $S(t)$, $C \in \mathcal{L}(W, \mathbb{C}^p)$ is an admissible output operator for $S(t)$, and $D \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^p)$. Notice that for every $x_0 \in W$ the output $y(t)$ given by (3b) is a continuous function on $[0,\infty)$ with values in $\mathbb{C}^p$. If $x_0 \in V$ we can make sense of $y(\cdot)$ as a function in $L^2_{loc}(0,\infty;\mathbb{C}^p)$ by applying Remark 4(iv).

Assumption (PS): For the rest of the paper we shall assume that the system given by (3) is a Pritchard-Salamon system.

Let $e_1, ..., e_m$ be the canonical basis of $\mathbb{C}^m$. We define the impulse response $R(\cdot)$ of (3) by setting $R(\cdot)e_i = \mathfrak{O}_\infty^V Be_i + \delta_0 De_i$, $i = 1, ..., m$. It follows from Remark 4(iv) that $R(\cdot) \in (L^2_{loc}(0,\infty;\mathbb{C}^{p\times m}) + \delta_0\mathbb{C}^{p\times m})$. In order to formulate the next result, it is useful to define

$$\Omega := \{u \in L^2_{loc}(0,\infty;\mathbb{C}^m) : u(\cdot)\exp(-\gamma\,\cdot) \in L^2(0,\infty;\mathbb{C}^m) \text{ for some } \gamma \in \mathbb{R}\}$$

Furthermore, if $u \in \Omega$, we set $\gamma(u) := \inf\{\gamma \in \mathbb{R} : u(\cdot)\exp(-\gamma\,\cdot) \in L^2(0,\infty;\mathbb{C}^m)\}$.

THEOREM 1. *Consider the Pritchard-Salamon system (3), suppose that $u \in \Omega$, and let λ and η be numbers which satisfy $\lambda > \max(\omega_W, \omega_V, \gamma(u))$ and $\eta > \max(\omega_W, \omega_V)$. Then the following statements hold true*

(i) $y(\cdot)\exp(-\lambda\,\cdot) \in L^1(0,\infty;\mathbb{C}^p) \cap L^2(0,\infty;\mathbb{C}^p)$ and $\hat{y}(s) = [C(sI - A^V)^{-1}B + D]\hat{u}(s)$ for all $s \in \mathbb{C}_\lambda$.

(ii) $R(\cdot)\exp(-\eta\,\cdot) \in (L^1(0,\infty;\mathbb{C}^{p\times m}) + \delta_0\mathbb{C}^{p\times m})$ and $\hat{R}(s) = \mathfrak{O}(s)B + D$ for all $s \in \mathbb{C}_\eta$.

(iii) $\hat{R}(s) = \mathfrak{O}(s)B + D = C(sI - A^V)^{-1}B + D$ for all $s \in \mathbb{C}_\eta$.

For the proof of the above theorem see [CLTZ92]. Statement (i) means that the transfer function matrix $G(s)$ of (3) is given by $G(s) = C(sI - A^V)^{-1}B + D$, while statement (ii) says that the Laplace transform of the impulse response $R(\cdot)$ equals $\mathfrak{D}(s)B + D$, which is also a "transfer function candidate" for the system (3). The third statement shows that the "two transfer functions" coincide and hence that it is justified to call $R(\cdot)$ the impulse response of system (3). Statement (iii) is the difficult part of Theorem 1[6]. It is easy to prove if $\mathrm{dom}(A^V) \subseteq W$.

Next we present a result on perturbations of (3) induced by admissible state-feedback and admissible output-injection. For the proof see [CLTZ92].

THEOREM 2. *(i) Let $F \in \mathcal{L}(W, \mathbb{C}^m)$ be an admissible output operator for $S(t)$. Then there exists a unique C_0-semigroup $S_{BF}(t)$ on W and V satisfying*

$$S_{BF}(t)x = S(t)x + \int_0^t S(t-\tau)BFS_{BF}(\tau)x\,d\tau \quad \text{for all } x \in W\,. \tag{4}$$

Moreover, B is an admissible input operator for $S_{BF}(t)$ and C and F are admissible output operators for $S_{BF}(t)$.

(ii) Let $H \in \mathcal{L}(\mathbb{C}^p, V)$ be an admissible input operator for $S(t)$. Then there exists a unique C_0-semigroup $\tilde{S}_{HC}(t)$ on W and V satisfying

$$\tilde{S}_{HC}(t)x = S(t)x + \int_0^t \tilde{S}_{HC}(t-\tau)HCS(\tau)x\,d\tau \quad \text{for all } x \in W\,. \tag{5}$$

Moreover, B and H are admissible input operators for $\tilde{S}_{HC}(t)$ and C is an admissible output operator for $\tilde{S}_{HC}(t)$.

(iii) If $BF = HC$ then $S_{BF}(t) = \tilde{S}_{HC}(t)$.

(iv) If $\mathrm{dom}(A^V) \subseteq W$ then the infinitesimal generators A^V_{BF} and $\tilde{A}^V_{HC}$ of $S_{BF}(t)$ and $\tilde{S}_{HC}(t)$ on V are given by $A^V_{BF} = A^V + BF$ and $\tilde{A}^V_{HC} = A^V + HC$, respectively, where $\mathrm{dom}(A^V_{BF}) = \mathrm{dom}(A^V)$ and $\mathrm{dom}(\tilde{A}^V_{HC}) = \mathrm{dom}(A^V)$.

The above result shows that the Pritchard-Salamon class is invariant under state-feedback and output-injection, provided the state-feedback and output-injection operators are admissible output operators and admissible input operators for $S(t)$, respectively. In particular Theorem 2 applies to perturbations of (3) induced by static output feedback, i.e. perturbations of the form BKC, where $K \in \mathbb{C}^{m \times p}$.

We are now in the position to define the concepts of admissible stabilizability and admissible detectability.

Definition 5 (i) System (3) is called admissibly stabilizable if there exists an admissible output operator $F \in \mathcal{L}(W, \mathbb{C}^m)$ for $S(t)$ such that the semigroup $S_{BF}(t)$ given by (4) is exponentially stable on W and V.

[6]Although statement (iii) seems to be a trivial fact, the reader should notice that $C(sI - A^V)^{-1}B$ makes sense because B is an admissible input operator (see Remark 4(v)), while the operator $\mathfrak{D}(s)$ can only be defined since C is an admissible output operator (see Remark 4(vi)).

(ii) System (3) is called admissibly detectable if there exists an admissible input operator $H \in \mathcal{L}(\mathbb{C}^p, V)$ such that the semigroup $\tilde{S}_{HC}(t)$ given by (5) is exponentially stable on W and V.

PROPOSITION 3. *If (3) is admissibly stabilizable or admissibly detectable, then*

$$C(sI - A^V)^{-1}B + D \in \mathcal{T}_3^{p\times m} = \hat{\mathcal{B}}^{p\times m}.$$

Proposition 3 shows that the state-space concept of a Pritchard-Salamon system fits nicely together with the frequency-domain set-up of Callier and Desoer described in Example 1(iii). Since the interplay of state-space and frequency-domain concepts is a central theme of this volume, we give a proof the above result.

Proof of Proposition 3: Suppose that (3) is admissibly stabilizable and let $F \in \mathcal{L}(W, \mathbb{C}^m)$ be an admissible output operator for $S(t)$ stabilizing (3) on W and V. The exponential growth constants of $S_{BF}(t)$ will be denoted by ω_W^{BF} and ω_V^{BF}. Moreover, let A_{BF}^W and A_{BF}^V denote the infinitesimal generators of $S_{BF}(t)$ on W and V, respectively. For $\mathrm{Re}(s) > \max(\omega_W, \omega_V, \omega_W^{BF}, \omega_V^{BF})$ we obtain from (4) via Laplace transformation that

$$(sI - A_{BF}^V)^{-1}x = (sI - A^V)^{-1}x + (sI - A^V)^{-1}BF(sI - A_{BF}^W)^{-1}x \text{ for all } x \in W. \quad (6)$$

By Theorem 2 the triple $(S_{BF}(t), B, F)$ is a Pritchard-Salamon system. Hence it follows from Remark 4(vi) that $F(sI - A_{BF}^W)^{-1}$ admits an extension $\mathfrak{D}_F(s) \in \mathcal{L}(V, \mathbb{C}^m)$. Using Remark 4(v) shows that $(sI - A_{BF}^V)^{-1}B \in \mathcal{L}(\mathbb{C}^m, W)$. Moreover, by Theorem 1(iii), we have that $\mathfrak{D}_F(s)B = F(sI - A_{BF}^V)^{-1}B$. As a consequence, we may conclude from (6), that for all $\mathrm{Re}(s) > \max(\omega_W, \omega_V, \omega_W^{BF}, \omega_V^{BF})$

$$C(sI - A_{BF}^V)^{-1}B = C(sI - A^V)^{-1}B[I + F(sI - A_{BF}^V)^{-1}B]. \quad (7)$$

Set $T(s) := I + F(sI - A_{BF}^V)^{-1}B$ and note that $T \in \hat{\mathcal{A}}_-^{m\times m}$ and $\det T \in \hat{\mathcal{A}}_-^\infty$, by Theorem 1. Furthermore, by Theorem 2, the triple $(S_{BF}(t), B, C)$ is a Pritchard-Salamon system, and hence using again Theorem 1 we obtain that $C(sI - A_{BF}^V)^{-1}B \in \hat{\mathcal{A}}_-^{p\times m}$. The claim follows now from (7). The proof is similar if we assume that (3) is admissibly detectable.□

Proposition 3 shows that the transfer matrix of an admissibly stabilizable and admissibly detectable Pritchard-Salamon system belongs to $\hat{\mathcal{B}}^{p\times m}$. However, not every element in $\hat{\mathcal{B}}^{p\times m}$ is the transfer function matrix of a Pritchard-Salamon system, see Example 6(iv) below. It is a difficult open problem to give a characterization of the Pritchard-Salamon class in input-output terms.

The following important result shows the equivalence of input-output and exponential stability for Pritchard-Salamon systems.

THEOREM 4. *Suppose that system (3) is admissibly stabilizable and admissibly detectable. Then the following statements are equivalent:*

(i) System (3) is exponentially stable on W and V.
(ii) $C(sI - A^V)^{-1}B + D \in (H^\infty)^{p\times m}$.
(iii) $C(sI - A^V)^{-1}B + D \in \hat{\mathcal{A}}_-^{p\times m}$.

For a proof of the above theorem see [CLTZ92]. Equivalence results similar to Theorem 4 have been proved by a number of authors for various classes of infinite-dimensional systems, see [Loge86c], [Loge87b], [BoCu88], [Curt88], [JaNe88], [YaHa88], [Yama91], [Reba91], and [YaHa92]. It seems that the result in [Reba91] is the most general one of its kind.

It follows from Proposition 3 that the transfer function matrix of a Pritchard-Salamon system has right and left-coprime factorizations, provided the system is admissibly stabilizable or detectable. The next result shows that under certain conditions the factors of a coprime factorization of the transfer matrix of system (3) can be expressed in terms of state-space data. It is proved in [NeJB84] for the case of finite-dimensional systems and was extended to the Pritchard-Salamon class in [Curt90].

PROPOSITION 5. *Suppose that system (3) is admissibly stabilizable and admissibly detectable and denote the transfer matrix of (3) by G. Moreover, let* $F \in \mathcal{L}(W, \mathbb{C}^m)$ *be an admissible output operator and let* $H \in \mathcal{L}(\mathbb{C}^p, V)$ *be an admissible input operator such that* $S_{BF}(t)$ *and* $\tilde{S}_{HC}(t)$ *given by (4) and (5) are exponentially stable on W and V, let* A^V_{BF} *and* $\tilde{A}^V_{HC}$ *be as in Theorem 2(iv), and set* $B_{HD} := B + HD$ *and* $C_{DF} := C + DF$. *Then the eight matrices*

$$
\begin{aligned}
N(s) &= D + C_{DF}(sI - A^V_{BF})^{-1}B\,, & \tilde{N}(s) &= D + C(sI - \tilde{A}^V_{HC})^{-1}B_{HD}\\
D(s) &= I + F(sI - A^V_{BF})^{-1}B\,, & \tilde{D}(s) &= I + C(sI - \tilde{A}^V_{HC})^{-1}H\\
X(s) &= -F(sI - \tilde{A}^V_{HC})^{-1}H\,, & \tilde{X}(s) &= -F(sI - A^V_{BF})^{-1}H\\
Y(s) &= I - F(sI - \tilde{A}^V_{HC})^{-1}B_{HD}\,, & \tilde{Y}(s) &= I - C_{DF}(sI - A^V_{BF})^{-1}H
\end{aligned}
$$

form a so-called doubly coprime factorization of G, i.e. $G(s) = N(s)D^{-1}(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ *and*

$$
\begin{pmatrix} Y(s) & -X(s) \\ -\tilde{N}(s) & \tilde{D}(s) \end{pmatrix}
\begin{pmatrix} D(s) & \tilde{X}(s) \\ N(s) & \tilde{Y}(s) \end{pmatrix}
= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.
$$

As in the finite-dimensional case stabilizing feedback operators and stabilizing output injections can be found by solving algebraic operator Riccati equations, see [PrSa87].

Examples

In this subsection we mention a few examples of different types of systems which occur frequently in the applications, some of which fit into the frequency-domain set-up and/or the state-space set-up presented in the previous two subsections and some of which do not. It is intended as an illustrative rather than a comprehensive list.

Example 6 (i) *Retarded systems:* All retarded systems (with delays in the input and output variables) can be reformulated as Pritchard-Salamon systems (see e.g. [Sala84] and [PrSa85]) and the entries of their transfer matrices belong to the Callier-Desoer ring $\hat{\mathcal{B}}$ (see e.g. [Loge86b]). As a specific example consider

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= ax_2(t-h) + bu(t-h_u) \\ y(t) &= cx_1(t-h_y), \end{aligned}$$

where $a, b, c \in \mathbb{R}$ and $h, h_u, h_y \geq 0$. The transfer function $G(s)$ is given by

$$G(s) = \frac{cbe^{-(h_u+h_y)s}}{s(s-ae^{-hs})},$$

which is clearly an element in $\hat{\mathcal{B}}$.

(ii) *A neutral system* (see [Loge87b]): Consider the neutral system

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + u(t) \\ \dot{x}_2(t) - \dot{x}_2(t-h) &= x_1(t) - ax_2(t) \\ y(t) &= x_2(t), \end{aligned}$$

where $a, h > 0$. The transfer function of this system is given by

$$G(s) = \frac{1}{(s+1)(s(1-e^{-hs})+a)}.$$

It is clear that G is in $\mathcal{T}_5 = \mathcal{Q}(\hat{\mathcal{A}})$ and in [Loge87b] it is shown that G belongs to H^∞. However, the system has an infinite root chain s_n in the open left half-plane such that $\mathrm{Re}(s_n) \to 0$ as $n \to \infty$. Since the generalized Hautus conditions are satisfied in the whole complex plane it follows that s_n is a pole of G for all n. As a consequence we have that $G \notin \hat{\mathcal{B}}$ and $G \notin \mathcal{T}_4$. The above system admits an abstract semigoup description with bounded control and observation operators and hence is clearly a Pritchard-Salamon system. We mention that a large class of neutral systems with delays in the input and the output variables can be described within the Pritchard-Salamon set-up, see e.g. [Sala84].

(iii) *Heat equation with Neumann boundary control and distributed observation:* Consider the following partial differential equation for $(x,t) \in (0,1) \times (0,\infty)$

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial^2 z}{\partial x^2}(x,t)\,; \; \frac{\partial z}{\partial x}(0,t) = 0, \; \frac{\partial z}{\partial x}(1,t) = u(t) \text{ for all } t > 0 \\ y(t) &= \frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon} z(x,t)\,dx\,, \text{ where } x_0 \in (0,1) \text{ and } \varepsilon > 0\,. \end{aligned}$$

This system is in the Pritchard-Salamon class (see [PrSa87]) and its transfer function is given by

$$G(s) = \frac{1}{2\varepsilon}\frac{\sinh[\sqrt{s}(x_0+\varepsilon)] - \sinh[\sqrt{s}(x_0-\varepsilon)]}{s\sinh(\sqrt{s})} \in \hat{\mathcal{B}}\,.$$

(iv) *Heat equation with Dirichlet boundary control and point observation:* Consider the partial differential equation

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial^2 z}{\partial x^2}(x,t)\,; \; z(0,t) = 0, \; z(1,t) = u(t) \text{ for all } t > 0 \\ y(t) &= z(x_0,t)\,, \; x_0 \in (0,1) \end{aligned}$$

for $(x,t) \in (0,1) \times (0,\infty)$. It cannot be described as a Pritchard-Salamon system. Its transfer function is

$$G(s) = \frac{\sinh(\sqrt{s}\,x_0)}{\sinh(\sqrt{s})} \in \hat{\mathcal{A}}_-$$

and hence belongs to the Callier-Desoer ring.

(v) *Wave equation with Dirichlet boundary control and distributed observation:* Consider for $(x,t) \in (0,1) \times (0,\infty)$

$$\frac{\partial^2 z}{\partial t^2}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t)\,;\; z(0,t) = u(t),\, z(1,t) = 0 \text{ for all } t > 0$$
$$y(t) = \frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon} z(x,t)\,dx\,, \text{ where } x_0 \in (0,1) \text{ and } \varepsilon > 0\,.$$

This system can be reformulated as a Pritchard-Salamon system, see [PrSa87]. Its transfer function is

$$G(s) = \frac{1}{\varepsilon s}\{\frac{\cosh[s(x_0+\varepsilon)] - \cosh[s(x_0-\varepsilon)]}{1-e^{2s}} + \frac{1}{2}(e^{-s(x_0-\varepsilon)} - e^{-s(x_0+\varepsilon)})\} \in \mathcal{T}_5 = \mathfrak{Q}(\hat{\mathcal{A}})\,.$$

Since G has infinitely many poles on the imaginary axis it is not an element in $\mathcal{T}_3 = \hat{\mathfrak{B}}$ or $\mathcal{T}_4$.

(vi) *Wave equation with Neumann boundary control and point observation in the velocity:* For $(x,t) \in (0,1) \times (0,\infty)$ consider

$$\frac{\partial^2 z}{\partial t^2}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t)\,;\; z(0,t) = 0,\, \frac{\partial z}{\partial x}(1,t) = u(t) \text{ for all } t > 0$$
$$y(t) = \frac{\partial z}{\partial t}(1,t)$$

This system is not in the Pritchard-Salamon class. Its transfer function is

$$G(s) = \frac{1-e^{-2s}}{1+e^{-2s}} \in \mathcal{T}_5 = \mathfrak{Q}(\hat{\mathcal{A}})\,.$$

Since G has infinitely many poles on the imaginary axis it does not belong to $\mathcal{T}_3 = \hat{\mathfrak{B}}$ or $\mathcal{T}_4$.

For further examples of systems belonging to the Callier-Desoer and/or Pritchard-Salamon class see e.g. [PrSa87], [BoCS88], [Curt88], [Curt89], [Bont89], and [LeKo89].

3 Closed-loop stability

This section is devoted to the stability of feedback systems. Among the many forms of performance specifications used in the design of control systems, the most important requirement is that the system is stable: First and foremost any feedback control scheme has to ensure closed-loop stability.

External closed-loop stability

Let $\mathcal{S}$ be an integral domain, let $\mathcal{D} \subseteq \mathcal{S}$ be a multiplicative subset with $1 \in \mathcal{D}$ and $0 \notin \mathcal{D}$ and set $\mathcal{T} = \mathcal{S}\mathcal{D}^{-1}$. Let $G \in \mathcal{T}^{p\times m}$ and $K \in \mathcal{T}^{m\times p}$ and consider the feedback system in Figure 1, which will be denoted by $\mathfrak{F}(G, K)$. We shall call the feedback system stable if every transfer function $u_i \mapsto y_j$ that occurs around the loop is stable. More precisely:

Definition 7 Let $G \in \mathcal{T}^{p\times m}$ and $K \in \mathcal{T}^{m\times p}$, where $\mathcal{T} = \mathcal{S}\mathcal{D}^{-1}$. The feedback system $\mathfrak{F}(G, K)$ is called $\mathcal{S}$-stable if $\det(I+GK) \neq 0$ and the closed-loop transfer function matrix

$$\mathfrak{CL}(G,K) = \begin{pmatrix} K(I+GK)^{-1} & -KG(I+KG)^{-1} \\ GK(I+GK)^{-1} & G(I+KG)^{-1} \end{pmatrix} \tag{8}$$

is in $\mathcal{S}^{(m+p)\times(m+p)}$.

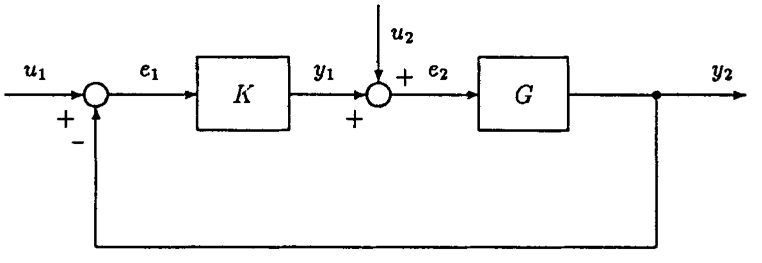

Figure 1: Closed-loop system

The above notion of external closed-loop stability was introduced in a finite-dimensional polynomial setting in [DeCh75], which also contains several examples showing that any three of the block entries of $\mathfrak{CL}(G, K)$ could be stable (in the sense that their entries are in $\mathcal{S}$) while the fourth is unstable. It is not difficult to show that we arrive at same concept of external closed-loop stability if we use the transfer matrix from (u_1, u_2) to (e_1, e_2) instead of $\mathfrak{CL}(G, K)$. When G and K admit coprime factorizations, then $\mathcal{S}$-stability of $\mathfrak{F}(G, K)$ can be characterized as follows (see [ViSF82]).

THEOREM 6. *Suppose that $G \in \mathcal{T}^{p\times m}$, $K \in \mathcal{T}^{m\times p}$, and let (N_G, D_G) and $(\tilde{D}_K, \tilde{N}_K)$ be a r.c.f. and a l.c.f. of G and K, respectively. Then the feedback system $\mathfrak{F}(G, K)$ is $\mathcal{S}$-stable if and only if the matrix $\tilde{N}_K N_G + \tilde{D}_K D_G$ is unimodular in $\mathcal{S}^{m\times m}$. A similar statement holds if G admits a l.c.f. and K admits a r.c.f.*

As a corollary we obtain for the rings $\mathcal{T}_i$, $i = 1, \ldots, 6$, presented in Example 1:

COROLLARY 7. *Suppose $G \in \mathcal{T}_i^{p\times m}$ and $K \in \mathcal{T}_i^{m\times p}$, $i = 1, \ldots, 6$, and let (N_G, D_G) and $(\tilde{D}_K, \tilde{N}_K)$ be a r.c.f. and a l.c.f. of G and K, respectively. The feedback system $\mathfrak{F}(G, K)$ is $\mathcal{S}_i$-stable if and only if*

$$\inf\{|\det[\tilde{N}_K(s)N_G(s) + \tilde{D}_K(s)D_G(s)]| : s \in \mathbb{C}_0\} > 0 \tag{9}$$

For the case of $\mathcal{T} = \mathcal{T}_3, \mathcal{T}_5$ Theorem 6 (and hence Corollary 7) was first proved in [CaDe76b] and [Vidy78].

External closed-loop stability and pole-zero cancellations

Although it is well-known (at least for the single-input single-output case) that pole-zero cancellations in the right-half plane lead to unstable closed-loop systems, only few rigorous results in this direction can be found in the literature, one of which will be described in the following. We restrict our attention to the rings $\mathcal{T}_i$, $i = 2,3,4$, although the results of this subsection remain true for the rings $\mathcal{T}_1$, $\mathcal{T}_5$, and $\mathcal{T}_6$, provided some suitable extra assumptions are made.

Suppose $G \in \mathcal{T}_i^{p\times m}$, $i = 2,3,4$, let $z \in \mathbb{C}_0^{cl}$ and let (N, D) be a r.c.f. of G. The complex number z is called a *pole* of G if $\det D(z) = 0$ and we set $\pi_z(G) := \min\{n \geq 0 : d^n/ds^n(\det D(s))|_{s=z} \neq 0\}$. The number $\pi_z(G)$ is called the *multiplicity* of the pole z. If $K \in \mathcal{T}_i^{m\times p}$ then it can be shown that $\pi_z(GK) \leq \pi_z(G) + \pi_z(K)$ (see [Loge86a], [LoOw87]).

Definition 8 Suppose $G \in \mathcal{T}_i^{p\times m}, K \in \mathcal{T}_i^{m\times p}$ and $z \in \mathbb{C}_0^{cl}$, $i = 2,3,4$. We say that GK contains a pole-zero cancellation at z if $\pi_z(GK) < \pi_z(G) + \pi_z(K)$. Otherwise (i.e. $\pi_z(GK) = \pi_z(G) + \pi_z(K)$) we say that GK contains no pole-zero cancellation at z.

In case that G and K are square, the following sufficient condition for the absence of pole-zero cancellations which resembles the single-input single-output case is given in [Loge86a] and [LoOw87].

PROPOSITION 8. *Let $G \in \mathcal{T}_i^{m\times m}$ and $K \in \mathcal{T}_i^{m\times m}$, $i = 2,3,4$, suppose that (N_G, D_G) and (N_K, D_K) are right-coprime factorizations of G and K, and let $z \in \mathbb{C}_0^{cl}$. Under these conditions GK contains no pole-zero cancellation at z if*

$$|\det N_G(z)| + |\det D_K(z)| > 0 \quad \textit{and} \quad |\det N_K(z)| + |\det D_G(z)| > 0\,. \tag{10}$$

The condition (10) is not necessary for the absence of pole-zero cancellations, see [Loge86a] or [LoOw87] for a counterexample. The next result gives a necessary and sufficient condition for $\mathcal{S}_i$-stability in terms of the transfer function matrix $GK(I+GK)^{-1}$ and pole-zero cancellations of GK in $\mathbb{C}_0^{cl}$.

THEOREM 9. *Let $G \in \mathcal{T}_i^{p\times m}$ and $K \in \mathcal{T}_i^{m\times p}$, $i = 2,3,4$, and suppose that* $\det(I+GK) \not\equiv 0$. *Then the feedback system $\mathfrak{F}(G,K)$ is $\mathcal{S}_i$-stable if and only if $GK(I+GK)^{-1} \in \mathcal{S}_i^{p\times p}$ and GK does not contain any pole-zero cancellations in $\mathbb{C}_0^{cl}$.*

The above theorem is proved in [Loge86a] and [LoOw87]. See also [AnGe81] for a similar result in a finite-dimensional discrete-time setting.

The Nyquist criterion

The famous Nyquist stability criterion is one of the basic tools in the frequency-domain approach to feedback control. It gives a necessary and sufficient condition for closed-loop stability, requiring for its application only open-loop data which can be deduced from frequency-response measurements. It is worthwhile mentioning that Nyquist's original paper [Nyqu32] on the stability of feedback amplifiers is not restricted to rational transfer functions, but includes a certain class of infinite-dimensional systems as well. In the last 30 years there has been a considerable interest in a rigorous treatment of Nyquist-type stability criteria for infinite-dimensional plants, see e.g. [Deso65], [Herz68], [Davi72], and [CaDe76a] for single-input single-output systems and [DeWa80], [VaHa80], [Moss80], [ChDe82], and[Loge86a] for multivariable systems.

In the following, if $a \in \mathbb{C}$ and φ is a closed curve in the complex plane not passing through a, let $\nu(\varphi, a)$ denote the winding number of φ around a. Moreover, let χ be a parametrization of the $\imath\omega$-axis such that $\chi(t)$ moves downwards from $\imath\infty$ to $-\imath\infty$, and for $G \in \mathcal{T}_i^{p\times m}$ $(i = 2,3,4)$ let $\varpi(G)$ denote the number of poles of G in $\mathbb{C}_0^{cl}$ (counting multiplicities).

LEMMA 10. *Suppose that* $G \in \mathcal{T}_i^{p\times m}$, $K \in \mathcal{T}_i^{m\times p}$ $(i = 2,3,4)$, (N_G, D_G) *is a r.c.f. of* G, *and* $(\tilde{D}_K, \tilde{N}_K)$ *is a l.c.f. of* K. *If* $\det D_G(\imath\omega) \neq 0$ *and* $\det \tilde{D}_K(\imath\omega) \neq 0$ *for all* $\omega \in \mathbb{R}$ *and* $\lim_{|s|\to\infty,\, s\in\mathbb{C}_0^{cl}} G(s)K(s) = (GK)(\infty)$ *exists, then*

$$\inf\{|\det[\tilde{N}_K(s)N_G(s) + \tilde{D}_K(s)D_G(s)]| : s \in \mathbb{C}_0\} > 0$$

if and only if the following two conditions are satisfied:

$$\det[(I + GK)(\imath\omega)] \neq 0 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\} \tag{11}$$

$$\nu(\det[(I + GK) \circ \chi], 0) = -[\varpi(G) + \varpi(K)]\,. \tag{12}$$

A proof of the above result can be found for example in [DeWa80], [ChDe82], and [Loge86a]. It is clear that we have to restrict our attention to the rings $\mathcal{T}_i$, $i = 2,3,4$, since the encirclement condition (12) makes no sense if G and/or K have a pole at ∞ or if they have infinitely many poles in $\mathbb{C}_0$. Combining Lemma 10 and Corollary 7 gives:

COROLLARY 11. *Under the conditions of Lemma 10 we have that* $\mathfrak{CL}(G,K) \in \mathcal{S}_i^{(m+p)\times(m+p)}$ *if and only if the conditions (11) and (12) hold.*

Corollary 11 is a graphical stability criterion which generalizes the classical scalar Nyquist criterion for finite-dimensional systems to a class of multivariable infinite-dimensional systems. However, if K is is of the form $K(s) = kK_0(s)$, where k is a real gain parameter and $K_0 \in \mathcal{T}_i^{m\times p}$ is fixed, Corollary 11 has the disadvantage that for *each* value of k a diagram has to be plotted in order to check closed-loop stability, while the scalar Nyquist criterion allows one to examine closed-loop stability for a *continuum* of gain parameter values by inspecting a single frequency response plot. This drawback can be overcome by introducing the notion of the *eigencontour* of a square transfer matrix $G \in \mathcal{T}_i^{m\times m}$ $(i = 2,3,4)$ with respect to a curve $\varphi : [0,1] \to \mathbb{C}_0^{cl}$, denoted by $\epsilon[G,\varphi]$, which is formed by the path of the eigenvalues of $G(\varphi(t))$ as t traverses the interval $[0,1]$.

THEOREM 12. *Under the conditions of Lemma (10) the closed-loop is stable in the sense that* $\mathfrak{CL}(G,K) \in \mathcal{S}_i^{(m+p)\times(m+p)}$ *if and only if* $-1 \notin \mathrm{image}(\epsilon[GK,\chi])$ *and* $\nu(\epsilon[GK,\chi],-1) = -[\varpi(G)+\varpi(K)]$.

Idea of the proof: It can be shown (either by making use of some elementary algebraic function theory or by making use of the approach in [DeWa80]) that $\epsilon[GK,\chi]$ is a closed chain and

$$\nu(\epsilon[GK,\chi],-1) = \nu(\det[(I+GK)\circ\chi],0)\,,$$

see [DeWa80] and [Loge86a] for details. Once the above equality is established the result follows from Corollary 11. □

The above theorem is an extension of the multivariable Nyquist criterion for finite-dimensional systems given in [PoMa79] (see also [Smit81]). In Lemma 10, Corollary 11, and Theorem 12 it is assumed that G and K have no poles on the $\imath\omega$-axis. The results remain true without making that assumption if we replace χ by a curve χ^* having indentations into the left-half plane whenever G or K have poles on the $\imath\omega$-axis.

Equivalence of external and internal closed-loop stability

We consider the closed-loop configuration of two Pritchard-Salamon systems

$$x_p(t) = S_p(t)x_{p0} + \int_0^t S_p(t-\tau)B_p e_p(\tau)\,d\tau\,, \quad e_p = u_2 + y_c \tag{13a}$$

$$y_p(t) = C_p x_p(t) \tag{13b}$$

and

$$x_c(t) = S_c(t)x_{c0} + \int_0^t S_c(t-\tau)B_c e_c(\tau)\,d\tau\,, \quad e_c = u_1 - y_p \tag{14a}$$

$$y_c(t) = C_c x_c(t) + D_c e_c(t) \tag{14b}$$

with state spaces $W_p \hookrightarrow V_p$ and $W_c \hookrightarrow V_c$ (cf. Figure 1, where now plant and compensator are given by (13) and (14), respectively, and $y_1 = y_c$, $y_2 = y_p$, $e_1 = e_c$, and $e_2 = e_p$). Moreover, we assume that (13) and (14) have finite-dimensional input and output spaces. Define $W_e := W_c \oplus W_p$, $V_e := V_c \oplus V_p$, and

$$S_e(t) := \begin{pmatrix} S_c(t) & 0 \\ 0 & S_p(t) \end{pmatrix},\ B_e := \begin{pmatrix} B_c & 0 \\ 0 & B_p \end{pmatrix},\ C_e := \begin{pmatrix} C_c & 0 \\ 0 & C_p \end{pmatrix},\ K_e := \begin{pmatrix} 0 & -I \\ I & -D_c \end{pmatrix}.$$

It is clear that $(S_e(t), B_e, C_e)$ is a Prichard-Salamon system. Hence there exists a unique C_0-semigroup $S_{cl}(t)$ on V_e and W_e (i.e. $S_{cl}(t)$ is a C_0-semigroup on V_e which restricts to a C_0-semigroup on W_e) satisfying

$$S_{cl}(t)x_{e0} = S_e(t)x_{e0} + \int_0^t S_e(t-\tau)B_eK_eC_eS_{cl}(\tau)x_{e0}\,d\tau \quad \text{for all } x_{e0} \in W_e\,,$$

see Theorem 2. Setting $x_{cl}(0) := (x_{c0}, x_{p0})^T$, $u_{cl} := (u_1, u_2)^T$, $y_{cl} := (y_c, y_p)^T$ and

$$B_{cl} := \begin{pmatrix} B_c & 0 \\ B_p D_c & B_p \end{pmatrix}, \; C_{cl} := \begin{pmatrix} C_c & -D_c C_p \\ 0 & C_p \end{pmatrix}, \; D_{cl} := \begin{pmatrix} D_c & 0 \\ 0 & 0 \end{pmatrix},$$

the closed-loop system given by (13) and (14) can be written as

$$x_{cl}(t) = S_{cl}(t) x_{cl}(0) + \int_0^t S_{cl}(t-\tau) B_{cl} u_{cl}(\tau)\, d\tau \tag{15a}$$

$$y_{cl}(t) = C_{cl} x_{cl}(t) + D_{cl} u_{cl}(t)\,. \tag{15b}$$

Since B_{cl} is an admissible input operator for $S_e(t)$ and C_{cl} is an admissible output operator for $S_e(t)$, an application of Theorem 2 shows that (15) is again a Pritchard-Salamon system. Hence we have proved that the Pritchard-Salamon class is closed under the operation of feedback interconnection. If we denote the transfer function matrices of (13) and (14) by G and K, respectively, then the transfer function matrix of (15) is given by $\mathfrak{CL}(G, K)$. Moreover, it follows from results in [CLTZ92] that the closed-loop system (15) is admissibly stabilizable and detectable if the same is true for the plant (13) and the compensator (14). Using Theorem 4 we arrive at the following result.

THEOREM 13. *Suppose that the plant (13) and the compensator (14) are both admissibly stabilizable and detectable. Then the following statements are equivalent:*

(i) The closed-loop system (15) is exponentially stable on V_e and W_e.
(ii) The entries of $\mathfrak{CL}(G, K)$ are in H^∞.
(iii) The entries of $\mathfrak{CL}(G, K)$ are in $\hat{\mathcal{A}}_-$.

Results similar to Theorem 13 have been proved by a number of authors for various classes of infinite-dimensional systems, see [Loge86a], [Loge86c], [Curt88], [JaNe88], and [YaHa92]. The above result seems to be the most general one of its kind.

Closed-loop stability and the existence of coprime factorizations

The following question was posed in [ViSF82]. Suppose that $\mathcal{S}$ is an integral domain and denote its quotient field by $\mathfrak{Q}(\mathcal{S})$. Let $G \in \mathfrak{Q}(\mathcal{S})^{p\times m}$ and suppose that $K \in \mathfrak{Q}(\mathcal{S})^{m\times p}$ stabilizes G in the sense that $\mathfrak{CL}(G, K) \in \mathcal{S}^{(m+p)\times(m+p)}$. Is it true that G has right and left coprime factorizations? In general the answer is no, as was shown in [Anan85], where a counterexample is given for the case of $\mathcal{S} = \mathbb{Z}[\sqrt{-5}]$. However, there are interesting cases were the above question has a positive answer. The following result can be found in [Vidy85].

PROPOSITION 14. *Let $G \in \mathfrak{Q}(\mathcal{S})^{p\times m}$ and $K \in \mathfrak{Q}(\mathcal{S})^{m\times p}$. If K has a r.c.f. (l.c.f.) and $\mathfrak{CL}(G, K) \in \mathcal{S}^{(m+p)\times(m+p)}$ then G admits a l.c.f. (r.c.f.).*

Applied to infinite-dimensional systems the above result gives a necesary condition for the existence of finite-dimensional stabilizing compensators, see Section 4. For the case of $\mathcal{S} = H^\infty$ the following theorem is proved in [Inou88] and [Smit89].

THEOREM 15. *Let $G \in \mathfrak{Q}(H^\infty)^{p\times m}$ and suppose there exists $K \in \mathfrak{Q}(H^\infty)^{m\times p}$ such that $\mathfrak{CL}(G,K) \in (H^\infty)^{(m+p)\times(m+p)}$. Then G has right and left coprime factorizations.*

It is well-known that H^∞ is not a Bezout ring (see Section 2). Hermite rings[7] are the next best thing to Bezout rings, at least as far as feedback stabilization and synthesis is concerned. We claim that Theorem 15 implies in particular that H^∞ is a Hermite ring. Let $G \in \mathfrak{Q}(H^\infty)^{p\times m}$ and note that by theorem 8.1.66 in [Vidy85] it is sufficient to show that the existence of a r.c.f. for G implies the existence of a l.c.f. for G. But if G admits a r.c.f, then by Theorem 6 there exists a stabilizing compensator $K \in \mathfrak{Q}(H^\infty)^{m\times p}$, which in turn implies via Theorem 15 that G has a l.c.f. as well. To the best of the author's knowledge it is not known whether $\hat{A}$ is a Hermite ring. It seems to be difficult to exploit the fact that H^∞ is a Hermite ring in order to show that this is also true for $\hat{A}$.

Parametrization of all stabilizing compensators

Let $\mathcal{S}$ be an integral domain, suppose that $\mathcal{D}$ is a multiplicative saturated subset of $\mathcal{S}$ with $1 \in \mathcal{D}$ and $0 \notin \mathcal{D}$, and set $\mathcal{T} = \mathcal{S}\mathcal{D}^{-1}$. For $G \in \mathcal{T}^{p\times m}$ define the set $\mathfrak{S}(G) \subseteq \mathcal{T}^{m\times p}$ of all stabilizing compensators for G by

$$\mathfrak{S}(G) := \{K \in \mathcal{T}^{m\times p} : \det(I+GK) \neq 0 \text{ and } \mathfrak{CL}(G,K) \in \mathcal{S}^{(m+p)\times(m+p)}\}.$$

The following fundamental result gives a complete parametrization of the set $\mathfrak{S}(G)$ for a given plant G.

THEOREM 16. *Suppose $G \in \mathcal{T}^{p\times m}$ has a r.c.f. (N,D) and a l.c.f. $(\tilde{D},\tilde{N})$. Let $X,\tilde{X} \in \mathcal{S}^{m\times p}$, $Y \in \mathcal{S}^{m\times m}$, and $\tilde{Y} \in \mathcal{S}^{p\times p}$ be such that $XN+YD = I_m$ and $\tilde{N}\tilde{X}+\tilde{D}\tilde{Y} = I_p$. Then*

$$\begin{aligned}\mathfrak{S}(G) &= \{(Y-S\tilde{N})^{-1}(X+S\tilde{D}) : S \in \mathcal{S}^{m\times p} \text{ and } \det(Y-S\tilde{N}) \in \mathcal{D}\}\\ &= \{(\tilde{X}+DS)(\tilde{Y}-NS)^{-1} : S \in \mathcal{S}^{m\times p} \text{ and } \det(\tilde{Y}-NS) \in \mathcal{D}\}\end{aligned}$$

Theorem 16 characterizes the set of all compensators that stabilize a given plant in terms of the "free" parameter S[8]. The correspondence between the parameter and the compensator is injective in the following sense: Suppose G is a given transfer function matrix, choose a particular r.c.f. (N,D) of G, a particular l.c.f. $(\tilde{D},\tilde{N})$ of G and select particular matrices $X, Y, \tilde{X}$ and $\tilde{Y}$ with entries in $\mathcal{S}$ such that $XN+YD = I$ and $\tilde{N}\tilde{X}+\tilde{D}\tilde{Y} = I$, then for each $K \in \mathfrak{S}(G)$ there exists a unique matrix S_1 over $\mathcal{S}$ such that $\det(Y-S_1\tilde{N}) \in \mathcal{D}$ and $K = (Y-S_1\tilde{N})^{-1}(X+S_1\tilde{D})$, as well as a unique matrix S_2 over $\mathcal{S}$ such that $\det(\tilde{Y}-NS_2) \in \mathcal{D}$ and $K = (\tilde{X}+DS_2)(\tilde{Y}-NS_2)^{-1}$. By substituting the parametrization into the expression for $\mathfrak{CL}(G,K)$ we obtain a parametrization of all stable closed-loop transfer function matrices achievable by feedback. For example, the first equation in Theorem 16 gives

$$\mathfrak{CL}(G,K) = \begin{pmatrix} D(X+S\tilde{D}) & D(Y-S\tilde{N})-I \\ N(X+S\tilde{D}) & N(Y-S\tilde{N}) \end{pmatrix}.$$

[7] An integral domain $\mathcal{S}$ is called a *Hermite ring* if any unimodular row $(a_1 \cdots a_n) \in \mathcal{S}^{1\times n}$ (i.e. any row such that $a_1,\dots,a_n$ generate $\mathcal{S}$) can be complemented to a unimodular matrix $A \in \mathcal{S}^{n\times n}$.

[8] The parameter S is not entirely free, because of the constraints $\det(Y-S\tilde{N}) \in \mathcal{D}$ and $\det(\tilde{Y}-NS) \in \mathcal{D}$. In case that $\mathcal{T} = \mathcal{T}_i$, $i = 1,\dots,6$, these constraints will be satisfied if $G(s) \to 0$ as $|s| \to \infty$ in $\mathbb{C}_0$.

Note that this parametrization is affine-linear in the parameter S. The only assumption in Theorem 16 is that the plant has a r.c.f. and a l.c.f. So, it covers all systems which belong to $\mathcal{T}_i^{p\times m}$, $i = 1, ..., 4$. In order to apply Theorem 16 to a plant in $\mathcal{T}_5^{p\times m}$ or $\mathcal{T}_6^{p\times m}$ we have to assume that G has a r.c.f. and a l.c.f. However, this is not a serious limitation, because it follows from Corollary 17 in Section 4, that if G does not have a r.c.f. (l.c.f.), then it cannot be stabilized by any controller that has a l.c.f. (r.c.f.). In particular, it cannot be stabilized by a lumped compensator.

Theorem 16 is the basis for any systematic feedback control synthesis procedure, because first and foremost a feedback system must be stable. The parametrization of all stabilizing compensators given by the above theorem sets the stage for the choice of a compensator which apart from stabilizing the plant achieves a number of prespecified design constraints: strong stabilization (i.e. stabilization by a stable compensator), tracking of prescribed reference trajectories, rejection of a given class of disturbance signals, robustness etc. Finally, if any remaining design latitude exists after these goals have been met it may be used to optimize some measure of performance, e.g. sensitivity, stability robustness or energy consumption. In H^∞-control the above parametrization has been used for the reformulation of various H^∞-control problems as a model-matching problem, see e.g. [Fran87]. In general it is a difficult problem to express design constraints and/or performance specifications in terms of the parameter S and a lot of more work needs to be done in this direction.

Theorem 16 was first proved by Youla, Bongiorno, and Jabr in a finite-dimensional polynomial setting (see [YoBJ76] for the single-input single-output case and [YoJB76] for the multivariable case). The above general version of the result is due to [DLMS80]. See also [Vidy85] for a detailed treatment of the above parametrization (which is sometimes called the Youla-Bongiorno-Jabr parametrization) and its applications to control system synthesis. A tutorial introduction into these issues for the class of finite-dimensional single-input single-output systems is given in [SMCKI82] and [SMCKI83]. Theorem 16 deals with unity-gain feedback systems. Extensions to more general closed-loop configurations may be found e.g. in [Vidy85] and [Nett86].

4 Finite-dimensional stabilization

Practical feedback control of infinite-dimensional systems must be accomplished with a finite (small) number of actuators and sensors and a control algorithm which can be implemented by an one-line digital computer. Therefore the controller should be finite-dimensional, and this has motivated much of the work on stabilization of distributed parameter systems by finite-dimensional output feedback.

Existence of finite-dimensional stabilizing controllers

Since any rational transfer function matrix admits right and left-coprime factorizations, we obtain the following necessary condition for the existence of a stabilizing finite-dimensional compensator from Proposition 14. For the case of $\mathcal{T} = \mathcal{T}_6$ it follows also from Theorem 15.

COROLLARY 17. *Suppose that $G \in \mathcal{T}_i^{p\times m}$, $i = 5,6$, and there exists a proper rational compensator $K \in \mathcal{T}_2^{m\times p}$ such that $\mathfrak{CL}(G,K) \in \mathcal{S}_i^{(m+p)\times(m+p)}$. Then G admits a r.c.f. and a l.c.f.*

For the following it is useful to introduce the ring H_c^∞ consisting of all those functions in H^∞ which admit a continuous extension to $\mathbb{C}_0^{cl} \cup \{\infty\}$, more precisely

$$H_c^\infty := \{f \in C(\mathbb{C}_0^{cl}) \cap H^\infty : \lim_{|s|\to\infty,\, s\in\mathbb{C}_0^{cl}} f(s) \text{ exists in } \mathbb{C}\}.$$

It is clear that H_c^∞ is a closed subring of H^∞. Let Δ denote the so-called disc algebra, i.e. the algebra of all holomorphic functions on the open unit disc $\mathbb{D}$ which admit a continuous extension to $\mathbb{D}^{cl}$. Defining the canonical bijection $\iota : \mathbb{C}_0^{cl} \cup \{\infty\} \to \mathbb{D}^{cl}$, $s \mapsto (s-1)/(s+1)$, where $\iota(\infty) := 1$, it is clear that the map $\vartheta : H_c^\infty \to \Delta$, defined by $(\vartheta f)(z) = f(\iota^{-1}(z))$, is an isometric isomorphism of rings. Hence, the following proposition is an easy consequence of the fact that the polynomials form a dense subset of the disc algebra (see [Rudi74], p. 397).

PROPOSITION 18. *The closure of the ring of proper stable rational transfer functions with respect to the norm $\|\cdot\|_\infty$ is given by H_c^∞, i.e. $(\mathbb{C}(s) \cap H^\infty)^{cl} = H_c^\infty$.*

Applying the matrix-valued corona theorem (see theorem 14.10 in [Fuhr81]) and observing that this result is also true for H_c^∞ yields:

LEMMA 19. *If $N \in (H_c^\infty)^{p\times m}$ and $D \in (H_c^\infty)^{p\times p}$ are right-coprime over H^∞, then they are right-coprime over H_c^∞. An analogous statement holds for left-coprime factorizations.*

As a corollary we obtain from Proposition 18, Lemma 19 and Theorem 6 the following sufficient condition for the existence of finite-dimensional stabilizing compensators for plants in $\mathfrak{Q}(H^\infty)^{p\times m}$.

COROLLARY 20. *Suppose that $G \in \mathfrak{Q}(H^\infty)^{p\times m}$ admits a r.c.f. $(N,D) \in (H_c^\infty)^{p\times m} \times (H_c^\infty)^{m\times m}$. Then there exists a proper rational compensator K such that $\mathfrak{CL}(G,K) \in (H^\infty)^{(m+p)\times(m+p)}$.*

The following example shows that the condition in the above corollary (which implies in particular that G is the limit of a sequence of lumped plants with respect to the graph topology[9]) is not necessary for the existence of a finite-dimensional stabilizing controller.

Example 9 Consider the transfer function $G(s) = (1-e^{-2s})/(1+e^{-2s})$ of Example 6(vi). For any $k > 0$ the compensator $K(s) \equiv k$ stabilizes G, with $\mathfrak{CL}(G,K)$ not only in $(H^\infty)^{2\times 2}$, but also in $\hat{\mathcal{A}}_-^{2\times 2}$. However, it is clear that G does not satisfy the assumption in Corollary 20.

[9]Suppose that $G, G_n \in \mathcal{T}_6^{p\times m} = \mathfrak{Q}(H^\infty)^{p\times m}$, $n \in \mathbb{N}$, have right-coprime factorizations. The plants G_n converge to G in the *graph topology* as $n \to \infty$ if there exist a r.c.f. (N,D) of G and a sequence (N_n, D_n) of r.c.f.'s of G_n such that $N_n \to N$ and $D_n \to D$ in the H^∞-norm, see [ViSF82] and [Vidy85] for details.

The next result shows that any plant $G \in \mathcal{T}_i^{p\times m}$, $i = 3,4$, can be stabilized by finite-dimensional compensators.

THEOREM 21. *If $G \in \mathcal{T}_i^{p\times m}$, $i = 3,4$, then there exists a strictly proper rational compensator $K \in \mathcal{T}_2^{m\times p}$ such that $\mathfrak{CL}(G,K) \in \mathcal{S}_i^{(m+p)\times(m+p)}$.*

Proof: Let (N, D) be a r.c.f. of $[1/(s+1)]G(s)$ and choose matrices X and Y with entries in $\mathcal{S}_i$ such that $XN + YD = I$. Without loss of generality we may assume that D is rational (see Example 1) and that $D(\infty) = I$. It is clear that $N(s) \to 0$ as $|s| \to \infty$ in $\mathbb{C}_0^{cl}$. Setting

$$N_G(s) := (s+1)N(s),\ D_G(s) := D(s),\ X_G(s) := \frac{1}{s+1}X(s),\ Y_G(s) := Y(s)\,,$$

we see that $G = N_G D_G^{-1}$ and $X_G N_G + Y_G D_G = I$, i.e. (N_G, D_G) is a r.c.f. of G. Next note that $\lim_{|s|\to\infty, s\in\mathbb{C}_0^{cl}} X_G(s) = 0$ and $\lim_{|s|\to\infty, s\in\mathbb{C}_0^{cl}} Y_G(s) = I$. By Proposition 18 there exists a sequence of proper stable rational matrices $\tilde{N}_n$ and $\tilde{D}_n$ such that $\lim_{n\to\infty} \tilde{N}_n = X_G$ and $\lim_{n\to\infty} \tilde{D}_n = Y_G$ in the H^∞-norm. Moreover, without loss of generality, we may assume that $\tilde{N}_n(\infty) = 0$ and $\tilde{D}_n(\infty) = I$. Realizing that the matrix $\tilde{N}_n N_G + \tilde{D}_n D_G$ will be unimodular over $\mathcal{S}_i$ for all suffciently large n, it follows from Theorem 6 that the compensator $K_n := \tilde{D}_n^{-1}\tilde{N}_n$ stabilizes G for all sufficiently large n. The claim now follows, since by construction the K_n are strictly proper rational matrices. □

As an immediate consequence of the previous theorem, Proposition 3, and Theorem 13 we obtain:

COROLLARY 22. *Suppose that the Pritchard-Salamon (3) is admissibly stabilizable and admissibly detectable, then there exists a strictly proper finite-dimensional compensator (A_c, B_c, C_c) with state-space $\mathbb{C}^n$ such that the closed-loop system given by (15) is exponentially stable on $W \oplus \mathbb{C}^n$ and $V \oplus \mathbb{C}^n$*

Theorem 21 (and its proof) is due to [Nett84], see also [NeJB83] and [Loge84]. It was reproved in [CuGl86] in a slightly different way. Corollary 22 was proved in [JaNe88] for systems with bounded control and observation, and in [KaKT85], [KaKT86] and [Loge86b] for certain classes of retarded and neutral systems (with delays in the internal, control and observation variables)[10]. Although the above theorem and its corollary are not particularly deep results, they seem to be the most general ones on the *existence* of finite-dimensional stabilizing controllers. In particular, Corollary 22 extends the existence results of the state-space approaches presented in [Schu83a], [Bala84], [Bala86], and [Ito90], which all assume the input and output operators to be bounded. State-space based treatments of the finite-dimensional stabilization problem for systems with unbounded conrol and observation can be found in e.g. in [Curt84] and [CuSa86]. Although the results in these two papers have a large overlap with Corollary 22, they are neither completely contained in it nor do they contain Corollary 22.

[10]In this case the stabilizability and detectability assumptions are satisfied if and only if the generalized Hautus conditions hold in the closed right-half plane, see [Sala84], [PrSa85], and [PrSa87].

The proof of Theorem 21 indicates how to compute a finite-dimensional compensator for a given plant G in $\mathcal{T}_i^{p\times m}$, $i = 3,4$:
Step 1: Compute a r.c.f. or a l.c.f. of G and solve the corresponding Bezout equation. If the plant is a Pritchard-Salamon system this can be accomplished by solving two operator Riccati equations, and then applying Proposition 5. In case that the plant is a retarded system with commensurate delays an alternative (constructive) procedure is given in [KaKT86].
Step 2: Approximate the solutions of the Bezout equation by rational matrices. The most straightforward procedure is to convert the problem into one which consists of the polynomial approximation of $m(m+p)$ functions f_j belonging to the disc algebra. Polynomial approximations with respect to the H^∞-norm are given by the Cesáro means of f_j (see [Hoff62], pp. 16), which require for its calculation the computation of the Fourier coefficients of $f_j(e^{\imath\omega})$. This method has the disadvantage that no error bounds are available. More sophisticated rational approximation schemes can be found in the literature, see e.g. [GlCP88], [GuLK89], [Maki90], [GlLP90], and [GlLP91].
Step 3: Apply any suitable robustness test (see e.g. [ChDe82], [Nett84], and [CuGl86]) in order to ensure that the finite-dimensional compensator obtained in Step 2 is stabilizing. This requires the computation of an H^∞-norm.

Let G be an irrational transfer function matrix. Theorem 21 says in particular that the condition $G \in \hat{\mathfrak{B}}^{p\times m}$ is sufficient for the existence of a stabilizing strictly proper finite-dimensional controller. We are going to show that that for a large class of transfer matrices this condition is also necessary. In order to define this class, let $\mathcal{A}_\infty$ denote the convolution ring of all distributions f with support contained in $[0,\infty)$ such that $f\exp(-\mu\,\cdot) \in \mathcal{A}$ for some $\mu = \mu(f) \in \mathbb{R}$. Clearly, all f in $\mathcal{A}_\infty$ are Laplace transformable and we set $\hat{\mathcal{A}}_\infty := \{\hat{f} : f \in \mathcal{A}_\infty\}$. Note that $\hat{\mathfrak{B}}$ is contained in $\hat{\mathcal{A}}_\infty$. A transfer matrix $G \in \hat{\mathcal{A}}_\infty{}^{p\times m}$ is called *strictly proper* if there exists $\alpha \in \mathbb{R}$ such that $G(s) \to 0$ as $|s| \to \infty$ in $\mathbb{C}_\alpha$, i.e. $G(s)$ "rolls off" in some half-plane. This does not imply that G will necessarily "roll off" in $\mathbb{C}_0$. In particular G may not be bandlimited in the sense that $G(\imath\omega) \to 0$ as $|\omega| \to \infty$. Strictly proper transfer matrices $G \in \hat{\mathcal{A}}_\infty{}^{p\times m}$ correspond to systems which do not instantaneously respond to applied inputs, a behaviour which is exhibited by all physical devices. The following proposition is a special case of a result in [HeJN91].

PROPOSITION 23. *Suppose that $G \in \hat{\mathcal{A}}_\infty{}^{p\times m}$ and $K \in \hat{\mathcal{A}}_\infty{}^{m\times p}$. If $\mathfrak{CL}(G,K) \in \hat{\mathcal{A}}_-^{(m+p)\times(m+p)}$ and GK (or KG) is strictly proper, then $G \in \hat{\mathfrak{B}}^{p\times m}$.*

Roughly speaking, the Callier-Desoer ring $\hat{\mathfrak{B}}$ is restricted to systems with at most finitely many unstable poles. While this is a limitation from a theoretical point of view, the above result indicates that from a practical synthesis point of view it is not such a restriction.

Example 10 Consider once again the transfer function $G(s) = (1 - e^{-2s})/(1 + e^{-2s})$ of Example 6(vi) and Example 9. Since G is in $\hat{\mathcal{A}}_\infty$, but not in $\hat{\mathfrak{B}}$, it follows from Proposition 23 that G is not stabilizable by a strictly proper compensator in $\hat{\mathcal{A}}_\infty$.

Combining Theorem 21 and Proposition 23 yields

COROLLARY 24. *Suppose that $G \in \hat{\mathcal{A}}_{\infty}{}^{p\times m}$. There exists a strictly proper rational matrix K such that $\mathfrak{CL}(G,K) \in \hat{\mathcal{A}}_{-}^{(m+p)\times(m+p)}$ if and only if $G \in \hat{\mathfrak{B}}^{p\times m}$.*

The main assumption in Theorem 21 is that the plant has at most finitely many unstable poles. Example 9 shows that this condition is not necessary for the existence of finite-dimensional stabilizing compensators (note that G has infinitely many poles on the imaginary axis). For plants in $\mathfrak{Q}(\hat{\mathcal{A}})^{p\times m}$ a general solution to this problem is given by the following result from [ViAn89].

THEOREM 25. *Suppose that $G \in \mathfrak{Q}(\hat{\mathcal{A}})^{p\times m}$ has right and left coprime factorizations, select a r.c.f. (N, D), and let $\check{N}$ and $\check{D}$ denote the inverse Laplace transforms of N and D, respectively. Then G can be stabilized by a proper finite-dimensional controller if and only if there exisits a matrix $M \in \mathbb{C}^{m\times(m+p)}$ such that*

$$M \begin{bmatrix} \check{N} \\ \check{D} \end{bmatrix}_{pa}$$

is a unimodular matrix in $\mathcal{A}^{m\times m}$. Here $[\cdot]_{pa}$ denotes the purely atomic part.

If the plant G is assumed to be in $\mathfrak{T}_3{}^{p\times m} = \hat{\mathfrak{B}}^{p\times m}$, then Theorem 21 is contained in Theorem 25 as a special case.

An important problem is the parametrization of all finite-dimensional stabilizing compensators of a given plant G. Suppose that G is in $\hat{\mathfrak{B}}^{p\times m}$ and $\lim_{|s|\to\infty,\, s\in\mathbb{C}_0^{cl}} G(s) = 0$. Choose a r.c.f. (N, D) and a l.c.f. $(\tilde{D}, \tilde{N})$ of G. Then, clearly, $N(s) \to 0$ and $\tilde{N}(s) \to 0$ as $|s| \to \infty$ in $\mathbb{C}_0$, and without loss of generality we may assume that D and $\tilde{D}$ are rational matrices satisfying $D(\infty) = I_m$ and $\tilde{D}(\infty) = I_p$. Moreover, select matrices $\tilde{X}$ and $\tilde{Y}$ with entries in $\hat{\mathcal{A}}_-$ such that $\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I_p$ and $\tilde{X}(s) \to 0$ as $|s| \to \infty$ in $\mathbb{C}_0$. Finally, introduce the linear-fractional map $\Sigma : \hat{\mathcal{A}}_- \cap H_c^\infty \to \mathfrak{S}(G)$, $S \mapsto (\tilde{X} + DS)(\tilde{Y} - NS)^{-1}$. Denoting the set of all proper finite-dimensional stabilizing compensators of G by $\mathfrak{S}_f(G)$ and using the Youla-Bongiorno-Jabr parametrization (see Theorem 16) it is not difficult to show that $\mathfrak{S}_f(G)$ is densely contained in ran(Σ) (with respect to the graph topology), see [Nett84]. So far, a complete solution of the parametrization problem has not been found.

Strong stabilization

This subsection deals with the problem of strong (finite-dimensional) stabilization, i.e. the problem of when it is possible to stabilize an infinite-dimensional plant with a stable (finite-dimensional) compensator. An investigation of stabilizability by stable compensators is important, since it plays an essential role in many synthesis problems, such as simultaneous stabilization of two (or a finite number) of plants, two-stage compensation, and reliable stabilization, see [Vidy85] for a detailed discussion of the finite-dimensional case.

In the following we restrict our attention to plants in $\hat{\mathfrak{B}}_r^{p\times m}$, where $\hat{\mathfrak{B}}_r^{p\times m}$ denotes the subring of all functions $f \in \hat{\mathfrak{B}}$ satisfying $\bar{f}(s) = f(\bar{s})$ for all $s \in \mathbb{C}_0$ (cf. Section 2 for remarks on the "real" versions $\mathfrak{T}_{ir}$ of the rings $\mathfrak{T}_i$, $i = 1, ..., 6$). The "real" version $\hat{\mathcal{A}}_{-,r}$ of $\hat{\mathcal{A}}_-$ is defined in an analogous way. Note that a transfer matrix $G \in \hat{\mathcal{A}}_-^{p\times m}$ is strictly proper if and only if $\check{G}(\cdot)\exp(\varepsilon\cdot) \in L^1(0,\infty;\mathbb{C}^{p\times m})$ for some $\varepsilon > 0$, where $\check{G}$ denotes the the inverse Laplace transform of G.

Let G be a plant in $\hat{\mathfrak{B}}_r^{p\times m}$ and choose a r.c.f. (N, D) of G. Without loss of generality we may assume that D is rational. In order to stabilize G with a strictly proper stable (real) compensator one has to show, that there exists $K \in \hat{\mathcal{A}}_{-,r}{}^{m\times p}$ with $K(s) \to 0$ as $|s| \to \infty$ in $\mathbb{C}_0^{cl}$ such that $\det(K(s)N(s) + D(s)) \neq 0$ for all $s \in \mathbb{C}_0^{cl}$ (this follows from Corollary 7 and the fact that $\det D(s)$ is bounded away from 0 at ∞ in $\mathbb{C}_0$). There is an obvious condition that must be satisfied for this to be possible. It is customary to call a point $z \in \mathbb{C}$ a *blocking zero* of G if $G(z) = 0$, or equivalently, $N(z) = 0$, i.e. all entries of N vanish at z. In addition, the point at infinity should be considered as a blocking zero too, since $K(\infty) = 0$, and hence $(KN)(\infty) = 0$ for all strictly proper $K \in \hat{\mathcal{A}}_{-,r}{}^{m\times p}$. Clearly, at each blocking zero z, $\det(K(z)N(z)+D(z)) = \det D(z)$. Since both $\det(K(s)N(s)+D(s))$ and $\det D(s)$ are real on the real axis, and $\det(K(s)N(s) + D(s))$ is not allowed to have any zeros in $\mathbb{C}_0^{cl}$, we conclude that necessarily $\det D(s)$ must have the same sign at each blocking zero of G that belongs to the interval $[0,\infty]$ (in particular, this set of blocking zeros includes the point at infinity). Another way to say this is that the sum of the MacMillan degrees of the real poles of G between consecutive real blocking zeros of G must be even. This condition is usually referred to as the *parity interlacing condition*. The following theorem shows that the parity interlacing condition is also sufficient for the existence of a stabilizing strictly proper rational compensator.

THEOREM 26. *For a plant $G \in \hat{\mathfrak{B}}_r^{p\times m}$ there exists a strictly proper stable compensator $K \in \hat{\mathcal{A}}_{-,r}{}^{m\times p}$ such that $\mathfrak{CL}(G,K) \in \hat{\mathcal{A}}_{-,r}^{(m+p)\times(m+p)}$ if and only if G satisfies the parity interlacing condition. Moreover, whenever the parity interlacing condition holds then there exists a strictly proper stable rational controller which stabilizes G.*

For lumped plants the above result was first proved in [YoBL74], see [Vidy85] for a detailed treatment. The infinite-dimensional version was proved in [Staf85] for single-input single-output plants. It was extended to the multivariable case in [Staf92]. The results in [Staf85] and [Staf92] cover a class of transfer functions which is larger than $\hat{\mathfrak{B}}_r$ in the sense that the inverse Laplace transform of the numerator of a transfer function is merely supposed to be a bounded measure on $[0,\infty)$. Note that the hypothesis in Theorem 26 does not exclude the possibility that the plant has infinitely many real blocking zeros in $[0,\infty)$. If there are infinitely many blocking zeros in $[0,\infty)$ then they cluster at ∞, provided that $G(s) \not\equiv 0$. Since G has at most finitely many poles in $[0,\infty)$ it follows that only finitely many blocking zeros have to be taken into account in order to check the parity interlacing condition.

5 Regulation by output feedback

One of the most important applications of feedback is to achieve servoaction, that is to obtain a closed-loop system that tracks a prespecified class of refererence signals and rejects a given class of external disturbances with zero asymptotic error. In this section we will survey some of the results on infinite-dimensional servomechanisms which can be found in the literature.

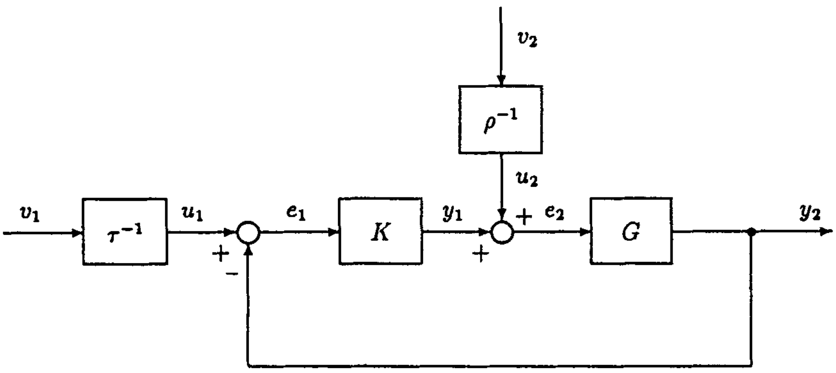

Figure 2: The servo problem

The internal model principle

Consider Figure 2, where as before $G \in \mathcal{T}_i^{p\times m}$ and $K \in \mathcal{T}_i^{m\times p}$, $i = 1, \ldots .6$, while τ and ρ belong to $\mathcal{D}_i$, i.e. τ and ρ are in $\mathcal{S}_i$ such that τ^{-1} and ρ^{-1} belong to $\mathcal{T}_i$. The output y_2 is required to "track" any reference signal u_1 generated through τ^{-1} by $v_1 \in \mathcal{S}_i^p$. More precisely, e_1 should be in $\mathcal{S}_i^p$ whenever v_1 is in $\mathcal{S}_i^p$ and $v_2 = 0$. Similarly, any plant input disturbance u_2 generated through ρ^{-1} by a v_2 in $\mathcal{S}_i^m$ is to be "rejected" at the output y_2. Specifically, y_2 should be in $\mathcal{S}_i^m$ whenever v_2 is in $\mathcal{S}_i^m$ and $v_1 = 0$. Setting

$$\mathfrak{R}(G,\tau,\rho) := \{K \in \mathfrak{S}(G) : \tau^{-1}(I+GK)^{-1} \in \mathcal{S}_i^{p\times p} \text{ and } \rho^{-1}G(I+KG)^{-1} \in \mathcal{S}_i^{p\times m}\}$$

we say that a compensator $K \in \mathcal{T}_i^{m\times p}$ is a solution of the (τ,ρ)-*servoproblem* for G if $K \in \mathfrak{R}(G,\tau,\rho)$. If $K \in \mathfrak{R}(G,\tau,1)$ ($K \in \mathfrak{R}(G,1,\rho)$) we say that K solves the τ-*tracking problem* (ρ-*disturbance rejection problem*) for G. Furthermore, if $K \in \mathfrak{R}(G,\tau,\rho)$ and there exists a neighbourhood $\mathcal{N}_G$ of G with respect to the graph topology such that $K \in \mathfrak{R}(G',\tau,\rho)$ for all $G' \in \mathcal{N}_G$ then K is called a ***robust*** solution of the (τ,ρ)-servoproblem for G. Let the set of all such controllers be denoted by $\mathfrak{R}_{ro}(G,\tau,\rho)$. The elements of $\mathfrak{R}(G,\tau,\rho)$ ($\mathfrak{R}_{ro}(G,\tau,\rho)$) are also called (robust) (τ,ρ)-*regulators* for G.

We remark that *asymptotic* tracking and disturbance rejection are not necessarily implied by the above requirements. This is due to the fact that the inverse Laplace transform of an element in $\mathbb{S}_i$ does not necessarily approach zero asymptotically in time. Nothing is lost, however, if we make the following assumptions:

Assumption (S1): $G \in \mathcal{T}_3{}^{p\times m} = \hat{\mathfrak{B}}^{p\times m}$.
Assumption (S2): The entries of v_1 and v_2 are strictly proper stable rational functions.

If (S1) and (S2) are satisfied then it is clear that any $K \in \mathfrak{R}(G, \tau, \rho)$ will achieve *asymptotic* tracking and disturbance rejection for the reference and disturbance signals given by $\tau^{-1}v_1$ and $\rho^{-1}v_2$, respectively[11]. Moreover, note that all command inputs and disturbance signals occuring in practice can be generated under the constraint (S2). For example, suppose that the plant has two inputs and two outputs and that the closed-loop system is required to track the command input $(\theta(t), \sin(t))$ asymptotically in time (here $\theta(\cdot)$ denotes the Heaviside function). Setting

$$\tau(s) = \frac{s(s^2+1)}{(s+1)^3},\ v_1^1(s) = \frac{s^2+1}{(s+1)^3},\ v_1^2(s) = \frac{s}{(s+1)^3},$$

we see that $\tau \in \mathcal{D}_3 = \hat{\mathcal{A}}_-^\infty$, v_1^1 and v_1^2 are strictly proper rational functions, and $v_1^1\tau^{-1}$ and $v_1^2\tau^{-1}$ coincide with the Laplace transforms of $\theta(\cdot)$ and $\sin(\cdot)$, respectively.

We are now in the position to formulate the so-called *internal model principle.*

THEOREM 27. *Let $G \in \hat{\mathfrak{B}}^{p\times m}$, $K \in \hat{\mathfrak{B}}^{m\times p}$ (where $m \geq p$), τ, $\rho \in \hat{\mathcal{A}}_-^\infty$, and suppose that (N_K, D_K) is a r.c.f. of K and that μ is a least common multiple of τ and ρ in $\hat{\mathcal{A}}_-^\infty$* [12]. *Under these conditions K is in $\mathfrak{R}_{ro}(G, \tau, \rho)$ if and only if $K \in \mathfrak{S}(G)$ and $\mu^{-1}D_K \in \hat{\mathcal{A}}_-{}^{p\times p}$.*

The internal model principle says (roughly speaking) that a controller which achieves robust servoaction necessarily contains a duplicate of the dynamics of the reference and disturbance signals. The assumption in Theorem 27 that $m \geq p$ is not restrictive, since it can be shown that robust tracking is only possible if the number of plant inputs is greater or equal to the number of plant outputs. Using the internal model principle it is not difficult to prove that the robust servoproblem is equivalent to a stabilization problem:

THEOREM 28. *Let $G \in \hat{\mathfrak{B}}^{p\times m}$ (where $m \geq p$), τ, $\rho \in \hat{\mathcal{A}}_-^\infty$, let $(\tilde{D}_G, \tilde{N}_G)$ be a l.c.f. of G and let μ denote a least common multiple of τ and ρ in $\hat{\mathcal{A}}_-^\infty$. There exists a robust solution of the (τ, ρ)-servoproblem for G if and only if μI_p and $\tilde{N}_G$ are left-coprime. If this is the case, then $\mathfrak{R}_{ro}(G, \tau, \rho) = \mu^{-1}\mathfrak{S}(\mu^{-1}G)$.*

It follows from Theorem 28 that the Youla-Bongiorno-Jabr parametrization of all stabilizing controllers of a given plant (see Theorem 16) induces a parametrization of the set $\mathfrak{R}_{ro}(G, \tau, \rho)$. Moreover, by Theorem 21, if the robust (τ, ρ)-servoproblem admits a solution at all, then it can be solved by a finite-dimensional compensator. Theorem 27 and Theorem 28 can be found in [Nett84]. For the finite-dimensional case similar results are

[11] Recall that if k is a convolution kernel in $\mathcal{A}$ and $u \in L^\infty(0,\infty)$ then $k * u \in L^\infty(0,\infty)$. Under the extra assumption that u converges to 0 asymptotically or exponentially, the same is true for $k * u$.

[12] Note that there exists such a least common multiple, because τ and ρ belong to $\hat{\mathcal{A}}_-^\infty$ and hence have at most finitely many zeros in $\mathbb{C}_\alpha$ for some $\alpha < 0$.

given in [FrVi83] and in section 7.5 of [Vidy85] (an inspection of the proofs shows that they carry over to infinite-dimensional systems with transfer functions in $\hat{\mathfrak{B}}^{p\times m}$). In the above results it is assumed that the reference and disturbance signals are generated by τ^{-1} and ρ^{-1}, where τ and ρ are in $\hat{\mathcal{A}}_-^\infty$. Extensions to multivariable reference and disturbance signal generators can be derived as in the finite-dimensional case, see [Vidy85]. The servoproblem has been investigated for various classes of distributed parameter systems, see [Fran77], [DeWa79], [CaDe80b], [SaMu81], [FeCa82], and [YaHa88]. All these papers are written from an input-output point of view and come to conclusions which are closely related to the above results. References which investigate tracking and disturbance rejection problems with state-space methods include [Koba83], [Schu83b], [Curt83], and [UkIw90].

Trivially, the internal model principle remains sufficient for the solvability of the servoproblem without robustness. However, as the following example shows, it is not necessary in the nonrobust case.

Example 11 (see [Fran77]) Set $G(s) = 1/s$, $K(s) \equiv 1$, $\tau(s) = s/(s+1)$, and $\rho(s) \equiv 1$. Trivially, $(N_K(s), D_K(s)) \equiv (1,1)$ is coprime factorization of K. An easy computation shows that $K \in \mathfrak{R}(G,\tau,\rho)$. But $\mu^{-1}D_K = \tau^{-1}D_K \notin \hat{\mathcal{A}}_-$, and hence, by Theorem 27, $K \notin \mathfrak{R}_{ro}(G,\tau,\rho)$. Indeed, let $\varepsilon_n > 0$ be a such that $\lim_{n\to\infty}\varepsilon_n = 0$, and set $G_n(s) = 1/(s-\varepsilon_n)$. Then G_n converges to G in the graph topology as $n \to \infty$, but

$$\tau^{-1}(s)(I + G_n(s)K(s))^{-1} = \frac{s+1}{s}\frac{s-\varepsilon_n}{s-\varepsilon_n+1} \notin \hat{\mathcal{A}}_- \text{ for all } n \in \mathbb{N},$$

which shows that K is not a robust (τ,ρ)-regulator for G.

PI-control of uncertain infinite-dimensional systems

In the following we apply the internal model principle to robust low-gain and high-gain control problems. First we consider the low-gain situation, where a low-gain PI-controller is applied to an uncertain stable plant in order to achieve asymptotic tracking of step commands and asymptotic rejection of step-disturbances. The following result is proved in [LoOw89].

THEOREM 29. *Let $G \in \hat{\mathcal{A}}_-^{p\times m}$, suppose that* $\operatorname{rank} G(0) = p$, *and choose a matrix $K_P \in \mathbb{C}^{m\times p}$ such that $\mathfrak{F}(G, K_P)$ is stable.*[13] *Then there exists a matrix $K_I \in \mathbb{C}^{m\times p}$ satisfying*

$$\operatorname{spec}((I + G(0)K_P)^{-1}G(0)K_I) \subset \mathbb{C}_0 .$$

For each such K_I there exists a number $k^* > 0$ such that for all $0 \le k < k^*$ the controller

$$K_k(s) := \frac{1}{s}kK_I + K_P \tag{16}$$

achieves closed-loop stability in the sense that $\mathfrak{CL}(G, K_k) \in \hat{\mathcal{A}}_-^{(m+p)\times(m+p)}$ for all $k \in [0, k^*)$.

[13] Note that by the small-gain theorem (see e.g. [DeVi75]) the closed-loop system $\mathfrak{F}(G, K_P)$ is stable for any $K_P \in \mathbb{C}^{m\times p}$ satisfying $\|K_P\| < 1/\|G\|_\infty$.

Note that exact knowledge of G is not required. For pure integral control (i.e. $K_P = 0$) it is sufficient to know $G(0)$. This information can be deduced from plant step data. If proportional action is required then some extra information is needed (e.g. an upper bound on $\|G\|_\infty$).

Setting $r_{step}(s) = \rho_{step}(s) = s/(s+1)$ we obtain by combining Theorem 29 and Theorem 27:

COROLLARY 30. *Under the conditions of Theorem 29 the controller K_k given by (16) is in $\mathfrak{R}_{ro}(G, r_{step}, \rho_{step})$ for all $k \in (0, k^*)$.*

If in Figure 2 the signals v_1 and v_2 are given by $v_1(s) = (r_1, ..., r_p)^T(1/s+1)$, $r_i \in \mathbb{C}$ $(i = 1, ..., p)$, and $v_2(s) = (d_1, ..., d_m)^T(1/s+1)$, $d_i \in \mathbb{C}$ $(i = 1, ...m)$, we see that the closed-loop system $\mathfrak{F}(G, K_k)$ asymptotically tracks step commands of the form $(r_1, ..., r_p)^T\theta(t)$ while it asymptotically rejects step-disturbances of the form $(d_1, ..., d_m)^T\theta(t)$.

In a finite-dimensional state-space setting results similar to Theorem 29 and Corollary 30 can be found in [Davi76] (see also [Mora85] and [Lunz89], chapter 10). They were extended to various classes of infinite-dimensional systems in [Pohj82], [PoLa83], [Pohj85], [KoPo85], [JuKo87], and [LoBO88]. Corollary 30 seems to be the most general result of its kind. The above references (apart from [Davi76]) deal with step commands and step disturbances. It is shown in [HaPo91] that asymptotic tracking and asymptotic disturbance rejection of a more general class of reference and disturbance signals can be achieved if the controller (16) is replaced by a proportional-plus-multiintegral controller.

We now turn our attention to a high-gain control problem. It is "dual" to the low-gain case in the sense that the plant is assumed to be minimum-phase, while it is allowed to be unstable. A high-gain PI-controller is applied in order to achieve stabilization, asymptotic tracking of step commands and asymptotic rejection of step disturbances. Let G be a square transfer matrix of size $m \times m$ which is meromorphic on $\mathbb{C}_\alpha$ for some $\alpha < 0$. In the following we make the assumption

Assumption (HG): There exist $\Gamma \in \mathbb{C}^{m\times m}$, $\det\Gamma \neq 0$, and $H \in (H^\infty_-)^{m\times m}$ such that

$$G^{-1}(s) = s\Gamma + H(s) \tag{17}$$

Of course, (17) is equivalent to

$$G(s) = (I + \frac{1}{s}\Gamma^{-1}H(s))^{-1}\frac{1}{s}\Gamma^{-1},$$

i.e. G can be written as the feedback interconnection of an integrator in the forward loop and a stable infinite-dimensional system in the feedback loop.

Remark 12 (i) It is not difficult to show that any meromorphic transfer matrix satisfying the assumption (HG) is in $\hat{\mathfrak{B}}^{m\times m}$.
(ii) A characterization of the condition (HG) in terms of the zeros of G and the behaviour of $sG(s)$ as $|s| \to \infty$ in $\mathbb{C}_\alpha$ (for some suitable $\alpha < 0$) is given in [LoZw92].

Consider the PI-controller

$$C_k(s) := \tilde{\Gamma}\operatorname{diag}_{1\le i\le m}\left(\frac{k_i}{s} + k_i + 1\right), \tag{18}$$

where $\tilde{\Gamma} \in \mathbb{C}^{m\times m}$, $\det\tilde{\Gamma} \neq 0$, $k = (k_1, ..., k_m)^T$, and $k_i > 0$ for all $i = 1, ..., m$. The above controller (18) was investigated in [OwCh82] when applied to finite-dimensional systems, see also [Lunz89], section 6.4. The infinite-dimensional case is studied in [LoOw87] and [LoZw92]. The following result can be found in [LoOw87].

THEOREM 31. *Suppose that $G \in \hat{\mathfrak{B}}^{m\times m}$ satisfies (HG). Then $\mathfrak{CL}(G, C_k) \in \hat{\mathcal{A}}_-^{2m\times 2m}$ for all sufficiently large k_i, $i = 1, ..., m$, provided that $\|\tilde{\Gamma}^{-1}(\tilde{\Gamma} - \Gamma)\| < 1$.*

Notice that C_k does not depend on H. The condition involving $\tilde{\Gamma}$ is trivially satisfied if $\tilde{\Gamma} = \Gamma$. However, Γ might not be known exactly to the designer.

By combining Theorem 31 and Theorem 27 we obtain

COROLLARY 32. *Under the conditions of Theorem 31 the controller C_k given by (18) is in $\mathfrak{R}_{ro}(G, \tau_{step}, \rho_{step})$ for all sufficiently large k_i ($i = 1, ..., m$) provided that $\|\tilde{\Gamma}^{-1}(\tilde{\Gamma} - \Gamma)\| < 1$.*

As a consequence the closed-loop system $\mathfrak{F}(G, C_k)$ achieves asymptotic tracking of step commands and asymptotic rejection of step disturbances if the gains k_i are sufficiently high.

Example 13 Consider the retarded system

$$\dot{x}(t) = \int_{-h}^{0} dA(\tau)x(t+\tau) + Bu(t) \tag{19a}$$

$$y(t) = Cx(t), \tag{19b}$$

where $A \in BV(-h, 0; \mathbb{R}^{n\times n})$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{m\times n}$. Setting $\hat{A}(s) := \int_{-h}^{0} dA(\tau)\exp(s\tau)$ the transfer matrix $G(s)$ of (19) is given by $G(s) = C(sI - \hat{A}(s))^{-1}B$. If $\det(CB) \neq 0$ and

$$\det\begin{pmatrix} sI - \hat{A}(s) & B \\ C & 0 \end{pmatrix} \neq 0 \text{ for all } s \in \mathbb{C}_0^{cl},$$

then G satisfies assumption (HG) with $\Gamma = (CB)^{-1}$, see [LoZw92]. Hence Theorem 31 and Corollary 32 can be applied to the retarded system (19). An analogous result holds true for a class of Volterra integrodifferential systems, see [LoZw92].

Conditions in state-space terms for (HG) to be satisfied are given in [LoZw92].

6 Conclusions

In this paper we have surveyed a number of frequency-domain results on stabilization and regulation of infinite-dimensional systems, which have been obtained within the fractional representation approach to feedback system analysis and synthesis, and we have shown how they are linked to the Pritchard-Salamon class of state-space systems. It is clear that the fractional representation approach to infinite-dimensional feedback system analysis and synthesis and its links to state-space descriptions of distributed parameter systems is an area of research in which a lot more work needs to be done. Amongst the topics requiring further investigation are:

- Computational issues: Reliable and efficient algorithms for the computation of coprime factorizations and the solutions of Bezout equations are required in order to increase the applicability of the theory. Rational approximation schemes for bounded holomorphic functions need further investigation from a computational point of view.
- Synthesis of state-space and frequency-domain methods: In Section 2 we have presented a number of results relating state-space and frequency-domain properties of Pritchard-Salamon systems. As already mentioned, there are many interesting and important systems described by partial differential equations which do not belong to the Pritchard-Salamon class. It is a challenging problem to achieve a synthesis of state-space and frequency-domain methods for a set-up which is more general than the one of Pritchard and Salamon. First steps in this direction have been taken for example in [Weis90b] and [Reba91], where the class of the so-called regular state-space systems is investigated.
- Infinitely many unstable poles: The system in Example 6(vi) has infinitely many poles on the imaginary axis. As already mentioned, it can be stabilized by proportional output feedback of the type $u = -ky$ for all $k > 0$. However, it is known from [DaLP86] that the resulting closed-loop system can be destabilized by *arbitrary* small delays in the feedback loop. Given a plant with infinitely many unstable poles, it is an interesting problem, if it is possible to construct (finite-dimensional proper) stabilizing compensators which have the property that the closed-loop system is robust with respect to small time-delays.
- Infinite-dimensional compensators: In this paper we have considered almost exclusively finite-dimensional compensators. Due to the progress of the VLSI technology, and, to a lesser extent, computer technology in general, a future exclusive emphasis on finite-dimensional stabilization and regulation seems unnatural.
- Time-varying and/or nonlinear infinite-dimensional plants: A challenging problem is the generalization of fractional representation theory to time-varying and/or nonlinear infinite-dimensional systems. Although a frequency-domain point of view is for time-varying and/or nonlinear systems no longer appropriate, it is in many cases possible to model the system as a "ratio" of two bounded causal operators on a Hilbert space. First steps in this direction have been taken for example in

[FeSa82] (chapters 8 and 9), [DeKa88], [Verm88], and [Fein92]. Notice that now the ring S of "stable" systems is in general not commutative anymore. Moreover, in the nonlinear case S fails to be right-distributive, and hence S is no longer a ring.

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